Non-commutative Hilbert Modular Symbols

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Abstract

The main goal of this paper is to construct non-commutative Hilbert modular symbols. However, we also construct commutative Hilbert modular symbols. Both the commutative and the non-commutative Hilbert modular symbols are generalizations of Manin's classical and non-commutative modular symbols. We prove that many cases of (non-)commutative Hilbert modular symbols are periods in the sense on Kontsevich-Zagier. Hecke operators act naturally on them.

Manin defines the non-commutative modilar symbol in terms of iterated path integrals. In order to define non-commutative Hilbert modular symbols, we use a generalization of iterated path integrals to higher dimensions, which we call iterated integrals on membranes. Manin examines similarities between non-commutative modular symbol and multiple zeta values both in terms of infinite series and in terms of iterated path integrals. Here we examine similarities in the formulas for non-commutative Hilbert modular symbol and multiple Dedekind zeta values, recently defined by the author, [H3], both in terms of infinite series and in terms of iterated integrals on membranes.

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Contents

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0 Introduction

Classical elliptic modular symbols were introduced by Birch [Bi] and Manin [M1] in connection with Birch - Swinnerton-Dyer conjecture for certain congruence subgroups of $SL_2(\mathbb{Z})$. We recall that a modular symbol $\{p,q\}$ is associated to a pair of cusp points $p,q\in\mathbb{P}^1(\mathbb{Q})$ on the completed upper half plane $\mathbb{H}^1\cup\mathbb{P}^1(\mathbb{Q})$. One can think of the modular symbol $\{p,q\}$ as a homology class of the geodesic connecting p and q, in $H_1(X_\Gamma,\{cusps\})$, where X_Γ is the modular curve associated to a congruence subgroup of $SL_2(\mathbb{Z})$. One can pair $\{p,q\}$ with a cusp form f by

$$\{p,q\}\times f\mapsto\int_p^q f dz$$

If f is a cusp form of weight 2 then fdz can be considered as cohomology class in $H^1(X_\Gamma)$. This gives a pairing between homology (Betti) and cohomology (de Rham) that leads to periods. Modular symbols are a useful tool applied to L-functions and computation of cohomology groups. For a review of such topics, one can consult [M3].

Their theory was developed by Manin, Drinfeld, Shokurov, Mazur, [M1, Dr, Shok, Maz]. Later the theory was extended to higher ranks by A. Ash, Rudolf, A. Borel, Gunnels [AR, AB, Gu].

Elliptic modular symbols are important tool in the study of modular forms. They are particularly useful in computations with modular forms. J. Cremona designed algorithms for computations with elliptic curves, based on modular symbols ("modular symbol algorithm"), see [Cre]. Some of the applications include computations of homology and cohomology. Also, the study of special values of L-functions became a vast area of applications of classical modular symbols, see [MS, KMS].

Later W. Stein also has contributed to the difficult area of computations with modular forms. See his excellent book [Ste], which contains both theory and computational methods. For higher rank groups one can consult the Appendix by P. Gunnels in the same book.

Manin's non-commutative modular symbol [M2] is a generalization of both the classical modular symbol and of multiple zeta values in terms of Chen's iterated integral theory in the holomorphic setting. Manin shows that the non-commutative modular symbol is a non-commutative 1-coclycle. He also shows that each of the iterated integrals on Hecke eigenforms that enter in the non-commutative modular symbol are periods.

The main goal of this paper is to construct non-commutative Hilbert modular symbols. However, we also construct an analog of the classical modular symbol for Hilbert modular varieties. Both symbols are generalizations of the corresponding Manin's constructions.

We compute explicit integrals in terms of the non-commutative Hilbert modular symbol of type **b** and present similar formulas for the recently defined multiple Dedekind zeta values (see [H3]). We prove that the iterated integrals on membranes that enter in the non-commutative modular symbol of type **c** are periods. We also give some explicit and some categorical arguments in support of a conjecture that a certain type of non-commutative Hilbert modular symbol satisfies a non-commutative 2-cocycle condition.

Before describing our results let us recall the non-commutative modular symbol of Manin [M2]. Let $\nabla = d - \sum_{i=1}^{m} X_i f_i dz$ be a connection on the upper half plane, where $f_1, \ldots, f_m$ are cusp forms and $X_1, \ldots, X_m$ are formal variables. One can think of $X_1, \ldots, X_m$ as constant square matrices of the same size.

Let J_a^b be the parallel transport of the identity matrix 1 at the point a to the matrix J_a^b at the point b. Alternatively, J_b^a can be written as a generating series of iterated path integrals of the forms $f_1 dz, \ldots, f_m dz$, (see [Ch] and [M2]), namely,

$$J_a^b = 1 + \sum_{i=1}^{m} X_i \int_a^b f_i dz + \sum_{i,j=1}^{m} X_i X_j \int_a^b f_i dz \cdot f_j dz + \cdots$$

Then $J_a^b J_b^c = J_a^c$. This property leads to the 1-cocycle $c_a^1(\gamma) = J_{\gamma a}^a$, which is the non-commutative modular symbol (see [M2] and Section 1 of this paper). If $f_1, \ldots, f_m$ are normalized cusp Hecke eigenforms then each iterated integral appearing in the generating series J_a^b is a period. In this paper we introduce both commutative and non-commutative modular symbols for Hilbert modular surfaces. As it turned out we need some new tools in comparison to the classical modular symbols. In particular for the non-commutative Hilbert modular symbol we need iterated integrals in dimension higher than one. We introduce them and study their properties in the special case of Hilbert modular surfaces. In this paper, we construct both commutative and non-commutative modular symbols for the Hilbert modular group $SL_2(\mathcal{O}_K)$. For the Hilbert modular group, one may consult [B] and [F]. In the case of Hilbert modular surface, it is not possible to repeat Manin's constructions for the non-commutative modular symbols, since the integration domain is two-dimensional over the complex numbers. Instead, we develop a new approach (Section 2), which we call iterated integrals on membranes. This is a higher dimensional analogue of iterated path integrals. In Subsection 3.7, we explore similar relations between non-commutative Hilbert modular symbols and multiple Dedekind zeta values (see [H3]).

In Section 3, we associate modular symbols for $SL_2(\mathcal{O}_K)$ to geodesic triangles and geodesic diangles (2-cells whose boundary has two vertices and two edges, which are geodesics.) We are going to explain how the geodesic triangles and the geodesic diangles are constructed. Consider 4 cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. We can map every three of them to 0, 1 and ∞ with a linear fractional transformation $\gamma \in GL_2(K)$. There is a diagonal map $\mathbb{H}^1 \to \mathbb{H}^2$, whose image Δ contains 0, 1 and ∞. We can take a pull-back of Δ with respect to the map γ in order to obtain a holomorphic (or anti-holomorphic) curve that passes through the given three points. If $\det \gamma$ is totally positive or totally negative then $\gamma^*\Delta$ is a holomorphic curve in $\mathbb{H}^2$. If $\det \gamma$ is not totally positive or totally negative (that is in one of the real embedding it is positive and in the other it is negative) then $\gamma^*\Delta$ is anti-holomorphic. This means that it is holomorphic in $\mathbb{H}^2$ if we conjugate the complex structure in one of the copies of $\mathbb{H}^1$. The same type of change of the complex structure is considered in [F].

On each holomorphic, or anti-holomorphic curve $\gamma^*\Delta$, there is a unique geodesic triangle connecting the three given points. However, if we take two of the points, we see that they belong to two geodesic triangles. Thus they belong to two holomorphic, (or anti-holomorphic) curves. Therefore, there are two geodesic connecting the two points - each lying on different holomorphic (or anti-holomorphic) curves, as faces of the corresponding geodesic triangles, defining the curves. There are two pairings that we consider: the first one is an integral of a cusp form over a geodesic triangle and the second one is an integral of a cusp form over a geodesic diangle. If we integrate a holomorphic 2-form coming from a cusp form over a geodesic triangle, we obtain 0, if the triangle lies on an holomorphic curve. Thus the only non-zero pairings come from integration of a cusp form over a diangle or over a triangle, lying on an anti-holomorphic curve.

Now let us look again at the four cusp points together with the geodesics that we have just described. We obtain four geodesic triangles, corresponding to each triple of points among the four points, and six diangles, corresponding to the six "edges" of a tetrahedron with vertices the four given points. Thus, we obtain a "tetrahedron" with thickened edges. We will use tetrahedrons with thickened edges as an intuition for a non-commutative 2-coclycle relation (see Conjecture 3.15) for the non-commutative Hilbert modular symbol, which is an analogue of Manin's non-commutative 1-cocycle relation for the non-commutative modular symbol.

Usually, the four vertices are treated as a tetrahedron and a 2-cocycle is functional on the faces, considered as 2-chains. The boundary is defined as a sum of the 2-cocycles on each of the faces (which are triangles). The boundary of the tetrahedron gives a boundary relation of a 2-cocycle.

In our case the analogue of a 2-cocycle is a functional on diangles and on triangles. And the boundary map is a sum over the faces of the thickened tetrahedron. Thus, the faces of the thickened tetrahedron are four triangles and six diangles, corresponding to the six edges of a tetrahedron.

We show that the geodesics on the boundary of a diangle or of a geodesic triangle lie on a holomorphic curves $\gamma^*\Delta$ for various elements γ with totally positive or totally negative determinant. This implies that when we take the quotient by a Hilbert modular group the holomorphic curve $\gamma^*\Delta$ becomes Hirzebruch-Zagier divisor [HZ]. Then we prove that the commutative Hilbert modular symbols paired with a cusp forms of weight $(2, 2)$ give periods in the sense of [KZ].

In order to construct a non-commutative Hilbert modular symbol, first we define a suitable generalization of iterated path integrals, which we call iterated integrals on membranes (see Section 2). We choose the word "membrane" since such integrals are invariant under suitable variation of the domain of integration.

There is a topological reason for considering non-commutative Hilbert modular symbol as opposed to only commutative one. Let us first make such comparison for the case of $SL_2(\mathbb{Z})$. The commutative modular symbol captures $H_1(X_\Gamma)$, while the non-commutative symbol captures the rational homotopy type of the modular curve X_Γ. Now, let $\tilde{X}$ be a smooth Hilbert modular surface, by which we mean the minimal desingularization of the Borel-Baily compactification due to Hirzebruch. Then the rational fundamental group of a Hilbert modular surface vanishes, $\pi_1(\tilde{X})_{\mathbb{Q}} = 0$, (see [B]). The non-commutative Hilbert modular symbol is an attempt to capture more from the rational homotopy type compared to what $H_2(\tilde{X})$ captures.

For convenience of the reader, first we define type **a** iterated integrals on membranes (Definition 2.3). They are simpler to define and more intuitive. However, they do not have enough properties. (For example, they do not have an integral shuffle relation.) Then we define the type **b** iterated integrals on membranes (Definition 2.4), which involves two permutations. Type **b** has integral shuffle relation (Theorem 2.21 (i)) and type **a** is a particular case of type **b**.

We are mostly interested in iterated integrals of type **b**. If there is no index specifying the type of iterated integral over membranes, we assume that it is of type **b**.

Similarly to Manin's approach, we define a generating series of iterated integrals over membrane of type **b** over U, which we denote by $J(U)$. We also define a shuffle product of generating series of iterated integrals over membranes of type **b** (see Theorem 2.21 part (iii)),

$$\phi(J(U_1) \times_{Sh} J(U_2)) = J(U_1 \cup U_2)$$

for disjoint manifolds with corners of dimension 2, U_1 and U_2, as subsets of $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, (see [BS]). This shuffle product generalizes the composition of generating series of iterated path integrals, namely, $J_a^b J_b^c = J_a^c$, to dimension 2. Note that similar definition is also possible in higher dimensions. Also $J(U)$ is invariant under homotopy. This allows to consider cocycles and coboundaries, where the relations use homotopy invariance and the values at the different cells can be composed via the shuffle product.

We define non-commutative Hilbert modular symbols which we call c^1 and c^2. Then c^1 is the functional J on certain geodesic diangles and c^2 is the functional J on geodesic triangles. Conjecturally, c^1 is a 1-cocycle such that if we change the base point of c^1 then c^1 is modified by a coboundary. Also conjecturally, c^2 is a 2-cocycle up to finitely many multiples of different values of c^1. Also, if we change the base point of c^2 then c^2 is modified by a coboundary up to a finitely many multiples of different values of c^1. In Subsection 3.5 we give explicit formulas in support of the interpretation of the non-commutative symbols as co-cycles.

In Subsection 3.6, we give a categorical construction, which might help proving that the non-commutative symbols as co-cycles.

In Subsection 3.7, we define multiple L-values associated to cusp forms and we compare them to multiple Dedekind zeta values (see [H3]).

We also briefly recall the Riemann zeta values and multiple zeta values (MZVs). The Riemann zeta values are defined as

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k},$$

where n is an integer. MZVs are defined as

$$\zeta(k_1,\ldots,k_m)=\sum_{0<n_1<\cdots<n_m}\frac{1}{n_1^{k_1}\ldots n_m^{k_m}},$$

where $n_1,\ldots,n_m$ are integers. The above MZV is of depth m. Riemann zeta values $\zeta(k)$ and MZV $\zeta(k_1,\ldots,k_m)$ were defined by Euler [Eu] for $m=1,2$.

The common feature of MZVs and the non-commutative modular symbol is that they both can be written as iterated path integrals (see [G1], [G2]). Moreover, Manin's non-commutative modular symbol resembles the generating series of MZV, which is the Drinfeld associator. Let us recall that the Drinfeld associator is a generating series of iterated integrals of the type J_a^b associated to the connection

$$\nabla = d - A\frac{dx}{x} - B\frac{dx}{1-x}$$

on $Y_{\Gamma(2)}=\mathbb{P}^1-\{0,1,\infty\}$. One can think of $Y_{\Gamma(2)}$ as the modular curve associated to the congruence subgroup $\Gamma(2)$ of $SL_2(\mathbb{Z})$. Then the differential forms $\frac{dx}{x}$ and $\frac{dx}{1-x}$ are Eisenstein series of weight 2 on the modular curve $Y_{\Gamma(2)}$.

Relations between MZV and modular forms have been examined by many authors. For example, Goncharov has considered a mysterious relation between MZV (multiple zeta values) of given weight and depth 3 and cohomology of $GL_3(\mathbb{Z})$ (see [G2] and [G3]), which is closely related to the cohomology of $SL_3(\mathbb{Z})$. In pursue for such a relation in depth 4, Goncharov has suggested and the author has computed the group cohomology of $GL_4(\mathbb{Z})$ with coefficients in a family of representations, [H2]. Another relation between modular forms and MZV is presented in [GKZ].

Similarly to Manin's approach, we explore relations between the non-commutative Hilbert modular symbols and multiple Dedekind zeta values (see [H3]). Let us recall multiple Dedekind zeta values. Let each of $C_1,\ldots,C_m$ be a suitable subset of the ring of integers $\mathcal{O}_K$ of a number field K. We call each of $C_1,\ldots,C_m$ a cone. Then **multiple Dedekind zeta values** are defined as

$$\zeta_{K;C_1,\ldots,C_m}(k_1,\ldots,k_m)=\sum_{\alpha_i\in C_i\text{ for }i=1,\ldots,m}\frac{1}{N(\alpha_1)^{k_1}N(\alpha_1+\alpha_2)^{k_2}\cdots N(\alpha_1+\cdots+\alpha_m)^{k_m}}.$$

The connection between non-commutative Hilbert modular symbols and multiple Dedekind zeta values is both in similarities in the infinite sum formulas and in the definition in terms of iterated integrals on membranes (see [H3]).

We consider a non-commutative Hilbert modular symbol of type **b** over one diangle and compare it with (multiple) Dedekind zeta values with summation over one discrete cone [H3]. However, in this case the two series look very different. We obtain that the multiple L-values are non- commutative modular symbols defined as J evaluated at an infinite union of diangles. We obtain that such L-values are very similar to the sum of multiple Dedekind zeta values, in the same way as the integrals in the Manin's non-commutative modular symbol are similar to the multiple zeta values (MZV). Then the sum of the multiple Dedekind zeta values is over an infinite union of cones. The idea to consider cones originated by Zagier [Z] and more generally by Shintani [Sh].

Classical or commutative modular symbols for $SL_3(\mathbb{Z})$ and $SL_4(\mathbb{Z})$ were constructed by Ash and Borel [AB] and Ash and Gunnels [AG]. For $GL_2(\mathcal{O}_K)$, where K is a real quadratic field, Gunnels and Yasaki have defined a modular symbol based on Voronoi decomposition of a fundamental domain, in order to compute the 3-rd cohomology group of $GL_2(\mathcal{O}_K)$, (see [GY]). (For the Hilbert modular group, $SL_2(\mathcal{O}_K)$, one may consult [B] and [F].) In contrast, here we use a geodesic triangulation of $\mathbb{H}^2/SL_2(\mathcal{O}_K)$. We are interested mostly in 2-cells, whose boundaries are geodesics. One of the (commutative) symbols that we define here, resembles combinatorially the symplectic modular symbol of Gunnells, [Gu]. However, the meanings of the two types of symbols and their approaches are different.

There are several different directions for further work on Hilbert modular symbols. First of all, the commutative Hilbert modular symbols have good behavior with respect to Hecke operators. It will be interesting to extend the Hecke operators to the cases of higher equal weight case (k, k). To apply Hecke operators to the Hilbert modular groups one either assumes a trivial narrow class group or one has to work with adeles. Another possible continuation of the coronet work is to extend commutative Hilbert modular symbols to the adelic setting. Then, one may try to extend these properties - higher equal weight cusp forms and Hecke operators in the adelic setting to the non-commutative Hilbert modular symbols. Hopefully, the abelian Hilbert modular symbol would lead to computational tools for cohomology of some Hilbert modular groups with coefficients in various representations.

For the non-commutative Hilbert modular symbols we expect that some of the continuations would be establishing the 2-categorical framework that define non-abelian 2-cohomology set. Such a task would also have applications to non-commutative reciprocity laws on algebraic surfaces. In dimension 1, we have a non-commutative reciprocity law as a reciprocity law for a generating series of iterated path integrals on a complex curve [H4]. In dimension 2 we have proven both the Parshin reciprocity and a new reciprocity for a 4-function local symbols [H5] defined by the author, which are particular cases in the generating series. A 2-categorical 2-nd cohomology set would capture algebraically the generating series of iterated integrals on membranes needed for the general reciprocity on algebraic surfaces.

Finally, we expect that the (non-)commutative Hilbert modular symbols would be used for studying L-functions and multiple L-functions together with their special values.

1 Manin's non-commutative modular symbol

In this section we would like to recall the definition and the main properties of the Manin's non-commutative modular symbol, (see [M2]). In his paper [M2], Manin uses iterated path integrals on a modular curve and on its universal cover - the upper half plane. Our main constructions are parallel to some extend to Manin's approach and for that reason we recall it below. However, instead of iterated path integrals we introduce a new tool - iterated integrals on membranes (see Section 2). Only this notion is adequate for studying non-commutative Hilbert modular symbols by generalizing the iteration process to higher dimensions.

1.1 Iterated path integrals

Here we recall iterated path integrals (see also [P], [Ch], [G1], [M2]). In the next Section, we are going to generalize them to iterated integrals over membranes.

Definition 1.1 *Let $\omega_1, \ldots, \omega_m$ be m holomorphic 1-forms on the upper half plane together with the cusps, $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. Let*

$$g : [0,1] \to \mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q}),$$

be a piece-wise smooth path. We define an iterated integral

$$\int_g \omega_1 \ldots \omega_m = \int \cdots \int_{0<t_1<t_2\cdots<t_m<1} g^*\omega_1(t_1) \wedge \cdots \wedge g^*\omega_m(t_m).$$

Let $X_1, \ldots, X_m$ be formal variables. Consider the differential equation

$$dF(\Omega) = F(\Omega)(X_1\omega_1 + \ldots X_n\omega_m) \tag{1.1}$$

with values in the associative but non-commutative ring of formal power series in the non-commuting variables $X_1, \ldots, X_m$ over the ring of holomorphic functions on the upper half plane. There is a unique solution with initial condition for $F(\Omega)(g(0)) = 1$, at the starting point $g(0)$, equal to 1. Then $F(\Omega)$ at the end of the path, that is at the point $g(1)$, has the value

$$F_g(\Omega) = 1 + \sum_{i=1}^m X_i \int_g \omega_i + \sum_{i,j=1}^m X_iX_j \int_g \omega_i\omega_j + \sum_{i,j,k=1}^m X_iX_jX_k \int_g \omega_i\omega_j\omega_k + \ldots \tag{1.2}$$

Using the Solution (1.2) to Equation (1.1), we prove the following theorem.

Theorem 1.2 *Let g_1 and g_2 be two paths such that the end of g_1, $g_1(1)$ is equal to the beginning of g_2, $g_2(0)$. Let g_1g_2 denote the concatenation of g_1 and g_2. Then*

$$F_{g_1g_2}(\Omega) = F_{g_1}(\Omega)F_{g_2}(\Omega).$$

Proof. The left hand side is the value of the solution of the linear first order ordinary differential equation at the point $g_2(1)$. From the uniqueness of the solution, we have that the solution along g_2 gives the same result, when the initial condition at $g_2(0)$ is $F_{g_1}(\Omega)$. That result is $F_{g_1}(\Omega)F_{g_2}(\Omega)$.

The same result can be proven via product formula for iterated integrals. We need this alternative proof in order to use it for generalization to higher dimensions.

Lemma 1.3 *(Product Formula) Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and g_1, g_2 be two paths such that the end of g_1 is the beginning of g_2, that is $g_1(1) = g_2(0)$. As before we denote by g_1g_2 the concatenation of the paths g_1 and g_2. Then*

$$\int_{g_1g_2} \omega_1 \cdots \omega_m = \sum_{i=0}^m \int_{g_1} \omega_1 \cdots \omega_i \int_{g_2} \omega_{i+1} \cdots \omega_m.$$

Proof. Let $g_1 : [0,1] \to \mathbb{C}$ and let $g_2 : [1,2] \to \mathbb{C}$. We consider the concatenation g_1g_2 as a map $g_1g_2 : [0,2] \to \mathbb{C}$ such that its restriction to the interval $[0,1]$ gives the path g_1 and its restriction to the interval $[1,2]$ gives g_2. From Definition 1.1, we have that

$$\int_{g_1g_2} \omega_1 \cdots \omega_m = \int \cdots \int_{0<t_1<\cdots<t_m<2} (g_1g_2)^*\omega_1(t_1) \wedge \cdots \wedge (g_1g_2)^*\omega_m(t_m).$$

In the domain of integration $0 < t_1 < \cdots < t_m < 2$ insert the number 1. Geometrically, we cut the simplex $0 < t_1 < \cdots < t_m < 2$ into a disjoint union of products of pairs of simplexes such that its $t_k \in [0,1]$ for $k \leq i$ and $t_k \in [1,2]$ for $k > i$. Thus, the union is over distinct values of i for $i = 0, \ldots, m$. And for each fixed i the two simplexes are $0 < t_1 < \cdots < t_i < 1$ and $1 < t_{i+1} < \cdots < t_m < 2$. Then we have

$$\begin{aligned}
&\int_{g_1g_2} \omega_1 \cdots \omega_m = \\
&= \sum_{i=0}^{n} \int \cdots \int_{0<t_1<\cdots<t_i<1;\, 1<t_{i+1}<\cdots<t_m<2} (g_1g_2)^*\omega_1(t_1) \wedge \cdots \wedge (g_1g_2)^*\omega_m(t_m) = \\
&= \sum_{i=0}^{n} \left(\int \cdots \int_{0<t_1<\cdots<t_i<1} g_1^*\omega_1(t_1) \wedge \cdots \wedge g_1^*\omega_i(t_i) \right) \times \\
&\quad \times \left(\int \cdots \int_{1<t_{i+1}<\cdots<t_m<2} g_2^*\omega_{i+1}(t_{i+1}) \wedge \cdots \wedge g_2^*\omega_m(t_m) \right) = \\
&= \sum_{i=0}^{m} \int_{g_1} \omega_1 \cdots \omega_i \int_{g_2} \omega_{i+1} \cdots \omega_m.
\end{aligned}$$

Definition 1.4 *The set of all shuffles $sh(i,j)$ is a subset of all permutations of the set $\{1, 2, \ldots, i+j\}$ such that*

$$\rho(1) < \cdots < \rho(i)$$

and

$$\rho(i+1) < \cdots < \rho(i+j).$$

Such a permutation σ is called a shuffle.

Lemma 1.5 *(Shuffle Relation) Let $\omega_1, \ldots, \omega_m$ be holomorphic 1-forms on $\mathbb{C}$ and let g be a path. Then*

$$\int_g \omega_1 \cdots \omega_i \int_g \omega_{i+1} \cdots \omega_m = \sum_{\sigma \in sh(i,m-i)} \int_g \omega_{\rho(1)} \cdots \omega_{\rho(m)},$$

where $sh(i,j)$ is the set of shuffles from Definition 1.4.

1.2 Manin's non-commutative modular symbol

Now let g be a geodesic connecting two cusps a and b in the completed upper half plane $\mathbb{H}^1 \cup \mathbb{P}^1(\mathbb{Q})$. Let $\Omega = \{f_1dz, \ldots, f_mdz\}$ be a finite set of holomorphic forms with respect to a congruence subgroup Γ of $SL_2(\mathbb{Z})$, such that $f_1, \ldots, f_m$ are cusp forms of weight 2. Let

$$J_a^b = F_g(\Omega).$$

As a reformulation of Theorem 1.2, we obtain the following.

Lemma 1.6

$$J_a^b J_b^c = J_a^c.$$

We give a direct consequence of it.

Corollary 1.7

$$J_b^a = (J_a^b)^{-1}.$$

Now we are ready to define Manin's non-commutative modular symbol. Note that there is a natural action of Γ on J_a^b. If $\gamma \in \Gamma$ then γJ_a^b is defined as $J_{\gamma a}^{\gamma b}$. If $f_1, \ldots, f_m$ are cusp forms of weight 2. Then $\omega_1 = f_1 dz, \ldots, \omega_m = f_m dz$ are forms of weight 0, that is, they are invariant forms with respect to the group Γ. Then

$$\gamma J_a^b = F_{\gamma g}(\omega_1, \ldots, \omega_m) = F_g(g^*\omega_1, \ldots, g^*\omega_m) = F_g(\omega_1, \ldots, \omega_m) = J_a^b.$$

Let Π be a subgroup of the invertible elements $\mathbb{C} \ll X_1, \ldots, X_m \gg$ with constant term 1. We extend action of Γ on J_a^b to a trivial action of Γ on Π.

Following Manin, we present the key Theorem for and Definition of the non-commutative modular symbol.

Theorem 1.8 *Put*

$$c_a^1(\gamma) = J_{\gamma a}^a.$$

Then c_a^1 represent a cohomology class in $H^1(\Gamma, \Pi)$ independent of the base point a

Proof. First c_a^1 is a cocycle:

$$dc_a^1(\beta, \gamma) = J_{\beta a}^a (\beta \cdot J_{\gamma a}^a)(J_{\beta\gamma a}^a)^{-1} = J_{\beta a}^a J_{\beta\gamma a}^{\beta a} J_a^{\beta\gamma a} = 1.$$

Second, c_a^1 and c_b^1 are homologous:

$$c_a^1(\gamma) = J_{\gamma a}^a = J_b^a J_{\gamma b}^b J_{\gamma a}^{\gamma b} = J_b^a c_b^1(\gamma)(\gamma \cdot J_b^a)^{-1}. \ \square$$

Definition 1.9 *A non-commutative modular symbol is a non-abelian cohomology class in $H^1(\Gamma, \Pi)$, with representative*

$$c_a^1(\gamma) = J_{\gamma a}^a,$$

2 Iterated integrals on membranes

Iterated integrals on membranes are a higher dimensional analogue of iterated path integrals. This technical tool was used in [H3] for constructing multiple Dedekind zeta values and in [H5] for proving new and classical reciprocity laws on algebraic surfaces. It appeared first in the author preprint [H1] for the purpose of non-commutative Hilbert modular symbols.

2.1 Definition and properties

Let $\mathbb{H}^1$ be the upper half plane. Let $\mathbb{H}^2$ be a product of two upper half planes. We are interested in the action of $GL_2(K)$, where K is a real quadratic field. This group acts on $\mathbb{H}^2$ by linear fractional transforms. It is convenient to introduce cusp points $\mathbb{P}^1(K)$ as boundary points of $\mathbb{H}^2$.

Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$, which are continuous at the cusps $\mathbb{P}^1(K)$. Let

$$g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a continuous map, which is smooth almost everywhere. Denote by F^1 and F^2 the following coordinate-wise foliations: For any $a \in [0,1]$, define the leaves

$$F^1_a = \{(t_1, t_2) \in [0,1]^2 \mid t_1 = a\}.$$

and

$$F^2_a = \{(t_1, t_2) \in [0,1]^2 \mid t_2 = a\}.$$

Definition 2.1 *We call the above map $g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ a membrane on $\mathbb{H}^2$ if it is continuous and piecewise differentiable map such that $g(F^1_a)$ and $g(F^2_a)$ belong to a finite union of holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ for all constants a.*

Similarly, we define a membrane of a Hilbert modular variety. Let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $Y_\Gamma = \mathbb{H}^2/\Gamma$, which are continuous at the cusps $\mathbb{P}^1(K)/\Gamma$. Let

$$g : [0,1]^2 \to X_\Gamma$$

be a continuous map, which is smooth almost everywhere, where $X_\Gamma = \mathbb{H}^2 \cup \mathbb{P}^1(K)$ Let $f_i : X_\Gamma \to \mathbb{P}^1(\mathbb{C})$ for $i = 1, 2$ be two algebraically independent rational functions on the Hilbert modular surface X_Γ. Denote by F^1 and F^2 the following coordinate-wise foliations: For any $a \in [0,1]$, define the leaves

$$F^1_a = \{(t_1, t_2) \in [0,1]^2 \mid t_1 = a\}.$$

and

$$F^2_a = \{(t_1, t_2) \in [0,1]^2 \mid t_2 = a\}.$$

Let also

$$P^1_x = \{P \in X_\Gamma \mid f_1(P) = x\}.$$

and

$$P^2_x = \{P \in X_\Gamma \mid f_2(P) = x\}.$$

Definition 2.2 *We call the above map $g : [0,1]^2 \to X_\Gamma$ on X_Γ if it is continuous and piecewise differentiable map such that for each a there are x_1 and x_2 such that $g(F^1_a) \subset P^1_{x_1}$ and $g(F^2_a) \subset P^2_{x_2}$.*

We define three types of iterated integrals over membranes - type **a**, type **b** and type **c**. Type **a** consists of linear iterations and type **b** is more general and involves permutations. Type **a** is less general, but more intuitive. The advantage of type **b** is that it satisfies integral shuffle relation (Theorem 2.21). In other words a product of two integrals of type **b** can be expresses as a finite sum of iterated integrals over membranes of type **b**. However, one might not be able to express a product of two integrals of type **a** as a sum of finitely many integrals of type **a**. Both type **a** and type **b** are defined on a product of two upper half planes. Type **c** is defined on a Hilbert modular surface; that is, on a quotient of a product of upper half planes by an arithmetic group, which is commensurable to $SL_2(\mathcal{O}_K)$. Type **c** also satisfies a shuffle product, that is a product of two integrals of this type can be expresses a finite sum of such.

Definition 2.3 *(Type* **a***, ordered iteration over membranes) Let*

$$g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a membrane on $\mathbb{H}^2 \cup \mathbb{P}^1(K)$*.*

$$\int_g \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^*\omega_i(t_{1,j}, t_{2,j}),$$

where

$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \ | \ 0 \leq t_{1,1} \leq \cdots \leq t_{1,m} \leq 1, \ 0 \leq t_{2,1} \leq \cdots \leq t_{2,m} \leq 1\}.$$

Definition 2.4 *(Type* **b***, 2 permutations) Let*

$$g : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$$

be a membrane on $\mathbb{H}^2 \cup \mathbb{P}^1(K)$*. Let* ρ_1, ρ_2 *be two permutations of the set* $\{1, 2 \dots, m\}$*.*

$$\int_g^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^*\omega_j(t_{1,\rho_1(j)}, t_{2,\rho_2(j)}),$$

where

$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \ | \ 0 \leq t_{1,1} \leq \cdots \leq t_{1,m} \leq 1, \ 0 \leq t_{2,1} \leq \cdots \leq t_{2,m} \leq 1\}.$$

Definition 2.5 *(Type* **c***, 2 permutations) Let*

$$g : [0,1]^2 \to X_\Gamma$$

be a membrane on the Hilbert modular surface $X_\Gamma = (\mathbb{H}^2 \cup \mathbb{P}^1(K))/\Gamma$*. Let* ρ_1, ρ_2 *be two permutations of the set* $\{1, 2 \dots, m\}$*.*

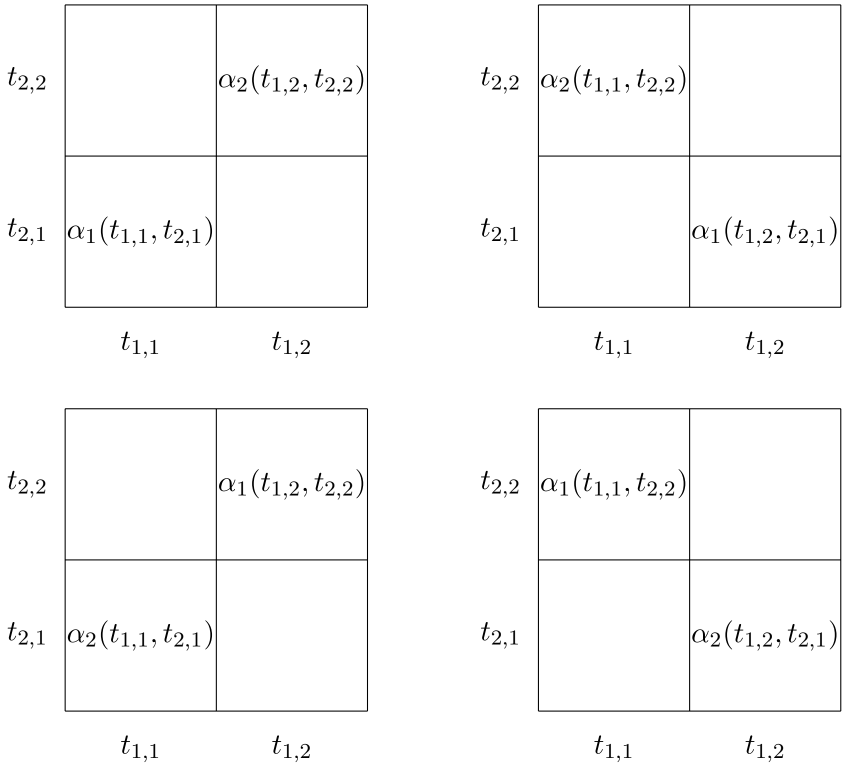
$$\int_g^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_D \bigwedge_{j=1}^m g^*\omega_j(t_{1,\rho_1(j)}, t_{2,\rho_2(j)}),$$

where

$$D = \{(t_{1,1}, \dots, t_{2,m}) \in [0,1]^{2m} \ | \ 0 \leq t_{1,1} \leq \cdots \leq t_{1,m} \leq 1, \ 0 \leq t_{2,1} \leq \cdots \leq t_{2,m} \leq 1\}.$$

Examples of iterated integral of type b: Let $\alpha_i(t_1, t_2) = g^*\omega_i(t_1, t_2)$. Denote by (1) the trivial permutation and by (12) the permutation exchanging 1 and 2.

1. The following 4 diagrams



correspond, respectively, to the integrals

$$\int_g^{(1),(1)} \omega_1 \cdot \omega_2, \qquad \int_g^{(12),(1)} \omega_1 \cdot \omega_2,$$

$$\int_g^{(12),(12)} \omega_1 \cdot \omega_2, \qquad \int_g^{(1),(12)} \omega_1 \cdot \omega_2,$$

2. The following diagram

$t_{2,3}$			$\alpha_2(t_{1,3}, t_{2,3})$
$t_{2,2}$	$\alpha_2(t_{1,1}, t_{2,2})$		
$t_{2,1}$		$\alpha_1(t_{1,2}, t_{2,1})$	
	$t_{1,1}$	$t_{1,2}$	$t_{1,3}$

corresponds to the integral

$$\int_g^{(12),(1)} \omega_1 \cdot \omega_2 \cdot \omega_3.$$

Remark 2.6 *Let us give more intuition about Definition 2.4. Each of the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ has two arguments. Consider the set of first arguments for each of the differential forms $g^*\omega_1, \ldots, g^*\omega_m$. They are ordered as follows*

$$0 < t_{1,1} < t_{1,2} < \cdots < t_{1,m} < 1, \tag{2.3}$$

(They are the coordinates of the domain D.) Since $g^*\omega_j$ depends on $t_{1,\rho_1(j)}$, we have that $t_{1,k}$ is an argument of $g^*\omega_{\rho_1^{-1}(k)}$, where $k = \rho_1(j)$. Then we can order the differential forms $g^*\omega_1, \ldots, g^*\omega_m$ according to the order of their first arguments given by the Inequalities (2.3)*, which is*

$$g^*\omega_{\rho_1^{-1}(1)}, g^*\omega_{\rho_1^{-1}(2)}, \ldots, g^*\omega_{\rho_1^{-1}(m)}$$

Similarly, we can order the differential forms $g^*\omega_1, \ldots, g^*\omega_m$, with respect to the order of their second arguments

$$g^*\omega_{\rho_2^{-1}(1)}, g^*\omega_{\rho_2^{-1}(2)}, \ldots, g^*\omega_{\rho_2^{-1}(m)}$$

We call the first ordering **horizontal** *and the second ordering* **vertical**.

Now we are going to examine homotopy of a domain of integration and how that reflects on the integral. Let $g_s : [0,1]^2 \to \mathbb{H}^2 \cup \mathbb{P}^1(K)$ be a family of membranes such that $g_s(0,0) = \infty$ and $g_s(1,1) = 0$. Assume that the parameter s is in the interval $[0,1]$.

Put $h(s, t_1, t_2) = g_s(t_1, t_2)$ to be a homotopy between g_0 and g_1. Let

$$G_s : [0,1]^{2m} \to (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m,$$

be the map

$$G_s(t_{1,1}, \ldots, t_{2,m}) = \left(g_s\left(t_{1,\sigma_1(1)}, t_{2,\sigma_2(1)}\right), g_s\left(t_{1,\sigma_1(2)}, t_{2,\sigma_2(2)}\right), \ldots, g_s\left(t_{1,\sigma_1(m)}, t_{2,\sigma(m)}\right)\right).$$

Let H be the induced homotopy between G_0 and G_1, defined by

$$H(s, t_{1,1}, \ldots, t_{2,m}) = G_s(t_{1,1}, \ldots, t_{2,m}).$$

We define diagonals in the domain $D \subset (0,1)^{2m}$, where

$$D = \{(t_{1,1}, t_{2,1}, \ldots, t_{1,m}, t_{2,m}) \in (0,1)^{2m} \mid 0 \le t_{1,1} \le t_{1,2} \le \cdots \le t_{1,m} \le 1, \\ \text{and } 0 \le t_{2,1} \le t_{2,2} \le \cdots \le t_{2,m} \le 1\}.$$

We define $D_{1,k}$ for $k = 0, \ldots, m$ as $D_{1,0} = D|_{t_{1,1}=0}$, $D_{1,k} = D|_{t_{1,k}=t_{1,k+1}}$, for $k = 1, \ldots, m-1$ and $D_{1,m} = D|_{t_{1,m}=1}$. Similarly, we define $D_{2,k}$ for $k = 0, \ldots, m$ as $D_{2,0} = D|_{t_{2,1}=0}$, $D_{2,k} = D|_{t_{2,k}=t_{2,k+1}}$, for $k = 1, \ldots, m-1$ and $D_{2,m} = D|_{t_{2,m}=1}$.

For iterated integrals of types **a** and **b**, we define diagonals in $V = (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$. We denote a generic coordinate of $V = (\mathbb{H}^2 \cup \mathbb{P}^1(K))^m$ by $(z_{1,1}, z_{2,1}, \ldots, z_{1,m}, z_{2,m})$ For $k = 1, \ldots, m-1$, let $V_{1,k} = V|_{z_{1,k}=z_{1,k+1}}$. Let also, $V_{1,0} = V|_{z_{1,1}=0}$ and $V_{1,m} = V|_{z_{1,m}=1}$. Similarly, for $k = 1, \ldots, m-1$, let $V_{2,k} = V|_{z_{2,k}=z_{2,k+1}}$. Let also, $V_{2,0} = V|_{z_{2,1}=0}$ and $V_{2,m} = V|_{z_{2,m}=1}$.

For iterated integrals of type **c**, we define "diagonals" as fibers product of schemes corresponding to certain varieties. (for fiber product of schemes one may look at the book [Har]. Occasionally, it will be more natural to realize the multiple fiber products as a finite limit in the category of schemes of finite type over $\mathbb{C}$. Let $X_{i,j} = X_\Gamma$ for $i, j = 1, \ldots, n$. Let V be the universal scheme (finite limit) that maps to X_{ij} for each i and j as a part of a commutative diagram. The commutative diagram is defined as follows: $X_{i,j}$ and $X_{i+1,j}$ both map to $\mathbb{P}^1(\mathbb{C})$ via the morphism f_1 for $1 \le i \le n-1$ and all j and $X_{i,j}$ and $X_{i,j+1}$ both map to $\mathbb{P}^1(\mathbb{C})$ via the morphism f_2 for $1 \le j \le n-1$ and all i. Let $V_{1,0}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{1,j}$, so that $f_1 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and and the corresponding $f_2 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted. Let $V_{2,0}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{i,1}$, so that $f_2 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and and the corresponding $f_1 : X_{1,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted. Let $V_{1,n}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{n,j}$, so that $f_1 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and and the corresponding $f_2 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted. Let $V_{2,n}$ be the subscheme of V defined by putting $\mathbb{P}^1(\mathbb{C})$ in the place of $X_{i,n}$, so that $f_2 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is replaced by the identity map and and the corresponding $f_1 : X_{n,j} \to \mathbb{P}^1(\mathbb{C})$ is deleted. Let also $V_{1,i}$ be the subscheme of V obtained by replacing each factor $X_{i,j} \times_{\mathbb{P}^1(\mathbb{C})} X_{i+1,j}$ by the corresponding diagonal for fixed i and for all j. And finally, let $V_{2,j}$ be the subscheme of V obtained by replacing each factor $X_{i,j} \times_{\mathbb{P}^1(\mathbb{C})} X_{i,j+1}$ by the corresponding diagonal for fixed j and all i.

Theorem 2.7 *(Homotopy Invariance Theorem I) The iterated integrals on membranes from Definition 2.4 (of type* **b***) are homotopy invariant, when the homotopy preserves the boundary of the membrane.*

Proof. Let

$$\Omega = \bigwedge_{j=1}^{m} \omega_j(z_{1,\sigma_1(j)}, z_{2,\sigma_2(j)}).$$

Note that Ω is a closed form, since ω_i is a form of top dimension. By Stokes Theorem, we have

$$0 = \int_{s=0}^{s=1} \int_D H^* d\Omega = \tag{2.4}$$

$$= \int_D G_1^* \Omega - \int_D G_0^* \Omega \pm \tag{2.5}$$

$$\pm \int_{s=0}^{s=1} \sum_{k=1}^{m-1} \left(\int_{D_{1,k}} \pm \int_{D_{2,k}} \right) H^* \Omega \tag{2.6}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,0}} \pm \int_{D_{2,0}} \right) H^* \Omega \tag{2.7}$$

$$\pm \int_{s=0}^{s=1} \left(\int_{D_{1,m}} \pm \int_{D_{2,m}} \right) H^* \Omega \tag{2.8}$$

We want to show that the difference in the terms in (2.5) is zero. It is enough to show that each of the terms (2.6), (2.7) and (2.8) are zero. If $z_{1,k} = z_{1,k+1}$ for types **a** and **b** (or on $V_{1,k}$ for type **c**), then the wedge of the corresponding differential forms will vanish. Thus the terms in (2.6) are zero. If $z_1 = 0$ then $dt_1 = 0$, defined via the pull-back H^*. Then the terms (2.7) are equal to zero. Similarly, we obtain that the last integral (2.8) vanishes. □

Let A be a manifold with corners of dimension 2 in $[0,1]^2$. We recall the domain of integration

$$D = \{(t_{1,1}, \ldots, t_{2,m}) \in [0,1]^{2m} \mid 0 \leq t_{1,1} \leq \cdots \leq t_{1,m} \leq 1, \ 0 \leq t_{2,1} \leq \cdots \leq t_{2,m} \leq 1\}.$$

Let us define

$$A^D = \{(t_{1,1}, \ldots, t_{2,m}) \in D \mid (t_{1,i}, t_{2,j}) \in A \text{ for } i, j = 1, \ldots, m\}$$

Let ρ_1 and ρ_2 be two permutations of m elements. We define

$$G(t_{1,1}, \ldots, t_{2,m}) = \left(g\left(t_{1,\rho_1(1)}, t_{2,\rho_2(1)}\right), g\left(t_{1,\rho_1(2)}, t_{2,\rho_2(2)}\right), \ldots, g\left(t_{1,\rho_1(m)}, t_{2,\rho(m)}\right)\right).$$

as a function on A^D. Recall

$$\Omega = \bigwedge_{j=1}^{m} \omega_j(z_{1,\rho_1(j)}, z_{2,\rho_2(j)}).$$

Definition 2.8 *With the above notation, we define an iterated integral over a membrane of type* **b** *restricted to a domain U, where $U = g(A)$ as*

$$^b\int_{g,U}^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_{A^D} G^* \Omega.$$

Now we are going to define iterated integrals of type **c**.

Definition 2.9 *Let* $\Omega_0 = \bigwedge_{i,j=1}^{m} \Omega_{i,j}$*, where* $\Omega_{i,j} = \omega_i \delta_{i,j}$ *on* $X_{i,i} \equiv X_\Gamma$*, and where* $\Omega_{i,j} = 1$ *for* $i \neq j$*. Let* $in : X \to \prod_{i,j=1}^{n} X_{i,j}$ *be the inclusion of the finite limit into the product of the schemes* $X_{i,j}$*. Let* $\Omega = in^* \Omega_0$*.*

With this definition of Ω*, we define iterated integrals of type* **c** *restricted to a domain* U*, where* $U = g(A)$ *as*

$$^{c}\int_{g,U}^{\rho_1,\rho_2} \omega_1 \cdots \omega_m = \int_{A^D} G^* \Omega.$$

Let A_1 and A_2 be two manifolds with corners, with a common component of the boundary as subsets of $[0,1]^2$. Let $A = A_1 \cup A_2$. Let s be a map of sets with values 1 or 2,

$$s : \{1, \ldots, m\} \to \{1, 2\}.$$

We define a certain set A_s^D as a subset of A^D in the following way: Consider the image of the map G. It has m coordinates. The first coordinate, $g\left(t_{1,\rho_1(1)}, t_{2,\rho_2(1)}\right)$, will be restricted to the set $A_{s(1)}$. The second coordinate, $g\left(t_{1,\rho_1(2)}, t_{2,\rho_2(2)}\right)$, will be restricted to $A_{s(2)}, \ldots$ and the last m coordinate $g\left(t_{1,\rho_1(m)}, t_{2,\rho_2(m)}\right)$ will be restricted to $A_{s(m)}$. Formally, this can be written as

$$A_s^D = \{(t_{1,1}, \ldots, t_{2,m}) \in A^D \mid (t_{1,\rho_1(i)}, t_{2,\rho_2(i)}) \in A_{s(i)} \text{ for } i = 1, \ldots, m\}.$$

Note that the image of the map s is 1 or 2.

Definition 2.10 *With the above notation, we define an iterated integral of type* **b** *or* **c** *over two domains* U_1 *and* U_2*, where* $U_i = g(A_i)$ *and* $U = U_1 \cup U_2$ *by*

$$\int_{g,U,s}^{\rho_1\rho_2} \omega_1 \cdots \omega_m = \int_{A_s^D} G^* \Omega. \tag{2.9}$$

For type **b** *we have that* U *is in* $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ *and for type* **c** *we have that* U *is in* $X_\Gamma = (\mathbb{H}^2 \cup \mathbb{P}^1(K))/\Gamma$*.*

Again we examine homotopy of iterated integrals on membranes. Now we restrict the domain of integration to a manifold with corners A as a subset of $[0,1]^2$. Assume that for the boundary of a domain A, denoted by ∂A, we have that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$ for type **b** and in X_Γ for type **c**. We call a *complex boundary* of $g(\partial A)$ the minimal union of complex analytic (holomorphic) curves such that $g(\partial A)$ belongs to a finite union of complex analytic curves in $\mathbb{H}^2$ for type **b** and in X_Γ for type **c**.

Theorem 2.11 *(Homotopy Invariance Theorem II) Iterated integrals over membranes are homotopy invariant with respect to a homotopy that changes the boundary* ∂U *of the domain of integration* U*, so that the boundary varies on a finite union of complex analytic curves.*

Proof. Assume that $g_0(\partial A)$ and $g_1(\partial A)$ have the same complex boundary. Let h be a homotopy between g_0 and g_1, such that for each value of s we have that $h(s, \partial A)$ has the same complex boundary as $h(0, \partial A) = g_0(\partial A)$. Let $A \subset B$ be a strict inclusion of disks.

Identify $B - A^\circ$ with $A \times [0,1]$. Let $i : B - A^\circ \to [0,1] \times \partial A$. Here A° is the interior of A and ∂A is the boundary of A. Let $\tilde{g}_0$ be a map from B to $\mathbb{H}^2$ so that $\tilde{g}_0(a) = g_0(a)$ for $a \in A$ and $\tilde{g}_0(b) \in h(i(b))$. Since the restriction of pull-back $(\tilde{g}_0^*\omega_i)|_{B-A} = 0$ is mapped to a finite union of complex curves, then it vanishes. Therefore

$$\int_A g_0^*\Omega = \int_B \tilde{g}_0^*\Omega. \tag{2.10}$$

Let $\tilde{g}_1$ be a membrane from B defined by $\tilde{g}_1(a) = g_1(a)$ for $a \in A$ and $\tilde{g}_1(b) = \tilde{g}_1(a)$ for i(b)=(s,a). (Note that $i(b) \in [0,1] \times \partial A$.) Again

$$\int_A g_1^*\Omega = \int_B \tilde{g}_1^*\Omega. \tag{2.11}$$

However, the boundary of B is mapped to the same set (point-wise) by both $\tilde{g}_0$ and $\tilde{g}_1$. Moreover, the homotopy between g_0 and g_1 extends to a homotopy between $\tilde{g}_0$ and $\tilde{g}_1$ that respects the inclusion into the complex boundary. Thus by Theorem 2.7, we have that

$$\int_B \tilde{g}_0^*\Omega = \int_B \tilde{g}_1^*\Omega.$$

Using Equations (2.10) and (2.11), we complete the proof of this theorem. □

2.2 Generating series

We are going to define two types of generating series - type **a** and type **b**, corresponding to the iterated integrals on membranes of type **a** and type **b**.

Definition 2.12 *(Type a) Let A be a domain in $\mathbb{R}^2$. Let g be a membrane. Let $U = g(A) \subset \mathbb{H}^2$. And let $\omega_1, \ldots, \omega_m$ be holomorphic 2-forms on $\mathbb{H}^2$. We define a generating series of type* **a** *by*

$$J^a(U) = 1 + \sum_{k=1}^{\infty} \sum_{c:\{1,\ldots,k\}\to\{1,\ldots,m\}} X_{c(1)} \otimes \cdots \otimes X_{c(k)} \int_{g,U} \omega_{c(1)} \ldots \omega_{c(k)},$$

where $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ be a map of sets.

Consider a map of sets $c : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and two permutations ρ_1, ρ_2 of $\{1, 2, \ldots, k\}$. We call two triples (c', ρ_1', ρ_2') and $(c'', \rho_1'', \rho_2'')$ equivalent if they are in the same orbit of the permutation group S_k. That is, $(c'', \rho_1'', \rho_2'') \sim (c', \rho_1', \rho_2')$ if for some $\tau \in S_k$ we have $c'' = c'\tau^{-1}$, $\rho_1'' = \rho_1'\tau^{-1}$ and $\rho_2'' = \rho_2'\tau^{-1}$. Then for each equivalence class of a triple (c, ρ_1, ρ_2), we can associate a unique pair $(c \circ \rho_1, c \circ \rho_2)$, (which are precisely the indices of the X-variables and Y-variables in (2.14) and (2.15), respectively.) The reason for using such an equivalence is that the integral in (2.15) is invariant by the above action of $\tau \in S_k$ on the triple (c, ρ_1, ρ_2).

Definition 2.13 *(Ring R, values of the generating series) The values of the generation series of iterated integrals on membranes will be in a ring R, which we define as follows. Let R_0 be the quotient of the ring of formal power series*

$$R_0 = \mathbb{C} \ll X_1, Y_1, \ldots, X_m, Y_m \gg /I$$

modulo the two-sided ideal I generated by $X_iY_j - Y_jX_i$ for $i,j = 1,\dots,m$. Let $R \subset R_0$ be the subring of formal power series whose monomials have the following property: in every monomial of R', X_i occurs as many times as Y_i.

Definition 2.14 *(Type b) We define the generating series of type* **b** *on U by*

$$J^b(U) = 1 + \sum_{k=1}^{\infty} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1))} \otimes \dots \otimes X_{c(\rho_1^{-1}(k))} \otimes \tag{2.12}$$

$$\otimes Y_{c(\rho_2^{-1}(1))} \otimes \dots \otimes Y_{c(\rho_2^{-1}(k))} \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \dots \omega_{c(k)}, \tag{2.13}$$

where the second summation is over all maps of sets $c : \{1,\dots,k\} \to \{1,\dots,m\}$ and all permutations ρ_1, ρ_2 of k elements, up to the above equivalence.

Let Y_Γ be a Hilbert modular surface. Let α and β be two rational functions on Y_Γ. We denote by D the union of the divisors $(\alpha)_\infty$ and $(\beta)_\infty$ at infinity. Let $F : Y_\Gamma - D \to \mathbb{C}^2$ be defined as $F(y) = (\alpha(y), \beta(y))$. Let $g : (0,1)^2 \to Y_\Gamma - D$ be a membrane, so that the composition $F \circ g$ respects the coordinate-wise foliations. Consider the differential forms ω_i from the Definition of type b. They are invariant under the action of the arithmetic group Γ. Thus, we can treat them as differential forms on the Hilbert modular variety Y_Γ.

Definition 2.15 *(Type c) With the new definition of a membrane g, and a domain $U \subset Y_\Gamma$, we define the generating series of type* **c** *by*

$$J^c(U) = 1 + \sum_{k=1}^{\infty} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1))} \otimes \dots \otimes X_{c(\rho_1^{-1}(k))} \otimes \tag{2.14}$$

$$\otimes Y_{c(\rho_2^{-1}(1))} \otimes \dots \otimes Y_{c(\rho_2^{-1}(k))} \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \dots \omega_{c(k)}, \tag{2.15}$$

where the second summation is over all maps of sets $c : \{1,\dots,k\} \to \{1,\dots,m\}$ and all permutations ρ_1, ρ_2 of k elements, up to the above equivalence.

Definition 2.16 *(Ring R', genrating series $J(U_1,U_2)$) We define a generating series of iterated integrals on two disjoint domain U_1 and U_2 (see Definition 2.10). Let $U_i = g(A_i)$.*

$$J(U_1,U_2) = 1 + \sum_{k=1}^{\infty} \sum_{s:\{1,\dots,k\}\to\{1,2\}} \sum_{(c,\rho_1,\rho_2)/\sim} X_{c(\rho_1^{-1}(1)),s(1)} \otimes \dots \otimes X_{c(\rho_1^{-1}(k)),s(k)} \otimes \tag{2.16}$$

$$\otimes Y_{c(\rho_2^{-1}(1)),s(1)} \otimes \dots \otimes Y_{c(\rho_2^{-1}(k)),s(k)} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \dots \omega_{c(k)}, \tag{2.17}$$

The generating series takes values in a ring R' defined as follows. Let R'_0 be a quotient of the ring of formal power series

$$R'_0 = \mathbb{C} \ll X_{1,1}, X_{1,2}, Y_{1,1}, Y_{1,2}, \dots, X_{m,1}, X_{m,2}, Y_{m,1}, Y_{m,2} \gg /I',$$

where I' is the two-sided ideal generated by Lie commutators of X with any subscript and Y with any subscript. Let R' be a subring of R'_0 with the property: in every monomial of R', $X_{i,j}$ occurs as many times as $Y_{i,j}$.

Lemma 2.17 *Let $\phi : R' \to R$ be a homomorphism of rings defined by $\phi(X_{i,1}) = \phi(X_{i,2}) = X_i$ and $\phi(Y_{i,1}) = \phi(Y_{i,2}) = Y_i$. If $U = U_1 \cup U_2$ is in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ then*

$$\phi(J(U_1, U_2)) = J^b(U).$$

If $U = U_1 \cup U_2$ is in X_Γ then

$$\phi(J(U_1, U_2)) = J^c(U).$$

Proof. After applying the homomorphism ϕ the formal variables on the left hand side become independent of the map s. Therefore, we have to examine what happens when we sum over all possible maps s. The map $s(i)$ is 1 or 2. Their meaning is the following: If $s(i) = 1$ then we restrict the form $g^*\omega_{c(i)}$ to A_1 (instead of to A). Similarly, if $s(i) = 2$, we restrict $g^*\omega_{c(i)}$ to A_2. If we add both choices, restriction to A_1 and restriction to A_2, then we obtain restriction of $g^*\omega_{c(i)}$ to $A = A_1 \cup A_2$. Thus, we obtain the formula

$$\sum_{s:\{1,\dots,k\}\to\{1,2\}} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \dots \omega_{c(k)} = \int_{g,U}^{\rho_1,\rho_2} \omega_{c(1)} \dots \omega_{c(k)}.$$

We do the same for every monomial in R. That proves the above Lemma for the generating series. $\square$

2.3 Shuffle product of generating series

The regions of integration that we are mostly interested in will be ideal diangles, that is, a 2-cell whose boundary has two vertices and two edges, and ideal triangles. All other regions that we will deal with are going to be a finite union of ideal diangles and ideal triangles. The first type of decomposition is based on a union of two diangles with a common vertex. The second type of decomposition will be based on two of the cells (diangles, or triangles) with a common edge.

Let g_1 and g_2 be two membranes. Let $P = (0, 0)$ and $Q = (1, 1)$ be the vertices of a diangle A and $Q = (1, 1)$ and $R = (2, 2)$ are the two points of a diangle B as subsets of $\mathbb{R}^2$. Assume that A lies within the rectangle with vertices $(0, 0), (0, 1), (1, 1), (1, 0)$. Similarly, assume that B lies within the rectangle $(1, 1), (1, 2), (2, 2), (2, 1)$. Let $U = g(A)$ and $V = g(B)$.

Theorem 2.18 *(i)*

$$\int_{g,U\cup V} \omega_1 \dots \omega_m = \sum_{j=0}^{m} \int_{g,U} \omega_1 \dots \omega_j \int_{g,V} \omega_{j+1} \dots \omega_m;$$

(ii) The generating series of type **a** *from Definition 2.12 satisfies the following property:*

$$J^a(g; A \cup B; \Omega) = J^a(g; A; \Omega) J^a(g; B; \Omega).$$

The proof of the first statement is essentially the same as the combinatorial proof for composition of path, when one considers iterated path integrals (see Lemma 1.3). The second statement is combining all compositions into generating series (see Definition 2.12), resembling the Manin's approach for non-commutative modular symbol.

For generating series of type **b**, we have a similar statement.

Definition 2.19 *Let ρ' and ρ'' be two permutations of the sets $\{1,\dots,i\}$, $\{i+1,\dots,i+j\}$, respectively. We define the permutation $\rho'^{-1}\cup\rho''^{-1}$ of $\{1,\dots,i+j\}$, which acts on $\{1,\dots,i\}$ as ρ'^{-1} and on $\{i+1,\dots,i+j\}$ as ρ''^{-1}. We define the set of shuffles of two given permutations, denoted by $sh(\rho',\rho'')$, as the set of all permutations ρ of the set $\{1,2,\dots,i+j\}$ such that ρ^{-1} is the composition of a shuffle of sets $\tau\in sh(i,j)$ (see Definition 1.4) and with $\rho'^{-1}\cup\rho''^{-1}$. That is,*

$$\rho^{-1}=\tau\circ(\rho'^{-1}\cup\rho''^{-1}).$$

Definition 2.20 *We define a shuffle of two monomials*

$$M'=X_{c'(\rho_1'^{-1}(1))}\otimes\cdots\otimes X_{c'(\rho_1'^{-1}(i))}\otimes Y_{c'(\rho_2'^{-1}(1))}\otimes\cdots\otimes Y_{c'(\rho_2'^{-1}(i))}\int_{g,U'}^{\rho_1',\rho_2'}\omega_{c'(1)}\dots\omega_{c'(i)}$$

and

$$M''=X_{c''(\rho_1''^{-1}(1))}\otimes\cdots\otimes X_{c''(\rho_1''^{-1}(j))}\otimes Y_{c''(\rho_2''^{-1}(1))}\otimes\cdots\otimes Y_{c''(\rho_2''^{-1}(j))}\int_{g,U''}^{\rho_1'',\rho_2''}\omega_{c''(i+1)}\dots\omega_{c''(i+j)},$$

where ρ_1' and ρ_2' are permutations of $\{1,\dots,i\}$ and c' is a map of sets $c':\{1,\dots,i\}\to\{1,\dots,m\}$, and ρ_1'' and ρ_2'' are permutations of $\{i+1,\dots,i+j\}$ and c'' is a map of sets $c'':\{i+1,\dots,i+j\}\to\{1,\dots,m\}$. By a shuffle product of the monomials M' and M'', we mean the following sum

$$\begin{aligned}M'\times_{Sh}M''=\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}&X_{c(\rho_1^{-1}(1)),s(1)}\otimes\cdots\otimes X_{c(\rho_1^{-1}(i+j)),s(i+j)}\otimes\\&\otimes Y_{c(\rho_2^{-1}(1)),s(1)}\otimes\cdots\otimes Y_{c(\rho_2^{-1}(i+j)),s(i+j)}\int_{g,U'\cup U'',s}^{\rho_1,\rho_2}\omega_{c(1)}\dots\omega_{c(i+j)},\end{aligned}$$

where $c:\{1,\dots,i+j\}\to\{1,\dots,m\}$ such that the map c restricted to the first i elements is c' and c restricted to the last j elements is c''. Here the maps s takes the value 1 on the set $c^{-1}\{1,\dots,i\}=c'^{-1}\{1,\dots,i\}$ and it takes the value 2 on the set $c^{-1}\{i+1,\dots,i+j\}=c''^{-1}\{i+1,\dots,i+j\}$.

Theorem 2.21 *(Shuffle product) For iterated integrals of type* **b** *and the corresponding generating series, we have the following shuffle relations:*

(i)

$$\int_{g,U}^{\rho_1',\rho_2'}\omega_1\dots\omega_j\int_{g,U}^{\rho_1'',\rho_2''}\omega_{j+1}\dots\omega_m=\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}\int_{g,U}^{\rho_1,\rho_2}\omega_1\cdots\omega_m \tag{2.18}$$

(ii)

$$\int_{g,U'}^{\rho_1',\rho_2'}\omega_1\dots\omega_j\int_{g,U''}^{\rho_1'',\rho_2''}\omega_{j+1}\dots\omega_m=\sum_{\rho_1\in sh(\rho_1',\rho_1''),\rho_2\in sh(\rho_2',\rho_2'')}\int_{g,U,s}^{\rho_1,\rho_2}\omega_1\cdots\omega_m, \tag{2.19}$$

where s is a map from $\{1,\dots,m\}$ to $\{1,2\}$ so that $\{1,\dots,j\}$ are mapped to 1 and the remaining elements are mapped to 2.

(iii)

$$\phi(J^b(U') \times_{Sh} J^b(U'')) = J^b(U' \cup U'') \tag{2.20}$$

(iv)

$$\phi(J^c(U') \times_{Sh} J^c(U'')) = J^c(U' \cup U'') \tag{2.21}$$

Proof. For part (i), it is useful to consider the two orders of differential forms, given in Remark 2.6. Note that we need to order the forms both horizontally and vertically in the terminology of Remark 2.6. Let us consider first the horizontal order. That is the order with respect to the first variables of the differential forms $g^*\omega_{\rho_1'^{-1}(1)}, \ldots, g^*\omega_{\rho_1'^{-1}(j)}$ and $g^*\omega_{\rho_1''^{-1}(j+1)}, \ldots, g^*\omega_{\rho_1''^{-1}(m)}$, corresponding to the two integrals on the left hand side of Equation (2.18). In order to arrange both of the above orderings in one sequence of increasing first arguments, we need to shuffle them (similarly to a shuffle of a deck of cards.) That leads to $\rho_1 \in sh(\rho_1', \rho_1'')$ (see Definition 2.19). We proceed similarly, with the second arguments and the permutations ρ_2', ρ_2'' and ρ_2.

For part (ii) apply the equality from part (i) when the differential forms $g^*\omega_1, \ldots, g^*\omega_j$ are multiplied by the function $\mathbf{1_{A'}}$ defined by

$$\mathbf{1_{A'}}(x) = \begin{cases} 1 & \text{for } x \in A' \\ 0 & \text{for } x \notin A' \end{cases}$$

and the differential forms $g^*\omega_{j+1}, \ldots, g^*\omega_m$ are multiplied by $\mathbf{1_{A''}}$. For part (iii), we are going to establish similar relation among generating series as elements of R'. Applying the homomorphism $\phi : R' \to R$ from Lemma 2.17, we obtain desired equality. Every monomial from $J(U_1)$ is of the form

$$M' = X_{c'(\rho_1'^{-1}(1))} \otimes \cdots \otimes X_{c'(\rho_1'^{-1}(i))} \otimes Y_{c'(\rho_2'^{-1}(1))} \otimes \cdots \otimes Y_{c'(\rho_2'^{-1}(i))} \int_{g,U'}^{\rho_1',\rho_2'} \omega_{c'(1)} \ldots \omega_{c'(i)}$$

and similarly every monomial from $J(U_2)$ is of the form

$$\begin{aligned} M'' = & X_{c''(\rho_1''^{-1}(1))} \otimes \cdots \otimes X_{c''(\rho_1''^{-1}(j))} \otimes \\ & \otimes Y_{c''(\rho_2''^{-1}(1))} \otimes \cdots \otimes Y_{c''(\rho_2''^{-1}(j))} \int_{g,U''}^{\rho_1'',\rho_2''} \omega_{c''(i+1)} \ldots \omega_{c''(i+j)}, \end{aligned}$$

where ρ_1' and ρ_2' are permutations of $\{1, \ldots, i\}$ and c' is a map of sets $c' : \{1, \ldots, i\} \to \{1, \ldots, m\}$, and ρ_1'' and ρ_2'' are permutations of $\{i+1, \ldots, i+j\}$ and c'' is a map of sets $c'' : \{i+1, \ldots, i+j\} \to \{1, \ldots, m\}$. We take the shuffle product of the monomials M' and M'' (see Definition 2.20)

$$\begin{aligned} M' \times_{Sh} M'' = & \sum_{\rho_1 \in sh(\rho_1',\rho_1''), \rho_2 \in sh(\rho_2',\rho_2'')} X_{c(\rho_1^{-1}(1)),s(1)} \otimes \cdots \otimes X_{c(\rho_1^{-1}(i+j)),s(i+j)} \otimes \\ & \otimes Y_{c(\rho_2^{-1}(1)),s(1)} \otimes \cdots \otimes Y_{c(\rho_2^{-1}(i+j)),s(i+j)} \int_{g,U,s}^{\rho_1,\rho_2} \omega_{c(1)} \ldots \omega_{c(i+j)}, \end{aligned}$$

where the map s takes the value 1 on the set $c^{-1}\{1, \ldots, i\}$ and takes the value 2 on the set $c^{-1}\{i+1, \ldots, i+j\}$. It determines the map s uniquely.

In order to complete the proof, we have to show that every monomial in $J(U_1,U_2)$ can be obtained in exactly one way as a result (on the right hand side) of a shuffle product of a pair of monomials (M_1,M_2) from $J(U_1)$ and $J(U_2)$. Every monomial from $J(U_1,U_2)$ is characterized by two permutation ρ_1,ρ_2, and two maps of sets $c:\{1,\dots,k\}\rightarrow\{1,\dots,m\}$ and $s:\{1,\dots,k\}\rightarrow\{1,2\}$. Let i be the number of elements in $s^{-1}(1)$ and j be the number of elements in $s^{-1}(2)$. Then $i+j=k$. Then i is the number of differential forms among $g^*\omega_{c(1)},\cdots,g^*\omega_{c(k)}$, which are restricted to the set A_1. The remaining j differential forms are restricted to A_2. Also, every permutation ρ_1 can be written in an unique way as a composition of a shuffle $\tau_1\in sh(i,j)$ and two disjoint permutations ρ_1' and ρ_1'' of i and of j elements, respectively (see Definition 2.19). Similarly, ρ_2 can be written in a unique way as a product of a shuffle $\tau_2\in sh(i,j)$ and two disjoint permutation ρ_2' and ρ_2''. The map of sets c_1 is defined as a restriction of the map c to the image of ρ_1'. Similarly, the map c_2 is defined as a restriction of the map c to the image of ρ_1''. Now we can define the monomials M' and M'' in $J(U_1)$ and $J(U_2)$, based on the triples ρ_1',ρ_2',c' and ρ_1'',ρ_2'',c'', respectively. Such monomials are unique. One can show that the shuffle product of M' and M'' contains the monomial in $J(U_1,U_2)$, that we started with, exactly once. The proof of part (iii) is complete after applying Lemma 2.17. $\square$

3 Hilbert modular symbols

In this Section, we recall the Hilbert modular group and its action on the product of two upper half planes. Then we define commutative Hilbert module symbol, (Subsection 3.1) and its pairing with the cohomology of the Hilbert modular surface, (Subsection 3.2). In Subsections 3.3 and 3.4, we define the non-commutative Hilbert module symbols (Definition 3.13) as a generating series of iterated integrals over membranes of type **b**. We also examine relations among the non-commutative Hilbert modular symbols (Theorem 3.12), which we interpret as cocycle conditions or as a difference by a coboundary (Theorem 3.14). In Subsection 3.5, we consider a two-category C with a sheaf J on C. Then the non-commutative Hilbert modular symbol a sheaf on a two-category. This is done in order to give a plausible approach to defining a suitable non-commutative cohomology set. In Subsection 3.6, we make explicit computations and compare them to computations for multiple Dedekind zeta values.

3.1 Commutative Hilbert modular symbols

In this Subsection, we define a commutative Hilbert modular symbol, using geodesics, geodesic triangles and geodesic diangles. Then, we prove certain relations among the commutative Hilbert modular symbols, which are generalized to relations among non-commutative Hilbert modular symbols (Subsection 3.4).

Let $K=\mathbb{Q}(\sqrt{d})$ be a real quadratic extension of $\mathbb{Q}$. Then the ring of integers in K is

$$\mathcal{O}_K=\begin{cases}\mathbb{Z}[\frac{1+\sqrt{d}}{2}] & \text{for } d=1 \mod 4,\\ \\ \mathbb{Z}[\sqrt{d}] & \text{for } d=2,3 \mod 4.\end{cases}$$

Then $\Gamma = SL_2(\mathcal{O}_K)$ is called a Hilbert modular group. Let $\gamma \in \Gamma$. We recall the action of γ on a product of two upper half planes $\mathbb{H}^2$. Let

$$\gamma = \gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

Let a_2, b_2, c_2, d_2 be the Galois conjugate of a_1, b_1, c_1, d_1, respectively. Let us define γ_2 by

$$\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Let $z = (z_1, z_2)$ be any point of the product of two upper half planes $\mathbb{H}^2$.

For an element $\gamma \in GL_2(K)$, we define the following action: If $\det\gamma$ is totally positive, that is $\det\gamma_1 > 0$ and $\det\gamma_2 > 0$, then the action of γ on $z = (z_1, z_2) \in \mathbb{H}^2$ is essentially the same as for $\gamma \in SL_2(K)$, namely,

$$\gamma z = (\gamma_1 z_1, \gamma_2 z_2),$$

where

$$\gamma_1 z_1 = \frac{a_1 z_1 + b_1}{c_1 z_1 + d_1} \quad \text{and} \quad \gamma_2 z_2 = \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}$$

are linear fractional transforms. If $\det\gamma$ is totally negative, that is, $\det\gamma_1 < 0$ and $\det\gamma_2 < 0$, then we define

$$\gamma z = \left(-\frac{a_1 \overline{z}_1 + b_1}{c_1 \overline{z}_1 + d_1}, -\frac{a_2 \overline{z}_2 + b_2}{c_2 \overline{z}_2 + d_2}\right).$$

Similarly if one of the embeddings of $\det\gamma$ is positive and the other is negative, for example, $\det\gamma_1 > 0$ and $\det\gamma_2 < 0$, such as $\det\gamma = \sqrt{d}$, then

$$\gamma z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, -\frac{a_2 \overline{z}_2 + b_2}{c_2 \overline{z}_2 + d_2}\right).$$

We add cusp points $\mathbb{P}^1(K)$ to $\mathbb{H}^2$. Then the quotient $SL_2(\mathcal{O}_K)\backslash(\mathbb{P}^1(K) \cup \mathbb{H}^2)$ is compact.

We are going to examine carefully geodesics joining the cusps 0, 1 and ∞.

Let z_0, z_1, z_∞ be three distinct cusp points. There is a unique element $\gamma \in PGL_2(K)$ such that that send z_0, z_1 and z_∞ to $0, 1$ and ∞, respectively.

Let

$$i : \mathbb{H} \to \mathbb{H}^2$$

$$i(x) = (x, x)$$

be the diagonal map and Δ be its image. Consider the Hirzebruch-Zagier divisor $X = \gamma^*\Delta$. It is an analytic curve that passes through the points z_0, z_1 and z_∞. Then X is a holomorphic curve in $\mathbb{H}^2$ if $\det\gamma$ is totally positive or totally negative. If $\det\gamma$ is not totally positive or totally negative, then X is a holomorphic curve in $\mathbb{H}^1 \times \overline{\mathbb{H}}^1 \cup \mathbb{P}^1(K)$, in other words it is anti-holomorphic curve in $\mathbb{H}^2$, such as $z_1 = -\overline{z}_2$. Let $\Delta_X = \gamma^*\Delta$ be the pull-back of the geodesic triangle Δ between the points $0, 1, \infty$ in the analytic curve X.

Given four points on the boundary in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, we are tempted to consider them as vertices of a geodesic tetrahedron in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$, whose faces are triangles of the type Δ_X. However, there is one problem that we encounter: Two distinct cusps could be connected by two different geodesics in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. In particular, two triangles from the faces of the "tetrahedron" might not have a common edge, but only two common vertices. Thus, we are led to consider a *thickened tetrahedron* with two types of faces on the boundary: the first type is an ideal triangle that we have just defined and the other type is an ideal diangle - a union of geodesics connecting two fixed points, which has the homotopy type of a disc with two vertices and two edges. The two edges of an ideal diangle in the boundary of a thickened tetrahedron correspond to the two geodesics connecting the same two cusps, where two geodesics belong to the geodesic triangles that have the two cusps in common.

Let us describe a diangle $D_{0,\infty;1,\alpha}$ whose two vertices are 0 and ∞ and whose two sides are geodesics that belong to each of the ideal triangles $0, 1, \infty$ and $0, \alpha, \infty$. The geodesic l_0 between the points 0 and ∞ that lie on the geodesic triangle $0, 1, \infty$ can be parametrized in the following way: $\{(it, it) \,|\, t \in \mathbb{R},\, t \geq 0\} \subset Im(\mathbb{H}) \times Im(\mathbb{H})$. Here by $Im(\mathbb{H})$ we mean the imaginary part of the upper half plane. The element $\gamma \in \Gamma$ that sends $0, \alpha, \infty$ to $0, 1, \infty$ is $\gamma = \left(\begin{array}{cc} \alpha^{-1} & 0 \\ 0 & 1 \end{array} \right)$ Then $(\alpha^{-1})^*(it, it) = (|\alpha_1| it, |\alpha_2| it)$. Therefore, the geodesic l_α between the points 0 and ∞ that lie on the geodesic triangle $0, \alpha, \infty$ can be parametrized in the following way $\{(|\alpha_1| it, |\alpha_2| it) \,|\, t \in \mathbb{R},\, t \geq 0\} \subset Im(\mathbb{H}) \times Im(\mathbb{H})$. Then, we define the diangle $D_{0,\infty;1,\alpha}$ as the two dimensional region in $Im(\mathbb{H}) \times Im(\mathbb{H})$ between the lines l_0 and l_α. We also consider the diangle with orientation. If $|\alpha_1| > |\alpha_2|$ then it is positively oriented. If the inequality is reversed then the diangle is negatively oriented; if $|\alpha_1| = |\alpha_2|$ then it is a degenerate diagle, which consists of a single geodesic. All other diangles that we will consider are translates of $D_{0,\infty;1,\alpha}$ via the action of any element $\gamma \in PGL_2(K)$.

Lemma 3.1 *(i) Each geodesic triangle Δ_X lies either on a holomorphic curve or on an anti-holomorphic curve.*

(ii) Each geodesic in a geodesic triangle Δ_X belongs both to a holomorphic curve and to an anti-holomorphic curve.

Part (i) follows from the construction of a geodesic triangle before the lemma. For part (ii), consider the following: Let $\Delta(0, 1, \infty)$ be the geodesic triangle in the diagonal of $\mathbb{H}^2$ connecting the points 0, 1 and ∞. It is a holomorphic curve. Thus, a geodesic $\{(it, it) \in \mathbb{H}^2 \mid t > 0\}$, connecting the points 0 and ∞ as a face of the geodesic triangle $\Delta(0, 1, \infty)$ lies on a holomorphic curve. Now consider the geodesic triangle $D(0, \sqrt{d}, \infty)$. It lies on an anti-holomorphic curve in $\mathbb{H}^2$, by which we mean a complex curve in $\mathbb{H}^2$, where we have taken the complex conjugate complex structure in one of the upper half planes. Since, the linear fractional transform that sends $D(0, \sqrt{d}, \infty)$ to $D(0, 1, \infty)$ does not have totally positive (or totally negative) determinant. Explicitly, the linear fractional transform that sends $(0, \sqrt{d}, \infty)$ to $(0, 1, \infty)$ is

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}$$

Then

$$(\gamma_1, \gamma_2) = \left(\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{d} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \right)$$

We have $\gamma_1(it) = \frac{1}{\sqrt{d}} it$ and $\gamma_2(it) = -\frac{1}{\sqrt{d}} \overline{it} = \gamma_1(it)$. Then the same geodesic (it, it) belongs to the anti-holomorphic curve given by the pull-back of the diagonal with respect to the linear fractional map γ. Thus, we obtain that the geodesic (it, it), connecting 0 and ∞, belongs to both a holomorphic curve and an anti-holomorphic curve. Similarly, any translate of the geodesic (it, it) via a linear fractional map from $GL_2(K)$ would belong to both a holomorphic curve and an anti-holomorphic curve. That proves part (ii). $\Box$

Definition 3.2 *Let p_1, p_2, p_3, p_4 be cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. To each triple of points p_1, p_2, p_3, we associate the geodesic triangle $\{p_1, p_2, p_3\}$ with coefficient 1 as an element of the singular chain complex in $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$. Also, to each quadruple of points p_1, p_2, p_3, p_4, we associate the geodesic diangle between the two geodesic connecting p_1 and p_2 so that the first geodesic is a face of the geodesic triangle $\{p_1, p_2, p_3\}$ and the second geodesic is a face of the geodesic triangle $\{p_1, p_2, p_4\}$. We denote such diangle by $\{p_1, p_2; p_3, p_4\}$. We call the geodesic triangle $\{p_1, p_2, p_3\}$ and the geodesic diangle $\{p_1, p_2; p_3, p_4\}$, considered as elements of $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$,* **commutative Hilbert modular symbols**.

Theorem 3.3 *The commutative Hilbert modular symbols modulo the boundary of singular 3-chains, $\partial C_3(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$ satisfy the following properties:*

1. If σ is a permutation of the set $\{1, 2, 3\}$ then

$$\{p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}\} = sign(\sigma)\{p_1, p_2, p_3\}.$$

2. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve of the type $\gamma^*\Delta$ then

$$\{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} = \{p_1, p_2, p_4\} + \{p_1, p_3, p_4\}.$$

To each four points p_1, p_2, p_3, p_4, we associate a diangle with vertices p_1 and p_2. Let $\{p_1, p_2; p_3, p_4\}$ be the corresponding symbol.

3. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve of the type $\gamma^*\Delta$ then

$$0 = \{p_1, p_2; p_3, p_4\}.$$

4. For every district four points p_1, p_2, p_3, p_4, we have the following relations based on the orientation of the domain

$$\{p_2, p_1; p_3, p_4\} = \{p_1, p_2; p_4, p_3\} = -\{p_2, p_1; p_4, p_3\} = -\{p_1, p_2; p_3, p_4\}.$$

5. For every five points p_1, p_2, p_3, p_4, p_5, we have

$$\{p_1, p_2; p_3, p_4\} + \{p_1, p_2; p_4, p_5\} = \{p_1, p_2; p_3, p_5\}.$$

6. We also have relation between the two types of commutative Hilbert modular symbols. For every four distinct points $p_1, p_2; p_3, p_4$, we have

$$\begin{aligned} 0 =& \{p_1, p_2, p_3\} + \{p_2, p_3, p_4\} - \\ & - \{p_1, p_2, p_4\} - \{p_1, p_3, p_4\} + \\ & + \{p_1, p_2; p_3, p_4\} + \{p_2, p_3; p_1, p_4\} + \{p_3, p_1; p_2, p_4\} + \\ & + \{p_3, p_4; p_1, p_2\} + \{p_1, p_4; p_2, p_3\} + \{p_2, p_4; p_3, p_1\}. \end{aligned}$$

Proof. Part 1 follows from orientation of the simplex in singular homology. Part 2 is an equality induced by two different triangulations on a holomorphic (or anti-holomoprhic) curve with 4 vertices. In that setting the diangles are trivial, which proves Part 3. Part 4 follows from orientation of the diangle. Part 5 corresponds to a union of two geodesic diangles with a common face, given by a third geodesic diangle. Part 5 will be used for a non-commutative 1-cocycle relation for the non-commutative Hilbert modular symbol (see Conjecture 3.14). Part 6 is a boundary relation for the boundary of a thickened tetrahedron. By a thickened tetrahedron we mean a union of four geodesic triangles corresponding to each triple of points among the four points p_1, p_2, p_3, p_4 together with six geodesic diangles that correspond to the area between the faces of the geodesic triangles. They correspond exactly to the thickening of the six edges of a tetrahedron. □

Part 6 will be used to derive explicit formulas for the non-commutative Hilbert modular symbol of type **c'** resembling a non-commutative 2-cocycle relation (see Conjecture 3.15).

3.2 Pairing of the modular symbols with cohomology

In this subsection, we consider pairings between commutative Hilbert modular symbols and cusp forms. In some cases, we prove that such pairings give periods in the sense of [KZ].

We are interested in holomorphic cusp forms with respect to Γ. Equivalently, we can consider the holomorphic 2-forms on $\tilde{X}$, which is the minimal smooth algebraic compactification of X [Hirz]. At this point we should distinguish between geodesic triangles p_1, p_2, p_3 that lie on a holomorphic curve or on anti-holomorphic curve. The reason for distinguishing is that a holomorphic 2-form restricted to a holomorphic curve vanishes. The way to distinguish the two type of geodesic triangles is the following: Let γ be a linear fractional transform that sends the points p_1, p_2, p_3 to $0, 1, \infty$. If $\det\gamma$ is totally positive or totally negative then the geodesic triangle p_1, p_2, p_3 lies on a holomorphic curve. If $\det\gamma$ is not totally positive nor totally negative then the geodesic triangle p_1, p_2, p_3 lies on an anti-holomorphic curve.

Definition 3.4 *Let $M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$ be the span of the Hilbert modular symbols $\{p_1, p_2, p_3\}$ and $\{p_1, p_2; p_3, p_4\}$ as a subspace of the singular chain $C_2(\mathbb{H}^2 \cup \mathbb{P}^1(K), \mathbb{Q})$.*

We define the following pairing

$$<,>: M_2(\mathbb{H}^2 \cup \mathbb{P}^1(K)) \times S_{2,2}(\Gamma) \to \mathbb{C},$$

by setting

$$<\{p_1,p_2,p_3\}, f dz_1 \wedge dz_2> = \int_{\{p_1,p_2,p_3\}} f dz_1 \wedge dz_2$$

for geodesic triangles and

$$<\{p_1,p_2;p_3,p_4\}, f dz_1 \wedge dz_2> = \int_{\{p_1,p_2;p_3,p_4\}} f dz_1 \wedge dz_2$$

for geodesic diangles.

We are going to use that a Hilbert modular surface $X(\mathbb{C})$ can be realized as the complex points of an arithmetic surface defined over a number field F.

Theorem 3.5 *The image of the above pairing is a period over a number field F, when we integrate a normalized cusp Hecke eigenform f of weight $(2,2)$; (for Hecke eigenforms, see [Shi], [BBDDDV]).*

Proof. From Lemma 3.1 (ii), the boundary of the geodesic triangles of the diangles are geodesics that lie on holomorphic curves in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$. Therefore, in the quotient by the congruence group Γ, the geodesic lie in Hirzebruch-Zagier divisor on the Hilbert modular surface. Thus, we integrate a closed algebraic differential 2-form, (that is, a global differential 2-form with algebraic coefficients), on the Hilbert modular surface, with boundaries Hirzebruch-Zagier divisors.

Conjecture 3.6 *Let $f \in S_{k,k}(\Gamma)$ be a normalized cusp Hecke eigenform of weight (k,k). Then*

$$\int_{\{p_1,p_2,p_3\}} f dz_1 \wedge dz_2$$

for geodesic triangles and

$$\int_{\{p_1,p_2;p_3,p_4\}} f dz_1 \wedge dz_2$$

for geodesic diangles are periods.

Theorem 3.5 is a proof of Conjecture 3.6 for the case of cusp form of weight $(2,2)$.

3.3 Iteration - revisited

We have defined iterated integrals on diangles in Definitions 2.14, 2.15. However, these definitions have to be extended to other domains of integration in order to consider iterated integrals on geodesic triangles.

A consequence of the results from this Subsection is the following:

Theorem 3.7 *Iterated integrals of type* **c** *on a geodesic diangle and on a geodesic triangle of algebraic differential 2-forms on a Hilbert modular surface are periods in the sense of Kontsevich-Zagier.*

Before giving the proof, we need definitions of several objects as well as their properties. In the process, we will be able to extend the definition on iterated integral on membrane when the domain of integration is a geodesic triangle.

For type **b**, in Definition 2.14, we have a map $g : U \to \mathbb{H}^2$ that sends the two $\mathbb{R}$-foliations on U into two coordinate-wise $\mathbb{C}$-foliations of $\mathbb{H}^2$. The same definition does not work when the domain U is a geodesic triangle. The reason is that a geodesic triangle is either a holomorphic curve or (an anti-holomorphic curve). In both cases, a pull-back of one leaf to the geodesic triangle is a point not a line, (which is the case for the diangles).

In order to extend Definitions 2.14, 2.15 to the case when the domain U is a geodesic triangle, we are going to construct a new spaces using fiber product multiple times.

Now, we are going to define a space Y_n associated to an iterated integral on n 2-forms on $\mathbb{H}^2$. We are going to use fiber products (see [Har]). Let p_1 and p_2 be the projections of $\mathbb{H}^2$ on the first and the second component, respectively. Define $X_{ij} = \mathbb{H}^2$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. (One should think of the component X_{ij} as the complexification of the real coordinated (s_i, t_j).) Let $C_i = \mathbb{H}$ for $1 \leq i \leq n$ and $C'_j = \mathbb{H}$ for $1 \leq j \leq n$. Let

$$X_j = X_{1j} \times_{C'_j} X_{2j} \times_{C'_j} \cdots \times_{C'_j} X_{nj}.$$

(X_j corresponds to the variable t_j) Then

$$X_j \subset X_{1j} \times X_{2j} \times \cdots \times X_{nj}.$$

Let also

$$P_j = (p_1, \cdots, p_1) : X_{1j} \times X_{2j} \times \cdots \times X_{nj} \to C_1 \times \cdots \times C_n$$

Let $P_j^\circ = P_j|_{X_j}$ be the restriction of P_j to the subset X_j. We define Y_n as the fiber product of $X_1, \ldots, X_n$ with respect to the morphisms $P_1^\circ, \ldots, P_n^\circ$ over the base $C_1 \times \cdots \times C_n$, namely

$$Y_n = X_1 \times_C \cdots \times_C X_n, \tag{3.22}$$

where $C = C_1 \times \cdots \times C_n$. Note that X_j is isomorphic to X_{j+1}. Let Z_j be the subspace of Y_n defined by setting the j- and the $(j+1)$-component of $Y_n = X_1 \times_C \cdots \times_C X_n$ to be equal. (The space Z_j corresponds to a boundary components obtained by letting $t_j = t_{j+1}$.) Similarly, we could have defined Y_n by defining first

$$X'_i = X_{i1} \times_{C_i} X_{i2} \times_{C_i} \cdots \times_{C_i} X_{in}$$

(X'_i corresponds to s_i) so that

$$X'_i \subset X_{i1} \times X_{i2} \times \cdots \times X_{in}$$

Let

$$P'_i = (p_2, \ldots, p_2) : X_{i1} \times X_{i2} \times \cdots \times X_{in} \to C'_1 \times \cdots \times C'_n$$

Define $P_i'^\circ = P'_i|_{X'_i}$ to be the restriction of P'_i to X'_i. We define Y_n as the fiber product of $X'_1, \ldots, X'_n$ with respect to the morphisms $P_1'^\circ, \ldots, P_n'^\circ$ over the base $C'_1 \times \cdots \times C'_n$, namely

$$Y_n = X'_1 \times_{C'} \cdots \times_{C'} X'_n, \tag{3.23}$$

where $C' = C'_1 \times \cdots \times C'_n$. Similarly we define Z'_i to be the subspace of Y_n defined by setting the i- and the $(i+1)$-component of $Y_n = X'_1 \times_{C'} \cdots \times_{C'} X'_n$ to be equal. (The space Z'_i corresponds to a boundary components obtained by letting $s_i = s_{i+1}$.)

We have given two Definitions 3.22 and 3.23 of the space Y_n. In the two definitions we have only exchanged the role of p_1 and p_2. We will prove that both definitions lead to the same object in the case of $n = 2$. The general case is left to the reader.

Lemma 3.8 *For $n = 2$, the two Definitions 3.22 and 3.23 define isomorphic objects Y_2.*

Proof. The space Y_2 can be defined as a finite limit (in a categorial sense) of a diagram in the following way. Consider the commutative diagram

$$\begin{tikzcd}
 & & C'_1 & & \\
 & X_{11} \arrow[ur] \arrow[dl] & & X_{12} \arrow[ul] \arrow[dr] & \\
C_1 & & & & C_2 \\
 & X_{21} \arrow[ul] \arrow[dr] & & X_{22} \arrow[ur] \arrow[dl] & \\
 & & C'_2 & &
\end{tikzcd}$$

For any space W such that

$$\begin{tikzcd}
 & & C'_1 & & \\
 & X_{11} \arrow[ur] \arrow[dl] & & X_{12} \arrow[ul] \arrow[dr] & \\
C_1 & & W \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & C_2 \\
 & X_{21} \arrow[ul] \arrow[dr] & & X_{22} \arrow[ur] \arrow[dl] & \\
 & & C'_2 & &
\end{tikzcd} \tag{3.24}$$

commutes, we have that the maps $f_{ij} : W \to X_{ij}$ factor through $g_{ij} : Y_2 \to X_{ij}$ so that $f_{ij} = g_{ij} \circ h$, for some $h : W \to Y_2$ and Y_2 is part of the commutative diagram

$$
\begin{tikzcd}
 & & C_1' & & \\
 & X_{11} \arrow[ur] \arrow[dl] & & X_{12} \arrow[ul] \arrow[dr] & \\
C_1 & & Y_2 \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & C_2 \\
 & X_{21} \arrow[ul] \arrow[dr] & & X_{22} \arrow[ur] \arrow[dl] & \\
 & & C_2' & &
\end{tikzcd}
$$

In order to prove this universal property of Y_2 we follow the first definition of Y_2. It leads to the commutative diagram

$$
\begin{tikzcd}
 & & C_1' & & \\
 & X_{11} \arrow[ur] \arrow[dl] & X_{11} \times_{C_1'} X_{12} \arrow[l] \arrow[r] & X_{12} \arrow[ul] \arrow[dr] & \\
C_1 & & W \arrow[u] \arrow[d] & & C_2 \\
 & X_{21} \arrow[ul] \arrow[dr] & X_{21} \times_{C_2'} X_{22} \arrow[l] \arrow[r] & X_{22} \arrow[ur] \arrow[dl] & \\
 & & C_2' & &
\end{tikzcd}
$$

Then we have that $X_1 = X_{11} \times_{C_1'} X_{12}$ maps to $C = C_1 \times C_2$ and also $X_2 = X_{21} \times_{C_2'} X_{22}$ maps to $C = C_1 \times C_2$. Thus the maps from W to any element of the diagram factors through $Y_2 = X_1 \times_C X_2$. Similarly, W factors through $X_1' \times_{C'} X_2'$, where $X_1' = X_{11} \times_{C_1} X_{21}$, $X_2' = X_{12} \times_{C_2} X_{22}$ and $C' = C_1' \times C_2'$. Since both $X_1 \times_C X_2$ and $X_1' \times_{C'} X_2'$ are universal objects with respect to the diagram (3.24), we have that they are isomorphic. □

Now, we return to the initial question of this subsection, namely, how to iterated over a geodesic triangle so that it is consistent with the current definition of iteration over a diangle.

For an n-fold iteration of n 2-forms of types **b** or **c**, we have to specify a domain $U \subset \mathbb{H}^2$, $dim_{\mathbb{R}} U = 2$ and a pair of permutations ρ_1 and ρ_2 of n elements. We make an essential assumption that the boundary of $U \subset \mathbb{H}^2$, denoted by ∂U, projected onto the Hilbert modular surface Y_Γ lies on a finite union of Hirzebruch-Zagier divisors. We will denote the finite union of such Hirzebruch-Zagier divisors by HZ.

Let

$$P_{\rho_1,\rho_2} : X_{11} \times \cdots \times X_{nn} \to X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}$$

be a projection to n of the factors. Let $U_{ij} \cong U$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Let

$$I : U_{\rho_1(1)\rho_2(1)} \times \cdots \times U_{\rho_1(n)\rho_2(n)} \to X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}$$

be the induced from the product of inclusion of the domains $U \to X$. We will use the following notation

$$\underline{U}^\rho = U_{\rho_1(1)\rho_2(1)} \times \cdots \times U_{\rho_1(n)\rho_2(n)}$$

and

$$\underline{X}^\rho = X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}.$$

Then the map I becomes

$$I : \underline{U}^\rho \to \underline{X}^\rho$$

Let

$$J : Y_n \to \underline{X}^\rho$$

be the composition of the natural inclusion $Y_n \to X_{11} \times \cdots \times X_{nn}$ and the projection P_{ρ_1,ρ_2}. Then we define the domain of integration to be

$$\underline{U}^\rho_{Y_n} = \underline{U}^\rho \times_{\underline{X}^\rho} Y_n,$$

which is the fiber product of the maps I and J. Since $I : \underline{U}^\rho \to \underline{X}^\rho$ is an inclusion, we have that the induced map

$$\underline{U}^\rho_{Y_n} \to Y_n$$

is an inclusion.

In the above constructions, we have used a parallel between type **b** and type **c** of iterated integrals on membranes. THe following definition allows us to extend in some sense the two types when the domain of integration is an ideal triangle

Definition 3.9 *(iterated integrals on membranes of types* **b'** *or* **c'***) For any manifold with corners of dimension* 2 *on the Hilbert modular variety, we define an iterated integral*

$$\int_U^{\Sigma_n(\rho_1,\rho_2)} (f_1 dz_1 \wedge dz_2) \cdots (f_n dz_1 \wedge dz_2) = \int_{\underline{U}^\rho_{Y_n}} J^*(f_1 dz_1 \wedge dz_2, \ldots, f_n dz_1 \wedge dz_2), \quad (3.25)$$

where $f_k dz_1 \wedge dz_2$ is a form defined on $X_{\rho_1(k)\rho_2(k)}$, for $1 \leq k \leq n$. If Y_n and $\underline{U}^\rho_{Y_n}$ are constructed in the setting of type **b** *iterated integrals on membranes, then the above definition is of iterated integral on membranes of type* **b'**. *Similarly, if Y_n and $\underline{U}^\rho_{Y_n}$ are constructed in the setting of type* **c** *iterated integrals on membranes, then the above definition is of iterated integral on membranes of type* **c'**.

If U is a diangle, then the relation of the above integral to the ones defined by iterated integrals over membranes is the following. The integral

$$\int_U^{\Sigma_n(\rho_1,\rho_2)} (f_1 dz_1 \wedge dz_2) \cdots (f_n dz_1 \wedge dz_2)$$

is the sum of the integrals from Definitions 2.14 or 2.15, namely, the sum

$$\sum_{\rho\in\Sigma_n}\int_U^{(\rho\rho_1,\rho\rho_2)}(f_1dz_1\wedge dz_2)\cdots(f_ndz_1\wedge dz_2)$$

over the orbit of the diagonal action of the permutation group Σ_n on any chosen pair of permutations (ρ_1,ρ_2).

Proposition 3.10 *(Properties of the iterated integral 3.25) (a) the iterated integral 3.25 is well-defined when U is an ideal triangle both for types* **b** *and* **c**;

(b) the iterated integral 3.25 for type **c** *is a period if U is an ideal triangle or an ideal diangle, when $f_1,\dots,f_n$ are normalized Hecke eigenforms of weight $(2,2)$.*

(c) the iterated integral 3.25, both for types **b** *and* **c**, *is homotopy invariant with respect to a homotopy that varies within the divisors*

$$J^{-1}(X_{\rho_1(1)\rho_2(1)}\times\cdots\times p_1^{-1}(q_i)\times\cdots\times X_{\rho_1(n)\rho_2(n)}),$$

where q_i is a point of $X_{\rho_1(i)\rho_2(i)}$ for fixed i and $p_1: X_{\rho_1(i)\rho_2(i)}\to C$; or a homotopy that varies within the divisors

$$J^{-1}(X_{\rho_1(1)\rho_2(1)}\times\cdots\times p_2^{-1}(q_i)\times\cdots\times X_{\rho_1(n)\rho_2(n)}),$$

where q_i is a point of $X_{\rho_1(i)\rho_2(i)}$ for fixed i and $p_2: X_{\rho_1(i)\rho_2(i)}\to C'$.

Proof. (a) The integral 3.25 is well defined for any two dimensional submanifold with corners of the Hilbert modular variety. ([BS])

(b) The iterated integral 3.25 is a period since:

(1) a Hilbert modular variety can be defined over a number field

(2) the normalized Hecke eigenforms $f_1,\dots,f_n$ of weight (2,2) can be realized as algebraic differential forms on the Hilbert modular variety;

(3) the boundary of the region of integration $\overline{U}^{\rho}_{Y_n}$ is a divisor on Y_n, namely,

$$\bigcup_{i=1}^{n} HZ_i,$$

where

$$HZ_i=J^{-1}(X_{\rho_1(1)\rho_2(1)}\times\cdots\times HZ\times\cdots\times X_{\rho_1(n)\rho_2(n)})$$

is a divisor of Y_n obtained by a pull-back of a divisor whose i-th component is a Hirzebruch-Zagier divisor HZ and the rest of the factors are $X_{\rho_1(k)\rho_2(k)}$ for $k\neq i$.

(c) The proof is essentially the same as the one of Theorem 2.7. □

The domain $\underline{U}_{Y_n}$ might be cut into disconnected components by the divisors Z_i's and Z_j''s. In order to choose a connected component we need to define another region of integration. Recall: For the case of intreated integrals on membranes of type **b**, $p_1:\mathbb{H}^2\to C$ and $p_2:\mathbb{H}^2\to C'$ be projections onto the first and the second component with $C\cong\mathbb{H}$ and $C'\cong\mathbb{H}$.

For the case of iterated integrals on membranes of type **c**, $p_1 = \alpha_1 \circ \pi$ and $p_2 = \alpha_2 \circ \pi$ are compositions of the map from the universal cover to the Hilbert modular surface

$$\pi : \mathbb{H}^2 \to X_\Gamma$$

and

$$\alpha_1, \alpha_2 : X_\Gamma \to \mathbb{P}^1$$

be two algebraically independent rational functions on the Hilbert modular surface and $C_i \cong \mathbb{P}^1$ and $C'_j \cong \mathbb{P}^1$ for $1 \leq i \leq n$ and $1 \leq j \leq n$.

Let $q_0, q_1, r_0, r_1 \in \mathbb{P}^1$ be points. Let each of Q_0, Q_1, R_0 and R_1 be a connected component of $p_1^{-1}(q_0)$, of $p_1^{-1}(q_1)$, of $p_2^{-1}(r_0)$, and of $p_2^{-1}(r_1)$, respectively.

Let $V \to \mathbb{H}^2$ be a domain in $\mathbb{H}^2$ such that its boundaries lie on the union

$$Q_0 \cup Q_1 \cup R_0 \cup R_1,$$

and its interior does not meet the union $Q_0 \cup Q_1 \cup R_0 \cup R_1$. We define the divisors Z_0, Z_n, Z'_0, Z'_n of Y_n, which will have the following meaning: Z_0 will be the beginning of the integration of the t_1 variable ($t_1 = 0$), Z_n will be the end of the integration of the t_n variable ($t_n = 1$), Z'_0 will be the beginning of the integration of the s_1 variable ($s_1 = 0$), and Z'_n will be the end of the integration of the s_n variable ($s_n = 1$). We define them as the a fiber product

$$Z_0 = Q_0 \times_C X_2 \times_C \cdots \times_C X_n,$$

$$Z_n = X_1 \times_C \cdots \times_C X_{n-1} \times_C Q_1,$$

$$Z'_0 = R_0 \times_{C'} X'_2 \times_{C'} \cdots \times_{C'} X'_n$$

and

$$Z'_n = X'_1 \times_{C'} \cdots \times_{C'} X'_{n-1} \times_{C'} R_1.$$

We will use the following notation

$$\underline{V}^\rho = V_{\rho_1(1)\rho_2(1)} \times \cdots \times V_{\rho_1(n)\rho_2(n)}$$

and

$$\underline{X}^\rho = X_{\rho_1(1)\rho_2(1)} \times \cdots \times X_{\rho_1(n)\rho_2(n)}.$$

Then the map I' becomes

$$I' : \underline{V}^\rho \to \underline{X}^\rho$$

Let

$$J' : Y_n \to \underline{X}^\rho$$

be the composition of the natural inclusion $Y_n \to X_{11} \times \cdots \times X_{nn}$ and the projection P_{ρ_1,ρ_2}. Then we define the domain of integration to be

$$\underline{V}^\rho_{Y_n} = \underline{V}^\rho \times_{\underline{X}^\rho} Y_n,$$

which is the fiber product of the maps I' and J'. Since $I' : \underline{U}^\rho \to \underline{X}^\rho$ is an inclusion, we have that the induced map

$$\underline{V}^\rho_{Y_n} \to Y_n$$

is an inclusion.

Then the divisors $Z_0, Z_1, \ldots, Z_{n-1}, Z_n$ and $Z'_0, Z'_1, \ldots, Z'_{n-1}, Z'_n$ cut out from $\underline{V}_{Y_n}$ a product of two n-simplices, corresponding to the region where $\{0 \leq s_1 \leq \cdots \leq s_n \leq 1\} \times \{0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$ is embedded. Denote by $\overline{V}^{\rho}_{Y_n}$ the connected components of $\overline{V}^{\rho}_{Y_n}$ that contains the image of $\{0 \leq s_1 \leq \cdots \leq s_n \leq 1\} \times \{0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$ under the map g from Definition 2.5.

Proof. (of Theorem 3.7) We consider the type of iterated integrals defined in Definition 2.15. Using the above notation the domain of integration is U, where $U \subset V$. We define

$$\overline{U}^{\rho}_{Y_n} = \underline{U}^{\rho}_{Y_n} \cap \overline{V}^{\rho}_{Y_n}$$

Then the boundary of $\overline{U}^{\rho}_{Y_n}$ lies on the union of divisors

$$\partial \overline{U}^{\rho}_{Y_n} \subset \left(\bigcup_{i=1}^{n} Z_i \right) \cup \left(\bigcup_{j=1}^{n} Z'_j \right).$$

The normalized Hecke eigenforms of weight (2,2) can be realized as algebraic differential forms on the Hilbert modular variety. Then the iterated integrals on a membrane of type **c** over the domain U is a period since:

(1) a Hilbert modular variety can be defined over a number field

(2) the normalized Hecke eigenforms $f_1, \ldots, f_n$ of weight (2,2) can be realized as algebraic differential forms on the Hilbert modular variety;

(3) the boundary of the region of integration $\overline{U}^{\rho}_{Y_n}$ is a divisor on Y_n, namely,

$$\left(\bigcup_{i=1}^{n} Z_i \right) \cup \left(\bigcup_{j=1}^{n} Z'_j \right).$$

□

3.4 Generating series and relations

In this Subsection we examine the generating series of iterated integrals on membranes (of types **b'** or **c'**), evaluated at geodesic triangles and geodesic diangles. We prove relations among them. Most importantly, the generating series J will be used in Subsection 3.5 for Defining non-commutative Hilbert modular symbols. Moreover, the relations that we prove in this Section, will be interpreted as cocycles or as coboundaries of the the non-commutative Hilbert modular symbols satisfy in Subsection 3.5.

Definition 3.11 *Let $f_1, \ldots, f_m$ be m cusp forms with respect to a Hilbert modular group Γ. Let $f_1 dz_1 \wedge dz_2, \ldots, f_m dz_1 \wedge dz_2$ be the corresponding differential forms, defining the generating series. Let $J(p_1, p_2, p_3)$ be the generating series J evaluated at the geodesic triangle with vertices p_1, p_2, p_3. Let $J(p_1, p_2; p_3, p_4)$ be the generating series J evaluated at the geodesic diangle $\{p_1, p_2; p_3, p_4\}$.*

Both $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ will be called non-commutative Hilbert modular symbols after the action of the arithmetic group is included (see Definition 3.13).

Theorem 3.12 *The generating series J is one of the following types* **b**, **c**, **b'** *or* **c'**. *Note that $J(p_1, p_2; p_3, p_4)$ is defined for all the types, while $J(p_1, p_2, p_3)$ is defined only for types* **b'** *or* **c'**. *Then the generating series $J(p_1, p_2, p_3)$ and $J(p_1, p_2; p_3, p_4)$ satisfy the following relations:*

1. If σ is a permutation of the set $\{1, 2, 3\}$ then

$$J(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}) = J^{sign(\sigma)}(p_1, p_2, p_3).$$

2. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve of the type $\gamma^*\Delta$ then

$$\begin{aligned} 1 =& J(p_1, p_2, p_3)J(p_2, p_3, p_4) \\ & J(p_2, p_1, p_4)J(p_1, p_4, p_3) \end{aligned}$$

and

3. If p_1, p_2, p_3, p_4 are four points on the same holomorphic (or anti-holomorphic) curve of the type $\gamma^*\Delta$ then

$$1 = J(p_1, p_2; p_3, p_4).$$

4. For every four points p_1, p_2, p_3, p_4, we have the following relation based on the orientation of the domain

$$\begin{aligned} J(p_2, p_1; p_3, p_4) =& J(p_1, p_2; p_4, p_3) = \\ =& J^{-1}(p_2, p_1; p_4, p_3) = \\ =& J^{-1}(p_1, p_2; p_3, p_4). \end{aligned}$$

5. For every five points p_1, p_2, p_3, p_4, p_5, we have

$$J(p_1, p_2; p_3, p_4)J(p_1, p_2; p_4, p_5) = J(p_1, p_2; p_3, p_5).$$

6. For every four points p_1, p_2, p_3, p_4, we have the following relation, based on the boundary of a thickened tetrahedron,

$$\begin{aligned} 1 =& J(p_1, p_2, p_3)J(p_2, p_3, p_4) \\ & J(p_2, p_1, p_4)J(p_1, p_4, p_3) \\ & J(p_1, p_2; p_3, p_4)J(p_2, p_3; p_1, p_4)J(p_3, p_1; p_2, p_4) \\ & J(p_3, p_4; p_1, p_2)J(p_1, p_4; p_2, p_3)J(p_2, p_4; p_3, p_1). \end{aligned}$$

Proof. For part 1, let σ be an odd permutation. Let U be an union of two triangles along one of their the edges. Let the first triangle be with vertices p_1, p_2, p_3 and the second triangle be with vertices p_3, p_2, p_1 with the opposite orientation. We can glue the two triangles along the edge p_1p_2. (Glueing along any other edge would lead to the same result for the corresponding generating series.) From the shuffle product formula Theorem 2.21 (iii), it follows that $J(U) = J(p_1, p_2, p_3)J(p_3, p_2, p_1)$. (Note that the product is not the product in the ring R. It is induced by a shuffle product of iterated integrals on membranes.) From the second homotopy invariance theorem (Theorem 2.11) it follows that the generating series $J(U)$ depends on U up to homotopy, which keeps the boundary components p_2p_3, p_3p_2, p_1p_3 and p_3p_1 on fixed union holomorphic curves. We can contract U to its boundaries ∂U so that the contracting homotopy "keeps the boundary components" on a fixed union of holomorphic curves. Therefore, $J(U) = J(\partial U) = 1$.

Parts 2, 4 and 5 can be proven similarly.

For part 3, if p_1, p_2, p_3, p_4 belong to the same holomorphic (or anti-holomorphic) curve then the corresponding diangle has no interior, since the two edges will coincide. Recall that the edges of the diangle are defined via unique geodesic triangles lying on a holomorphic (or anti-holomorphic) curve.

The proof of part 6 is essentially the same as the one for part 1; however, we will prove it independently, since it is a key property of the non-commutative Hilbert modular symbol. Consider a thickened tetrahedron with vertices p_1, p_2, p_3, p_4. The faces of the thickened tetrahedron are precisely the ones listed in the product of property 6. The whole product is equal to $J(V)$, where V = *union of all faces of the thickened tetrahedron.* From the second homotopy invariance theorem it follows that the generating series $J(V)$ depends on V up to homotopy, which keeps the boundary components on a fixed union holomorphic curves. Since V bounds a contractible 3-dimensional region (a thickened tetrahedron), from Theorem 2.11, it follows that $J(V) = J(point) = 1$. $\square$

3.5 Definition of non-commutative Hilbert modular symbols

In this Subsection we define non-commutative Hilbert modular symbols. They are analogues of Manin's non-commutative modular symbol (see [M2]), applicable to the Hilbert modular group. Instead of iterated path integrals that Manin uses, we use a higher dimensional analogue, defined in Section 2.

Usually, a modular symbol represents a cohomology class. Manin's non-commutative modular symbol represent a non-commutative 1-st cohomology class. We would like to say that the non-commutative Hilbert modular symbols represent non-commutative cohomology classes, which we formulate as Conjectures.

After defining the non-commutative Hilbert modular symbols, we prove some of their properties. These properties will be interpreted intuitively as co-cycle conditions or as co-boundary conditions. The approach in this Subsection is more geometric. The purpose for presenting them here is to give many examples of relations and to help establishing a suitable cohomology theory that would truly capture these relations in a more structured way.

The cocycle interpretation is only for intuition it is not precise. The formula holds for geometric reasons. Note that the composition is not the multiplication in the ring R, it is given by the shuffle product (see Theorem 2.21), which works for the generating series on iterated integrals on membranes. The multiplication is written linearly as we would multiply several elements in a group or in a ring; however, the multiplication is two-dimensional among regions with common boundaries.

In the next Subsection will give some intuition about higher categories for the purpose of giving more structure to the non-commutative Hilbert modular symbols and for a possible approach to defining such a 1-st and 2-nd non-commutative cohomology class.

For definitions of iterated integrals on membranes of types **b**, **c**, **b'** and **c'** see Definitions 2.4, 2.5) for types **b**, **c**, and 3.9 for types **b'** and **c'**.

Definition 3.13 *We define non-commutative Hilbert modular symbols as a generating series of iterated integrals on membranes of types* **b**, *or* **c**, *or* **b'**, *or* **c'** *over a geodesic diangle by*

$$c^1_{p_1,p_2;p_3}(\gamma) = J(p_1,p_2;p_3,\gamma p_3).$$

We also define non-commutative Hilbert modular symbols as a generating series of iterated integrals on membranes of types **b'** *or* **c'** *over a geodesic triangle by*

$$c^2_p(\gamma,\delta) = J(p,\gamma p,\gamma\delta p),$$

where p, p_1, p_2, p_3 are cusp points in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ and $\beta,\gamma,\delta \in SL_2(\mathcal{O}_K)$.

We are going to define an action of $Mat_2(\mathcal{O}_K)^+$ on the generating series J^c, where $Mat_2(\mathcal{O}_K)^+$ is the semi-group of 2×2 matrices with a totally positive determinant.

In order to interpret $c^1(\gamma)$ and $c^2(\gamma,\delta)$ as cocycles, we are going to define an action of the semi-group $Mat_2(\mathcal{O}_K)^+$ on the whole ring R, where the generating series take values. Such an action can be given via Hecke operators.

For simplicity, we shall assume that $\mathcal{O}_K$ has narrow class number 1. We consider all Hecke eigenforms of weight $(2,2)$ with respect to $Mat_2(\mathcal{O}_K)^+$. Now, let u be a unit such that $u_1 > 0$ and $u_2 < 0$, where u_1 and u_2 are the images of u under the two real embeddings of K into $\mathbb{R}$. It exists, since the narrow class group is trivial. (For example $K = \mathbb{Q}(\sqrt{2})$ is such a field.) We define an action of $\gamma \in Mat_2(\mathcal{O}_K)$ on the ring R (Definition 2.13), where the generating series takes values. We define

$$\gamma \bullet f \mapsto T_\gamma(f)$$

if $\gamma \in Mat_2(\mathcal{O}_K)^+$. Let $f_1,\dots,f_m$ be a basis of Hecke eigenforms of the space of the cusp form of weight $(2,2)$. To each f_i we associate two of the generators of R X_i and Y_i. Let $X_1,Y_1,\dots,X_m,Y_m$ be generators of R. Then the action of $\gamma \in Mat_2(\mathcal{O}_K)^+$ is given by $T_\gamma(X_i) = c(\gamma,f_i)X_i$ and $T_\gamma(Y_i) = Y_i$, where $c(\gamma,f_i)$ the eigenvalue of the Hecke operator.

In this setting the group action, namely, the action of the Hilbert modular group is trivial. This trivial action extend to the the action of $T_1 = id$ on the whole ring R. In fact, for an element $\beta \in SL_2(\mathcal{O}_K)$, the trivial action on $c^1_{p_1,p_2;p_3}$ and on c^2_p can be realized as follows:

$$(\beta c^1_{p_1,p_2;p_3})(\gamma) = c^1_{\beta p_1,\beta p_2;\beta p_3}(\beta\gamma)$$

and

$$(\beta c^2_p)(\gamma,\delta) = c^2_{\beta p}(\beta p,\beta\gamma p,\beta\gamma\delta p).$$

The last two relations hold, since for a cusp form of weight $(2,2)$ the differential form $f dz_1 \wedge dz_2$ is invariant under the action of the Hilbert modular group Γ. Algebraically, for any geodesic diangle, we have

$$\beta J(p_1,p_2;p_3,p_4) = J(p_1,p_2;p_3,p_4) = J(\beta p_1,\beta p_2;\beta p_3,\beta p_4).$$

similarly for a geodesic triangle,

$$\beta J(p_1, p_2, p_3) = J(p_1, p_2, p_3) = J(\beta p_1, \beta p_2, \beta p_3).$$

The relations among the symbols are based on two properties: composition via shuffle product Theorem 2.21 (iii) and the homotopy invariance (Theorems 2.7 and 2.11).

Conjecture 3.14 *The non-commutative Hilbert modular symbol $c^1_{p_1,p_2;p_3}$ is a 1-cocycle. Moreover, if we change the point p_3 to q_3, then the cocycle changes by a coboundary.*

Property 5 of Theorem 3.12 can be interpreted as a 1-cocycle relation. Consider the analogy with non-commutative 1-cocycle of a group acting on a non-commutative ring, we define the boundary of $c^1_{p_1,p_2;p_3}$ by

$$dc^1_{p_1,p_2;p_3}(\beta,\gamma) = c^1_{p_1,p_2;p_3}(\beta)(\beta c^1_{p_1,p_2;p_3})(\gamma)(c^1_{p_1,p_2;p_3}(\beta\gamma))^{-1}$$

The action of β on the cocycle is given in Definition 3.13. In contrast to a 1-st non-commutative cocycle (see for example Kenneth Brown [Br]), here we have two-dimensional composition of symbols, that is, one can compose the symbols as two-morphisms in a two-category.

Then

$$\begin{aligned} dc^1_{p_1,p_2;p_3}(\beta,\gamma) &= J(p_1,p_2;p_3,\beta p_3)(\beta J(p_1,p_2;p_3,\gamma p_3))J^{-1}(p_1,p_2;p_3,\beta\gamma p_3) \\ &= J(p_1,p_2;p_3,\beta p_3)J(p_1,p_2;\beta p_3,\beta\gamma p_3)J^{-1}(p_1,p_2;p_3,\beta\gamma p_3) \\ &= 1. \end{aligned} \tag{3.26}$$

If we change p_3 to q_3 then the cocycle changes by a coboundary. Let $b^0 = J(p_1, p_2; p_3, q_3)$ be a 0-cochain. Then

$$\begin{aligned} c^1_{p_1,p_2;q_3}(\gamma) &= J(p_1,p_2;p_3,\gamma p_3) \\ &= J(p_1,p_2;p_3,q_3)J(p_1,p_2;q_3,\gamma q_3)J(p_1,p_2;\gamma q_3,\gamma p_3) \\ &= J(p_1,p_2;p_3,q_3)J(p_1,p_2;q_3,\gamma q_3)(\gamma J(p_1,p_2;p_3,q_3))^{-1} \\ &= b^0 c^1_{p_1,p_2;q_3}(\gamma)(\gamma b^0)^{-1} \end{aligned} \tag{3.27}$$

Conjecture 3.15 *The non-commutative Hilbert modular symbol $c^2_p(\beta,\gamma)$ satisfies a 2-cocycle relation. Moreover, if we change the point p to q, then the cocycle changes by a coboundary up to terms involving c^1.*

Recall

$$c^2_p(\beta,\gamma) = J(p, \beta p, \beta\gamma p).$$

Then c^2_p satisfies a 2-cocycle condition up to a multiple of the 1-cocycle $c^1_{q_1,q_2;q_3}$ for various points q_1, q_2, q_3. For the 2-cocycle relation, we compute $dc^2_p(\beta,\gamma,\delta)$.

$$\begin{aligned} dc^2_p(\beta,\gamma,\delta) =& c^2_p(\beta,\gamma)c^2(\beta,\gamma\delta)(c^2(\beta\gamma,\delta))^{-1}(\beta\cdot c^2(\gamma,\delta))^{-1} = \\ =& J(p,\beta p,\beta\gamma p)J(p,\beta p,\beta\gamma\delta p)\times \\ & \times J(p,\beta\gamma p,\beta\gamma\delta p)^{-1}J(\beta p,\beta\gamma p,\beta\gamma\delta p)^{-1} \end{aligned} \tag{3.28}$$

In order to have $dc^2_p(\beta,\gamma,\delta) = 1$, we must multiply by suitable values of c^1, corresponding to edges of a certain thickened tetrahedron. Then

$$\begin{aligned}
dc^2_p(\beta,\gamma,\delta)\times[&c^1_{p,\beta p;\beta\gamma p}((\beta\gamma)\delta(\beta\gamma)^{-1})c^1_{\beta p,\beta\gamma p;p}(\beta\gamma\delta)c^1_{\beta\gamma p,p;\beta p}((\beta)\gamma\delta\beta^{-1})\times\\
&\times c^1_{\beta\gamma p,\beta\gamma\delta p;p}(\beta)c^1_{p,\beta\gamma\delta p;\beta p}(\beta\gamma\beta^{-1})c^1_{\beta p,\beta\gamma\delta p;\beta\gamma p}((\beta\gamma)^{-1})] =\\
=&[c^2_p(\beta,\gamma)c^2(\beta,\gamma\delta)(c^2(\beta\gamma,\delta))^{-1}(\beta\cdot c^2(\gamma,\delta))^{-1}]\times\\
&\times [c^1_{p,\beta p;\beta\gamma p}((\beta\gamma)\delta(\beta\gamma)^{-1})c^1_{\beta p,\beta\gamma p;p}(\beta\gamma\delta)c^1_{\beta\gamma p,p;\beta p}((\beta)\gamma\delta\beta^{-1})\times\\
&\times c^1_{\beta\gamma p,\beta\gamma\delta p;p}(\beta)c^1_{p,\beta\gamma\delta p;\beta p}(\beta\gamma\beta^{-1})c^1_{\beta p,\beta\gamma\delta p;\beta\gamma p}((\beta\gamma)^{-1})] =\\
=&[J(p,\beta p,\beta\gamma p)J(p,\beta p,\beta\gamma\delta p)\times\\
&\times J(p,\beta\gamma p,\beta\gamma\delta p)^{-1}J(\beta p,\beta\gamma p,\beta\gamma\delta p)^{-1}]\times\\
&\times [J(p,\beta p;\beta\gamma p,\beta\gamma\delta p)J(\beta p,\beta\gamma p;p,\beta\gamma\delta p)J(\beta\gamma p,p;\beta p,\beta\gamma\delta p)\times\\
&\times J(\beta\gamma p,\beta\gamma\delta p;p,\beta p)J(p,\beta\gamma\delta p;\beta p,\beta\gamma p)J(\beta p,\beta\gamma\delta p;\beta\gamma p,p)] =\\
=&1.
\end{aligned}$$

The first equality follows from Equation (3.28). The second equality follows from the definition of the symbols. And the last equality follows from Property 6 of Theorem 3.12 with $(p_1,p_2,p_3,p_4) = (p,\beta p,\beta\gamma p,\beta,\gamma\delta p)$. Therefore, we obtain that $dc^2_p(\beta,\gamma,\delta)$ is 1 up to values of the 1-cocycle c^1.

Now we would like to have that c^2_p and c^2_q are homologous.

Conjecture 3.16 *The conjectural cocycles c^2_p and c^2_q are homologous*

$$c^2_p(\beta,\gamma) = c^2_q(\beta,\gamma)[db^1_{pq}(\beta,\gamma)]\prod_i J(D_i),$$

up to a product of $J(D_i)$,where D_i are geodesic diangles.

Before we proceed, we would like to make an analogy between 1-dimensional and 2-dimensional cocycles. For the 1-dimensional cocycle, the property that it is a cocycle uses the geometry of a triangle where the faces of the triangle are essentially the 1-cocycle. We want commutativity of the triangular diagram. We think of the commutativity of the diagram as follows: consider the interior of the triangle as a homotopy of paths and we think of the 1-cocycle as a homotopy invariant function. For the 2-cocycle, the 2-cocycle relation is represented by the faces of a tetrahedron. By a 'commutativity' of the diagram, we mean a homotopy invariant 2-cocycle and a homotopy from one of the faces to the union of the other three faces.

The comparison that $c^1_{p_1,p_2;p_3}$ and $c^1_{p_1,p_2;q_3}$ are homologous is given by a square-shaped diagram. The analogy with dimension 2 is that the cocycles c^2_p and c^2_q are two faces of an octahedron. The vertices associated to $c^2_p(\beta,\gamma)$ are $(p,\beta p,\beta\gamma p)$ and the vertices associated to c^2_q are $(q,\beta q,\beta\gamma q)$. The two faces will be opposite to each other on the octahedron Oct so that the three pair of opposite vertices are $(p,\beta\gamma q)$, $(\beta p,q)$ and $(\beta\gamma p,\beta q)$. The remaining 6 faces are combined into two triples. Each of them corresponds to a coboundary of a 1-chain.

Let

$$b^1_{p,q}(\beta) = [J(p,q,\beta p)J(q,\beta q,\beta p)][J(q,\beta p;p,\beta q)].$$

Consider the action of $\gamma \in \Gamma$ on b^1, by action on each point in the argument of J, denoted as before by $\gamma \cdot b^1$. Then, we define

$$db^1_{p,q}(\beta,\gamma) = b^1_{p,q}(\beta)[\beta \cdot b^1_{p,q}(\gamma)][b^1_{p,q}(\beta\gamma)]^{-1},$$

where $\beta \cdot b^1_{p,q}(\gamma) = [J(\beta p, \beta q, \beta\gamma p)J(\beta q, \beta\gamma q, \beta\gamma p)][J(\beta q, \beta\gamma p; p; \beta\gamma q)]$.

Consider the above octahedron Oct. Remove from it the tetrahedron T with vertices $(p, q, \beta\gamma q, \beta\gamma p)$. Then the triangles of the remaining geometric figure are precisely the triangles in the definitions of $c^2_p(\beta,\gamma)$, $c^2_q(\beta,\gamma)$ and $db^1_{p,q}(\beta,\gamma)$. Now, consider thickening of the edges, which are common for two triangles. It can be done in the following way. Instead of any triangle, we can take a geodesic triangle. The two triangles that had a common edge might have only two common vertices. Then the region between the two geodesic, one for each of the geodesic triangles, forms the induced diangle. Take J of the induces diangles from the octahedron Oct and J^{-1} of the induced diangles from the tetrahedron T. Their product gives $\prod_i J(D_i)$. The equality holds because we apply J to the union of the faces of the thickened $Oct - T$, which gives 1.

3.6 A two-category

Why do we need a two-category? Is there an example of a sheaf on this category/topology? How does the non-commutative Hilbert modular symbols represents a sheaf?

The ideas presented in this Subsection will be developed in a follow-up paper. Here we present the basic constructions that give justification for the conjectures that the non-commutative Hilbert modular symbols c^1 and c^2 are co-cycles in some categoric and sheaf-theoretic setting. For sheaves on 2-categories one may consult Street [St]. Since our 2-morphisms are invertible one may also use Lurie's [L] constructions of sheaves on higher categories.

We are going to construct a 2-category C and a sheaf J on the 2-category C. We define p to be an object of the 2-category C if p is a cusp point, that is $p \in \mathbb{P}^1(K)$. We define 1-morphisms in the following way. Let σ be the geodesic connecting 0 and ∞ that lies on the diagonal $\Delta = i(\mathbb{H}) \subset \mathbb{H} \times \mathbb{H}$. There is unique such geodesic. All geodesics $\gamma^*\sigma$ together with a choice of orientation are defined to be 1-morphisms, where $\gamma \in PGL_2(K)$. We define the 1-morphisms of C to be a finite concatenation of geodesics of the type $\gamma^*\sigma$ or the trivial path whose image coincides with a cusp point. Consider ideal triangles and ideal diangles as cells from which we build manifolds with corners. A 2-morphism is a finite union of manifolds with corners, made from finitely many ideal dangles and ideal triangles, which is path connected and has orientation.

The boundary of a 1-morphism is a union of two object objects - the starting point and the ending point of the directed path. The boundary of a 2-morphism (a 2-manifold with corners) is a finite union of 1-morphisms (oriented loops), where the orientation of the loops on the boundary is induced by the orientation of the 2-manifold with corners.

Now we are going to define a 2-sheaf J, whose values on a 2-morphism will be in a subset of the ring R and whose values of an object and on a 1-morphism will be a subset of a countable product of the ring R with itself.

As always, $S_{2,2}(\Gamma)$ denotes the space of cusp forms of weight $(2, 2)$ with respect to the group Γ. Here we will consider this space as the space of holomorphic 2 forms on

$\mathbb{H}\times\mathbb{H}$, which vanish on the cusps and which can dessend to the Hilbert modular surface $X_\Gamma = \Gamma\backslash(\mathbb{H}^2 \cup \mathbb{P}^1(K))$. Every n-tuple of such holomorphic forms $\Omega \in (S_{2,2}(\Gamma))^n$ defines a value on a 2-morphism f in C. Let this values be the generating series $J_f(\Omega)$. Let J_f be the collection of all values $J_f(\Omega)$ for all $\Omega \in (S_{2,2}(\Gamma))^n$. Let e be a 1-morphism. We say that e is in the boundary of a 2-morphism f, denoted by $e \subset \partial f$, if the image of the loop e is in the boundary of the image of the membrane f together with the induced orientation on e from f. We say that an object p is in the boundary of a 1-morphism e, denoted by $p \in \partial e$, if p is a source or a target of e. We define the values of J on a 1-morphism e to be the product

$$\prod_{e\subset\partial f} J_f \subset \prod_{e\subset\partial f} R.$$

We define values of J on objects p to be

$$\prod_{p\in\partial e;\, e\subset\partial f} J_f \subset \prod_{p\in\partial e;\, e\subset\partial f} R.$$

The sheaf conditions for 1-morphisms and the sheaf conditions for 2-morphisms resemble the classical conditions for a presheaf to be a sheaf.

Let $f_i : A_i \to [0,1]^2$ be a finite collection of disjoint 2-morphisms, whose union is a morphism $f : A \to [0,1]^2$. We define a finite collection f_{ij}^k of 1-morphisms and 0-morphisms (objects) such that the union $\bigcup_k Im(f_{ij}^k) = Im(f_i)\cap Im(f_j)$ is a disjoint union of the intersection.

Then the equalizer

$$J_f \to \prod_i J_{f_i} \rightrightarrows \prod_{ijk} J_{f_{ij}^k}$$

is exact (for a definition of equalizer one may consult [Bor]).

Similarly, let e be a 1-morphism and let $\{e_i\}_i$ be a finite set of disjoint 1-morphisms such that the union $\bigcup_i Im(e_i) = Im(e)$. We can write the intersection $Im(e_i)\cap Im(e_j)$ as a finite union of 0-morphisms $\bigcup_k Im(e_{ij}^k)$, for some 1-morphisms e_{ij}^k.

Then the equalizer

$$J_e \to \prod_i J_{e_i} \rightrightarrows \prod_{ijk} J_{e_{ij}^k}$$

is exact.

The cochain is defined as

$$\prod_{p:\ 0\text{-morph}} J_p \to \prod_{e:\ 1\text{-morph}} J_e \to \prod_{f:\ 2\text{-morph}} J_f \to \prod_{\substack{g\,:\,2\text{-morph}\\ \partial g=\emptyset}} J_g$$

The maps $J_e \to J_p$ and $J_p \to J_f$ are surjective when they are defined, resembling flabby sheaves. Thus, we should have trivial 0-th or 1-st cohomology set. The only non-trivial cohomology will be 2-nd cohomology set. The cocycle conditions for both non-commutative Hilbert modular symbols c^1 and c^2 can be interpreted as a particular case of maps

$$\prod_{f:\ \text{2-morph}} J_f \to \prod_{\substack{g\,:\,\text{2-morph}\\ \partial g=\emptyset}} J_g$$

For c^1, the boundary condition is that a union of two diangles with a common edge is a third diangle. One can think of the these three diangles as a boundary of a degenerate 3-dimensional region. One can realize this cocycle condition as a sheaf-theoretic one by modifying the above definition so that the 2-morphisms consists of a finite union of ideal diangles (without using the ideal triangles). Then the sheaf-theoretic 2-nd cocycle condition is the one for non-commutative Hilbert modular symbol c^1.

If we are able to quotient the 2-category described in the beginning of this subsection by the 2-morphisms generated by dianlges, then we have only two morphisms generated by ideal triangles. The non-commutative Hilbert modular symbol c^2 is exactly the one that considers ideal triangles. Note that its cocycle relation for c^2 is satisfied up to 2-morphisms generated by diangles.

3.7 Explicit Computations. Multiple Dedekind Zeta Values

In this Subsection, we make explicit computations of some ingredients in the non-commutative Hilbert modular symbol. In [M2], Manin compares explicit formulas of integrals in the non-commutative modular symbol to multiple zeta values. The similarities are both in terms of infinite series formulas and in terms of formulas via iterated path integrals. Here we compare certain integrals in the non-commutative Hilbert modular symbol to multiple Dedkeind zeta values (for multiple Dedekind zeta values, see [H3]). Again the similarities are both in terms of infinite series formulas and in terms of formulas via iterated integrals over membranes.

We are going to consider the Fourier expansion of two Hilbert cusp forms f and g. Let $\omega_f = f dz_1 \wedge dz_2$, $\omega_g = g dz_1 \wedge dz_2$ and $\omega_0 = dz_1 \wedge dz_2$. We are going to associate L-values to iterated integrals of the forms ω_f and ω_g. The L-values would be iterated integrals over an union of diangles. One can think of a diangle connecting 0 and ∞ as a segment or a real cone. The union will be a disjoint union of all such real cones connecting 0 and ∞ or simply $\mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H})$. We also recall the definition of a multiple Dedekind zeta values via (discrete) cone. Finally, we show analogous formulas for iterated L-values associated to Hilbert cusp forms and for multiple Dedekind zeta values.

We will be mostly interested in the modular symbol associated to a diangle. Let us recall what we mean by a diangle.

Let p_1, p_2, p_3, p_4 be four cusp points. Let $\gamma_1 \in GL_2(K)$ be a linear fractional transform that sends $\gamma_1(p_1) = 0$, $\gamma_1(p_2) = \infty$, $\gamma_1(p_3) = 1$. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0,\infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_1(p_1, p_2) = \gamma_1^{-1}\lambda(0,\infty)$$

be the pull-back of the geodesic λ to a geodesic connecting p_1 and p_2.

Now consider the triple p_1, p_2 and p_4. Let $\gamma_2 \in GL_2(K)$ be a linear fractional transform that sends $\gamma_2(p_1) = 0$, $\gamma_2(p_2) = \infty$ and $\gamma_2(p_4) = 1$. Let Δ be the image of the diagonal embedding of $\mathbb{H}^1$ into $\mathbb{H}^2$. Then 0, 1 and ∞ are boundary points of Δ. Let $\lambda(0,\infty)$ be the unique geodesic in Δ that connects 0 and ∞. And let

$$\lambda_2(p_1, p_2) = \gamma_2^{-1}\lambda(0,\infty)$$

be the pull-back of the geodesic λ to a geodesic connecting p_1 and p_2.

By a diangle, we mean a region in $\mathbb{H}^2 \cup \mathbb{P}^1(K)$ of homotopy type of a disc, bounded by the geodesics $\lambda_1(0,\infty)$ and $\lambda_2(0,\infty)$.

We are going to present a computation for the diangle D_u defined by the points $(0, \infty, u^1, u^{-1})$, where u is a generator for the group of units modulo ± 1 in K. Let (1) be the trivial permutation.

Lemma 3.17 *Let u be a totally positive unit. Then*

$$\int\int_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)} dz_1 \wedge dz_2 = \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)}$$

Proof. Let u_1 and u_2 be the two embeddings of u into $\mathbb{R}$. Then $(0, \infty, u)$ can be mapped to $(0, \infty, 1)$ by $\gamma_1 = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. The geodesic $\lambda(0,\infty)$ can be parametrized by (it, it) for $t \in \mathbb{R}$. Then the geodesic $\lambda_1(0,\infty)$ on the geodesic triangle $(0,\infty,u)$ can be parametrized by $\{(iu_1 t, iu_2 t) \mid t > 0\}$. Similarly, the geodesic $\lambda_2(0,\infty)$ on the geodesic triangle $(0, \infty, u^{-1})$ can be parametrized by $\{(iu_2 t, iu_1 t) \mid t > 0\}$. Then the diangle D_u can be parametrized by

$$D_u = \{(z_1, z_2) \in \mathbb{H}^2 \,|\, Re(z_1) = Re(z_2) = 0, Im(z_1) \in \left(\frac{u_1}{u_2}t, \frac{u_2}{u_1}t\right), Im(z_2) = t \in (0,\infty)\}.$$

Then we have

$$\begin{aligned}
\int\int_{D_u}^{(1)(1)} e^{2\pi i(\alpha_1 z_1 + \alpha_2 z_2)} dz_1 \wedge dz_2 &= \int_\infty^0 \left(\int_{\frac{u_2}{u_1}t}^{\frac{u_1}{u_2}t} e^{2\pi i(\alpha_1 z_1 + \alpha_2 t)} dz_1 \right) dt = \\
&= \frac{1}{2\pi i \alpha_1} \int_\infty^0 \left(e^{\alpha_1 \frac{u_1}{u_2} t + \alpha_2 t} - e^{\alpha_1 \frac{u_2}{u_1} t + \alpha_2 t} \right) dt = \\
&= \frac{1}{(2\pi i)^2} \frac{1}{\alpha 1} \left(\frac{1}{\alpha_1 \frac{u_1}{u_2} + \alpha_2} - \frac{1}{\alpha_1 \frac{u_2}{u_1} + \alpha_2} \right) = \\
&= \frac{1}{(2\pi i)^2} \frac{u_2^2 - u_1^2}{(\alpha_1 u_1 + \alpha_2 u_2)(\alpha_1 u_2 + \alpha_2 u_1)}
\end{aligned}$$

Therefore, one term of the Fourier expansion of a Hilbert cusp form paired with a symbol given by one diangle does not resemble a norm of an algebraic integer. However, if we integrate over an infinite union of diangles then a similarity with Dedekind zeta and with multiple Dedekind zeta occurs.

Consider the limit when $n \to \infty$ of D_{u^n}. It is the product of the two imaginary axes of the two upper half planes. Denote by

$$\mathrm{Im}(\mathbb{H}^2) = \mathrm{Im}(\mathbb{H}) \times \mathrm{Im}(\mathbb{H}).$$

One can think of this region as an infinite union of diangles.

Denote by αz the sum of products $\alpha_1 z_1 + \alpha_2 z_2$. Using the methods of [H3], Section 1, we obtain

$$\frac{(2\pi i)^{-2}}{N(\alpha)N(\alpha+\beta)} = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i \alpha z} dz_1 \wedge dz_2 \cdot e^{2\pi i \beta z} dz_1 \wedge dz_2$$

and

$$\frac{1}{(2\pi i)^2} \frac{1}{N(\alpha)^3 N(\alpha+\beta)^2} = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} e^{2\pi i \alpha z} dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2) \cdot (dz_1 \wedge dz_2) \cdot e^{2\pi i \beta z} dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2)$$

Let f and g be two cusp form of wights $(2k, 2k)$ and $(2l, 2l)$, respectively. Consider the Fourier expansion of both of the cusp forms. Let

$$f = \sum_{\alpha >> 0} a_\alpha e^{2\pi i \alpha z}$$

and let

$$g = \sum_{\beta >> 0} b_\beta e^{2\pi i \beta z}.$$

Since f is of weight $(2k, 2k)$, we have that $a_{u\alpha} = a_\alpha$, where u is a unit. For such a modular form the modular factor with respect to the transformation $z \to uz$ is 1. The L-values of f is

$$L_f(n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{\alpha \in \mathcal{O}_K^+/U^+} a_\alpha e^{2\pi i \alpha z} dz_1 \wedge dz_2 \cdot (dz_1 \wedge dz_2)^{\cdot(n-1)} = \frac{1}{(2\pi i)^{2n}} \sum_{\alpha \in \mathcal{O}_K^+/U^+} \frac{a_\alpha}{N(\alpha)^n}.$$

Here $\mathcal{O}_K^+$ denotes the totally positive algebraic integers in K and U^+ denotes the totally positive units.

We recall some of the definitions from [H3]. We fix a positive cone C in $\mathcal{O}_K$, by which we mean

$$C = \mathbb{N} \cup \{\alpha \in \mathcal{O}_K \mid a + b\epsilon,\ a, b \in \mathbb{N}\},$$

where ϵ is a generator of the group of totally positive units. By $\epsilon^k C$, we mean the collection of products $\epsilon^k \alpha$, where α varies in the cone C.

The following infinite sum is an example of a multiple Dedekind zeta value

$$\zeta_{K;C,\epsilon^k C}(m, n) = \sum_{\alpha \in C} \sum_{\beta \in \epsilon^k C} \frac{1}{N(\alpha)^m N(\alpha+\beta)^n}.$$

Let $Z(m, n) = \sum_{k \in \mathbb{Z}} \zeta_{K;C,\epsilon^k C}(m, n)$, where C is any set representing the totally positive algebraic integers $\mathcal{O}_K^+$ modulo totally positive units U^+.

Lemma 3.18 *The values $Z(m,n)$ are finite for $m > n > 1$.*

Proof. Let ϵ be a generators of the group of totally positive units U^+ in K. For the two real embeddings ϵ_1 and ϵ_2 of ϵ, we can assume that $\epsilon_1 > 1 > \epsilon_2$. Otherwise we can take its reciprocal.

$$Z(m,n) = \sum_{k\in Z}\sum_{\alpha,\beta\in C}\frac{1}{N(\alpha)^m N(\alpha+\epsilon^k\beta)^n} < \tag{3.29}$$

$$< \sum_{\alpha,\beta\in C}\frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n}+\right. \tag{3.30}$$

$$\left.+\sum_{k=1}^{\infty}\frac{2^n}{\epsilon_1^k}\left(\frac{1}{\alpha_1^n\beta_2^n}+\frac{1}{\alpha_2^n\beta_1^n}\right)\right) <$$

$$< \sum_{\alpha,\beta\in C}\frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n}+\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{2}{\epsilon_1^k}\left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n}\right)\right) = \tag{3.31}$$

$$= \sum_{\alpha,\beta\in C}\frac{1}{N(\alpha)^m}\left(\frac{1}{N(\alpha+\beta)^n}+\right. \tag{3.32}$$

$$\left.+\frac{2}{\epsilon_1 - 1}\left(\frac{N(\alpha+\beta)^n - N(\alpha)^n}{N(\alpha+\beta)^n}\right)\right) =$$

$$= \sum_{\alpha,\beta\in C}\frac{1}{N(\alpha)^m N(\alpha+\beta)^n}+ \tag{3.33}$$

$$+\frac{2}{\epsilon_1-1}\frac{1}{N(\alpha)^n} - \frac{2}{N(\alpha)^{m-n}N(\alpha+\beta)^n} =$$

$$=\zeta_K(C;m,n) - \frac{2}{\epsilon_1-1}\left(\zeta_K(C;n)+\zeta_K(C;m-n,n)\right). \tag{3.34}$$

Equation (3.29) is the definition. Inequality (3.30) is based on the following: $\epsilon_2 < 1$ is replaced with 1 then $k > 0$. For $k < 0$ we use $\epsilon_2^k = \epsilon_1^{-k}$. We put 1 for ϵ_1^k for $k < 0$. The case $k = 0$ is treated separately. Finally we group the terms with equal powers of ϵ_1. In the Inequality (3.31) we estimate the mixed terms in the brackets. In Equation (3.32) we take the sum of the geometric series in ϵ_1^{-1}. Then in Equation (3.33) we open the brackets. And finally, in Equation (3.34), we express the sums as a finite linear combinations of a Dedekind zeta value and multiple Dedekind zeta values.

The following definition of an iterated L-value is the coefficient of one monomial from the non-commutative Hilbert modular symbol of type **b**.

Definition 3.19 *For a pair of Hilbert cusp forms f and g with Fourier expansion*

$$f = \sum_{\alpha>>0} a_\alpha e^{2\pi i\alpha z} \text{ and } g = \sum_{\beta>>0} b_\beta e^{2\pi i\beta z},$$

we define the following iterated L-values

$$L_{f,g}(m,n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} (a_\alpha e^{2\pi i\alpha z} dz_1 \wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(m-1)}\cdot$$
$$\cdot (b_\beta e^{2\pi i\beta z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(n-1)}.$$

Theorem 3.20 *Using the above definition we have*

$$L_{f,g}(m,n) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in C,\beta\in\epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}$$

Proof.

$$L_{f,g}(m,n) = \int_{\mathrm{Im}(\mathbb{H}^2)}^{(1)(1)} \sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} (a_\alpha e^{2\pi i\alpha z} dz_1 \wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(m-1)}\cdot$$
$$\cdot (b_\beta e^{2\pi i\beta z} dz_1\wedge dz_2)\cdot(dz_1\wedge dz_2)^{\cdot(n-1)} =$$
$$= \sum_{(\alpha,\beta)\in(\mathcal{O}_K^+,\mathcal{O}_K^+)/U} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n} =$$
$$= \sum_{k\in\mathbb{Z};\alpha,\beta\in C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\epsilon^k\beta)^n} =$$
$$= \sum_{k\in\mathbb{Z},} \sum_{\alpha\in C} \sum_{\beta\in\epsilon^k C} \frac{a_\alpha b_\beta}{N(\alpha)^m N(\alpha+\beta)^n}.$$

□

We would like to bring to the attention of the reader the Definition 3.19 of the multiple L- values. More specifically, we would like to point out that the region of integration is an infinite union of diangles, (or equivalently an infinite union of real cones; see the beginning of this Section.) Note also that in Theorem 3.20 the values of the multiple L-functoins are expressed as an infinite sums over different discrete cones, namely, over $\epsilon^k C$, for $k\in\mathbb{Z}$. However, a single real cone D_u as in Lemma 3.18, does not correspond to a single discrete cone. Only a good union of real cones $\mathrm{Im}(\mathbb{H})\times\mathrm{Im}(\mathbb{H})$ corresponds to a good union of discrete cones $\bigcup_{k\in\mathbb{Z}}(C,\epsilon^k C)$ as a fundamental domain of $(\mathcal{O}_K^+,\mathcal{O}_K^+)/U^+$.

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