Irreducible projective characters of wreath products

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Abstract. The irreducible character values of the spin wreath products $\widetilde{\Gamma}_n = \Gamma \wr \widetilde{S}_n$ of the symmetric group and a finite group Γ are completely determined for arbitrary Γ.

1. Introduction

The symmetric group has been one of the core materials in mathematics and theoretical physics since Frobenius developed representation theory of finite groups. Using the duality between S_n and GL_m Schur found the complete set of invariants for the general linear group and revolutionized the theory of symmetric functions using the Schur functions in his dissertation. In the thirties Specht [**13**] generalized Schur and Frobenius' theory to the wreath products of the symmetric group and any finite group. More recently Macdonald reformulated Specht's theory in the classic monograph [**6**].

The double covering groups $\widetilde{S}_n$ of the symmetric group have many similar and interesting properties as shown by Schur in [**11**]. In that seminal paper Schur generalized Frobenius's theory and introduced the famous Schur Q-functions. Schur's character theory of $\widetilde{S}_n$ consists of two parts. The first part of the character values on conjugacy classes associated to partitions with odd integer parts are exactly given by an analogous Frobenius formula in terms of the Schur Q-functions; the second part of character values on strict partitions was solved with the help of twisted tensor products of Clifford algebras. In the same direction, Morris [**7**] formulated an iterative rule for computing the spin character values of the symmetric group, and Nazarov [**9**] constructed all irreducible representations of the spin group. Józefiak [**5**] also computed the projective character values for a related double covering group of the hyperoctahedral group using similar techniques.

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In [**1**] the second author and collaborators determined all irreducible character values on *even* conjugacy classes of the spin wreath products $\widetilde{\Gamma}_n$ by vertex operator calculus in the context of the spin McKay correspondence. The character values generalize the first part of Schur's theory on $\widetilde{S}_n$. The key was to show that the character values at conjugacy classes of even colored partitions (with odd integer parts) are given by matrix coefficients of certain products of twisted vertex operators. However, the other character values on *odd* strict partition-valued functions could not be explained in the context of the McKay correspondence. To the authors' knowledge, other available methods such as the Hopf algebraic approach [**2**] seem not helpful either.

The knowledge of the remaining character values for $\widetilde{\Gamma}_n$ would be an analog of the second part in Schur's pioneering work [**11**]. In the case of Γ being abelian, we solved the problem in [**3**] by using the Mackey-Wigner method of little groups (cf. [**12**]) to construct all irreducible spin representations of the wreath products (see also [**8**] for some cases). However the method of little groups does not work in the most general case for arbitrary finite group Γ. It seems that a new method is needed for determination of the irreducible characters.

The purpose of this paper is to complete the character theory of the spin wreath products $\widetilde{\Gamma}_n$ and compute the missing part of the characters table of $\widetilde{\Gamma}_n$ for any finite group Γ. We will construct all irreducible characters and in particular provide explicit formulas for character values on the conjugacy classes of the second type. It turns out that spin character values in this part can be non-zero for more than two conjugacy classes in contrast of Schur's case, nevertheless they are still sparsely zero. We note that the exhaustion method is used to pick up all non-zero character values, which bears some similarity to Schur's original method.

2. Projective representations of $\widetilde{\Gamma}_n$

2.1. Spin wreath products. According to Schur, there are two non-isomorphic double covering groups of the symmetric group S_n when $n \geq 4$ and $n \neq 6$. But their representations are in complete one-to-one correspondence. We fix one of them, and let the spin symmetric group $\widetilde{S}_n$ be the finite group generated by z and t_i, $(i = 1, \cdots, n-1)$ with the relations:

$$z^2 = 1,\ t_i^2 = (t_i t_{i+1})^3 = z,\ z t_i = t_i z,\ t_i t_j = z t_j t_i, \quad (|i-j| > 1). \tag{2.1}$$

Let θ_n be the homomorphism from $\widetilde{S}_n$ to S_n sending t_i to the transposition $(i, i+1)$ and z to 1. This says that $\widetilde{S}_n$ is a central extension of S_n by the cyclic group $\mathbb{Z}_2$.

The spin group $\widetilde{S}_n$ has a cycle presentation *à* la Conway [**14**]. For $i < j$, the transposition $[ij]$ is deinfed as $[ij] = t_{j-1} \cdots t_{i+1} t_i t_{i+1} \cdots t_{j-1}$. Here if $j = i+1$, we take $t_i = [i, i+1]$. Then for $i_1 < i_2 < \cdots < i_k \leq n$, we define $[i_1 i_2 \cdots i_k] = [i_1 i_2][i_2 i_3] \cdots [i_{k-1} i_k]$. Finally for a permutation $\sigma \in S_k$

we define

$$[i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(k)}] = z^{l(\sigma)}[i_1i_2\cdots i_k]. \tag{2.2}$$

Given a permutation $w \in S_n$. We can fix its cycle product as follows. First each cycle is written as a word lexicographically by rotating its content, then we rearrange the order of the cycles lexicographically to obtain a unique presentation $w = \prod_{i=1}^{l}(a_{i1}\cdots a_{i\lambda_i})$, where λ_i are the lengths of the cycles. We then define the element $t_w = \prod_{i=1}^{l}[a_{i1}\cdots a_{i\lambda_i}] \in \widetilde{S}_n$. Note that $\theta_n^{-1}(w) = z^p t_w$. One also has that $t_{w_1}t_w t_{w_1}^{-1} = z^p t_{w_1 w w_1^{-1}}$ for $w, w_1 \in S_n$, where $p = 0$ or 1. Similarly for a partition ρ we denote $t_\rho = t_{w(\rho)}$, where $w(\rho) = (1\cdots\rho_1)\cdots(n-\rho_l+1\cdots n)$.

For a positive integer n and a finite group Γ, let $\Gamma^n = \Gamma\times\cdots\times\Gamma$ be the n-fold direct product of Γ, and let $\Gamma^0 = 1$. The spin group $\widetilde{S}_n$ acts on Γ^n by permuting the components:

$$t_w(g_1,\cdots,g_n) = (g_{w^{-1}(1)},\cdots,g_{w^{-1}(n)}),\ z(g_1,\cdots,g_n) = (g_1,\cdots,g_n). \tag{2.3}$$

The spin wreath product $\widetilde{\Gamma}_n = \Gamma\wr\widetilde{S}_n$ is the semi-direct product

$$\widetilde{\Gamma}_n = \Gamma^n \rtimes \widetilde{S}_n = \{(g,t)|g=(g_1,\cdots,g_n)\in\Gamma^n, t\in\widetilde{S}_n\}$$

with the multiplication $(g,t)\cdot(h,s) = (gt(h), ts)$. The quotient group $\widetilde{\Gamma}_n/\langle z\rangle$ is isomorphic to the semidirect product $\Gamma_n = \Gamma^n \rtimes S_n$, and the canonical homomorphism θ_n from $\widetilde{\Gamma}_n$ to Γ_n sends $(g,t_i)\mapsto(g,(i,i+1))$ and $(g,z)\mapsto(g,1)$. For simplicity we have used the same symbol for the homomorphism θ_n for the wreath product.

We can define a parity p for $\widetilde{\Gamma}_n$ by (here $g\in\Gamma^n, t_i\in\widetilde{S}_n$)

$$p(g,t_i) = 1\,(1\le i\le n-1), \qquad p(g,z) = 0.$$

This agrees with the usual parity for $\widetilde{S}_n$ when Γ is the trivial group. Therefore the group algebra $\mathbb{C}[\widetilde{\Gamma}_n]$ has a superalgebra structure, and $\mathbb{C}[\Gamma\wr\widetilde{A}_n]$ is the even subspace.

2.2. Conjugacy classes. Let $\Gamma_* = \{c^i|i=0,1,\ldots,r\}$ be the set of conjugacy classes of Γ and denote by $\Gamma^* = \{\gamma_i|i=0,1,\cdots,r\}$ the set of irreducible characters of Γ. Let ζ_c be the order of the centralizer of an element in the conjugacy class $c\in\Gamma_*$, then the order of the conjugacy class c is $|\Gamma|/\zeta_c$. Here for a finite set X we denote by $|X|$ its cardinality. In the following we follow Macdonald's notations [**6**].

A partition-valued function $\rho = (\rho(c))_{c\in\Gamma_*}$ defined on Γ_* consists of $|\Gamma_*|$ partitions indexed by conjugacy classes $c\in\Gamma_*$. The weight of ρ is defined by $||\rho|| = \sum_{c\in\Gamma_*}|\rho(c)|$, and the length is given by $l(\rho) = \sum_{c\in\Gamma_*} l(\rho(c))$. It helps to visualize ρ as a colored partition in which each sub-partition $\rho(c)$ is colored by c. Let $\mathcal{P}(\Gamma_*)$ be the set of partition-valued functions indexed by Γ_*. It is well-known that the conjugacy classes of Γ_n are parameterized by $\mathcal{P}(\Gamma_*)$. For an element $(g,\sigma)\in\Gamma_n$, the permutation σ gives rise to a cycle partition $\lambda = (\lambda_1,\lambda_2,\ldots)$. For each part $\lambda_i = k$, which corresponds to the cycle $(i_1i_2\cdots i_k)$, we associate the cycle-product $g_{i_k}g_{i_{k-1}}\cdots g_{i_1}\in\Gamma$. If the cycle-product belongs to the conjugacy class c, then we color this part λ_i by c which turns λ into a colored partition. In this way we get the parametrization of conjugacy classes of Γ_n by $\mathcal{P}(\Gamma_*)$. For $\rho=(\rho(c))_{c\in\Gamma_*}\in\mathcal{P}_n(\Gamma_*)$, let C_ρ be the corresponding conjugacy class in Γ_n.

We denote by $\mathcal{SP}(\Gamma_*)$ the set of partition-valued functions $(\rho(c))_{c\in\Gamma_*}$ in $\mathcal{P}(\Gamma_*)$ such that each partition $\rho(c)$ is *strict*, i.e. $\rho(c)$ has distinct parts. Let $\mathcal{OP}(\Gamma_*)$ be the set of partition-valued functions $(\rho(c))_{c\in\Gamma_*}$ on Γ_* such that all parts of the partitions $\rho(c)$ are odd integers.

For each partition λ we define the *parity* $d(\lambda)=|\lambda|-l(\lambda)$. Similarly, for a partition-valued function $\rho=(\rho(c))_{c\in\Gamma_*}$, we define $d(\rho)=\|\rho\|-l(\rho)$. Then ρ is *even* (resp. *odd*) if $d(\rho)$ is even (resp. odd). We set $\mathcal{P}_n^0(\Gamma_*)$ (resp. $\mathcal{P}_n^1(\Gamma_*)$) to be the collections of even (resp. odd) partition-valued functions ρ on Γ_* such that $||\rho||=n$. As a convention we denote $\mathcal{SP}_n^i(\Gamma_*)=\mathcal{P}_n^i(\Gamma_*)\cap\mathcal{SP}(\Gamma_*)$ and $\mathcal{OP}_n(\Gamma_*)=\mathcal{P}_n(\Gamma_*)\cap\mathcal{OP}(\Gamma_*)$ for $i\in\{0,1\}$. When Γ_* just consists of a single element, $\mathcal{P}(\Gamma_*)$ will be simply written as $\mathcal{P}$. Similarly we have notations such as $\mathcal{OP}_n$, $\mathcal{SP}_n$, and $\mathcal{SP}_n^i$.

2.3. Split conjugacy classes. An element $\widetilde{x}\in\widetilde{\Gamma}_n$ is called *non-split* if $\widetilde{x}$ is conjugate to $z\widetilde{x}$. Otherwise $\widetilde{x}$ is said to be *split*. An element $x\in\Gamma_n$ is called split if $\theta_n^{-1}(x)$ is split. A conjugacy class of $\widetilde{\Gamma}_n$ is called split if its elements are split. It is known that the conjugacy class C_ρ of Γ_n splits if and only if the preimage $\theta_n^{-1}(C_\rho)=:D_\rho$ splits into two conjugacy classes in $\widetilde{\Gamma}_n$.

For a partition $\lambda=(1^{m_1}2^{m_2}3^{m_3}\cdots)$ of n, we denote by $z_\lambda=\prod_{i\geq1}i^{m_i}m_i!$ the order of the centralizer of the permutation with cycle type λ in S_n. For each partition-valued function $\rho=(\rho(c))_{c\in\Gamma_*}$, we define

$$Z_\rho=\prod_{c\in\Gamma_*}z_{\rho(c)}\zeta_c^{l(\rho(c))},$$

which is the order of the centralizer of an element of conjugacy type $\rho=(\rho(c))_{c\in\Gamma_*}$ in Γ_n. The order of the centralizer of an element of conjugacy type ρ in $\widetilde{\Gamma}_n$ is given by

$$\widetilde{Z}_\rho=\begin{cases}2Z_\rho, & C_\rho\text{ is split,}\\ Z_\rho, & C_\rho\text{ is non-split.}\end{cases}$$

For each split conjugacy class C_ρ in Γ_n, we define the conjugacy class D_ρ^+ in $\widetilde{\Gamma}_n$ to be the conjugacy class containing the element (g,t_ρ) and define $D_\rho^-=zD_\rho^+$, then $D_\rho=D_\rho^+\cup D_\rho^-$.

As usual a representation π of $\widetilde{\Gamma}_n$ is called *spin* if $\pi(z)=-id$, then its character is a projective character of Γ_n. It is clear that the character of a spin representation of $\widetilde{\Gamma}_n$ are determined by its values on split conjugacy classes. By a standard result [**10, 1**] the conjugacy class of $\widetilde{\Gamma}_n$ is split in $\widetilde{\Gamma}_n$ if and only if either $\rho\in\mathcal{OP}_n(\Gamma_*)$ or $\rho\in\mathcal{SP}_n^1(\Gamma_*)$. In particular the split conjugacy classes of $\widetilde{S}_n$ (i.e. when Γ is the trivial group) are parameterized either by partitions with odd integers or by odd strict partitions. By Euler's theorem the number of strict partitions is equal to the number of odd partitions, therefore the total number of such conjugacy classes of $\widetilde{\Gamma}_n$ are given by $|\mathcal{SP}^0_n(\Gamma_*)| + 2|\mathcal{SP}^1_n(\Gamma_*)|$.

3. The irreducible spin character table of $\widetilde{\Gamma}_n$

3.1. Schur's theory of $\widetilde{S}_n$. Like the symmetric group S_n, nontrivial projective (spin) characters of $\widetilde{S}_n$ are parameterized by strict partitions λ of n. One can classify spin characters into the so-called *double spin* and *associate spin* characters. The associated character χ' of a spin character is defined to be $\chi' = sgn \cdot \chi$, where sgn is the sign character. If $\chi' = \chi$, then we say χ is a double spin character (or self-associated). For $\nu \in \mathcal{SP}_n$ and $d(\nu) = n - l(\nu)$ even, there corresponds a unique irreducible (double) spin character Δ_ν; for $d(\nu)$ odd, there corresponds a pair of irreducible (associate) spin characters $\Delta^{\pm}_\nu$ for $\widetilde{S}_n$. Schur [**11**] showed that there was an analogous Frobenius formula for the spin character values at the even conjugacy classes indexed by partitions with odd integers. For $\nu \in \mathcal{SP}$ the Schur Q-function Q_ν is defined by

$$Q_\nu(x_1, \cdots, x_n) = 2^l \sum_{w \in S_n/S_{n-l}} x^{\nu_1}_{w(1)} \cdots x^{\nu_n}_{w(n)} \prod_{i<j} \frac{x_{w(i)} + x_{w(j)}}{x_{w(i)} - x_{w(j)}},$$

where $n \geq l = l(\nu)$ and S_{n-l} acts on $x_{l+1}, \cdots, x_n$. It is known that Q_ν is a polynomial in power sum symmetric functions $p_1, p_3, p_5, \cdots$. As usual we will denote $p_\alpha = p_{\alpha_1} p_{\alpha_2} \cdots$ for a partition α.

Schur showed that nontrivial values of the spin character Δ_ν at even conjugacy classes are given by

$$Q_\nu = \sum_{\alpha \in \mathcal{OP}_n} 2^{[\frac{l(\nu)+l(\alpha)+\bar{d}(\nu)}{2}]} z_\alpha^{-1} \Delta_\nu(\alpha) p_\alpha, \tag{3.1}$$

where $[a]$ denotes the largest integer $\leq a$ and $\bar{d}(\nu)$ is equal to 0 (resp. 1) if $d(\nu)$ is even (resp. odd). Schur further proved the following results.

THEOREM 3.1. [**11**] *(i) For $n - l(\nu)$ even, the character Δ_ν of $\widetilde{S}_n$ is determined by $\{\Delta_\nu(\alpha) | \alpha \in \mathcal{OP}_n\}$ (given by Eq.(3.1)) and $\Delta_\nu(\mu) = 0$ for $\mu \notin \mathcal{OP}_n$.*

(ii) For $n - l(\nu)$ odd, the character Δ_ν of $\widetilde{S}_n$ is determined by $\{\Delta_\nu(\alpha) | \alpha \in \mathcal{OP}_n\}$ (given by Eq.(3.1)) and $\Delta_\nu(\nu) = (\sqrt{-1})^{(n-l(\nu)+1)/2} \sqrt{\nu_1 \cdots \nu_k/2}$ for $\nu = (\nu_1, \cdots, \nu_k)$; $\Delta_\nu(\mu) = 0$ for $\mu \neq \nu$ and $\mu \in \mathcal{SP}^1_n$. Moreover, $(\Delta_\nu)'(\alpha) = \Delta_\nu(\alpha)$ for $n - l(\alpha)$ even and $(\Delta_\nu)'(\alpha) = -\Delta_\nu(\alpha)$ for $n - l(\alpha)$ odd.

For $\widetilde{\Gamma}_n$, the values of nontrivial spin characters at the conjugacy classes associated to ρ for $\rho \in \mathcal{OP}_n(\Gamma_*)$ are given by Frenkel-Jing-Wang [**1**].

LEMMA 3.2. *For an irreducible spin $\widetilde{\Gamma}_n$-character χ_λ with type $\lambda \in \mathcal{SP}^1_n(\Gamma^*)$ and a subset G of $(\widetilde{\Gamma}_n)_*$, set $\langle \chi, \chi\rangle_G = \sum_{D_\rho\in G} \frac{1}{|\widetilde{Z}_\rho|}|\chi(D_\rho)|^2$, then $\langle \chi, \chi\rangle_{\mathcal{OP}_n(\Gamma_*)} = \langle \chi, \chi\rangle_{\mathcal{SP}^1_n(\Gamma_*)} = \frac{1}{2}$ (see [**1**]).*

PROOF. Since $\chi^{'}_\lambda(x) = (-1)^{deg(x)}\chi_\lambda(x)$ for $x \in \widetilde{\Gamma}_n$, we have that

$$
\begin{aligned}
2 =&\langle \chi_\lambda + \chi^{'}_\lambda, \chi_\lambda + \chi^{'}_\lambda\rangle_{\widetilde{\Gamma}_n}\\
=&\Big(\sum_{\rho\in\mathcal{OP}_n(\Gamma_*)} + \sum_{\rho\in\mathcal{SP}^1_n(\Gamma_*)}\Big)\frac{1}{\widetilde{Z}_\rho}|(\chi_\lambda + \chi^{'}_\lambda)(D_\rho)|^2 \qquad (3.2)\\
=&\sum_{\rho\in\mathcal{OP}_n(\Gamma_*)} \frac{1}{\widetilde{Z}_\rho}\cdot 4|\chi_\lambda(D_\rho)|^2
\end{aligned}
$$

Therefore, $\sum_{\rho\in\mathcal{OP}_n(\Gamma_*)} \frac{1}{\widetilde{Z}_\rho}|\chi_\lambda(D_\rho)|^2 = \sum_{\rho\in\mathcal{SP}^1_n(\Gamma_*)} \frac{1}{\widetilde{Z}_\rho}|\chi_\lambda(D_\rho)|^2 = \frac{1}{2}$. □

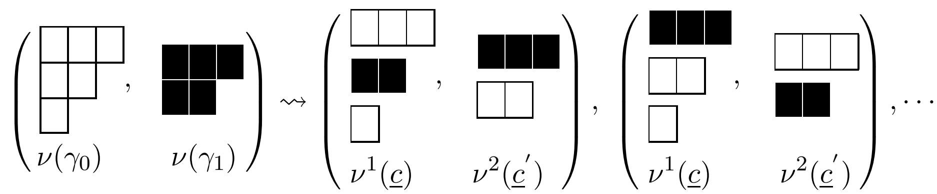
Table 1 shows the status of character values. Part D indicates what we will compute in this paper: $\chi(D^\pm_\rho)$ for $\rho \in \mathcal{SP}^1_n(\Gamma_*)$.

TABLE 1. The spin character table for $\widetilde{\Gamma}_n$

Character \ Class :	$\rho \in \mathcal{OP}_n(\Gamma_*)$	$\rho \in \mathcal{SP}^1_n(\Gamma_*)$
$\lambda \in \mathcal{SP}^0_n(\Gamma_*)$, χ_λ:	A: known	C: 0
$\langle \chi_\lambda, \chi_\lambda\rangle$:	1	0
$\lambda \in \mathcal{SP}^1_n(\Gamma_*)$, χ_λ:	B: known	D: this paper
$\langle \chi_\lambda, \chi_\lambda\rangle$:	1/2	1/2

3.2. Decomposition of colored partitions. When Γ is a finite abelian group, we used the Mackey-Wigner method of little groups to decompose the action of S_n on the characters of Γ^n. It turns out that the invariant subgroup of each S_n-orbit is a Young subgroup of S_n and vice versa. Then we can construct all spin irreducible representations indexed by strict partition-valued functions by induction, and show that the character values are sparsely zero and the non-zero values are given according to how the partitions are supported on various conjugacy classes (see [**3**]). This method is no longer available when Γ is an arbitrary finite group. Next we use a different method to compute spin character values on odd strict partition-valued functions for a general finite group Γ.

We discuss the conjugacy classes generated by Young subgroups. For $\nu = (\nu_\gamma)_{\gamma\in\Gamma^*} \in \mathcal{SP}(\Gamma^*)$, we first erase the coloring of ν_{γ_i} and reassign colors arbitrarily from Γ_*. Suppose $\nu_{\gamma_i} = (\nu^i_1, \nu^i_2, \cdots, \nu^i_s)$ as an ordinary strict partition. Then for any composition $c(\underline{i}) = (c^{i_1}, \cdots, c^{i_s})$ from $\Gamma_* = \{c^0, \cdots, c^r\}$, we assign the colors $c^{i_1}, \cdots, c^{i_s}$ consecutively to the parts of ν_{γ_i} to get a colored partition $\nu^i(\underline{c})$ of the shape ν^i. The resulted multi-colored partition will be denoted by ν^I, where $I \in \{0, \cdots, r\}^{l(\nu)}$. We let $[\nu]$ be the collection of all these multi-colored partitions, which are parameterized by mappings $I : \{1, 2, \cdots, l(\nu)\} \to \{0, \cdots, r\}$. Therefore the cardinality of $[\nu]$ is $(r+1)^{l(\nu)}$. See the following example.



3.3. Spin supermodules vs. spin modules. A spin $\widetilde{\Gamma}_n$-module V becomes a spin supermodule when $ch_V(x) = 0$ for all odd elements x. According to [**5**] there are two basic types of simple supermodules: type M or Q, corresponding to our double spin and a pair of associated spin modules when forgetting the $\mathbb{Z}_2$-gradation. Moreover all double spin and associate spin modules are realized in this way.

For a strict partition-valued function $\lambda = (\lambda_\gamma)_{\gamma \in \Gamma^*} \in \mathcal{SP}_n(\Gamma^*)$, let $J_\lambda = \{\gamma \in \Gamma^* | \lambda_\gamma \text{ is odd strict}\}$ and $J_\lambda' = \Gamma^* - J_\lambda = \{\gamma \in \Gamma^* | \lambda_\gamma \text{ is even strict}\}$. Let U_γ be the irreducible Γ-module associated with the irreducible character $\gamma \in \Gamma^*$. Suppose V_{λ_γ} is the irreducible spin $\widetilde{S}_{|\lambda_\gamma|}$-supermodule determined by the strict partition λ_γ, then $U_\gamma^{\otimes|\lambda_\gamma|} \otimes V_{\lambda_\gamma}$ is a spin $\widetilde{\Gamma}_{|\lambda_\gamma|}$-supermodule. Let $\widetilde{\Gamma}_\lambda$ be the non-trivial double cover of the Young subgroup $\Gamma^n \rtimes S_{(|\lambda_{\gamma_0}|, \cdots, |\lambda_{\gamma_r}|)}$. Then by [**1**], the super tensor product

$$\hat{\bigotimes}_{\gamma \in \Gamma^*} (U_\gamma^{\otimes|\lambda_\gamma|} \otimes V_{\lambda_\gamma})$$

decomposes completely into $2^{[\frac{|J_\lambda|}{2}]}$ copies of an irreducible spin $\widetilde{\Gamma}_\lambda$-supermodule. Denote this irreducible supermodule by W_λ.

PROPOSITION 3.3. *The underlying $\widetilde{\Gamma}_n$-module of the induced supermodule $Ind_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n} W_\lambda$ is an irreducible double spin $\widetilde{\Gamma}_n$-module or a direct sum of two associated irreducible $\widetilde{\Gamma}_n$-modules according to the supermodule is of type M or Q respectively. In terms of partitions this corresponds to whether $d(\lambda) = n - l(\lambda)$ is even or odd.*

PROOF. This is true in a more general context. Each irreducible $\widetilde{\Gamma}_n$-supermodule of type M (resp. Q) is an irreducible double spin (resp. a pair of associated spin) $\widetilde{\Gamma}_n$-module(s). Suppose that the underlying $\widetilde{\Gamma}_n$-module of our irreducible $\widetilde{\Gamma}_n$-supermodule $V = \mathrm{Ind}_{\widetilde{\Gamma}_\lambda}^{\widetilde{\Gamma}_n} W_\lambda$ decomposes into a direct sum of irreducible $\widetilde{\Gamma}_n$-modules:

$$V = \sum_{i=1}^{m} V_i \oplus \sum_{j=1}^{q} (W_j \oplus W_j'),$$

where V_i are irreducible double spin modules, and W_j and W'_j are irreducible associate spin modules. It follows from general theory [**4**] of double spin and associate spin modules that, as an $\Gamma \wr \widetilde{A}_n$-module, $Res(V_i)$ decomposes into $V'_i \oplus V''_i$, while $Res(W_j)$ or $Res(W'_j)$ remains irreducible. Thus we will have

$$\langle V, V \rangle_{\Gamma \wr \widetilde{A}_n} = 2m + 2q.$$

On the other hand we know that $\langle V, V \rangle_{\widetilde{\Gamma}_n} = 1$ or 2 according to the spin supermodule V being of type M or Q by vertex operator calculus [**1**]. This means that $\langle V, V \rangle_{\Gamma \wr \widetilde{A}_n} = 2$, so we must have that either $m = 1$ or $q = 1$. □

3.4. Construction of irreducible spin characters. Let $\tilde{\Delta}_{\lambda_\gamma}$ be the character of the supermodule V_{λ_γ}, then $\hat{\bigotimes}_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|} \otimes \tilde{\Delta}_{\lambda_\gamma}))$ is the character of super tensor product $\hat{\bigotimes}_{\gamma\in\Gamma^*}(U_\gamma^{\otimes|\lambda_\gamma|} \otimes V_{\lambda_\gamma})$. In order to consider the ordinary spin characters of $\widetilde{\Gamma}_\lambda$, Schur defined the starred tensor product $\circledast_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma})$ to be an underlying ordinary irreducible component of $\hat{\bigotimes}_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|} \otimes \tilde{\Delta}_{\lambda_\gamma})$ (see [**11**]). Let $\tilde{\chi}_\lambda$ (resp. $\tilde{\Delta}_{\lambda_\gamma}$) be the character of irreducible supermodule W_λ (resp. V_{λ_γ}). It is known that $\tilde{\chi}_\lambda$ (resp. $\tilde{\Delta}_{\lambda_\gamma}$) is also an irreducible ordinary character when $d(\lambda)$ (resp. $d(\lambda_\gamma)$) is even. We denote it by χ_λ (resp. Δ_λ) when it is regarded as an ordinary character. While $\tilde{\chi}_\lambda$ (resp. $\tilde{\Delta}_{\lambda_\gamma}$) decomposes into two ordinary irreducible characters χ_λ and χ'_λ (resp. Δ_{λ_γ} and Δ'_{λ_γ}) when $d(\lambda)$ (resp. $d(\lambda_\gamma)$) is odd.

Remark 3.4. Let $\rho = \rho^0 \cup \cdots \cup \rho^r$ be a union of partition-valued functions on Γ_* whose parts are union of those of partition-valued functions $\rho^0, \cdots, \rho^r$ on Γ_*. Then D_ρ splits into two conjugacy classes of $\widetilde{\Gamma}_{(|\rho^0|,\cdots,|\rho^r|)}$ if and only if
(1) $\rho = \rho^0 \cup \cdots \cup \rho^r \in \mathcal{OP}_{|\rho^0|+\cdots+|\rho^r|}(\Gamma_*)$ or
(2) $\rho^i \in \mathcal{SP}_{|\rho^i|}(\Gamma_*)$ for $0 \le i \le r$ and $||\rho|| - l(\rho)$ is odd.

For brevity, we have used the numbers $0, 1, \cdots, r$ by $\gamma \in \Gamma^*$ to index each sub-partition-valued function in ρ. More precisely, we denote $\rho_{\gamma_j} := \rho^j$. For $(g, t_\rho) \in \widetilde{\Gamma}_\lambda$ with $\rho = \rho^0 \cup \cdots \cup \rho^r$, one has [**3**]

$$\begin{aligned} \chi_\lambda(g, t_\rho) &= \circledast_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma})(g, t_\rho) \\ &= 2^{[\frac{|J_\lambda|}{2}]}(\sqrt{-1})^{[\frac{|J_\lambda|}{2}]\cdot\prod_{\gamma\in J_\lambda} d(t_{\rho_\gamma})} \cdot \prod_{\gamma\in\Gamma^*} \gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(g, t_{\rho_\gamma}). \end{aligned} \tag{3.3}$$

Let $\tilde{\chi}_\lambda \uparrow$ be the induced character of $\tilde{\chi}_\lambda$ from $\widetilde{\Gamma}_\lambda$ to $\widetilde{\Gamma}_n$. We have $\tilde{\chi}_\lambda \uparrow = \chi_\lambda \uparrow$ when $d(\lambda)$ is even and $\tilde{\chi}_\lambda \uparrow = (\chi_\lambda + \chi'_\lambda) \uparrow = \chi_\lambda \uparrow + \chi'_\lambda \uparrow$ when $d(\lambda)$ is odd. If χ_λ is irreducible, then $\chi_\lambda \uparrow$ is irreducible by Prop. 3.3. Hence by Mackey's decomposition theorem and Frobenius reciprocity we obtain (cf. [**3**]):

$$\langle \chi_\lambda \uparrow, \chi_\lambda \uparrow \rangle_{\widetilde{\Gamma}_n} = \langle \chi_\lambda, \chi_\lambda \rangle_{\widetilde{\Gamma}_\lambda}. \tag{3.4}$$

3.5. Spin character values. Let $\lambda = (\lambda_{\gamma_0}, \cdots, \lambda_{\gamma_r})$ be a strict partition valued function on Γ^*. Set $\rho = \rho^0 \cup \cdots \cup \rho^r$ be the partition-valued function defined in Remark 3.4 such that $|\rho^j| = |\lambda_{\gamma_j}|$ for $j = 0, \cdots, r$. From Schur's results in Theorem 3.1, one sees that if $\Delta_{\lambda_{\gamma_j}}(t_{\rho^j})$ has a nonzero value then ρ^j must be in $[\lambda_{\gamma_j}]$ for $\gamma_j \in J_\lambda$, and ρ^j must lie in $\mathcal{OSP}_{|\lambda_{\gamma_j}|}(\Gamma_*) := \mathcal{OP}_{|\lambda_{\gamma_j}|}(\Gamma_*) \cap \mathcal{SP}_{|\lambda_{\gamma_j}|}(\Gamma_*)$ for $\gamma_j \in J_\lambda'$. then we have the following result.

Proposition 3.5. *Let $\lambda = (\lambda_\gamma)_{\gamma\in\Gamma^*} \in \mathcal{SP}_n^1(\Gamma^*)$. If $\rho = \cup_{j=0}^r \rho_{\gamma_j} \in \mathcal{SP}_n^1(\Gamma_*)$ such that ρ_{γ_j} lies in $[\lambda_{\gamma_j}]$ for $\gamma_j \in J_\lambda$, and ρ_{γ_j} is in $\mathcal{OSP}_{|\lambda_{\gamma_j}|}(\Gamma_*)$ for $\gamma_j \in J_\lambda'$, then*

$$\prod_{\gamma\in J_\lambda'} \Big(\sum_{\rho_\gamma \in \mathcal{OSP}_{|\rho_\gamma|}(\Gamma_*)} \frac{1}{Z_{\rho_\gamma}} \Big| \prod_{c\in\Gamma_*} \gamma(c)^{l(\rho_\gamma(c))} \Delta_{\lambda_\gamma}(t_{\rho_\gamma}) \Big|^2 \Big) = 1. \tag{3.5}$$

Proof. Write $m = |J_\lambda|$ then we have (Note: here m is odd)

$$\begin{aligned}
&\langle \chi_\lambda \uparrow, \chi_\lambda \uparrow \rangle_{\widetilde{\Gamma}_n - \Gamma\wr\widetilde{A}_n} \\
=&2^{-[\frac{m}{2}]\cdot 2} \sum_{\substack{\rho=\cup_{\gamma\in\Gamma^*}\rho_\gamma \in \mathcal{SP}_n^1(\Gamma_*)\\ \rho_\gamma\in\mathcal{SP}_{|\lambda_\gamma|}(\Gamma_*)}} \frac{1}{\widetilde{Z}_\rho} \Big| \big(\circledast_{\gamma\in\Gamma^*} \gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}\big)(D_\rho) \Big|^2 \\
=&2^{-m+1} \cdot \sum_{\substack{\rho=\cup_{\gamma_i\in\Gamma^*}\rho_{\gamma_i} \in \mathcal{SP}_n^1(\Gamma_*)\\ \rho_{\gamma_i}\in\mathcal{SP}_{|\lambda_{\gamma_i}|}(\Gamma_*)}} 2^{m-2} \prod_{\gamma\in\Gamma^*} \Big(\frac{1}{Z_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big) \qquad (3.6)\\
=&\frac{1}{2} \prod_{\gamma\in\Gamma_*} \Big(\sum_{\bar{d}(\rho_\gamma)=\bar{d}(\lambda_\gamma), \rho_\gamma\in\mathcal{SP}_{|\lambda_\gamma|}(\Gamma_*)} \frac{1}{Z_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big)
\end{aligned}$$

As $Z_\rho = \prod_{\gamma\in\Gamma^*} Z_{\rho_\gamma}$ and $\widetilde{Z}_{\rho_\gamma} = 2Z_{\rho_\gamma}$ for $\gamma \in J_\lambda$, so Eq. (3.6) becomes

$$\begin{aligned}
=&\frac{1}{2} \prod_{\gamma\in J_\lambda} \Big(\sum_{\rho_\gamma\in\mathcal{SP}^1_{|\lambda_\gamma|}(\Gamma_*)} \frac{2}{\widetilde{Z}_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big) \cdot \\
&\prod_{\gamma\in J_\lambda'} \Big(\sum_{\rho_\gamma\in\mathcal{SP}^0_{|\lambda_\gamma|}(\Gamma_*)} \frac{1}{Z_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big) \\
=&2^{m-1} \prod_{\gamma\in J_\lambda} \langle \gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}, \gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma} \rangle_{\mathcal{SP}^1_{|\lambda_\gamma|}(\Gamma_*)} \cdot \qquad (3.7)\\
&\prod_{\gamma\in J_\lambda'} \Big(\sum_{\rho_\gamma\in\mathcal{OSP}_{|\lambda_\gamma|}(\Gamma_*)} \frac{1}{Z_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big) \\
=&\frac{1}{2} \prod_{\gamma\in J_\lambda'} \Big(\sum_{\rho_\gamma\in\mathcal{OSP}_{|\lambda_\gamma|}(\Gamma_*)} \frac{1}{Z_{\rho_\gamma}} |\gamma^{\otimes|\lambda_\gamma|} \otimes \Delta_{\lambda_\gamma}(D_{\rho_\gamma})|^2 \Big).
\end{aligned}$$

Thus it follows from [**1**] and Lemma 3.2 that

$$\langle \chi_\lambda\uparrow, \chi_\lambda\uparrow\rangle_{\mathcal{OP}_n(\Gamma_*)} = \langle \chi_\lambda\uparrow, \chi_\lambda\uparrow\rangle_{\mathcal{SP}^1_n(\Gamma_*)} = \frac{1}{2}.$$

Eqs. (3.6) and (3.7) imply that

$$\prod_{\gamma\in J'_\lambda}\Big(\sum_{\rho_\gamma\in\mathcal{OSP}_{|\rho_\gamma|}(\Gamma_*)}\frac{1}{Z_{\rho_\gamma}}\Big|\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\Delta_{\lambda_\gamma}(z^p t_{\rho_\gamma})\Big|^2\Big) = 1. \tag{3.8}$$

□

The following theorem gives the remaining part of the spin character table for $\widetilde{\Gamma}_n$.

Theorem 3.6. *Let $\lambda = (\lambda_\gamma)_{\gamma\in\Gamma^*} \in \mathcal{SP}^1_n(\Gamma^*)$ and $\rho\in\mathcal{SP}^1_n(\Gamma_*)$. (i) If $\rho = \cup_{\gamma\in\Gamma^*}\rho_\gamma$ such that $\rho_\gamma\in[\lambda_\gamma]$ for $\gamma\in J_\lambda$ and $\rho_\gamma\in\mathcal{OSP}_{|\rho_\gamma|}(\Gamma_*)$ for $\gamma\in J'_\lambda$, then*

$$\chi_\lambda\uparrow(D^\pm_\rho) = \pm K_\rho\prod_{\gamma\in\Gamma^*}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\cdot\prod_{\gamma\in J'_\lambda}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})\cdot$$
$$(\sqrt{-1})^{\Sigma_{\gamma\in J_\lambda}\frac{|\lambda_\gamma|-l(\lambda_\gamma)+1}{2}}\sqrt{\frac{\prod_{\gamma\in J_\lambda}z_{\lambda_\gamma}}{2}},$$

*where the value of $\prod_{\gamma\in J'_\lambda}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})$ is given by Schur Q-functions (see [**1**]).*

(ii) $\chi_\lambda\uparrow(D^\pm_\rho) = 0$, otherwise.

Proof. (i) Let T be a left coset of $\widetilde{\Gamma}_\lambda$ in $\widetilde{\Gamma}_n$ and K_ρ is the number of left cosets T of $\widetilde{\Gamma}_\lambda$ in $\widetilde{\Gamma}_n$ such that $(g,t_\rho)T = T$. Then $t^{-1}(g,t_\rho)t\in\widetilde{\Gamma}_\lambda$ for any left coset representative $t\in T$. It follows from the formula of induced character that

$$\begin{aligned}\chi_\lambda\uparrow(D^\pm_\rho) &= \pm\frac{1}{|\widetilde{\Gamma}_\lambda|}\sum_{T\in\widetilde{\Gamma}_n/\widetilde{\Gamma}_\lambda}\Big(\sum_{t\in T}\circledast_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|}\otimes\Delta_{\lambda_\gamma})(t^{-1}(g,t_\rho)t)\Big)\\ &= \pm K_\rho\cdot\circledast_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|}\otimes\Delta_{\lambda_\gamma})(g,t_\rho)\\ &= \pm K_\rho\cdot 2^{\frac{|J_\lambda|-1}{2}}\prod_{\gamma\in\Gamma^*}(\gamma^{\otimes|\lambda_\gamma|}\otimes\Delta_{\lambda_\gamma})(D^+_{\rho_\gamma}).\end{aligned} \tag{3.9}$$

In the above we have used $\chi_\lambda\uparrow(D^-_\rho) = -\chi_\lambda\uparrow(D^+_\rho)$, and the second line vanishes if $t_\rho\notin\widetilde{S}_\lambda$. Moreover, by [**1**],

$$\gamma^{\otimes|\lambda_\gamma|}\otimes\Delta_{\lambda_\gamma}(D^+_{\rho_\gamma}) = \prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\cdot\Delta_{\lambda_\gamma}(t_{\rho_\gamma}).$$

So Eq. (3.9) is equal to

$$
\begin{aligned}
&= \pm K_\rho \cdot 2^{\frac{|J_\lambda|-1}{2}} \prod_{\gamma\in\Gamma^*}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\cdot\prod_{\gamma\in\Gamma^*}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})\\
&= \pm K_\rho \cdot 2^{\frac{|J_\lambda|-1}{2}} \prod_{\gamma\in\Gamma^*}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\cdot\prod_{\gamma\in J_\lambda'}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})\cdot\\
&\qquad \prod_{\gamma\in J_\lambda}(\sqrt{-1})^{\frac{|\rho_\gamma|-l(\rho_\gamma)+1}{2}}\sqrt{\frac{\prod_{c\in\Gamma_*}z_{\rho_\gamma(c)}}{2}}\\
&= \pm K_\rho \prod_{\gamma\in\Gamma^*}\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\cdot\prod_{\gamma\in J_\lambda'}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})\cdot\\
&\qquad (\sqrt{-1})^{\Sigma_{\gamma\in J_\lambda}\frac{|\lambda_\gamma|-l(\lambda_\gamma)+1}{2}}\sqrt{\frac{\prod_{\gamma\in J_\lambda}z_{\lambda_\gamma}}{2}}.
\end{aligned}
\tag{3.10}
$$

(ii) Note that we can assume that $\rho = \cup_{\gamma\in\Gamma^*}\rho_\gamma$, where $\rho_\gamma \in \mathcal{SP}_{|\lambda_\gamma|}(\Gamma_*)$. Otherwise it is easy to see that $\chi_\lambda\uparrow(\rho) = 0$.

(1) If $\rho_\gamma \in \mathcal{OSP}_{|\rho_\gamma|}(\Gamma_*)$ for $\gamma \in J_\lambda'$, we show that $\rho_\gamma \in [\lambda_\gamma]$ for $\gamma \in J_\lambda$. We denote $m = |J_\lambda|$. It follows from Proposition 3.5 that

$$
\begin{aligned}
&\langle\chi_\lambda\uparrow,\chi_\lambda\uparrow\rangle_{\mathcal{SP}^1_n(\Gamma_*)}\\
=&2^{m-1}\prod_{\gamma\in J_\lambda}\Big(\sum_{\rho_\gamma\in\mathcal{SP}^1_{|\lambda_\gamma|}(\Gamma_*)}\frac{1}{\widetilde{Z}_{\rho_\gamma}}\Big|\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\Delta_{\lambda_\gamma}(z^p t_{\rho_\gamma})\Big|^2\Big)\\
\geq&2^{m-1}\prod_{\gamma\in J_\lambda}\Big(\sum_{\rho_\gamma\in[\lambda_\gamma]}\frac{1}{Z_{\rho_\gamma}}\Big|\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}\Delta_{\lambda_\gamma}(t_{\rho_\gamma})\Big|^2\Big)\\
=&2^{m-1}\prod_{\gamma\in J_\lambda}\Big(\sum_{\rho_\gamma\in[\lambda_\gamma]}\Big(\prod_{c\in\Gamma_*}\frac{|\gamma(c)|^{2l(\rho_\gamma(c))}}{z_{\rho_\gamma(c)}\zeta_c^{l(\rho_\gamma(c))}}\Big)\left|\sqrt{\frac{\prod_{c\in\Gamma_*}z_{\rho_\gamma(c)}}{2}}\right|^2\Big)\\
=&2^{m-1}\prod_{\gamma\in J_\lambda}\Big(\sum_{\rho_\gamma\in[\lambda_\gamma]}\frac{1}{2}\frac{\prod_{c\in\Gamma_*}|\gamma(c)|^{2l(\rho_\gamma(c))}}{\prod_{c\in\Gamma_*}\zeta_c^{l(\rho_\gamma(c))}}\Big)
\end{aligned}
\tag{3.11}
$$

Since $\rho_\gamma \in [\lambda_\gamma]$, $l(\rho_\gamma) = l(\lambda_\gamma)$ and

$$\gamma(c)^{l(\rho_\gamma(c))} = \gamma^{\otimes l(\rho_\gamma(c))}\cdot(\underbrace{c,\cdots,c}_{l(\rho_\gamma(c))}),$$

hence we have that

$$
\begin{aligned}
&\prod_{c\in\Gamma_*}\gamma(c)^{l(\rho_\gamma(c))}=\gamma^{\otimes l(\rho_\gamma)}\cdot(\underbrace{c^0,\cdots,c^0}_{l(\rho_\gamma(c^0))},\cdots,\underbrace{c^r,\cdots,c^r}_{l(\rho_\gamma(c^r))})\\
&=\gamma^{\otimes l(\rho_\gamma(c^0))}\otimes\cdots\otimes\gamma^{\otimes l(\rho_\gamma(c^r))}\cdot(\underbrace{c^0,\cdots,c^0}_{l(\rho_\gamma(c^0))},\cdots,\underbrace{c^r,\cdots,c^r}_{l(\rho_\gamma(c^r))})\\
&=\gamma^{\otimes l(\rho_\gamma)}\cdot(c^{i_1},c^{i_2},\cdots,c^{i_{l(\rho_\gamma)}})
\end{aligned}\tag{3.12}
$$

which satisfies that the number of $c^j(j=0,\cdots,r)$ in $\{c^{i_1},c^{i_2},\cdots,c^{i_{l(\rho_\gamma)}}\}$ is equal to $l(\rho_\gamma(c^j))$. Clearly $C_{\rho_\gamma}=(c^{i_1},c^{i_2},\cdots,c^{i_{l(\rho_\gamma)}})$ is a conjugacy class of $\Gamma^{l(\lambda_\gamma)}$. Let $\zeta_{C_{\rho_\gamma}}$ be the order of the centralizer of an element in the conjugacy class C_{ρ_γ} of $\Gamma^{l(\lambda_\gamma)}$, then $\zeta_{C_{\rho_\gamma}}=\prod_{c\in\Gamma_*}\zeta_c^{l(\rho_\gamma(c))}$. As $\rho_\gamma\in[\lambda_\gamma]$, so C_{ρ_γ} can run through all conjugacy classes in $\Gamma^{l(\lambda_\gamma)}$. For example, if we assume $[\lambda_\gamma]=\{\rho_\gamma\in\mathcal{SP}^1_{11}(\Gamma_*)|\lambda_\gamma=(5,4,2)\}$, then $\rho_\gamma=(5_{\square_1},4_{\square_2},2_{\square_3})$ and each $\square_j$ can run from c^0 to c^r. So $C_{\rho_\gamma}=(\square_1,\square_2,\square_3)$ runs through all conjugacy classes of Γ^3. Subsequently Eq. (3.11) becomes

$$
\begin{aligned}
&=2^{m-1}\prod_{\gamma\in J_\lambda}(\sum_{C_{\rho_\gamma}\in(\Gamma^{l(\lambda_\gamma)})_*}\frac{1}{2\zeta_{C_{\rho_\gamma}}}\cdot\\
&\quad|\gamma^{\otimes l(\rho_\gamma(c^0))}\otimes\cdots\otimes\gamma^{\otimes l(\rho_\gamma(c^r))}\cdot(\underbrace{c^0,\cdots,c^0}_{l(\rho_\gamma(c^0))},\cdots,\underbrace{c^r,\cdots,c^r}_{l(\rho_\gamma(c^r))})|^2)\\
&=2^{m-1}\prod_{\gamma\in J_\lambda}(\sum_{C_{\rho_\gamma}\in(\Gamma^{l(\lambda_\gamma)})_*}\frac{1}{2\zeta_{C_{\rho_\gamma}}}\cdot|\gamma^{\otimes l(\rho_\gamma)}(c^{i_1},c^{i_2},\cdots,c^{i_{l(\rho_\gamma)}})|^2)\\
&=2^{m-1}\prod_{\gamma\in J_\lambda}(\frac{1}{2}\sum_{C_{\rho_\gamma}\in(\Gamma^{l(\lambda_\gamma)})_*}\frac{1}{\zeta_{C_{\rho_\gamma}}}|\gamma^{\otimes l(\lambda_\gamma)}(C_{\rho_\gamma})|^2)\\
&=2^{m-1}\prod_{\gamma\in J_\lambda}\frac{1}{2}\langle\gamma^{\otimes l(\lambda_\gamma)},\gamma^{\otimes l(\lambda_\gamma)}\rangle_{\Gamma^{l(\lambda_\gamma)}}=\frac{1}{2},
\end{aligned}\tag{3.13}
$$

which forces $\chi_\lambda\uparrow(D^{\pm}_\rho)=0$ if $\rho_\gamma\notin[\lambda_\gamma]$ for $\gamma\in J_\lambda$.

(2) If $\rho_\gamma\notin\mathcal{OSP}_{|\lambda_\gamma|}(\Gamma_*)$ for $\gamma\in J'_\lambda$, then there is at least one ρ_γ not in $\mathcal{OP}_{|\lambda_\gamma|}(\Gamma_*)$ for $\gamma\in J'_\lambda$. Meanwhile, Δ_{λ_γ} is a double spin character when $\gamma\in J'_\lambda$, so we have $\Delta_{\lambda_\gamma}(t_{\rho_\gamma})=0$, thus $\chi_\lambda\uparrow(D^{\pm}_\rho)=0$. This completes the proof. □

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