ASYMPTOTICS OF THE YANG-MILLS FLOW FOR HOLOMORPHIC VECTOR BUNDLES OVER KÄHLER MANIFOLDS: THE CANONICAL STRUCTURE OF THE LIMIT

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Abstract. In the following article we study the limiting properties of the Yang-Mills flow associated to a holomorphic vector bundle E over an arbitrary compact Kähler manifold (X, ω). In particular we show that the flow is determined at infinity by the holomorphic structure of E. Namely, if we fix an integrable unitary reference connection A_0 defining the holomorphic structure, then the Yang-Mills flow with initial condition A_0, converges (away from an appropriately defined singular set) in the sense of the Uhlenbeck compactness theorem to a holomorphic vector bundle E_∞, which is isomorphic to the associated graded object of the Harder-Narasimhan-Seshadri filtration of (E, A_0). Moreover, E_∞ extends as a reflexive sheaf over the singular set as the double dual of the associated graded object. This is an extension of previous work in the cases of 1 and 2 complex dimensions and proves the general case of a conjecture of Bando and Siu.

Contents

1. Introduction

This paper is a study of the Yang-Mills flow, the L^2-gradient flow of the Yang-Mills functional; and in particular its convergence properties at infinity. The flow is (after imposing the Coulomb gauge condition) a parabolic equation for a connection on a holomorphic vector bundle. Very soon after the introduction of the flow equations, Donaldson and Simpson proved that in the case of a stable bundle the gradient flow converges smoothly at infinity (see [DO1],[DO2],[SI]). In the unstable case the behaviour of the flow is more ambiguous. Nevertheless, even in the general case there is an appropriate notion of convergence (a version of Uhlenbeck's compactness theorem) that is always satisfied. The goal of this article is to prove that this notion depends only on the holomorphic structure of the original bundle.

We follow up on work whose origin lies in two principal directions, both related to stability properties of holomorphic vector bundles over compact Kähler manifolds. The first strain is the seminal work of Atiyah and Bott [AB], in which the authors study the moduli space of stable holomorphic bundles over Riemann surfaces. In particular, they computed the $\mathcal{G}^{\mathbb{C}}$-equivariant Betti numbers of this space in certain cases, where $\mathcal{G}^{\mathbb{C}}$ is the complex gauge group of a holomorphic vector bundle E (over a Riemann surface X) acting on the space $\mathcal{A}_{\mathrm{hol}}$ of holomorphic structures of E. Their approach was to stratify $\mathcal{A}_{\mathrm{hol}}$ by Harder-Narasimhan type. The type is a tuple of rational numbers $\mu = (\mu_1, \cdots, \mu_R)$ associated to a holomorphic structure $(E, \bar{\partial}_E)$, defined using a filtration of E by analytic subsheaves whose successive quotients are semi-stable, called the Harder-Narasimhan filtration. One of the resulting strata of $\mathcal{A}_{\mathrm{hol}}$ consists of the semi-stable bundles. Furthermore the action of $\mathcal{G}^{\mathbb{C}}$ preserves the stratification, and the main result that yields the computation of the equivariant Betti numbers is that the stratification by Harder-Narasimhan type is equivariantly perfect under this action.

Atiyah and Bott also noticed that the problem might be amenable to a more analytic approach. Specifically they considered the Yang-Mills functional YM on the space $\mathcal{A}_h$ of integrable, unitary connections with respect to a fixed hermitan metric on E. The space $\mathcal{A}_h$ may be identified with $\mathcal{A}_{\mathrm{hol}}$ by sending a connection ∇_A to its $(0,1)$ part $\bar{\partial}_A$. The Yang-Mills functional is defined by taking the L^2 norm of the curvature of ∇_A, and is a Morse function on $\mathcal{A}_h/\mathcal{G}$, where $\mathcal{G}$ is the unitary gauge group. Therefore this functional induces the usual stable-unstable manifold stratification on $\mathcal{A}_h$ (or equivalently $\mathcal{A}_{\mathrm{hol}}$) familiar from Morse theory. It is natural to conjecture that this analytic stratification is in fact the same as the algebraic stratification given by the Harder-Narasimhan type. The authors of [AB] stopped short of proving this statement, instead leaving it at the conjectural level, and working directly with the algebraic stratification. They noted however that a key technical point in proving the equivalence was to show the convergence of the gradient flow of the Yang-Mills functional at infinity. This was proven in [D] by Daskalopoulos (see also [R]). Specifically, in the case of Riemann surfaces, Daskalopoulos showed the asymptotic convergence of the Yang-Mills Flow, that there is indeed a well-defined stratification in the sense of Morse theory in this case, and that it coincides with the algebraic stratification (which makes sense in all dimensions).

When (X, ω) is a higher dimensional Kähler manifold, the Yang-Mills flow fails to converge in the usual sense. This brings us to the second strain of ideas of which the present paper is a continuation: the so-called "Kobayashi-Hitchin correspondences". These are statements (in various levels of generality) relating the existence of Hermitian-Einstein metrics on a holomorphic bundle E, to the stability of E. Namely, E admits an Hermitian-Einstein metric if and only if E is polystable. This was first proven in the case of a Riemann surface by Narasimhan and Seshadri. Their proof did not use differential geometry, and the condition that the bundle admits an Hermitian-Einstein connection was originally formulated purely in terms of representations of the fundamental group of the Riemann surface. It was Donaldson who gave the first proof using gauge theory, reformulating the statement in terms of a metric of constant central curvature. He initially did this in the case of a Riemann surface in [DO3] by considering sequences of connections in a complex gauge orbit that are minimising for a certain functional, which is analogous to our HYM_α functionals defined in Section 3.2. Shortly after this, Donaldson extended the result to the case of algebraic surfaces in [DO1], and later to the case of projective complex manifolds in [DO2]. In both [DO1] and [DO2] the idea of the proof was to reformulate the flow as an equivalent parabolic PDE, show long-time existence of the equation, and then prove that for a stable bundle, this modified flow indeed converges, the solution being the desired Hermitian-Einstein metric. This was generalised by Uhlenbeck and Yau in [UY] in the case of a compact Kähler manifold using different methods. Finally, in [BS], Bando and Siu extended the correspondence to coherent analytic sheaves on Kähler manifolds by considering what they called "admissible" hermitian metrics, which are metrics on the locally free part of the sheaf having controlled curvature. They also conjectured that there should also be a correspondence (albeit far less detailed) between the Yang-Mills flow and the Harder-Narasimhan filtration in higher dimensions despite the absence of a Morse theory for the Yang-Mills functional.

There are two main features that distinguish the higher dimensional case from the case of Riemann surfaces. As previously mentioned, the flow does not converge in general. However, the only obstruction to convergence is bubbling phenomena. Specifically, one of Uhlenbeck's compactness results (see [UY] Theorem 5.2) applies to the flow, which means that there are always subsequences that converge (in a certain Sobolev norm) away from a singular set of Hausdorff codimension at least 4 inside X (which we will denote by Z_{an}), to a connection on a possibly different vector bundle E_∞. A priori, this pair of a limiting connection and bundle depends on the subsequence. In the case of two complex dimensions, the singular set is a locally finite set of points (finite in the compact case) and by Uhlenbeck's removable singularities theorem E_∞ extends over the singular set as a vector bundle with a Yang-Mills connection. In higher dimensions, again due to a result of Bando and Siu, E_∞ extends over the singular set, but only as a reflexive sheaf. Although we will not use their result, Hong and Tian have proven in [HT] that in fact the convergence is in C^∞ on the complement of Z_{an} and that Z_{an} is a holomorphic subvariety.

A separate, but intimately related issue is the Harder-Narasimhan filtration. In the case of a Riemann surface the filtration is given by subbundles. In higher dimensions, it is only a filtration by subsheaves. Again however, away from a singular set Z_{alg}, which is a complex analytic subset of X of complex codimension at least 2, the filtration is indeed given by subbundles. Once more, in the case of a Kähler surface this is a locally finite set of points (finite in the compact case).

The main result of this paper (the conjecture of Bando and Siu), describes the relationship between the analytic and algebraic sides of the above picture. To state it, we recall that there is a refinement of the Harder-Narasimhan filtration called the Harder-Narasimhan-Seshadri filtration, which is a double filtration whose successive quotients are stable rather than merely semi-stable. Then if $(E, \bar\partial_E)$ is a holomorphic vector bundle where the operator $\bar\partial_E$ denotes the holomorphic structure, write $Gr_\omega^{HNS}(E, \bar\partial_E)$ for the associated graded object (the direct sum of the stable quotients) of the Harder-Narasimhan-Seshadri filtration. Notice that by the Kobayashi-Hitchin correspondence, $Gr_\omega^{HNS}(E, \bar\partial_E)$ also carries a natural Yang-Mills connection on its locally free part, given by the direct sum of the Hermitian-Einstein connections on each of the stable factors, and this connection is unique up to gauge. The main theorem says in particular that the limiting bundle along the flow is in fact independent of the subsequence chosen in order to employ Uhlenbeck compactness, and is determined entirely by the holomorphic structure $\bar\partial_E$ of E. Furthermore, the limiting connection is precisely the connection on $Gr_\omega^{HNS}(E, \bar\partial_E)$.

Theorem 1.1. *Let (X, ω) be a compact Kähler manifold, and $E \to X$ be an hermitian vector bundle. Let A_0 denote an integrable, unitary connection endowing E with a holomorphic structure $\bar\partial_E = \bar\partial_{A_0}$. Let A_∞ denote the Yang-Mills connection on $Gr_\omega^{HNS}(E, \bar\partial_E)$ restricted to $X - Z_{\mathrm{alg}}$ induced from the Kobayashi-Hitchin correspondence. Let A_t be the time t solution of the flow with initial condition A_0. Then as $t \to \infty$, $A_t \to A_\infty$ in the sense of Uhlenbeck, and on $X - Z_{\mathrm{alg}} \cup Z_{\mathrm{an}}$, the vector bundles $Gr_\omega^{HNS}(E, \bar\partial_E)$ and the limiting bundle E_∞ are holomorphically isomorphic. Moreover, E_∞ extends over Z_{an} as a reflexive sheaf to $\left(Gr_\omega^{HNS}(E, \bar\partial_E)\right)^{**}$.*

This theorem was proven in [DW1] by Daskalopoulos and Wentworth in the case when $\dim X = 2$. In this case, the filtration consists of vector bundles, whose successive quotients may have point singularities. As stated earlier, this means E_∞ extends as a vector bundle and [DW1] proves that this bundle is isomorphic to the vector bundle $\left(Gr_\omega^{HNS}(E, \bar\partial_E)\right)^{**}$.

We now give an overview of our proof, pointing out what goes through directly from [DW1] and where we require new arguments. Section 2 consists of the basic definitions we need from sheaf theory, including the Harder-Narasimhan and Harder-Narasimhan-Seshadri filtrations and their associated graded objects, as well as the corresponding types. We also discuss the Yang-Mills functional, the Hermitian-Yang-Mills functional and the version of the Uhlenbeck compactness result that we will need. Although we will primarily be concerned with the flow, the proof of Theorem 1.1 is set up to work for slightly more general sequences of connections, so we state the compactness theorem in this generality first, and specialise to the flow when appropriate. Lastly, we recall the notion of a weakly holomorphic projection operator associated to a subsheaf first introduced in [UY], the Chern-Weil formula, and a lemma on the boundedness of second fundamental forms from [DW1].

In Section 3 we introduce the Yang-Mills flow and its basic properties. We recast Uhlenbeck compactness in the context of the flow, which satisfies the boundedness conditions required to apply the general theorem. We recall one of the main results of [DW1], that the Harder-Narasimhan type of an Uhlenbeck limit is bounded from below by the type of the initial bundle with respect to the partial ordering on types. Finally, Section 3 ends with a discussion of Yang-Mills type functionals associated to Ad-invariant convex functions on the Lie algebra of the unitary group.

Section 4 details the main results we will need about resolution of singularities. This is the first place in which our presentation differs fundamentally from that of [DW1]. The main strategy of the proof is to eliminate the singular set of the Harder-Narasimhan-Seshadri filtration by blowing up, and doing all the necessary analysis on the blowup. In the two-dimensional case, since the singularities consist only of points, this can be done directly by hand as in [DW1] see also [BU1]. In the general case we must appeal to the resolution of singularities theorem of Hironaka see [H1] and [H2]. We consider the filtration as a rational section of a flag bundle, and apply the resolution of indeterminacy theorem for rational maps. If we write $\pi : \tilde{X} \to X$ for the composition of the blowups involved in resolution, the result is that the pullback bundle $\pi^*E \to \tilde{X}$ has a filtration by subbundles, which away from the exceptional divisor $\mathbf{E}$ is precisely the filtration on X.

We will need to consider a natural family of Kähler metrics ω_ε on $\tilde{X}$, which are perturbations of the pullback form $\pi^*\omega$ by the irreducible components of the exceptional divisor, and which are introduced in order to compensate for the fact that $\pi^*\omega$ fails to be a metric on $\mathbf{E}$. The filtration of π^*E by subbundles is not quite the Harder-Narasimhan-Seshadri filtration with respect to ω_ε but is closely related. In particular, the main result of this section is that the Harder-Narasimhan type of π^*E with respect to ω_ε converges to the type of E with respect to ω. This was proven in the surface case in [DW1] using an argument of Buchdahl from [BU1]. The proof contained in [DW1] seems to be insufficient in the higher dimensional case, so we give a rather different proof of this result. The main ingredient is a bound on the ω_ε degree of a subsheaf of π^*E with torsion-free quotient in terms of its pushforward sheaf that is uniform as $\varepsilon \to 0$. To prove this we use standard algebro-geometric facts together with a modification of an argument of Kobayashi [KOB] first used to prove the uniform boundedness of the degree of subsheaves of a vector bundle with respect to a fixed Kähler metric. In particular we prove the following theorem:

Theorem 1.2. *Let (X, ω) be a compact Kähler manifold and $\tilde{S}$ be a subsheaf (with torsion free quotient $\tilde{Q}$) of a holomorphic vector bundle $\tilde{E}$ on $\tilde{X}$, where $\pi : \tilde{X} \to X$ is given by a sequence of blowups along complex submanifolds of* codim ≥ 2*. Then then there is a uniform constant M such that the degrees of $\tilde{S}$ and $\tilde{Q}$ with respect to ω_ε satisfy:* $\deg(\tilde{S}, \omega_\varepsilon) \leq \deg(\pi_*\tilde{S}) + \varepsilon M$*, and* $\deg(\tilde{Q}, \omega_\varepsilon) \geq \deg(\pi_*\tilde{Q}) - \varepsilon M$.

Similar statements are proven in the case of a surface by Buchdahl [BU1] and for projective manifolds by Daskalopoulos and Wentworth see [DW3].

Section 5 is the technical heart of the proof. An essential fact needed to complete the proof of Theorem 1.1 is that the Harder-Narasimhan type of the limiting sheaf is in fact equal to the type of the initial bundle. This fact seems to be closely related to the existence of what is called an L^p-approximate critical hermitian structure. In rough terms this is an hermitian metric on a holomorphic vector bundle whose Hermitian-Einstein tensor is L^p-close to that of a Yang-Mills connection (a critical value) determined by the Harder-Narasimhan type of the bundle (see Definition 5.1). Since any connection on E has Hermitian-Yang-Mills energy bounded below by the type of E, and we have a monotonicity property along the flow, the result of Section 3 implies that the existence of an approximate structure then ensures that the flow starting from this initial condition realises the correct type in the limit. Then one shows that *any* initial condition flows to the correct type, essentially by proving that the set of such metrics is open and closed (and non-empty by the existence of an approximate structure) in the space of smooth metrics, and applying the connectivity of the latter space. This last argument appears in detail in [DW1] and we do not repeat it. The main theorem of Section 5 is the following:

Theorem 1.3. *Let $E \to X$ be a holomorphic vector bundle over a Kähler manifold with Kähler form ω. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

The method does not extend to $p = \infty$. This is straightforward in the case when the filtration is given by subbundles (even for $p = \infty$). Given an exact sequence of holomorphic vector bundles:

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

and hermitian metrics on S and Q, one can scale the second fundamental form $\beta \mapsto t\beta$ to obtain an isomorphic bundle whose Hermitian-Einstein tensor is close to the direct sum of those of S and Q. In general it seems difficult to do this directly. The problem here is that the filtration is not in general given by subbundles, and so the vast majority Section 5 is an argument needed to address this point. This is precisely where we need the resolution of the filtration obtained in Section 4. We first take the direct sum of the Hermitian-Einstein metrics on the stable quotients in the resolution by subbundles, which sits inside the pullback π^*E under the blowup map $\pi : \tilde{X} \to X$. Then the argument above shows that after modifying this metric by a gauge transformation, its Hermitian-Einstein tensor becomes close to the type in the L^p norm. We complete the proof by pushing this metric down to $E \to X$ using a cutoff argument.

In broad outline our discussion in Section 5 follows the ideas in [DW1]. The principal difference is that the authors of [DW1] were able to rely on the fact that the singular set was given by points when applying the cutoff argument, in particular they knew that there were uniform bounds on the derivatives of the cutoff function. We must allow for the fact that the singular set is higher dimensional, and therefore need to replace their arguments involving coverings of the singular set by disjoint balls of arbitrarily small radius by calculations in a tubular neighbourhood. We first assume Z_{alg} is smooth and that blowing up once along Z_{alg} resolves the singularities. The essential point is that the Hausdorff codimension of Z_{alg} is large enough to allow the arguments of [DW1] to go through in this case. We then reduce the general theorem to this case by applying an inductive argument on the number of blowups required to resolve the filtration. It is here that we crucially use the convergence of the Harder-Narasimhan type proven in section 4.

In Section 6, following Bando and Siu, we introduce a degenerate Yang-Mills flow on the composition of blowups $\tilde{X}$ with respect to the degenerate metric $\pi^*\omega$. We review some basic properties of this flow that are necessary for the proof of Theorem 1.1. In particular we show that a solution of this degenerate flow is in fact an hermitian metric, and solves the ordinary flow equations with respect to the metric $\pi^*\omega$ away from the exceptional divisor $\mathbf{E}$.

Section 7 completes the proof of the main theorem by showing the isomorphism of the limit E_∞ with $\left(Gr_\omega^{HNS}(E, \bar{\partial}_E)\right)^{**}$. The basic idea follows that of [DW1] which in turn is a generalisation of the argument of Donaldson in [DO1]. His idea is to construct a non-zero holomorphic map to the limiting bundle as the limit of the sequence of gauge transformations defined by the flow. In the case that the initial bundle is stable and has stable image, one may apply the basic fact that such a map is always an isomorphism. In general, the idea in [DW1] is simply to apply this argument to the first factor of the associated graded object (which is stable) and then perform an induction. The image of the first factor will be stable because of the result in Section 5 about the type of the limiting sheaf. The difficulty with this method is in proving that the limiting map is in fact non-zero. This follows directly from Donaldson's proof in the case of a single subsheaf, but it is more complicated to construct such a map on the entire filtration. The authors of [DW1] avoid applying Donaldson's method directly by appealing to a complex analytic argument involving analytic extension see also [BU2]. Arguing in this fashion makes the induction rather easier. However, this requires the complement of the singular set to have strictly pseudo-concave boundary, which is true in the case of surfaces, but is not guaranteed in higher dimensions.

Therefore we give a proof of a slightly more differential geometric character. Namely, in the case that the filtration is given by subbundles, we follow the argument of Donaldson, which goes through with modest corrections in higher dimensions, and does indeed suffice to complete the induction alluded to. In the general case, we must again appeal to a resolution of singularities of the filtration and apply the previous strategy to the pullback bundle over the composition of blowups $\tilde{X}$. The problem one encounters with this approach is that the induction breaks down due to the appearance of second fundamental forms of each piece of the filtration, which are not bounded in L^∞ with respect to the degenerate metric $\pi^*\omega$. To rectify this, we apply the degenerate flow of Section 6 for some fixed non-zero time t to each element of the sequence of connections, and this new sequence does have the desired bound. This is due to the key observation of Bando and Siu that the Sobolev constant of $\tilde{X}$ with respect to the metrics ω_ε is bounded away from zero. A theorem of Cheng and Li then implies uniform control over subsolutions to the heat equation, which is sufficient to understand the degenerate flow. One then has to show that the limit obtained from this new sequence of connections is independent of t and is the correct one. This section is an expanded and slightly modified account of an argument contained in the unpublished preprint [DW3].

We conclude the introduction with some general comments. First of all, as pointed out in [DW1], the proof of Theorem 1.1 is essentially independent of the flow, and one obtains a similar theorem by restricting to sequences of connections which are minimising with respect to certain Hermitian-Yang-Mills type functionals. Indeed, the statement appears explicitly as Theorem 7.1. Secondly, one expects that there should be a relationship between the two singular sets Z_{alg} and Z_{an}. Namely, in the best case Z_{alg} should be exactly the set of points where bubbling occurs. One always has containment $Z_{\text{alg}} \subset Z_{\text{an}}$, and in the separate article [DW2] Daskalopoulos and Wentworth have shown that in the surface case equality does in fact hold. We hope to be able to clarify this issue in higher dimensions in a future paper.

Finally, the author is aware of a recent series of preprints [J1],[J2],[J3] by Adam Jacob which collectively give a proof of Theorem 1.1 using different methods.

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2. Preliminary Remarks

2.1. **Subsheaves of Holomorphic Bundles and the HNS Filtration.** We now recall some basic sheaf theory. All of this material may be found in [KOB]. As stated in the introduction, the main obstacle we will face is that we must consider arbitrary subsheaves of a holomorphic vector bundle. Throughout, X will be a compact Kähler manifold (unless otherwise stated) with Kähler form ω, E a holomorphic vector bundle, and $S \subset E$ a subsheaf.

Recall that an analytic sheaf $\mathcal{E}$ on X is called torsion free if the natural map $\mathcal{E} \to \mathcal{E}^{**}$ is injective. We call $\mathcal{E}$ reflexive if this map is an isomorphism. Of vital importance is the fact that a torsion free sheaf is "almost a vector bundle" in the following sense. For $\mathcal{E}$ a sheaf on X recall that its singular set is $\mathrm{Sing}(\mathcal{E}) = \{x \in X \mid \mathcal{E}_x \textit{ is not free}\}$. Here $\mathcal{E}_x$ is the stalk of $\mathcal{E}$ over x. In other words $\mathrm{Sing}(\mathcal{E})$ is the set of points where $\mathcal{E}$ fails to be locally free, i.e., a vector bundle. The set $\mathrm{Sing}(\mathcal{E})$ is a closed complex analytic subvariety of X of codimension at least 2.

Recall that the saturation of a subsheaf $S \subset E$ is defined by $\mathrm{Sat}_E(S) = \ker(E \to Q/\operatorname{Tor}(Q))$ and that S is a subsheaf of $\mathrm{Sat}_E(S)$ with torsion quotient, and the quotient $E/\mathrm{Sat}_E(S)$ is torsion free. We also have the following lemma whose proof we omit.

Lemma 2.1. *Let E be a holomorphic vector bundle. Suppose $S_1 \subset S_2 \subset E$ are subsheaves with S_2/S_1 torsion. Then $\mathrm{Sat}_E(S_1) = \mathrm{Sat}_E(S_2)$.*

The ω-**slope** of a torsion free sheaf $\mathcal{E}$ on X is defined by:

$$\mu_\omega(\mathcal{E}) = \frac{\deg_\omega(\mathcal{E})}{\mathrm{rk}(\mathcal{E})} = \frac{1}{\mathrm{rk}(\mathcal{E})}\int_X c_1(\mathcal{E}) \wedge \omega^{n-1}.$$

Note that the right hand side is well defined independently of the representative for $c_1(\mathcal{E})$ since ω is closed. Throughout we will assume that the volume of X with respect to ω is normalised to be $\frac{2\pi}{(n-1)!}$, where $n = \dim_{\mathbb{C}} X$.

Definition 2.2. *We say that a torsion free sheaf $\mathcal{E}$ is ω-**stable** (ω-**semistable**) if for all proper subsheaves $S \subset \mathcal{E}$, $\mu_\omega(S) < \mu_\omega(\mathcal{E})$ ($\mu_\omega(S) \le \mu_\omega(\mathcal{E})$). Equivalently $\mu_\omega(Q) > \mu_\omega(\mathcal{E})$ ($\mu_\omega(Q) \ge \mu_\omega(\mathcal{E})$) for every torsion free quotient Q.*

We have the following important proposition.

Proposition 2.3. *There is an upper bound on the set of slopes $\mu_\omega(S)$ of subsheaves of a torsion free sheaf $\mathcal{E}$, and moreover this upper bound is realised by some subsheaf $\mathcal{E}_1 \subset \mathcal{E}$. Furthermore, we can choose $\mathcal{E}_1$ so that for any $S \subset \mathcal{E}$, if $\mu_\omega(S) = \mu_\omega(\mathcal{E}_1)$ then $\mathrm{rk}(S) \le \mathrm{rk}(\mathcal{E}_1)$. Moreover such a subsheaf is unique.*

For the proof see Kobayashi [KOB]. The sheaf $\mathcal{E}_1$ is called the **maximal destabilising subsheaf** of $\mathcal{E}$. This sheaf is also clearly semistable.

Remark 2.4. *If $S \subset \mathcal{E}$ is a subsheaf with torsion free quotient $Q = \mathcal{E}/S$, then $Q^* \hookrightarrow \mathcal{E}^*$ is a subsheaf and $\deg(Q^*) = -\deg(Q)$. By the above proposition $\mu_\omega(Q^*)$ is bounded from above, so $\mu_\omega(Q)$ is bounded from below.*

Remark 2.5. *Note also that the saturation of a sheaf has slope at least as large as the slope of the original sheaf. Therefore the maximal destabilising subsheaf is saturated by definition.*

Definition 2.6. *We will write $\mu^{\max}(\mathcal{E})$ for the maximal slope of a subsheaf, and $\mu^{\min}(\mathcal{E})$ for the minimal slope of a torsion free quotient. Clearly we have the equality $\mu^{\min}(\mathcal{E}) = -\mu^{\max}(\mathcal{E}^*)$.*

We now specialise to the case of a holomorphic vector bundle E, although the following all holds also for an arbitrary torsion-free sheaf.

Proposition 2.7. *There is a filtration:*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

such that the quotients $Q_i = E_i/E_{i-1}$ are torsion free and semistable, and $\mu_\omega(Q_{i+1}) < \mu_\omega(Q_i)$. Furthermore, the associated graded object: $Gr^{HN}_\omega(E) = \oplus_i Q_i$, is uniquely determined by the isomorphism class of E and is called the ***Harder-Narasimhan filtration****. The Harder-Narasimhan filtration is unique.*

In the sequel we will usually abbreviate this as the HN filtration, and we will write $\mathbb{F}^{HN}_i(E)$ for the i^{th} piece of the filtration. The previous proposition follows from Proposition 2. The maximal destabilising subsheaf is $\mathbb{F}^{HN}_1(E)$. Then consider the quotient $E/\mathbb{F}^{HN}_1(E)$ and its maximal destabilising subshseaf. Define $\mathbb{F}^{HN}_2(E)$ to be the pre-image of this subsheaf under the natural projection. Iterating this process gives the stated filtration, and one easily checks that it has the desired properties.

Another invariant of the isomorphism class of E is the collection of all slopes of the quotients Q_i.

Definition 2.8. *Let E have rank K. Then we form a K-tuple*

$$\mu(E) = (\mu(Q_1), \cdots, \mu(Q_1), \cdots, \mu(Q_i), \cdots, \mu(Q_i), \cdots \mu(Q_l), \cdots \mu(Q_l))$$

where $\mu(Q_i)$ is repeated $\mathrm{rk}(Q_i)$ times. Then $\mu(E)$ is called the ***Harder-Narasimhan*** *(or HN)* ***type*** *of E.*

We will also need a result describing the HN filtration of E in terms of the HN filtration of a subsheaf S and its quotient Q. The following lemma and its corollary are elementary and we omit the proofs.

Proposition 2.9. *Let $0 \to S \to E \to Q \to 0$ be an exact sequence of torsion free sheaves with E a holomorphic vector bundle such that $\mu^{\min}(S) > \mu^{\max}(Q)$. Then the HN filtration of E is given by:*

$$0 \subset \mathbb{F}^{HN}_1(S) \subset \cdots \subset \mathbb{F}^{HN}_k(S) = S \subset \mathbb{F}^{HN}_{k+1}(E) \subset \cdots \subset \mathbb{F}^{HN}_l(E) = E,$$

where $\mathbb{F}^{HN}_{k+i}(E) = \ker(E \to Q/\mathbb{F}^{HN}_i(Q))$, for $i = 0, 1, \cdots, l-k$. In particular, this means that $Q_i = \mathbb{F}^{HN}_{k+i}(E)/\mathbb{F}^{HN}_{k+i-1}(E) = \mathbb{F}^{HN}_i(Q)$ and therefore $Gr^{HN}(E) = Gr^{HN}(S) \oplus Gr^{HN}(Q)$.

Corollary 2.10. *Suppose that $0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E$ is a filtration of E by subbundles, and suppose that for each i, $\mu^{\min}(E_i) > \mu^{\max}(E/E_i)$. Then the Harder-Narasimhan filtration of E is given by:*

$$\begin{aligned} 0 &\subset \mathbb{F}^{HN}_1(E_1) \subset \cdots \subset \mathbb{F}^{HN}_{k_1}(E_1) = E_1 \subset \cdots \subset \mathbb{F}^{HN}_{k_1+\cdots+k_{l-1}}(E_{l-1}) = E_{l-1} \\ &\subset \mathbb{F}^{HN}_{k_1+\cdots+k_{l-1}+1}(E) \subset \cdots \subset \mathbb{F}^{HN}_{k_1+\cdots+k_l}(E) = E. \end{aligned}$$

Now we will define the double filtration that appears in the statement of the Main Theorem. Its existence follows from the existence of the HN filtration and the following proposition.

Proposition 2.11. *Let Q be a semi-stable torsion free sheaf on X. Then there is a filtration:*

$$0 \subset F_1 \subset \cdots \subset F_l = Q$$

such that F_i/F_{i-1} is stable and torsion-free. Also, for each i we have $\mu(F_i/F_{i-1}) = \mu(Q)$. The associated graded object:

$$Gr^S_\omega(Q) = \oplus_i F_i/F_{i-1}$$

is uniquely determined by the isomorphism class of Q, though the filtration itself is not. Such a filtration is called a ***Seshadri filtration*** *of Q.*

Proposition 2.12. *Let E be a holomorphic vector bundle on X. Then there is a double filtration $\{E_{i,j}\}$ with the following properties. If the HN filtration is given by:*

$$0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E,$$

then $E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,l_i} = E_i$, where the successive quotients $Q_{i,j} = E_{i,j}/E_{i,j-1}$ are stable and torsion-free. Furthermore: $\mu_\omega(Q_{i,j}) = \mu_\omega(Q_{i,j+1})$ for $j > 0$, and $\mu_\omega(Q_{i,j}) > \mu_\omega(Q_{i+1,j})$. The associated graded object $Gr^{HNS}_\omega(E) = \oplus_i \oplus_j Q_{i,j}$ is uniquely determined by the isomorphism class of E. This double filtration is called the ***Harder-Narasimhan-Seshadri filtration*** *(or HNS filtration) of E.*

Similarly, we can define an K-tuple:

$$\mu = (\mu(Q_{1,1}), \cdots, \mu(Q_{i,j}), \cdots, \mu(Q_{l,k_l}))$$

where each $\mu(Q_{i,j})$ is repeated according to $\mathrm{rk}(Q_{i,j})$. Note that this vector is exactly the same as the Harder-Narasimhan type of E (the slopes of a Seshadri filtration are all equal). Since each of the quotients $Q_{i,j}$ is torsion-free, $\mathrm{Sing}(Q_{ij})$ lies in codimension at least 2. We will write:

$$Z_{\mathrm{alg}} = \cup_{i,j} \mathrm{Sing}(E_{i,j}) \cup \mathrm{Sing}(Q_{i,j}).$$

This is a complex analytic subset of codimension at least two, and corresponds exactly to the set of points at which the HNS filtration fails to be given by subbundles. We will refer to it as the algebraic singular set of the filtration.

2.2. **The Yang-Mills Functional and Uhlenbeck Compactness.** Recall that for $E \to X$ a complex vector bundle, the set of holomorphic structures on E may be identified with the set of operators $\bar{\partial}_E$ satisfying the Leibniz rule and the integrability condition $\bar{\partial}_E \circ \bar{\partial}_E = 0$. When we wish to make the holomorphic structure explicit we will sometimes write $(E, \bar{\partial}_E)$.

In general we will represent a connection either abstractly by its covariant derivative ∇_A or in local coordinates by its connection 1-form A. We will be careless about this distinction and use whichever notation is more convenient. We will write $\bar{\partial}_A$ and ∂_A for the $(0,1)$ and $(1,0)$ parts of ∇_A respectively. If $(E, \bar{\partial}_E)$ is equipped with a smooth hermitian metric h, then there is a unique h-unitary connection ∇_A on E called the **Chern connection** that satisfies $\bar{\partial}_A = \bar{\partial}_E$. More specifically the local form of this connection in terms of h is: $A = \bar{h}^{-1}\partial\bar{h}$, with curvature $F_A = \bar{\partial}\left(\bar{h}^{-1}\partial\bar{h}\right)$. Sometimes we will denote this connection by $\nabla_A = (\bar{\partial}_E, h)$. Conversely, if we have in hand a unitary connection ∇_A whose curvature $F_A = \nabla_A \circ \nabla_A$ is of type $(1,1)$ (i.e. $F_A^{0,2} = 0$), then $\bar{\partial}_A$ defines a holomorphic structure on E by the Newlander-Nirenberg theorem, and $\nabla_A = (\bar{\partial}_A, h)$.

Let $\mathcal{A}_h$ denote the space of h-unitary connections ∇_A on E, and write $\mathcal{A}_h^{1,1}$ for the subset consisting of those with $F_A^{0,2} = 0$. The above discussion translates to the statement that there is a bijection between $\mathcal{A}_h^{1,1}$ and the space $\mathcal{A}_{\mathrm{hol}}$ of integrable $\bar{\partial}_E$ operators. We will write $\mathcal{G}$ for the set of unitary gauge transformations of E. The set $\mathcal{G}$ is a bundle of groups whose fibres are copies of $U(n)$, and $\mathcal{G}$ acts on $\mathcal{A}_h$ by the usual conjugation $g \cdot \nabla_A = g^{-1} \circ \nabla_A \circ g$. Moreover this induces an action on F_A, which is also by conjugation, so the subspace $\mathcal{A}_h^{1,1}$ is preserved. We will write:

$$\mathcal{B}_h = \mathcal{A}_h/\mathcal{G}\ ,\ \mathcal{B}_h^{1,1} = \mathcal{A}_h^{1,1}/\mathcal{G}$$

for the quotients.

Finally there is also an action of the full complex gauge group $\mathcal{G}^{\mathbb{C}}$ (the set of all complex gauge transformations of E) on $\mathcal{A}_{\mathrm{hol}}$ again by conjugation, i.e. $g \cdot \bar{\partial}_E = g^{-1} \circ \bar{\partial}_E \circ g$. The set of isomorphism classes of holomorphic structures on E is precisely the quotient $\mathcal{A}_{\mathrm{hol}}/\mathcal{G}^{\mathbb{C}}$, and via the bijection $\mathcal{A}_{\mathrm{hol}} \simeq \mathcal{A}_h^{1,1}$ we see that $\mathcal{G}^{\mathbb{C}}$ also acts on $\mathcal{A}_h^{1,1}$, extending the action of $\mathcal{G}$. Moreover, $\mathcal{G}^{\mathbb{C}}$ also acts on the space of hermitian metrics via $h \mapsto g \cdot h$ where $g \cdot h(s_1, s_2) = h(g(s_1), g(s_2))$. In matrix form this reads $g \cdot h = \bar{g}^T h g$.

Now, starting from a holomorphic bundle E with hermitian metric h and Chern connection $(\bar{\partial}_E, h)$, we may use a complex gauge transformation to perturb this connection in two different ways. We may either let g act on $\bar{\partial}_E$ or on h. If we write g^* for the adjoint of g with respect to h, then $g \cdot h(s_1, s_2) = h(g^* g(s_1), s_2)$. If we set $k = g^* g$, then the connection corresponding to h and $g \cdot h$ are related by:

$$\bar{\partial}_h = \bar{\partial}_{g \cdot h} \textit{ and } \partial_{g \cdot h} = k \circ \partial_h \circ k^{-1}.$$

Now note that the action of a complex gauge transformation g on a connection ∇_A is

$$g \cdot \nabla_A = g^* \circ \partial_A \circ (g^*)^{-1} + g^{-1} \circ \bar{\partial}_A \circ g,$$

so $g \circ \nabla_{g \cdot A} \circ g^{-1} = k \circ \partial_A \circ k^{-1} + \bar{\partial}_A = (\bar{\partial}_E, g \cdot h)$ or

$$\nabla_{g \cdot A} = (g \cdot \bar{\partial}_E, h) = g^{-1} \circ (\bar{\partial}_E, g \cdot h) \circ g.$$

Taking the square of this formula also gives the relation between the respective curvatures:

$$F_{(g \cdot \bar{\partial}_E, h)} = g^{-1} \circ F_{(\bar{\partial}_E, g \cdot h)} \circ g.$$

If we denote by $\mathfrak{u}((E,h)) \subset End(E)$ the subbundle of skew-hermitian endomorphisms, then for a section σ of $\mathfrak{u}(E)$, we will write $|\sigma|$ for its pointwise norm. This is defined as usual by

$$|\sigma| = \left(\sum_{i=1}^{K} |\lambda_i|^2\right)^{\frac{1}{2}}$$

where the λ_i are the eigenvalues of σ at a given point and K is the rank of E. Now we may define the **Yang-Mills functional** (YM functional) by:

$$YM(\nabla_A) = \int_X |F_A|^2 \, dvol.$$

If we assume that X is Kähler, we have:

$$YM(\nabla_A) = \int_X |F_A|^2 \frac{\omega^n}{n!}.$$

This functional is gauge invariant and so defines a map $YM : \mathcal{B}_h \to \mathbb{R}$. Its critical points are the so called **Yang-Mills connections** and satisfy the Euler-Lagrange equations for YM: $d_A^* F_A = 0$, where d_A is the covariant derivative induced on $End(E)$ valued 2-forms by ∇_A. If $\nabla_A \in \mathcal{A}_h^{1,1}$ then we may also define the **Hermitian-Yang-Mills functional**:

$$HYM(\nabla_A) = \int_X |\Lambda_\omega F_A|^2 \frac{\omega^n}{n!},$$

where Λ_ω is, as usual the adjoint of the Lefschetz operator, (which is given by wedging with the Kähler form). For a $(1,1)$ form $G = \sum G_{i,j} dz_i \wedge d\bar{z}_j$ this can be written explicitly in coordinates as

$$\Lambda_\omega G = -2\sqrt{-1}\left(g^{ij} G_{ij}\right)$$

where g^{ij} denotes the inverse of the metric. The quantity $\Lambda_\omega F_A$ is called the **Hermitian-Einstein tensor** of A. Again HYM is gauge invariant and so defines a functional $HYM : \mathcal{B}_h^{1,1} \to \mathbb{R}$. Critical points of the functional satisfy the Euler-Lagrange equations: $d_A \Lambda_\omega F_A = 0$. On the other hand, just as in the preceding discussion, we may regard the holomorphic stucture as being fixed and consider the space of $(1,1)$ connections as being the set of pairs $(\bar{\partial}_E, h)$ where h varies over all hermitian metrics. We may therefore think of HYM as a functional $HYM(h) = HYM(\bar{\partial}_E, h)$ on the space of hermitian metrics on E. A critical metric of HYM is referred to a **critical hermitian structure** on $(E, \bar{\partial})$.

An important fact that we will use is that when X is compact, there is a relation between the two functionals YM and HYM. Explicitly:

$$YM(\nabla_A) = HYM(\nabla_A) + \frac{4\pi^2}{(n-2)!} \int_X \left(2c_2(E) - c_1^2(E)\right) \wedge \omega^{n-2}$$

for any $A \in \mathcal{A}_h^{1,1}$. The second term depends only on the topology of E and the form ω, so YM and HYM have the same critical points on $\mathcal{A}_h^{1,1}$. Furthermore, ∇_A is a critical point of YM and HYM, if and only if h is a critical hermitian structure for the holomorphic stucture on E given by A.

For a Yang-Mills connection we have the following proposition.

Proposition 2.13. *Let $\nabla_A \in \mathcal{A}_h^{1,1}$ be a Yang-Mills connection on an hermitian vector bundle (E,h) over a Kähler manifold X. Then $\nabla_A = \oplus_{i=1}^{l} \nabla_{A_i}$ where $E = \oplus_{i=1}^{l} Q_i$ is an orthogonal splitting of E, and where $\sqrt{-1}\Lambda_\omega F_{A_i} = \lambda_i \mathbf{Id}_{Q_i}$, where λ_i are constant. If X is compact, then $\lambda_i = \mu(Q_i)$.*

The proof is simply the observation (stated above) that the Hermitian-Einstein tensor of a Yang-Mills connection is covariantly constant, and so has constant eigenvalues and eigenspaces of constant rank. Therefore E breaks up into a direct sum of the eigenspaces of this operator.

Definition 2.14. *Let $E \to (X, \omega)$ be a holomorphic bundle. Then a connection ∇_A such that there exists a constant λ with:*

$$\sqrt{-1}\Lambda_\omega F_A = \lambda \mathbf{Id}_E$$

*is called an **Hermitian-Einstein connection**. If A is the Chern connection of $(\bar{\partial}_E, h)$ for some hermitian metric h, then h is called an **Hermitian-Einstein metric**.*

The existence of such a metric is related to stability properties of E. This is the **Kobayashi-Hitchin correspondence** (or Donaldson-Uhlenbeck-Yau theorem).

Theorem 2.15. *A holomorphic vector bundle E on a compact Kähler manifold (X, ω), admits an Hermitian-Einstein metric if and only if E is polystable, i.e. E splits holomorphically into a direct sum of ω-stable bundles of the same ω-slope $\mu_\omega(E)$. Such a metric is unique up to a positive constant.*

For the proof in the case of projective surfaces and projective complex manifolds see [DO1] and [DO2] respectively. For the proof in the general case see [UY]. From the HYM equations it is clear that an Hermitian-Einstein connection is Hermitian-Yang-Mills (and so Yang-Mills).

Remark 2.16. *Note that if E is holomorphic and $\nabla_A = (\bar\partial_E, h)$ for some hermitian metric h, then the same argument shows that $(E, h) = \oplus_{i=1}^l (Q_i, h_i)$ where the h_i are Hermitian-Einstein metrics and the splitting is orthogonal with respect to h. Since the splitting is preserved by the Chern connection ∇_A, it is also holomorphic with respect to the holomorphic structure on E given by $\bar\partial_E$.*

We now give the statement of the general Uhlenbeck compactness theorem. Although we will be primarily concerned with the theorem as it applies to the Yang-Mills flow of the next section, the proof of the main theorem in Section 7 will also rely on this more general statement.

Theorem 2.17. *Let X be a Kähler manifold (not necessarily compact) and $E \to X$ a hermitian vector bundle with metric h. Fix any $p > n$. Let ∇_{A_j} be a sequence of integrable, unitary connections on E such that $\|F_{A_j}\|_{L^2(X)}$ and $\|\Lambda_\omega F_{A_j}\|_{L^\infty(X)}$ are uniformly bounded. Then there is a subsequence (still denoted A_j), a closed subset $Z_{\mathrm{an}} \subset X$ with Hausdorff codimension at least 4, and a smooth hermitian vector bundle (E_∞, h_∞) defined on the complement $X - Z_{\mathrm{an}}$ with a finite action Yang-Mills connection ∇_{A_∞} on E_∞, such that $\nabla_{A_j|X-Z_{\mathrm{an}}}$ is gauge equivalent to a sequence of connections that converges to ∇_{A_∞} weakly in $L^p_{1,loc}(X - Z_{\mathrm{an}})$.*

The statement of this version of Uhlenbeck compactness may be found for example in Uhlenbeck-Yau ([UY] Theorem 5.2). The proof is essentially contained in [U2] and the statement about the singular set follows from the arguments in [NA]. We will call such a limit ∇_{A_∞} an **Uhlenbeck limit**. Furthermore, we have the following crucial extension of this theorem due essentially to Bando and Siu.

Corollary 2.18. *If in addition to the assumptions in the previous theorem, we also require that:*

$$\|d_{A_j}\Lambda_\omega F_{Aj}\|_{L^2(X)} \longrightarrow 0,$$

then any Uhlenbeck limit ∇_{A_∞} is Yang-Mills. On $X - Z_{\mathrm{an}}$ we therefore have a holomorphic, orthogonal, splitting:

$$(E_\infty, h_\infty, \nabla_{A_\infty}) = \oplus_{i=1}^l (Q_{\infty,i}, h_{\infty,i}, \nabla_{A_{\infty,i}})$$

Moreover E_∞ extends to a reflexive sheaf (still denoted E_∞) on all of X.

Most of the content of this corollary resides in the last statement, which may be found in [BS] as Corollary 2. The proof presented there is based on results in the papers [B] and [SIU]. The statement about the splitting follows directly from the fact that an Uhlenbeck limit is Yang-Mills and Proposition 2.13. Therefore it only remains to prove that the stated condition implies the limit is Yang-Mills. For a proof of this see for example [DW1].

We will need the following simple corollary of Uhlenbeck compactness, which we will use repeatedly.

Corollary 2.19. *With the same assumptions as in Theorem 2.17, $\Lambda_\omega F_{A_j} \to \Lambda_\omega F_{A_\infty}$ in $L^p(X - Z_{\mathrm{an}})$ for all $1 \leq p < \infty$.*

For the proof see [DW1].

In general, if $\mathcal{E}$ is only a reflexive sheaf, Bando and Siu ([BS]) defined the notion of an **admissible hermitian metric**. This is an hermitian metric h on the locally free part of $\mathcal{F}$ such that:

· $\Lambda_\omega F_h \in L^\infty(X, \omega)$

· $F_h \in L^2(X, \omega)$.

Corollary 2.18 says that the limiting metric is an admissible hermitian metric on the reflexive sheaf E_∞ that is a direct sum of admissible Hermitian-Einstein metrics. We also point out the version of the Kobayashi-Hitchin correspondence for reflexive sheaves, due to Bando and Siu [BS].

Theorem 2.20. *(Bando-Siu) A reflexive sheaf $\mathcal{E}$ on a compact Kähler manifold (X,ω) admits an admissible Hermitian-Einstein metric if and only if it is polystable. Such a metric is unique up to a positive constant.*

Note that this theorem says the $(Gr_\omega^{HNS}(E))^{**}$ carries an admissible Yang-Mills connection (where admissible has the same meaning for connections), which is unique up to gauge.

2.3. **Weakly Holomorphic Projections/Second Fundamental Forms.** Let $S \subset E$ be a subsheaf with quotient Q. Then away from $\mathrm{Sing}(S) \cup \mathrm{Sing}(Q)$, S is a subbundle. If we fix an hermitian metric h on E, then we may think of S as a direct summand away from the singular set, and there is a corresponding smooth projection operator $\pi : E \to S$ depending on h. The condition of being a holomorphic subbundle almost everywhere can be shown to be equivalent to the condition: $(\mathbf{Id}_E - \pi)\,\bar\partial_E \pi = 0$. Since π is a projection operator we also have $\pi^2 = \pi = \pi^*$. Furthermore it can be shown that π extends to an L_1^2 section of $\mathrm{End}\,E$. Conversely it turns out that an operator with these properties determines a subsheaf.

Definition 2.21. *An element $\pi \in L_1^2(\mathrm{End}\,E)$ is called a weakly holomorphic projection operator if the conditions*

$$(\mathbf{Id}_E - \pi)\,\bar\partial_E \pi = 0 \text{ and } \pi_j^2 = \pi_j = \pi_j^* \qquad *$$

hold almost everywhere.

Theorem 2.22. *(Uhlenbeck-Yau) A weakly holomorphic projection operator π of a holomorphic vector bundle (E,h) with a smooth hermitian metric over a compact Kähler manifold (X,ω) determines a coherent subsheaf of E. That is, there exists a coherent subsheaf S of E together with a singular set $V \subset X$ with the following properties:*

$\cdot$ $\mathrm{Codim}\,V \geq 2$,
$\cdot \pi_{|X-V}$ *is C^∞ and satisfies $*$,*
$\cdot S_{|X-V} = \pi_{|X-V}(E_{|X-V}) \hookrightarrow E_{|X-V}$ *is a holomorphic subbundle.*

The proof of this theorem is contained in [UY]. From here on out we will identify a subsheaf with its weakly holomorphic holomorphic projection.

If $S \subset E$ is a subsheaf, then away from $\mathrm{Sing}(S) \cup \mathrm{Sing}(Q)$ there is an orthogonal splitting $E = S \oplus Q$. In general we may write the Chern connection $\nabla_{(\bar\partial_E,h)}$ connection on E as:

$$\nabla_{(\bar\partial_E,h)} = \begin{pmatrix} \nabla_{(\bar\partial_S,h_S)} & \beta \\ -\beta^* & \nabla_{(\bar\partial_Q,h_Q)} \end{pmatrix}$$

where $\nabla_{(\bar\partial_S,h_S)}$ and $\nabla_{(\bar\partial_Q,h_Q)}$ are the induced Chern connections on S and Q respectively, and β is the second fundamental form. Recall that $\beta \in \Omega^{0,1}(Hom(Q,S))$. More specifically, in terms of the projection operator, we have $\bar\partial_E \pi = \beta$ and $\partial_E \pi = \beta^*$. Also β extends to an L^2 section of $\Omega^{0,1}(Hom(Q,S))$ everywhere as $\bar\partial_E \pi$ since π is L_1^2. We also have the following well-known formula for the degree of a subsheaf in terms of its weakly holomorphic projection.

Theorem 2.23. *(Chern-Weil Formula) Let $S \subset E$ be a saturated subsheaf of a holomorphic vector bundle with hermitian metric h, and π the associated weakly holomorphic projection. Let $\bar\partial_E$ denote the holomorphic structure on E. Then we have:*

$$\deg S = \frac{1}{2\pi n}\int_X \mathrm{Tr}\left(\sqrt{-1}\Lambda_\omega F_{(\bar\partial_E,h)}\pi\right)\omega^n - \frac{1}{2\pi n}\int_X |\beta|^2\,\omega^n$$

The statement of this theorem as well as a sketch of the proof may be found in [SI]. This formula will also follow as a special case of our discussion in Section 4.

Clearly any sequence π_j of such projection operators is uniformly bounded in $L^\infty(X)$. As an immediate corollary of the Chern-Weil formula we have the following.

Corollary 2.24. *A sequence π_j of weakly holomorphic projection operators such that $\deg \pi_j$ is bounded from below is uniformly bounded in L_1^2. In particular, if $\deg \pi_j$ is constant then π_j is bounded in L_1^2.*

Now suppose ∇_{A_0} is a reference connection, $g_j \in \mathcal{G}^{\mathbb{C}}$ is a sequence of complex gauge transformations, and ∇_{A_j} is the sequence of integrable unitary connections on an hermitian vector bundle (E,h) given by $\nabla_{A_j} = g_j \cdot \nabla_{A_0}$, and assume as before that $\Lambda_\omega F_{A_j}$ is uniformly bounded in L^∞. Let $S \subset E$ be a subbundle with quotient Q. We have a sequence of projection operators π_j given by orthogonal projection onto $g_j(S)$

(with respect to the metric h) from E to holomorphic subbundles S_j (whose holomorphic structures are induced by the connections ∇_{A_j}) smoothly isomorphic to S. We will denote by Q_j the corresponding quotients. Each of these holomorphic subbundles has a second fundamental form which we will write as β_j. Assume that the β_j are also uniformly bounded in L^2 (this will later be a consequence of our hypotheses). Then with all of the above understood, we have the following result.

Lemma 2.25. *For any $1 \leq p < \infty$, the β_j are bounded in $L^p_{1,loc}(X - Z_{\text{an}})$, uniformly for all j. In particular the β_j are uniformly bounded on compact subsets of $X - Z_{\text{an}}$.*

The proof is the same as in [DW1] Section 2.2.

3. The Yang-Mills Flow and Basic Properties

3.1. **The Flow Equations/Lower Bound for the HN Type of the Limit.** As stated in the introduction, although many of our arguments are valid for minimising sequences of unitary connections, our primary interest will be in sequences obtained from the **Yang-Mills flow**. This is a sequence of integrable unitary connections A_t obtained as solutions of the L^2-gradient flow equations for the YM functional. Explicitly:

$$\frac{\partial A_t}{\partial t} = -d^*_{A_t} F_{A_t}, \quad A_0 \in \mathcal{A}^{1,1}_h.$$

It follows from [DO1] and [SI] that the above equations have a unique solution in $\mathcal{A}^{1,1}_h \times [0,\infty)$. Moreover, the flow preserves complex gauge orbits, that is, A_t lies in the orbit $\mathcal{G}^{\mathbb{C}} \cdot A_0$. This may be seen as follows. Instead of solving for the connection, fix A_0 so that $\bar{\partial}_{A_0} = \bar{\partial}_E$, and consider instead the family of hermitian metrics h_t satisfying the **Hermitian-Yang-Mills flow equations**:

$$h_t^{-1}\frac{\partial h_t}{\partial t} = -2\left(\sqrt{-1}\Lambda_\omega F_{h_t} - \mu_\omega(E) Id_E\right).$$

In the above, F_{h_t} is the curvature of $(\bar{\partial}_E, h_t)$. The Yang-Mills and Hermitian-Yang-Mills flow equations are equivalent up to gauge. If $A_t = g_t \cdot A_0$ is a solution of the Yang-Mills flow, then $h_t = h_0 g_t^* g_t$ is a solution of the Hermitian-Yang-Mills flow. Conversely, if $h_t = h_0 k_t$ (where $h_0 k_t(s_1, s_2) = h_0(k_t s_1, s_2)$) for a positive definite self-adjoint (with respect to h_0) endomorphism k_t, then $A_t = (k_t)^{\frac{1}{2}} A_0$ is real gauge equivalent to a solution of the Yang-Mills flow. To spell out the equivalence precisely, the map:

$$g_t : (E, \bar{\partial}_E, h_0 k_t) \longrightarrow (E, g_t(\bar{\partial}_E), h_0)$$

is a biholomorphism and an isometry, where $k_t = g_t^* g_t$. For a detailed discussion of this see [WIL] section 3.1 for details.

Lemma 3.1. *Let A_t be a solution of the YM flow. Then:*

(1)

$$\frac{\partial F_{A_t}}{\partial t} = -\triangle_{A_t} F_{A_t}$$

and therefore,

$$\frac{d}{dt}\left\|F_{A_t}\right\|^2_{L^2} = -2\left\|d^*_{A_t} F_{A_t}\right\|^2_{L^2} \leq 0.$$

Hence, $t \mapsto YM(A_t)$, and $t \mapsto HYM(A_t)$ are non-increasing.

(2) $\left|\Lambda_\omega F_{A_t}\right|^2$ *satisfies*

$$\frac{\partial \left|\Lambda_\omega F_{A_t}\right|^2}{\partial t} + \triangle\left|\Lambda_\omega F_{A_t}\right|^2 = -2\left|d^*_{A_t} F_{A_t}\right|^2 \leq 0,$$

so by the maximum principle $\sup\left|\Lambda_\omega F_{A_t}\right|^2$ *is decreasing in t.*

For the proof see [DOKR] Chapter 6.

Now we may apply the Uhlenbeck compactness theorem to a sequence of connections given by the flow.

Proposition 3.2. *Let X be a compact Kähler manifold. Let A_0 be any fixed connection, and A_t denote its evolution along the flow. Fix $p > n$. For any sequence $t_j \to \infty$ there is a subsequence (still denoted t_j), a closed subset $Z_{\text{an}} \subset X$ with Hausdorff codimension at least 4, and a smooth hermitian vector bundle (E_∞, h_∞) defined on the complement $X - Z_{\text{an}}$ with a finite action Yang-Mills connection A_∞ on E_∞, such that $A_{t_j|X-Z_{\text{an}}}$ is gauge equivalent to a sequence of connections that converges to A_∞ weakly in $L^p_{1,loc}(X - Z_{\text{an}})$.*

Away from $Z_{\rm an}$ there is a smooth splitting: $(E_\infty, A_\infty, h_\infty) = \oplus_{i=l}^{l}(Q_{\infty,i}, A_{\infty,i}, h_{\infty,i})$, where $A_{\infty,i}$ is the induced connection on Q_i, and $h_{\infty,,i}$ is an Hermitian-Einstein metric. Furthermore, E_∞ extends over $Z_{\rm an}$ as a reflexive sheaf (still denoted E_∞), so that the metrics $h_{\infty,i}$ are admissible Hermitian-Einstein metrics on the extension.

Proof. The functions $\|F_{A_t}\|_{L^2}$ and $\|\Lambda_\omega F_{A_t}\|_{L^\infty}$ are uniformly bounded by parts (1) and (2) of Lemma 3.1 respectively. By [DOKR] Proposition 6.2.14, $\lim_{t\to\infty}\|\nabla_{A_t}\Lambda_\omega F_{A_t}\|_{L^2} = 0$. The remaining statements follow from Corollary 2.18. □

Just as before we call A_∞ an Uhlenbeck limit of the flow.

Lemma 3.3. *If A_∞ is an Uhlenbeck limit of A_{t_j}, then $\Lambda_\omega F_{A_j} \to \Lambda_\omega F_{A_\infty}$ in $L^p(X - Z_{\rm an})$ for all $1 \le p < \infty$. Moreover,* $\lim_{t\to\infty} HYM(A_t) = HYM(A_\infty)$.

Proof. The first part is immediate from Corollary 2.19. The second statement is immediate from the facts that $t \to HYM(A_t)$ is non-increasing, and $HYM(A_{t_j}) \mapsto HYM(A_\infty)$. □

The set of all HN types of holomorphic bundles on X has a partial ordering due to Shatz [SH]. For a pair of K-tuples μ and λ with $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_K$ and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_K$ and $\sum_i \mu_i = \sum_i \lambda_i$, we write

$$\mu \le \lambda \Longleftrightarrow \sum_{j\le k}\mu_j \le \sum_{j\le k}\lambda_j \ for\ all\ k = 1, \cdots, K.$$

This partial ordering was originally used by Shatz to stratify the space of holomorphic structures on a complex vector bundle.

The first crucial step in [DW1] is to prove that the HN type of an Uhlenbeck limit is bounded below by the HN type μ_0 of E. For the proofs of this and its corollaries, we refer to [DW1] as the proof is unchanged in the general case.

Proposition 3.4. *Let A_j be a sequence of connections along the YM flow on a holomorphic vector bundle of rank K, with Uhlenbeck limit A_∞. Let μ_0 be the HN type of E with holomorphic structure $\bar\partial_{A_0}$. Let λ_∞ be the HN type of $\bar\partial_{A_\infty}$. Then $\mu_0 \le \lambda_\infty$.*

Corollary 3.5. *Let $\mu = (\mu_1, \cdots, \mu_K)$ be the HN type of a rank K holomorphic vector bundle $(E, \bar\partial_E)$ on X. Then*

$$\sum_{i=1}^K \mu_i^2 \le \frac{1}{2\pi n}\int_X |\Lambda_\omega F_A|^2\,\omega^n$$

and

$$\left(\sum_{i=1}^K \mu_i^2\right)^{\frac{1}{2}} \le \frac{1}{2\pi n}\int_X |\Lambda_\omega F_A|\,\omega^n$$

for all unitary connections ∇_A in the $\mathcal{G}^{\mathbb{C}}$ orbit of $(E, \bar\partial_E)$.

3.2. **Hermitian-Yang-Mills Type Functionals.** The YM and HYM functionals are not sufficient to distinguish different HN types in general. In other words there may be multiple connections with the same YM number, but which induce holomorphic structures with different HN types. In this subsection we introduce generalisations of the HYM functional that can be used to distinguish different types. This is only a technical device, but will be used essentially in Section 5.

Write $\mathfrak{u}(K)$ for the Lie algebra of the unitary group $U(K)$. Fix a real number $\alpha \ge 1$. Then for $\mathbf{v} \in \mathfrak{u}(K)$, a skew hermitian matrix with eigenvalues $\sqrt{-1}\lambda_1, \cdots, \sqrt{-1}\lambda_K$, let $\varphi_\alpha(\mathbf{v}) = \sum_{i=1}^K |\lambda_i|^\alpha$. It can be seen that there is a family $\varphi_{\alpha,\rho}$, $0 < \rho \le 1$, of smooth convex Ad-invariant functions such that $\varphi_{\alpha,\rho} \to \varphi_\alpha$ uniformly on compact subsets of $\mathfrak{u}(K)$. By Atiyah-Bott ([AB]), Proposition 12.16, φ_α is a convex function on $\mathfrak{u}(K)$. Now if E is a vector bundle of rank K equipped with an hermitian metric, we may consider a section $\sigma \in \Gamma(X, \mathfrak{u}(E))$ as collection of local sections $\{\sigma_\beta\}$ such that $\sigma_\beta = \mathrm{Ad}(g_{\beta\gamma})\sigma_\gamma$ where $g_{\beta\rho}$ are the transition functions for E. By the Ad-invariance of φ_α, $\varphi_\alpha(\sigma_\beta) = \varphi_\alpha(\sigma_\gamma)$, so φ_α induces a well-defined function Φ_α on $\mathfrak{u}(E)$. Then for a fixed real number N, define:

$$HYM_{\alpha,N}(A) = \int_X \Phi_\alpha(\Lambda_\omega F_A + \sqrt{-1}N\mathbf{Id}_E)dvol_\omega$$

and $HYM_\alpha(A) = HYM_{\alpha,0}(A)$. Note that $HYM = HYM_2$ is the usual HYM functional. In the sequel we will write:

$$\begin{aligned} HYM_{\alpha,N}(\mu) &= HYM_\alpha(\mu+N) = \frac{2\pi}{(n-1)!}\Phi_\alpha(\sqrt{-1}\,(\mu+N)), \\ \textit{where } \mu+N &= (\mu_1+N,\cdots,\mu_K+N) \end{aligned}$$

is identified with the matrix diag $(\mu_1+N,\cdots,\mu_K+N)$. Therefore:

$$HYM(\mu) = \frac{2\pi}{(n-1)!}\sum_{i=1}^{K}\mu_i^2.$$

We have the following elementary lemma whose proof we omit.

Lemma 3.6. *The functional* $\mathbf{v} \to \left(\int_X \Phi_\alpha(\mathbf{v})\right)^{\frac{1}{\alpha}}$ *is equivalent to the* $L^\alpha(\mathfrak{u}(E))$ *norm.*

The following three propositions will be crucial in Section 5. For the proofs see [DW1].

Proposition 3.7. (1) *If* $\mu \le \lambda$, *then* $\Phi_\alpha(\sqrt{-1}\mu) \le \Phi_\alpha(\sqrt{-1}\lambda)$ *for all* $\alpha \ge 1$.
(2) *Assume* $\mu_K \ge 0$ *and* $\lambda_K \ge 0$. *If* $\Phi_\alpha(\sqrt{-1}\mu) = \Phi_\alpha(\sqrt{-1}\lambda)$ *for all* α *in some set* $A \subset [1,\infty)$ *possessing a limit point, then* $\mu = \lambda$.

Proposition 3.8. *Let* A_t *be a solution of the* YM *flow. Then for any* $\alpha \ge 1$ *and any* N, $t \mapsto HYM_{\alpha,N}(A_t)$ *is non-increasing.*

Proposition 3.9. *Let* A_∞ *be a subsequential Uhlenbeck limit of* A_t *where* A_t *is a solution of the* YM *flow. Then for all* $\alpha \ge 1$, $\lim_{t\to\infty} HYM_{\alpha,N}(A_t) = HYM_{\alpha,N}(A_\infty)$.

4. Properties of Blowups and Resolution of the HNS Filtration

In this section we discuss the properties of blowups of complex manifolds along complex submanifolds that will be used in the subsequent discussion. Essentially all of this material is standard, but we review it carefully now because we will need to employ these facts often in the proofs of the main results.

4.1. **Resolution of Singularities Type Theorems.** The HNS filtration is in general only given by subsheaves, making it difficult to do analysis. We will therefore need some way of obtaining a filtration by subbundles, that is, a way of resolving the singularities. In two dimensions, when the singular set consists of point singularities this can be done by hand (see [BU1]), but in higher dimensions the only available tool seems to be the general resolution of singularities theorem of Hironaka. Specifically:

Theorem 4.1. *(Resolution of Singularities) Let* X *be a compact, complex space (or* $\mathbb{C}$*-scheme). Then there exists a finite sequence of blowups with smooth centres:*

$$\tilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

such that $\tilde{X}$ *is compact and non-singular (a complex manifold) and the centre* Y_{j-1} *of each blowup* π_j *is contained in the singular locus of* X_{j-1}.

For the proof see [H1] and [H2]. What we will actually use is the following corollary:

Corollary 4.2. *(Resolution of the Locus of Indeterminacy) Let* X *and* Y *be compact, complex spaces and let* $\varphi : X \dashrightarrow Y$ *be a rational (meromorphic) map. Then there exists a compact, complex space* $\tilde{X} \xrightarrow{\pi} X$ *obtained from* X *by a sequence of blowups with smooth centres and a holomorphic map* $\psi : \tilde{X} \to Y$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \widetilde{X} & & \\ \downarrow & \searrow & \\ X & \underset{\varphi}{\dashrightarrow} & Y \end{array}.$$

In our case both X and Y (and hence also $\tilde{X}$) will be complex manifolds. Note that in this case a blowup with "smooth centre" is the same as the blowup along a complex submanifold. We will apply the Corollary in the following way.

The HNS filtration of a bundle E, which in the sequel we will abbreviate for simplicity as:

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E$$

(i.e. we ignore the notation indicating that it is a double filtration), as stated previously, is in general a filtration only by subsheaves of E. We may think of a subbundle $S \subset E$ of rank k as a holomorphic section of the Grassmann bundle $Gr(k, E)$, the bundle whose fibre at each point is the set of k-dimensional complex subspaces of the fibre of E. Similarly a filtration by subbundles corresponds to a holomorphic section of the partial flag bundle $\mathbb{FL}(d_1, \cdots, d_l, E)$, the bundle whose fibre at each point is the set of l flags of type $(d_1, \cdots, d_l)$ where $d_i = \mathrm{rk}(E_i)$. On the other hand a filtration by subsheaves corresponds to a rational section $X \overset{\sigma}{\dashrightarrow} \mathbb{FL}(d_1, \cdots, d_l, E)$. The corollary says that by blowing up finitely many times along complex submanifolds, we obtain an honest section $\tilde{X} \to \mathbb{FL}(d_1, \cdots, d_l, \pi^*E)$. More explicitly, we have a diagram:

$$\begin{array}{ccc} \tilde{X} & \overset{\tilde{\sigma}}{\underset{}{\rightleftarrows}} & \mathbb{FL}(\pi^*E) \\ \downarrow & \searrow & \downarrow \\ X & \overset{\sigma}{\underset{p}{\dashrightarrow \atop \longleftarrow}} & \mathbb{FL}(E) \end{array}$$

where $\tilde{\sigma}$ will be constructed below. The outer square is just the pullback diagram for the map $\tilde{X} \overset{\pi}{\to} X$. First we claim that the triangle:

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow & \searrow & \\ X & \longleftarrow & \mathbb{FL}(E) \end{array}$$

commutes. If we write ψ for the desingularised map $\tilde{X} \longrightarrow \mathbb{FL}(E)$, then note that for a point $\tilde{x} \in \tilde{X} - \mathbf{E}$, we have $\psi(\tilde{x}) = \psi(\pi^{-1}(x))$ for $x \in Z_{\mathrm{alg}}$. Then we have: $p(\psi(\tilde{x})) = p(\sigma(\pi(\tilde{x}))) = x = \pi(\tilde{x})$ since σ is well-defined and a section away from Z_{alg} and we know the diagram:

$$\begin{array}{ccc} \tilde{X} & & \\ \downarrow & \searrow & \\ X & \overset{\sigma}{\dashrightarrow} & \mathbb{FL}(E) \end{array}$$

commutes. In other words on $\tilde{X} - \mathbf{E}$ we have $p \circ \psi = \pi$. But since both of these are holomorphic maps $\tilde{X} \longrightarrow X$, $p \circ \psi = \pi$ on $\tilde{X}$ by the uniqueness principle for holomorphic maps, since they agree on a non-empty open subset. Now $\mathbb{FL}(\pi^*E) = \pi^*\mathbb{FL}(E) = \{(\tilde{x}, \nu) \in \tilde{X} \times \mathbb{FL}(E) \mid \pi(\tilde{x}) = p(\nu)\}$. Now define $\tilde{\sigma} : \tilde{X} \longrightarrow \mathbb{FL}(\pi^*E)$ by $\tilde{\sigma}(\tilde{x}) = (\tilde{x}, \psi(\tilde{x}))$. Since $p \circ \psi = \pi$ this is indeed a map into $\mathbb{FL}(\pi^*E)$, and it is manifestly a section.

In other words there is a filtration of π^*E:

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \pi^*E$$

where the $\tilde{E}_i$ are subbundles.

Now note that we have the following diagram:

$$\begin{array}{ccc} & & \tilde{Q}_i \\ & & \uparrow \\ & & \pi^*E \\ & \nearrow & \uparrow \\ \pi^*E_i & \dashrightarrow & \tilde{E}_i \end{array}$$

where the dashed line is the rational map corresponding to the equality of π^*E_i and $\tilde{E}_i$ away from $\mathbf{E}$ (both are equal to E_i), and $\tilde{Q}_i$ is the quotient of π^*E by $\tilde{E}_i$. Then $\tilde{Q}_i$ is a vector bundle and in particular torsion free. On the other hand the image of π^*E_i under the composition $\pi^*E_i \to \pi^*E \to \tilde{Q}_i$ is torsion since it is supported on the divisor $\mathbf{E}$, and hence must be zero. If we write $\mathrm{Im}\,\pi^*E_i$ for the image of $\pi^*E_i \longrightarrow \pi^*E$, this means there is an actual inclusion of sheaves $\mathrm{Im}\,\pi^*E_i \hookrightarrow \tilde{E}_i$. The quotient sheaf $\tilde{E}_i / \mathrm{Im}\,\pi^*E_i$ is supported on $\mathbf{E}$, hence torsion and so it follows from Lemma 2.1 that $\tilde{E}_i = \mathrm{Sat}_{\pi^*E}(\mathrm{Im}\,\pi^*E_i)$.

Since $\pi_*\tilde{E}_i$ is equal to E_i away from $\mathrm{Sing}\,E_i$ there is a birational map $E_i \dashrightarrow \pi_*\tilde{E}_i$. Now consider the short exact sequence:

$$0 \longrightarrow \tilde{E}_i \longrightarrow \pi^*E \to \tilde{Q}_i \longrightarrow 0.$$

Pushing this sequence forward, and noting that $\pi_* \tilde{Q}_i$ is torsion free and hence injects into its double dual, we have an exact sequence:

$$0 \longrightarrow \pi_* \tilde{E}_i \longrightarrow E \to \left(\pi_* \tilde{Q}_i\right)^{**}.$$

Recall that a sheaf S is normal if for any analytic subset Z of codimension at least two, the map $S(U) \longrightarrow S(U - Z)$ on local sections is an isomorphism for any open set U. In other words, the local sections of a normal sheaf extend over codimension two subvarieties. Furthermore, recall that a sheaf is reflexive if and only if it is both torsion free and normal. Then $(\pi_* \tilde{Q}_i)^{**}$ and E are in particular both normal since they are reflexive. A simple diagram chase reveals that normality of these sheaves together with the exactness of this last sequence implies that $\pi_* \tilde{E}_i$ is also normal (and in fact reflexive, since it is also torsion free).

Because E_i is saturated by construction, it is also reflexive and therefore normal. It is easy to see from the definitions that a map between normal sheaves that is defined away from a codimension two subvariety extends to a map on all of X. Since $\operatorname{Sing} E_i$ has singular set of codimension at least three, the map $E_i \dashrightarrow \pi_* \tilde{E}_i$ extends to an isomorphism $E_i \cong \pi_* \tilde{E}_i$.

Similarly, if $\tilde{Q}_i = \tilde{E}_i / \tilde{E}_{i-1}$, then $\pi_* \tilde{Q}_i$ is equal to Q_i away from $\operatorname{Sing} Q_i$ so again we have a birational map $(Q_i)^{**} \dashrightarrow (\pi_* \tilde{Q}_i)^{**}$. Since the double dual is always reflexive, these sheaves are normal, so the map extends to an isomorphism. To summarise:

Proposition 4.3. *Let*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{l-1} \subset E_l = E$$

be a filtration of a holomorphic vector bundle $E \to X$ by saturated subsheaves and let $Q_i = E_i / E_{i-1}$. Then there is a finite sequence of blowups along complex submanifolds whose composition $\pi : \tilde{X} \to X$ enjoys the following properties. There is a filtration

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \tilde{E} = \pi^* E$$

by subbundles. If we write $\operatorname{Im} \pi^* E_i$ for the image of $\pi^* E_i \hookrightarrow \pi^* E_i$, then $\tilde{E}_i = \operatorname{Sat}_{\pi^* E} (\operatorname{Im} \pi^* E_i)$. If $\tilde{Q}_i = \tilde{E}_i / \tilde{E}_{i-1}$ then we have $\pi_* \tilde{E}_i = E_i$ and $Q_i^{**} = (\pi_* \tilde{Q}_i)^{**}$.

We will also have occasion to consider ideal sheaves $\mathcal{I} \subset \mathcal{O}_X$ whose vanishing set is a closed complex subspace $Y \subset X$. If Y is smooth for example then we may blowup along Y to obtain a smooth manifold $\pi : \tilde{X} \longrightarrow X$. Denote by $\pi^* \mathcal{I}$ is the ideal sheaf generated by pulling back local sections of $\mathcal{I}$, in other words the ideal sheaf in $\mathcal{O}_{\tilde{X}}$ generated by the image of $\pi^{-1} \mathcal{I}$ under the map $\pi^{-1} \mathcal{O}_X \longrightarrow \mathcal{O}_{\tilde{X}}$ where $\pi^{-1} \mathcal{I}$ and $\pi^{-1} \mathcal{O}_X$ are the inverse image sheaves. Note that this is not necessarily equal to the usual sheaf theoretic pullback of $\mathcal{I}$ which is given by $\pi^{-1} \mathcal{I} \otimes_{\pi^{-1} \mathcal{O}_X} \mathcal{O}_{\tilde{X}}$ and may for example have torsion. The sheaf $\pi^* \mathcal{I}$ is sometimes called the "inverse image ideal sheaf". If the order of vanishing of $\mathcal{I}$ along Y is m, then $\pi^* \mathcal{I} \subset \mathcal{O}_{\tilde{X}}(-m\mathbf{E})$, that is, every element of $\pi^* \mathcal{I}$ vanishes to order at least m along the smooth divisor $\mathbf{E}$. In this situation we will use this notation without further comment. In general Y is not smooth, so we appeal to the following resolution of singularities theorem, which is sometimes referred to as "principalisation of $\mathcal{I}$" or more specifically "monomialisation of $\mathcal{I}$" , and results of this type are usually used to prove resolution of singularities.

Theorem 4.4. *Let X be a complex manifold and Y a closed complex subspace. Then there is a finite sequence of blowups along smooth centres whose composition yields a map $\pi : \tilde{X} \to X$ such that $\pi : \tilde{X} - \mathbf{E} \to X - W$ is biholomorphic, $\mathbf{E} = \boldsymbol{\pi}^{-1}(W)$ is a normal crossings divisor, and $\pi^* \mathcal{I} = \mathcal{O}_{\tilde{X}}(-\sum_i m_i \mathbf{E}_i)$ where the $\mathbf{E}_i$ are the irreducible components of $\mathbf{E}$. Moreover, $\pi^* \mathcal{I}$ is locally principal (monomial) in the following sense: for any $x \in X$ there is a local coordinate neighbourhood $U \subset X$ containing x and a local section f_0 of $\mathcal{O}_{\tilde{X}}(-\sum_i m_i \mathbf{E}_i)$ over $\boldsymbol{\pi}^{-1}(U)$, such that if f_j is any local section of $\mathcal{I}$ over U, then $\pi^* f_j = f_0 f_j^{'}$ where $f_j^{'}$ is a non-vanishing holomorphic function on $\boldsymbol{\pi}^{-1}(U)$. Furthermore, if ξ_k are local normal crossings coordinates for $\mathbf{E}$, then there is a factorisation:*

$$f_0 = \prod_k \xi_k^{m_k}$$

so that we may write:

$$\pi^* f_j = \prod_k \xi_k^{m_k} \cdot f_j^{'}.$$

For the proof, see for example Kollár [KO].

4.2. **Metrics on Blowups and Uniform Bounds on the Degree.** Now we consider the case that the original manifold is Kähler. The following proposition says that this property is preserved under blowing up and is standard in Kähler geometry.

Proposition 4.5. *Let (X, ω) be a Kähler manifold, and Y a compact, complex submanifold. Then the blowup $\tilde{X} = Bl_Y X$ along Y is also Kähler. Moreover $\tilde{X}$ possesses a one-parameter family of Kähler metrics given by $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $\varepsilon > 0$, $\pi : \tilde{X} \to X$ is the blowup map and η is itself a Kähler form on $\tilde{X}$.*

For the proof see for example [VO].

We will need a bound on the ω_ε degree of an arbitrary subsheaf of a holomorphic vector bundle E that depends on ε in such a way that as $\varepsilon \to 0$ the degree converges to the degree of a subsheaf on the base (namely the pushforward). This will be a consequence of the following lemma.

Lemma 4.6. *Let X be a compact complex manifold of dimension n and let τ and η be closed $(1,1)$-forms with τ semi-positve and η a Kähler form. Let $E \to X$ a holomorphic vector bundle. Then there is a constant M such that for any subsheaf $S \subset E$ with torsion free quotient and any $0 < k \leq n-1$:*

$$\deg_k(S, \tau, \eta) \equiv \int_X c_1(S) \wedge \tau^{n-k-1} \wedge \eta^k \leq M.$$

Proof. Note that when $k = n-1$, $\deg_k(S, \tau, \eta)$ is the ordinary η degree of S. We follow Kobayashi's proof that the degree of an arbitrary subsheaf is bounded above. Fix an hermitian metric h on E. The general case will follow from the case when S is a line subbundle L. In this case we can use the formula: $F_L = \pi F_E \pi + \beta \wedge \beta^*$, where π is the orthogonal projection to L and β is the second fundamental form. Since $c_1(L) = \frac{i}{2\pi} F_L$ we have that:

$$\deg_k(L, \tau, \eta) = \frac{i}{2\pi} \int_X \pi F_E \pi \wedge \tau^{n-k-1} \wedge \eta^k + \frac{i}{2\pi} \int_X \beta \wedge \beta^* \wedge \tau^{n-k-1} \wedge \eta^k.$$

Since $\|\pi\|_{L^\infty(X)} \leq 1$, the first term is clearly bounded from above. Therefore we only need to check that the second term is non-positive. This is the case since β is a $(0,1)$ form, and therefore $i\beta \wedge \beta^* \leq 0$. Therefore $\deg_k(L, \tau, \eta) \leq M$, for a constant independent of L. To extend the result to all subbundles $F \subset E$, simply find such an M as above for each exterior power $\Lambda^p E$ for $p = 1, \cdots, \mathrm{rk}\, E$, and take the maximum. Then apply the above argument to the line bundle $L = \det F \hookrightarrow \Lambda^p E$.

In general $S \overset{\imath}{\hookrightarrow} E$ is not a subbundle but there is an inclusion of sheaves $\det S \hookrightarrow \Lambda^p E$ where p is the rank of S. If V is the singular set of S, then S is a subbundle away from V, and so the inclusion $\det S \overset{\imath}{\hookrightarrow} \Lambda^p E$ is a line subbundle away from V. Let σ be any local holomorphic frame for $\det S$. Now consider the set: $W = \{x \in X \mid \imath(\sigma)(x) = 0\}$. Since $\det S$ is a line bundle this is clearly independent of σ. Furthermore because $\imath$ is an injective bundle map away from V, any $x \in W$ must be in V; that is, $W \subset V$. Now write $H = \imath^*(\Lambda^p h)$. This is an hermitian metric on $\det S$ over $X - W$. On the other hand there is some hermitian metric G on $\det S$ over all of X. We would like to show that:

$$\deg_k(S, \tau, \eta) = \int_X c_1(\det S, G) \wedge \tau^{n-k-1} \wedge \eta^k = \int_{X-W} c_1(\det S, H) \wedge \tau^{n-k-1} \wedge \eta^k$$

Then applying the above reasoning, the last integral is bounded since just as before

$$\int_{X-W} c_1(\det S, H) \wedge \tau^{n-k-1} \wedge \eta^k = \int_{X-V} c_1(S, h_S) \wedge \tau^{n-k-1} \wedge \eta^k \leq \frac{i}{2\pi} \int_{X-V} \pi F_E \pi \wedge \tau^{n-k-1} \wedge \eta^k$$

where h_S is the metric on $S_{|X-V}$ induced by h. Again this is bounded independently of π.

We will construct a C^∞ function f on X such that $H = fG$ on $X - W$. Then the usual formula for the curvature of the associated Chern connections implies:

$$\begin{aligned} c_1(\det S, H) \quad &= \quad \frac{i}{2\pi} \bar{\partial}\partial \log H = \frac{i}{2\pi} \bar{\partial}\partial \log f + c_1(\det S, G) \\ &\implies \quad c_1(\det S, G) = c_1(\det S, H) - \frac{i}{2\pi} \bar{\partial}\partial \log f \ \textit{on}\ X - W. \end{aligned}$$

Finally we will show:

$$\int_{X-W} \frac{i}{2\pi} \bar{\partial}\partial \log f \wedge \tau^{n-k-1} \wedge \eta^k = 0.$$

To construct f, let σ be any local holomorphic frame for $\det S$. If $(e_{1,}\cdots,e_r)$ is a local holomorphic frame for E, then define: $\imath(\sigma)=\sum_I \sigma^I e_I$, where $e_I=e_{i_1}\wedge\cdots\wedge e_{i_p}$, with $i_1<\cdots<i_p$. Then let

$$f=H(\sigma,\sigma)/G(\sigma,\sigma)=\sum_{I,J}H_{IJ}\sigma^I\bar{\sigma}^J$$

where $H_{IJ}=\Lambda^p h(e_I,e_J)/G(\sigma,\sigma)$. Then one may check that f is well-defined independently of σ. It is a smooth non-negative function vanishing exactly on W. Since the matrix (H_{IJ}) is positive definite, f vanishes exactly where all the σ_I vanish. It is also clear that we have the equality $H=fG$.

To complete the argument we will show that $\frac{i}{2\pi}\bar{\partial}\partial\log f$ integrates to zero. Let $\mathcal{I}$ be the sheaf of ideals in $\mathcal{O}_X$ generated by $\{\sigma_I\}$. By Theorem 4.4 there is a sequence of smooth blowups $\pi:\tilde{X}\to X$ such that $\pi^*\mathcal{I}$, the inverse image ideal sheaf of $\mathcal{I}$, is the ideal sheaf of a divisor $\mathbf{E}=\sum_i m_i\mathbf{E}_i$ where the $\mathbf{E}_i$ are the irreducible components of the support of the exceptional divisor $\operatorname{supp}\mathbf{E}=\cup_i\,\mathbf{E}_i$. In other words $\pi^*\mathcal{I}=\mathcal{O}_{\tilde{X}}(-\sum_i m_i\mathbf{E}_i)$ for some natural numbers m_i. Furthermore, we have: $\pi^*\sigma^I=\rho^I\cdot\xi_{i_1}^{m_{i_1}}\cdots\xi_{i_s}^{m_{i_s}}$, where $\{\xi_{i_j}\}$ are normal crossings coordinates for $\mathbf{E}$ on an open set where $\pi^*\sigma^I$ is defined, and ρ^I is a non-vanishing holomorphic function. Therefore we may locally write: $\pi^*f=\chi\cdot\left|\xi_{i_1}\right|^{2m_{i_1}}\cdots\left|\xi_{i_s}\right|^{2m_{i_s}}$, where χ is a strictly positive C^∞ function defined on $\tilde{X}$. If we write $\Phi=\frac{i}{2\pi}\partial\log\chi$, and $T_{d\Phi}$ for the current defined by $d\Phi=\frac{i}{2\pi}\bar{\partial}\partial\log\chi$, then since by definition:

$$\begin{aligned}T_{d\Phi}\left(\pi^*(\tau^{n-k-1}\wedge\eta^k)\right)&=-dT_\Phi(\pi^*(\tau^{n-k-1}\wedge\eta^k))\\ T_\Phi(d(\pi^*(\tau^{n-k-1}\wedge\eta^k))&=0\end{aligned}$$

since $\pi^*(\tau^{n-k-1}\wedge\eta^k)$ is closed. Away from the exceptional set we may write locally:

$$\begin{aligned}\frac{i}{2\pi}\partial\log\pi^*f&=\frac{i}{2\pi}\left(\partial\log\chi+2m_{i_1}\partial\log\left|\xi_{i_1}\right|+\cdots+2m_{i_s}\partial\log\left|\xi_{i_s}\right|\right)\\&=\Phi+\frac{i}{2\pi}\left(\frac{m_{i_1}d\xi_{i_1}}{\xi_{i_1}}+\cdots+\frac{m_{i_s}d\xi_{i_s}}{\xi_{i_s}}\right).\end{aligned}$$

The second term is integrable on its domain of definition and so $\frac{i}{2\pi}\bar{\partial}\partial\log\pi^*f$ is a $(1,1)$ form with $L^1_{loc}(\tilde{X})$ coefficients, and so defines a current. On the other hand by the Poincaré-Lelong formula, $\bar{\partial}$ applied to the second term is equal to $\displaystyle\sum_{i_j}m_{i_j}T_{\mathbf{E}_{i_j}}$, in the sense of currents, where $T_{\mathbf{E}_{i_j}}$ is the current defined by the smooth hypersurface $\mathbf{E}_{i_j}$. Finally then:

$$\int_{X-W}\frac{i}{2\pi}\bar{\partial}\partial\log f\wedge\pi^*\tau^{n-k-1}\wedge\pi^*\eta^k=\int_{\tilde{X}-\mathbf{E}}\frac{i}{2\pi}\bar{\partial}\partial\log\pi^*f\wedge\pi^*\tau^{n-k-1}\wedge\pi^*\eta^k$$

$$=T_{\frac{i}{2\pi}\bar{\partial}\partial\log\pi^*f}(\pi^*\tau^{n-k-1}\wedge\pi^*\eta^k)=\left(\sum_i m_iT_{\mathbf{E}_i}\right)(\pi^*\tau^{n-k-1}\wedge\pi^*\eta^k)=\sum_i m_i\int_{\mathbf{E}_i}\pi^*\tau^{n-k-1}\wedge\pi^*\eta^k=0$$

since the image of $\mathbf{E}_i$ under π has codimension at least two. This completes the proof. □

Remark 4.7. *If $0\to S\to E\to Q\to 0$ is an exact sequence, where E is a vector bundle and Q is torsion free, then the dualised sequence $0\to Q^*\to E^*\to S^*$ is exact, and so as in the above lemma there is a constant M associated to E independent of Q so that*

$$-\int_X c_1(Q)\wedge\tau^{n-k-1}\wedge\eta^k=\int_X c_1(Q^*)\wedge\tau^{n-k-1}\wedge\eta^k\le M.$$

In other words there is a uniform constant M so that: $-M\le\int_X c_1(Q)\wedge\tau^{n-k-1}\wedge\eta^k$, where Q is any torsion-free quotient of E.

Remark 4.8. *In the case that $k=n-1$, $\deg_k(S,\tau,\eta)=\deg(S,\eta)$ and the above constitutes a proof of Simpson's degree formula.*

We note that if $\tilde{X}\to X$ is a composition of finitely many blowups then we also have a family of Kähler metrics on $\tilde{X}$ by iteratively applying Proposition 4.5. We would now like to compute the degree of an arbitrary torsion-free sheaf $\tilde{S}$ on $\tilde{X}$ with respect to each metric ω_ε on $\tilde{X}$.

Theorem 4.9. *Let $\tilde{S}$ be a subsheaf (with torsion free quotient $\tilde{Q}$) of a holomorphic vector bundle $\tilde{E}$ on $\tilde{X}$, where $\pi : \tilde{X} \to X$ is given by a sequence of blowups along complex submanifolds of* codim ≥ 2. *Then there is a uniform constant M independent of $\tilde{S}$ such that the degrees of $\tilde{S}$ and $\tilde{Q}$ with respect to ω_ε satisfy:* $\deg(\tilde{S}, \omega_\varepsilon) \leq \deg(\pi_* \tilde{S}) + \varepsilon M$, *and* $\deg(\tilde{Q}, \omega_\varepsilon) \geq \deg(\pi_* \tilde{Q}) - \varepsilon M$.

Proof. The general case will follow from the case when $\tilde{S}$ is a line bundle $\tilde{L}$ (perhaps not a line subbundle). Recall that the Picard group of the blowup $Pic(\tilde{X}) = Pic(X) \oplus \mathbb{Z}\mathcal{O}(\mathbf{E}_1) \oplus \cdots \oplus \mathbb{Z}\mathcal{O}(\mathbf{E}_m)$ where the $\mathbf{E}_i$ are the irreducible components of the exceptional divisor. That is, we may write an arbitrary line bundle as $\tilde{L} = \pi^* L \otimes \mathcal{O}_{\tilde{X}}(\sum_i m_i \mathbf{E}_i)$ where L is a line bundle on X. Then by definition:

$$\deg(\tilde{L}, \omega_\varepsilon) = \int_{\tilde{X}} c_1(\tilde{L}) \wedge \omega_\varepsilon^{n-1} = \int_{\tilde{X}} c_1(\tilde{L}) \wedge (\pi^*\omega + \varepsilon\eta)^{n-1}.$$

Then we have an expansion:

$$(\pi^*\omega + \varepsilon\eta)^{n-1} = \pi^*\omega^{n-1} + \varepsilon(n-1)\pi^*\omega^{n-2} \wedge \eta + \cdots + \varepsilon^{n-2}(n-1)\pi^*\omega \wedge \eta^{n-2} + \varepsilon^{n-1}\eta^{n-1}.$$

Note that $\int_{\tilde{X}} c_1(\mathcal{O}_{\tilde{X}}(\mathbf{E}_i)) \wedge \pi^*\omega^{n-1} = \int_{\mathbf{E}_i} (\pi^*\omega)^{n-1} = 0$, since the image in X of each $\mathbf{E}_i$ lives in codimension at least 2. Therefore we are left with

$$\begin{aligned}
\deg(\tilde{L}, \omega_\varepsilon) &= \int_{\tilde{X}} c_1(\tilde{L}) \wedge \pi^*\omega^{n-1} + \sum_k \varepsilon^k \binom{n-1}{k} \left(\int_{\tilde{X}} c_1(\tilde{L}) \wedge \pi^*\omega^{n-k-1} \wedge \eta^k \right) \\
&= \int_{\tilde{X}} \pi^* c_1(L) \wedge \pi^*\omega^{n-1} + \sum_i m_i \int_{\tilde{X}} c_1(\mathcal{O}_{\tilde{X}}(\mathbf{E}_i)) \wedge \pi^*\omega^{n-1} \\
&\quad + \sum_k \varepsilon^k \binom{n-1}{k} \int_{\tilde{X}} c_1(\tilde{L}) \wedge \pi^*\omega^{n-k-1} \wedge \eta^k \\
&= \deg(L, \omega) + \sum_k \varepsilon^k \binom{n-1}{k} \int_{\tilde{X}} c_1(\tilde{L}) \wedge \pi^*\omega^{n-k-1} \wedge \eta^k
\end{aligned}$$

By the previous lemma the terms $\int_{\tilde{X}} c_1(\tilde{L}) \wedge \pi^*\omega^{n-k-1} \wedge \eta^k$, are all bounded uniformly independently of ε since $\pi^*\omega$ is semi-positive and η is a Kähler form. Therefore we have: $\deg(\tilde{L}, \omega_\varepsilon) \leq \deg(L, \omega) + \varepsilon M$.

Now note that if $\tilde{X} = Bl_Y X$ then $\pi_*\mathcal{O}(m\mathbf{E}) = \mathcal{O}_X$ if $m \geq 0$ and $\pi_*\mathcal{O}(m\mathbf{E}) = I_Y^{\otimes m}$ if $m < 0$, where I_Y is the ideal sheaf of holomorphic functions on X vanishing on Y. The determinant of an ideal sheaf is trivial if Y has codimension at least 2, so we have $\det(\pi_*\tilde{L}) = \det(L)$ so finally: $\deg(\tilde{L}, \omega_\varepsilon) \leq \deg(\pi_*\tilde{L}) + \varepsilon M$.

Now for an arbitrary subsheaf $\tilde{S} \subset \tilde{E}$, by definition $\deg(\tilde{S}, \omega_\varepsilon) = \deg(\det(\tilde{S}), \omega_\varepsilon)$. When $\pi_*\tilde{S}$ is a vector bundle, that is, away from its algebraic singular set, we have an isomorphism $\det(\pi_*\tilde{S}) = \pi_* \det \tilde{S}$. Their determinants are therefore isomorphic away from this set, and so by Hartogs' theorem there is an isomorphism of line bundles: $\det(\pi_*\tilde{S}) = \det(\pi_* \det \tilde{S})$ on X. Therefore by the previous argument:

$$\deg(\tilde{S}, \omega_\varepsilon) = \deg(\det(\tilde{S}), \omega_\varepsilon) \leq \deg(\pi_* \det \tilde{S}) + \varepsilon M = \deg(\pi_*\tilde{S}) + \varepsilon M \ .$$

The exact same argument together with the previous remark proves the second inequality as well. □

4.3. **Stability on Blowups and Convergence of the HN Type.**

Proposition 4.10. *Let $\tilde{E} \to \tilde{X}$ a holomorphic vector bundle where $\tilde{X} \to X$ is a sequence of blowups. If $\pi_*\tilde{E}$ is ω-stable, then there is an ε_2 such that $\tilde{E}$ is ω_ε-stable for all $0 < \varepsilon \leq \varepsilon_2$.*

Proof. Suppose there is a destabilising subsheaf $\tilde{S}_\varepsilon \subset \tilde{E}$, i.e. $\mu_{\omega_\varepsilon}(\tilde{S}_\varepsilon) \geq \mu_{\omega_\varepsilon}(\tilde{E})$ for each ε. Now among all proper subsheaves of $\pi_*\tilde{E}$, the maximal slope is realised by some subsheaf $\mathcal{F}$, in other words:

$$\mu_\omega(\mathcal{F}) = \sup\{\mu_\omega(S) \mid S \subset \pi_*\tilde{E}\}.$$

Then by the previous theorem we have:

$$\mu_\omega(\pi_*\tilde{E}) - \varepsilon M \leq \mu_{\omega_\varepsilon}(\tilde{E}) \leq \mu_\omega(\pi_*\tilde{S}_\varepsilon) + \varepsilon M \leq \mu_\omega(\mathcal{F}) + \varepsilon M.$$

In other words:

$$\mu_\omega(\pi_*\tilde{E}) \leq \mu_\omega(\mathcal{F}) + 2\varepsilon M$$

Since $\pi_*\tilde{E}$ is ω-stable, $\mu_\omega(\mathcal{F}) < \mu_\omega(\pi_*\tilde{E})$. Since the constant M is independent of ε, when ε is sufficiently small (more specifically, when $\varepsilon < (\mu_\omega(\pi_*\tilde{E}) - \mu_\omega(\mathcal{F}))/2M$), we have

$$\mu_\omega(\pi_*\tilde{E}) \leq \mu_\omega(\mathcal{F}) + 2\varepsilon M < \mu_\omega(\pi_*\tilde{E}),$$

which is a contradiction. □

Remark 4.11. *This shows in particular that for any resolution of a HNS filtration, the quotients* $\tilde{Q}_i = \tilde{E}_i/\tilde{E}_{i-1}$ *are stable with respect to* ω_ε *for* ε *sufficiently small, since the double dual of the pushforward is the double dual of* Q_i *which is stable by construction. This fact will be important in Section* 5.

For each of the metrics ω_ε there is also an HNS filtration of the pullback π^*E. We will need information about what happens to the corresponding HN types as $\varepsilon \to 0$. Namely we have:

Proposition 4.12. *Let* $E \to X$ *be a holomorphic vector bundle and* $\pi : \tilde{X} \to X$ *be a finite sequence of blowups resolving the HNS filtration. Then the HN type* $(\mu_1^\varepsilon, \cdots, \mu_K^\varepsilon)$ *of* π^*E *with respect to* ω_ε *converges to the HN type* $(\mu_1, \cdots, \mu_K)$ *of* E *with respect to* ω *as* $\varepsilon \longrightarrow 0$.

Proof. Let

$$0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \tilde{E}_2 \subset \cdots \subset \tilde{E}_{l-1} \subset \tilde{E}_l = \pi^*E$$

be a resolution of the HNS filtration. Since all the information about the HN type is contained in the HN filtration

$$0 = \mathbb{F}_0^{HN} \subset \mathbb{F}_1^{HN}(E) \subset \mathbb{F}_2^{HN}(E) \subset \cdots \subset \mathbb{F}_l^{HN}(E) = E,$$

we will just regard this as a resolution of singularities of the HN filtration and forget about Seshadri filtrations for the rest of this proof.

We would like to relate the resolution of the HN filtration of (E, ω), to the HN filtration of $(\pi^*E, \omega_\varepsilon)$ for small ε. We claim that for all ε in a sufficient range we may arrange that $\mu_{\omega_\varepsilon}^{\min}(\tilde{E}_i) > \mu_{\omega_\varepsilon}^{\max}(\pi^*E/\tilde{E}_i)$. Let $\mathcal{F}_1 \subset \tilde{E}_i \subset \mathcal{F}_2 \subset \pi^*E$ be any subsheaves such that $\tilde{E}_i/\mathcal{F}_1$ is torsion free. Note that for $\tilde{x} \in \tilde{X}$ with $\pi(\tilde{x}) = x$, we always have maps on the stalks $(\pi_*\mathcal{F}_i)_x \to (\mathcal{F}_i)_{\tilde{x}}$. Since π is in particular a biholomorphism away from $\mathbf{E}$, when $\tilde{x} \in \tilde{X} - \mathbf{E}$ these maps are isomorphisms. In other words the sequences:

$$0 \longrightarrow \pi_*\mathcal{F}_1 \longrightarrow E_i \longrightarrow \pi_*\left(\tilde{E}_i/\mathcal{F}_1\right) \longrightarrow 0$$

and

$$0 \longrightarrow E_i \longrightarrow \pi_*\mathcal{F}_2 \longrightarrow \pi_*\left(\mathcal{F}_2/\tilde{E}_i\right) \longrightarrow 0$$

are exact away from the singular set Z_{alg}. In particular this means $E_i/\pi_*\mathcal{F}_1 \hookrightarrow \pi_*(\tilde{E}_i/\mathcal{F}_1)$ and $\pi_*\mathcal{F}_2/E_i \hookrightarrow \pi_*(\mathcal{F}_2/\tilde{E}_i)$ with torsion quotients, which implies $(E_i/\pi_*\mathcal{F}_1)^{**} = (\pi_*(\tilde{E}_i/\mathcal{F}_1))^{**}$ and $(\pi_*\mathcal{F}_2/E_i)^{**} = (\pi_*(\mathcal{F}_2/\tilde{E}_i))^{**}$. Then finally we have $\mu_\omega(E_i/\pi_*\mathcal{F}_1) = \mu_\omega(\pi_*(\tilde{E}_i/\mathcal{F}_1))$ and $\mu_\omega(\pi_*\mathcal{F}_2/E_i) = \mu_\omega(\pi_*(\mathcal{F}_2/\tilde{E}_i))$.

The above argument together with Theorem 4.9 now implies that $\mu_{\omega_\varepsilon}(\tilde{E}_i/\mathcal{F}_1) \geq \mu_\omega(E_i/\pi_*\mathcal{F}_1) - \varepsilon M$ and $\mu_{\omega_\varepsilon}(\mathcal{F}_2/\tilde{E}_i) \leq \mu_\omega(\pi_*\mathcal{F}_2/E_i) + \varepsilon M$. On the other hand: $\mu_\omega(E_i/\pi_*\mathcal{F}_1) \geq \mu_\omega(Q_i) > \mu_\omega(Q_{i+1}) \geq \mu_\omega(\pi_*\mathcal{F}_2/E_i)$, where we have used the facts that $\mu_\omega(Q_i) = \mu_\omega^{\min}(E_i)$ and $\mu_\omega(Q_{i+1}) = \mu_\omega^{\max}(E/E_i)$. Therefore we have:

$$\mu_{\omega_\varepsilon}(\tilde{E}_i/\mathcal{F}_1) - \mu_{\omega_\varepsilon}(\mathcal{F}_2/\tilde{E}_i) \geq (\mu_\omega(E_i/\pi_*\mathcal{F}_1) - \mu_\omega(\pi_*\mathcal{F}_2/E_i)) - 2\varepsilon M.$$

As we have shown, the first term on the right hand side is strictly positive, so when ε is sufficiently small the entire right hand side is strictly positive. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ were arbitrary, for ε small $\mu_{\omega_\varepsilon}^{\min}(\tilde{E}_i)$ must be strictly bigger than $\mu_{\omega_\varepsilon}^{\max}(\pi^*E/\tilde{E}_i)$.

Now it follows from Proposition 2.9 that the HN filtration of $(\pi^*E, \omega_\varepsilon)$ is:

$$\begin{aligned} 0 \quad &\subset \quad \mathbb{F}_1^{HN,\varepsilon}(\tilde{E}_1) \subset \cdots \subset \mathbb{F}_{k_1}^{HN,\varepsilon}(\tilde{E}_1) = \tilde{E}_1 \subset \cdots \subset \mathbb{F}_{k_1+\cdots+k_{l-1}}^{HN,\varepsilon}(\tilde{E}_{l-1}) = \tilde{E}_{l-1} \\ &\subset \quad \mathbb{F}_{k_1+\cdots+k_{l-1}+1}^{HN,\varepsilon}(\tilde{E}_l) \subset \cdots \subset \mathbb{F}_{k_1+\cdots+k_l}^{HN,\varepsilon}(\tilde{E}_l) = \pi^*E. \end{aligned}$$

That is, the resolution appears within the HN filtration with respect to ω_ε, and two successive subbundles in the resolution are separated by the HN filtration of the larger bundle. Then for any i we consider the following part of the above filtration:

$$\begin{aligned} \tilde{E}_{i-1} \quad &= \quad \mathbb{F}_{k_1+\cdots+k_{i-1}}^{HN,\varepsilon}(\tilde{E}_{i-1}) \subset \mathbb{F}_{k_1+\cdots+k_{i-1}+1}^{HN,\varepsilon}(\tilde{E}_i) \subset \\ \cdots \quad &\subset \quad \mathbb{F}_{k_1+\cdots+k_i-1}^{HN,\varepsilon}(\tilde{E}_i) \subset \mathbb{F}_{k_1+\cdots+k_i}^{HN,\varepsilon}(\tilde{E}_i) = \tilde{E}_i. \end{aligned}$$

We claim that:

$$\mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+j}(\tilde{E}_i)/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+j-1}(\tilde{E}_i)\right)\longrightarrow\mu_\omega(E_i/E_{i-1})=\mu_\omega(Q_i)$$

for each $1\leq j\leq k_i$. Then the proposition will follow immediately. The slopes of the quotients in the HN filtration are strictly decreasing so we have:

$$\begin{aligned}\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right) &< \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+j}(\tilde{E}_i)/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+j-1}(\tilde{E}_i)\right)\\ &< \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right).\end{aligned}$$

Therefore it suffices to prove convergence of

$$\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)\ and\ \mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)$$

to $\mu_\omega(Q_i)$ as $\varepsilon\to 0$. Note that just as before we may argue that

$$\mu_\omega\left(\pi_*\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)\right)=\mu_\omega\left(E_i/\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)$$

and

$$\mu_\omega\left(\pi_*\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)\right)=\mu_\omega\left(\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/E_{i-1}\right).$$

By Theorem 4.9 we have:

$$\begin{aligned}\mu_\omega(Q_i)-\varepsilon M &= \mu_\omega(\pi_*\tilde{Q}_i)-\varepsilon M\leq\mu_{\omega_\varepsilon}(\tilde{Q}_i)\leq\mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)\\ &\leq \mu_\omega\left(\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/E_{i-1}\right)+\varepsilon M\leq\mu_\omega\left(E_i/E_{i-1}\right)+\varepsilon M\\ &= \mu_\omega(Q_i)+\varepsilon M\end{aligned}$$

where we have used that $F^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})$ is maximally destabilising in $\pi^*E/\tilde{E}_{i-1}$ and E_i/E_{i-1} is maximally destabilising in E/E_{i-1}. So

$$\mu_{\omega_\varepsilon}\left(\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_{i-1}+1}(\tilde{E}_{i-1})/\tilde{E}_{i-1}\right)\longrightarrow\mu_\omega(Q_i).$$

Similiarly we have:

$$\begin{aligned}\mu_\omega\left(Q_i\right)-\varepsilon M &= \mu_\omega\left(E_i/E_{i-1}\right)-\varepsilon M\leq\mu_\omega\left(E_i/\pi_*\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)-\varepsilon M\\ &\leq \mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)\leq\mu_{\omega_\varepsilon}(\tilde{Q}_i)\leq\mu_\omega\left(\pi_*\tilde{Q}_i\right)+\varepsilon M\\ &= \mu_\omega\left(Q_i\right)+\varepsilon M\end{aligned}$$

where we have used that $\mu_\omega\left(E_i/E_{i-1}\right)=\mu^{\min}_\omega\left(E_i\right)$ and $\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)=\mu^{\min}_{\omega_\varepsilon}(\tilde{E}_i)$. Then taking limits implies $\mu_{\omega_\varepsilon}\left(\tilde{E}_i/\mathbb{F}^{HN,\varepsilon}_{k_1+\cdots+k_i-1}(\tilde{E}_i)\right)\to\mu_\omega(Q_i)$. This completes the proof. □

Remark 4.13. *Note that the argument of the above proof also shows that we have convergence:*

$$\left(\mu_{\omega_\varepsilon}(\tilde{Q}_1),\cdots,\mu_{\omega_\varepsilon}(\tilde{Q}_l)\right)\longrightarrow\left(\mu_\omega(Q_1),\cdots,\mu_\omega(Q_l)\right),$$

where as usual $\mu_{\omega_\varepsilon}(\tilde{Q}_i)$ is repeated $\mathrm{rk}(\tilde{Q}_i)$ times. We will use this fact in the following section.

5. Approximate Critical Hermitian Structures/HN Type of the Limit

In this section we accomplish two important aims. One is the construction of a certain canonical type of metric on a holomorphic vector bundle over a Kähler manifold called an L^p-approximate critical hermitian structure. The other is identifying the Harder-Narasimhan type of the limiting vector bundle E_∞ along the flow, namely we prove that this is the same as the type of the original bundle E. This latter fact will be a crucial element in the proof of the main theorem, whereas the former will play no role in the remainder of the proof. However we remark that these two theorems are, due to certain technical considerations to be discussed below, very much intertwined.

If we fix a holomorphic structure on E, then a critical point of the HYM functional, thought of as a function $h\mapsto HYM(\bar\partial_E,h)$ on the space of metrics, is called (see Kobayashi [KOB]) a *critical hermitian* *structure*. The Kähler identities imply that this happens exactly when the corresponding connection $(\bar{\partial}_E, h)$ is Yang-Mills, and hence in this case the Hermitian-Einstein tensor splits: $\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E,h)} = \mu_1 Id_{Q_1} \oplus \cdots \oplus \mu_l Id_{Q_l}$. Here the holomorphic structure $\bar{\partial}_E$ splits into the direct sum $\oplus_i Q_i$ and the metric induced on each summand is Hermitian-Einstein with constant factor μ_i.

In general, the holomorphic structure on E is not split, and of course the Q_i may not be subbundles as at all, so it is not the case that we always have a critical hermitian structure. We therefore need to define a correct approximate notion of a critical point. In the subsequent discussion we follow Daskalopoulos-Wentworth [DW1].

Let h be a smooth metric on E and $\mathcal{F} = \{F_i\}_{i=0}^l$ a filtration of E by saturated subsheaves. For every F_i we have the corresponding weakly holomorphic projection π_i^h. These are bounded, L_1^2 hermitian endomorphisms of E. Here $F_0 = 0$, and so $\pi_0^h = 0$. Given real numbers $\mu_1, \cdots, \mu_l$, define the following L_1^2 hermitian endomorphism of E :

$$\Psi(\mathcal{F}, (\mu_1, \cdots, \mu_l), h) = \sum_{i=1}^{l} \mu_i \left(\pi_i^h - \pi_{i-1}^h\right).$$

Notice that away from the singular set of the filtration (points where it is given by sub-bundles), the bundle E splits smoothly as $\oplus Q_i = \oplus_i E_i/E_{i-1}$, and with respect to the splitting, the endomorphism $\Psi(\mathcal{F}, (\mu_1, \cdots, \mu_l), h)$ is just the diagonal map $\mu_1 Id_{Q_1} \oplus \cdots \oplus \mu_l Id_{Q_l}$.

In the special case where E is a holomorphic vector bundle over a Kähler manifold (X, ω), we will write $\Psi_\omega^{HNS}(\bar{\partial}_E, h)$ when the filtration of E is the HNS filtration $F_i = \mathbb{F}_i^{HNS}(E)$ and $\mu_1, \cdots, \mu_l$ are the distinct slopes appearing the HN type.

Definition 5.1. *Fix $\delta > 0$ and $1 \leq p \leq \infty$. An L^p δ-approximate critical hermitian structure on a holomorphic bundle E is a smooth metric h such that:*

$$\left\| \sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E,h)} - \Psi_\omega^{HNS}(\bar{\partial}_E, h) \right\|_{L^p(\omega)} \leq \delta.$$

The following theorem first appeared in [DW1].

Theorem 5.2. *If the HNS filtration of E is given by subbundles, then for any $\delta > 0$, E has an L^∞ δ-approximate critical hermitian structure.*

We begin by giving a (very simple) proof of this theorem in the case that the HNS filtration has length two (the general case follows from an inductive argument). Namely we assume that there is an exact sequence of the form:

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

where S and Q are stable vector bundles. Then fix Hermitian-Einstein metrics h_S and h_Q on S and Q. There is a smooth splitting $E \simeq S \oplus Q$ and so we may fix the metric $h_E = h_S \oplus h_Q$ on E. Of course in general we there is no holomorphic splitting. The failure of the sequence to split holomorphically is determined by the second fundamental form $\beta \in \Omega^{0,1}(\mathrm{Hom}(Q, S))$, and the holomorphic structure of E may be written as:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_S & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}$$

and similarly

$$\partial_E = \begin{pmatrix} \partial_S & 0 \\ -\beta^* & \partial_Q \end{pmatrix}.$$

Now the curvature of the connection $(\bar{\partial}_E, h_E)$ is $F_{(\bar{\partial}_E,h_E)} = (\bar{\partial}_E, h_E) \circ (\bar{\partial}_E, h_E) = \bar{\partial}_E \circ \partial_E + \partial_E \circ \bar{\partial}_E$. Therefore we have:

$$F_{(\bar{\partial}_E,h_E)} = \begin{pmatrix} F_{(\bar{\partial}_S,h_S)} - \beta \wedge \beta^* & \partial_E \beta \\ -\bar{\partial}_E \beta^* & F_{(\bar{\partial}_Q,h_Q)} - \beta^* \wedge \beta \end{pmatrix}.$$

Now applying $\sqrt{-1}\Lambda_\omega$ and using the Kähler identities we have:

$$\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E,h_E)} = \begin{pmatrix} \sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_S,h_S)} - \sqrt{-1}\Lambda_\omega \left(\beta \wedge \beta^*\right) & -\left(\bar{\partial}_E\right)^* \beta \\ -\left(\left(\bar{\partial}_E\right)^* \beta\right)^* & \sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_Q,h_Q)} - \sqrt{-1}\Lambda_\omega \left(\beta^* \wedge \beta\right) \end{pmatrix}.$$

Therefore we have:

$$\begin{aligned}
& \left\| \sqrt{-1}\Lambda_\omega F_{(\bar\partial_E, h_E)} - \mu_\omega(S) Id_S \oplus \mu_\omega(Q) Id_Q \right\|_{L^\infty(X,\omega)} \\
\leq\ & \left\| \sqrt{-1}\Lambda_\omega F_{(\bar\partial_S, h_S)} - \mu_\omega(S) Id_S \right\|_{L^\infty(X,\omega)} + \left\| \sqrt{-1}\Lambda_\omega F_{(\bar\partial_Q, h_Q)} - \mu_\omega(Q) Id_Q \right\|_{L^\infty(X,\omega)} \\
& + 2C \sup \left(|\beta|^2 + \left| (\bar\partial_E)^* \beta \right|^2 \right) \\
=\ & 2C \sup \left(|\beta|^2 + \left| (\bar\partial_E)^* \beta \right|^2 \right),
\end{aligned}$$

where we have used that h_S and h_Q are Hermitian-Einstein as well as the fact that $\mathrm{Tr} - \sqrt{-1}\Lambda_\omega (\beta \wedge \beta^*) = |\beta|^2$. Now change the holomorphic structure on E by applying the complex gauge transformation $g_t = t^{-1} Id_S \oplus t Id_Q$, so that:

$$g_t(\bar\partial_E) = \begin{pmatrix} \bar\partial_S & t^2\beta \\ 0 & \bar\partial_Q \end{pmatrix}.$$

Then we have:

$$\left\| \sqrt{-1}\Lambda_\omega F_{(g_t(\bar\partial_E), h_E)} - \mu_\omega(S) Id_S \oplus \mu_\omega(Q) Id_Q \right\|_{L^\infty(X,\omega)} \leq 2Ct^4 \sup \left(|\beta|^2 + \left| (\bar\partial_E)^* \beta \right|^2 \right)$$

which goes to 0 as t goes to 0.

In general, we will not obtain an L^∞ approximate structure. In the remainder of this section we show that for an arbitrary holomorphic bundle we have such a metric for $1 \leq p < \infty$. We must modify the above approach in the general case, since the filtration is not given by subbundles. A simple example of where this can happen is as follows.

Example 5.3. *It can be shown (see* [OSS] *page* 103*) that for $k < 3$ there is a locally free representative of rank 2 in $Ext^1_{\mathbb{CP}^2}(\mathcal{I}_p, \mathcal{O}_{\mathbb{CP}^2}(-k))$, where $\mathcal{I}_p$ is the ideal sheaf of a point. In other words there is a short exact sequence:*

$$0 \longrightarrow \mathcal{O} \longrightarrow E \longrightarrow \mathcal{I}_p \otimes \mathcal{O}_{\mathbb{CP}^2}(k) \longrightarrow 0$$

where $\mathcal{O}$ is the trivial line bundle. Moreover, one can compute that $c_1(E) = k$. Therefore, if we take $k < 0$, then $\mu(E) < 0$. Since $\mu(\mathcal{O}) = 0$, the section given by $\mathcal{O} \longrightarrow E$ vanishing at p, is a destabilising subsheaf of E, so E is unstable in this case. Since $\mathcal{O}$ and $\mathcal{I}_p \otimes \mathcal{O}_{\mathbb{CP}^2}(k)$ are rank one and hence are stable, and the slopes are strictly decreasing ($0 = \mu(\mathcal{O}) > \mu(\mathcal{I}_p \otimes \mathcal{O}_{\mathbb{CP}^2}(k)) = k$) this sequence is precisely the Harder-Narasimhan filtration for E. On the other hand the quotient $\mathcal{I}_p \otimes \mathcal{O}_{\mathbb{CP}^2}(k)$ fails to be locally free at the point p, since the ideal sheaf of a point on a complex surface is not locally free. Generalisations of this example are given by replacing the point p in $\mathbb{CP}^2$ by a locally complete intersection in $\mathbb{CP}^n$ with $n > 2$, or replacing $\mathbb{CP}^2$ by a a Kähler surface X with $\dim H^2(X, \mathcal{O}_X) = 0$ for instance.

Example 5.4. *In the above example, the only singular point of the filtration is the point p. If we blowup the point p, and consider $Bl_p\ \mathbb{CP}^2 = \widetilde{\mathbb{CP}}^2 \xrightarrow{\pi} \mathbb{CP}^2$, then the exceptional divisor $\mathbf{E}$ in this case is just a copy of $\mathbb{CP}^1$. By construction π^*E is trivial over this $\mathbb{CP}^1$ and is equal to E away from it. Therefore, since E contains the trivial line bundle $\mathcal{O}$ as a subsheaf, π^*E contains as a subbundle a copy of the line bundle $\mathcal{O}(\mathbf{E})$. Since $\mathcal{O}(\mathbf{E}) = \mathcal{O}$ away from $\mathbf{E}$ and $\pi^*\mathcal{O} = \mathcal{O}$, there is an inclusion of sheaves $\mathcal{O} \hookrightarrow \mathcal{O}(\mathbf{E})$. Indeed, since the quotient is supported on $\mathbf{E}$ and therefore torsion, by Lemma 2.1, $\mathrm{Sat}_{\pi^*E}\,\mathcal{O} = \mathcal{O}(\mathbf{E})$. In other words, a single blowup of the point p, gives a resolution of singularities in this case, and the filtration by subbundles of π^*E is given by $\mathcal{O}(\mathbf{E}) \subset \pi^*E$. Therefore on $\widetilde{\mathbb{CP}}^2$ we have an exact sequence:*

$$0 \longrightarrow \mathcal{O}(\mathbf{E}) \longrightarrow \pi^*E \longrightarrow \mathcal{O}(-\mathbf{E}) \otimes \pi^*\mathcal{O}_{\mathbb{CP}^2}(k) \longrightarrow 0.$$

Therefore, in the general case we will need a more sophisticated argument to deal with the fact that the subsheaf S (and the quotient Q) can have singularities. We outline our argument as follows. First we pass to a resolution of singularities $\pi : \tilde{X} \longrightarrow X$ for the HNS filtration. The blowup $\tilde{X}$ is equipped with a family of Kähler metrics ω_ε as described in the previous section. Therefore, if we fix some value ε_1, then with respect to the metric ω_{ε_1} on the blowup $\tilde{X}$, we will be in the same situation as above, when the filtration is given by subbundles. Just as in that case, by scaling the extension classes we can produce a metric $\tilde{h}$ with the desired property on the pullback bundle $\pi^*E \longrightarrow \bar{X}$.

Of course this is not what we want, but we may use this metric to produce a metric on E via a cut-off argument. Namely, we first assume that the singular set is a complex submanifold, and that the resolution of singularities is achieved by performing one blowup operation. Then we choose a cut-off function ψ in a tubular neighbourhood of the singular set, and fix any smooth background metric H on this tubular neighbourhood. We define the metric on $E \longrightarrow X$ by $h = \psi H + (1-\psi)\tilde{h}$.

Now we can break the estimate up into three estimates on three different regions. We define ψ so that on a smaller neighbourhood of the singular set h is equal to H. The desired estimate will follow on this region by taking the radius of the neighbourhood to be arbitrarily small. Outside of the tubular neighbourhood, h is equal to $\tilde{h}$ and we can estimate as in the case of subbundles. Finally we must also estimate in the annulus defined by these two open sets. This can be achieved by defining ψ to have bounds on its first and second derivatives that depend on the reciprocal of the radius of the tubular neighbourhood and its square respectively. The Hermitian-Einstein tensor will depend on two derivatives of ψ on the annulus, so a pointwise estimate on this quantity will depend on this radius, but a simple argument using the fact that the Hausdorff codimension of the singular set is a least 4, shows we can also obtain the appropriate estimate in this region.

Strictly speaking, we need to estimate the difference of the Hermitian-Einstein tensor $\Lambda_\omega F_h$ of this metric with the endomorphism $\Psi^{HNS}(\mu_1, \cdots, \mu_l)$ constructed from the slopes obtained from the HNS filtration on $E \longrightarrow X$. On the other hand, h has been constructed from $\tilde{h}$, which has been defined so that the difference between its Hermitian-Einstein tensor $\Lambda_{\omega_{\varepsilon_1}} F_{\tilde{h}}$ and the corresponding endomorphism coming from the filtration (by subbundles) of $\pi^*E \longrightarrow \bar{X}$ can be estimated on $\tilde{X}$. Because ω_{ε_1} is a perfectly defined Kähler metric on X away from the singular set (which is where this estimate must be performed), one could try to do the estimate on this region directly, as described in the preceding paragraph, by first estimating $\Lambda_\omega F_{\tilde{h}}$ in terms of $\Lambda_{\omega_{\varepsilon_1}} F_{\tilde{h}}$ uniformly in ε_1 and the size of the neighbourhood, but attempts to do this were unsuccessful.

Therefore, in order to perform the estimate properly, we will need to work on the blowup. Namely, we estimate the Hermitian-Einstein tensor for π^*h with respect to the family of Kähler metrics ω_ε. Since this metric is a pullback, it suffices to show that we obtain estimates on the blowup that are uniform in ε. Then taking the limit as $\varepsilon \to 0$ will yield an estimate with respect to the metric ω on X. However, note again that the metric $\tilde{h}$ must be chosen at some point, and this requires fixing a value ε_1. Therefore, to imitate our argument above, we need to estimate the L^p norm of $\Lambda_{\omega_\varepsilon} F_{\tilde{h}}$ uniformly in ε in terms of $\Lambda_{\omega_{\varepsilon_1}} F_{\tilde{h}}$. Here we crucially use the fact that we are working on the blowup. Namely, all that is required is an estimate close to the exceptional divisor (since it is trivial on the complement of such a neighbourhood). The fact that the exceptional divisor has only normal crossings singularities is the key to proving that such an estimate holds.

Something very similar was done in [DW1]. The author has noticed an error in [DW1] on this point. In particular, Lemma 3.14 is slightly incorrect. Instead, the right hand side should have an additional term involving the L^2 norm of the full curvature. This does not essentially disrupt the proof, because the Yang-Mills and Hermitian-Yang-Mills functionals differ only by a topological term, but it has the effect of changing the logic of the argument somewhat, as well as increasing the technical complexity.

This is the reason behind most of the work done in this section. The precise proof, given below, is a delicate balancing act between the scaling parameter t, the parameter ε_1 used to define $\tilde{h}$, the radius R of the tubular neighbourhood, and the parameter $0 < \varepsilon \leq \varepsilon_1$ defining the family of Kähler metrics on $\tilde{X}$. Furthermore, the scheme explained above will only give the correct estimate in L^p for p sufficiently close to 1. On the other hand, such a metric is all that is required to prove that the Harder-Narasimhan type of the limiting sheaf E_∞ is the same as that of E. With this knowledge, it is in fact very easy to prove in turn that E has an L^p δ-approximate structure for all $1 \leq p < \infty$. This new metric depends on the value of p, and is in fact given by running the Yang-Mills flow for some finite time.

We begin with a preliminary technical lemma, which will be used repeatedly throughout this section. It will be used in conjunction with Hölder's inequality to show that certain quantities depending a priori on ε can in fact be estimated independently of ε in certain L^p spaces with p very close to 1. It is the use of this lemma that limits this particular method of constructing a δ-approximate structure to these particular values of p. We use this to prove the L^p bound on $\Lambda_{\omega_\varepsilon} F$ in terms of $\Lambda_{\omega_{\varepsilon_1}} F$ for any $(1,1)$-form F. The construction of the metric together with the estimate in L^p for p close to 1 is the substance of Proposition 5.7. We use this and the material in Section 3.2 to prove the statement concerning the HN type of the limit. Then we quote a result about convergence of the HN filtration along the flow from [DW1], and use this to prove the existence of an L^p structure for each $1 \leq p < \infty$. Finally, at the end of this section we do an inductive argument on the number of blowups required to resolve singularities in order to remove the restriction we put on the singular set. This argument actually uses the existence of an L^p structure for $p = 2$ (in the special case in which it has been proven).

Lemma 5.5. *Let X be a compact Kähler manifold of dimension n, and let $\pi : \tilde{X} \to X$ be a blowup along a complex submanifold Y of complex codimension k where $k \geq 2$. Consider the natural family $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $0 < \varepsilon \leq \varepsilon_1$ and η is a Kähler form on $\tilde{X}$. Then given any α and $\tilde{\alpha}$ such that $1 < \alpha < 1 + \frac{1}{2(k-1)}$, and $\frac{\alpha}{1-2(k-1)(\alpha-1)} < \tilde{\alpha} < \infty$, and if we let $s = \frac{\tilde{\alpha}}{\tilde{\alpha}-\alpha}$ then if we write g_ε for the Kähler metric associated to ω_ε, and g_ϖ for the hermitian metric associated to a fixed Kähler form ϖ on $\tilde{X}$, we have: $\det\left(g_\varepsilon^{-1} g_\varpi\right) \in L^{2(\alpha-1)s}(\tilde{X}, \varpi)$, and the value of the $L^{2(\alpha-1)s}$ norm is uniformly bounded in ε.*

Proof. Since g_ε converges to the Kähler metric $\pi^*\omega$ away from the exceptional divisor $\mathbf{E}$, on the complement of a neighbourhood of $\mathbf{E}$ there is always such a uniform bound (and on this set $(\det g_\varepsilon / \det g_\varpi)^{2(1-\alpha)s}$ is clearly integrable). It therefore suffices to prove the result in a neighbourhood of the exceptional divisor. Let $y \in Y$ and U be a local coordinate chart containing y consisting of coordinates $(z_1, \cdots, z_n)$. Now Y has codimension k so that locally Y is given by the slice coordinates $\{z_1 = z_2 = \cdots = z_k = 0\}$. Recall that on the blow-up $\tilde{X}$ we have explicit coordinate charts $\tilde{U}_m \subset \tilde{U} = \pi^{-1}(U)$ where $\tilde{U}_m = \{z \in U - Y \mid z_m \neq 0\} \cup \{(z, [\nu]) \in \mathbb{P}(\boldsymbol{\zeta})_{|Y \cap U} \mid \nu_m \neq 0\}$, where $\mathbb{P}(\boldsymbol{\zeta})$ is the projectivisation of the normal bundle of Y. Let $(\xi_1, \cdots, \xi_n)$ denote local coordinates on $\tilde{U}_m$. In these coordinates the map $\pi : \tilde{X} \to X$ is given by:

$$(\xi_1, \cdots, \xi_n) \longrightarrow (\xi_1\xi_m, \cdots, \xi_{s-1}\xi_m, \xi_m, \xi_{m+1}\xi_m, \cdots, \xi_k\xi_m, \xi_{k+1}, \cdots, \xi_n).$$

Now locally, we may write the Kähler form on X in terms of the associated metric g, as $\omega = \frac{i}{2} g_{ij} dz^i \wedge d\bar{z}^j$. Then the top power has the form: $\omega^n = n!(i/2)^n \det g_{ij}\, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$, and using this coordinate description we may compute: $\pi^*\omega^n = n!(i/2)^n \left(\pi^* \det g_{ij}\right) |\xi_m|^{2k-2}\, d\xi_1 \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_n$.

Note that $\pi^* \det g_{ij}$ is non-vanishing since $\det g_{ij}$ is non-vanishing, and so degeneracy of the pullback occurs only along the hypersurface defined by $\xi_m = 0$. In other words, $(\xi_1, \cdots, \xi_n)$ are normal crossings coordinates on the blow-up for the exceptional divisor $\mathbf{E}$, and locally $\mathbf{E}$ takes the form $\{\xi_m = 0\}$.

The top power of the Kähler form ω_ε is:

$$\omega_\varepsilon^n = \pi^*\omega^n + \varepsilon n \pi^*\omega^{n-1} \wedge \eta + .. + \varepsilon^l \binom{n}{l} \pi^*\omega^{n-l} \wedge \eta^l + \cdots + \varepsilon^{n-1} n \pi^*\omega \wedge \eta^{n-1} + \varepsilon^n \eta^n.$$

In the local coordinates $(\xi_1, \cdots, \xi_n)$ we have: $\omega_\varepsilon^n = n!(i/2)^n \det g_{ij}^\varepsilon d\xi_1 \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_n$. We may therefore obtain a lower bound (not depending on ε) on $\det g_{ij}^\varepsilon$ as follows. Note that $\eta > 0$. On the other hand, the only degeneracy of $\pi^*\omega$ is only on vectors tangent to the exceptional divisor (in other words, the restriction of $\pi^*\omega$ vanishes on $\mathbf{E}$), so $\pi^*\omega \geq 0$. Therefore $\pi^*\omega^l \wedge \eta^{n-l}$ is non-negative for every l.

Then comparing the two expressions for ω_ε^n, this implies that we have the lower bound: $\det g_{ij}^\varepsilon \geq C\,|\xi_m|^{2k-2}$, where $C = \inf \pi^* \det g_{ij}$ on $\tilde{U}_m$ for each $0 < \varepsilon \leq \varepsilon_1$. Taking the $2(1-\alpha)s$ power of both sides we see that

$$\int_{\tilde{U}_m} (\det g_\varepsilon / \det g_\varpi)^{2(1-\alpha)s} \varpi^n \leq C \int_{\tilde{U}_m} (\det g_{ij}^\varepsilon)^{2(1-\alpha)s} \leq C \int_{\tilde{U}_m} |\xi_m|^{4(1-\alpha)(k-1)s},$$

where the last two integrals are with respect to the standard Euclidean measure. Using the condition on $\tilde{\alpha}$ one computes that $4(1-\alpha)(k-1)s > -2$ and so the functions $|\xi_m|^{4(1-\alpha)s(k-1)}$, are integrable (as can be seen by computing the integral in polar coordinates), and the result follows. □

Lemma 5.6. *Let $\pi : \tilde{X} \to X$, the codimension k, and the family of metrics ω_ε be the same as in the previous lemma. Let $\tilde{B}$ be a holomorphic vector bundle on $\tilde{X}$ and F a $(1,1)$- form with values in the auxiliary vector bundle $End(\tilde{B})$. Let $1 < \alpha < 1 + \frac{1}{4k(k-1)}$ and $\frac{\alpha}{1-2(k-1)(\alpha-1)} < \tilde{\alpha} < 1 + \frac{1}{2(k-1)}$. Then there is a number κ_0 such that for any $0 < \kappa \leq \kappa_0$, there exists a constant C independent of ε, ε_1, and κ, and a constant $C(\kappa)$ such that:*

$$\|\Lambda_{\omega_\varepsilon} F\|_{L^\alpha(\tilde{X}, \omega_\varepsilon)} \leq C \left(\left\|\Lambda_{\omega_{\varepsilon_1}} F\right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})} + \kappa \|F\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \right) + \varepsilon_1 C(\kappa) \|F\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})}$$

Proof. In the following argument, out of convenience, we will engage in the slight (and quite orthodox) abuse of notation of dividing a top degree form by the volume form. Since the determinant line bundle of T^*X is trivial, any such form may be written as the product of some smooth function (or in this case endomorphism) with the volume form, and so dividing by the volume form simply returns this function (endomorphism).

Recall that $(\Lambda_{\omega_\varepsilon}F)\,\omega_\varepsilon^n = F\wedge\omega_\varepsilon^{n-1}$ and $\left(\Lambda_{\omega_{\varepsilon_1}}F\right)\omega_{\varepsilon_1}^n = F\wedge\omega_{\varepsilon_1}^{n-1}$ so that:

$$\Lambda_{\omega_\varepsilon}F = \frac{F\wedge\omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}, \Lambda_{\omega_{\varepsilon_1}}F = \frac{F\wedge\omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n}.$$

Note also that

$$\omega_\varepsilon^n = \frac{\det g^\varepsilon}{\det g^{\varepsilon_1}}\omega_{\varepsilon_1}^n$$

Now we write:

$$\begin{aligned}
\Lambda_{\omega_\varepsilon}F &= \frac{F\wedge\omega_\varepsilon^{n-1}}{\omega_\varepsilon^n} = \frac{F\wedge(\omega_{\varepsilon_1}^{n-1}+\omega_\varepsilon^{n-1}-\omega_{\varepsilon_1}^{n-1})}{\omega_\varepsilon^n}\\
&= \left(\frac{F\wedge\omega_{\varepsilon_1}^{n-1}+\sum_{l=1}^{n-1}(\varepsilon^l-\varepsilon_1^l)\binom{n-1}{l}F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_\varepsilon^n}\right).\\
&= \frac{\det g_{\varepsilon_1}}{\det g_\varepsilon}\left(\Lambda_{\omega_{\varepsilon_1}}F + \frac{\sum_{l=1}^{n-1}(\varepsilon^l-\varepsilon_1^l)\binom{n-1}{l}F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right).
\end{aligned}$$

Therefore:

$$|\Lambda_{\omega_\varepsilon}F|^\alpha \le C\left|\frac{\det g_{\varepsilon_1}}{\det g_\varepsilon}\right|^\alpha\left(\left|\Lambda_{\omega_{\varepsilon_1}}F\right|^\alpha + \sum_{l=1}^{n-1}\left|\varepsilon^l-\varepsilon_1^l\right|^\alpha\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^\alpha\right)$$

(by convexity of the function $|\cdot|^\alpha$ when $\alpha>1$). Again, we set $s=\frac{\tilde\alpha}{\tilde\alpha-\alpha}$ (note again that s is a conjugate variable to $\frac{\tilde\alpha}{\alpha}$). By the above expression and Hölder's inequality with respect to the metric ω_{ε_1}:

$$\begin{aligned}
\|\Lambda_{\omega_\varepsilon}F\|_{L^\alpha(\tilde X,\omega_\varepsilon)} &= \left(\int_{\tilde X}|\Lambda_{\omega_\varepsilon}F|^\alpha\,\omega_\varepsilon^n\right)^{\frac{1}{\alpha}} \le
\end{aligned}$$

$$C\left(\int_{\tilde X}\left(\frac{\det g_\varepsilon}{\det g_{\varepsilon_1}}\right)^{(1-\alpha)s}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\alpha s}}\left(\left(\int_{\tilde X}\left|\Lambda_{\omega_{\varepsilon_1}}F\right|^{\tilde\alpha}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\tilde\alpha}} + \left(\int_{\tilde X}\sum_{l=1}^{n-1}\left(\varepsilon_1^l\right)^{\tilde\alpha}\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^{\tilde\alpha}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\tilde\alpha}}\right).$$

By the previous lemma the factor

$$\left(\int_{\tilde X}\left(\frac{\det g_\varepsilon}{\det g_{\varepsilon_1}}\right)^{(1-\alpha)s}\omega_{\varepsilon_1}^n\right)^{\frac{1}{\alpha s}}$$

is uniformly bounded in ε.

Now we need to control the second term of the second factor above. We divide $\tilde X$ into two pieces: an arbitrarily small neighbourhood V_κ with $\mathrm{Vol}(V_\kappa,\omega_{\varepsilon_1}) = \kappa^{\frac{2}{2-\tilde\alpha}}$ of the exceptional divisor $\mathbf{E}$ and its complement. We will perform two separate estimates, one for each piece. Write the components of F in a local basis as $F_{\rho i\bar j}^\gamma$. At any point we may choose an orthonormal basis for the tangent space so that η is standard and $\pi^*\omega$ is diagonal. Then if we call this basis $\{e_i\}$, we have

$$\begin{aligned}
\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^{2\tilde\alpha} &= \left|\frac{\left(\sum_{i,j}F_{i\bar j}e_i\wedge\bar e_j\right)\wedge\left(\sum_i\pi^*g_{ii}e^i\wedge\bar e^i\right)^{(n-1)-l}\wedge\left(\sum_i e^i\wedge\bar e^i\right)^l}{\omega_{\varepsilon_1}^n}\right|^{2\tilde\alpha}\\
&\le \frac{C}{\left|\omega_{\varepsilon_1}^n\right|^{2\tilde\alpha}}\left(\sum_{i.j,\gamma,\rho}\left|F_{\rho i\bar j}^\gamma\right|^2\right)^{\tilde\alpha} = C\frac{|F|_\eta^{2\tilde\alpha}}{\left|\omega_{\varepsilon_1}^n\right|^{2\tilde\alpha}}.
\end{aligned}$$

Now note that on $\tilde X - V_\kappa$ the pullback $\pi^*\omega$ determines a metric, in other words $(\pi^*\omega)^n$ is non-vanishing, so since $\omega_{\varepsilon_1}^n \longrightarrow (\pi^*\omega)^n$, the quantity $\left|\omega_{\varepsilon_1}^n\right|^{2\tilde\alpha}$ is uniformly bounded away from 0. Therefore

$$\left|\frac{F\wedge\pi^*\omega^{(n-1)-l}\wedge\eta^l}{\omega_{\varepsilon_1}^n}\right|^{\tilde\alpha} \le C\,|F|_\eta^{\tilde\alpha}.$$

On the other hand, if we again choose a basis for which η is standard and such that ω_{ε_1} is diagonal, we have:

$$|F|^{2\tilde{\alpha}}_{\eta} = \left| \left(\sum_{i.j,\gamma,\rho} \left|F^{\gamma}_{\rho ij}\right|^2 \right) \right|^{\tilde{\alpha}} \leq C \left| \left(\sum_{i.j,\gamma,\rho} \frac{1}{g^{\varepsilon_1}_{ii} g^{\varepsilon_1}_{jj}} \left|F^{\gamma}_{\rho ij}\right|^2 \right) \right|^{\tilde{\alpha}} = C\, |F|^{2\tilde{\alpha}}_{\omega_{\varepsilon_1}}$$

since the product of the eigenvalues $g^{\varepsilon_1}_{ii} g^{\varepsilon_1}_{jj}$ is again uniformly bounded ($g^{\varepsilon_1}_{ii} g^{\varepsilon_1}_{jj} \to \pi^* g_{ii} \pi^* g_{jj}$ as $\varepsilon_1 \to 0$). Thus, on $\tilde{X} - V_\kappa$ we have the further pointwise bound: $|F|^{\tilde{\alpha}}_{\eta} \leq C\, |F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}}$. Therefore the integral on $\tilde{X} - V_\kappa$ is:

$$\begin{aligned} \left(\int_{\tilde{X}-V_\kappa} \left(\varepsilon_1^l\right)^{\tilde{\alpha}} \left| \frac{F \wedge \pi^*\omega^{(n-1)-l} \wedge \eta^l}{\omega^n_{\varepsilon_1}} \right|^{\tilde{\alpha}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} &\leq C\varepsilon_1 \left(\int_{\tilde{X}-V_\kappa} |F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} \\ &\leq C\varepsilon_1 \|F\|_{L^{\tilde{\alpha}}(\omega_{\varepsilon_1})} \leq C(\kappa)\varepsilon_1 \|F\|_{L^2(\omega_{\varepsilon_1})} \end{aligned}$$

since by assumption $\tilde{\alpha} < 2$.

Now we estimate this term on V_κ. Choose an orthonormal basis for the tangent space at a point in V_κ such that ω_{ε_1} is standard and η is diagonal. Then we have $g^{\varepsilon_1}_{ij} = \pi^* g_{ij} + \varepsilon_1 \eta_{ij}$, so if $i \neq j$, $\pi^* g_{ij} = 0$, and if $i = j$, $\eta_{ii} = \frac{1-\tilde{g}_{ii}}{\varepsilon_1}$. Note also that $0 \leq \tilde{g}_{ii} < 1$ since $0 < \eta_{ii}$. If we write Ω for the standard Euclidean volume form then:

$$\begin{aligned} & \sum_{l=1}^{n-1} \left(\varepsilon_1^l\right)^{\tilde{\alpha}} \left| \frac{F \wedge \pi^*\omega^{(n-1)-l} \wedge \eta^l}{\omega^n_{\varepsilon_1}} \right|^{\tilde{\alpha}} \\ = & \sum_{l=1}^{n-1} \left| \frac{\left(\sum_{i,j} F_{i\bar{j}} e_i \wedge \bar{e}_j\right) \wedge \left(\sum_i \pi^* g_{i\bar{i}} e^i \wedge \bar{e}^i\right)^{(n-1)-l} \wedge \left(\sum_i (1 - \pi^* g_{ii})\, e^i \wedge \bar{e}^i\right)^l}{\Omega} \right|^{\tilde{\alpha}} \\ \leq & \; C \left(\sum_{i.j,\gamma,\rho} \left|F^{\gamma}_{\rho i\bar{j}}\right| \right)^{\tilde{\alpha}} \leq C\, |F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}} . \end{aligned}$$

Therefore:

$$\begin{aligned} & \left(\int_{V_\kappa} \sum_{l=1}^{n-1} \left(\varepsilon_1^l\right)^{\tilde{\alpha}} \left| \frac{F \wedge \pi^*\omega^{(n-1)-l} \wedge \eta^l}{\omega^n_{\varepsilon_1}} \right|^{\tilde{\alpha}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} \\ \leq & \; C \left(\int_{V_\kappa} |F|^{\tilde{\alpha}}_{\omega_{\varepsilon_1}} \omega^n_{\varepsilon_1} \right)^{\frac{1}{\tilde{\alpha}}} \leq C\,\mathrm{Vol}(V_\kappa, \omega_{\varepsilon_1})^{1-\frac{\tilde{\alpha}}{2}} \|F\|_{L^2(V_\kappa, \omega_{\varepsilon_1})} \leq C\kappa \|F\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \;\; (H\ddot{o}lder). \end{aligned}$$

Now we obtain the desired estimate:

$$\|\Lambda_{\omega_\varepsilon} F\|_{L^\alpha(\tilde{X},\omega_\varepsilon)} \leq C \left(\left\|\Lambda_{\omega_{\varepsilon_1}} F\right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})} + \kappa \|F\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} \right) + \varepsilon_1 C(\kappa) \|F\|_{L^2(\tilde{X},\omega_{\varepsilon_1})} .$$

□

Proposition 5.7. *Let $E \to X$ be a holomorphic vector bundle of rank K over a compact Kähler manifold with Kähler form ω. Assume that E has Harder-Narasimhan type $\mu = (\mu_1, \cdots, \mu_K)$ that the singular set Z_{alg} of the HNS filtration is smooth, and furthermore that blowing up along the singular set resolves the singularities of the HNS filtration. There is an $\alpha_0 > 1$ such that the following holds: given any $\delta > 0$ and any N, there is an hermitian metric h on E such that $HYM^{\omega}_{\alpha,N}(\bar{\partial}_E, h) \leq HYM_{\alpha,N}(\mu) + \delta$, for all $1 \leq \alpha < \alpha_0$.*

Proof. As before, let $\pi : \tilde{X} \to X$ be a blow-up along a smooth, complex submanifold Y, and we assume that this resolves the singularities of the HNS filtration. In other words there is a filtration of $\tilde{E} = \pi^* E$ on $\tilde{X}$ that is given by sub-bundles and is equal to the HNS filtration of E away from the divisor $\mathbf{E}$. Let ω_ε denote the aforementioned family of Kähler metrics on $\tilde{X}$ given by $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ where $0 < \varepsilon \leq 1$ and η is a fixed Kähler metric on $\tilde{X}$. We will construct the metric h on E from an hermitian metric $\tilde{h}$ on $\pi^* E$ to be specified later.

Since Z_{alg} is a complex submanifold, we consider its normal bundle $\boldsymbol{\zeta}$, or more particularly the open subset: $\boldsymbol{\zeta}_R = \{(x,\nu) \in \boldsymbol{\zeta} \mid |\nu| < R\}$. By the tubular neighbourhood theorem, for R sufficiently small this set is diffeomorphic to an open neighbourhood U_R of Z_{alg}. We choose a background metric H on this open set.

Let ψ be a smooth cut-off function supported in U_1 and and identically 1 on $U_{1/2}$ and such that $0 \leq \psi \leq 1$ everywhere. Then if we define $\psi_R(x,\nu) = \psi(x, \frac{\nu}{R})$, ψ_R is identically 1 on $U_{R/2}$ and supported in U_R with $0 \leq \psi_R \leq 1$ and furthermore there are bounds on the derivatives:

$$\left|\frac{\partial \psi_R}{\partial z_i}\right| \leq \frac{C}{R} \qquad , \qquad \left|\frac{\partial}{\partial \bar{z}_i}\frac{\partial \psi_R}{\partial z_i}\right| \leq \frac{C}{R^2}$$

where the constant C does not depend on R. Suppose for the moment that we have constructed an hermitian metric $\tilde{h}$ on π^*E. If we continue to denote by H and ψ_R their pullbacks to $\tilde{X}$, then we may define the following metric on π^*E :

$$h_{\psi_R} := \psi_R H + (1-\psi_R)\tilde{h}$$

Observe that on $X - U_R$ we have $h_{\psi_R} = \tilde{h}$ and on $U_{R/2}$, $h_{\psi_R} = H$.

Now we need to estimate the difference:

$$\begin{aligned} &\left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\tilde{E}}, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\ = &\left|\int_{\tilde{X}} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha\left(\sqrt{-1}(\mu_1+N), \cdots, \sqrt{-1}(\mu_K+N)\right)\right| \end{aligned}$$

where Φ_α is the convex functional on $\mathfrak{u}(\tilde{E})$ given as in Section 3.2 by $\Phi_\alpha(a) = \sum_{j=1}^k |\lambda_j|^\alpha$, where the $\sqrt{-1}\lambda_j$ are the eigenvalues of a. From here on out we will write $\sqrt{-1}(\mu+N)$ in place of $(\sqrt{-1}(\mu_1+N), \cdots, \sqrt{-1}(\mu_K+N))$. Therefore we have:

$$\begin{aligned} &\left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\tilde{E}}, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\ \leq &\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\ + &\left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\ = &\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\ + &\left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \end{aligned}$$

where the last equality comes from the fact that h_{ψ_R} is equal to H on $U_{R/2}$. Dividing the first integral further we have:

$$\begin{aligned} &\left|HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_E, h_{\psi_R}) - HYM_{\alpha,N}(\mu)\right| \\ \leq &\left|\int_{\pi^{-1}(U_R-U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}})\right| \\ + &\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}}+N))\right| \\ + &\left|\int_{\tilde{X}-\pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}}+N)) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \\ + &\left|\int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1}N\mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\sqrt{-1}(\mu+N))\right| \end{aligned}$$

where in the first integral on the right hand side we have used the fact that outside of U_R the metrics h_{ψ_R} and $\tilde{h}$ agree. Here, $\mu_{\omega_{\varepsilon_1}}$ denotes the usual K-tuple of rational numbers made from the ω_{ε_1} slopes of the quotients of the resolution.

Recall that the norm on $L^\alpha(\mathfrak{u}(\tilde{E}))$, $a \mapsto \left(\int_M \Phi_\alpha(a)\right)^{1/\alpha}$ is equivalent to the L^α norm and so there is a universal constant C independent of R and ε such that:

$$\begin{aligned}
&\left|\int_{\pi^{-1}(U_R - U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} + \sqrt{-1} N \mathbf{I}_{\tilde{E}}) - \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1} N \mathbf{I}_E)\right| \\
&+ \left|\int_{\tilde{X} - \pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_{\tilde{h}} + \sqrt{-1} N \mathbf{I}_E) - \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N))\right| \\
\leq\ & C\left(\left\|\Lambda_{\omega_\varepsilon} F_{h_{\psi_R}} - \Lambda_{\omega_\varepsilon} F_{\tilde{h}}\right\|^\alpha_{L^\alpha(\pi^{-1}(U_R - U_{R/2}), \omega_\varepsilon)} + \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}\right\|^\alpha_{L^\alpha(\tilde{X} - \pi^{-1}(U_{R/2}), \omega_\varepsilon)}\right).
\end{aligned}$$

First we dispose of

$$\left|\int_{\tilde{X} - \pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N)) - \Phi_\alpha(\sqrt{-1}(\mu + N))\right|$$

by choosing ε_1 close to zero and using Remark 4.13. That is, we may choose ε_1 small enough so that

$$\left|\int_{\tilde{X} - \pi^{-1}(U_{R/2})} \Phi_\alpha(\sqrt{-1}(\mu_{\omega_{\varepsilon_1}} + N)) - \Phi_\alpha(\sqrt{-1}(\mu + N))\right| < \frac{\delta}{2}$$

Next will will bound:

$$\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{h}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\right\|^\alpha_{L^\alpha(\tilde{X} - \pi^{-1}(U_{R/2}), \omega_\varepsilon)}.$$

Note that at this point we have not specified the metric $\tilde{h}$ on $\pi^* E$. We will do so now. Each of the ω-stable quotients Q_i of the Harder-Narasimhan-Seshadri filtration remains stable on the blowup with respect to the metrics ω_ε with ε sufficiently small (see Remark 4.11), so that the quotients $\tilde{Q}_i$ are also ω_{ε_1}-stable and admit a unique Hermitian-Einstein metric $\tilde{G}_i^{\varepsilon_1}$. The prototype for our metric $\tilde{h}$ will be the metric $\tilde{G}_{\varepsilon_1} = \oplus_i \tilde{G}_i^{\varepsilon_1}$. Just as in the beginning of this section, we need to modify $\tilde{G}_{\varepsilon_1}$ by a gauge transformation in order to obtain the appropriate bound. More precisely, since holomorphic structures on the bundle $\tilde{E}$ are equivalent to integrable unitary connections, this is the same as showing that if we fix the metric $\tilde{G}_{\varepsilon_1}$, there is a complex gauge transformation $\tilde{g}$ of $\tilde{E}$ such that $\left\|\Lambda_{\omega_\varepsilon} F_{(\tilde{g}(\bar{\partial}_{\tilde{E}}), \tilde{G}_{\varepsilon_1})} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\right\|_{L^\alpha(\tilde{X} - \pi^{-1}(U_{R/2}), \omega_\varepsilon)}$ is small. When we take the direct sum, the second fundamental form enters into the curvature and so we ask that there is a gauge transformation making this contribution small.

We will do this iteratively. If we write $\tilde{S} = \tilde{E}_1 = \tilde{Q}_1 \subset \pi^* E$ for the first sub-bundle in the filtration of $\pi^* E$ on $\tilde{X}$, then the discussion at the beginning of this section applies in exactly the same way to the exact sequence:

$$0 \longrightarrow \tilde{S} \longrightarrow \pi^* E \longrightarrow \tilde{Q} \longrightarrow 0$$

where $\tilde{Q} = \oplus_{i=2}^l \tilde{Q}_i$. Therefore if we apply the gauge transformation $g_t = t^{-1} Id_{\tilde{S}} \oplus t Id_{\tilde{Q}}$ to the operator $\bar{\partial}_{\tilde{E}}$ as before, the curvature may be written as:

$$F_{(g_t(\bar{\partial}_{\tilde{E}}), \tilde{G}_{\varepsilon_1})} = \begin{pmatrix} F_{(\bar{\partial}_{\tilde{S}}, \tilde{G}_1^{\varepsilon_1})} - t^4 \beta \wedge \beta^*_{\tilde{S}} & t^2 \partial_E \beta_{\tilde{S}} \\ -t^2 \bar{\partial}_E \beta^*_{\tilde{S}} & F_{(\bar{\partial}_Q, \oplus_{i=2}^l \tilde{G}_i^{\varepsilon_1})} - t^4 \beta^*_{\tilde{S}} \wedge \beta_{\tilde{S}} \end{pmatrix}.$$

Taking $\Lambda_{\omega_\varepsilon}$, we obtain terms involving:

$$\begin{aligned}
t^4 \Lambda_{\omega_\varepsilon} \beta \wedge \beta^*_{\tilde{S}} &= t^4 \frac{\beta \wedge \beta^*_{\tilde{S}} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}, t^2 \Lambda_{\omega_\varepsilon} \partial_E \beta_{\tilde{S}} = t^2 \frac{\partial_E \beta_{\tilde{S}} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}, \\
t^2 \Lambda_{\omega_\varepsilon} \bar{\partial}_E \beta^*_{\tilde{S}} &= t^2 \frac{\bar{\partial}_E \beta^*_{\tilde{S}} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}, t^4 \Lambda_{\omega_\varepsilon} \beta^*_{\tilde{S}} \wedge \beta_{\tilde{S}} = t^4 \frac{\beta^*_{\tilde{S}} \wedge \beta_{\tilde{S}} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}.
\end{aligned}$$

Recalling also that

$$\omega_\varepsilon^n = \left|\frac{\det g_\varepsilon}{\det g_\eta}\right| \eta^n,$$

and applying Hölder's inequality we see that

$$\begin{aligned}
&\left\|\Lambda_{\omega_\varepsilon} F_{(\tilde{g}_t(\bar{\partial}_{\tilde{E}}),\tilde{G}_{\varepsilon_1})} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
\leq\ &\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_1^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{S}) Id_{\tilde{Q}_1}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
&+\left\|\Lambda_{\omega_\varepsilon} F_{\oplus_{i=2}^l \tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\oplus_{i=2}^l \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}i}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
+&\left(\int_{\tilde{X}-\pi^{-1}(U_{R/2})}\left|\frac{\det g_\varepsilon}{\det g_\eta}\right|^{(1-\alpha)s}\eta^n\right)^{\frac{1}{\alpha s}}\left(\int_{\tilde{X}-\pi^{-1}(U_{R/2})}\left(\left|t^4\frac{\beta\wedge\beta_{\tilde{S}}^*\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}+\left|t^2\frac{\partial_E\beta_{\tilde{S}}\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}\right)\eta^n\right)^{\frac{1}{\tilde{\alpha}}}\\
+&\left(\int_{\tilde{X}-\pi^{-1}(U_{R/2})}\left|\frac{\det g_\varepsilon}{\det g_\eta}\right|^{(1-\alpha)s}\eta^n\right)^{\frac{1}{\alpha s}}\left(\int_{\tilde{X}-\pi^{-1}(U_{R/2})}\left(\left|t^2\frac{\bar{\partial}_E\beta_{\tilde{S}}^*\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}+\left|t^4\frac{\beta_{\tilde{S}}^*\wedge\beta_{\tilde{S}}\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}\right)\eta^n\right)^{\frac{1}{\tilde{\alpha}}}
\end{aligned}$$

where $\tilde{\alpha}$ and s are as in Lemma 5.5 (recall that s and $\frac{\tilde{\alpha}}{\alpha}$ are a conjugate pair). By the lemma, the last two terms above are bounded uniformly in ε. Therefore, the contribution of these terms can be made small by making t sufficiently small.

Similarly, we can apply the same argument to the extensions:

$$0\longrightarrow \tilde{E}_i/\tilde{E}_{i-1}=\tilde{Q}_i\longrightarrow \pi^*E/\tilde{E}_{i-1}\longrightarrow \pi^*E/\tilde{E}_i=\oplus_{j=i+1}^l\tilde{Q}_j\longrightarrow 0$$

using the gauge transformation $g_t = Id_{\tilde{Q}_1}\oplus\cdots\oplus Id_{\tilde{Q}_{l-1}}\oplus t^{-1}Id_{\tilde{Q}_i}\oplus_{j=i+1}^l tId_{\tilde{Q}_j}$. Such an extension will give a further second fundamental form $\beta_{\tilde{Q}_i}$, and its contribution can be estimated in exactly the same way as above.

Continuing in this way, we see that there is a complex gauge transformation g of π^*E such that:

$$\begin{aligned}
\left\|\Lambda_{\omega_\varepsilon} F_{(\tilde{g}(\bar{\partial}_{\tilde{E}}),\tilde{G}_{\varepsilon_1})} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}} Id_{\tilde{E}}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} &\leq \left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_1^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_1) Id_{\tilde{Q}_1}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
&+\cdots+\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_l^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_l) Id_{\tilde{Q}_l}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)} + \Theta(t,\beta_{\tilde{Q}_1},\cdots,\beta_{\tilde{Q}_l})
\end{aligned}$$

where $\Theta(t,\beta_{\tilde{Q}_1},\cdots,\beta_{\tilde{Q}_l})\to 0$ as $t\to 0$. Therefore we have reduced this estimate to an estimate on each of the terms:

$$\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}.$$

On the other hand we have:

$$\begin{aligned}
&\left\|\Lambda_{\omega_\varepsilon} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
\leq&\left\|\Lambda_{\omega_\varepsilon}\left(F_{\tilde{G}_i^{\varepsilon_1}} - \frac{\sqrt{-1}}{n}\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right)\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
+&\left\|\frac{\sqrt{-1}}{n}\Lambda_{\omega_\varepsilon}(\omega_{\varepsilon_1}-\omega_\varepsilon)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}
\end{aligned}$$

where we have used the fact that $\Lambda_{\omega_\varepsilon}\omega_\varepsilon = n$. Now by Lemma 5.6 we have:

$$\begin{aligned}
&\left\|\Lambda_{\omega_\varepsilon}\left(F_{\tilde{G}_i^{\varepsilon_1}} - \frac{\sqrt{-1}}{n}\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right)\right\|_{L^\alpha(\tilde{X}-\pi^{-1}(U_{R/2}),\omega_\varepsilon)}\\
&\qquad\leq C\left(\left\|\Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}) Id_{\tilde{Q}_i}\right\|_{L^{\tilde{\alpha}}(\tilde{X},\omega_{\varepsilon_1})}\right)\\
&\qquad+\kappa C\left(\left\|F_{\tilde{G}_i^{\varepsilon_1}}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}+\frac{1}{n}\left\|\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right)\\
&\qquad+\varepsilon_1 C(\kappa)\left(\left\|F_{\tilde{G}_i^{\varepsilon_1}}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}+\frac{1}{n}\left\|\omega_{\varepsilon_1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i}\right\|_{L^2(\tilde{X},\omega_{\varepsilon_1})}\right)
\end{aligned}$$

and

$$\begin{aligned} &\left\| \frac{\sqrt{-1}}{n} \Lambda_{\omega_\varepsilon}(\omega_{\varepsilon_1} - \omega_\varepsilon)\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^\alpha(\tilde{X} - \pi^{-1}(U_{R/2}), \omega_\varepsilon)} \\ \leq\ & \frac{\varepsilon_1}{n} C \left(\left\| \left(\Lambda_{\omega_{\varepsilon_1}} \eta\right) \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})} + \kappa \left\| \eta \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \right) \\ & + \frac{\varepsilon_1^2}{n} C(\kappa) \left\| \eta \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} \end{aligned}$$

again using Lemma 5.6. Here we have used the fact that $\omega_{\varepsilon_1} - \omega_\varepsilon = (\varepsilon_1 - \varepsilon)\eta$ in the second inequality. Of course, $\left\| \Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})} = 0$, by the construction of $G_i^{\varepsilon_1}$. On the other hand:

$$\begin{aligned} \left\| F_{\tilde{G}_i^{\varepsilon_1}} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} &= \left\| \Lambda_{\omega_{\varepsilon_1}} F_{\tilde{G}_i^{\varepsilon_1}} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} + \pi^2 n(n-1) \int_{\tilde{X}} \left(2c_2(\tilde{Q}_i) - c_1^2(\tilde{Q}_i) \right) \wedge \omega_{\varepsilon_1}^{n-2} \\ &= \left\| \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})} + \pi^2 n(n-1) \int_{\tilde{X}} \left(2c_2(\tilde{Q}_i) - c_1^2(\tilde{Q}_i) \right) \wedge \omega_{\varepsilon_1}^{n-2} \end{aligned}$$

which is bounded. Likewise the terms $\left\| \omega_{\varepsilon_1} \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})}$ and $\left\| \eta \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^2(\tilde{X}, \omega_{\varepsilon_1})}$ are bounded. The only remaining issue is: $\left\| \left(\Lambda_{\omega_{\varepsilon_1}} \eta\right) \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})}$. But writing

$$\left| \Lambda_{\omega_{\varepsilon_1}} \eta \right|^{\tilde{\alpha}} = \left| \frac{\eta \wedge \omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n} \right|^{\tilde{\alpha}} = \left| \frac{\eta \wedge \omega_{\varepsilon_1}^{n-1}}{\eta^n} \right|^{\tilde{\alpha}} \left| \frac{\det g_\eta}{\det g_{\varepsilon_1}} \right|^{\tilde{\alpha}}$$

and

$$\omega_{\varepsilon_1}^n = \left| \frac{\det g_{\varepsilon_1}}{\det g_\eta} \right| \eta^n$$

$$\begin{aligned} &\left\| \left(\Lambda_{\omega_{\varepsilon_1}} \eta\right) \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right\|_{L^{\tilde{\alpha}}(\tilde{X}, \omega_{\varepsilon_1})} \\ \leq\ & C \left(\int_{\tilde{X}} \left| \frac{\det g_{\varepsilon_1}}{\det g_\eta} \right|^{(1-\tilde{\alpha})\tilde{s}} \eta^n \right)^{\frac{1}{\tilde{\alpha}\tilde{s}}} \left(\int_{\tilde{X}} \left| \frac{\eta \wedge \omega_{\varepsilon_1}^{n-1}}{\eta^n} \right|^\beta \left| \mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{Q}_i} \right|^\beta \eta^n \right)^{\frac{1}{\beta}} \end{aligned}$$

by Hölder's inequality with respect to the metric η. Here again $\tilde{\alpha}$ is as in Lemma 5.6 and $\tilde{s} = \frac{\beta}{\beta - \tilde{\alpha}}$ where $\frac{\tilde{\alpha}}{1 - 2(k-1)(\tilde{\alpha}-1)} < \beta < \infty$. By Lemma 5.5 this is uniformly bounded in ε_1 since we also have $\omega_{\varepsilon_1}^{n-1} \longrightarrow \pi^* \omega^{n-1}$.

Therefore we may choose t, κ, and ε_1 so that

$$\left\| \Lambda_{\omega_\varepsilon} F_{\tilde{G}_{\varepsilon_1}} - \sqrt{-1}\mu_{\omega_{\varepsilon_1}}(\tilde{Q}_i) Id_{\tilde{G}_{\varepsilon_1}} \right\|_{L^\alpha(\tilde{X} - \pi^{-1}(U_{R/2}), \omega_\varepsilon)} < \frac{\delta}{4}$$

for all ε and all α sufficiently close to 1. We will now fix these values of t,κ, and ε_1.

The term

$$\left| \int_{\pi^{-1}(U_{R/2})} \Phi_\alpha(\Lambda_{\omega_\varepsilon} F_H + \sqrt{-1} N \mathbf{I}_E) - \Phi_\alpha(\sqrt{-1}(\mu + N)) \right|$$

is bounded by:

$$C \left\| \Lambda_{\omega_\varepsilon} F_H - \sqrt{-1}\mu \right\|_{L^\alpha(\pi^{-1}(U_{R/2}), \omega_\varepsilon)}.$$

Now write

$$\left| \Lambda_{\omega_\varepsilon} F_H \right|^\alpha = \left| \frac{F_H \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^{n-1}} \right|^\alpha = \left| \frac{F_H \wedge \omega_\varepsilon^{n-1}}{\eta^n} \right|^{\tilde{\alpha}} \left| \frac{\det g_\eta}{\det g_\varepsilon} \right|^{\tilde{\alpha}}$$

and

$$\omega_\varepsilon^n = \left| \frac{\det g_\varepsilon}{\det g_\eta} \right| \eta^n,$$

we have

$$\begin{aligned}\left\|(\Lambda_{\omega_\varepsilon}F_H-\sqrt{-1}\mu\right\|_{L^\alpha(\pi^{-1}(U_{R/2}),\omega_\varepsilon)} &\leq C_1\left\|\Lambda_{\omega_\varepsilon}F_H\right\|_{L^\alpha(\pi^{-1}(U_{R/2}),\omega_\varepsilon)}+C_2\operatorname{Vol}(U_{R/2},\omega)\leq \\ C_1\left(\int_{\pi^{-1}(U_{R/2})}\left|\frac{\det g_\varepsilon}{\det g_\eta}\right|^{(1-\alpha)s}\eta^n\right)^{\frac{1}{s\tilde{\alpha}}}&\left(\int_{\pi^{-1}(U_{R/2})}\left|\frac{F_H\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^{\tilde{\alpha}}\eta^n\right)^{\frac{1}{\tilde{\alpha}}}+C_2\operatorname{Vol}(U_{R/2},\omega),\end{aligned}$$

where α and s are as in Lemma 5.5. By that lemma, the factor

$$\left(\int_{\pi^{-1}(U_{R/2})}\left|\frac{\det g_\varepsilon}{\det g_\eta}\right|^{(1-\alpha)s}\eta^n\right)$$

is uniformly bounded, and so the result is that there is an R such that

$$\left|\int_{\pi^{-1}(U_{R/2})}\Phi_\alpha(\Lambda_{\omega_\varepsilon}F_H+\sqrt{-1}N\mathbf{I}_E)-\Phi_\alpha(\sqrt{-1}(\mu+N))\right|<\frac{\delta}{8}.$$

Therefore the only remaining estimates required are on: $\left\|\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\psi_R}}-\Lambda_{\omega_\varepsilon}F_{\tilde{h}}\right\|_{L^\alpha(\pi^{-1}(U_R-U_{R/2}),\omega_\varepsilon)}$. If we let k_{ψ_R} be an endomorphism such that $\tilde{h}=k_{\psi_R}h_{\psi_R}$. Then

$$F_{h_{\psi_R}}-F_{\tilde{h}}=\bar{\partial}_{\tilde{E}}(k_{\psi_R}^{-1}\partial_{\tilde{h}}k_{\psi_R})$$

where $\partial_{\tilde{h}}$ is the $(1,0)$ part of the Chern connection for $\tilde{h}$. The expression on the right hand side involves only two derivatives of ψ_R, and so, using the bound on the derivatives of ψ_R, there is a bound of the form:

$$\left|F_{h_{\psi_R}}-F_{\tilde{h}}\right|\leq C_1+\frac{C_2}{R^2}.$$

where C_1 and C_2 are independent of both ε and R. Now as usual we have:

$$\begin{aligned}\left|\Lambda_{\omega_\varepsilon}\left(F_{\tilde{h}_{\psi_R}}-F_{\tilde{h}}\right)\right|^\alpha &= \left|\frac{\left(F_{\tilde{h}_{\psi_R}}-F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}\right|^\alpha=\left|\frac{\left(F_{\tilde{h}_{\psi_R}}-F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^\alpha\left|\frac{\det g_\eta}{\det g_\varepsilon}\right|^\alpha \\ \text{and } \omega_\varepsilon^n &= \frac{\det g_\varepsilon}{\det g_\eta}\eta^n.\end{aligned}$$

Then we compute:

$$\begin{aligned}&\left\|\Lambda_{\omega_\varepsilon}F_{h_{\psi_R}}-\Lambda_{\omega_\varepsilon}F_{\tilde{h}}\right\|_{L^\alpha(\pi^{-1}(U_R-U_{R/2}),\omega_\varepsilon)} \\ = &\left(\int_{\pi^{-1}(U_R-U_{R/2})}\left|\frac{\left(F_{\tilde{h}_{\psi_R}}-F_{\tilde{h}}\right)\wedge\omega_\varepsilon^{n-1}}{\eta^n}\right|^\alpha\left|\frac{\det g_\eta}{\det g_\varepsilon}\right|^\alpha\frac{\det g_\varepsilon}{\det g_\eta}\eta^n\right)^{\frac{1}{\alpha}} \\ \leq &\left(\int_{\pi^{-1}(U_R-U_{R/2})}\left(\frac{\det g_\varepsilon}{\det g_\eta}\right)^{(1-\alpha)s}\eta^n\right)^{\frac{1}{\alpha s}}\left(\int_{\pi^{-1}(U_R-U_{R/2})}\left(C_1+\frac{C_2}{R^{2\tilde{\alpha}}}\right)\eta^n\right)^{\frac{1}{\tilde{\alpha}}}.\end{aligned}$$

Here s and $\tilde{\alpha}$ are as in Lemma 5.5 and we have applied Hölder's inequality to the conjugate pair s and $\frac{\tilde{\alpha}}{\alpha}$. By that lemma, the first factor is uniformly bounded in ε. We must therefore show that as $R\to 0$, the first factor can be made arbitrarily small. To do this we note that the open set U_R may be covered by a union of balls $\cup_j B_r^j$. Therefore:

$$\int_{\pi^{-1}(U_R-U_{R/2})}C_1+C_2R^{-2\tilde{\alpha}}\leq\sum_j(C_1+C_2R^{-2\tilde{\alpha}})vol(B_r^j)$$

and up to a constant $vol(B_r^j)=r^{2n}$ where n is the complex dimension of X.

The key observation is now that the singular set Z_{alg} is a complex submanifold of X and has complex codimension at least 2, in other words it is of real dimension at most $2n-4$. This implies that Z_{alg} has Hausdorff dimension at most $2n-4$, i.e. it has zero d-dimensional Hausdorff measure for $d<2n-4$. In other words, for each $0 \leq d < 4$, and a given $\delta > 0$, there is a cover of Z_{alg} and an $r > 0$ such that $\sum_j r^{2n-d} < \delta$. Now assume that we have chosen $R = r$. Then then the cover described above is also a cover for U_R so

$$\int_{\pi^{-1}(U_R - U_{R/2})} C_1 + C_2 R^{-2\tilde{\alpha}} \leq \sum_j (C_1 r^{2n} + C_2 r^{2n-2\tilde{\alpha}}).$$

Note that by assumption $\tilde{\alpha} < 2$. In other words, we may select R so that:

$$\left\| \Lambda_{\omega_\varepsilon} F_{\tilde{h}_{\psi_R}} - \Lambda_{\omega_\varepsilon} F_{\tilde{h}} \right\|_{L^\alpha(\pi^{-1}(U_R - U_{R/2}), \omega_\varepsilon)} < \frac{\delta}{16}.$$

Thus choosing ε_1 and R in the manner specified above gives us for each ε a bound on the difference of the HYM functionals: $\left| HYM^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_E, \tilde{h}_{\psi_R}) - HYM_{\alpha,N}(\mu) \right| \leq \delta$. Now sending $\varepsilon \to 0$ we finally see that there exists a metric h with $\left| HYM^{\omega}_{\alpha,N}(\bar{\partial}_E, h) - HYM_{\alpha,N}(\mu) \right| < \delta$, for all N and all α sufficiently close to 1. □

Lemma 5.8. *Let $E \to X$ and α_0 be the same as in the proposition. Let h be any smooth hermitian metric on E and A_t a solution of the Yang-Mills flow whose initial condition is $(\bar{\partial}_E, h)$. Let μ_0 denote the Harder-Narasimhan type of E. Then $\lim_{t\to\infty} HYM_{\alpha,N}(A_t) = HYM_{\alpha,N}(\mu_0)$, for all $1 \leq \alpha \leq \alpha_0$ and all N.*

As a consequence, if A_∞ is an Uhlenbeck limit along the flow: $HYM_{\alpha,N}(A_\infty) = HYM_{\alpha,N}(\mu_0)$, since $HYM_{\alpha,N}(A_\infty) = \lim_{t\to\infty} HYM_{\alpha,N}(A_t)$. The proof of Lemma 5.8 is exactly the same as in [DW1]. It uses Proposition 5.7. One easily shows that for any initial metric such that the conclusion of Proposition 5.7 holds, the property $\lim_{t\to\infty} HYM_{\alpha,N}(A_t) = HYM_{\alpha,N}(\mu_0)$ holds. The fact that this is true for any metric then follows from a distance decreasing argument.

We can now identify the Harder-Narasimhan type of the limit.

Proposition 5.9. *Let $E \to X$ have the same properties as before. Let A_t be a solution to the YM flow with initial condition A_0 whose limit along the flow is A_∞. Let E_∞ be the corresponding holomorphic vector bundle defined away from Z_{an}. Then the HN type of (E_∞, A_∞) is the same as that of (E_0, A_0).*

Proof. Let $\mu_0 = (\mu_1, \cdots, \mu_K)$ and $\mu_\infty = (\mu_1^\infty, \cdots, \mu_K^\infty)$ be the HN types of (E_0, A_0) and (E_∞, A_∞). A restatement of the above lemma is that $\Phi_\alpha(\mu_0 + N) = \Phi_\alpha(\mu_\infty + N)$ for all $1 \leq \alpha \leq \alpha_0$ and all N. Choose N to be large enough so that $\mu_K + N \geq 0$. Then we also have $\mu_K^\infty + N \geq 0$ by Proposition 3.4, and therefore $\mu_K + N = \mu_K^\infty + N$ by Proposition 3.7, so $\mu_K = \mu_K^\infty$. □

Let $(E, \bar{\partial}_{A_0})$ be a holomorphic bundle, and A_0 an initial connection, and A_{t_j} its evolution along the flow for a sequence of times t_j. Then we have the following.

Lemma 5.10. (1) *Let $\{\pi_j^{(i)}\}$ be the HN filtration of $(E, \bar{\partial}_{A_{t_j}})$ and $\{\pi_\infty^{(i)}\}$ the HN filtration of $(E_\infty, \partial_{A_\infty})$. Then after passing to a subsequence, $\pi_j^{(i)} \to \pi_\infty^{(i)}$ strongly $L^p \cap L^2_{1,loc}$ for all $1 \leq p < \infty$ and all i.*

(2) *Assume the original bundle $(E, \bar{\partial}_{A_0})$ is semi-stable and $\{\pi_{ss,j}^{(i)}\}$ are Seshadri filtrations of $(E, \bar{\partial}_{A_{t_j j}})$. Without loss of generality assume the ranks of the subsheaves $\pi_{ss,j}^{(i)}$ are constant in j. Then there is a filtration $\{\pi_{ss,\infty}^{(i)}\}$ of $(E, \bar{\partial}_{A_\infty})$ such that after passing to a subsequence $\{\pi_{ss,j}^{(i)}\} \to \{\pi_{ss,\infty}^{(i)}\}$ strongly in $L^p \cap L^2_{1,loc}$ for all $1 \leq p < \infty$ and all i. The rank and degree of $\pi_{ss,\infty}^{(i)}$ is equal to the rank and degree of $\pi_{ss,j}^{(i)}$ for all i and j.*

For the proof see [DW1] Lemma 4.5. It uses Proposition 5.9.

Proposition 5.11. *Assume as before that $E \to X$ is a holomorphic vector bundle such that Z_{an} is smooth and that blowing up once resolves the singularities of the HNS filtration. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

Proof. Let A_t be a solution to the YM flow with initial condition $A_0 = (\bar{\partial}_E, h)$, and let A_∞ be the limit along the flow for some sequence A_{t_j}. Then we may apply the previous lemma to conclude that $\Psi_\omega^{HNS}(\bar{\partial}_{A_{t_j}}, h) \xrightarrow{L^p} \Psi_\omega^{HNS}(\bar{\partial}_{A_\infty}, h_\infty)$ after passing to another subsequence if necessary. Since A_∞ is a Yang-Mills connection, $\sqrt{-1}\Lambda_\omega F_{A_\infty} = \Psi_\omega^{HN}(\bar{\partial}_{A_\infty}, h_\infty)$. Therefore:

$$\begin{aligned} \left\| \sqrt{-1}\Lambda_\omega F_{A_{t_j}} - \Psi_\omega^{HNS}(\bar{\partial}_{A_{t_j}}, h) \right\|_{L^p(\omega)} &\leq \\ \left\| \Lambda_\omega F_{A_{t_j}} - \Lambda_\omega F_{A_\infty} \right\|_{L^p(\omega)} + \left\| \Psi_\omega^{HNS}(\bar{\partial}_{A_{t_j}}, h) - \Psi_\omega^{HNS}(\bar{\partial}_{A_\infty}, h_\infty) \right\|_{L^p(\omega)} &\longrightarrow 0 \end{aligned}$$

where we have also used Lemma 3.3. □

Now we would like to eliminate the assumptions that Z_{an} is smooth and that blowing up once resolves the singularities of the HNS filtration.

Theorem 5.12. *Let $E \to X$ be a holomorphic vector bundle over a Kähler manifold with Kähler form ω. Then given $\delta > 0$ and any $1 \leq p < \infty$, E has an L^p δ-approximate critical hermitian structure.*

Proof. By 4.3, we know that we can resolve the singularities of the HNS filtration by blowing up finitely many times. Moreover, the i^{th} blowup is obtained by blowing up along a complex submanifold contained in the singular set associated to the pullback bundle over the manifold produced at the $(i-1)st$ stage of the process. In other words there is a tower of blow-ups:

$$\tilde{X} = X_m \xrightarrow{\pi_m} X_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

such that if $E = E_0$ is the original bundle, and $E_i = \pi_i^*(E_{i-1})$, then there is a filtration of $\tilde{E} = \pi_m^*(E_{m-1})$ that is given by sub-bundles and isomorphic to the HNS filtration of E away from $\mathbf{E}$. Note that on each blowup X_i we have a family of Kähler metrics defined iteratively by $\omega_{\varepsilon_1,\cdots,\varepsilon_i} = \pi^*\omega_{\varepsilon_1,\cdots,\varepsilon_{i-1}} + \varepsilon_i\eta_i$, where η_i is any Kähler form on X_i. Then consider $\omega_{\varepsilon_1,\cdots,\varepsilon_m}$ on $\tilde{X}$ to be a fixed metric for specified values of $\varepsilon_1, \cdots, \varepsilon_m < 1$, and fix $\delta > 0$. Fix δ_0 to be a number that is very small with respect to δ. By the previous proposition, for every p there is a δ_0-approximate critical hermitian structure on E_{n-1}. In particular there is such a metric for $p = 2$. In other words there is a metric h_{m-1} so that:

$$\left\| \sqrt{-1}\Lambda_{\omega_{\varepsilon_1,\cdots\varepsilon_{m-1}}} F_{(\bar{\partial}_{E_{m-1}},h_{m-1})} - \Psi^{HNS}_{\omega_{\varepsilon_1,\cdots\varepsilon_{m-1}}}(\bar{\partial}_{E_{m-1}}, h_{m-1}) \right\|_{L^2(\omega_{\varepsilon_1,\cdots\varepsilon_{m-1}})} < \delta_0.$$

By construction this metric depends on the values of $\varepsilon_1, \cdots, \varepsilon_m$, since it is constructed from a metric on the blowup which itself is constructed using the notion of stability with respect to $\omega_{\varepsilon_1,\cdots,\varepsilon_m}$.

We prove the result by induction on the number of blowups. Assume that we have an L^2 δ_0-approximate critical hermitian structure for each of the bundles $E_i \to X_i$ for $1 \leq i \leq m-2$. Then in particular, with respect to the metric ω_{ε_1} on X_1, we have a metric h_1 on $E_1 \to X_1$ such that:

$$\left\| \sqrt{-1}\Lambda_{\omega_{\varepsilon_1}} F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_{E_1}, h_1) \right\|_{L^2(\omega_{\varepsilon_1})} < \delta_0.$$

Since X_1 is obtained from X by blowing up along a smooth, complex submanifold, we may use the exact same cut-off argument, choosing a cutoff function with respect to a neighbourhood U_R as in Proposition 5.7 to construct a metric h_R on the bundle $E \to X$ which depends on the value of ε_1. In the following we will continue to denote by h_R its pullback to X_1. As in the proof of Proposition 5.7 we have $h_R = h_1$ outside of the set $\pi_1^{-1}(U_R)$. We divide the proof into two steps.

(Step 1) There is an L^p δ-approximate critical hermitian structure for p close to 1

First let us assume that p satisfies the hypotheses of Lemma 5.6. In other words, substitute p for α in the statement. Similarly, substitute $\tilde{p}$ for $\tilde{\alpha}$. We will show that a single metric, namely h_R, gives an L^p δ-approximate critical hermitian structure for all p within this range. We need to estimate the difference

$$\left\| \sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R) \right\|_{L^p(\omega_\varepsilon)}$$

where $\tilde{h} = \pi_1^* h$. Now:

$$\begin{aligned} \left\| \sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R) \right\|_{L^p(\omega_\varepsilon)} &\leq \\ \left\| \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_R)} - \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_1)} \right\|_{L^p(\omega_\varepsilon)} &+ \left\| \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1) - \Psi^{HNS}_{\omega}(\bar{\partial}_E, h_R) \right\|_{L^p(\omega_\varepsilon)} \\ &+ \left\| \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar{\partial}_E, h_1) \right\|_{L^p(\omega_\varepsilon)}. \end{aligned}$$

We can make the second term smaller than $\frac{\delta}{3}$ by choosing ε_1 small and using the convergence of the HN types. The third term is bounded by two applications of Lemma 5.6 as follows:

$$\begin{aligned}
&\left\|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^p(\omega_\varepsilon)} \leq \\
&\left\|\Lambda_{\omega_\varepsilon}\left(F_{(\bar\partial_{E_1},h_1)} - \frac{1}{n}\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right)\right\|_{L^p(\omega_\varepsilon)} + \left\|\frac{1}{n}\Lambda_{\omega_\varepsilon}\left(\omega_{\varepsilon_1} - \omega_\varepsilon\right)\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^p(\omega_\varepsilon)} \\
\leq\quad & C\left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\omega_{\varepsilon_1})} \\
&+\kappa C\left(\left\|F_{(\bar\partial_{E_1},h_1)}\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})} + \frac{1}{n}\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}\right) \\
&+\varepsilon_1 C(\kappa)\left(\left\|F_{(\bar\partial_{E_1},h_1)}\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})} + \frac{1}{n}\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}\right) \\
&+\frac{\varepsilon_1^2}{n}C(\kappa)\left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})} \\
&+\frac{\varepsilon_1}{n}C\left(\left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\tilde X,\omega_{\varepsilon_1})} + \kappa\left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}\right).
\end{aligned}$$

Recall from the statement of Lemma 5.6 that none of the above constants depends on ε_1. All terms with a κ in front and no $C(\kappa)$ can be made small by choosing κ small, so these terms can be ignored. Clearly the terms

$$\left\|\omega_{\varepsilon_1}\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}, \left\|\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}$$

are bounded independently of ε_1 since the HN type converges. Therefore we need only show that

$$\left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\omega_{\varepsilon_1})}, \left\|F_{(\bar\partial_{E_1},h_1)}\right\|_{L^2(\tilde X,\omega_{\varepsilon_1})}, \left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\tilde X,\omega_{\varepsilon_1})}$$

are uniformly bounded in ε_1. Then we can choose κ first and then ε_1 so that:

$$\left\|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^p(\omega_\varepsilon)} < \frac{\delta}{3}.$$

Firstly we have:

$$\left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\omega_{\varepsilon_1})} \leq C\left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar\partial_{E_1},h_1)} - \Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\omega_{\varepsilon_1})} < \delta_0$$

by Hölder's inequality (since $\tilde p < 2$), and the induction hypothesis. Note that the constant above is independent of ε_1 since the ω_{ε_1} volume is bounded. Also, the following bound:

$$\begin{aligned}
\left\|F_{(\bar\partial_{E_1},h_1)}\right\|_{L^2(\omega_{\varepsilon_1})} &= \left\|\Lambda_{\omega_{\varepsilon_1}} F_{(\bar\partial_{E_1},h_1)}\right\|_{L^2(\omega_{\varepsilon_1})} + \pi^2 n(n-1)\int_{\tilde X}\left(2c_2(E_1) - c_1^2(E_1)\right)\wedge\omega_{\varepsilon_1}^{n-2} \\
&\leqslant \left\|\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^2(\omega_{\varepsilon_1})} + \delta_0 + \pi^2 n(n-1)\int_{\tilde X}\left(2c_2(E_1) - c_1^2(E_1)\right)\wedge\omega_{\varepsilon_1}^{n-2}
\end{aligned}$$

obtained from the usual relationship between the Hermitian-Einstein tensor and the full curvature in L^2, together with the induction hypothesis, shows that this term is bounded in ε_1 as well. Finally, writing

$$\begin{aligned}
\Lambda_{\omega_{\varepsilon_1}}\eta &= \frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\omega_{\varepsilon_1}^n} = \frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\frac{\det g_\eta}{\det g_{\varepsilon_1}} \\
\omega_{\varepsilon_1}^n &= \frac{\det g_{\varepsilon_1}}{\det g_\eta}\eta^n
\end{aligned}$$

then by Hölder's inequality we have:

$$\left\|\Lambda_{\omega_{\varepsilon_1}}\eta\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right\|_{L^{\tilde p}(\tilde X,\omega_{\varepsilon_1})} \leq \left(\int_{\tilde X}\left|\frac{\det g_{\varepsilon_1}}{\det g_\eta}\right|^{(1-\tilde p)(\tilde s)}\eta^n\right)^{\frac{1}{\tilde p\tilde s}}\left(\int_{\tilde X}\left|\frac{\eta\wedge\omega_{\varepsilon_1}^{n-1}}{\eta^n}\right|^w\left|\Psi^{HNS}_{\omega_{\varepsilon_1}}(\bar\partial_E, h_1)\right|^w\eta^n\right)^{\frac{1}{w}}$$

where $\tilde s = \frac{w}{w-\tilde p}$ and $\frac{\tilde p}{1-2(k-1)(\tilde p-1)} < w < \infty$. By Lemma 5.5 this is bounded in ε_1.

We have already seen that

$$\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1}, h_R)} - \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{E_1}, h_1)}\right\|_{L^p(\omega_\varepsilon)}$$

can be estimated, since it is 0 outside of U_R and the same argument as in the proof of Proposition 5.7, shows that by making R sufficiently small, we can make the contribution from this term over U_R less than $\frac{\delta}{3}$. Therefore the estimate on $\left\|\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E, h)} - \Psi^{HNS}_\omega(\bar{\partial}_E, h)\right\|_{L^p(\omega)}$ for these values of p follows by sending $\varepsilon \to 0$.

Step 2 (Extending to all p)

Repeating the arguments of Lemma 5.8, Proposition 5.9, Lemma 5.10, and Proposition 5.11, now gives the existence of an L^p δ-approximate critical hermitian structure on E for each p. This metric will depend on p. □

Notice that during the course of the above proof we have also proven the following:

Theorem 5.13. *Let $E \to X$ be a holomorphic vector bundle over a Kähler manifold. Let A_t be a solution to the YM flow with initial condition A_0 whose limit along the flow is A_∞. Let E_∞ be the corresponding holomorphic vector bundle defined away from Z_{an}. Then the HN type of (E_∞, A_∞) is the same as (E_0, A_0).*

6. The Degenerate Yang-Mills Flow

In this section we introduce a version of the Yang-Mills flow with respect to the degenerate metric $\omega_0 = \pi^*\omega$ on a sequence of blowups $\pi : \tilde{X} \to X$ along complex submanifolds. This flow will solve the usual Hermitian-Yang-Mills flow equations on $\tilde{X} - \mathbf{E}$ with respect the metric ω. It will be useful in the proof of the main theorem, because we will again need to desingularise the HNS filtration, and consider a sequence of blowups. The discussion in this section is an extension of ideas in [BS].

Let $\pi : \tilde{X} \to X$ be a sequence of smooth blowups, and let ω_ε be the usual family of Kähler metrics on $\tilde{X}$. We will write $L^p_k(\tilde{X}, \omega_\varepsilon)$ for the corresponding Sobolev spaces. The following lemma is clear.

Lemma 6.1. *Fix a compact subset $W \subset\subset \tilde{X} - \mathbf{E}$. Let $\tilde{E}$ be a vector bundle. Then there exists a family of constants $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that for any r-form $F \in \Omega^r(\tilde{X} - \mathbf{E}, \tilde{E})$*

$$(1 - C(\varepsilon))\,\|F\|_{L^p_k(W,\omega_0)} \le \|F\|_{L^p_k(W,\omega_\varepsilon)} \le (1 + C(\varepsilon))\,\|F\|_{L^p_k(W,\omega_0)}\,.$$

Throughout this section $\tilde{E} \to \tilde{X}$ will be a holomorphic vector bundle of rank K, equipped with a smooth hermitian metric $\tilde{h}_0$. Although later we will mainly be interested in the case where $\tilde{E} = \pi^*E$, we do not assume this here.

Note that $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$ is uniformly bounded in ε, since for any fixed Kähler form ϖ on $\tilde{X}$ we have:

$$\begin{aligned} \left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right| &= \left|\frac{F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\omega_\varepsilon^n}\right| = \left|\frac{F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\varpi^n}\right| \left|\frac{\det g_\varpi}{\det g_\varepsilon}\right|, \\ \omega_\varepsilon^n &= \frac{\det g_\varepsilon}{\det g_\varpi}\varpi^n \end{aligned}$$

so

$$\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)} = \int_{\tilde{X}} \left|\frac{F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)} \wedge \omega_\varepsilon^{n-1}}{\varpi^n}\right| \varpi^n$$

which is clearly bounded uniformly in ε. Write $\tilde{h}_{\varepsilon,t}$ for the evolution of $\tilde{h}_0$ under the HYM flow with respect to the metric ω_ε.

Lemma 6.2. (1) *Let $t_0 > 0$. Then $\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})}\right|$ is uniformly bounded for all $t \ge t_0 > 0$ and all $\varepsilon > 0$. The bound depends only on t_0 and the uniform bound on $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$.*

(2) *$\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon,t})}\right|$ is bounded uniformly on compact subsets of $\tilde{X} - \mathbf{E}$ for all $t \ge 0$ and all $\varepsilon > 0$. The bound depends only on the local bound on $\left|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right|$ and the uniform bound on $\left\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_0)}\right\|_{L^1(\omega_\varepsilon)}$.*

Proof. By Lemma 3.1 (2), the pointwise norm $\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_{\varepsilon,t})}\right|$ is a subsolution of the heat equation on $(\tilde X,\omega_\varepsilon)$ (see also [BS] equation 3.3). If $K^\varepsilon_t(x,y)$ is the heat kernel for the ω_ε Laplacian on $\tilde X$ then

$$\int_{\tilde X} K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y)$$

is a solution of the heat equation and therefore:

$$\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_{\varepsilon,t})}\right|(x)-\int_{\tilde X} K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y)$$

is also a subsolution. Because

$$\int_{\tilde X} K^\varepsilon_0(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y)=\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(x),$$

the maximum principle for the heat equation now implies that

$$\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|(x)\le\int_{\tilde X} K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y).$$

By [BS] Lemma 4, there is a bound: $K^\varepsilon_t(x,y)\le C\left(1+1/t^n\right)$ for some constant C independent of ε. Part (1) now follows.

For part (2), let $\Omega_1\subset\subset\Omega\subset\subset\tilde X-\mathbf{E}$, and let ψ be a smooth cut-off function supported in Ω and identically 1 in a neighbourhood of $\bar\Omega_1$. Then just as in part (1) we have:

$$\begin{aligned}\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|(x) &\le \int_{\tilde X} K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y)\\ &= \int_{\tilde X}\psi K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y)\\ &\quad+\int_{\tilde X}(1-\psi)\,K^\varepsilon_t(x,y)\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)dvol_{\omega_\varepsilon}(y).\end{aligned}$$

By the maximum principle, the first term on the right hand side is bounded from above by: $\sup\left\{\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E},\tilde h_0)}\right|(y)\mid y\in\Omega\right\}$. Since $\Omega\subset\subset\tilde X-\mathbf{E}$, the function $1/\det g^\varepsilon_{ij}$ is uniformly bounded in ε, so this sup and hence the first integral above are uniformly bounded in ε. By [GR] Theorem 3.1, there are positive constants δ, C_1, C_2, independent of t and ε, such that for $x\ne y$,

$$K^\varepsilon_t(x,y)\le C_1\left(1+\frac{1}{\delta^2t^2}\right)\exp\left(-\frac{\left(d_{\omega_\varepsilon}(x,y)\right)^2}{C_2t}\right).$$

where d_{ω_ε} is the distance function on $\tilde X$ with respect to the Riemannian metric induced by ω_ε. Of course $d_{\omega_\varepsilon}(x,y)$ is bounded from below for $x\in\Omega_1$ and $y\in\mathrm{supp}(1-\psi)$ uniformly in ε. Therefore, $K^\varepsilon_t(x,y)$ is uniformly bounded in ε and t, for these values of x and y. Then the second term on the right is uniformly bounded in terms of $\left\|\left|\Lambda_\omega F_{(\bar\partial_{\tilde E}\tilde h_0)}\right|\right\|_{L^1(\omega_\varepsilon)}$, so $\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|$ is uniformly bounded on Ω_1. □

If we write $\tilde h_{\varepsilon,t}=\tilde k_{\varepsilon,t}\tilde h_0$, then it follows from the HYM flow equations and the second part of the previous lemma that both $\tilde k_{\varepsilon,t}$ and $\tilde k^{-1}_{\varepsilon,t}$ are uniformly bounded on compact subsets of $\tilde X-\mathbf{E}$ for $0\le t\le t_0$ (one sees easily that their determinant and trace are bounded, which is enough). The statement that $\left|\Lambda_{\omega_\varepsilon} F_{(\bar\partial_{\tilde E}\tilde h_{\varepsilon,t})}\right|$ is uniformly bounded on compact subsets of $\tilde X-\mathbf{E}$ translates to the statement that there is a section $f_{\varepsilon,t}\in\mathfrak{u}(\tilde E)$, uniformly bounded on compact subsets of $\tilde X-\mathbf{E}$, such that:

$$\sqrt{-1}\Lambda_{\omega_\varepsilon}\bar\partial_{A_0}\left(\tilde k^{-1}_{\varepsilon,t}\partial_{A_0}\tilde k_{\varepsilon,t}\right)=f_{\varepsilon,t},$$

where A_0 is the connection $(\bar\partial_E,\tilde h_0)$. It therefore follows from [BS] Proposition 1, that $\tilde k_{\varepsilon,t}$ has a uniform $C^{1,\alpha}$ bound (for any $0<\alpha<1$) on compact subsets of $\left(\tilde X-\mathbf{E}\right)\times[0,\infty)$. Furthermore, we may write:

$$\begin{aligned}\sqrt{-1}\Lambda_{\omega_\varepsilon}\bar\partial_{A_0}\left(\tilde k^{-1}_{\varepsilon,t}\partial_{A_0}\tilde k_{\varepsilon,t}\right) &= \tilde k^{-1}_{\varepsilon,t}\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\partial_{A_0}\tilde k_{\varepsilon,t}\right)+\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\tilde k^{-1}_{\varepsilon,t}\right)\left(\partial_{A_0}\tilde k_{\varepsilon,t}\right)\\ &= \tilde k^{-1}_{\varepsilon,t}\triangle_{(\bar\partial_{A_0},\omega_\varepsilon)}\tilde k_{\varepsilon,t}-\tilde k^{-1}_{\varepsilon,t}\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar\partial_{A_0}\tilde k_{\varepsilon,t}\right)\tilde k^{-1}_{\varepsilon,t}\left(\partial_{A_0}\tilde k_{\varepsilon,t}\right),\end{aligned}$$

where in the last equality we have used the Kähler identities and the expression for $\bar{\partial}_{A_0}\tilde{k}_{\varepsilon,t}^{-1}$. Therefore we have:

$$\triangle_{(\bar{\partial}_{A_0},\omega_\varepsilon)}\tilde{k}_{\varepsilon,t}-\sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar{\partial}_{A_0}\tilde{k}_{\varepsilon,t}\right)\tilde{k}_{\varepsilon,t}^{-1}\left(\partial_{A_0}\tilde{k}_{\varepsilon,t}\right)=\tilde{k}_{\varepsilon,t}f_{\varepsilon,t}.$$

By elliptic regularity, this yields a uniform L^p_2 bound (for $1<p<\infty$) on $\tilde{k}_{\varepsilon,t}$ on compact subsets of $\left(\tilde{X}-\mathbf{E}\right)\times[0,\infty)$. It now follows from the HYM the flow equations, that $\frac{\partial\tilde{h}_{\varepsilon,t}}{\partial t}$ has a uniform L^p bound on compact subsets of $\left(\tilde{X}-\mathbf{E}\right)\times[0,\infty)$, and so for any $W\subset\subset\left(\tilde{X}-\mathbf{E}\right)$ and $T\geq 0$, there is a uniform $L^p_{2/1}(W\times[0,T))$ bound on $\tilde{h}_{\varepsilon,t}$, where the 2/1 in the previous notation refers to the fact that there is 1 derivative in the time variable and 2 derivatives in the space variables. By weak compactness, there is a subsequence $\varepsilon_j\to 0$, so that $\tilde{h}_{\varepsilon_j,t}\to\tilde{h}_t$ weakly in $L^p_{2/1}$ on compact subsets. By the Sobolev imbedding theorem, $\tilde{h}_{\varepsilon_j,t}\to\tilde{h}_t$ in $C^{1/0}$ on compact subsets. By a further diagonalisation as $T\to\infty$, $\tilde{h}_{\varepsilon_j,t}\to\tilde{h}_t$ for all $t\geq 0$.

Definition 6.3. *We will refer to the resulting limit $\tilde{h}_t$ corresponding to the initial metric $\tilde{h}_0$ and the degenerate metric ω_0 as the **degenerate Hermitian-Yang-Mills flow**.*

Of course a priori $\tilde{h}_t$ may depend on the subsequence ε_j. It is possible to show that under the assumption that $\Lambda_{\omega_\varepsilon}F_{\tilde{h}_0}$is uniformly bounded in L^∞, $\tilde{h}_t$ is unique. This assumption will not be satisfied in our case. We will show however that $\tilde{h}_t$ solves the HYM equations on $\tilde{X}-\mathbf{E}$ with respect to the metric ω_0 with initial condition $\tilde{h}_0$.

Lemma 6.4. *Let $\tilde{h}_t$ be defined as above. Then $\tilde{h}_t$ is an hermitan metric on $\tilde{E}\to\tilde{X}-\mathbf{E}$ for all $t\geq 0$, and solves the HYM equations on $\tilde{X}-\mathbf{E}$:*

$$\tilde{h}_t^{-1}\frac{\partial\tilde{h}_t}{\partial t}=-2\left(\Lambda_{\omega_0}F_{\tilde{h}_t}-\mu_{\omega_0}(E)\mathbf{Id}_E\right).$$

Proof. Clearly $\tilde{h}_t$ is positive semi-definite since it is a limit of metrics. Therefore we only need to check that $\det\tilde{h}_t$ is positive. Taking the trace of both sides of the HYM equations for the metric ω_ε, we get:

$$\frac{\partial}{\partial t}\left(\log\det\tilde{h}_{\varepsilon,t}\right)=-2\,\mathrm{Tr}\left(\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}-\mu_{\omega_\varepsilon}(E)\mathbf{Id}_E\right)$$

integrating both sides:

$$\left|\log\left(\frac{\det\tilde{h}_{\varepsilon,T}}{\det\tilde{h}_0}\right)\right|=2\left|\int_0^T\mathrm{Tr}\left(\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}-\mu_{\omega_\varepsilon}(E)\mathbf{Id}_E\right)\right|.$$

By the previous lemma, the right hand side is bounded uniformly in ε, so $\det\tilde{h}_T=\lim_{\varepsilon_j\to 0}\det\tilde{h}_{\varepsilon_j,T}$ must be positive. Since $\tilde{h}_{\varepsilon_j,t}\to\tilde{h}_t$ weakly in $L^p_{2/1}$ and $C^{1/0}$ it follows that $\tilde{h}_t$ solves the HYM equations on $\tilde{X}-\mathbf{E}$. □

For the remainder of this section, we will write $F(-)$ for the curvature of a metric in order to avoid a preponderance of subscripts.

Lemma 6.5. *$\left\|F(\tilde{h}_t)\right\|_{L^2(\tilde{X},\omega_0)}$ and $\left\|\Lambda_{\omega_0}F(\tilde{h}_t)\right\|_{L^\infty(\tilde{X},\omega_0)}$ are uniformly bounded for all $t\geq t_0>0$. The bound depends only on t_0 and the uniform bound on $\left\|\Lambda_{\omega_\varepsilon}F(\tilde{h}_0)\right\|_{L^1(\omega_\varepsilon)}$.*

Proof. Let $W\subset\subset\tilde{X}-\mathbf{E}$ be a compact subset. By construction $F(\tilde{h}_{\varepsilon,t})\to F(\tilde{h}_t)$ weakly in $L^2(W,\omega_0)$. Applying Lemma 6.1 and the relation between $F(\tilde{h}_{\varepsilon,t})$ and $\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})$ in L^2 we have:

$$\begin{aligned}\left\|F(\tilde{h}_t)\right\|_{L^2(W,\omega_0)} &\leq \liminf_{\varepsilon\to 0}\left\|F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(W,\omega_0)}\leq C_1\liminf_{\varepsilon\to 0}\left\|F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(W,\omega_\varepsilon)}\\ &\leq C_1\liminf_{\varepsilon\to 0}\left\|F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)}\leq C_1\liminf_{\varepsilon\to 0}\left\|\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)}+C_2\\ &\leq C_3\liminf_{\varepsilon\to 0}\left\|\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|_{L^\infty(\tilde{X})}+C_2,\end{aligned}$$

where C_3 is independent of W, and C_2 is the product of C_1 with a topological constant. The bound in L^2 now follows from Lemma 6.2 (1).

For the second part again fix $W \subset\subset \tilde{X} - \mathbf{E}$. We claim that for fixed t and W, as $\varepsilon \to 0$ there is a uniform bound

$$\left\|\Lambda_{\omega_0} F(\tilde{h}_{\varepsilon,t})\right\|_{L^p(W,\omega_0)} \leq \left\|\Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|_{L^p(W,\omega_0)} + 1.$$

Otherwise, there is a sequence ε_j such that:

$$\left\|\Lambda_{\omega_0} F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^p(W,\omega_0)} \geq \left\|\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^p(W,\omega_0)} + 1.$$

Then

$$\left|\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right| \left\|F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^p(W,\omega_0)} \geq \left\|\left(\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right)\left(F(\tilde{h}_{\varepsilon_j,t})\right)\right\|_{L^p(W,\omega_0)} \geq 1,$$

where $\left|\Lambda_{\omega_0} - \Lambda_{\omega_{\varepsilon_j}}\right|$ denotes the operator norm. Now we have $\tilde{h}_{\varepsilon_j,t} \to \tilde{h}_t$ weakly in $L^p_2(\omega_0, W)$, so $\left\|F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^p(W,\omega_0)}$ is uniformly bounded. Since $\Lambda_{\omega_{\varepsilon_j}} \to \Lambda_{\omega_0}$ on W, this is a contradiction, and so we have proved the claim. Therefore:

$$\begin{aligned}\left\|\Lambda_{\omega_0} F(\tilde{h}_t)\right\|_{L^p(W,\omega_0)} &\leq \liminf_{\varepsilon\to 0} \left\|\Lambda_{\omega_0} F(\tilde{h}_{\varepsilon,t})\right\|_{L^p(W,\omega_0)} \leq \liminf_{\varepsilon\to 0} \left\|\Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|_{L^p(W,\omega_0)} + 1 \\ &\leq C \liminf_{\varepsilon\to 0} \left\|\Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|_{L^\infty(\tilde{X})} + 1.\end{aligned}$$

Taking $p \to \infty$, the lemma now follows from Lemma 6.2. □

Proposition 6.6. *For almost all $t \geq t_0 > 0$, we have:*

$$\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_0} F(\tilde{h}_t)\right\|_{L^2(\tilde{X},\omega_0)} \leq \liminf_{\varepsilon\to 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)} < \infty.$$

As will be seen in the course of the proof, this implies that: $\displaystyle\int_{t_0}^{\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_0} F(\tilde{h}_t)\right\|_{L^2(\omega_0)} dt < \infty.$

Proof. By Lemma 3.1 (1) we have:

$$\frac{d}{dt} \left\|F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} = -2 \left\|d^*_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} = -2 \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\omega_\varepsilon)}.$$

Then:

$$2\int_{t_0}^{\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} dt \leq \left\|F(\tilde{h}_{\varepsilon,t_0})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} \leq \left\|\Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t_0})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} + C.$$

By Lemma 6.2 (1) the right hand side is uniformly bounded as $\varepsilon \to 0$. Then by Fatou's lemma we have:

$$\begin{aligned}2\int_{t_0}^{\infty} \liminf_{\varepsilon\to 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} dt &\leq 2 \liminf_{\varepsilon\to 0} \int_{t_0}^{\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} dt \\ &\leq \liminf_{\varepsilon\to 0} \left\|\Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t_0})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} + C < \infty.\end{aligned}$$

Therefore, for almost all $t \geq t_0$, we have:

$$\liminf_{\varepsilon\to 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} < \infty.$$

Now we prove the first inequality:

$$\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)} \Lambda_{\omega_0} F(\tilde{h}_t)\right\|_{L^2(\tilde{X},\omega_0)} \leq \liminf_{\varepsilon\to 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)}.$$

It is enough to show this for an arbitrary compact subset $W \subset\subset \tilde{X} - \mathbf{E}$. For almost all $t \geq t_0$, we may choose a sequence $\varepsilon_j \to 0$ such that

$$\lim_{j\to\infty} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})} \Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2(W,\omega_{\varepsilon_j})} = \liminf_{\varepsilon\to 0} \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})} \Lambda_{\omega_\varepsilon} F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(W,\omega_\varepsilon)} = b < \infty.$$

Since $\tilde{h}_{\varepsilon_j,t} \to \tilde{h}_t$ weakly in $L^p_2(\tilde{W})$, we have $\Lambda_{\omega_0} F_{\tilde{h}_{\varepsilon_j,t}} \to \Lambda_{\omega_0} F_{\tilde{h}_t}$ weakly in $L^p(\tilde{W})$, and $\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})} \to \nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}$ in $C^0(W)$. It follows by the triangle inequality and Lemma 6.1, that

$$\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^2(W,\omega_0)} \leq (1+C_j)\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})}\Lambda_{\omega_{\varepsilon_j}}F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^2(W,\omega_{\varepsilon_j})} + c_j$$

where C_j and $c_j \to 0$. Then, $\left\|\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|_{L^2_1(W,h_t,\omega_0)}$ is uniformly bounded as $j \to \infty$. Choose a subsequence (still written j) such that $\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})$ converges weakly in $L^2_1(W,\omega_0)$. By Rellich compactness we also have strong convergence $\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0}F(\tilde{h}_t)$ in $L^2(W)$. By the choice of ε_j and the previous inequality, we have $\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2(W,\omega_0)} \to b$. Then finally:

$$\begin{aligned}
\left\|\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2_1(W,h_t,\omega_0)} &\leq \liminf_{j\to\infty}\left\|\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2_1(W,h_t,\omega_0)} \\
&\leq \liminf_{j\to\infty}\left(\left\|\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2(W,\omega_0)} + \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2(W,\omega_0)}\right) \\
&\leq \left\|\Lambda_{\omega_0}F_{\tilde{h}_t}\right\|^2_{L^2(W,\omega_0)} + b.
\end{aligned}$$

Since $\left\|\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2_1(W,h_t,\omega_0)} = \left\|\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2(W,\omega_0)} + \left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2(W,\omega_0)}$, we have

$$\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2(W,\omega_0)} \leq b = \liminf_{\varepsilon\to 0}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(W,\omega_\varepsilon)}.$$

The second statement in the proposition now follows since:

$$\begin{aligned}
\int_{t_0}^{\infty}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\Lambda_{\omega_0}F(\tilde{h}_t)\right\|^2_{L^2(\tilde{X},\omega_0)}dt &\leq \int_{t_0}^{\infty}\liminf_{\varepsilon\to 0}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)}dt \\
(Fatou's\ lemma) \quad &\leq \liminf_{\varepsilon\to 0}\int_{t_0}^{\infty}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|_{L^2(\tilde{X},\omega_\varepsilon)}dt \\
&\leq \liminf_{\varepsilon\to 0}\left\|\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t_0})\right\|^2_{L^2(\tilde{X},\omega_\varepsilon)} + C < \infty.
\end{aligned}$$

□

The following is an immediate consequence.

Corollary 6.7. *There is a sequence $t_j \to \infty$ such that $\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{t_j})}\Lambda_{\omega_0}F(\tilde{h}_t)\right\|_{L^2(\tilde{X},\omega_0)} \to 0$.*

One result of all this discussion is the following corollary, which follows from the previous corollary, Lemma 6.5, and Corollary 2.18. Although we will not use it in the sequel, we feel it is worth stating explicitly.

Corollary 6.8. *Let $t_j \longrightarrow \infty$ as in the previous corollary. Consider the sequence $\tilde{A}_{t_j} = (\bar{\partial}_{\tilde{E}}, \tilde{h}_{t_j})$ of connections defined over $\tilde{X} - \mathbf{E} = X - Z_{\mathrm{alg}}$. Then there is a further subsequence (still denoted t_j) such that $\tilde{A}_{t_j}$ has an Uhlenbeck limit $\tilde{A}_\infty$ on a reflexive sheaf $\tilde{E}_\infty$, which is a vector bundle away from a set $\tilde{Z}_{\mathrm{an}}$ of Hausdorff codimension at least 4. The connection $\tilde{A}_\infty$ is Yang-Mills.*

In the next section we will also need the following proposition.

Proposition 6.9. *For almost all $t > 0$, there is a sequence $\varepsilon_j(t) \to 0$ such that $\Lambda_{\omega_{\varepsilon_j}}F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0}F(\tilde{h}_t)$ in L^p for all $1 \leq p \leq \infty$. In particular: $HYM^{\omega_{\varepsilon_j}}_\alpha\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})}\right) \to HYM^{\omega_0}_\alpha\left(\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_t)}\right)$ for all α.*

Proof. Fix $\delta > 0$. Let $\tilde{U}$ be an open set containing $\mathbf{E}$ with $\mathrm{vol}(\tilde{U}) < \frac{\delta}{3C}$ where C is an upper bound on $\left|\Lambda_{\omega_\varepsilon}F_{\tilde{h}_{\varepsilon,t}}\right|$ which exists by Lemma 6.2. Now let t, ε_j be such that

$$\lim_{j\to\infty}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon_j,t})}\Lambda_{\omega_{\varepsilon_j}}F(\tilde{h}_{\varepsilon_j,t})\right\|^2_{L^2(W,\omega_{\varepsilon_j})} = \liminf_{\varepsilon\to 0}\left\|\nabla_{(\bar{\partial}_{\tilde{E}},\tilde{h}_{\varepsilon,t})}\Lambda_{\omega_\varepsilon}F(\tilde{h}_{\varepsilon,t})\right\|^2_{L^2(W,\omega_\varepsilon)} < \infty$$

as in the proof of the previous proposition, where $W = \tilde{X} - \tilde{U}$. Therefore, by the same argument as in the above proof we have strong convergence $\Lambda_{\omega_0} F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0} F(\tilde{h}_t)$ in $L^2(W, \omega_0)$. Therefore the same is true for $\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t})$. In particular there exists a J such that for $j, k \geq J$, we have:

$$\left\| \Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t}) - \Lambda_{\omega_{\varepsilon_k}} F(\tilde{h}_{\varepsilon_k,t}) \right\|_{L^2(W,\omega_0)} \leq \frac{\delta}{3}.$$

By the choice of $\tilde{U}$, it follows that for $j, k \geq J$:

$$\left\| \Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t}) - \Lambda_{\omega_{\varepsilon_k}} F(\tilde{h}_{\varepsilon_k,t}) \right\|_{L^2(\tilde{X},\omega_0)} \leq \delta.$$

Since $\Lambda_{\omega_{\varepsilon_j}} F_{\tilde{h}_{\varepsilon_j,t}}$ is a Cauchy sequence it converges strongly in $L^2(\tilde{X}, \omega_0)$. Since $\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0} F(\tilde{h}_t)$ weakly in $L^2_{loc}(\tilde{X} - \mathbf{E}, \omega_0)$, it follows that $\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0} F(\tilde{h}_t)$ strongly in $L^2(\tilde{X} - \mathbf{E}, \omega_0)$. Since both $\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t})$ and $\Lambda_{\omega_0} F(\tilde{h}_t)$ are bounded in L^∞ (see Lemma 6.2 and Lemma 6.5) it follows that $\Lambda_{\omega_{\varepsilon_j}} F(\tilde{h}_{\varepsilon_j,t}) \to \Lambda_{\omega_0} F(\tilde{h}_t)$ strongly in $L^p(\tilde{X} - \mathbf{E}, \omega_0)$ for all p. By Lemma 6.2 and Lemma 3.6 we have:

$$HYM_\alpha^{\omega_{\varepsilon_j}} \left(\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_{\varepsilon_j,t})} \right) \longrightarrow HYM_\alpha^{\omega_0} \left(\nabla_{(\bar{\partial}_{\tilde{E}}, \tilde{h}_t)} \right).$$

□

7. Proof of the Main Theorem

In this section we complete the proof of the main theorem. The result is a direct corollary of the following theorem.

Theorem 7.1. *Let A_0 be an integrable, unitary connection on a holomorphic, hermitian vector bundle E , μ_0 the Harder-Narasimhan type of $(E, \bar{\partial}_{A_0})$, and $A \subset [1, \infty)$ be any set containing an accumulation point. Let A_j be a sequence of integrable, unitary connections on E such that:*
- *$(E, \bar{\partial}_{A_j})$ is holomorphically isomorphic to $(E, \bar{\partial}_{A_0})$ for all i;*
- *$HYM_{\alpha,N}(A_j) \longrightarrow HYM_{\alpha,N}(\mu_0)$ for all $\alpha \in A \cup \{2\}$ and all $N > 0$.*

Then there is a Yang-Mills connection A_∞ on a bundle E_∞ defined outside a a closed subset of Hausdorff codimension at least 4 such that:

(1) $(E_\infty, \bar{\partial}_{A_\infty})$ is isomorphic to $Gr^{HNS}(E, \bar{\partial}_{A_0})$ as a holomorphic bundle on $X - Z_{\text{an}}$;
(2) After passing to a subsequence, $A_j \to A_\infty$ in $L^2_{loc}(X - Z_{\text{an}})$;
(3) There is an extension of the bundle E_∞ to a reflexive sheaf (still denoted E_∞) such that $E_\infty \approxeq Gr^{HNS}(E, \bar{\partial}_{A_0})^{**}$.

The proof will be a modification of Donaldson's argument from [DO1] that there is a non-zero holomorphic map $(E, \bar{\partial}_{A_0}) \to (E_\infty, \bar{\partial}_{A_\infty})$ in the case that $(E, \bar{\partial}_{A_0})$ is semi-stable. If the bundles in question are actually stable, we may then apply the elementary fact that a non-zero holomorphic map between stable bundles with the same slope is necessarily an isomorphism. Of course in our case $(E, \bar{\partial}_{A_0})$ is not necessarily semi-stable so the argument must be modified. We first construct such a map on the maximal destabilising subsheaf $S \subset E$ (which is semi-stable). If we assume that S is stable (in other words if we construct the map on the first piece of the HNS filtration) this identifies S with a subsheaf of the limiting sheaf E_∞. We then use an inductive argument to identify each the successive quotients with a direct summand of E_∞. This is relatively straightforward in the case that the HNS filtration is given by subbundles, but in the general case technical complications arise. Therefore, to clearly illustrate our technique, we will first present an exposition of the simpler case where there are no singularities, and then explain the modifications necessary to complete the argument.

7.1. The Subbundles Case.

We begin with the following proposition.

Proposition 7.2. *Let E be a holomorphic, hermitian vector bundle and $A_j = g_j(A_0)$ be a sequence of integrable, unitary connections on E. Let $A \subset [1, \infty)$ be any set containing an accumulation point. Assume that $HYM_{\alpha,N}(A_j) \to HYM_{\alpha,N}(\mu_0)$ for all $N > 0$ and all $\alpha \in A \cup \{2\}$. Let $S \subset (E, \bar{\partial}_{A_0})$ be a holomorphic subbundle. Then there is closed subset Z_{an} of Hausdorff codimension at least 4, a reflexive sheaf E_∞ which is an hermitian vector bundle away from Z_{an} and a Yang-Mills connection A_∞ on E_∞ such that:*

(1) After passing to a subsequence $A_j \to A_\infty$ in $L^2_{loc}(X - Z_{\text{an}})$;
(2) The Harder-Narasimhan type of $(E_\infty, \bar{\partial}_{A_\infty})$ is the same as that of $(E, \bar{\partial}_{A_0})$;
(3) There is a non-zero holomorphic map $g^S_\infty : S \longrightarrow (E_\infty, \bar{\partial}_{A_\infty})$.

Proof. We first reduce to the case where the Hermitian-Einstein tensors $\Lambda_\omega F_{A_j}$ are uniformly bounded. Write $A_{j,t}$ for the time t solution to the YM flow equations with initial condition A_j. By Lemma 3.1, $\left|\Lambda_\omega F_{A_{j,t}}\right|^2$ is a sub-solution of the heat equation. Then for each $t > 0$ and each $x \in X$:

$$\left|\Lambda_\omega F_{A_{j,t}}\right|^2(x) \le \int_X K_t(x,y)\left|\Lambda_\omega F_{A_{j,t}}\right|^2(y)dvol_\omega(y).$$

Here $K_t(x,y)$ is the heat kernel on X. By a theorem of Cheng and Li (see [CHLI]) there is a bound:

$$0 < K_t(x,y) \le C\left(1 + \frac{1}{t^n}\right),$$

and so for any fixed $t_0 > 0$, $\left\|\Lambda_\omega F_{A_{j,t_0}}\right\|_{L^\infty(X,\omega)}$ is uniformly bounded in terms of $\left\|\Lambda_\omega F_{A_j}\right\|_{L^2(X,\omega)}$. Since we assume in particular that $HYM(A_j) \to HYM(\mu_0)$ we know that $\left\|\Lambda_\omega F_{A_j}\right\|_{L^2(X,\omega)}$ is uniformly bounded independently of j, and therefore $\left\|\Lambda_\omega F_{A_{j,t_0}}\right\|_{L^\infty(X,\omega)}$ is uniformly bounded.

For the remainder of the argument we would like to replace A_j with A_{j,t_0}, so that we may assume in the sequel that we have the above bound. In order to do this we must know that the Uhlenbeck limit of the new sequence A_{j,t_0} is the same as that of A_j. We argue as follows:

$$\begin{aligned}\|A_{j,t_o} - A_j\|_{L^2} &\overset{Minkowski}{\le} \int_0^{t_0}\left\|\frac{\partial A_{j,s}}{\partial s}\right\|_{L^2} ds \le \sqrt{t_0}\left(\int_0^{t_0}\left\|d^*_{A_{j,s}}F_{A_{j,s}}\right\|^2_{L^2} ds\right)^{\frac{1}{2}} \\ &= \sqrt{t_0}\left(\int_0^{t_0} -\frac{1}{2}\frac{d}{ds}\left\|F_{A_{j,s}}\right\|^2_{L^2} ds\right)^{\frac{1}{2}} = \sqrt{\frac{t_0}{2}}\left(YM(A_j) - YM(A_{j,t_0})\right)^{\frac{1}{2}} \longrightarrow 0\end{aligned}$$

because A_j is minimising for the YM functional and YM is non-increasing along the flow. This shows that the two limits are equal, and moreover the proof also shows that $\left\|d^*_{A_{j,s}}F_{D_{j,s}}\right\|_{L^2} \to 0$ for almost all s, so this limit is a Yang-Mills connection. Since we have assumed additionally that $HYM_{\alpha,N}(A_j)$ (and hence $HYM_{\alpha,N}(A_{j,t_0})$) is minimising for $\alpha \in A$, it follows from Propositions 3.7 (2) and 3.9 that the HN type of (E_∞, A_∞) is the same as that of (E_0, A_0).

We may therefore assume from here on out that the Hermitian-Einstein tensors $\Lambda_\omega F_{A_j}$ are uniformly bounded independently of j. Note that we have already proven both (1) and (2) above. It remains to construct the non-zero holomorphic map.

Observe that for any holomorphic section σ of a holomorphic vector bundle $V \longrightarrow (X,\omega)$ equipped with an hermitian metric $\langle -,-\rangle$, and whose Chern connection is A, we have that

$$\begin{aligned}\sqrt{-1}\bar{\partial}\partial\left|\sigma\right|^2 &= \sqrt{-1}\bar{\partial}\partial\left\langle\sigma,\sigma\right\rangle = \sqrt{-1}\left(\left\langle\partial_A\sigma,\partial_A\sigma\right\rangle + \left\langle\sigma,\bar{\partial}_A\partial_A\sigma\right\rangle\right) \\ &= \sqrt{-1}\left(\left\langle\partial_A\sigma,\partial_A\sigma\right\rangle + \left\langle\sigma,F_A\sigma\right\rangle\right)\end{aligned}$$

since σ is holomorphic. Applying Λ_ω and using the Kähler identities, we have:

$$\triangle_\partial\left|\sigma\right|^2 = \sqrt{-1}\Lambda_\omega\bar{\partial}\partial\left|\sigma\right|^2 = -\left|\partial_A\sigma\right|^2 + \left\langle\sigma,\sqrt{-1}\left(\Lambda_\omega F_A\right)\sigma\right\rangle.$$

Now let $g^S_j : S \to (E, \bar{\partial}_{A_j})$ be given by the restriction of g_j to S. By definition, this is a holomorphic section of $\mathrm{Hom}(S,E)$, whose Chern connection is $A_0^* \otimes A_j$. Then applying the above formula to g^S_j and writing $k^S_j = (g^S_j)^*({g_j}^S)$, and h^S and h_j for the metrics corresponding to $A_{0|S}$ and A_j, we have

$$\triangle_\partial \operatorname{Tr} k^S_j + \left|\partial_{A_0^*\otimes A_j} g^S_j\right|^2 = \left\langle g^S_j, \sqrt{-1}\left(\Lambda_\omega F_{h_j} g^S_j - g^S_j \Lambda_\omega F_{h^S}\right)\right\rangle,$$

and so

$$\triangle_\partial(\operatorname{Tr} k^S_j) \le (\operatorname{Tr} k^S_j)\left(\left|\Lambda_\omega F_{h_j}\right| + \left|\Lambda_\omega F_{h^S}\right|\right).$$

Now we use the bound on $\left|\Lambda_\omega F_{h_j}\right|$. Let $C_1 = \sup_j \left\|\Lambda_\omega F_{h_j}\right\|_{L^\infty(X,\omega)}$ and $C_2 = \|\Lambda_\omega F_{h^S}\|_{L^\infty(X)}$. Multiplying both sides of the above inequality by $\operatorname{Tr} k_j^S$ and integrating by parts shows:

$$\int_X \left|\nabla \operatorname{Tr} k_j^S\right|^2 dvol_\omega \leq (C_1 + C_2) \int_X \left|\operatorname{Tr} k_j^S\right|^2 dvol_\omega.$$

By the Sobolev imbedding $L_1^2 \hookrightarrow L^{\frac{2n}{n-1}}$ the previous inequality gives a bound

$$\left\|\operatorname{Tr} k_j^S\right\|_{L^{\frac{2n}{n-1}}(X,\omega)} \leq C \left\|\operatorname{Tr} k_j^S\right\|_{L^2(X,\omega)}$$

where C depends only on C_1,C_2 and the Sobolev constant of (X, ω). A standard Moser iteration gives a bound: $\left\|\operatorname{Tr} k_j^S\right\|_{L^\infty(X,\omega)} \leq C \left\|\operatorname{Tr} k_j^S\right\|_{L^2(X,\omega)}$.

At this point we may repeat Donaldson's argument (appropriately modified for higher dimensions). For the reader's convenience we reproduce it here. By definition $\operatorname{Tr}(k_j^S) = \left|g_j^S\right|^2$. Since non-zero constants act trivially on $\mathcal{A}^{1,1}$ we may normalise the g_j^S so that $\left\|g_j^S\right\|_{L^4(X)} = \left\|\operatorname{Tr}(k_j^S)\right\|_{L^2(X)} = 1$. The above bound implies that there is a subsequence of the g_j^S that converges to a limiting gauge transformation g_∞^S weakly in every L_2^p for example. Since $Z_{\rm an}$ has Hausdorff codimension at least 4, we may of course find a covering of $Z_{\rm an}$ by balls $\{B_i^r\}_i$ of radius r such that: $C\left(\sum_i Vol(B_i^r)\right) < 1/2$. If we write $K_r = X - \cup_i B_i^r \cup \operatorname{Sing}(E_\infty)$, then our L^∞ bound implies that: $\left\|g_j^S\right\|_{L^4(K_r)} \geq 1/2$ for all j. This implies that g_∞^S is non-zero. We now show g_∞^S is holomorphic.

If we denote by $\bar\partial_{A_0 \otimes A_\infty}$ the $(0,1)$ part of the connection on $E^* \otimes E_\infty = Hom(E, E_\infty)$ induced by the connections A_0 and A_∞. We will identify E and E_∞ on K_r. Then by definition we have:

$$\bar\partial_{A_0^* \otimes A_\infty} g_j^S = \left(g_j^S A_0 - A_\infty g_j^S\right) = (g_j^S A_0 (g_j^S)^{-1} - A_\infty) g_j^S = (A_j - A_\infty) g_j^S.$$

Since $A_0 \to A_\infty$ in $L^2(K_r)$ this implies $\bar\partial_{A_0 \otimes A_\infty} g_\infty^S = 0$, in other words g_∞^S is holomorphic on K_r. Since this argument works for any choice of r, and the K_r give an exhaustion of $X - Z_{\rm an} \cup \operatorname{Sing}(E_\infty)$, g_∞^S is holomorphic on $X - Z_{\rm an} \cup \operatorname{Sing}(E_\infty)$. By a version of Hartogs theorem (see [SHI] Lemma 3) there is an extension of g_∞^S to $X - \operatorname{Sing}(E_\infty)$. Finally, by normality of these sheaves (both are reflexive) there is an extension to a non-zero map $g_\infty^S : S \to E_\infty$. □

We are now ready to perform the induction, and therefore prove the main theorem in the case when the HNS filtration is given by subbundles. We first assume the quotients $Q_i = E_i/E_{i-1}$ in the Harder-Narasimhan filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l = (E, \bar\partial_{A_0})$ are stable (so the HN and HNS filtrations are the same). From Proposition 2.13, E_∞ has a holomorphic splitting $E_\infty = \oplus_{i=1}^{l'} Q_{\infty,i}$. By Theorem 5.13 the HN types of E and E_∞ are the same, so $l = l'$ and $\mu(E_1) = \mu(Q_{\infty,1}) > \mu(Q_{\infty,i})$ for $i = 2, \cdots, l$. By the above proposition there is a non-zero holomorphic map $g_\infty : E_1 \to E_\infty$. Since we are assuming E_1 is stable, and the $Q_{\infty,i}$ $(i > 1)$ have slope strictly smaller than E_1, the induced map onto these summands is 0 and hence $g_\infty : E_1 \to Q_{\infty,1}$. Again by stability of E_1 and $Q_{\infty,1}$ and the fact that E_1 and $Q_{\infty,1}$ have the same rank and degree, this map is an isomorphism. This is the first step in the induction.

The inductive hypothesis will be that the connections A_j restricted to E_{i-1} converge to connections on the bundle $Gr(E_{i-1})$, in other words $Gr(E_{i-1}) \subset E_\infty$. Let $E_{\infty,i} = \oplus_{j \leq i} Q_{\infty,j}$ and set: $E_\infty = Gr(E_{i-1}) \oplus R$, and consider the short exact sequence of bundles: $0 \to E_{i-1} \to E_i \to Q_i \to 0$. Since $Gr(E_i) = Gr(E_{i-1}) \oplus Q_i$, to complete the induction we need only show that Q_i is a direct summand of R. The sequence of connections on E_i^* induced by A_j satisfy the hypotheses of the proposition, so we may apply this result to the dual exact sequence: $0 \to Q_i^* \to E_i^* \to E_{i-1}^* \to 0$, and therefore obtain a holomorphic map $Q_i^* \to (E_{\infty,i})^*$. Because Q_i^* is the maximal destabilising subsheaf of $(E_{\infty,i})^*$ this implies that Q_i^* is isomorphic to a summand of R^*. This completes the proof under the assumption that the quotients are stable.

To extend this to the general case, it suffices to consider the case that the original bundle $(E, \bar\partial_{A_0})$ is semi-stable. In other words the filtration is a Seshadri filtration of E. Then as in the above argument we may conclude that E_1 is isomorphic to a factor of E_∞ we also again obtain a non-zero holomorphic map $g_\infty : Q_i^* \to (E_{\infty,i})^*$. However, the Seshadri quotients all have the same slope, so we do not know via slope considerations that Q_i^* maps into R^*. On the other hand we know that the weakly holomorphic projections converge. If $\pi_j^{(i-1)}$ denotes the sequence of projections to $g_j(E_{i-1})$ and $\pi_\infty^{(i-1)}$ the projection onto $E_{\infty,i-1}$, then $\pi_j^{(i-1)} \to \pi_\infty^{(i-1)}$ by the proof of Lemma 4.5 of [DW1]. If we denote by $\breve\pi_j^{(i-1)}$ the dual projection, then for each j, the image of Q_i^* is in the kernel of $\check{\pi}_j^{(i-1)}$. In other words the image $g_\infty(Q_i^*)$ lies in the kernel of $\check{\pi}_j^{(i-1)}$. Therefore since we have convergence, the image of $g_\infty(Q_i^*)$ lies in the kernel of $\check{\pi}_\infty^{(i-1)}$ which is in R^*. Therefore Q_i^* is isomorphic to a factor of R^* and this completes the proof.

7.2. **The General Case.** In general the HNS filtration is not given by subbundles. The argument we have given in Proposition 7.2 for the construction of the holomorphic map $S \to E_\infty$ remains valid if S is an arbitrary torsion free subsheaf since the connections in question are all defined a priori on the ambient bundle E, and since the second fundamental form β of S drops out of the estimates, there is no problem obtaining a uniform bound on the Hermitian-Einstein tensors. On the other hand, when we try to run the inductive argument, the restrictions of the connections A_j to the pieces E_i of the HNS filtration only make sense on the locally free part of these subsheaves. This prevents us from applying the argument of Proposition 7.2 in the inductive step because to do so requires global L^∞ bounds on the appropriate Hermitian-Einstein tensors, which we do not have, since the restrictions of the A_j do not extend over the singular set Z_{alg}.

The strategy for proving the main theorem in the general case mirrors our method in Section 4. Roughly speaking we proceed as follows. Let $A_j = g_j(A_0)$ be a sequence of connections. First we pass to an arbitrary resolution $\pi : \tilde{X} \to X$ of singularities of the HNS filtration. Then we construct an isomorphism from the associated graded object of the filtration for the pullback bundle π^*E (away from the exceptional set $\mathbf{E}$) to the Uhlenbeck limit of the sequence π^*A_j on the Kähler manifold $(\tilde{X} - \mathbf{E}, \omega_0) = (X - Z_{\text{alg}}, \omega)$ where $\omega_0 = \pi^*\omega$. Then we will use the fact that these bundles extend as reflexive sheaves over Z_{alg} to the double dual of the associated graded object of E and the Uhlenbeck limit of A_j respectively, and hence by normality of these sheaves, the isomorphism extends as well.

The outline of the proof given above has to be modified somewhat for technical reasons which we will now explain. Just as for the case of subbundles, by first running the YM flow for finite time we may assume there is a uniform bound $\left\|\Lambda_\omega F_{A_j}\right\|_{L^\infty(X)}$ or equivalently on $\left\|\Lambda_{\omega_0} F_{\tilde{A}_j}\right\|_{L^\infty(\tilde{X}-\mathbf{E})}$ where $\tilde{A}_j = \pi^*A_j$. As usual we will denote by A_∞ the Uhlenbeck limit of A_j on (X, ω) and we have $A_j \to A_\infty$ in $L^p_{1,loc}(X - Z_{\text{an}})$ for $p > n$. The proof of the proposition proves all but (3) of Theorem 7.1. Let $E_i \subset E$ be a factor of the HNS filtration and $A_j^{(i)} = \pi_j^{(i)} A_j$ be the connections on $g_j(E_i)$ induced from A_j, and $A_\infty^{(i)} = \pi_\infty^{(i)} A_\infty$. By Lemma 5.10 it follows that $A_j^{(i)} \to A_\infty^{(i)}$ weakly in $L^p_{1,loc}(X - Z_{\text{an}} \cup Z_{\text{alg}})$.

If $\pi : \tilde{X} \to X$ is the aforementioned resolution of singularities then the filtration of $\pi^*E = \tilde{E}$ is given by subbundles $\tilde{E}_i \subset \tilde{E}$, isomorphic to E_i away from the exceptional divisor $\mathbf{E}$. Write $\tilde{g}_j = g_j \circ \pi$ and let $\tilde{A}_j^{(i)}$ be the connection induced by $\tilde{A}_j = \pi^*A_j$ on $\tilde{g}_j(\tilde{E}_i)$. We will write $\tilde{\pi}_j$ for the projection to $\tilde{g}_j(\tilde{E}_i)$ and $\tilde{\beta}_j$ for the second fundamental forms for the connections $\tilde{A}_j$ with respect to the subbundles $\tilde{E}_i$; in other words these are sections of the bundle $\Omega^{0,1}\left(\tilde{X}, Hom(\tilde{Q}_i, \tilde{E}_i)\right)$ for an auxiliary bundle $\tilde{Q}_i$. Then this sequence of connections satisfies the following:

(1) There is a closed subset $\tilde{Z}_{\text{an}} \subset \tilde{X} - \mathbf{E}$ of Hausdorff codimension at least 4
and a Yang-Mills connection $\tilde{A}_\infty^{(i)}$ defined on a reflexive sheaf $\tilde{E}_{\infty,i} \to \tilde{X} - \mathbf{E}$ (which is a bundle on $\tilde{X} - (\tilde{Z}_{\text{an}} \cup \mathbf{E})$), such
that $\tilde{A}_j^{(i)} \to \tilde{A}_\infty^{(i)}$ weakly in $L^p_{1,loc}\left(\tilde{X} - (\tilde{Z}_{\text{an}} \cup \mathbf{E})\right)$.

(2) We have the standard formula for the curvature:

$$\sqrt{-1}\Lambda_{\omega_0} F_{\tilde{A}_j^{(i)}} = \sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\pi}_j F_{\tilde{A}_j} \tilde{\pi}_j\right) + \sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\beta}_j \wedge \tilde{\beta}_j^*\right).$$

Also:

- The $\tilde{\beta}_j$ are locally bounded on $\tilde{X} - (\tilde{Z}_{\text{an}} \cup \mathbf{E})$ uniformly in j (Lemma 2.25)
- The $\tilde{\beta}_j \to 0$ in $L^2(\omega_0)$. In particular, they are uniformly bounded in $L^2(\omega_0)$ (see the proof of [DW1] Lemma 4.5).

Note that the term

$$\sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\pi}_j F_{\tilde{A}_j} \tilde{\pi}_j\right)$$

is bounded in $L^\infty(\tilde{X},\omega_0)$ since $\tilde{A}_j = \pi^* A_j$. The key point here is that term

$$\sqrt{-1}\Lambda_{\omega_0}\left(\tilde{\beta}_j \wedge \tilde{\beta}_j^*\right)$$

is not bounded in $L^\infty(\tilde{X},\omega_0)$ since it may be written as

$$\sqrt{-1}\frac{\left(\tilde{\beta}_j \wedge \tilde{\beta}_j^*\right)\wedge \omega_0^{n-1}}{\omega_0^n}$$

which blows up near $\mathbf{E}$. This is a problem because in order to carry out the induction in the preceding sub-section we had to consider exact sequences of the form:

$$0 \longrightarrow \tilde{Q}_i^* \longrightarrow \tilde{E}_i^* \longrightarrow \tilde{E}_{i-1}^* \longrightarrow 0$$

(here $\tilde{Q}_i = \tilde{E}_i/\tilde{E}_{i-1}$) and apply Proposition 7.2 to construct a non-zero holomorphic map $\tilde{Q}_i^* \to \tilde{E}_{\infty,i}^*$. This involved knowing that there was a uniform L^∞ bound on the Hermitian-Einstein tensors of the induced connections $(\tilde{A}_j^{(i)})^*$ and $(\tilde{A}_{j,Q}^{(i)})^*$ on $\tilde{E}_i^*$ and $\tilde{Q}_i^*$. Since this is not the case we cannot apply this argument directly. On the other hand we do know that for all positive times $t>0$, the degenerate Yang-Mills flow of Section 6 gives connections $\tilde{A}_{j,t}^{(i)}$ such that $\Lambda_{\omega_0}F_{\tilde{A}_{j,t}^{(i)}}$ is uniformly bounded (see Lemma 6.5). For each t the deformed sequence of connections has an Uhlenbeck limit $\tilde{A}_{\infty,t}^{(i)}$ on a reflexive sheaf $\tilde{E}_{\infty,i}^t$ which a priori depends on t.

There are now two points to address. In parallel to Proposition 7.2 we will show that after resolving the singularities of the maximal destabilising subsheaf S to a bundle $\tilde{S}$ there is a non-zero holomorphic map $\tilde{S}\to\tilde{E}_\infty^t$ (where $\tilde{E}_\infty^t$ is an Uhlenbeck limit of $\tilde{A}_{j,t}$) away from $\mathbf{E}$. This is not automatic from the proof of Proposition 7.2 because the connections $\tilde{A}_{j,t}$ do not extend smoothly across $\mathbf{E}$, so the integration by parts involved in the proof is not valid. We will instead derive this map as a limit of the maps produced from the corresponding argument for the family of Kähler manifolds $(\tilde{X},\omega_\varepsilon)$. Secondly we need to know that the Uhlenbeck limits $(\tilde{E}_\infty^t,\tilde{A}_{\infty,t})$ are independent of t and are all equal to $(\tilde{E}_\infty,\tilde{A}_\infty)$. Again, this does not follow from our previous argument since, as we have noted, the second fundamental forms of the restricted connections are only bounded in L^2 and therefore the curvatures are only bounded in L^1. In particular we do not have that $\tilde{A}_j^{(i)}$ is minimising for the functional YM. Establishing these two facts will complete the proof of the main theorem, since then we may use induction just as for the case when the HNS filtration is given by subbundles.

We begin with the first point.

Proposition 7.3. *Let $\tilde{E}\to\tilde{X}$ be a vector bundle with an hermitian metric $\tilde{h}$. Let $\tilde{A}_j=\tilde{g}_j(\tilde{A}_0)$ be a sequence of unitary connections on $\tilde{E}$, and assume $\Lambda_{\omega_0}F_{\tilde{A}_j}$ is bounded uniformly in j in $L^1(\tilde{X},\omega_0)$. Let $\tilde{A}_{j,t}$ be the solution of the degenerate YM flow at time t with initial condition $\tilde{A}_j$, and suppose that this sequence has an Uhlenbeck limit $(\tilde{E}_\infty^t,\tilde{A}_{\infty,t})$. Finally let $\tilde{S}\subset\tilde{E}$ be a subbundle of $(\tilde{E},\tilde{A}_0)$. Then there is a non-zero holomorphic map $\tilde{g}_\infty:\tilde{S}\to\tilde{E}_\infty^t$ on $\tilde{X}-\mathbf{E}$. Furthermore, assume that $(\tilde{E}_\infty^t,\tilde{A}_{\infty,t})$ has an extension $(E_\infty^t,A_{\infty,t})$ as a reflexive sheaf over Z_{alg} to X, assume $\tilde{S}$ also extends to a reflexive sheaf S on X. Then $\tilde{g}_\infty$ induces a non-zero holomorphic map $g_\infty:S\to E_\infty^t$.*

Proof. Let ω_ε be the standard family of Kähler metrics on $\tilde{X}$ and fix $t>0$. Let $\varepsilon_i\to 0$ be a sequence as in Section 6, i.e. if $\tilde{A}_{j,t}^{\varepsilon_i}$ is the time t YM flow on $(\tilde{X},\omega_{\varepsilon_i})$, then $\tilde{A}_{j,t}^{\varepsilon_i}\to\tilde{A}_{j,t}$ in $C^{1/0}$ on compact subsets of $\tilde{X}-\mathbf{E}$. Choose a family of metrics $\tilde{h}_{\varepsilon_i}^{\tilde{S}}$ on $\tilde{S}$ converging uniformly on compact subsets of $\tilde{X}-\mathbf{E}$ to a metric $\tilde{h}_0^{\tilde{S}}$ defined away from $\mathbf{E}$, and such that $\sup\left|\Lambda_{\omega_{\varepsilon i}}F_{\tilde{h}_{\varepsilon_i}^{\tilde{S}}}\right|$ is uniformly bounded as $\varepsilon_i\to 0$ (take for example the time 1 HYM flow of $\tilde{h}$ with respect ω_ε). For each j and each $\varepsilon_i>0$, we have a non-zero holomorphic map $\tilde{g}_{\varepsilon_i,j}^{\tilde{S}}:\tilde{S}\to(\tilde{E},\bar{\partial}_{\tilde{A}_{j,t}^{\varepsilon_i}})$. Just as in Section 7.1, we set $k_{\varepsilon_i,j}^{\tilde{S}}=\left(\tilde{g}_{\varepsilon_i,j}^{\tilde{S}}\right)^*\tilde{g}_{\varepsilon_i,j}^{\tilde{S}}$. As in Proposition 7.2 we have the inequality:

$$\Delta_{(\partial,\omega_\varepsilon)}(\operatorname{Tr}\tilde{k}_{\varepsilon_i,j}^{\tilde{S}})\le(\operatorname{Tr}\tilde{k}_{\varepsilon_i,j}^{\tilde{S}})\left(\left|\Lambda_{\omega_{\varepsilon_i}}F_{\tilde{A}_{j,t}^{\varepsilon_i}}\right|+\left|\Lambda_{\omega_{\varepsilon i}}F_{\tilde{h}_{\varepsilon_i}^{\tilde{S}}}\right|\right).$$

Both factors on the right are uniformly bounded as $\varepsilon_i\to 0$ by assumption. It follows that we have the inequality: $\|\operatorname{Tr}\tilde{k}_{\varepsilon_i,j}^{\tilde{S}}\|_{L^\infty(\tilde{X})}\le C\|\operatorname{Tr}\tilde{k}_{\varepsilon_i,j}^{\tilde{S}}\|_{L^2(\tilde{X},\omega_\varepsilon)}$, where the constant C depends only on these uniform bounds and the Sobolev constant of $(\tilde{X}, \omega_{\varepsilon_i})$ is also uniformly bounded away from zero by [BS] Lemma 3. As in the proof of Proposition 7.2 we rescale $\tilde{g}^{\tilde{S}}_{\varepsilon_i,j}$ so that $\|\tilde{g}^{\tilde{S}}_{\varepsilon_i,j}\|_{L^4(\tilde{X},\omega_\varepsilon)} = 1$. A diagonalisation argument for an exhaustion of $\tilde{X} - \mathbf{E}$ together with the sup bound gives a sequence of non-zero holomorphic maps $\tilde{g}^{\tilde{S}}_j : \tilde{S} \to (\tilde{E}, \bar{\partial}_{\tilde{A}_{j,t}})$ defined on $\tilde{X} - \mathbf{E}$ with $\tilde{g}^{\tilde{S}}_{\varepsilon_i,j} \to \tilde{g}^{\tilde{S}}_j$ uniformly on compact subsets as $\varepsilon_i \to 0$ such that: $\|\tilde{g}^{\tilde{S}}_j\|_{L^\infty} \le C$, and $\|\tilde{g}^{\tilde{S}}_j\|_{L^4(\omega_0)} = 1$. Repeating the proof of Proposition 7.2 yields a nonzero limit $\tilde{g}^{\tilde{S}}_\infty : \tilde{S} \to (\tilde{E}^t_\infty, \tilde{A}_{\infty,t})$. The last statement follows from the normality of the sheaves in question. □

Secondly we have:

Proposition 7.4. *Let $\tilde{E} \to \tilde{X}$ be a Hermitian vector bundle with a unitary integrable connection $\tilde{A}_0$. We assume that the holomorphic bundle $(\tilde{E}, \bar{\partial}_{A_0})$ restricted to $\tilde{X} - \mathbf{E} = X - Z_{\mathrm{alg}}$ extends to a holomorphic bundle $(E, \bar{\partial}_E)$ on X with Harder-Narasimhan type $\mu = (\mu_1, \cdots, \mu_R)$. Let $\tilde{A}_j = \tilde{g}_j(\tilde{A}_0)$ be a sequence of unitary connections on $\tilde{E}$, and assume there is a subset $\tilde{Z}_{\mathrm{an}} \subset \tilde{X} - \mathbf{E}$ of Hausdorff codimension at least 4, and a YM connection $\tilde{A}_\infty$ on a bundle $\tilde{E}_\infty \to \tilde{X} - (\tilde{Z}_{\mathrm{an}} \cup \mathbf{E})$ such that $\tilde{A}_j \to \tilde{A}_\infty$ weakly in $L^p_{1,loc}$ (where $p > n$) on compact subsets of $\tilde{X} - (\tilde{Z}_{\mathrm{an}} \cup \mathbf{E})$. We assume that the constant eigenvalues of $\sqrt{-1}\Lambda_{\omega_0} F_{\tilde{A}_\infty}$ are given by the vector μ. Finally assume $\Lambda_{\omega_0} F_{\tilde{A}_j} \to \Lambda_{\omega_0} F_{\tilde{A}_\infty}$ in $L^1(\omega_0)$. Then there is a subsequence such that for almost all $t > 0$, $\tilde{A}_{j,t} \to \tilde{A}_\infty$ in $L^p_{1,loc}$ away from $\tilde{Z}_{\mathrm{an}} \cup \mathbf{E}$ where $\tilde{A}_{j,t}$ is the time t degenerate YM flow with initial condition $\tilde{A}_j$.*

This will follow from a sequence of lemmas.

Lemma 7.5. *For any $t > 0$, $\|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\|_{L^\infty(\tilde{X}-\mathbf{E})}$ is uniformly bounded in j. Moreover, for almost all $t > 0$, $\lim_{j\to\infty} HYM^{\omega_0}\left(\tilde{A}_{j,t}\right) = HYM(\mu)$.*

Proof. The first statement follows from Lemma 6.5. By assumption, we have $\Lambda_{\omega_0} F_{\tilde{A}_j} \to \Lambda_{\omega_0} F_{\tilde{A}_\infty}$ in L^1, and $\Lambda_{\omega_0} F_{\tilde{A}_\infty}$ has constant eigenvalues $\mu_1, \cdots, \mu_K$. Set $M^2 = \sum_{i=1}^K \mu_i^2 = \frac{HYM(\mu)}{2\pi}$. Also let $\mu_{1,\varepsilon}, \cdots, \mu_{K,\varepsilon}$ be the HN type of $(E, \bar{\partial}_{\tilde{A}_0})$ with respect to ω_ε, and set $\tilde{M}^2_\varepsilon = \sum_{i=1}^K \mu^2_{i,\varepsilon}$. By Corollary 3.5 we know:

$$\tilde{M}_\varepsilon \le \frac{1}{2\pi} \int_{\tilde{X}} \left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right| dvol_{\omega_\varepsilon}.$$

By Proposition 6.9 , for almost all t, we can find a sequence $\varepsilon_i = \varepsilon_i(t) \to 0$ such that $\Lambda_{\omega_{\varepsilon_i}} F_{\tilde{A}^{\varepsilon_i}_{j,t}} \to \Lambda_{\omega_0} F_{\tilde{A}_{j,t}}$ in any $L^p(\omega_0)$. Let $\varepsilon_i \to 0$ and using the convergence of the HN type:

$$M \le \frac{1}{2\pi} \int_{\tilde{X}} \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}$$

for all j and almost all $t \ge 0$. We also have:

$$\begin{aligned} \left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right|(x) &\le \int_{\tilde{X}} K^\varepsilon_t(x,y) \left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right|(y) dvol_{\omega_\varepsilon}(y) \\ &= M + \int_{\tilde{X}} K^\varepsilon_t(x,y) \left(\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right| - M\right) dvol_{\omega_\varepsilon} \end{aligned}$$

where $K^\varepsilon_t(x,y)$ is the heat kernel on $(\tilde{X}, \omega_\varepsilon)$ (since $K^\varepsilon_t(x,y)$ has integral equal to 1). Since we have the bound: $K^\varepsilon_t(x,y) \le C(1 + 1/t^n)$, there is a constant $C(t)$ independent of ε such that:

$$\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}\right|(x) \le M + C \left\|\left|\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}\right| - M\right\|_{L^1(\tilde{X},\omega_\varepsilon)}.$$

Then just as above we have:

$$\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|(x) \le M + C \left\|\left|\Lambda_{\omega_0} F_{\tilde{A}_j}\right| - M\right\|_{L^1(\tilde{X},\omega_0)}$$

for almost all $x \in \tilde{X} - \mathbf{E}$ and almost all $t > 0$. Since $\left|\Lambda_{\omega_0} F_{\tilde{A}_j}\right| \to \left|\Lambda_{\omega_0} F_{\tilde{A}_\infty}\right| = M$ in L^1, we have

$$\lim_{j\to\infty} \sup \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|(x) \le M$$

for almost all $x \in \tilde{X} - \mathbf{E}$ and almost all $t > 0$. On the other hand since $\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}$ is uniformly bounded in j, we can use the lower bound for

$$\frac{1}{2\pi}\int_{\tilde{X}} \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}$$

and Fatou's Lemma to show:

$$M \leq \int_{\tilde{X}} \lim_{j\to\infty} \sup \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0}.$$

It follows that $\lim_{j\to\infty} \sup\left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right|^2 = M^2$ almost everywhere. By Fatou's lemma we therefore have:

$$\begin{aligned} HYM(\mu) &\leq \lim_{j\to\infty} \inf HYM^{\omega_0}(\tilde{A}_{j,t}) \leq \lim_{j\to\infty} \sup HYM^{\omega_0}(\tilde{A}_{j,t}) \\ &= \lim_{j\to\infty} \sup \int_{\tilde{X}} \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0} \leq \int_{\tilde{X}} \lim_{j\to\infty} \sup \left|\Lambda_{\omega_0} F_{\tilde{A}_{j,t}}\right| dvol_{\omega_0} \\ &= 2\pi M^2 = HYM(\mu). \end{aligned}$$

□

Lemma 7.6. *For almost all $t_0 > 0$, $\left\|\tilde{A}_{j,t} - \tilde{A}_{j,t_0}\right\|_{L^2(\tilde{X},\omega_0)} \to 0$, uniformly for almost all $t \geq t_0$.*

Proof. As before let $\varepsilon_i \to 0$ be a sequence such that $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ and $\tilde{A}^{\varepsilon_i}_{j,t_0} \to \tilde{A}_{j,t_0}$ in C^0_{loc}. Then we again have:

$$\begin{aligned} \left\|\tilde{A}^{\varepsilon_i}_{j,t} - \tilde{A}^{\varepsilon_i}_{j,t_0}\right\|_{L^2} &\overset{Minkowski}{\leq} \int_{t_0}^{t} \left\|\frac{\partial \tilde{A}^{\varepsilon_i}_{j,s}}{\partial s}\right\|_{L^2} \\ \leq \sqrt{t}\left(\int_{t_0}^{t} \left\|d^*_{A_{j,s}} F_{\tilde{A}^{\varepsilon_i}_{j,s}}\right\|^2_{L^2}\right)^{\frac{1}{2}} &= \sqrt{t}\left(\int_{t_0}^{t} -\frac{1}{2}\frac{d}{ds}\left\|F_{\tilde{A}^{\varepsilon_i}_{j,s}}\right\|^2_{L^2} ds\right)^{\frac{1}{2}} \\ = \sqrt{\frac{t}{2}}\left(YM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - YM(\tilde{A}^{\varepsilon_i}_{j,t})\right)^{\frac{1}{2}} &= \sqrt{\frac{t}{2}}\left(HYM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - HYM(\tilde{A}^{\varepsilon_i}_{j,t})\right)^{\frac{1}{2}} \\ \leq \sqrt{\frac{t}{2}}\left(HYM(\tilde{A}^{\varepsilon_i}_{j,t_0}) - HYM(\mu_{\varepsilon_i})\right)^{\frac{1}{2}}. \end{aligned}$$

Using Proposition 6.9 and Proposition 4.12 this yields:

$$\left\|\tilde{A}_{j,t} - \tilde{A}_{j,t_0}\right\|_{L^2(\tilde{X},\omega_0)} \leq \sqrt{\frac{t}{2}}\left(HYM(\tilde{A}_{j,t_0}) - HYM(\mu)\right)^{\frac{1}{2}}$$

The result follows by applying the previous lemma. □

Lemma 7.7. *There is a YM connection $\tilde{A}_{\infty,*}$ on a reflexive sheaf $\tilde{E}_{\infty,*} \to \tilde{X} - \mathbf{E}$ with the following property: for almost all $t > 0$ there is a subsequence and a closed subset $\tilde{Z}^t_{\text{an}} \subset \tilde{X} - \mathbf{E}$, possibly depending on t and the choice of subsequence, such that $\tilde{A}_{j,t} \to \tilde{A}_{\infty,*}$ in $L^p_{1,loc}$ ($p > n$) away from $\operatorname{Sing}(\tilde{E}_{\infty,*}) \cup \tilde{Z}^t_{\text{an}} \cup \mathbf{E}$.*

Proof. As in Proposition 6.6 and using Proposition 6.9 we have:

$$HYM(\tilde{A}_{j,t_1}) - HYM(\tilde{A}_{j,t_2}) \geq 2\int_{t_1}^{t_2} \left\|d^*_{A_{j,s}} F_{\tilde{A}_{j,s}}\right\|_{L^2(\omega_0)} ds$$

for almost all $t_2 \geq t_1 > 0$. It follows from Lemma 7.5 and Fatou's lemma that:

$$\lim_{j\to\infty} \inf \left\|d^*_{A_{j,s}} F_{\tilde{A}_{j,s}}\right\|^2_{L^2(\omega_0)} = 0,$$

for almost all t. Choose a sequence t_k of such t with $t_k \to 0$. For each k there is a subsequence $j_m(t_k)$, a YM connection $\tilde{A}_{\infty,t_k}$, and a set of Hausdorff codimension at least 4 which we denote by $\tilde{Z}^{t_k}_{\text{an}}$, depending on the choice of subsequence such that $\tilde{A}_{j_m,t_k} \to \tilde{A}_{\infty,t_k}$ in $L^p_{1,loc}$ away from $\tilde{Z}^{t_k}_{\text{an}}$. By a diagonalisation argument, assume without loss of generality that the original sequence satisfies $\tilde{A}_{j,t_k} \to \tilde{A}_{\infty,t_k}$ for all t_k. On the other hand, by Lemma 7.6, $\tilde{A}_{\infty,t_k} = \tilde{A}_{\infty,*}$ is independent of t_k. For any t, there is a k with $t \geq t_k$, so Lemma 7.6 also implies $\tilde{A}_{j,t} \to \tilde{A}_{\infty,*}$ in L^2_{loc} for almost all $t > 0$. Hence, any Uhlenbeck limit of $\tilde{A}_{j,t}$ coincides with $\tilde{A}_{\infty,*}$. □

The proof of Proposition 7.4 will be complete if we can show $\tilde{A}_\infty = \tilde{A}_{\infty,*}$. First we will need:

Lemma 7.8. *$\Lambda_{\omega_\varepsilon} F_{\tilde{A}_{j,t}}$ is bounded on compact subsets of $\tilde{X} - \mathbf{E}$, uniformly for all j, all $t \geq 0$, and all $\varepsilon > 0$.*

Proof. By our assumptions it follows that $\Lambda_{\omega_\varepsilon} F_{\tilde{A}_j}$ are uniformly bounded in L^1 and that they are uniformly locally bounded. The result now follows just as in the proof of Lemma 6.2(2). □

Corollary 7.9. *$\left|\tilde{A}_{j,t} - \tilde{A}_\infty\right|$ is bounded in any $L^p_{1,loc}$ away from $\tilde{Z}_{\mathrm{an}} \cup \mathbf{E}$, uniformly for all j and all $0 \leq t \leq t_0$. In particular, the singular set $\tilde{Z}^t_{\mathrm{an}}$ is independent of t and is equal to $\tilde{Z}_{\mathrm{an}}$.*

Proof. Since $\tilde{A}_j \to \tilde{A}_\infty$ in $L^p_{1,loc}$, it suffices to prove that $\left|\tilde{A}_{j,t} - \tilde{A}_j\right|$ is bounded in C^1_{loc}. Choose a sequence ε_i such that $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ in C^1_{loc}. It suffices to prove $\left|\tilde{A}^{\varepsilon_i}_{j,t} - \tilde{A}_j\right|$ is bounded in C^1_{loc} uniformly in ε_i. Write $\tilde{A}^{\varepsilon_i}_{j,t} = \tilde{g}^{\varepsilon_i}_{j,t}(\tilde{A}_j)$ and $\tilde{k}^{\varepsilon_i}_{j,t} = (\tilde{g}^{\varepsilon_i}_{j,t})^* \tilde{g}^{\varepsilon_i}_{j,t}$. It suffices to show that $(\tilde{k}^{\varepsilon_i}_{j,t})^{-1}$ is bounded and $\tilde{k}^{\varepsilon_i}_{j,t}$ has bounded derivatives, locally with respect to a trivialisation of $\tilde{E}$. The local boundedness of $\tilde{k}^{\varepsilon_i}_{j,t}$ and $(\tilde{k}^{\varepsilon_i}_{j,t})^{-1}$ follows from the flow equations and the preceding lemma. Namely, it is easy to see that the determinant and trace of these endomorphisms are bounded, and this easily implies the boundedness of the endomorphisms themselves. The boundedness of the derivatives follows from [BS] Proposition 1 applied to the equation

$$\triangle_{(\bar{\partial}_{A_0},\omega_\varepsilon)} \tilde{k}_{\varepsilon,t} - \sqrt{-1}\Lambda_{\omega_\varepsilon}\left(\bar{\partial}_{A_0}\tilde{k}_{\varepsilon,t}\right)\tilde{k}^{-1}_{\varepsilon,t}\left(\partial_{A_0}\tilde{k}_{\varepsilon,t}\right) = \tilde{k}_{\varepsilon,t} f_{\varepsilon,t}.$$

□

Now we can complete the proof of Proposition 7.4. Fix a smooth test form $\phi \in \Omega^1(\tilde{X}, \mathfrak{u}(E))$, compactly supported away from $\tilde{Z}_{\mathrm{an}} \cup \mathbf{E}$. Choose $0 < \delta \leq 1$. For $\varepsilon > 0$ we have:

$$\begin{aligned}
\int_{\tilde{X}} \left\langle \phi, \tilde{A}^\varepsilon_{j,\delta} - A_j \right\rangle dvol_{\omega_\varepsilon} &= \int_0^\delta dt \int_{\tilde{X}} \left\langle \phi, \frac{\partial \tilde{A}^\varepsilon_{j,t}}{\partial t} \right\rangle dvol_{\omega_\varepsilon} \\
(flow\ equations) &= -\int_0^\delta dt \int_{\tilde{X}} \left\langle \phi, \left(d_{\tilde{A}^\varepsilon_{j,t}}\right)^* F_{\tilde{A}^\varepsilon_{j,t}} \right\rangle dvol_{\omega_\varepsilon} \\
(K\ddot{a}hler\ identities) &= \sqrt{-1}\int_0^\delta dt \int_{\tilde{X}} \left\langle \phi, \left(\partial_{\tilde{A}^\varepsilon_{j,t}} - \bar{\partial}_{\tilde{A}^\varepsilon_{j,t}}\right) \Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}} \right\rangle dvol_{\omega_\varepsilon} \\
&= \sqrt{-1}\int_0^\delta dt \int_{\tilde{X}} \left\langle \left(\partial_{\tilde{A}^\varepsilon_{j,t}} - \bar{\partial}_{\tilde{A}^\varepsilon_{j,t}}\right)^* \phi, \Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}} \right\rangle dvol_{\omega_\varepsilon}.
\end{aligned}$$

By Lemma 7.8, $\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}$ is bounded on the support of ϕ for all j, all $\varepsilon > 0$, and all $0 \leq t \leq \delta$, and the bound may be taken to be independent of δ. Therefore:

$$\int_{\tilde{X}} \left\langle \phi, \tilde{A}^\varepsilon_{j,\delta} - A_j \right\rangle dvol_{\omega_\varepsilon} \leq C \int_0^\delta dt \left\| \left(\partial_{\tilde{A}^\varepsilon_{j,t}} - \bar{\partial}_{\tilde{A}^\varepsilon_{j,t}}\right)^* \phi \right\|_{L^1(\omega_0)}.$$

Applying this inequality to a sequence, $\tilde{A}^{\varepsilon_i}_{j,t} \to \tilde{A}_{j,t}$ in C^1_{loc},

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{j,\delta} - A_j \right\rangle dvol_{\omega_0} \right| \leq C \int_0^\delta dt \left\| \left(\partial_{\tilde{A}_{j,t}} - \bar{\partial}_{\tilde{A}_{j,t}}\right)^* \phi \right\|_{L^1(\omega_0)}.$$

By the Corollary 7.9, $\left|\tilde{A}_{j,t} - \tilde{A}_\infty\right|$ is locally bounded in any L^p independently of j. Then

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{j,\delta} - A_j \right\rangle dvol_{\omega_0} \right| \leq C\delta$$

where C depends only on the L^1 norm of $\partial_{\tilde{A}_\infty}\phi$, $\bar{\partial}_{\tilde{A}_\infty}\phi$ and the bounds on $\Lambda_{\omega_\varepsilon} F_{\tilde{A}^\varepsilon_{j,t}}$ and $\left|\tilde{A}_{j,t} - \tilde{A}_\infty\right|$. In particular C is independent of j. Taking limits as $j \to \infty$ we have:

$$\left| \int_{\tilde{X}} \left\langle \phi, \tilde{A}_{\infty,\delta} - A_\infty \right\rangle dvol_{\omega_0} \right| \leq C\delta$$

and since δ and was arbitrary and $\tilde{A}_{\infty,\delta} = \tilde{A}_{\infty,*}$ for almost all small δ, this implies $\tilde{A}_{\infty,*} = A_\infty$. This concludes the proof of Proposition 7.4 and hence the proof of the main theorem.

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