ON "OBSERVABLE" LI-YORKE TUPLES FOR INTERVAL MAPS

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Abstract. In this paper we study the set of Li-Yorke d-tuples and its d-dimensional Lebesgue measure for interval maps $T\colon [0,1] \to [0,1]$. If a topologically mixing T preserves an absolutely continuous probability measure 9with respect to Lebesgue), then the d-tuples have Lebesgue full measure, but if T preserves an infinite absolutely continuous measure, the situation becomes more interesting. Taking the family of Manneville-Pomeau maps as example, we show that for any $d \geq 2$, it is possible that the set of Li-Yorke d-tuples has full Lebesgue measure, but the set of Li-Yorke $d+1$-tuples has zero Lebesgue measure.

1. Introduction

Abundance of Li-Yorke pairs (see [26] and Definition 1.1 below) is a frequently used criterion for declaring that a dynamical system $T\colon X \to X$ on a compact metric space (X,ρ) is chaotic. A milestone was the result that positive topological entropy implies the existence of an uncountable scrambled set, *i.e.,* a set in which every pair of distinct points has the Li-Yorke property [6]. However, 2^∞ maps (*i.e.,* with periodic points of period 2^k for each $k \geq 0$, but no other periods) have zero entropy, but they can still have uncountable scrambled sets [37], so that Li-Yorke pairs give a slightly more refined view on mathematical chaos than the condition that $h_{top}(T) > 0$.

Multimodal interval maps never have closed invariant scrambled sets, and it can be proved that C^3 multimodal interval maps with nonflat critical points only have scrambled sets of Lebesgue measure zero, see [12]. In view of these results, when speaking about "observable chaos" it is more reasonable to consider the size of the set of Li-Yorke pairs, than scrambled sets themselves. This idea comes from Lasota, and was first employed by Piórek in [34].

Various papers (see *e.g.* [4, 12]) comment on the measure-theoretic properties of Li-Yorke pairs and scrambled sets. In particular, [12] gives a comprehensive account in the setting of smooth multimodal interval maps $T\colon [0,1] \to [0,1]$, on questions whether Li-Yorke pairs have full two-dimensional Lebesgue measure $\lambda_2 := \lambda \times \lambda$ in $[0,1]^2$. Under certain mixing conditions of λ (exactness suffices but we use the weaker condition of $\limsup$ full, see Definition 1.2) this is indeed the case. One question left open in [12] is whether there are smooth conservative maps for which λ_2-a.e. pair (x,y) is Li-Yorke and additionally its orbit under $T \times T$ is not dense in $[0,1]^2$.

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The topic of this paper is the d-fold measure of Li-Yorke d-tuples. We give the definitions first. Let (X,ρ) be a compact metric space and $T\colon X\to X$ a continuous map acting on it.

Definition 1.1. *A d-tuple $\underline{x}=(x_1,\dots,x_d)$ is called:*

(1) asymptotic *if* $\lim_n \max_{i,j}\rho(T^n(x_i),T^n(x_j))=0$*;*
(2) proximal *if* $\liminf_n \max_{i,j}\rho(T^n(x_i),T^n(x_j))=0$*;*
(3) δ-separated *if* $\limsup_n \min_{i\neq j}\rho(T^n(x_i),T^n(x_j))>\delta$*; if $\underline{x}$ is δ-separated for some $\delta>0$, we call it* separated*;*
(4) Li-Yorke (LY for short) *if it proximal and separated, that is:*

$$\begin{cases}\liminf_n \max_{i,j}\rho(T^n(x_i),T^n(x_j))=0,\\ \limsup_n \min_{i\neq j}\rho(T^n(x_i),T^n(x_j))>0.\end{cases}$$

(5) δ-Li-Yorke (or simply, δ-LY) *for some $\delta>0$, if $\underline{x}$ is LY and:*

$$\limsup_n \min_{i\neq j}\rho(T^n(x_i),T^n(x_j))>\delta.$$

We denote by LY_d and LY_d^δ the set of all LY and δ-LY d-tuples, respectively. Also we use T_d as abbreviation for the d-fold product map $T\times\cdots\times T$ on X^d.

It is known that a transitive system with a fixed point has a LY d-tuple for any $d\geq 2$ [43], and consequently, every totally transitive system with dense periodic points (hence topologically weakly mixing) must have such tuples. It is also not hard to see that T has LY d-tuples if and only if T^n has them for every $n\geq 1$. Therefore, each transitive map on the interval must have LY d-tuples. In fact, every topologically mixing map on an infinite space has a dense Mycielski set M such that any $d\geq 2$ distinct points in M form a LY d-tuple (see *e.g.* [24]). On the other hand, maps of the interval with zero topological entropy never have LY 3-tuples [23]. However, there are dynamical systems on the Cantor set such that each d-tuple of distinct points is LY, but no uncountable set with this property for $(d+1)$-tuples exists [24]. In fact, the system need not have LY $(d+1)$-tuples at all [15].

The main motivation of the paper is the following question.

Question: *How large is the set of LY d-tuples?*

Obviously, there is no one good answer to this question without specifying what "large" means. In purely topological case it would be a residual set. But in many cases, *e.g.* on the interval, there is a natural reference measure such as Lebesgue measure. Even with this natural tool at hand, the answers depend on the degree of smoothness of the map. The smoother the map is, the better the proposed method of measurement is appropriate.

Let λ be Lebesgue measure, or more generally a non-singular Borel reference measure. We will assume that λ is fully supported, *i.e.,* $\lambda(U)>0$ for every open $U\subset X$, or otherwise assume that λ is non-atomic and restrict T to $\mathrm{supp}(\lambda)$. Let $\lambda_d=\lambda\times\cdots\times\lambda$ denote the d-fold product measure on X^d.

The following definitions come from [3] and [35] respectively:

Definition 1.2. *Let λ be a non-singular probability measure on X. Then λ is called:*

(1) $\limsup$ full *if* $\lambda(A) > 0$ *implies that* $\limsup_{n\to\infty} \lambda(T^n(A)) = 1$*;*
(2) full[1] *if* $\lambda(A) > 0$ *implies that* $\lim_{n\to\infty} \lambda(T^n(A)) = 1$.

When $d \geq 3$, then λ being $\limsup$ full is no longer sufficient to guarantee that λ_d-a.e. d-tuple is Li-Yorke. Instead, if λ admits an equivalent weak mixing T-invariant probability measure μ, then this holds, see [4] and Lemma 3.2. We show

Theorem A. *Let λ be a non-singular, fully supported, Borel probability measure, and denote by λ_d the d-fold product measure.*

(1) If λ is $\limsup$ *full then $\lambda_2(LY_2^\delta) = 1$ for any $\delta < diam(X)/2$,*
(2) If λ is full then $\lambda_d(LY_d^\delta) = 1$ for every $d \geq 2$ and some $\delta > 0$. If X is additionally connected then $\lambda_d(LY_d^\delta) = 1$ for every $\delta < diam(X)/2(d-1)$.

Remark 1.3. *Without the connectedness assumptions, a bound $\delta < diam(X)/2(d-1)$ cannot work. For instance, if X is a union of two small intervals but placed at long distance, then $diam(X)$ is large but two points in any triple have to be very close.*

For a smooth topologicallly mixing interval map T preserving a probability measure $\mu \ll \lambda$, the above theorem supplies an abundance of Li-Yorke d-tuples. If the T-invariant measure μ is only σ-finite, then the difficulty in showing the abundance of Li-Yorke d-tuples for $d \geq 3$ lies in the separation along a subsequence. Under mild conditions, any two points in a d-tuple separate infinitely often, but it is difficult to show that three or more points separate at the same time. The family of *Manneville-Pomeau maps* $T_\alpha : [0,1] \to [0,1]$ defined by

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}), \\ 2x-1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

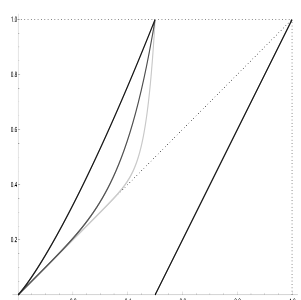

FIGURE 1. Graph of $T_{\frac{1}{2}}$, T_4 and T_{15}.

illustrates this perfectly.

The point 0 is a neutral fixed point, weakly expanding, but the speed at which points move away from 0 is slower as α increases. The typical situation can thus be that one point in a d-tuple separates itself from the rest, while the other $d-1$ points linger in a neighbourhood of 0. Using known techniques of renewal theory to a first return induced map we prove the following theorem.

[1]Compared to $\limsup$ full, lim full seems a more logical name for this property, but it is called *full* by Proppe & Boyarski in [35], hence we follow this terminology.

Theorem B. *For the Manneville-Pomeau map T_α, the following conditions hold:*

(1) λ_2-a.e. pair is $\frac{1}{3}$-Li-Yorke,

(2) and for $d \geq 3$:

(i) if $\alpha \leq \frac{d-1}{d-2}$ and $\delta < 1/2(d-2)$, then λ_d-a.e. d-tuple is δ-Li-Yorke;

(ii) if $\alpha > \frac{d-1}{d-2}$ and $\varepsilon > 0$, then λ_d-a.e. d-tuple is not ε-separated; in particular λ_d-a.e. d-tuple is not Li-Yorke.

Remark 1.4. *For $\alpha > 2$, the product system is λ_2-dissipative, whence orbits of typical pairs are not dense in* $[0,1]^2$*, however still λ_2-a.e. pair is LY. This addresses a question posed in* [12, p. 527].

If $\frac{d}{d-1} < \alpha \leq \frac{d-1}{d-2}$, then λ_d-a.e. $\underline{x}$ is LY in $[0,1]^2$*, however λ_{d+1}-a.e $\underline{x}$ is not LY, nor asymptotic (*i.e., *$T^n_{\alpha,d}(\underline{x}) \not\to \Delta$, the diagonal).*

When measuring tuples using Lebesgue measure as reference, much may depend on the class of maps we are considering. In the class of continuous maps we can quite easily perturb the dynamics, and hence topological conjugacy can completely change qualitative description of the map in terms of reference measure. We can prove the following:

Theorem C. *There exist pairwise topologically conjugate maps $P, S, T \colon [0,1] \to [0,1]$ and sets K, L, M such that:*

(1) K, L, M have full Lebesgue measure;

(2) there are positive numbers δ_d such that every d-tuple of distinct points in L (resp. in M) is δ_d-LY for S (resp. T);

(3) $\omega(x, S) = [0,1]$ for every $x \in L$;

(4) there exists a Cantor minimal set A of positive measure such that $\omega(x, T) = A$ for every $x \in M$.

(5) none of the pairs $(x, y) \in K \times K$ is LY for P; in particular the set of LY pairs has zero Lebesgue measure.

The maps P, S, T are the same from dynamical point of view, however the size of the set of LY tuples detected by Lebesgue measure is completely different in each case.

The paper is organised as follows. Section 2 gives preliminary information from ergodic theory. Section 3 shows the results on LY d-tuples that can be derived from the assumption that λ is full or lim sup full. In Section 4 we present and prove our results for the C^0 setting. Finally, in Section 5 we discuss the situation of σ-finite measure (Manneville-Pomeau) which provides the most interesting examples where the theory of Section 3 break down.

2. Preliminaries from Measure Theory

Let X be a compact topological space with Borel σ-algebra $\mathcal{B}$ and let $T \colon X \to X$ be a Borel measurable map. We will only consider measures λ for which every $B \in \mathcal{B}$ is measurable, and such that $\lambda(U) > 0$ for every open set, *i.e.,* λ is fully supported Borel measure. In particular, this means that if $\varepsilon > 0$, then $g(\varepsilon) := \inf_{x \in X} \lambda(B(x; \varepsilon)) > 0$. Indeed, if $g(\varepsilon) = 0$, then, due to compactness of X, we can find a convergent sequence $x_n \to x$ such that $\lambda(B(x_n; \varepsilon)) \to 0$. But then $B(x; \varepsilon/2) \subset B(x_n; \varepsilon)$ for n sufficiently large, so $\lambda(B(x; \varepsilon/2)) \leq \lim_n \lambda(B(x_n; \varepsilon)) = 0$, in contradiction to $\operatorname{supp}(\lambda) = X$.

Definition 2.1. *A measure λ on $\mathcal{B}$ is (with respect to T):*

(1) non-singular *provided that $\lambda(A) = 0$ if $\lambda(T^{-1}(A)) = 0$,*
(2) conservative *if for every set $\lambda(A) > 0$ there is $n > 0$ such that $\lambda(A \cap T^n(A)) > 0$,*
(3) ergodic[2] *if $A = T^{-1}(A) \pmod{\lambda}$ implies that $\lambda(A) = 0$ or $\lambda(A^c) = 0$,*
(4) exact *if the* tail-field $\bigvee_n T^{-n}\mathcal{B}$ *is trivial, or equivalently $\lambda(A)$ or $\lambda(A^c) = 0$ whenever $T^{-n} \circ T^n(A) = A \pmod{\lambda}$ for all $n \in \mathbb{N}$.*

Remark 2.2. *If λ is conservative, then $\lambda(A \cap T^n(A)) > 0$ for some $n > 0$ and so there is $m > 0$ such that*

$$\lambda(A \cap T^{n+m}(A)) \geq \lambda(A \cap T^n(A) \cap T^m(A \cap T^n(A))) > 0.$$

In other words (by induction), for every $n > 0$ there exists $k > n$ such that $\lambda(A \cap T^k(A)) > 0$, provided that $\lambda(A) > 0$, and hence it follows that λ-a.e. $x \in A$ returns to A infinitely often.

With any non-singular measure λ w.r.t. T we can associate the Perron-Frobenius operator $\mathcal{L}\colon L^1(\lambda) \to L^1(\lambda)$, uniquely defined by the formula

$$\int_A \mathcal{L} f d\lambda = \int_{T^{-1}(A)} f d\lambda \quad \text{for all } f \in L^1(\lambda) \text{ and all } A \in \mathcal{B}.$$

Remark 2.3. *An equivalent property to exactness (assuming that λ is non-singular) is that $\int |\mathcal{L}^n f| d\lambda \to 0$ as $n \to \infty$ for any $f \in L^1(\lambda)$ with $\int f d\mu = 0$, where $\mathcal{L}$ is the Perron-Frobenius operator (see* [25] *or* [1, Theorem 1.3.3.]*).*

Lemma 2.4. *If λ is a non-singular, σ-finite and exact Borel measure, then d-dimensional direct product measure λ_d is ergodic for every integer $d \geq 1$.*

Proof. Parallel to Lemma 18 of [12]. □

We repeat the following fact after Thaler [39].

Lemma 2.5. *Let λ be an exact non-singular Borel measure and $\mu \ll \lambda$ an infinite σ-finite T-invariant measure. Then*

$$\int_A \mathcal{L}^n f d\lambda \to 0 \qquad \text{as } n \to \infty,$$

for all $A \in \mathcal{B}$, $\mu(A) < \infty$ and all $f \in L^1(\lambda)$.

Proof. Fix $B \in \mathcal{B}$, $0 < \mu(B) < \infty$ and denote

$$g = \frac{\int_X f d\lambda}{\mu(B)} 1_B\, h$$

where $h = \frac{d\mu}{d\lambda}$ and observe that $\int_X (f - g) d\lambda = 0$. Then

$$\begin{aligned} \int_A \mathcal{L}^n f d\lambda &= \int_A \mathcal{L}^n (f-g) d\lambda + \int_{T^{-n}(A)} g d\lambda \\ &= \int_A \mathcal{L}^n (f-g) d\lambda + \frac{\int_X f d\lambda}{\mu(B)} \mu(T^{-n}(A) \cap B) \end{aligned}$$

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[2] If Borel sets A, B are such that $\lambda(A \setminus B) = 0$ then we write $A \subset B \pmod{\lambda}$. Similarly, when $A \subset B \pmod{\lambda}$ and $B \subset A \pmod{\lambda}$ then we denote this fact by $A = B \pmod{\lambda}$.

Since λ is exact (see Remark 2.3), the first term tends to 0. By invariance of μ we obtain $\mu(T^{-n}(A) \cap B) \leq \mu(T^{-n}(A)) = \mu(A)$ and hence

$$\limsup_{n\to\infty} \int_A \mathcal{L}^n f d\lambda \leq \frac{\mu(A)}{\mu(B)} \int_X f d\lambda.$$

But μ is infinite and σ-finite, so we can start with arbitrarily large $\mu(B)$ which completes the proof. $\square$

Lemma 2.6. *Let λ be an exact non-singular Borel measure and $\mu \ll \lambda$ an infinite σ-finite T-invariant measure. Then λ is not full.*

Proof. Since μ is σ-finite, there is $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$, in particular $\lambda(A) > 0$. While $\mu(X) = \infty$, there are countably many pairwise disjoint sets F_i such that $\mu(F_i) < \infty$ and $\bigcup_{i=1}^{\infty} F_i = X$. Fix any $k > 0$. Since $1 = \lambda(X) = \sum_{i=1}^{\infty} \lambda(F_i)$ there is $r > 0$ such that $\sum_{i=r+1}^{\infty} \lambda(F_i) < 2^{-k-1}$. Hence, for every $n > 0$ we have

$$\lambda(T^{-n}(A)) = \sum_{i=1}^{\infty} \lambda(T^{-n}(A) \cap F_i) \leq 2^{-k-1} + \sum_{i=1}^{r} \lambda(T^{-n}(A) \cap F_i).$$

But, for any i we obtain by Lemma 2.5 that

$$\lambda(T^{-n}(A) \cap F_i) = \int_{T^{-n}(A)} 1_{F_i} d\lambda = \int_A \mathcal{L}^n(1_{F_i}) d\lambda \to 0.$$

Hence, there is $n_k > 0$ such that $\lambda(T^{-n_k}(A) \cap F_i) < 2^{-k-1}/r$ for each $1 \leq i \leq r$ and consequently $\lambda(T^{-n_k}(A)) < 2^{-k}$. In particular, if we set $B := \cup_{k\geq 1} T^{-n_k}(A)$ then $0 < \lambda(B) < 1$. The proof is finished by the fact, that $\lambda(B^c) > 0$, but $T^{n_k}(B^c) \cap A = \emptyset$ for all $k \geq 1$, which shows that λ is not full. $\square$

Note that none of above properties required that λ be *invariant i.e.,* $\lambda(A) = \lambda(T^{-1}(A))$ for any $A \in \mathcal{B}$. A definition that requires invariance is *weak mixing*: for every $A, B \in \mathcal{B}$, there is a sequence $n_k \to \infty$ such that $\mu(T^{-n_k}(A) \cap B) \to \mu(A)\mu(B)$ as $k \to \infty$. It is known that if μ is weak mixing, then for every $A, B \in \mathcal{B}$, there is a set $\mathcal{N} \subset \mathbb{N}$ of full density such that $\lim_{\mathcal{N} \ni n \to \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B)$.

Let us recall two facts, which are Theorem 23 and Proposition 24 from [12], respectively. We write $C^3_{nf}([0,1])$ for the collection of C^3 multimodal interval maps $f\colon [0,1] \to [0,1]$ where all critical points c are *non-flat*, *i.e.,* there is $\ell_c \in (1,\infty)$ such that $|T(x) - T(c)|/|x-c|^{\ell_c}$ is bounded and bounded away from zero for all x sufficiently close to c.

For interval maps, we call a closed invariant set $A \subset [0,1]$ an *attractor* (see [29]) if its basin $\{x \in [0,1] : \omega(x) \subset A\}$ has positive Lebesgue measure, and no proper subset of A has this property. Examples of attractors A which are also Cantor sets are the Feigenbaum attractor and the "wild" attractor of a Fibonacci unimodal map of sufficiently high critical order, see [13].

Theorem 2.7. *Let $T \in C^3_{nf}([0,1])$ be a topologically mixing map having no Cantor attractors. If λ is conservative, then it is* $\limsup$ *full.*

Theorem 2.8. *Let $T \in C^3_{nf}([0,1])$ be topologically mixing and denote by λ the Lebesgue measure on $[0,1]$. Then the following statements are equivalent:*

1. *there exists an invariant probability measure $\mu \ll \lambda$,*
2. $\liminf_{n\to\infty} \lambda(T^n(A)) > 0$ *for every measurable set $A \in \mathcal{B}$, $\lambda(A) > 0$,*
3. *λ is full for T .*

This shows that Lebesgue measure is lim sup full but not full (for $T \in C^3_{nf}([0,1])$) precisely when it is conservative but admits no absolutely continuous probability measure. The first such examples (within the quadratic family) were constructed by Johnson [21], and more detailed constructions can be found in [20, 8]. Whether there exists a quadratic map for which λ does not even admit a σ-finite invariant probability measure is unknown. However, there are cases where a σ-finite measure $\mu \ll \lambda$ exists such that $\mu(J) = \infty$ for all intervals $J \subset [0,1]$, see [8, 2, 11].

Lemma 2.9. *Let λ be a fully supported non-singular probability measure which is* lim sup *full. If λ admits an equivalent T-invariant probability measure μ, then λ is full.*

Proof. Let $g^+(\varepsilon) = \sup\{\mu(A) : \lambda(A) \le \varepsilon\}$ and $g^-(\varepsilon) = \sup\{\lambda(A) : \mu(A) \le \varepsilon\}$. Since $\mu \ll \lambda$ and both μ, λ are probability measures, we easily obtain that $g^+(\varepsilon) \to 0$ as $\varepsilon \to 0$, and since $\lambda \ll \mu$ also $g^-(\varepsilon) \to 0$ as $\varepsilon \to 0$. Let B be an arbitrary measurable set with $\lambda(B) > 0$, and let (n_k) be a sequence such that $\lambda(T^{n_k}(B)) > 1 - 1/k$. Then $\mu(T^{n_k}(B)) \ge 1 - g^+(1/k)$ and by invariance, also $\mu(T^n(B)) \ge 1 - g^+(1/k)$ for all $n \ge n_k$. Using the definition of g^-, we find $\lambda(T^n(B)) \ge 1 - g^-(g^+(1/k))$ for all $n \ge n_k$. Since $\lim_{\varepsilon\to 0} g^+(\varepsilon) = \lim_{\varepsilon \to 0} g^-(\varepsilon) = 0$, it follows that λ is full. $\square$

3. Li-Yorke tuples and d-fold product measures

Another way of expressing the LY_d-tuples is

$$LY_d = \Big(\bigcap_{k=1}^{\infty}\bigcap_{r=1}^{\infty}\bigcup_{n>r} A_{k,n}\Big) \cap \Big(\bigcup_{k=1}^{\infty}\bigcap_{r=1}^{\infty}\bigcup_{n>r} D_{k,n}\Big),$$

where

$$\begin{cases} A_{k,n} = \{\underline{x} : \max_{i,j} \rho(T^n(x_i), T^n(x_j)) < 1/k\}, \\ D_{k,n} = \{\underline{x} : \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) > 1/k\}. \end{cases}$$

Similarly, we can write

$$LY_d^\delta = \Big(\bigcap_{k=1}^{\infty}\bigcap_{r=1}^{\infty}\bigcup_{n>r} A_{k,n}\Big) \cap \Big(\bigcap_{r=1}^{\infty}\bigcup_{n>r} D_n^\delta\Big),$$

where

$$D_n^\delta = \{\underline{x} : \min_{i\neq j} \rho(T^n(x_i), T^n(x_j)) > \delta\}.$$

Since $A_{k,n}$ and $D_{k,n}$ and D_n^δ are open, LY_d and LY_d^δ are Borel sets (in fact G_δ-sets), hence their indicator function is measurable w.r.t. Borel measures on X^d, and we can use Fubini's theorem.

The following result is an extension of a result in [12] which states (for interval maps, but the argument is general) that if λ is lim sup full, then the LY-pairs have full measure w.r.t. λ_2.

Proof of Theorem A. We argue by induction. The initiation step is for δ-LY-pairs with $\delta < \operatorname{diam}(X)/2$. Given $x \in X$, let $LY_x = \{y \in X : (x,y) \in LY_2^\delta\}$. First we argue that λ-a.e. y is δ-separated in pair with x. If this was the case, the set of points which are not δ-separated in pair with x has positive measure, *i.e.,*

$$\lambda\left(\bigcup_{r=1}^{\infty}\bigcap_{n>r}\{y : \rho(T^n(x), T^n(y)) \le \delta\}\right) > 0.$$

Therefore, we can find $r \in \mathbb{N}$ such that $A := \cap_{n>r}\{y : \rho(T^n(x), T^n(y)) \leq \delta\}$ has positive measure. Fix $0 < \gamma < \mathrm{diam}(X) - 2\delta$ and observe that $\mathrm{diam}(T^n(A)) \leq 2\delta < \mathrm{diam}(X) - \gamma$ for every $n > r$ and hence there exists x_n such that $T^n(A) \cap B(x_n; \gamma) = \emptyset$. In particular $\lambda(T^n(A)) < 1 - g(\gamma) < 1$ for all $n > r$. This contradicts the assumption that λ is lim sup full.

Similarly, we argue that λ-a.e. y is proximal w.r.t. x. Indeed, if not, then we can choose $k \in \mathbb{N}$ sufficiently large such that the set $D := \cap_{n\geq 0}\{y \in X : \rho(T^n(x), T^n(y)) > 1/k\}$ has positive measure. But then $T^n(D) \cap B(T^n(x); 1/k) = \emptyset$, and thus $\lambda(T^n(D)) < 1 - g(1/k) < 1$ for all n, so again λ is not lim sup full.

The set of δ-LY-pairs can be written as $LY_2^\delta = \cup_x LY_x$, so by Fubini's theorem, it has full measure. This completes the first step of induction and proves also that if λ is lim sup full then $\lambda_2(LY_2^\delta) = 1$ for any $\delta < \mathrm{diam}(X)/2$.

We continue the induction, fixing $d \geq 2$ and assuming that LY_d^ε has full d-fold product measure for any $\varepsilon < \mathrm{diam}(X)/2(d-1)$ when X is connected, and for some $\varepsilon > 0$ otherwise (hence every ε sufficiently small). If X is connected, then we fix any $0 < \delta < \mathrm{diam}(X)/2d$. If X is not connected, then we fix any distinct points $a_1, \dots, a_{d+1}$ and put $\delta = \min\{\min_{s\neq t} \rho(a_s, a_t)/6, \varepsilon\}$.

Take $\underline{x} \in LY_d^\delta$, let $(m_u)_{u\geq 1}$ and $(n_u)_{u\geq 1}$ be sequences along which $\underline{x}$ is asymptotic, resp. separated, that is

$$\begin{cases} \liminf_u \max_{i,j} \rho(T^{m_u}(x_i), T^{m_u}(x_j)) = 0, \\ \limsup_u \min_{i\neq j} \rho(T^{n_u}(x_i), T^{n_u}(x_j)) > \delta. \end{cases}$$

and let

$$LY_{\underline{x}} = \left\{ y \in X : \begin{array}{l} \liminf_u \max_i \rho(T^{m_u}(x_i), T^{m_u}(y)) = 0, \\ \limsup_u \min_i \rho(T^{n_u}(x_i), T^{n_u}(y)) > \delta. \end{array} \right\}$$

We show that λ-a.e. y is δ-separated with $\underline{x}$ along the subsequence (m_u). If the set of non δ-separated points has positive measure, then there is $r \in \mathbb{N}$ such that $A := \bigcap_{u>r}\{y : \min_i \rho(T^{m_u}(x_i), T^{m_u}(y)) \leq \delta\}$ has positive measure. But then $T^{m_u}(A) \subset \bigcup_{i=1}^d B(T^{m_u}(x_i); \delta)$. If X is connected, then take any $\xi > 0$ such that $2d(\delta + 2\xi) < \mathrm{diam}(X)$ and in the other case put $\xi = \delta$.

We claim that $\bigcup_{i=1}^d B(T^{m_u}(x_i); \delta + \xi) \neq X$. First, we consider the case that X is connected. If the claim does not hold then, since X is connected, for every points $p, q \in X$ there are pairwise distinct numbers $i_1, \dots, i_k$, where $k \leq d$, such that $p \in B(T^{m_u}(x_{i_1}); \delta + \xi)$, $q \in B(T^{m_u}(x_{i_1}); \delta + \xi)$ and $B(T^{m_u}(x_{i_j}); \delta + \xi) \cap B(T^{m_u}(x_{i_{j+1}}); \delta + \xi) \neq \emptyset$. But then

$$\begin{aligned} \rho(p, q) &\leq \delta + \xi + 2(k-1)(\delta + \xi) + \delta + \xi \leq 2d\delta + 2d\xi \\ &\leq 2d(\delta + 2\xi) - 2d\xi < \mathrm{diam}(X) - 2\xi. \end{aligned}$$

which is a contradiction, since p, q were arbitrary. Indeed, the claim holds. Similarly, if X is not connected, then by the definition of δ, each ball $B(T^{m_u}(x_{i_{j+1}}); \delta+\xi)$ can contain at most one point a_s, and hence also in this case $\bigcup_{i=1}^d B(T^{m_u}(x_i); \delta + \xi) \neq X$. Indeed, the claim holds.

By the above claim, for every u there is a point q_u such that

$$B(q_u; \xi) \cap \bigcup_{i=1}^d B(T^{m_u}(x_i); \delta) = \emptyset$$

which in particular implies that $\lambda(T^{m_u}(A)) < 1-g(\xi)$ for all $u > r$. This contradicts that λ is full. This proves that λ-a.e. y is δ-separated with $\underline{x}$ along the subsequence (m_u).

Similarly, we argue that λ-a.e. y is proximal w.r.t. $\underline{x}$ along the subsequence (n_u). Indeed, otherwise we can choose $k \in \mathbb{N}$ sufficiently large such that the set $D := \bigcap_{u=0}^{\infty}\{y \in X : \max_i \rho(T^{n_u}(x_i), T^{n_u}(y)) \geq 1/k\}$ has positive measure. Passing to a subsequence of n_u if necessary we may assume that for every u there is j such that if $y \in D$ then $\rho(T^{n_u}(x_j), T^{n_u}(y)) \geq 1/k$. But then $T^{n_u}(D) \cap (B(T^{n_u}(x_j); 1/k)) = \emptyset$ and so $\lambda(T^{n_u}(D)) < 1 - g(1/k) < 1$ for all u, therefore again λ is not full.

The set of δ-LY $(d+1)$-tuples contains the set $LY^{\delta}_{d+1} \supset \bigcup_{\underline{x}\in LY^{\delta}_d} LY_{\underline{x}}$, so again by Fubini's theorem, it has full measure. This completes the proof. □

Lemma 3.1. *If the d-fold product (X^d, T_d) has a conservative non-atomic product measure λ_d, then the set of d-tuples that are not asymptotic has full λ_d-measure.*

Proof. Assume on the contrary, that the set of asymptotic d-tuples has positive measure, *i.e.,*

$$\lambda_d(\bigcap_{k>0}\bigcup_{r>0}\bigcap_{n>r} A_{k,n}) = \alpha > 0. \tag{3.1}$$

where $A_{k,n} = \{\underline{x} : \max_{i,j} \rho(T^n(x_i), T^n(x_j)) < 1/k\}$. Take $\varepsilon > 0$ sufficiently small, such that if we denote by Δ_ε the ε-neighbourhood of the diagonal $\Delta = \{\underline{x} : x_i = x_j \text{ for all } 1 \leq i,j \leq d\}$ then $\lambda_d(\Delta_\varepsilon) < \alpha/2$. Fix any $k \in \mathbb{N}$ such that $1/k < \varepsilon$. By (3.1) there is $m > 0$ such that

$$\lambda_d(\bigcup_{r=1}^{m}\bigcap_{n>r} A_{k,n}) > \alpha/2.$$

Denote $A = \bigcup_{r=1}^{m}\bigcap_{n>r} A_{k,n} = \bigcap_{n>m} A_{k,n}$. Note that $T_d^n(A) \subset \Delta_\varepsilon$ for all $n \geq m$ and $\lambda_d(A') > 0$ for $A' = A \setminus \Delta_\varepsilon$. By conservativity, $\lambda(T_d^n(A') \cap A') > 0$ for some $n > m$ which contradicts the definition of A. Therefore the set of asymptotic d-tuples is of measure zero and the lemma follows. □

The following is a simple extension of a well-known fact for LY-pairs (see *e.g.* [4]):

Lemma 3.2. *If X has at least two points and μ is a weakly mixing, fully supported, T-invariant Borel measure, then there is $\delta_d > 0$ such that the δ_2-LY d-tuples have full μ_d-measure for any integer $d > 1$.*

Proof. We start with proximality. Assume by contradiction that there is a set $A \subset X^d$ with $\mu_d(A) > 0$, and $m \in \mathbb{N}$, $\varepsilon > 0$ such that $T_d^n(A) \cap \Delta_\varepsilon = \emptyset$ for every $n \geq m$. Since μ is fully supported, $\mu_d(\Delta_\varepsilon) > 0$. But μ_d is weak mixing, hence the product measure μ_d is weak mixing too, and so there is a subsequence (n_k) such that $\mu_d(T_d^{-n_k}(\Delta_\varepsilon) \cap A) \to \mu_d(\Delta_\varepsilon)\mu_d(A) > 0$. This implies that there is $n > m$ such that $T_d^n(A) \cap \Delta_\varepsilon \neq \emptyset$, contradicting the definition of A. Therefore μ_d-a.e. $\underline{x}$ is proximal along a subsequence.

Since X has at least two points and μ is weakly mixing, X is infinite. In particular, there exists an open set $U \subset X^d$ such that $\overline{U} \cap \Delta_{i,j} = \emptyset$ for every $i \neq j$, where $\Delta_{i,j} := \{\underline{x} \in X^d : x_i = x_j\}$. Take any $0 < \delta < \inf_{\underline{x}\in U} \min_{i\neq j} \rho(x_i, x_j)$. We use the same argument, to show that a.e. $\underline{x}$ visits U infinitely often under action of T_d. Combining the two, we obtain that μ_d-a.e. d-tuple is δ-LY. □

Remark 3.3. *It is clear from the proof that the statement of Lemma 3.2 holds for any $\delta_2 < diam(X)$.*

4. Li-Yorke tuples in the continuous setting

Let T be a continuous map on compact metric space (X,ρ). We say that T is *topologically mixing* if for every pair of open sets $U, V \subset X$, $T^n(U) \cap V \neq \emptyset$ for all n sufficiently large. We say that T is *topologically weak mixing* if $T \times T$ is transitive on $X \times X$.

The following fact is standard and its utility to Li-Yorke chaos dates back at least to works of Iwanik [19].

Lemma 4.1. *Let (X,ρ) be a compact metric space with at least two points and let $T\colon X \to X$ be topologically weakly mixing. For every $d > 1$ there is $\delta_d > 0$ such that the set of δ_d-LY d-tuples is residual.*

Proof. Since X is not a singleton, topological mixing implies that X is an infinite set without isolated points. Fix a sequence of pairwise distinct points $\{a_i\}_{i=1}^{\infty}$ and let $\delta_d = \inf_{1\leq i<j\leq d} \rho(a_i,a_j)/3$. Now, any d-tuple with orbit dense in X^d is δ_d-LY and the set of such tuples is residual by transitivity of T_d. □

Let us recall an important fact which can be derived from works of Kuratowski and Mycielski (see *e.g.* [31]). We recall that $M \subset X$ is called a *Mycielski set* if it is a countable union of Cantor sets.

Theorem 4.2 (Kuratowski-Mycielski)**.** *Let X be a perfect complete metric space, and assume that R_k is a residual subset of X^{n_k}, where $n_k \geq 2$ for each $k \in \mathbb{N}$. Then there exists a Mycielski set M dense in X such that for each $k \in \mathbb{N}$ if points $x_1, \cdots, x_{n_k} \in M$ are pairwise distinct then $(x_1, \cdots, x_{n_k}) \in R_k$.*

The following fact is known, but since the proof is simple, we sketch it for completeness.

Lemma 4.3. *Let (X,ρ) be a compact metric space with at least two points and let $T\colon X \to X$ be topologically weakly mixing. There is a sequence $\{\delta_d\}_{d=2}^{\infty} \subset (0,1)$ and a Mycielski set $M \subset X$ such that for any $d \geq 2$, any d pairwise distinct points $x_1,\dots,x_d \in M$ and any integers $s_1,\dots,s_d \geq 0$ the d-tuple $(T^{s_1}(x_1),\dots,T^{s_d}(x_d))$ is δ_d-LY. Additionally, $\omega(x,T) = X$ for every $x \in M$.*

Proof. Combine the technique from the proof of Lemma 4.1 with Theorem 4.2 and the fact that the set of points with dense orbit is residual in X to obtain a desired Mycielski set. □

Let $\Sigma_2^+ = \{0,1\}^{\mathbb{N}}$ be endowed with the standard prefix metric, and let σ be the left shift on Σ_2^+. The following fact was proved by Moothathu in [32] (which extends the existence of horseshoes argument from Misiurewicz & Szlenk [30] to the C^0 setting ensuring that the map π is really injective everywhere).

Lemma 4.4. *If a map T acting on the unit interval has positive topological entropy, then there exist $n > 0$, a T^n-invariant closed set Λ and a homeomorphism $\pi\colon \Lambda \to \Sigma_2^+$ such that $\pi \circ (T^n|_\Lambda) = \sigma \circ \pi$.*

Theorem 4.5. *Let $T\colon [0,1] \to [0,1]$ be topologically mixing. There exist dense Mycielski sets M_1, M_2, M_3 and numbers $\delta_n > 0$ such that:*

(1) *each n-tuple consisting of $n \geq 2$ distinct points in M_i is δ_n-LY, where $i = 1, 2$,*
(2) *every point in M_1 has dense orbit in $[0,1]$,*
(3) *there exists a minimal Cantor set A such that $\omega(x,T) = A$ for every $x \in M_2$ and $M_2 \cap A$ contains a Cantor set,*
(4) *$M_3 \times M_3$ contains no Li-Yorke pairs.*

Proof. The set M_1 is obtained by a direct application of Theorem 4.3. Constructing the set M_2 requires a little more work.

It is known that every mixing map on the unit interval has positive topological entropy, so by Lemma 4.4 we can find $n > 0$ and a T^n-invariant set Λ where the dynamics is conjugated with the full one-sided shift Σ_2^+. Clearly, we may assume that $\Lambda \subset (0,1)$. Take any minimal weakly mixing subset of Σ_2^+ (*e.g.* one-sided version of the Chacón flow [5]) and let us denote it by A_n. Passing through conjugating homeomorphism, we may assume that $A_n \subset \Lambda$. Clearly it is a Cantor set as a perfect subset of the Cantor set Σ_2^+ (up to conjugating homeomorphism) and also it is not hard to see that $A = \bigcup_{j=0}^{n-1} T^j(A_n)$ is a minimal subsystem of T. Let us apply Theorem 4.3 to the dynamical system $T^n|_{A_n}$, and let numbers $\delta_d > 0$ and a Cantor set $C \subset A_n \subset A$ be such that for any $d > 1$, any d pairwise distinct points $x_1, \ldots, x_d \in C$ and any integers $s_1, \ldots, s_d \in n\mathbb{N} \cup \{0\}$ the tuple $(T^{s_1}(x_1), \ldots, T^{s_d}(x_d))$ is δ_d-LY for T^n (clearly, it is also δ_d-LY for T).

Since C is homeomorphic to C^2 we can find pairwise disjoint Cantor sets $\{C_i\}_{i=0}^{\infty}$ such that $\cup_i C_i \subset C$. There is also $\varepsilon > 0$ such that $C \subset (\varepsilon, 1-\varepsilon)$. Let $\{U_i\}_{i=1}^{\infty}$ be a sequence composed of all open subintervals of $[0,1]$ with rational endpoints. Since T is mixing, for every i there is k_i such that $T^{nk_i}(U_i) \cap [0,\varepsilon] \neq \emptyset$ and $T^{nk_i}(U_i) \cap [1-\varepsilon, 1] \neq \emptyset$. It is known that if an image of a compact set via a continuous map is uncountable then there is a Cantor set on which this map is one-to-one (see *e.g.* [38, Remark 4.3.6]). Hence, for every i there is a Cantor set $D_i \subset U_i$ such that $T^{nk_i}|_{D_i}$ is one-to-one and $T^{nk_i}(D_i) \subset C_i$. Denote $M_2 = C_0 \cup \bigcup_{i=1}^{\infty} D_i$. Since for every $x \in M_2$ there is $j \geq 0$ such that $T^j(x) \in A$ and A is a minimal set, we immediately obtain that $\omega(x,T) = A$ for every $x \in M_2$.

Let us take any tuple $(x_1, \ldots, x_d)$ of pairwise distinct points in M_2. There are numbers $i_1, \ldots, i_d$ such that $x_j \in D_{i_j}$. Denote $z_j = T^{nk_{i_j}}(x_j) \in C_j$ and observe that points z_j are also pairwise distinct. Let $m = \max_j k_{i_j}$ and denote $s_j = m - k_{i_j}$. Now, it is enough to note that

$$\begin{aligned}\rho(T^{nm}(x_p), T^{nm}(x_q)) &= \rho(T^{n(s_p+k_{i_p})}(x_p), T^{n(s_q+k_{i_q})}(x_q)) \\ &= \rho(T^{ns_p}(z_p), T^{ns_q}(z_q))\end{aligned}$$

The construction of M_2 completed by the definition of the set C.

To construct M_3, let us first recall that every Sturmian minimal system does not contain Li-Yorke pairs [6, Example 3.15] (it is a so-called almost distal system) and Σ_2^+ contains an uncountable family of pairwise disjoint Sturmian systems [22, Proposition 4.44]. Repeating conjugacy argument from the previous step we may view Sturmian minimal subshift M (which is a Cantor set) as a subset of $[0,1]$. Note that since M is an infinite minimal system for T^n without Li-Yorke pairs, set $\hat{M} = \bigcup_{j=0}^{n-1} T^j(M)$ is also minimal, and cannot contain Li-Yorke pairs (sets $T^j(M), T^i(M)$ for $i \neq j$ are either disjoint of equal).

Proceed the same way as in the construction of M_2 to obtain a dense Mycielski set M_3 such that every point $x \in M_3$ is eventually transformed into a point $z_x \in M$.

Now, take any $x, y \in M_3$ and k sufficiently large, so that $T^k(x), T^k(y) \in \hat{M}$. If $T^k(x) \neq T^k(y)$ then (x, y) is not Li-Yorke pair by the definition of $\hat{M}$ and if $T^k(x) = T^k(y)$ then (x, y) is not Li-Yorke pair neither. $\square$

As a direct consequence of theorem by Oxtoby & Ulam (see [33, Thm. 9]) we obtain that if $B \subset [0, 1]$ is a dense Mycielski set then there exists a homeomorphism $\phi\colon [0, 1] \to [0, 1]$ such that $\phi(B)$ has full Lebesgue measure.

In fact, if $A, B \subset [0, 1]$ are dense in $[0, 1]$ and either of them is the union of pairwise disjoint Cantor sets, then there is an increasing homeomorphism $\varphi\colon [0, 1] \to [0, 1]$ such that $\varphi(A) \subseteq B$.

Note that the dense Mycielski sets M_1, M_2, M_3 in Theorem 4.5 are pairwise disjoint. In particular, if one of them has full Lebesgue measure, the other have measure zero. As a direct consequence of Theorem 4.5 we obtain Theorem C.

The above theorem shows that in interval dynamics Cantor attractor and no Cantor attractor cases always coexist, but their "physical" visibility depends on the special structure of the map. In fact, using the sets M_1, M_2, M_3 from Theorem 4.5, we can distribute Lebesgue measure in any proportion between Li-Yorke pairs made of points with dense orbits, Li-Yorke pairs with a Cantor attractor, or without Li-Yorke pairs.

Remark 4.6. *If $f \in C^3_{nf}(I)$ then by Theorem E in* [42], *any minimal set must have zero Lebesgue measure. Hence, the situation described in Theorem C(4) can never occur for these maps.*

5. The Manneville-Pomeau map

5.1. **Inducing.** Suppose that we are in the case when every non-singular Borel measure $\mu \ll \lambda$ for T is σ-finite and infinite. Then Theorem 2.8 indicates that λ is not full, so the results of Section 3 are not applicable in this case. Another approach to address the question of Li-Yorke d-tuples is via inducing. The idea is to choose an appropriate subset $Y \subset X$ and consider the (first return) induced map (Y, F), such that Y can be decomposed in countably many subsets $\mathcal{Z} := \{Y_j\}_{j\in\mathbb{N}}$ with $\lambda(Y \setminus \cup_j Y_j) = 0$ and such that $F(Y_j) = T^{\tau_j}(Y_j) = Y$ where the first return time $\tau(x) = \min\{n \geq 1 : T^n(x) \in Y\}$ is constant τ_j of Y_j. Assume that the *distortion* of the Jacobian is bounded uniformly in the iterate n *i.e.,*

$$\sup_{n\in\mathbb{N}} \sup_{Z\in\bigvee_{i=0}^{n-1} F^{-i}\mathcal{Z}} \sup_{x,y\in Z} \frac{J_{F^n}(x)}{J_{F^n}(y)} < \infty. \tag{5.1}$$

We take the Jacobian w.r.t. Lebesgue measure λ: For any j, the push-forward measure $\lambda(F(A))$ for $A \subset Y_j$ is well-defined and absolutely continuous with respect to λ. Hence, $J_F = \frac{d\mu\circ F}{d\lambda}$ is well-defined for λ-a.e. $x \in Y_j$. Similarly, we can define J_{F^n} on any set $Z \in \bigvee_{i=0}^{n-1} F^{-i}\mathcal{Z}$. By the definition $Y = \cup_j Y_j$ (mod 0) and sets Y_j are in practice intervals, hence we may view each Jacobian J_{F^n} as a function defined on Y.

In the case of bounded distortion (Y, F) preserves an absolutely continuous invariant probability measure. Let us call this measure ν; its density $\frac{d\nu}{d\lambda}$ is bounded and bounded away from 0. It projects to a T-invariant measure

$$\mu(A) = \sum_n \sum_{i=0}^{n-1} \nu(T^{-i}(A) \cap \{y \in Y : \tau(y) = n\}), \tag{5.2}$$

which can be normalised if the normalising constant $\int_Y \tau \, d\nu < \infty$, but if not, then μ is σ-finite, see *e.g.* [7, Chapter 6]. The measure ν is ergodic and exact, and and these properties carry over to (X, T, μ) (for exactness to carry over, we need the additional condition that $gcd(\tau) = 1$). In fact, $\mu|_Y$ and $\nu|_Y$ differ by a fixed constant, regardless whether $\int_Y \tau \, d\nu < \infty$ or not, because $F : Y \to Y$ is the first return map, see (5.2).

If the tail $\nu(\{y : \tau(y) > n\})$ is sufficiently heavy (and regular), then the probability of two independently chosen initial points to return to Y at the same time infinitely often under iteration of T can be zero.

Let us denote

$$u_n = \lambda(\{y \in Y : T^n(y) \in Y\})$$

and observe that

$$\lambda_d(\{y \in Y^d : T_d^n(y) \in Y^d\}) = u_n^d.$$

Since $\frac{d\mu}{d\lambda}$ is bounded and bounded away from zero on Y, we can freely interchange μ and λ in these formulas. Therefore, if

$$\sum_n u_n^d < \infty, \tag{5.3}$$

then the Borel-Cantelli Lemma gives

$$\begin{aligned} 0 &= \mu_d(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{y \in Y^d : T_d^n(y) \in Y^d\}) \\ &= \mu_d(\{x \in Y^d : T_d^n(x) \in Y^d \text{ infinitely often}\}). \end{aligned}$$

The following recurrence lemma seems to be standard (see [1, Proposition 1.2.2]). However we could not find exact reference in the literature and hence decided to provide a proof for completeness.

Lemma 5.1. *Let μ be a σ-finite, non-singular and T-invariant Borel measure. If there exists $Y \in \mathcal{B}$ such that $0 < \mu(Y) < \infty$ and*

$$\sum_{n \geq 0} \mathcal{L}^n 1_Y = \infty \qquad \mu\text{-a.e. on } Y \tag{5.4}$$

(where $\mathcal{L}$ is the Perron-Frobenius operator), then Y is a recurrent set in the sense that

$$Y \subseteq \bigcup_{n \geq 1} T^{-n}(Y) \pmod{\mu}. \tag{5.5}$$

Moreover, if μ is ergodic and T-invariant, then μ is conservative.

Proof. First, we need to show that the set $A := Y \setminus \bigcup_{n \geq 1} T^{-n}(Y)$ of points which never return to Y has measure zero. We must have $A \cap T^{-j}(A) = \emptyset$ for every $j \geq 1$ because $A \subseteq Y$. This immediately implies that A is a wandering set for T, that is, the preimages $T^{-n}(A)$, $n \geq 0$, are pairwise disjoint. Directly from the definition of Perron-Frobenius operator $\mathcal{L}$ we have $\int_A \mathcal{L}^n 1_Y \, d\mu = \mu(Y \cap T^{-n}(A))$, hence

$$\int_A \sum_{n \geq 0} \mathcal{L}^n 1_Y \, d\mu = \sum_{n \geq 0} \mu(Y \cap T^{-n}(A)) \leq \mu(Y) < \infty,$$

By assumption (5.4) we obtain that $\mu(A) = 0$ which ends the proof of (5.5).

As a consequence of (5.5) we see that the set $E := \bigcup_{n\geq 0} T^{-n}(Y)$ satisfies $T^{-1}(E) = E \pmod{\mu}$. Since $Y \subset E$, we have $\mu(E) > 0$. If T is ergodic, then we can conclude that $X = \bigcup_{n\geq 0} T^{-n}(Y) \pmod{\mu}$. This, since T is measure preserving transformation, implies that assumptions of Maharam's Recurrence Theorem are satisfied (see *e.g.* [1, Thm 1.1.7]) which thus ensures that T is conservative. □

Proposition 5.2. *Let us assume that T preserves a non-singular exact Borel probability measure μ equivalent to λ. If there exists a set $Y \subset X$ recurrent in the sense of* (5.5) *such that $\mu(Y) > 0$ and the first return map (Y, F) has only onto branches with bounded distortion (in the sense of* (5.1)*) then the following conditions are equivalent for every integer $d \geq 1$:*

(1) $\sum_n u_n^d = \infty$, where $u_n = \lambda(\{y \in Y : T^n(y) \in Y\})$,
(2) d-fold product measure λ_d is ergodic and conservative.

Proof. To prove (2) $\Longrightarrow$ (1) let us first observe, that if λ_d is conservative then

$$\begin{aligned} 0 < \lambda_d(Y^d) &= \lambda_d(\{x \in Y^d : T_d^n(x) \in Y^d \text{ infinitely often}\}) \\ &= \lambda_d(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{y \in Y^d : T_d^n(y) \in Y^d\}) \end{aligned}$$

and hence by the Borel-Cantelli Lemma (see (5.3)) we must have $\sum_n u_n^d = \infty$.

To prove (1) $\Longrightarrow$ (2), denote by $\mathcal{L}$ and $\mathcal{P}$ the Perron-Frobenius operator for T and F, respectively, both w.r.t. λ. Then for every $n > 0$ and a.e. $x \in Y$ we have

$$\mathcal{P}^n f(x) = \sum_{y \in F^{-n}(x)} \frac{f(y)}{J_{F^n}(y)}, \tag{5.6}$$

where J_{F^n} is the Jacobian w.r.t. λ. The bounded distortion of J_{F^n} applied to (5.6) allows us to verify that there is a constant $\kappa > 0$ such that for every measurable $A \subset X$ and $n > 0$ we have

$$\inf_{x\in A} \sum_{k=0}^{n} \mathcal{P}^k 1_Y(x) \geq \kappa \cdot \sup_{x\in A} \sum_{k=0}^{n} \mathcal{P}^k 1_Y(x).$$

A similar estimate for the d-fold product system with Perron-Frobenius operator $\mathcal{P}_d$,

$$\inf_{\underline{x}\in A} \sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x}) \geq \kappa^d \cdot \sup_{\underline{x}\in A} \sum_{k=0}^{n} \mathcal{P}_d^k 1_{Y^d}(\underline{x}).$$

Hence, if there is a set $A \subset Y$ of positive measure such that $\sum_{k=0}^n \mathcal{P}_d^k 1_{Y^d}(\underline{x})$ is uniformly bounded for all $n \geq 0$ and $\underline{x} \in A$, then $\sum_{k=0}^n \mathcal{P}_d^k 1_{Y^d}(\underline{x})$ is uniformly bounded for all $n \geq 0$ and λ-a.e. $\underline{y} \in Y$.

Observe that for any Borel set $A \subset Y^d$ we have

$$\sum_{k=0}^{n} \int_A \mathcal{L}_d^k 1_{Y^d}\, d\lambda_d \leq \sum_{k=0}^{n} \int_A \mathcal{P}_d^k 1_{Y^d}\, d\lambda_d \leq \sum_{k=0}^{\infty} \int_A \mathcal{L}_d^k 1_{Y^d}\, d\lambda_d.$$

If there exists a set $A \subset Y^d$ with $\lambda_d(A) > 0$ and $M > 0$ such that $\sum_{k\geq 0} \mathcal{L}_d^k 1_{Y^d}(\underline{x}) < M$ for every $\underline{x} \in A$ then there is $c > 0$ such that

$$M \geq \sum_{k\geq 0} \int_A \mathcal{P}_d^k 1_{Y^d}\, d\lambda_d \geq c \sum_{k\geq 0} \int_{Y^d} \mathcal{P}_d^k 1_{Y^d}\, d\lambda_d \geq c \sum_{k\geq 0} \int_{Y^d} \mathcal{L}_d^k 1_{Y^d}\, d\lambda_d.$$

Consequently, divergence of $\sum_{k\geq 0} u_k^d = \sum_{k\geq 0} \int_{Y^d} \mathcal{L}_d^k 1_{Y^d}\, d\lambda_d$ implies that

$$\sum_{k\geq 0} \mathcal{L}_d^k 1_{Y^d} = \infty \qquad \lambda\text{-a.e. on } Y^d.$$

By Lemma 2.4, d-dimensional direct product measure λ_d, is ergodic with respect to T_d for every $d \geq 1$. Recall that μ_d is a T_d-invariant, non-singular and σ-finite measure. But λ_d is ergodic and μ_d is equivalent to λ_d, hence μ_d is ergodic. Applying Lemma 5.1 to μ_d and T_d, we obtain that μ_d is conservative, and using once again equivalence of μ_d and λ_d we conclude that λ_d is conservative, which completes the proof. □

5.2. **Manneville-Pomeau maps.** The classical example from interval dynamics where the u_n can be computed is the Manneville-Pomeau family, where Lebesgue measure λ will be our reference measure for μ. These are interval maps with a neutral fixed point and the inducing is with respect to a set Y bounded away from this fixed point. For us, it is convenient to use the family $T_\alpha : [0,1] \to [0,1]$ defined by

$$T_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } x \in [\frac{1}{2}, 1] =: Y. \end{cases}$$

It has an indifferent fixed point at 0, and the first return map $F : Y \to Y$ is uniformly expanding with bounded distortion (uniformly in all iterates, in the sense of (5.1), see *e.g.* [28]), $F : Y \to Y$ preserves an ergodic measure which pull back to an ergodic and conservative T_α-invariant measure μ, see Theorem 1 and Corollary 2 in [40].

Remark 5.3. *More precise estimates on the invariant density h were given by Thaler* [40, Corollary 1]*), who showed that $h(x)x^\alpha$ is bounded and bounded away from* 0*. In addition (see* [27, Lemma 2.3]*) for $\alpha \in (0,1)$ the density $h = \frac{d\mu}{d\lambda}$ is Lipschitz outside a neighbourhood of* 0 *and μ is mixing (see* e.g. [28, Theorem 7]*).*

If $y_0 = \frac{1}{2}$ and y_{k+1} is the unique point in $T_\alpha^{-1}(y_k) \cap [0, y_n]$, then $y_n \searrow 0$ such that

$$y_n \sim n^{-\beta} \qquad \text{for } \beta = \frac{1}{\alpha},$$

where $a_n \sim b_n$ stands for $\lim_n a_n/b_n \in (0,\infty)$, see [14]. If $y'_{n+1} \in [\frac{1}{2}, 1]$ is the other preimage of y_n, then $[\frac{1}{2}, y'_{n+1}) = \{y \in Y : \tau(y) \geq n+2\}$. By Remark 5.3, $\mu(\{y \in Y : \tau(y) \geq n\}) \sim |y'_{n-2} - \frac{1}{2}| = \frac{1}{2} y_{n-1} \sim n^{-\beta}$ and therefore

$$\begin{aligned} \int \tau d\mu &= \sum_n n\mu(\{y \in Y : \tau(y) = n\}) \\ &= \sum_n \mu(\{y \in Y : \tau(y) \geq n\}) \begin{cases} < \infty & \text{if } \alpha \in (0,1); \\ = \infty & \text{if } \alpha \geq 1. \end{cases} \end{aligned} \tag{5.7}$$

Theorem 5.4. *Assume that $\alpha > 1$ and write $\beta = 1/\alpha \in (0,1)$. Then $u_n \sim n^{\beta-1}$. In particular, $\sum_n u_n^d = \infty$ if and only if $\alpha \leq \frac{d}{d-1}$.*

Proof. From the above considerations, we have for the Manneville-Pomeau map that

$$\mu(x \in Y : \tau(x) = n) \sim \lambda([y_n, y_{n-1}]) \sim (n-1)^{-\beta} - n^{-\beta} \sim n^{-(1+\beta)}.$$

Then the estimate on u_n is a special case of the results of Gouëzel [18], partly correcting results from Doney [17] (see [18, Section 1.3]). More precisely, [18, Proposition 1.7] applied to $u = v = 1_Y$ gives $u_n = \int_Y u \cdot v \circ T^n d\mu \sim n^{\beta-1}$. Now it follows immediately that $\sum_n u_n^d = \infty$ if and only if $\alpha \leq \frac{d}{d-1}$. □

Proof of Theorem B. Recall that by Remark 5.3 for $\alpha < 1$ the map T_α preserves a mixing absolutely continuous probability measure, and hence the result follows by Lemma 3.2 (see also Remark 3.3).

If $\alpha \geq 1$, then there is an infinite σ-finite measure $\mu \sim \lambda$ (by Remark 5.3, the density of μ is bounded and bounded away from 0). More precisely, using Remark 5.3 again, $\mu([y_k, 1]) < \infty$ for fixed k, but $\mu([y_l, y_k]) \to \infty$ as $l \to \infty$. This means that given a neighbourhood U of the fixed point 0, Lebesgue-a.e. point x spends almost every iterate in U (*i.e.,* $N(x, U) := \{n \geq 0 : T^n(x) \in U\}$. has density 1 for ν-a.e. x). Indeed, for $U = [0, y_k)$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \#\{0 \leq i < n : T^i(x) \in [y_k, 1]\} &\leq \lim_{n\to\infty} \frac{\#\{0 \leq i < n : T^i(x) \in [y_k, 1]\}}{\#\{0 \leq i < n : T^i(x) \in [y_l, 1]\}} \\ &= \frac{\mu([y_k, 1])}{\mu([y_l, 1])} \to 0 \text{ as } l \to \infty, \end{aligned}$$

by the Ratio Ergodic Theorem (see *e.g.* [1, Theorem 2.2.5.]). It follows that λ_d-a.e. d-tuple is proximal along a subsequence.

Next, we claim that every pair is LY. This follows easily from the fact that T_α is expanding away from 0. More precisely, for every two $x, y \in [0, 1]$ which are not eventually mapped to one another, $\rho(T_\alpha^i(x), T_\alpha^i(y)) \leq \frac{1}{3}$ implies that $\rho(T_\alpha^{i+1}(x), T_\alpha^{i+1}(y)) > \rho(T_\alpha^i(x), T_\alpha^i(y))$. Therefore (x, y) is $\frac{1}{3}$-separated.

Now let $d \geq 3$, $1 \leq \alpha < \frac{d-1}{d-2}$ and choose $\delta < 1/2(d-2)$. By Theorem 5.4 applied to $d-1$ coordinates, $\sum_n u_n^{d-1} = \infty$ and hence, $([0, 1]^{d-1}, \lambda_{d-1}, T_{d-1})$ is conservative by Proposition 5.2. Therefore, taking $A \subset Y^{d-1}$ with $\mu_{d-1}(A) > 0$, for μ_{d-1}-a.e. $\underline{x}$, there is a sequence (t_n) such that $(T_{d-1})^{t_n}(x_i) \in A$ for all $n \in \mathbb{N}$. The set

$$A = [\frac{1}{2}, \frac{1}{2} + \eta] \times [\frac{1}{2} + \frac{1}{2(d-2)}, \frac{1}{2} + \frac{1}{2(d-2)} + \eta] \times \cdots \times [1 - \eta, 1]$$

has positive λ_{d-1}-measure, and clearly points $\underline{y} \in A$ are $\frac{1}{2(d-2)} - 2\eta$-separated $d-1$-tuples. Choosing η so small that $\frac{1}{2(d-2)} - 2\eta > \delta$ then shows that the $d-1$-tuple $\underline{x}$ is δ-separated along a subsequence (t_n).

We claim that the remaining coordinate x_d is close to 0, or more precisely: for λ-a.e. x_d we have $T_\alpha^{t_n}(x_d) < \frac{1}{2} - \delta$ infinitely often. Indeed, take $U_N := \{x_d \in [0, 1] : T_\alpha^{t_n}(x_d) \geq \frac{1}{2} - \delta$ for all $n \geq N\}$ and let $x_d^* \in U_N$ be arbitrary. Let $H_n \ni x_d^*$ be the maximal interval such that $T_\alpha^{t_n}(H_n) = [\frac{1}{2}(\frac{1}{2} - \delta), 1]$ and $H_n' \subset H_n$ is such that $T_\alpha^{t_n}(H_n') = [\frac{1}{2}(\frac{1}{2} - \delta), \frac{1}{2} - \delta)$. Since t_n is increasing, we clearly have $\lim_{n\to\infty} |H_n| = 0$. The maps $T_\alpha^{t_n} : H_n \to [\frac{1}{2}(\frac{1}{2} - \delta), 1]$ have uniformly bounded distortion (this is proved in virtually the same way as the distortion bound for the branches of F^k is proven, cf. [28]). Therefore there is $K > 0$ such that $|H_n'| > |H_n|/K$ for all $n \in \mathbb{N}$. Since $U_N \cap H_n' = \emptyset$ for $n \geq N$, it follows that x_d^* cannot be a Lebesgue density point of U_N. This means that $\lambda(U_N) = 0$ and hence $\lambda(\cup_N U_N) = 0$ as well, proving the claim. Using Fubini's Theorem, we conclude that λ_d-a.e. d-tuple is indeed δ-separated along a subsequence.

Finally, if $\alpha > \frac{d-1}{d-2}$, typical $d-1$-tuples $\underline{x} = (x_1, \dots x_{d-1})$ visit Y simultaneously only finitely often. Now let $\varepsilon = y_k$ and assume by contradiction that $\underline{x}$ visits $[\varepsilon, 1]^{d-1}$ infinitely often for a set of $\underline{x}$ of positive λ_{d-1}-measure. Hence, there are integers $a_1, \dots a_{d-1} \in \{0, 1, \dots, k\}$ such that $T^{n+a_1}(x_1), \dots, T^{n+a_{d-1}}(x_{d-1}) \in Y$ for infinitely many n, still for a set U of positive λ_{d-1}-measure. Take $a = \max_i a_i$ and for each $\underline{x} \in U$ take a $d-1$-tuple $\underline{y}$ with coordinates $y_i \in T^{a_i - a}(x_i) \cap Y$. Since T is non-singular (and the Cartesian product $\prod_{i=1}^{d-1} T^{a-a_i}$ is non-singular too), we have $\lambda_{d-1}(\{\underline{y} : \underline{x} \in U\}) > 0$. Now for every $\underline{y}$, there is an infinite sequence $(n_k)_{k\in\mathbb{N}}$ (depending on $\underline{y}$ but not on the index $i = 1, \dots, d-1$), such that $T^{n_k}(y_i) = T^{n_k + a_i - a}(x_i) \in Y$ for each i and k. This contradicts the first statement of this paragraph.

Coming back to a typical d-tuple $(x_1, \dots, x_d)$, the above argument shows that no matter how we select a $d-1$-tuple $\underline{x}$ from it, for all sufficiently large n, at least one coordinate $T^n(x_i) \in [0, \varepsilon)$. Therefore at least two coordinates of the d-tuple belong to $[0, \varepsilon)$. Since $\varepsilon > 0$ can be taken arbitrary small, the proof is complete. □

Remark 5.5. *We can replace the right branch of the Manneville-Pomeau map by* $2(1-x)$ *if a continuous map is preferred. The dynamical properties that we are concerned with remain the same. The same is true, if we view Manneville-Pomeau map as a continuous map on the unit circle.*

Remark 5.6. *One can increase the number of neutral fixed point,* e.g. *define the map*

$$T_{\alpha,\beta}(x) = \begin{cases} x + 2^\alpha x^{1+\alpha} & \text{if } x \in [0, \frac{1}{2}), \\ x - 2^\beta (1-x)^{1+\beta} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

see Figure 2, and consider the Li-Yorke behavior of tuples for this map. If $\alpha > \beta > 1$, *then neutral fixed point* 0 *dominates, and we expect the same behaviour as in Theorem B. For the case* $\alpha = \beta$, *we expect that* λ_3*-a.e.* 3*-tuple is Li-Yorke (where for typical triples* (x_1, x_2, x_3), *there are infinitely many* n *with* $T^n(x_1) \approx 0$, $T^n(x_2) \approx 1$ *and* $T^n(x_3) \in [\frac{1}{3}, \frac{2}{3}]$, *as well as are infinitely many* m *with* $T^m(x_1) \approx T^m(x_2) \approx T^m(x_3) \approx 0$. *Conjecturally, for* $d \geq 4$, *typical* d*-tuples are Li-Yorke if and only if* $\alpha \leq \frac{d-2}{d-3}$.

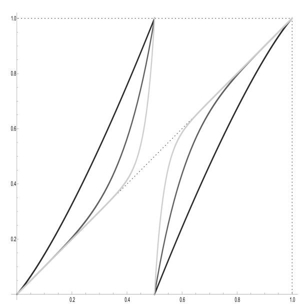

FIGURE 2. Graph of $T_{\alpha,\beta}$ for $\alpha = \beta = \frac{1}{2}, 4, 15$.

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