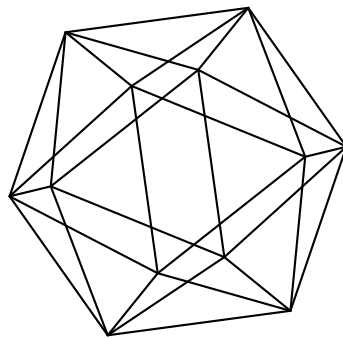


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Three faces of  $R_\infty$

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# THREE FACES OF $R_\infty$

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ABSTRACT. A (countable discrete) group  $G$  has the property  $R_\infty$ , if for any its automorphism  $\phi$  the number of twisted conjugacy classes is infinite. We study the following three aspects of the  $R_\infty$  property: relation to nonabelian cohomology, relation to isogredience classes, and relation to representation theory.

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## 1. INTRODUCTION

The following two interrelated problems are among the principal ones in the theory of twisted conjugacy (Reidemeister) classes in infinite discrete groups. The first one is the 20-years-old conjecture on existence of an appropriate twisted Burnside-Frobenius theory (TBFT), i.e. identification of the number  $R(\phi)$  of Reidemeister classes and the number of fixed points of the induced homeomorphism  $\hat{\phi}$  on an appropriate dual object (supposing  $R(\phi) < \infty$ ). The second one is the problem to outline the class of  $R_\infty$  groups (that is  $R(\phi) = \infty$  for any  $\phi$ ).

The interest in twisted conjugacy relations has its origins, in particular, in the Nielsen-Reidemeister fixed point theory (see, e.g. [30, 31, 7]), in Selberg theory (see, e.g. [45, 1]), Algebraic Geometry (see, e.g. [27]), and Galois cohomology (see, e.g. [44]). In representation theory twisted conjugacy probably occurs first in [21] (see, e.g. [46, 41]).

The TBFT was proved consequently for finite, Abelian [9], almost Abelian [11], and almost polycyclic groups [18, 12] (see also [17]).

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The problem of determining, which classes of groups have  $R_\infty$  property, is an area of active research initiated in [9]. It was shown by various authors that the following groups have the  $R_\infty$ -property:

- (1) non-elementary Gromov hyperbolic groups [8, 35]; relatively hyperbolic groups [13];
- (2) Baumslag-Solitar groups  $BS(m, n)$  except for  $BS(1, 1)$  [14], generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic [34]; the solvable generalization  $\Gamma$  of  $BS(1, n)$  given by the short exact sequence  $1 \rightarrow \mathbb{Z}[\frac{1}{n}] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$ , as well as any group quasi-isometric to  $\Gamma$  [47];
- (3) a wide class of saturated weakly branch groups (including the Grigorchuk group [26] and the Gupta-Sidki group [28]) [10], Thompson's group  $F$  [3]; generalized Thompson's groups  $F_{n,0}$  and their finite direct products [23];
- (4) symplectic groups  $Sp(2n, \mathbb{Z})$ , the mapping class groups  $Mod_S$  of a compact oriented surface  $S$  with genus  $g$  and  $p$  boundary components,  $3g + p - 4 > 0$ , and the full braid groups  $B_n(S)$  on  $n > 3$  strings of a compact surface  $S$  in the cases where  $S$  is either the compact disk  $D$ , or the sphere  $S^2$  [15], some classes of Artin groups of infinite type [32];
- (5) extensions of  $SL(n, \mathbb{Z})$ ,  $PSL(n, \mathbb{Z})$ ,  $GL(n, \mathbb{Z})$ ,  $PGL(n, \mathbb{Z})$ ,  $Sp(2n, \mathbb{Z})$ ,  $PSp(2n, \mathbb{Z})$ ,  $n > 1$ , by a countable abelian group, and normal subgroup of  $SL(n, \mathbb{Z})$ ,  $n > 2$ , not contained in the center [38];
- (6)  $GL(n, K)$  and  $SL(n, K)$  if  $n > 2$  and  $K$  is an infinite integral domain with trivial group of automorphisms, or  $K$  is an integral domain, which has a zero characteristic and for which  $\text{Aut}(K)$  is torsion [40];
- (7) irreducible lattice in a connected semi simple Lie group  $G$  with finite center and real rank at least 2 [39];
- (8) some metabelian groups of the form  $\mathbb{Q}^n \rtimes \mathbb{Z}$  and  $\mathbb{Z}[1/p]^n \rtimes \mathbb{Z}$  [16]; lamplighter groups  $\mathbb{Z}_n \wr \mathbb{Z}$  if and only if  $2|n$  or  $3|n$  [24]; free nilpotent group  $N_{rc}$  of rank  $r = 2$  and class  $c \geq 9$  [25],  $N_{rc}$  of rank  $r = 2$  or  $r = 3$  and class  $c \geq 4r$ , or rank  $r \geq 4$  and class  $c \geq 2r$ , any group  $N_{2c}$  for  $c \geq 4$ , every free solvable group  $S_{2t}$  of rank 2 and class  $t \geq 2$  (in particular the free metabelian group  $M_2 = S_{22}$  of rank 2), any free solvable group  $S_{rt}$  of rank  $r \geq 2$  and class  $t$  big enough [43]; some crystallographic groups [6, 37].

In the present paper we study the following three aspects of the  $R_\infty$  property: relation to nonabelian cohomology, relation to isogredience classes, and relation to representation theory.

In Section 2 we establish a natural bijective correspondence between the sets of Reidemeister classes and the nonabelian cohomology  $H^1(\mathbb{Z}, G)$ . As an application we obtain a new proof of 8-term exact sequence for fixed points and Reidemeister classes of an extension.

In Section 3 we introduce and investigate a new class of groups:  $S_\infty$  groups. These are the groups with infinite set of isogredience classes for any outer automorphism. A relation of  $S_\infty$  and  $R_\infty$  is established in Theorem 3.4.

In Section 4 we describe the connection of (ir)rational (finite in the terminology of [36]) representations with twisted conjugacy classes. In particular we obtain a sufficient condition for  $R_\infty$  property in Theorem 4.4.

We finish the paper with a couple of conjectures.

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## 2. NONABELIAN COHOMOLOGY

**2.1. Main identifications.** Suppose, a discrete group  $\Gamma$  acts on a discrete group  $G$ . We denote the action  $\sigma$  of  $s \in \Gamma$  on  $g \in G$  by  $\sigma(s)(g) := {}^s g$ . A map  $g : \Gamma \rightarrow G$ ,  $g : s \mapsto g_s$ , is called a *cocycle*, if

$$g_{st} = g_s \cdot {}^s g_t.$$

Two cocycles  $g$  and  $g'$  are *cohomological* if there exists  $b \in G$  such that

$$(1) \quad g_s = b^{-1} \cdot g'_s b.$$

The corresponding quotient set is  $H^1(\Gamma, G) = H^1(\Gamma, G, \sigma)$ .

Consider  $\Gamma = \mathbb{Z}$ , and denote  $\varphi := \sigma(1)$ . Then any cocycle  $g$  is determined by  $g_1 = g(1)$  by induction. Indeed,

$$\begin{aligned} g_2 &= g_1 \cdot {}^1 g_1 = g_1 \cdot \varphi(g_1), & g_3 &= g_2 \cdot {}^2 g_1 = g_2 \cdot \varphi^2(g_1) = g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1), \\ g_4 &= g_3 \cdot {}^3 g_1 = g_3 \cdot \varphi^3(g_1) = g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \cdot \varphi^3(g_1), \dots \end{aligned}$$

and for negative  $s$ :

$$\begin{aligned} g_1 &= g_{-m} \cdot {}^{-m} g_{m+1} = g_{-m} \varphi^{-m}(g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^m(g_1)) = g_{-m} \cdot \varphi^{-m}(g_1) \cdot \varphi^{-m+1}(g_1) \dots g_1, \\ g_{-m} &= (\varphi^{-m}(g_1) \cdot \varphi^{-m+1}(g_1) \dots \varphi^{-1}(g_1))^{-1}. \end{aligned}$$

If we define  $g : \mathbb{Z} \rightarrow G$  by these formulas, we obtain a cocycle: for  $m > n > 0$  we have

$$\begin{aligned} g_{m+n} &= g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{m+n-1}(g_1), \\ g_m \cdot {}^m g_n &= g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{m-1}(g_1) \cdot \varphi^m(g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{n-1}(g_1)) = g_{m+n}, \\ g_{m-n} &= g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{m-n-1}(g_1), \\ g_m \cdot {}^m g_{-n} &= g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{m-1}(g_1) \varphi^m(\varphi^{-1}(g_1^{-1}) \cdot \varphi^{-2}(g_1^{-1}) \dots \varphi^{-n}(g_1^{-1})) = g_{m-n}, \end{aligned}$$

and similarly for other cases. Thus, we have a bijection between cocycles and elements of  $G$ .

Suppose, the relation (1) keeps for  $s = 1$ . Then for  $s > 0$

$$\begin{aligned} g_s &= g_1 \cdot \varphi(g_1) \cdot \varphi^2(g_1) \dots \varphi^{s-1}(g_1) = \\ &= b^{-1} g'_1 \varphi(b) \cdot \varphi(b^{-1} g'_1 \varphi(b)) \dots \varphi^{s-1}(b^{-1} g'_1 \varphi(b)) = b^{-1} g'_s \varphi^s(b). \end{aligned}$$

Similarly, for  $s < 0$ . Thus this cocycles are cohomological. But the equality  $g_1 = b^{-1} g'_1 \varphi(b)$  means that  $g_1 \sim_\varphi g'_1$ . Thus we have proved the following statement:

**Theorem 2.1.** *Reidemeister classes of  $\varphi$  are in bijective correspondence with elements of  $H^1(\mathbb{Z}, G, \sigma)$ , in particular,  $R(\varphi) = \#H^1(\mathbb{Z}, G, \sigma)$ .*

Now we will indicate one more parallel with the study of Reidemeister classes. One of the tools which is used for this is a natural identification of the Reidemeister classes of  $\phi : G \rightarrow G$  with that ordinary conjugacy classes of the group  $\Gamma := G \rtimes_\phi \mathbb{Z}$ , which are in the 1-coset  $G \rtimes_\phi \{1\} \subset \Gamma$ . This identification was successfully used in [12, 17] in the (second) proof of TBFT for almost polycyclic groups.

For nonabelian cohomology the identification exists even in a more general situation, more or less known to specialists. More precisely, consider the semidirect product  $G \rtimes_\sigma \Gamma$  with the commutation relation

$$(2) \quad tat^{-1} = \sigma(t)a = {}^t a.$$

Denote by  $\pi$  the natural (well-defined) projection

$$\pi : G \rtimes_\sigma \Gamma \rightarrow \Gamma.$$

A map  $a : \Gamma \rightarrow G$  can be identified with its graph

$$a_s s \in G \rtimes_\sigma \Gamma, \quad s \in \Gamma.$$

If this map is a cocycle, then we have

$$a_{st}st = a_s^s a_t st = a_s s a_t s^{-1} st = a_s s a_t t,$$

i.e. its graph is a homomorphism  $\gamma_a : \Gamma \rightarrow G \rtimes_\sigma \Gamma$ . Conversely, let  $\gamma : G \rtimes_\sigma \Gamma$  be a homomorphism such that  $\pi \circ \gamma = \text{Id}_\Gamma$ , i.e. “of graph type”. Thus, it has the form  $s \mapsto \tilde{\gamma}_s s$  for some  $\tilde{\gamma} : \Gamma \rightarrow G$ . Then  $\tilde{\gamma}$  is a cocycle. Indeed,  $\tilde{\gamma}_{st}st = \tilde{\gamma}_s s \tilde{\gamma}_t t$ . Then

$$\tilde{\gamma}_{st} = \tilde{\gamma}_s s \tilde{\gamma}_t s^{-1} = \tilde{\gamma}_s^s \tilde{\gamma}_t.$$

We have proved the following statement (well known to specialists, we guess).

**Lemma 2.2.** *Cocycles of  $H^1(\Gamma, G, \sigma)$  can be identified with homomorphisms  $\gamma : \Gamma \rightarrow G \rtimes_\sigma \Gamma$  such that  $\pi \circ \gamma = \text{Id}_\Gamma$ .*

**Remark 2.3.** These homomorphisms are completely determined by their graphs — some specific subsets of  $G \rtimes_\sigma \Gamma$ .

**Lemma 2.4.** *Two cocycles are cohomologous if and only if the corresponding homomorphisms are conjugate by an element of  $G$ .*

*Proof.* Indeed,  $a'_s = b^{-1} a_s^s b$  for any  $s$  if and only if

$$a'_s s = b^{-1} a_s^s b s = b^{-1} a_s s b s^{-1} s = b^{-1} (a_s s) b.$$

□

**Definition 2.5.** Let us call the *support* of a class  $\alpha \in H^1(\Gamma, G, \sigma)$  a union of corresponding graphs of its cocycles in  $G \rtimes_\sigma \Gamma$ . Denote it by  $S(\alpha)$ .

**Remark 2.6.** Not every point of  $G \rtimes_\sigma \Gamma$  belongs to the graph of a cocycle. For example, let  $\Gamma = G = \mathbb{Z}$  and  $\sigma$  be trivial, i.e. the semi-direct product is the direct product. Then  $(2, 3)$  does not belong to any graph, because if  $(m, 1)$  belongs, then we should have  $(2, 3) = (3m, 3)$ .

**2.2. Exact 8-term sequences.** Consider a group extension respecting homomorphism  $\phi$ :

$$(3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H & \longrightarrow & 0 \\ & & \phi' \downarrow & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{p} & G/H & \longrightarrow & 0, \end{array}$$

where  $H$  is a normal subgroup of  $G$ . Denote  $\bar{G} := G/H$ . Then the induced mapping of Reidemeister classes is an epimorphism, because  $p(\tilde{g})p(g)\bar{\phi}(p(\tilde{g}^{-1})) = p(\tilde{g}g\phi(\tilde{g}^{-1}))$ .

In [29] the following exact sequence (as a particular case of the main theorem there) was obtained

$$(4) \quad 1 \rightarrow \text{Fix}(\phi') \rightarrow \text{Fix}(\phi) \rightarrow \text{Fix}(\bar{\phi}) \xrightarrow{\delta} \mathcal{R}(\phi') \rightarrow \mathcal{R}(\phi) \rightarrow \mathcal{R}(\bar{\phi}) \rightarrow 1,$$

where all morphisms are quite evident except of  $\delta$ , which is defined as follows:

$$(5) \quad \delta(\bar{\beta}) = \{\beta^{-1}\phi(\beta)\}_{\phi'}, \quad \beta \in G, p(\beta) = \bar{\beta}.$$

The definition does not depend on the choice of  $\beta$  with this property, because

$$\{(\beta h)^{-1}\phi(\beta h)\}_{\phi'} = \{h^{-1}(\beta^{-1}\phi(\beta))\phi(h)\}_{\phi'} = \{\beta^{-1}\phi(\beta)\}_{\phi'}.$$

We will give a short proof of this statement using our interpretation.



For this purpose consider the diagram

$$(6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \text{Fix}(\phi') & \longrightarrow & \text{Fix}(\phi) & \longrightarrow & \text{Fix}(\bar{\phi}) & \xrightarrow{\delta} & \mathcal{R}(\phi') & \longrightarrow & \\ & & \parallel & & \parallel & & \parallel & & \updownarrow & & \\ 1 & \longrightarrow & H^0(\mathbb{Z}, H) & \longrightarrow & H^0(\mathbb{Z}, G) & \longrightarrow & H^0(\mathbb{Z}, \bar{G}) & \xrightarrow{d} & H^1(\mathbb{Z}, H) & \longrightarrow & \\ & & & & & & & & \longrightarrow & \mathcal{R}(\phi) & \longrightarrow & \mathcal{R}(\bar{\phi}) & \longrightarrow & 1 \\ & & & & & & & & \updownarrow & & \updownarrow & & & \\ & & & & & & & & \longrightarrow & H^1(\mathbb{Z}, G) & \longrightarrow & H^1(\mathbb{Z}, \bar{G}) & \dashrightarrow & 1 \end{array}$$

where the bottom exact row can be found e.g. in [44, Sect. I.5]. First of all, we should remark that it can be extended to the right by the trivial homomorphism in our case (of  $\mathbb{Z}$ -action), because in this case any cocycle is defined by its value at 1 (as it is explained above). Second, the diagram is commutative: due to naturality only the middle square may cause doubts. But the map  $d$  is defined as

$$d(\bar{\beta}) = \{s \mapsto \beta^{-1s} \beta\}, \text{ where } p(\beta) = \bar{\beta}$$

(see [44, Sect. I.5.5]) and by (5) it commutes.

Our TBFT theorem for an almost polycyclic group  $G$  [12, 17] can be inserted in this context as a version of ‘‘Poincaré–Pontryagin–Tate duality’’:

$$(7) \quad \#H^1(\mathbb{Z}, G, \sigma) = \#H^0(\mathbb{Z}, \widehat{G}_f, \widehat{\sigma}),$$

where  $\widehat{G}_f$  is the finite-dimensional part of the unitary dual (a more delicate result will be discussed in the last section) and one of the sides of this equality is finite. Here  $\widehat{\sigma}(1)[\rho] := [\rho \circ \sigma]$ .

An interpretation of Reidemeister theory in terms of principal homogeneous spaces (or torsors) (see [44, Sect. 5.2, 5.3]) seems also prospective.

### 3. ISOGREDIENCE CLASSES

**Definition 3.1.** (see [35] and also [4]) Suppose,  $\Phi \in \text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ . We say that  $\alpha, \beta \in \Phi$  are *isogredient* (or *similar*) if  $\beta = \tau_h \circ \alpha \circ \tau_h^{-1}$  for some  $h \in G$ , where  $\tau_h(g) := ghg^{-1}$ .

Let  $\mathcal{S}(\Phi)$  be the set of isogredience classes of  $\Phi$ . If  $\Phi = \text{Id}_G$ , then above  $\alpha$  and  $\beta$  are inner, say  $\alpha = \tau_g, \beta = \tau_s$ . Since elements of center  $Z(G)$  give trivial inner automorphisms, we may suppose  $g, s \in G/Z(G)$ . Then the equivalence relation takes the form  $\tau_s = \tau_{ghg^{-1}}$ , i.e.,  $s$  and  $g$  are conjugate in  $G/Z(G)$ . Thus,  $\mathcal{S}(\text{Id})$  is the set of conjugacy classes of  $G/Z(G)$ .

Denote by  $S(\Phi)$  the cardinality of  $\mathcal{S}(\Phi)$ .

For a topological motivation of the above definition of the isogredience suppose that  $G \cong \pi_1(X)$ ,  $X$  is a compact space, and  $\Phi$  is induced by a continuous map  $f : X \rightarrow X$ . Let  $p : \widetilde{X} \rightarrow X$  be the universal covering of  $X$ ,  $\tilde{f} : \widetilde{X} \rightarrow \widetilde{X}$  a lifting of  $f$ , i.e.  $p \circ \tilde{f} = f \circ p$ . Two liftings  $\tilde{f}$  and  $\tilde{f}'$  are called *isogredient* or conjugate if there is a  $\gamma \in G$  such that  $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$ . Different lifting may have very different properties. Nielsen observed (see [30]) that conjugate lifting of homeomorphism of surface have similar dynamical properties. This led him to the definition of the isogredience of liftings in this case. Later Reidemeister and Wecken succeeded in generalizing the theory to continuous maps of compact polyhedra (see [30]).

The set of isogredience classes of automorphisms representing a given outer automorphism and the notion of index  $\text{Ind}(\Phi)$  defined via the set of isogredience classes are strongly related to important structural properties of  $\Phi$  (see [20]), for example in another (with respect to Bestvina–Handel [2]) proof of the Scott conjecture [19].

One of the main results of [35] is that for any non-elementary hyperbolic group and any  $\Phi$  the set  $\mathcal{S}(\Phi)$  is infinite, i.e.,  $S(\Phi) = \infty$ . We will extend this result. First, we introduce an appropriate definition.

**Definition 3.2.** A group  $G$  is an  $S_\infty$ -group if for any  $\Phi$  the set  $\mathcal{S}(\Phi)$  is infinite, i.e.,  $S(\Phi) = \infty$ .

Thus, the above result from [35] says: any non-elementary hyperbolic group is an  $S_\infty$ -group. On the other hand, finite and Abelian groups are evidently non  $S_\infty$ -groups.

Now, let us generalize the above calculation for  $\Phi = \text{Id}$  to a general  $\Phi$  (see [35, p. 512]). Two representatives of  $\Phi$  have form  $\tau_s \circ \alpha$ ,  $\tau_q \circ \alpha$ , with some  $s, q \in G$ . They are isogredient if and only if

$$\begin{aligned} \tau_q \circ \alpha &= \tau_g \circ \tau_s \circ \alpha \circ \tau_g^{-1} = \tau_g \circ \tau_s \circ \tau_{\alpha(g^{-1})} \circ \alpha, \\ \tau_q &= \tau_{gs\alpha(g^{-1})}, \quad q = gs\alpha(g^{-1})c, \quad c \in Z(G). \end{aligned}$$

So, the following statement is proved.

**Lemma 3.3.**  $S(\Phi) = R_{G/Z(G)}(\bar{\alpha})$ , where  $\alpha$  is any representative of  $\Phi$  and  $\bar{\alpha}$  is induced by  $\alpha$  on  $G/Z(G)$ .

Since  $Z(G)$  is a characteristic subgroup, we obtain from Lemma 3.3 and exact sequence (6) the following statement (in one direction it was discussed in [22, Remark 2.1]).

**Theorem 3.4.** Suppose,  $|Z(G)| < \infty$ . Then  $G$  is an  $R_\infty$ -group if and only if  $G$  is an  $S_\infty$ -group.

**Remark 3.5.** Of course, this argument is applicable to an individual  $\Phi$  as well.

Now we can give a more advanced example of a non  $S_\infty$ -group. Namely, consider Osin's group [42]. This is a non-residually finite exponential growth group with two conjugacy classes. Since it is simple, it is not  $S_\infty$  by Theorem 3.4.

#### 4. RATIONAL POINTS

In this section we show that not every finite-dimensional representation can be fixed by  $\widehat{\phi}$  if  $R(\phi) < \infty$ .

**Definition 4.1.** Let  $[\rho] \in \widehat{G}_f$ ,  $g \mapsto T_g$  be a (class of a) finite-dimensional representation. We say that  $\rho$  is *rational* if the number of distinct  $T_g$ 's is finite, and *irrational* otherwise.

**Remark 4.2.** Evidently,  $\rho$  is rational if and only if it can be factorized through a homomorphism  $G \rightarrow F$  on a finite group. An interesting research related rational points can be found in [36], where they are called *finite*.

We will need the following result from [11, 12].

**Lemma 4.3.** Let  $\rho$  be a finite dimensional irreducible representation of  $G$  on  $V_\rho$ , and  $\phi : G \rightarrow G$  is an automorphism.

1). There exists a twisted invariant function  $\omega : G \rightarrow \mathbb{C}$  being a matrix coefficient of  $\rho$  if and only if  $\widehat{\phi}[\rho] = [\rho]$ .

2). In this case such  $\omega$  is unique up to scaling.

3). If we have several distinct  $\widehat{\phi}$ -fixed representations, then the correspondent twisted invariant functions are independent.

*Proof.* Let us sketch a proof, the details can be found in [11, 12, 48]. Matrix coefficients of a finite dimensional representation  $\rho$  arise from functionals on  $\text{End } V_\rho$  and can be written as  $g \mapsto \text{Trace}(a\rho(g))$  for some matrix  $a \in \text{End } V_\rho$ . Since the equality

$$0 = \text{Trace}(ab) - \text{Trace}(a\rho(h)b\rho(\phi(h^{-1}))) = \text{Trace}((a - \rho(\phi(h^{-1}))a\rho(h))b)$$

for any  $b$  and  $h$  implies  $\rho(\phi(h))a = a\rho(h)$ , the above matrix coefficient is twisted invariant if and only if  $a$  is an intertwining operator between  $\rho$  and  $\phi \circ \rho$ . This gives 1), and Schur's lemma gives 2). Finally, matrix coefficients of distinct finite-dimensional representations are linear independent and 3) follows e.g. from [5, Corollary (27.13)].  $\square$

**Theorem 4.4.** *Let  $G$  be a finitely generated group and for an automorphism  $\phi$  at least one of the following two conditions holds:*

- 1). *There exist infinitely many finite-dimensional representation classes in  $\widehat{G}$  fixed by  $\widehat{\phi}$ .*
- 2). *There exists an irrational representation  $\rho$  fixed by  $\widehat{\phi}$ .*

*Then  $R(\phi) = \infty$ .*

*In particular, if we have one of these conditions for every automorphism  $\phi$ , then  $G$  has  $R_\infty$  property.*

*Proof.* 1) This follows immediately from Lemma 4.3.

2) Suppose that  $f_\rho(g) = \text{Trace}(a\rho(g))$  (see Lemma 4.3) takes finitely many values and will arrive to a contradiction. Indeed,  $f_\rho$  is a non-trivial matrix coefficient. Hence, (see, e.g. [33]) its left translations generate a finite-dimensional representation, which is equivalent to a direct sum of several copies of  $\rho$ . The space  $W$  of this representation has a basis  $L_{g_1}f_\rho, \dots, L_{g_k}f_\rho$ . Thus, all functions from  $W$  take only finitely many values (with level sets of the form  $\cap_i g_i U_j$ , where  $U_j$  are the level sets of  $f_\rho$ ). Taking unions of these sets (if necessary) we can form a finite partition  $G = V_1 \sqcup \dots \sqcup V_m$  such that elements of  $W$  are constant on the elements of the partition and for each pair  $V_i \neq V_j$  there exists a function from  $W$  taking distinct values on them. Thus any left translation maps  $V_i$  onto each other and the representation  $G$  on  $W$  factorizes through (a subgroup of) the permutation group on  $m$  elements, i.e. a finite group. The same is true for its subrepresentation  $\rho$ , thus it is rational. A contradiction.  $\square$

**Concluding remarks.** We would like to finish this paper with the following to interrelated conjectures, motivated by known examples and theorems.

**CONJECTURE R.** Let  $G$  be a finitely generated residually finite group. Either  $G$  is  $R_\infty$ , or  $G$  is solvable-by-finite.

By Theorem 3.4 Conjecture R implies the following

**CONJECTURE S.** Let  $G$  be a finitely generated residually finite group with finite center. Either  $G$  is  $S_\infty$ , or  $G$  is solvable-by-finite.

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