Rectification of enriched ∞-categories

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We prove a rectification theorem for enriched ∞-categories: If $\mathbf{V}$ is a nice monoidal model category, we show that the homotopy theory of ∞-categories enriched in $\mathbf{V}$ is equivalent to the familiar homotopy theory of categories strictly enriched in $\mathbf{V}$. It follows, for example, that ∞-categories enriched in spectra or chain complexes are equivalent to spectral categories and dg-categories. A similar method gives a comparison result for enriched Segal categories, which implies that the homotopy theories of n-categories and (∞, n)-categories defined by iterated ∞-categorical enrichment are equivalent to those of more familiar versions of these objects. In the latter case we also include a direct comparison with complete n-fold Segal spaces. Along the way we prove a comparison result for fibrewise simplicial localizations potentially of independent use.

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1 Introduction

In [13], David Gepner and I set up a general theory of "weakly enriched categories" — more precisely, we introduced a notion of ∞-categories *enriched* in a monoidal ∞-category, and constructed an ∞-category of these objects where the equivalences are the natural analogue of fully faithful and essentially surjective functors in this context. In this paper we are interested in the situation where the monoidal ∞-category we enrich in can be described by a monoidal model category — this applies to many, if not most, interesting examples of monoidal ∞-categories. If $\mathbf{V}$ is a model category, then inverting the weak equivalences W gives an ∞-category $\mathbf{V}[W^{-1}]$; if $\mathbf{V}$ is a monoidal model category, then $\mathbf{V}[W^{-1}]$ inherits a monoidal structure, so our theory produces an ∞-category of $\mathbf{V}[W^{-1}]$-enriched ∞-categories. On the other hand, there is also often a model structure on ordinary $\mathbf{V}$-enriched categories (cf. [19, 6, 32, 24]) where the weak equivalences are the so-called *DK-equivalences*, namely the functors that are *weakly fully faithful* (i.e. given by weak equivalences in $\mathbf{V}$ on morphism objects), and essentially surjective (up to homotopy). Our main goal in this paper is to prove a rectification theorem in this setting:

Theorem 1.1 *If* $\mathbf{V}$ *is a nice monoidal model category, then the homotopy theory of* ∞*-categories enriched in* $\mathbf{V}[W^{-1}]$ *is equivalent to the homotopy theory of ordinary* $\mathbf{V}$*-enriched categories with respect to the DK-equivalences.*

In particular, $\mathbf{V}[W^{-1}]$-enriched ∞-categories can be rectified to $\mathbf{V}$-categories: every $\mathbf{V}[W^{-1}]$-enriched ∞-category is equivalent to one coming from a category enriched in $\mathbf{V}$. We will state and prove a precise version of this result in §5. The precise meaning of "nice" required applies, for example, to the category of chain complexes over a ring with the usual projective model structure, and certain model structures on symmetric spectra. We can therefore conclude that the ∞-category of spectral categories is equivalent to that of spectral ∞-categories, and the ∞-category of dg-categories to that of ∞-categories enriched in the derived ∞-category of abelian groups.

If $\mathbf{V}$ is a nice Cartesian model category, i.e. a monoidal model category with respect to the Cartesian product, then the theory of $\mathbf{V}$-enriched Segal categories, as defined by Lurie [20] and Simpson [30], gives an alternative notion of "weakly $\mathbf{V}$-enriched categories". Using a similar proof strategy we also prove a comparison result in this setting:

Theorem 1.2 *If* $\mathbf{V}$ *is a nice Cartesian model category, then the homotopy theory of* ∞*-categories enriched in* $\mathbf{V}[W^{-1}]$ *is equivalent to the homotopy theory of* $\mathbf{V}$*-enriched Segal categories.*

We will prove a precise version of this theorem in §6. From this we can conclude that the homotopy theories of n-categories and (∞, n)-categories constructed in [13, §6.1] using iterated enrichment are equivalent to those constructed as iterated Segal categories, starting with sets or simplicial sets, respectively. These are due to Tamsamani and Pellissier-Hirschowitz-Simpson, and are constructed as model categories in [30].

Our last main result, which we will prove in §7, is a more direct comparison with (∞, n)-categories, generalizing that between ∞-categories enriched in spaces and Segal spaces in [13, §4.4]:

Theorem 1.3 *The homotopy theory of* (∞, n)*-categories obtained by iterated* ∞*-categorical enrichment is equivalent to that of complete* n*-fold Segal spaces.*

We now outline the proof of Theorem 1.1 and the organization of the paper. In [13] we defined enriched ∞-categories in a monoidal ∞-category $\mathcal{V}$ as "many-object associative algebras" in $\mathcal{V}$, or more precisely as algebras for a "many-object associative operad" Δ^{op}_X, where X is a space. In § 2 we briefly review this definition and the context in which it takes places, namely the theory of non-symmetric ∞-operads.

The first step in the proof of our rectification theorem is to show that for X a set and $\mathbf{V}$ a nice monoidal model category, the ∞-category $\mathrm{Alg}_{\Delta^{\mathrm{op}}_X}(\mathbf{V}[W^{-1}])$ of Δ^{op}_X-algebras in $\mathbf{V}[W^{-1}]$ is equivalent to the ∞-category obtained by inverting the weakly fully faithful functors in the category $\mathrm{Cat}_X(\mathbf{V})$ of $\mathbf{V}$-categories with a fixed set of objects X. To see this, we first (in § 3) review Lurie's rectification theorem for associative algebras (Theorem 4.1.4.4 of [21]) and observe that it generalizes to associative algebras in certain non-symmetric monoidal model categories.

Next, we wish to combine these equivalences to an equivalence of ∞-categories where the sets of objects are allowed to vary. In [13] we combined the ∞-categories $\mathrm{Alg}_{\Delta^{\mathrm{op}}_X}(\mathcal{V})$ for all spaces X to an ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ of *categorical algebras*. Here, we consider the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$ of categorical algebras with *sets* of objects. We will prove that if $\mathbf{V}$ is a nice monoidal model category, then $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$ is equivalent to the ∞-category obtained from the category $\mathrm{Cat}(\mathbf{V})$ of $\mathbf{V}$-categories by inverting those morphisms that are weakly fully faithful and bijective on sets of objects. To see this we need a technical result about ∞-categorical localizations of fibrations of categories, which we prove in § 4.

The "correct" ∞-category of $\mathcal{V}$-∞-categories is not $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, but rather the ∞-category obtained from this by inverting the fully faithful and essentially surjective functors. One of the main results of [13] was that this is equivalent to the full subcategory $\mathrm{Cat}^{\mathcal{V}}_{\infty}$ of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ spanned by those $\mathcal{V}$-∞-categories that are *complete* in the sense that their space of objects is equivalent to their classifying space of equivalences. We also showed, in [13, Theorem 5.2.17], that inverting the fully faithful and essentially surjective morphisms in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$ is equivalent to inverting them in $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$. Since the DK-equivalences in $\mathrm{Cat}(\mathbf{V})$, if $\mathbf{V}$ is a nice monoidal model category, correspond to the fully faithful and essentially surjective functors in $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$, we conclude that the ∞-category obtained from $\mathrm{Cat}(\mathbf{V})$ by inverting the DK-equivalences is equivalent to $\mathrm{Cat}^{\mathbf{V}[W^{-1}]}_{\infty}$. We will give the details of the proof we have just sketched in § 5, after the technical preliminaries of § 3 and § 4. We then prove the comparison with Segal categories using a similar proof in § 6 and the comparison with n-fold Segal spaces in § 7.

1.1 Notation

Much of this paper is based on work of Lurie in [19, 21]; we have generally kept his notation and terminology. In particular, by an ∞-*category* we mean a quasicategory, i.e. a simplicial set satisfying certain horn-filling properties. However, in the few cases where the notation of [13] differs from that of Lurie we have kept that of the latter. Here are some hopefully useful reminders:

- Generic categories are generally denoted by single capital bold-face letters ($\mathbf{A}, \mathbf{B}, \mathbf{C}$) and generic ∞-categories by single caligraphic letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}$). Specific categories and ∞-categories both get names in the normal text font: thus the category of small $\mathbf{V}$-categories is denoted $\mathrm{Cat}(\mathbf{V})$ and the ∞-category of small $\mathcal{V}$-∞-categories is denoted $\mathrm{Cat}_\infty^{\mathcal{V}}$.
- Δ is the simplicial indexing category, i.e. the category with objects the non-empty ordered sets $[n] = \{0, 1, \ldots, n\}$ and order-preserving maps as morphisms.
- A model category is *tractable* if it is combinatorial and there exists a set of generating cofibrations that consists of morphisms between cofibrant objects.
- Set_Δ is the category of simplicial sets, and Set_Δ^+ is the category of *marked* simplicial sets, i.e. simplicial sets equipped with a collection of 1-simplicies including the degenerate ones.
- If $\mathcal{C}$ is an ∞-category, we write $\iota\mathcal{C}$ for the *interior* or *underlying space* of $\mathcal{C}$, i.e. the largest subspace of $\mathcal{C}$ that is a Kan complex.
- If $f \colon \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $g \colon \mathcal{D} \to \mathcal{C}$, we will refer to the adjunction as $f \dashv g$.
- $\mathcal{S}$ is the ∞-category of spaces (in the sense of homotopy types or ∞-groupoids), and Cat_∞ is the ∞-category of ∞-categories.
- If $\mathbf{C}$ is a model category, we write $\mathbf{C}^{\mathrm{cof}}$ for the full subcategory of $\mathbf{C}$ spanned by the cofibrant objects.

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2 Brief Review of Non-Symmetric ∞-Operads and Enriched ∞-Categories

To orient the reader, we begin with a brief review of the non-symmetric version of Lurie's ∞-operads and the definition of enriched ∞-categories. We focus on the essential ideas and do not give complete technical details of definitions or results; for a more detailed introduction we refer the reader to [13, §2].

The starting point for the theory of non-symmetric ∞-operads is the *category of operators* of a non-symmetric operad (originally introduced by May and Thomason for symmetric operads):

Definition 2.1 Let $\mathbf{O}$ be a coloured non-symmetric operad (or in other words a *multi-category*). Its *category of operators* $\mathbf{O}^{\otimes}$ has objects (possibly empty) lists $(X_1, \ldots, X_n)$ of objects of $\mathbf{O}$, and a morphism $(X_1, \ldots, X_n) \to (Y_1, \ldots, Y_m)$ is given by a morphism $\phi\colon [m] \to [n]$ in Δ and for each $i = 1, \ldots, m$ a multimorphism in $\mathbf{O}$ from $(X_{\phi(i-1)+1}, \ldots, X_{\phi(i)})$ to Y_i.

There is an obvious projection $\pi\colon \mathbf{O}^{\otimes} \to \Delta^{\mathrm{op}}$, with the following properties:

(a) Recall that a morphism $\phi\colon [n] \to [m]$ in Δ is *inert* if it is the inclusion of a subinterval, i.e. if $\phi(i) = \phi(0) + i$ for all $i = 0, \ldots, n$. For every inert morphism $\phi\colon [n] \to [m]$ and every object $X \in \mathbf{O}^{\otimes}$ with $\pi(X) = [m]$, there exists a π-coCartesian morphism $X \to \phi_! X$ over ϕ.

(b) Let $\rho_i\colon [1] \to [n]$ denote the inert morphism in Δ that sends 0 to $i-1$ and 1 to i. The functors $\mathbf{O}^{\otimes}_{[n]} \to \mathbf{O}^{\otimes}_{[1]}$ induced by the coCartesian morphisms over ρ_i combine to give an equivalence of categories

$$\mathbf{O}^{\otimes}_{[n]} \xrightarrow{\sim} \prod_{i=1}^{n} \mathbf{O}^{\otimes}_{[1]}.$$

(c) Given objects $X \in \mathbf{O}^{\otimes}_{[n]}$, $Y \in \mathbf{O}^{\otimes}_{[m]}$, and a morphism $\phi\colon [m] \to [n]$ in Δ, the inert maps $Y \to \rho_{i,!} Y$ induce an isomorphism

$$\mathrm{Hom}^{\phi}_{\mathbf{O}^{\otimes}}(X, Y) \xrightarrow{\sim} \prod_{i=1}^{m} \mathrm{Hom}^{\rho_i \circ \phi}_{\mathbf{O}^{\otimes}}(X, \rho_{i,!} Y),$$

where $\mathrm{Hom}^{\phi}_{\mathbf{O}^{\otimes}}(X, Y)$ denotes the set of morphisms $X \to Y$ in $\mathbf{O}^{\otimes}$ that map to ϕ in Δ^{op}.

It is not hard to see that these three properties *characterize* the categories of operators of coloured non-symmetric operads:

Proposition 2.2 *Any functor $\pi\colon \mathbf{C} \to \Delta^{\mathrm{op}}$ that satisfies (a)–(c) determines a coloured non-symmetric operad that has $\mathbf{C}$ as its category of operators. Moreover, under this identification morphisms of operads correspond precisely to functors over Δ^{op} that preserve the coCartesian morphisms over the inert maps in Δ^{op}.*

Properties (a)–(c) have precise analogues in the theory of ∞-categories, and a *non-symmetric ∞-operad* is precisely a functor of ∞-categories $\mathcal{O} \to \Delta^{\mathrm{op}}$ with these properties. If $\mathcal{O}$ and $\mathcal{P}$ are non-symmetric ∞-operads in this sense, it is also easy to define the ∞-category of $\mathcal{O}$*-algebras* in $\mathcal{P}$:

Definition 2.3 The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\mathcal{P})$ of $\mathcal{O}$*-algebras* in $\mathcal{P}$ is the full subcategory of the functor ∞-category $\mathrm{Fun}_{\Delta^{\mathrm{op}}}(\mathcal{O}, \mathcal{P})$ of functors from $\mathcal{O}$ to $\mathcal{P}$ over Δ^{op} spanned by those functors that preserve the coCartesian morphisms over inert maps in Δ^{op}.

The simple definition of the homotopically correct category of algebras is one of the key advantages of the theory of ∞-operads over operads enriched in topological spaces or simplicial sets.

An important source of non-symmetric ∞-operads are non-symmetric operads enriched in simplicial sets or topological spaces: if $\mathbf{O}$ is a coloured non-symmetric operad enriched in simplicial sets, all of whose mapping spaces are Kan complexes, then its simplicial category of operators (defined completely analogously to the set-based version discussed above) is fibrant, and its coherent nerve $\mathrm{N}\mathbf{O}^{\otimes} \to \Delta^{\mathrm{op}}$ is an ∞-operad; for operads enriched in topological spaces, we simply take the singular simplicial sets of the mapping spaces first. For example, the associative operad just gives the identity map $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}}$, which is easily seen to be equivalent to the ∞-operad associated to an A_∞-operad. This should not be surprising: in the ∞-categorical setting it does not make sense to talk about "strict" associative algebras, the only meaningful notion is that of an algebra associative up to coherent homotopies, and this notion is already encoded in algebras for the associative ∞-operad.

We can also recognize monoidal categories from the category of operators perspective: they are precisely those categories of operators $\mathbf{C} \to \Delta^{\mathrm{op}}$ that are Grothendieck opfibrations. Analogously we can define a monoidal ∞-category to be a non-symmetric ∞-operad that is also a coCartesian fibration, but this can also be reformulated more simply:

Definition 2.4 A *monoidal ∞-category* is a coCartesian fibration $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ such that for each $[n] \in \Delta$ the functor $\mathcal{V}^{\otimes}_{[n]} \to \prod_{i=1}^{n} \mathcal{V}^{\otimes}_{[1]}$ induced by the coCartesian morphisms over the inert maps $\rho_i \colon [1] \to [n]$ is an equivalence of ∞-categories.

Using the correspondence between coCartesian fibrations and functors to the ∞-category Cat_∞ of ∞-categories, we get an equivalence between monoidal ∞-categories and *associative monoids* in Cat_∞:

Definition 2.5 Let $\mathcal{C}$ be an ∞-category with products. An *associative monoid* in $\mathcal{C}$ is a functor $\mu \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ that satisfies the *Segal condition*: for any $[n] \in \Delta$ the map $\mu([n]) \to \prod_{i=1}^{n} \mu([1])$ induced by the maps $\mu(\rho_i)$ is an equivalence.

There is also an equivalence between associative monoids in $\mathcal{C}$ and algebras for the associative ∞-operad in $\mathcal{C}$ (equipped with the monoidal structure given by the Cartesian product). In particular, we have:

Proposition 2.6 *There are equivalences of ∞-categories between associative algebras in Cat_∞, associative monoids in Cat_∞, and monoidal ∞-categories.*

What we have discussed so far is the non-symmetric variant of ∞-operads. Lurie's original theory, developed in [21], concerns symmetric ∞-operads. This has a completely analogous motivation, the only difference is that in the definition of the category of operators the category Δ^{op} is replaced by the category Γ^{op} of pointed finite sets. In the ∞-categorical setting this leads to Lurie's definitions of symmetric ∞-operads and symmetric monoidal ∞-categories. As the non-symmetric theory is the one relevant to the present paper, we refer the reader to [21] for more details and do not discuss this further here.

Instead, we turn to a brief summary of the theory of enriched ∞-categories as introduced in [13]. Recall that if $\mathbf{V}$ is a monoidal category, then $\mathbf{V}$-enriched categories with a fixed set X of objects can be regarded as the algebras for a certain non-symmetric coloured operad $\mathbf{O}_X$:

Definition 2.7 If X is a set, the multicategory $\mathbf{O}_X$ has $X \times X$ as its set of objects, and the multimorphism sets are defined by

$$\mathbf{O}_X((x_0,y_1),(x_1,y_2),\ldots,(x_{n-1},y_n);(y_0,x_n)) := \begin{cases} *, & \text{if } y_i = x_i,\ i = 0,\ldots,n, \\ \emptyset, & \text{otherwise.} \end{cases}$$

This suggests that if $\mathcal{V}$ is a monoidal ∞-category then we can define $\mathcal{V}$-enriched ∞-categories with set of objects X to be algebras in $\mathcal{V}$ for (the non-symmetric ∞-operad associated to) $\mathbf{O}_X$. This is indeed a correct definition, but it turns out not to be the most convenient to work with — for instance, we get a much better-behaved theory of enriched ∞-categories if we allow them to have *spaces* of objects, which is more easily accomplished with an alternative definition.

We therefore consider *generalized non-symmetric ∞-operads* — these are what we obtain by relaxing condition (b) for a category of operators above to allow $\mathbf{O}^{\otimes}_{[0]}$ to not be just a point, and instead require $\mathbf{O}^{\otimes}_{[n]}$ to be an iterated fibre product of $\mathbf{O}^{\otimes}_{[1]}$ over $\mathbf{O}^{\otimes}_{[0]}$. (The objects that have such categories of operators in the setting of ordinary categories have been studied under the names **fc**-*multicategories* by Leinster and *virtual double categories* by Cruttwell and Shulman.) For each set X we can define such a category of operators whose algebras in a monoidal category (i.e. functors over Δ^{op} that preserve coCartesian morphisms over inert maps) are precisely enriched categories with set of objects X:

Definition 2.8 Let X be a set. The category Δ^{op}_X has objects lists $(x_0, \ldots, x_n)$ of elements $x_i \in X$, and a unique morphism $(x_0, \ldots, x_n) \to (x_{\phi(0)}, \ldots, x_{\phi(m)})$ for each map $\phi\colon [m] \to [n]$ in Δ.

There is an obvious projection $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$, and if $\mathbf{V}$ is a monoidal category, then Δ^{op}_X-algebras in the category of operators $\mathbf{V}^{\otimes}$ are precisely $\mathbf{V}$-enriched categories with set of objects X. This leads to our definition of enriched ∞-categories:

Definition 2.9 If $\mathcal{V}^{\otimes} \to \Delta^{\mathrm{op}}$ is a monoidal ∞-category, then a *$\mathcal{V}$-enriched ∞-category* with set of objects X is an algebra for the generalized non-symmetric ∞-operad Δ^{op}_X in $\mathcal{V}^{\otimes}$.

The projection $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$ is the Grothendieck opfibration associated to the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ that sends $[n]$ to $X^{\times(n+1)}$ and $\phi\colon [m] \to [n]$ in Δ to the map $X^{\times(n+1)} \to X^{\times(m+1)}$ that takes $(x_0, \ldots, x_n)$ to $(x_{\phi(0)}, \ldots, x_{\phi(m)})$. This has an obvious generalization where we let X be a space: we simply take the coCartesian fibration $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$ of the analogous functor $\Delta^{\mathrm{op}} \to \mathcal{S}$ that sends $[n]$ to $X^{\times(n+1)}$.

When X is a set, both $\mathbf{O}_X$-algebras and Δ^{op}_X-algebras in a monoidal category $\mathbf{V}$ are equivalent to $\mathbf{V}$-categories with X as their set of objects. Similarly, algebras for the non-symmetric ∞-operad $\mathbf{O}^{\otimes}_X$ and the generalized non-symmetric ∞-operad Δ^{op}_X are equivalent, with the equivalence induced by a map of generalized ∞-operads (this is a special case of [13, Corollary 4.2.8]):

Proposition 2.10 *Suppose X is a set. There is an obvious functor ν_X from Δ_X^{op} to $\mathbf{O}_X^{\otimes}$ that sends the list $(x_0, \dots, x_n)$ to the list $((x_0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n))$. If $\mathcal{V}$ is a monoidal ∞-category, then the functor from $\mathrm{Alg}_{\mathbf{O}_X}(\mathcal{V})$ to $\mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathcal{V})$ given by composition with ν_X is an equivalence of ∞-categories.*

3 Rectifying Associative Algebras

In [21, §4.1.4] Lurie proves a rectification result for associative algebras: if $\mathbf{V}$ is a nice symmetric monoidal model category, then the ∞-category of (∞-categorical) associative algebras in $\mathbf{V}[W^{-1}]$, i.e. the ∞-category of algebras for the non-symmetric ∞-operad Δ^{op}, is equivalent to that associated to the model category of (strictly) associative algebras in $\mathbf{V}$, as constructed by Schwede and Shipley [29]. This is proved by showing that both sides are equivalent to the ∞-category of algebras for the free associative algebra monad on $\mathbf{V}[W^{-1}]$. In this section we review this result, and observe that it generalizes slightly to the setting of non-symmetric monoidal model categories; we will apply this to enriched categories in §5.

3.1 Review of Monoidal Model Categories

In this subsection we briefly review the construction of a monoidal ∞-category from a monoidal model category; the full details can be found in [21, §4.1.3].

If $\mathbf{V}$ is a simplicial model category, then one way of constructing an ∞-category from $\mathbf{V}$ is to regard the full subcategory $\mathbf{V}^{\circ}$ of fibrant-cofibrant objects as a simplicial category. This is fibrant in the model structure on simplicial categories, and so its coherent nerve $\mathrm{N}\mathbf{V}^{\circ}$ is an ∞-category. However, this construction does not work well with respect to monoidal structures. We will therefore instead use a more general, but less explicit, construction, that does not require $\mathbf{V}$ to have a simplicial enrichment:

Definition 3.1 Recall that there is a model structure (constructed in [19, §3.1.3]) on the category $\mathrm{Set}_{\Delta}^{+}$ of marked simplicial sets that is Quillen equivalent to the Joyal model structure on Set_{Δ}. In this model category all objects are cofibrant and the fibrant objects are precisely those marked simplicial sets (X, S) where X is a quasicategory and S is the collection of equivalences in X. If $\mathcal{C}$ is an ∞-category and W is a collection of morphisms in $\mathcal{C}$, then a fibrant replacement for the marked simplicial set $(\mathcal{C}, W)$ in this model structure gives the universal ∞-category $\mathcal{C}[W^{-1}]$ obtained from $\mathcal{C}$ by inverting the morphisms in W.

If $\mathbf{V}$ is a model category, and W is the class of weak equivalences in $\mathbf{V}$, we can therefore define the ∞-category $\mathbf{V}[W^{-1}]$ associated to the model category to be a fibrant replacement for the marked simplicial set $(\mathrm{N}\mathbf{V}, W)$ in this model structure on $\mathrm{Set}_{\Delta}^{+}$. Equivalently, we can restrict ourselves to cofibrant, fibrant, or fibrant-cofibrant objects and the weak equivalences between them. To get monoidal structures on the localization it is convenient to consider the cofibrant objects; since this gives an ∞-category equivalent to $\mathbf{V}[W^{-1}]$ we will use this notation also in this case, despite the slight ambiguity this introduces.

Definition 3.2 Let $\mathbf{V}$ be a model category equipped with a biclosed monoidal structure. We say that $\mathbf{V}$ is a *monoidal model category* if the unit of the monoidal structure is cofibrant and the tensor product functor $\otimes\colon \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ is a left Quillen bifunctor.

Remark 3.3 Let $\mathbf{V}$ be a model category equipped with a biclosed monoidal structure whose unit is cofibrant. If $f\colon A \to B$ and $g\colon A' \to B'$ are morphisms in $\mathbf{V}$, let $f \Box g$ be the induced morphism

$$A \otimes B' \amalg_{A \otimes A'} B \otimes A' \to B \otimes B';$$

this is the *pushout-product* of f and g. Then $\mathbf{V}$ is a monoidal model category if and only if $f \Box g$ is a cofibration whenever f and g are both cofibrations, and a trivial cofibration if either f or g is also a weak equivalence.

Lurie shows in [21, Proposition 4.1.3.2] that the functor that takes a pair $(\mathcal{C}, W)$ consisting of an ∞-category $\mathcal{C}$ and a collection of morphisms W to the localization $\mathcal{C}[W^{-1}]$ preserves products. It follows that this functor preserves $\mathcal{O}$-algebra structures for any ∞-operad $\mathcal{O}$. If $\mathbf{V}$ is a monoidal model category with weak equivalences W, then $(\mathrm{N}\mathbf{V}^{\mathrm{cof}}, W)$ is an associative algebra in the ∞-category of such pairs, and so, since a monoidal ∞-category is the same thing as an algebra for the associative ∞-operad in Cat_∞, we obtain the following key special case of this result:

Proposition 3.4 ([21, Example 4.1.3.6]) *Let $\mathbf{V}$ be a monoidal model category. Then $\mathbf{V}[W^{-1}]$ inherits the structure of a monoidal ∞-category.*

Remark 3.5 The requirement that the unit be cofibrant is often not taken as part of the definition of a monoidal model category, as there are important examples of model categories with monoidal structures where the unit is not cofibrant, but the other requirements for a monoidal model category as we have defined it are satisfied. We therefore point out that the assumption that $\mathbf{V}$ has a cofibrant unit is not essential for Proposition 3.4 to hold. If we drop this assumption then $(N\mathbf{V}, W)$ is still a non-unital associative algebra, and so $\mathbf{V}[W^{-1}]$ inherits a non-unital monoidal ∞-category structure. It is easy to see that a cofibrant replacement for the unit of the monoidal structure in $\mathbf{V}$ gives a *quasi-unit* in the sense of [21, Definition 5.4.3.5] — roughly speaking, this is an object I such that $X \otimes I \simeq X \simeq I \otimes X$ for every object X, but we are not given coherent associativity data for combinations of multiple such equivalences. A non-unital monoidal ∞-category with a quasi-unit can be extended to a full monoidal structure with this as unit in an essentially unique way by [21, Theorem 5.4.3.8], and so a monoidal model category without a cofibrant unit still induces a monoidal ∞-category structure on its associated ∞-category.

3.2 Model Categories of Associative Algebras

In this subsection we briefly recall the construction of a model structure on associative algebras, due to Schwede and Shipley, and observe that it generalizes to non-symmetric monoidal model categories satisfying an appropriate version of the monoid axiom. First we recall an observation of Schwede and Shipley on model structures for algebras over monads:

Definition 3.6 Let T be a monad on a model category $\mathbf{C}$. We say that T is an *admissible* monad if there exists a model structure on the category $\mathrm{Alg}(T)$ of T-algebras where a morphism is a weak equivalence or fibration if and only if the underlying morphism in $\mathbf{C}$ is a weak equivalence or fibration.

Write $F_T \colon \mathbf{C} \rightleftarrows \mathrm{Alg}(T) \colon U_T$ for the associated adjunction. If $\mathbf{C}$ is a combinatorial model category with sets I and J of generating cofibrations and trivial cofibrations, we say that T is *combinatorially admissible* if it is admissible and the model structure on $\mathrm{Alg}(T)$ is combinatorial with $F_T(I)$ and $F_T(J)$ as sets of generating cofibrations and trivial cofibrations.

Remark 3.7 Given a monad T on $\mathbf{C}$, a model structure on $\mathrm{Alg}(T)$ where a morphism is a weak equivalence or a fibration if and only if its underlying morphism in $\mathbf{C}$ is one is unique if it exists. Clearly, the existence of such a model structure implies certain restrictions on T — for example, it must preserve weak equivalences between cofibrant objects — but we will not attempt to describe these here, as we will only need the following admissibility criterion of Schwede and Shipley:

Theorem 3.8 (Schwede-Shipley, [29, Lemma 2.3]) *Suppose* $\mathbf{C}$ *is a combinatorial model category and* T *is a filtered-colimit-preserving monad on* $\mathbf{C}$*, and let* J *be a set* *of generating trivial cofibrations for* $\mathbf{C}$. *If the underlying morphism in* $\mathbf{C}$ *of every morphism in the weakly saturated class generated by* $F_T(J)$ *in* $\mathrm{Alg}(T)$ *is a weak equivalence, then* T *is combinatorially admissible.*

Remark 3.9 Since weak equivalences in $\mathbf{C}$ are closed under retracts and transfinite composites, the weakly saturated class generated by $F_T(J)$ will be contained in the weak equivalences provided the pushout of any morphism in $F_T(J)$ along any morphism in $\mathrm{Alg}(T)$ is a weak equivalence.

In [29], Schwede and Shipley analyze such pushouts in the case of associative algebras. They show that the pushout is a transfinite composite of pushouts of certain maps, as follows:

Theorem 3.10 (Schwede-Shipley [29, § 6]) *Suppose* $\mathbf{C}$ *is a combinatorial biclosed monoidal model category. Write* $\mathrm{Alg}(\mathbf{C})$ *for the category of associative algebra objects of* $\mathbf{C}$ *and* $F\colon \mathbf{C} \rightleftarrows \mathrm{Alg}(\mathbf{C}) : U$ *for the free algebra functor and forgetful functor. Let* $f\colon X \to Y$ *be a morphism in* $\mathbf{C}$ *and* $g\colon F(X) \to A$ *be a morphism in* $\mathrm{Alg}(\mathbf{C})$. *If*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[f']{} & B \end{array}$$

is a pushout diagram in $\mathrm{Alg}(\mathbf{C})$, *then there is a sequence of morphisms in* $\mathbf{C}$

$$A = B_0 \xrightarrow{\phi_1} B_1 \xrightarrow{\phi_2} B_2 \cdots$$

such that $B = \mathrm{colim}_t B_t$ *and* ϕ_t *is a pushout of* $(j\Box f)^{\Box n}\Box j$, *where* j *is the unique morphism* $\emptyset \to A$.

Based on this result Schwede and Shipley give a condition — the *monoid axiom* — for the hypothesis of Theorem 3.8 to hold, when the monoidal structure on the model category $\mathbf{C}$ is *symmetric*, which is true in most of the interesting examples. However, in the next section we wish to consider associative algebras in functor cateogories $\mathrm{Fun}(X \times X, \mathbf{V})$ (where X is a set), equipped with the non-symmetric "matrix multiplication" tensor product, for which associative algebras are precisely $\mathbf{V}$-categories with X as their set of objects. As noted by Muro [23], the following non-symmetric version of the monoid axiom applies in this context:

Definition 3.11 Suppose $\mathbf{C}$ is a monoidal model category, and let $\mathfrak{U}$ be the set of morphisms in $\mathbf{C}$ of the form $f_1 \Box \cdots \Box f_n$ where each f_i is either a trivial cofibration or of the form $\emptyset \to X_i$ for some cofibrant $X_i \in \mathbf{C}$, with at least one f_i being a trivial cofibration. We say that $\mathbf{C}$ satisfies the *monoid axiom* if the weakly saturated class $\overline{\mathfrak{U}}$ generated by $\mathfrak{U}$ is contained in the weak equivalences in $\mathbf{C}$.

Remark 3.12 Since the pushout-product $(\emptyset \to A) \Box f$ is just the tensor product $A \otimes f$ for any morphism f, the morphisms in $\mathfrak{U}$ are all trivial cofibrations in $\mathbf{C}$.

Remark 3.13 If $\mathbf{C}$ is *symmetric* monoidal, then we can use the symmetry to move all the morphisms of the form $\emptyset \to A$ in an element of $\mathfrak{U}$ to one side. Thus, since the pushout product of trivial cofibrations in $\mathbf{C}$ is a trivial cofibration by Remark 3.3, in the symmetric case the monoid axiom is equivalent to the corresponding statement where $\mathfrak{U}$ consists of morphisms of the form $f \otimes X$ with f a trivial cofibration and X a cofibrant object of $\mathbf{C}$. This is the original form of the monoid axiom, due to Schwede and Shipley.

Corollary 3.14 *Let $\mathbf{C}$ be a combinatorial biclosed monoidal model category that satisfies the monoid axiom. Then the free associative algebra monad on $\mathbf{C}$ is combinatorially admissible.*

Proof By Remark 3.9 it suffices to show that if $f \colon X \to Y$ is a trivial cofibration in $\mathbf{C}$, $g \colon F(X) \to A$ is a morphism in $\mathrm{Alg}(\mathbf{C})$, and

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ A & \xrightarrow[f']{} & B \end{array}$$

is a pushout diagram in $\mathrm{Alg}(\mathbf{C})$, then f' is a weak equivalence in $\mathbf{C}$. Since $\mathbf{C}$ satisfies the monoid axiom, it suffices to show that f' is contained in the weakly saturated class $\overline{\mathfrak{U}}$ generated by the class $\mathfrak{U}$ from Definition 3.11.

By Theorem 3.10, the morphism f' is a transfinite composite of pushouts of morphisms of the form $(j \Box f)^{\Box n} \Box j$, where j is the unique morphism $\emptyset \to A$, so to show that f' is contained in $\overline{\mathfrak{U}}$ it suffices to observe that these morphisms are contained in $\mathfrak{U}$ by definition. □

We will also need the following result of Schwede and Shipley:

Corollary 3.15 *Let* $\mathbf{C}$ *be a combinatorial biclosed monoidal model category that satisfies the monoid axiom. Then the forgetful functor* $\mathrm{Alg}(\mathbf{C}) \to \mathbf{C}$ *preserves cofibrant objects.*

3.3 Rectifying Algebras

We now observe that Lurie's rectification result for associative algebras also holds for non-symmetric monoidal model categories. To state the result, we first make the following definition:

Definition 3.16 Let $\mathbf{C}$ be a left proper tractable biclosed monoidal model category that satisfies the monoid axiom. By Corollary 3.15, the forgetful functor from associative algebras in $\mathbf{C}$ to $\mathbf{C}$ preserves cofibrant objects, so we have a natural functor $\mathrm{Alg}(\mathbf{C})^{\mathrm{cof}} \to \mathrm{Alg}(\mathbf{C}^{\mathrm{cof}}$. It is immediate from the construction of the monoidal ∞-category structure on $\mathbf{C}[W^{-1}]$ in Proposition 3.4, where W denotes the weak equivalences in $\mathbf{C}$, that there is a monoidal functor $\mathbf{C}^{\mathrm{cof}} \to \mathbf{C}[W^{-1}]$, which induces a functor of ∞-categories $\mathrm{Alg}(\mathbf{C}^{\mathrm{cof}}) \to \mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathbf{C}[W^{-1}])$. The composite functor $\mathrm{Alg}(\mathbf{C})^{\mathrm{cof}} \to \mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathbf{C}[W^{-1}])$ clearly takes weak equivalences of algebras to equivalences, and so induces a functor

$$\mathrm{Alg}(\mathbf{C})[\hat{W}^{-1}] \to \mathrm{Alg}_{\Delta^{\mathrm{op}}}(\mathbf{C}[W^{-1}])$$

is an equivalence, where $\hat{W}$ denotes the weak equivalences in the model structure on $\mathrm{Alg}(\mathbf{C})$.

Theorem 3.17 (Lurie) *Let* $\mathbf{C}$ *be a left proper tractable biclosed monoidal model category that satisfies the monoid axiom. Then the functor of* ∞*-categories*

$$\mathrm{Alg}(\mathbf{C})[\hat{W}^{-1}] \to \mathrm{Alg}(\mathbf{C}[W^{-1}])$$

defined above is an equivalence.

The proof is exactly the same as the proof of [21, Theorem 4.1.4.4]; in particular, the key technical result [21, Lemma 4.1.4.13] generalizes to this context:

Definition 3.18 Suppose $\mathbf{C}$ is a left proper tractable biclosed monoidal model category that satisfies the monoid axiom. Then the forgetful functor $U\colon \mathrm{Alg}(\mathbf{C}) \to \mathbf{C}$ takes weak equivalences to weak equivalences, by definition of the model structure on $\mathrm{Alg}(\mathbf{C})$. The composite functor of ∞-categories $\mathrm{Alg}(\mathbf{C}) \to \mathbf{C} \to \mathbf{C}[W^{-1}]$ thus takes the morphisms in $\hat{W}$ to equivalences in $\mathbf{C}[W^{-1}]$ and so factors through a unique functor $U_\infty\colon \mathrm{Alg}(\mathbf{C})[\hat{W}^{-1}] \to \mathbf{C}[W^{-1}]$ — this is the functor of ∞-categories associated to the right Quillen functor U.

Lemma 3.19 (Lurie) *Suppose $\mathbf{C}$ is a left proper tractable biclosed monoidal model category that satisfies the monoid axiom and $\mathbf{I}$ is a small category such that $\mathrm{N}\mathbf{I}$ is sifted. Then the forgetful functor $U_\infty\colon \mathrm{Alg}(\mathbf{C})[\hat{W}^{-1}] \to \mathbf{C}[W^{-1}]$ preserves $\mathrm{N}\mathbf{I}$-indexed colimits.*

We omit the proof, as it is exactly the same as that of [21, Lemma 4.1.4.13]. We will make use of Lemma 3.19 in the case of enriched categories, for which we have the following observation:

Lemma 3.20 *If $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom and X is a set, then there is a combinatorial model category structure on the category $\mathrm{Cat}_X(\mathbf{V})$ such that a morphism is a fibration or weak equivalence if and only if its image in $\mathrm{Fun}(X \times X, \mathbf{V})$ is. Moreover, if $\mathbf{I}$ is a small category such that $\mathrm{N}\mathbf{I}$ is sifted then the forgetful functor*

$$\mathrm{Cat}_X(\mathbf{V})[\mathrm{FF}_X^{-1}] \to \mathrm{Fun}(X \times X, \mathbf{V})[W_X^{-1}]$$

preserves $\mathrm{N}\mathbf{I}$-indexed colimits, where W_X denotes the class of natural transformations that are weak equivalences objectwise.

Proof Recall that if $\mathbf{V}$ is a biclosed monoidal category and X is a set then there is a monoidal structure on $\mathrm{Fun}(X \times X, \mathbf{V})$, given by

$$(F \otimes G)(x,y) = \coprod_{z \in X} F(x,z) \otimes G(z,y),$$

such that an associative algebra object in $\mathrm{Fun}(X \times X, \mathbf{V})$ is precisely a $\mathbf{V}$-category with objects X. By [23, Proposition 10.3], if $\mathbf{V}$ is a monoidal model category satisfying the monoid axiom, then so is $\mathrm{Fun}(X \times X, \mathbf{V})$ equipped with this monoidal structure. Applying Corollary 3.14 and Lemma 3.19 to $\mathrm{Fun}(X \times X, \mathbf{V})$ then implies the result. □

4 Fibrewise Localization

Suppose we have a functor of ordinary categories $F\colon \mathbf{C} \to \mathrm{Cat}$ together with a collection W_C of weak equivalences in each category $F(C)$ that is preserved by the functors $F(f)$. Then we have two ways to construct an ∞-category over $\mathbf{C}$ where these weak equivalences are inverted: On the one hand we can invert the weak equivalences in each category $F(C)$ to get a functor $\mathbf{C} \to \mathrm{Cat}_\infty$ that sends C to $F(C)[W_C^{-1}]$, which corresponds to a coCartesian fibration $\mathcal{E} \to \mathbf{C}$. On the other hand, if $\mathbf{E} \to \mathbf{C}$ is a

Grothendieck opfibration corresponding to F then there is a natural collection W of weak equivalences in $\mathbf{E}$ induced by those in the fibres, and we can invert these to get an ∞-category $\mathbf{E}[W^{-1}]$. Our main goal in this section is to prove that in this situation the natural map $\mathbf{E}[W^{-1}] \to \mathcal{E}$ is an equivalence of ∞-categories.

We will do this in two steps: in §4.1 we show that the ∞-category $\mathcal{E}$ here is a fibrant replacement in the coCartesian model structure on $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$ for $\mathrm{N}\mathbf{E}$ marked by the edges in W, then in §4.2 we use an explicit model for $\mathbf{E}[W^{-1}]$ to show that this, equipped with a natural choice of marked edges, is also weakly equivalent to $(\mathrm{N}\mathbf{E}, W)$. In addition, we prove in §4.3 that when the weak equivalences in each category $F(C)$ come from a (combinatorial) model structure, then there is a (combinatorial) model structure on $\mathbf{E}$ whose weak equivalences are the morphisms in W.

Remark 4.1 Fibrewise localization has also recently been studied by Hinich in [15]. His approach is quite different from ours, but allows him to prove a comparison analogous to ours also in the more general case where the base $\mathbf{C}$ is itself equipped with a class of weak equivalences.

4.1 The Relative Nerve

Recall that a *relative category* is a category $\mathbf{C}$ equipped with a collection of "weak equivalences", i.e. a subcategory W containing all objects and isomorphisms. Write RelCat for the obvious category of relative categories; this has been studied as a model for the theory of $(\infty, 1)$-categories by Barwick and Kan [4]. The usual nerve functor from categories to simplicial sets extends to a functor $L\colon \mathrm{RelCat} \to \mathrm{Set}_\Delta^+$ that sends $(\mathbf{C}, W)$ to $(\mathrm{N}\mathbf{C}, \mathrm{N}W_1)$. In [19, §3.1.3] Lurie constructs a model structure on Set_Δ^+ where a fibrant replacement for $L(\mathbf{C}, W)$ is precisely an ∞-categorical localization of $\mathbf{C}$ that inverts the morphisms in W (marked by the equivalences).

If $\mathbf{C}$ is a category, there is a model structure on $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$ where a fibrant object is a coCartesian fibration marked by its coCartesian morphisms, constructed in [19, §3.1.3], and in [19, §3.5.2] Lurie describes a right Quillen equivalence $\mathrm{N}_{\mathbf{C}}^+$ from the projective model structure on $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta^+)$ to this model structure on $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$. Given a functor $F\colon \mathbf{C} \to \mathrm{RelCat}$ we therefore have two reasonable ways of constructing a fibrant object of $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$:

(i) Find a fibrant replacement $\overline{F}$ for the functor $LF\colon \mathbf{C} \to \mathrm{Set}_\Delta^+$, and then form $\mathrm{N}_{\mathbf{C}}^+\overline{F}$.

(ii) Construct a Grothendieck opfibration $\mathbf{E} \to \mathbf{C}$ associated to F, regarded as a functor to categories, and write S for the collection of 1-simplices in $\mathrm{N}\mathbf{E}$ that correspond to composites of (fibrewise) weak equivalences and coCartesian morphisms. Then find a fibrant replacement in $(\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$ for $(\mathrm{N}\mathbf{E}, S) \to \mathrm{N}\mathbf{C}$.

Our main goal in this subsection is to prove that these give weakly equivalent objects. We begin by reviewing the definition of the functor $\mathrm{N}^+_{\mathbf{C}}$:

Definition 4.2 Let $\mathbf{C}$ be a category. Given a functor $F\colon \mathbf{C} \to \mathrm{Set}_\Delta$, we define $\mathrm{N}_{\mathbf{C}}F$ to be the simplicial set characterized by the property that a morphism $\Delta^I \to \mathrm{N}_{\mathbf{C}}F$, where I is a partially ordered set, is determined by:

(1) a functor $\sigma\colon I \to \mathbf{C}$,

(2) for every non-empty subset $J \subseteq I$ with maximal element j, a map $\tau_J\colon \Delta^J \to F(\sigma(j))$,

such that for all subsets $K \subseteq J \subseteq I$ with maximal elements $k \in K$ and $j \in J$, the diagram

$$\begin{array}{ccc} \Delta^K & \xrightarrow{\tau_K} & F(\sigma(k)) \\ \downarrow & & \downarrow \\ \Delta^J & \xrightarrow[\tau_J]{} & F(\sigma(j)) \end{array}$$

commutes. This defines a functor $\mathrm{N}_{\mathbf{C}}\colon \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta) \to (\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}}$.

The functor $\mathrm{N}_{\mathbf{C}}$ has a left adjoint, which we denote

$$\mathfrak{F}_{\mathbf{C}}\colon (\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}} \to \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta).$$

Proposition 4.3 *Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a functor. Then $\mathfrak{F}_{\mathbf{C}}\mathrm{N}\mathbf{E}$ is isomorphic to the functor $O_\pi\colon \mathbf{C} \to \mathrm{Set}_\Delta$ defined by $C \mapsto \mathrm{N}\mathbf{E}_{/C}$.*

Proof We must show that there is a natural isomorphism $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(\{\})) \cong \mathrm{Hom}(O_\pi, \{\})$; we will do this by defining explicit natural transformations

$$\phi\colon \mathrm{Hom}(O_\pi, \{\}) \to \mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(\{\}))$$

and

$$\psi\colon \mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}(\{\})) \to \mathrm{Hom}(O_\pi, \{\})$$

that are inverse to each other.

Given $X\colon \mathbf{C} \to \mathrm{Set}_\Delta$ and a natural transformation $\eta\colon O_\pi \to X$, define $\phi(\eta)\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_\mathbf{C}X$ to be the morphism that sends a simplex $\sigma\colon \Delta^I \to \mathrm{N}\mathbf{E}$ (which we can identify with a functor $I \to \mathbf{E}$) to the simplex of $\mathrm{N}_\mathbf{C}X$ determined by

- the composite functor $I \to \mathbf{E} \to \mathbf{C}$,
- for $J \subseteq I$ with maximal element j, the composite

$$\Delta^J \to \mathrm{N}\mathbf{E}_{/\pi(\sigma(j))} \xrightarrow{\eta_{\pi(\sigma(j))}} X(\pi(\sigma(j))).$$

Conversely, given a map $G\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_\mathbf{C}X$ of simplicial sets over $\mathrm{N}\mathbf{C}$, let $\psi(G)$ be the natural transformation $O_\pi \to X$ determined as follows: for $C \in \mathbf{C}$, the morphism $\psi(G)_C\colon \mathrm{N}\mathbf{E}_{/C} \to X(C)$ sends a simplex $\sigma\colon \Delta^I \to \mathrm{N}\mathbf{E}_{/C}$, where I has maximal element i, to the composite

$$\Delta^I \xrightarrow{\tau} X(\pi\sigma(i)) \xrightarrow{X(f)} X(C)$$

where

- τ is the I-simplex determined by the image under G of the I-simplex σ' of $\mathrm{N}\mathbf{E}$ underlying σ,
- f is the morphism $\pi(\sigma(i)) \to C$ in $\mathbf{C}$ from σ.

The remaining data in $G \circ \sigma'$ implies that this defines a map of simplicial sets $\mathrm{N}\mathbf{E}_{/C} \to X(C)$, and it is also easy to see that $\psi(G)$ is natural in C.

Both ϕ and ψ are obviously natural in X, and expanding out the definitions we see that $\phi\psi = \mathrm{id}$ and $\psi\phi = \mathrm{id}$, so we have the required natural isomorphism. □

Definition 4.4 Let $\mathbf{C}$ be a category. Given a functor $\overline{F}\colon \mathbf{C} \to \mathrm{Set}_\Delta^+$ we define $\mathrm{N}_\mathbf{C}^+\overline{F}$ to be the marked simplicial set $(\mathrm{N}_\mathbf{C}F, M)$ where F is the underlying functor $\mathbf{C} \to \mathrm{Set}_\Delta$ of $\overline{F}$, and M is the set of edges $\Delta^1 \to \mathrm{N}_\mathbf{C}F$ determined by

- a morphism $f\colon C \to C'$ in $\mathbf{C}$,
- a vertex $X \in F(C)$,
- a vertex $X' \in F(C')$ and an edge $F(f)(X) \to X'$ that is marked in $\overline{F}(C')$.

This determines a functor $\mathrm{N}_\mathbf{C}^+\colon \mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta^+) \to (\mathrm{Set}_\Delta^+)_{/\mathrm{N}\mathbf{C}}$.

The functor $\mathrm{N}_\mathbf{C}^+$ has a left adjoint, which we denote $\mathfrak{F}_\mathbf{C}^+$.

Corollary 4.5 *Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a functor, and let M be a set of edges of $\mathrm{N}\mathbf{E}$ that contains the degenerate edges. Then $\mathfrak{F}_\mathbf{C}^+(\mathrm{N}\mathbf{E}, M)$ is isomorphic to the functor $\overline{O}_\pi$ defined by $C \mapsto (\mathrm{N}\mathbf{E}_{/C}, M_C)$, where M_C is the collection of edges determined by $E \to E'$ in $\mathbf{E}$ and $\pi(E) \to \pi(E') \to C$ in $\mathbf{C}$ such that $\pi(E') \cong C$ and $E \to E'$ is in M.*

Proof We must show that there is a natural isomorphism

$$\mathrm{Hom}((\mathrm{N}\mathbf{E}, M), \mathrm{N}^+_{\mathbf{C}}(\{)) \cong \mathrm{Hom}(\overline{O}_\pi, \{).$$

Given $\overline{X}\colon \mathbf{C} \to \mathrm{Set}^+_\Delta$, with underlying functor $X\colon \mathbf{C} \to \mathrm{Set}_\Delta$, and a morphism $G\colon \mathrm{N}\mathbf{E} \to \mathrm{N}_{\mathbf{C}}X$, it is immediate from the definitions that G takes an edge $\sigma\colon E \to E'$ of $\mathrm{N}\mathbf{E}$ lying over $C \to C'$ in $\mathbf{C}$ to a marked edge of $\mathrm{N}^+_{\mathbf{C}}\overline{X}$ if and only if $\phi(G)_{C'}$ takes σ, regarded as an edge of $\mathrm{N}\mathbf{E}_{/C'}$, to a marked edge of $\overline{X}(C')$. Thus the natural isomorphism $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}X) \cong \mathrm{Hom}(O_\pi, X)$ of Proposition 4.3 identifies $\mathrm{Hom}((\mathrm{N}\mathbf{E}, M), \mathrm{N}^+_{\mathbf{C}}\overline{X})$, regarded as a subset of $\mathrm{Hom}(\mathrm{N}\mathbf{E}, \mathrm{N}_{\mathbf{C}}X)$, with $\mathrm{Hom}(\overline{O}_\pi, \overline{X})$, regarded as a subset of $\mathrm{Hom}(O_\pi, X)$. ◻

Theorem 4.6 (Lurie, [19, Proposition 3.2.5.18])

(i) *The adjunction $\mathfrak{F}_{\mathbf{C}} \dashv \mathrm{N}_{\mathbf{C}}$ is a Quillen equivalence between $(\mathrm{Set}_\Delta)_{/\mathrm{N}\mathbf{C}}$ equipped with the covariant model structure and $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_\Delta)$ equipped with the projective model structure.*

(ii) *The adjunction $\mathfrak{F}^+_{\mathbf{C}} \dashv \mathrm{N}^+_{\mathbf{C}}$ is a Quillen equivalence between $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ equipped with the coCartesian model structure and $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$ equipped with the projective model structure.*

Remark 4.7 By [19, Lemma 3.2.5.17], the functor $\mathfrak{F}^+_{\mathbf{C}}$ is naturally weakly equivalent to the straightening functor defined in [19, §3.2.1], which takes a fibrant functor $\mathbf{C} \to \mathrm{Set}^+_\Delta$ to the associated coCartesian fibration.

Recall that if $\mathcal{C}$ is an ∞-category we write $\mathcal{C}^\natural$ for the marked simplicial set given by $\mathcal{C}$ marked by the equivalences, and that if $\mathcal{E} \to \mathrm{N}\mathbf{C}$ is a coCartesian fibration we write $\mathcal{E}^\natural$ for the object of $(\mathrm{Set}^+_\Delta)_{/\mathrm{N}\mathbf{C}}$ given by $\mathcal{E}$ marked by the coCartesian morphisms.

Lemma 4.8 *Let $F\colon \mathbf{C} \to \mathrm{Cat}$ be a functor. Write $\pi\colon \mathbf{E} \to \mathbf{C}$ for the Grothendieck opfibration associated to F, so that $\mathbf{E}$ has objects pairs $(C \in \mathbf{C}, X \in F(C))$ and a morphism $(C, X) \to (D, Y)$ in $\mathbf{E}$ is given by a morphism $f\colon C \to D$ in $\mathbf{C}$ and a morphism $F(f)(X) \to Y$ in $F(D)$. Then:*

(i) $\mathrm{N}_{\mathbf{C}}(\mathrm{N}F) \to \mathrm{N}\mathbf{C}$ *is isomorphic to* $\mathrm{N}\pi$.

(ii) $\mathrm{N}^+_{\mathbf{C}}(\mathrm{N}F^\natural) \to \mathrm{N}\mathbf{C}$ *is isomorphic to* $(\mathrm{N}\mathbf{E})^\natural \to \mathrm{N}\mathbf{C}$.

Proof It is clear from the definition of $\mathrm{N}_{\mathbf{C}}$ that there is a natural isomorphism between n-simplices of $\mathrm{N}_{\mathbf{C}}(\mathrm{N}F)$ and n-simplices of $\mathrm{N}\mathbf{E}$, which proves (i). By definition, an edge of $\mathrm{N}^+_{\mathbf{C}}(\mathrm{N}F^\natural)$ is marked if it is given by $f\colon C \to C'$ in $\mathbf{C}$, $X \in F(C)$, and $F(f)(X) \to X'$ an isomorphism in $F(C')$. Under the identification with edges of $\mathrm{N}\mathbf{E}$, such edges precisely correspond to the coCartesian edges. This proves (ii). ◻

Proposition 4.9 *Given* $F\colon \mathbf{C} \to \mathrm{RelCat}$*, the counit map* $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}LF \to LF$ *is a weak equivalence in* $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^{+}_{\Delta})$.

Proof Since $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^{+}_{\Delta})$ is equipped with the projective model structure, it suffices to show that for all $C \in \mathbf{C}$ the morphism $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}LF(C) \to LF(C)$ is a weak equivalence in $\mathrm{Set}^{+}_{\Delta}$. Let F_0 be the underlying functor $\mathbf{C} \to \mathrm{Cat}$, and let $\mathbf{E} \to \mathbf{C}$ be the canonical Grothendieck opfibration associated to F_0. Then by Lemma 4.8 we can identify $\mathrm{N}^{+}_{\mathbf{C}}\mathrm{N}F_0^{\natural}$ with $\mathrm{N}\mathbf{E}^{\natural}$, and so by Corollary 4.5 we can identify $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}\mathrm{N}F_0^{\natural}(C)$ with $\mathrm{N}\mathbf{E}_{/C}$, marked by the set M_C of coCartesian morphisms $E \to E'$ such that $\pi(E') = C$.

The adjunction $\mathfrak{F}^{+}_{\mathbf{C}} \dashv \mathrm{N}^{+}_{\mathbf{C}}$ is a Quillen equivalence, so since $\mathrm{N}F_0^{\natural}$ is fibrant and every object of $(\mathrm{Set}^{+}_{\Delta})_{/\mathrm{N}\mathbf{C}}$ is cofibrant, the counit $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}\mathrm{N}F_0^{\natural} \to \mathrm{N}F_0^{\natural}$ is a weak equivalence in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^{+}_{\Delta})$. In particular, $(\mathrm{N}\mathbf{E}_{/C}, M_C) \to \mathrm{N}F_0(C)^{\natural}$ is a weak equivalence.

Let M'_C be the set of edges of $\mathrm{N}\mathbf{E}_{/C}$ corresponding to weak equivalences in $F(C)$. Then we have a pushout diagram

$$\begin{array}{ccc} (\mathrm{N}\mathbf{E}_{/C}, M_C) & \longrightarrow & \mathrm{N}F_0(C)^{\natural} \\ \downarrow & & \downarrow \\ (\mathrm{N}\mathbf{E}_{/C}, M_C \cup M'_C) & \longrightarrow & LF(C), \end{array}$$

since both vertical maps are pushouts along $\coprod_{f \in M'_C} \Delta^1 \hookrightarrow \coprod_{f \in M'_C} (\Delta^1)^{\sharp}$. As the model structure on $\mathrm{Set}^{+}_{\Delta}$ is left proper, it follows that $(\mathrm{N}\mathbf{E}_{/C}, M_C \cup M'_C) \to LF(C)$ is a weak equivalence.

By Corollary 4.5 we can identify $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}LF(C)$ with the simplicial set $\mathrm{N}\mathbf{E}_{/C}$, marked by the set M''_C of morphisms $E \to E'$ with $\pi(E') = C$ such that given a coCartesian factorization $E \to \overline{E} \to E'$ the morphism $\overline{E} \to E'$ is a weak equivalence in $LF(C)$. The obvious map $(\mathrm{N}\mathbf{E}_{/C}, M_C \cup M'_C) \to \mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}LF(C)$ is therefore marked anodyne, since the edges in M''_C are precisely the composites of edges in M_C and M'_C. In particular this is also a weak equivalence, and so by the 2-out-of-3 property the map $\mathfrak{F}^{+}_{\mathbf{C}}\mathrm{N}^{+}_{\mathbf{C}}LF(C) \to LF(C)$ is a weak equivalence, as required. □

Corollary 4.10 *Given* $F\colon \mathbf{C} \to \mathrm{RelCat}$*, let* $LF \to \overline{F}$ *be a fibrant replacement in the projective model structure on* $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^{+}_{\Delta})$. *Then* $\mathrm{N}^{+}_{\mathbf{C}}LF \to \mathrm{N}^{+}_{\mathbf{C}}\overline{F}$ *is a coCartesian equivalence in* $(\mathrm{Set}^{+}_{\Delta})_{/\mathrm{N}\mathbf{C}}$.

Proof The adjunction $\mathfrak{F}^{+}_{\mathbf{C}} \dashv \mathrm{N}^{+}_{\mathbf{C}}$ is a Quillen equivalence, so since $\overline{F}$ is fibrant and every object of $(\mathrm{Set}^{+}_{\Delta})_{/\mathrm{N}\mathbf{C}}$ is cofibrant, the morphism $\mathrm{N}^{+}_{\mathbf{C}}LF \to \mathrm{N}^{+}_{\mathbf{C}}\overline{F}$ is a weak equivalence if and only if the adjunct morphism $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \to \overline{F}$ is a weak equivalence. This follows by the 2-out-of-3 property, since in the commutative diagram

$$\begin{array}{ccc} \mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF & \longrightarrow & LF \\ & \searrow \quad \swarrow & \\ & \overline{F} & \end{array}$$

the morphism $LF \to \overline{F}$ is a weak equivalence by assumption, and $\mathfrak{F}^+_{\mathbf{C}}\mathrm{N}^+_{\mathbf{C}}LF \to LF$ is a weak equivalence by Proposition 4.9. ◻

Using Lemma 4.8 we can equivalently state this as:

Corollary 4.11 *Given $F\colon \mathbf{C} \to \mathrm{RelCat}$, suppose $\pi\colon \mathbf{E} \to \mathbf{C}$ is a Grothendieck opfibration corresponding to the underlying functor $\mathbf{C} \to \mathrm{Cat}$. Let M be the set of morphisms $f\colon E \to E'$ in $\mathbf{E}$ such that given a coCartesian factorization $E \to \pi(f)_!E \to E'$, the morphism $\pi(f)_!E \to E'$ is a weak equivalence in $F(\pi(E'))$. Then if $LF \to \overline{F}$ is a fibrant replacement in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}^+_\Delta)$, there is a coCartesian equivalence $(\mathrm{N}\mathbf{E}, M) \to \mathrm{N}^+_{\mathbf{C}}\overline{F}$.*

4.2 The Hammock Localization

Consider a functor $F\colon \mathbf{C} \to \mathrm{RelCat}$, and let $\pi\colon \mathbf{E} \to \mathbf{C}$ be an opfibration associated to the underlying functor $\mathbf{C} \to \mathrm{Cat}$. Our main goal in this subsection is to prove that inverting the collection W of fibrewise weak equivalences in $\mathbf{E}$ gives a coCartesian fibration $\mathbf{E}[W^{-1}] \to \mathbf{C}$. As a corollary, we will also see that $\mathbf{E}[W^{-1}]$ is the total space of the coCartesian fibration associated to the functor obtained from F by inverting the weak equivalences in the relative categories $F(C)$. We will prove this result by analyzing an explicit model for $\mathbf{E}[W^{-1}]$ as a simplicial category, namely the *hammock localization*. We begin by recalling the definition of this, specifically the version defined in [7, §35], and its basic properties:

Definition 4.12 A *zig-zag type* $Z = (Z_+, Z_-)$ consists of a decomposition $\{1, \ldots, n\} = Z_+ \amalg Z_-$. The *zig-zag category* $\mathbf{ZZ}$ is the category with objects zig-zag types and morphisms $Z \to Z'$ given by order-preserving morphisms $f\colon \{1, \ldots, n\} \to \{1, \ldots, n'\}$ such that $f(Z_+) \subseteq Z'_+$ and $f(Z_-) \subseteq Z'_-$. If Z is a zig-zag type, the associated zig-zag category $|Z|$ is the category with objects $0, \dots, n$ and

$$|Z|(i,j) = \begin{cases} *, & i \leq j, k \in Z_+ \text{ for } k = i+1, \dots, j, \\ *, & i \geq j, k \in Z_- \text{ for } k = j+1, \dots, i, \\ \emptyset, & \text{otherwise}. \end{cases}$$

This clearly gives a functor $|\{|\colon \mathbf{ZZ} \to \mathrm{Cat}$. If n is an odd integer, we abbreviate

$$\langle n \rangle := (\{2, 4, \dots, n-1\}, \{1, 3, \dots, n\})$$

and if n is an even integer we abbreviate

$$\langle n \rangle := (\{1, 3, \dots, n-1\}, \{2, 4, \dots, n\}).$$

Definition 4.13 Suppose $(\mathbf{C}, W)$ is a relative category. For $x, y \in \mathbf{C}$ and $Z \in \mathbf{ZZ}$ we define $\mathbf{L}_W\mathbf{C}_Z(x,y)$ to be the subcategory of $\mathrm{Fun}(|Z|, \mathbf{C})$ whose objects are the functors $F\colon |Z| \to \mathbf{C}$ such that $F(0) = x$, $F(n) = y$, and $F(i \to (i-1))$ is in W for all $i \in Z_-$, and whose morphisms are the natural transformations $\eta\colon F \to G$ such that $\eta_0 = \mathrm{id}_x$, $\eta_n = \mathrm{id}_y$, and η_i is in W for all i. We write $\mathcal{L}_W\mathbf{C}_Z(x,y) := \mathrm{N}\mathbf{L}_W\mathbf{C}_Z(x,y)$.

This construction gives a functor $\mathbf{ZZ}^{\mathrm{op}} \to \mathrm{Cat}$; we let $\mathbf{L}_W\mathbf{C}(x,y) \to \mathbf{ZZ}$ be the fibration associated to it by the Grothendieck construction. Using concatenation of zig-zags we get a strict 2-category $\mathbf{L}_W\mathbf{C}$ with the same objects as $\mathbf{C}$ and with mapping categories $\mathbf{L}_W\mathbf{C}(x,y)$; taking nerves, this gives a simplicial category $\mathcal{L}_W\mathbf{C}$ whose mapping spaces are $\mathcal{L}_W\mathbf{C}(x,y) := \mathrm{N}\mathbf{L}_W\mathbf{C}(x,y)$. This simplicial category is the *hammock localization* of $(\mathbf{C}, W)$.

Theorem 4.14 (Dwyer-Kan) *Let $(\mathbf{C}, W)$ be a relative category. Then:*

(i) The diagram

$$\begin{array}{ccc} W & \longrightarrow & \mathcal{L}_W W \\ \downarrow & & \downarrow \\ \mathbf{C} & \longrightarrow & \mathcal{L}_W\mathbf{C} \end{array}$$

is a homotopy pushout square in simplicial categories.

(ii) If $\mathcal{L}_W W \to \overline{\mathcal{L}}_W W$ is a fibrant replacement in simplicial categories, then $\mathrm{N}\overline{\mathcal{L}}_W W$ is a Kan complex and $\mathrm{N}W \to \mathrm{N}\overline{\mathcal{L}}_W W$ is a weak equivalence of simplicial sets.

Proof (i) follows by combining [7, Proposition 35.7], [8, Proposition 2.2], and [9, §4.5] (observe that a cofibration in the model structure on simplicial categories with a fixed set of objects described in [9, §7] is also a cofibration in the model structure on simplicial categories).

To prove (ii), we first observe that it follows from [9, §9.1] that $\mathcal{L}_W W$ is a simplicial groupoid. If $\mathcal{L}_W W \to \overline{\mathcal{L}}_W W$ is a fibrant replacement in simplicial categories, then $\mathrm{N}\overline{\mathcal{L}}_W W$ is the nerve of a fibrant simplicial groupoid, hence a Kan complex by [10, Theorem 3.3]. Let $\mathfrak{G}$ denote the left adjoint to the nerve of simplicial groupoids, as defined in [10, §3.1]; by [10, Theorem 3.3] the morphism $\mathrm{N}W \to \mathrm{N}\overline{\mathcal{L}}_W W$ is a weak equivalence if and only if the adjunct $\mathfrak{G}\mathrm{N}W \to \overline{\mathcal{L}}_W W$ is a weak equivalence of simplicial groupoids. This follows from [9, §5.5], since this implies that the mapping spaces in both are the appropriate loop spaces of $\mathrm{N}W$. □

Corollary 4.15 *Let $(\mathbf{C}, W)$ be a relative category. Suppose $\mathcal{L}_W\mathbf{C} \to \overline{\mathcal{L}}_W\mathbf{C}$ is a fibrant replacement in the model category of simplicial categories. Then*

$$L(\mathbf{C}, W) \to \mathrm{N}\overline{\mathcal{L}}_W\mathbf{C}^\natural$$

is a weak equivalence in Set_Δ^+.

Proof We must show that for every ∞-category $\mathcal{D}$, the induced map

$$\mathrm{Map}_{\mathrm{Set}_\Delta^+}(\mathrm{N}\overline{\mathcal{L}}_W\mathbf{C}^\natural, \mathcal{D}^\natural) \to \mathrm{Map}_{\mathrm{Set}_\Delta^+}(L(\mathbf{C}, W), \mathcal{D}^\natural)$$

is a weak equivalence of simplicial sets. Observe that

$$\mathrm{Map}_{\mathrm{Set}_\Delta^+}(L(\mathbf{C}, W), \mathcal{D}^\natural) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}\mathbf{C}, \mathcal{D}) \times_{\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W, \mathcal{D})} \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W, \iota\mathcal{D})$$

and $\mathrm{Map}_{\mathrm{Cat}_\infty}(\mathrm{N}W, \iota\mathcal{D}) \simeq \mathrm{Map}_{\mathcal{S}}(\overline{\mathrm{N}W}, \iota\mathcal{D}) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty}(\overline{\mathrm{N}W}, \mathcal{D})$, where $\mathrm{N}W \to \overline{\mathrm{N}W}$ denotes a fibrant replacement in the usual model structure on simplicial sets, so this is equivalent to requiring

$$\begin{array}{ccc} \mathrm{N}W & \longrightarrow & \overline{\mathrm{N}W} \\ \downarrow & & \downarrow \\ \mathrm{N}\mathbf{C} & \longrightarrow & \mathrm{N}\overline{\mathcal{L}}_W\mathbf{C} \end{array}$$

to be a homotopy pushout square. Theorem 4.14(i) implies that

$$\begin{array}{ccc} \mathrm{N}W & \longrightarrow & \mathrm{N}\overline{\mathcal{L}}_W W \\ \downarrow & & \downarrow \\ \mathrm{N}\mathbf{C} & \longrightarrow & \mathrm{N}\overline{\mathcal{L}}_W\mathbf{C} \end{array}$$

is a homotopy pushout square, since N is a right Quillen equivalence and all the objects are fibrant. By Theorem 4.14(ii) we also have that $\mathrm{N}W \to \mathrm{N}\overline{\mathcal{L}}_W W$ is a fibrant replacement in the usual model structure on simplicial sets, so the result follows. □

We now fix a functor $F\colon \mathbf{C} \to \mathrm{RelCat}$, and let $\pi\colon \mathbf{E} \to \mathbf{C}$ be a Grothendieck opfibration associated to the underlying functor $\mathbf{C} \to \mathrm{Cat}$. We say a morphism $\bar{f}\colon X \to Y$ in $\mathbf{E}$ lying over $f\colon A \to B$ in $\mathbf{C}$ is a *weak equivalence* if f is an isomorphism and $f_!X \to F$ is a weak equivalence in $F(B)$; write W for the subcategory of $\mathbf{E}$ whose morphisms are the weak equivalences. Our goal is to show that the nerve of $\mathcal{L}_W\mathbf{E} \to \mathbf{C}$ is (equivalent to) a coCartesian fibration. To prove this we need a technical hypothesis on the relative categories $F(C)$:

Definition 4.16 A relative category $(\mathbf{C}, W)$ satisfies the *two-out-of-three property* if given morphisms $r\colon A \to B$ and $s\colon B \to C$ such that two out of $r, s, s \circ r$ are in W, then so is the third.

Definition 4.17 We say that a relative category $\overline{\mathbf{C}} = (\mathbf{C}, W)$ is a *partial model category* if $\overline{\mathbf{C}}$ satisfies the two-out-of-three property and $\overline{\mathbf{C}}$ admits a *three-arrow calculus*, i.e. there exist subcategories $U, V \subseteq W$ such that

(i) for every zig-zag $A' \overset{u}{\leftarrow} A \overset{f}{\to} B$ in $\mathbf{C}$ with $u \in U$, there exists a functorial zig-zag
$$A' \overset{f'}{\to} B' \overset{u'}{\leftarrow} B$$
with $u' \in U$ such that $u'f = f'u$ and u' is an isomorphism if u is,

(ii) for every zig-zag $X \overset{g}{\to} Y' \overset{v}{\leftarrow} Y$ in $\mathbf{C}$ with $v \in V$, there exists a functorial zig-zag
$$X \overset{v'}{\leftarrow} X' \overset{g'}{\to} Y$$
with $v' \in V$ such that $gv' = vg'$ and v' is an isomorphism if v is,

(iii) every map $w \in W$ admits a functorial factorization $w = vu$ with $u \in U$ and $v \in V$.

Remark 4.18 If $\mathbf{M}$ is a model category (with functorial factorizations), then the relative category obtained by equipping $\mathbf{M}$ with the weak equivalences in the model structure is a partial model category. Similarly, the relative categories obtained from the full subcategories $\mathbf{M}^{\mathrm{cof}}$ of cofibrant objects, $\mathbf{M}^{\mathrm{fib}}$ of fibrant objects, and $\mathbf{M}^{\circ}$ of fibrant-cofibrant objects together with the weak equivalences between these objects are all partial model categories. The term "partial model category" is taken from [3], but we use the more general definition of [7, 36.1] since the more restrictive definition of Barwick and Kan does not include what is for us the key example, namely $\mathbf{M}^{\mathrm{cof}}$ for $\mathbf{M}$ a model category.

Theorem 4.19 (Dwyer-Kan) *Suppose $(\mathbf{C}, W)$ is a partial model category. Then for every pair of objects $X, Y \in \mathbf{C}$, the morphism $\mathcal{L}_W\mathbf{C}_{\langle n\rangle}(X,Y) \to \mathcal{L}_W\mathbf{C}(X,Y)$ is a weak equivalence of simplicial sets for all $n \geq 3$.*

Proof For $n = 3$ this is [8, Proposition 6.2(i)]; the general case follows similarly. □

Proposition 4.20 *Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. Let $\phi\colon A \to B$ be a morphism in $\mathbf{C}$, and let X and Y be objects of $\mathbf{E}_A$ and $\mathbf{E}_B$, respectively. Write $\mathcal{L}_W\mathbf{E}(X,Y)_\phi$ for the subspace of $\mathcal{L}_W\mathbf{E}(X,Y)$ over ϕ. The morphism*

$$\overline{\phi}^*\colon \mathcal{L}_W\mathbf{E}_B(\phi_!X, Y) \to \mathcal{L}_W\mathbf{E}(X,Y)_\phi$$

given by composition with a coCartesian morphism $\overline{\phi}\colon X \to \phi_!X$ is a weak equivalence of simplicial sets.

Proof It is easy to see that $\mathbf{E}$ is also a partial model category. It therefore follows from Theorem 4.19 that the maps $\mathcal{L}_W\mathbf{E}_{\langle 4\rangle}(X,Y)_\phi \to \mathcal{L}_W\mathbf{E}(X,Y)_\phi$ and $\mathcal{L}_W(\mathbf{E}_B)_{\langle 4\rangle}(\phi_!X,Y) \to \mathcal{L}_W\mathbf{E}_B(\phi_!X,Y)$ are weak equivalences. Since composition with $\overline{\phi}$ gives a functor

$$\overline{\phi}^*\colon \mathbf{L}_B := \mathbf{L}_W(\mathbf{E}_B)_{\langle 4\rangle}(\phi_!X,Y) \to \mathbf{L}_W\mathbf{E}_{\langle 4\rangle}(X,Y)_\phi =: \mathbf{L}$$

it therefore suffices to prove that this gives a weak equivalence upon taking nerves.

We will prove this in two steps. Let $\mathbf{L}^1$ denote the full subcategory of $\mathbf{L}$ spanned by objects

$$X = X_0 \xrightarrow{f_1} X_1 \xleftarrow{f_2} X_2 \xrightarrow{f_3} X_3 \xleftarrow{f_4} X_4 = Y$$

such that $X_i \in \mathbf{E}_B$ for $i \geq 1$ and f_i lies over id_B in $\mathbf{C}$ for $i \geq 2$; then $\overline{\phi}^*$ factors as

$$\mathbf{L}_B \xrightarrow{f} \mathbf{L}^1 \xrightarrow{i} \mathbf{L}.$$

We will show that each of these functors induces a weak equivalence of nerves.

First we consider $f\colon \mathbf{L}_B \to \mathbf{L}^1$, given by composition with $\overline{\phi}$. Define $q\colon \mathbf{L}^1 \to \mathbf{L}_B$ by sending a zig-zag

$$X \xrightarrow{g} Z \leftarrow Z' \to Y' \leftarrow Y$$

in $\mathbf{L}^1$ to

$$\phi_!X \xrightarrow{g'} Z \leftarrow Z' \to Y' \leftarrow Y$$

where $X \xrightarrow{\overline{\phi}} \phi_! X \xrightarrow{g'} Z$ is the coCartesian factorization of g (which exists since the other maps lie over id_B). Then it is clear that $qf \simeq \mathrm{id}$ and $fq \simeq \mathrm{id}$, so f is an equivalence of categories.

Next we want to define a functor $p \colon \mathbf{L} \to \mathbf{L}^1$. Given a zig-zag

$$X \xrightarrow{g} Z' \leftarrow Z \xrightarrow{h} Y' \leftarrow Y$$

in $\mathbf{L}$, this lies over

$$A \to C' \xleftarrow{\gamma} C \to B' \xleftarrow{\beta} B$$

where γ and β are isomorphisms, since weak equivalences in $\mathbf{E}$ map to isomorphisms in $\mathbf{C}$. Thus the coCartesian maps $Z' \to \gamma_!^{-1} Z'$ and $B' \to \beta_!^{-1} B'$ are isomorphisms, and our zig-zag is isomorphic to the zig-zag

$$X \to \gamma_!^{-1} Z' \leftarrow Z \to \beta_!^{-1} Y' \leftarrow Y.$$

To define p we may therefore assume that β and γ are identities, in which case p sends

$$X \xrightarrow{f} Z' \leftarrow Z \xrightarrow{g} Y' \leftarrow Y$$

lying over

$$A \xrightarrow{\alpha} C \xleftarrow{\mathrm{id}} C \xrightarrow{\psi} B \xleftarrow{\mathrm{id}} B$$

to

$$X \to \psi_! Z' \leftarrow \psi_! Z \to Y' \leftarrow Y$$

in $\mathbf{L}^1$; this is clearly functorial.

We wish to prove that p gives an inverse to i after taking nerves. It is obvious that $p \circ i \simeq \mathrm{id}$, so it suffices to show that $i \circ p$ is homotopic to the identity after taking nerves. To see this we consider the natural transformation $\eta \colon \mathbf{L} \to \mathrm{Fun}([1], \mathbf{L}_W \mathbf{E}_{\langle 6 \rangle}(x,y)_\phi)$ that sends our zig-zag to the diagram

$$\begin{array}{ccccccccccccc}
X & \longrightarrow & Z' & \longleftarrow & Z & \longrightarrow & \psi_! Z & \xleftarrow{\mathrm{id}} & \psi_! Z & \longrightarrow & Y' & \longleftarrow & Y \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \longrightarrow & Z' & \xleftarrow[\mathrm{id}]{} & Z' & \longrightarrow & \psi_! Z' & \longleftarrow & \psi_! Z & \longrightarrow & Y' & \longleftarrow & Y,
\end{array}$$

After composing with the inclusion $\mathbf{L}_W \mathbf{E}_{\langle 6 \rangle}(x,y)_\phi \to \mathbf{L}_W \mathbf{E}(x,y)_\phi$ the functor η_0 is clearly linked to the inclusion $\mathbf{L} \to \mathbf{L}_W \mathbf{E}(x,y)_\phi$ by a sequence of natural transformations, and similarly η_1 is linked to the composite of $i \circ p$ with this inclusion. Since natural transformations give homotopies of the induced maps between nerves it follows from Theorem 4.19 that the morphism on nerves induced by $i \circ p$ is homotopic to the identity. This completes the proof. □

Theorem 4.21 *Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. There is an ∞-category $\mathbf{E}[W^{-1}]$ such that $L(\mathbf{E}, W) \to \mathbf{E}[W^{-1}]^{\natural}$ is a weak equivalence in $\mathrm{Set}_{\Delta}^{+}$, and $\mathbf{E}[W^{-1}] \to \mathrm{N}\mathbf{C}$ is a coCartesian fibration.*

Proof Let $\mathcal{L}_W\mathbf{E} \to \overline{\mathcal{L}}_W\mathbf{E} \to \mathbf{C}$ denote a factorization of $\mathcal{L}_W\mathbf{E} \to \mathbf{C}$ as a trivial cofibration followed by a fibration in the model category of simplicial categories. Then $(\mathrm{N}\overline{\mathcal{L}}_W\mathbf{E})^{\natural}$ is a fibrant replacement for $L(\mathbf{E}, W)$ in $\mathrm{Set}_{\Delta}^{+}$. By [19, Proposition 2.4.4.3] to prove that $\mathrm{N}\overline{\mathcal{L}}_W\mathbf{E} \to \mathrm{N}\mathbf{C}$ is equivalent to a coCartesian fibration it suffices to show that for each morphism $f\colon C \to D$ in $\mathbf{C}$ and each X in $\mathbf{E}_C$ we have a homotopy pullback square of simplicial sets

$$\begin{array}{ccc} \mathcal{L}_W\mathbf{E}(f_!X, Y) & \xrightarrow{\overline{f}^*} & \mathcal{L}_W\mathbf{E}(X, Y) \\ \downarrow & & \downarrow \\ \mathbf{C}(D, B) & \xrightarrow[f^*]{} & \mathbf{C}(C, B) \end{array}$$

for all $B \in \mathbf{C}$ and $Y \in \mathbf{E}_B$, where $\overline{f}\colon X \to f_!X$ denotes a coCartesian morphism in $\mathbf{E}$ over f.

Since the inclusion of a point in a discrete simplicial set is a Kan fibration and the model structure on simplicial sets is right proper, given $g\colon D \to B$ the fibres at $\{g\}$ and $\{g \circ f\}$ in this diagram are homotopy fibres. To see that the diagram is a homotopy pullback square it thus suffices to show that composition with $\overline{f}$ induces a weak equivalence

$$\mathcal{L}_W\mathbf{E}(f_!X, Y)_g \to \mathcal{L}_W\mathbf{E}(X, Y)_{gf}$$

for all $g\colon D \to B$. But by Proposition 4.20, in the commutative diagram

$$\begin{array}{ccc} & \mathcal{L}_W\mathbf{E}_B((gf)_!X, Y) & \\ \swarrow & & \searrow \\ \mathcal{L}_W\mathbf{E}(f_!X, Y)_g & \longrightarrow & \mathcal{L}_W\mathbf{E}(X, Y)_{gf} \end{array}$$

the diagonal morphisms are both weak equivalences, hence by the 2-out-of-3 property so is the horizontal morphism. □

Corollary 4.22 *Suppose $F\colon \mathbf{C} \to \mathrm{RelCat}$ is a functor such that $F(C)$ is a partial model category for each $C \in \mathbf{C}$. Let $LF \to \overline{F}$ be a fibrant replacement in $\mathrm{Fun}(\mathbf{C}, \mathrm{Set}_{\Delta}^{+})$. Then there is a weak equivalence $L(\mathbf{E}, W) \to (\mathrm{N}_{\mathbf{C}}\overline{F})^{\natural}$ in $\mathrm{Set}_{\Delta}^{+}$.*

Proof By Theorem 4.21, there exists a coCartesian fibration $\mathbf{E}[W^{-1}] \to \mathrm{N}\mathbf{C}$ with a map
$$\phi\colon L(\mathbf{E}, W) \to \mathbf{E}[W^{-1}]^{\natural}$$
that is a weak equivalence in $\mathrm{Set}_{\Delta}^{+}$. The map ϕ is also a weak equivalence when regarded as a morphism in the over-category model structure on $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\natural}}$. Let
$$p_{!}\colon (\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\natural}} \rightleftarrows (\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\sharp}} : p^{*}$$
be the adjunction where $p_{!}$ is the identity on the underlying marked simplicial sets, and p^{*} forgets the marked edges that do not lie over isomorphisms in $\mathbf{C}$. If we equip $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\natural}}$ with the over-category model structure and $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\sharp}}$ with the coCartesian model structure, then this is a Quillen adjunction by [19, Proposition B.2.9], since these functors clearly come from a map of categorical patterns. Since all objects in $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\natural}}$ are cofibrant, the functor $p_{!}$ preserves weak equivalences, and so ϕ is also a weak equivalence when regarded as a morphism of $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\sharp}}$.

Let M' be the set of edges of $\mathrm{N}\mathbf{E}$ corresponding to coCartesian morphisms in $\mathbf{E}$, and let $\mathbf{E}[W^{-1}]^{+}$ denote the marked simplicial set obtained from $\mathbf{E}[W^{-1}]^{\natural}$ by also marking the morphisms in the image of M'. We have a pushout diagram
$$\begin{array}{ccc} L(\mathbf{E}, W) & \longrightarrow & \mathbf{E}[W^{-1}]^{\natural} \\ \downarrow & & \downarrow \\ (\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') & \longrightarrow & \mathbf{E}[W^{-1}]^{+}, \end{array}$$
as both vertical maps are pushouts along $\coprod_{f \in M'} \Delta^1 \hookrightarrow \coprod_{f \in M'} (\Delta^1)^{\sharp}$. Since the model structure on $(\mathrm{Set}_{\Delta}^{+})_{/\mathrm{N}\mathbf{C}^{\sharp}}$ is left proper, it follows that $(\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') \to \mathbf{E}[W^{-1}]^{+}$ is a weak equivalence.

Let $\mathbf{E}[W^{-1}]^{*}$ denote $\mathbf{E}[W^{-1}]$, marked by the coCartesian morphisms. These are composites of equivalences and morphisms in the image of M', so $\mathbf{E}[W^{-1}]^{+} \to \mathbf{E}[W^{-1}]^{*}$ is marked anodyne. Moreover, it follows as in the proof of Lemma 4.8 that $\mathrm{N}\mathbf{E}$ marked by the composites of morphisms in $\mathrm{N}W_1$ and M' is precisely $\mathrm{N}_{\mathbf{C}}^{+}LF$, so $(\mathrm{N}\mathbf{E}, \mathrm{N}W_1 \cup M') \to \mathrm{N}_{\mathbf{C}}^{+}LF$ is also marked anodyne. By the 2-out-of-3 property we therefore have a weak equivalence $\mathrm{N}_{\mathbf{C}}^{+}LF \to \mathbf{E}[W^{-1}]^{*}$. Thus $\mathbf{E}[W^{-1}]^{*}$ and $\mathrm{N}_{\mathbf{C}}^{+}\overline{F}$ are both fibrant replacements for $\mathrm{N}_{\mathbf{C}}^{+}LF$, and so are linked by a zig-zag of weak equivalences between fibrant objects.

This implies that the underlying ∞-categories $\mathbf{E}[W^{-1}]$ and $\mathrm{N}_{\mathbf{C}}\overline{F}$ are equivalent, and so by the 2-out-of-3 property the map $(\mathrm{N}\mathbf{E}, W) \to (\mathrm{N}_{\mathbf{C}}\overline{F})^{\natural}$ is a weak equivalence in $\mathrm{Set}_{\Delta}^{+}$, as required. □

4.3 Total Space Model Structures

As before we consider a functor $F\colon \mathbf{C} \to \mathrm{RelCat}$ and let $\mathbf{E} \to \mathbf{C}$ be an opfibration associated to F. Although not strictly necessary for the applications we are interested in below, in this subsection we show that if the functor F is obtained from a suitable functor from $\mathbf{C}$ to the category of combinatorial model categories, then the relative category structure on $\mathbf{E}$ considered above also comes from a combinatorial model category.

Definition 4.23 Let $\mathrm{ModCat}^{\mathrm{R}}$ be the category of model categories and right Quillen functors. A *right Quillen presheaf* on a category $\mathbf{C}$ is a functor $\mathbf{C}^{\mathrm{op}} \to \mathrm{ModCat}^{\mathrm{R}}$. A right Quillen presheaf is *combinatorial* if it factors through the full subcategory of combinatorial model categories.

Definition 4.24 Suppose $\mathbf{C}$ is a κ-accessible category. A right Quillen presheaf on $\mathbf{C}$ is κ*-accessible* if for each κ-filtered diagram $i\colon \mathbf{I} \to \mathbf{C}$ with colimit X, the category $F(X)$ is the limit of the categories $F(i(\alpha))$, and the model structure on $F(X)$ is induced by those on $F(i(\alpha))$ in the sense that a map $f\colon A \to B$ in $F(X)$ is a (trivial) fibration if and only if $F(g_\alpha)(f)$ is a (trivial) fibration in $F(i(\alpha))$ for all $\alpha \in \mathbf{I}$, where g_α is the canonical morphism $i(\alpha) \to X$. We say a right Quillen presheaf F on an accessible category $\mathbf{C}$ is *accessible* if there exists a cardinal κ such that $\mathbf{C}$ and F are κ-accessible.

Proposition 4.25 *Suppose $\mathbf{C}$ is a complete and cocomplete category and F is a right Quillen presheaf on $\mathbf{C}$. Let $\pi\colon \mathbf{E} \to \mathbf{C}$ be the Grothendieck fibration corresponding to F. Then there exists a model structure on $\mathbf{E}$ such that a morphism $\phi\colon X \to Y$ with image $f\colon A \to B$ in $\mathbf{C}$ is*

- *(W) a weak equivalence if and only if f is an isomorphism in $\mathbf{C}$ and the morphism $f_!X \to Y$ is a weak equivalence in $F(b)$.*
- *(F) a fibration if and only if $X \to f^*Y$ is a fibration in $F(a)$.*
- *(C) a cofibration if and only if $f_!X \to Y$ is a cofibration in $F(b)$.*

Moreover, if $\mathbf{C}$ is a presentable category and F is an accessible and combinatorial right Quillen presheaf, then this model structure on $\mathbf{E}$ is combinatorial.

Remark 4.26 If $f\colon A \to B$ is an isomorphism in $\mathbf{C}$, then $f^* = F(f)$ is an isomorphism of model categories with inverse $f_!$. Thus if $\phi\colon X \to Y$ is a morphism in $\mathbf{E}$ such that $f = \pi(\phi)$ is an isomorphism in $\mathbf{C}$, then $f_!X \to Y$ is a weak equivalence in $\mathbf{E}_B$ if and only if $X \to f^*Y$ is a weak equivalence in $\mathbf{E}_A$.

Remark 4.27 This model category structure is a particular case of that constructed by Roig [27] (and corrected by Stanculescu [31]), though he does not consider the combinatorial case. Roig's construction has also recently been significantly generalized by Harpaz and Prasma [14]. We include a proof for completeness.

Proof Limits in $\mathbf{E}$ are computed by first taking Cartesian pullbacks to the fibre over the limit of the projection of the diagram to $\mathbf{C}$, and then taking the limit in that fibre. Since all the fibres $\mathbf{E}_B$ have limits, it is therefore clear that $\mathbf{E}$ has limits. Similarly, since each functor ϕ^* for ϕ in $\mathbf{C}$ has a left adjoint, and each of the fibres $\mathbf{E}_B$ has all colimits, it is clear that $\mathbf{E}$ has colimits.

To show that $\mathbf{E}$ is a model category we must now prove that the weak equivalences satisfy the 2-out-of-3 property, and the cofibrations and trivial fibrations, as well as the trivial cofibrations and fibrations, form weak factorization systems. We check the 2-out-of-3 property first: Suppose we have morphisms $\overline{f}\colon X \to Y$ and $\overline{g}\colon Y \to Z$ in $\mathbf{E}$ lying over $f\colon A \to B$ and $g\colon B \to C$ in $\mathbf{C}$. If two out of the three morphisms $\overline{f}$, $\overline{g}$ and $\overline{g}\overline{f}$ are weak equivalences, it is clear that f and g must be isomorphisms. Thus $g_!$ is an isomorphism of model categories, and $g_!f_!X \to g_!Y$ is a weak equivalence in $\mathbf{E}_C$ if and only if $f_!X \to Y$ is a weak equivalence in $\mathbf{E}_B$. Combining this with the 2-out-of-3 property for weak equivalences in $\mathbf{E}_C$ gives the 2-out-of-3 property for $\mathbf{E}$.

We now prove that the cofibrations and trivial fibrations form a weak factorization system:

(1) *Any morphism has a factorization as a cofibration followed by a trivial fibration:* Given $\overline{f}\colon X \to Y$ in $\mathbf{E}$ lying over $f\colon a \to b$ in $\mathbf{C}$, choose a factorization $f_!X \to Z \to Y$ of $f_!X \to Y$ as a cofibration followed by a trivial fibration in $\mathbf{E}_b$. Then by definition $X \to Z$ is a cofibration and $Z \to Y$ is a trivial fibration in $\mathbf{E}$.

(2) *A morphism that has the left lifting property with respect to all trivial fibrations is a cofibration:* Suppose $\overline{f}\colon X \to Y$, lying over $f\colon A \to B$ in $\mathbf{C}$, has the left lifting property with respect to all trivial fibrations. Then in particular there exists a lift in all diagrams

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

where $X' \to Y'$ is a trivial fibration in $\mathbf{E}_B$. By the universal property of coCartesian morphisms, this clearly implies that $f_!X \to Y$ has the left lifting property with respect to trivial fibrations in $\mathbf{E}_B$, and so is a cofibration in $\mathbf{E}_B$. Thus $\overline{f}$ is a cofibration.

(3) *Cofibrations have the left lifting property with respect to trivial fibrations:* Suppose $\overline{f}\colon X \to Y$, lying over $f\colon A \to B$ in $\mathbf{C}$, is a cofibration, and $\overline{g}\colon X' \to Y'$, lying over $g\colon A' \to B'$, is a trivial fibration. Given a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\overline{\alpha}} & X' \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ Y & \xrightarrow[\overline{\beta}]{} & Y' \end{array}$$

lying over

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

we must show there exists a lift $Y \to X'$. Since $\overline{g}$ is a trivial fibration, g is an isomorphism. Pulling back along g^{-1} and pushing forward along $g\alpha = \beta f$ and β gives a diagram

$$\begin{array}{ccccccc} X & \longrightarrow & \beta_! f_! X & \longrightarrow & (g^{-1})^* X' & \longrightarrow & X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & \beta_! Y & \longrightarrow & Y' & \longrightarrow & Y' \end{array}$$

Here $\beta_! f_! X \to \beta_! Y$ is a cofibration in $\mathbf{E}_{B'}$ since $f_! X \to Y$ is a cofibration in $\mathbf{E}_B$ and $\beta_!$ is a left Quillen functor, and $(g^{-1})^* X' \to (g^{-1})^* g^* Y = Y$ is a trivial fibration in $\mathbf{E}_{B'}$ since $X \to g^* Y$ is a trivial fibration in $\mathbf{E}_{Af'}$ and $(g^{-1})^*$ is a right Quillen functor. Thus there exists a lift $\beta_! Y \to (g^{-1})^* X'$ which gives the desired lift $Y \to X'$.

(4) *A morphism that has the right lifting property with respect to all cofibrations is a trivial fibration:* Suppose $\overline{g}\colon X' \to Y'$, lying over $g\colon A' \to B'$ in $\mathbf{C}$, has the right lifting property with respect to all cofibrations. Then in particular there

exists a lift in all diagrams

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array}$$

where $X \to Y$ is a cofibration in $\mathbf{E}_{A'}$. By the universal property of Cartesian morphisms, this clearly implies that $X' \to g^*Y'$ has the right lifting property with respect to cofibrations in $\mathbf{E}_{A'}$, and so is a trivial fibration in $\mathbf{E}_{A'}$. On the other hand, there exists a lift in the diagram

$$\begin{array}{ccc} X' & \longrightarrow & X' \\ \downarrow & & \downarrow \\ g_!X' & \longrightarrow & Y', \end{array}$$

and projecting this down to $\mathbf{C}$ we see that g must be an isomorphism. Thus $\overline{g}$ is a trivial fibration in $\mathbf{E}$.

The proof that trivial cofibrations and fibrations form a weak factorization system is dual to that for cofibrations and trivial fibrations, so we omit the details. This completes the proof that $\mathbf{E}$ is a model category.

Now suppose the right Quillen presheaf F is combinatorial and accessible. It follows from [22, Theorem 5.3.4] that the category $\mathbf{E}$ is accessible, and the functor π is accessible, thus $\mathbf{E}$ is a presentable category since we already proved that it has small colimits.

Let κ be a cardinal such that $\mathbf{C}$ is κ-accessible and $\mathbf{E}_X$ is κ-accessible for each κ-compact object X in $\mathbf{C}$. For $X \in \mathbf{C}$, let I_X and J_X be sets of generating cofibrations and trivial cofibrations for $\mathbf{E}_X$. Let I and J be the unions of I_X and J_X, respectively, over all κ-compact objects $X \in \mathbf{C}$; then I and J are sets.

Suppose a morphism $\overline{f}\colon X \to Y$, lying over $f\colon A \to B$ in $\mathbf{C}$, has the right lifting property with respect to the morphisms in J; then $X \to f^*Y$ is a fibration in $\mathbf{E}_A$: To see this let $\mathbf{K} \to \mathbf{C}$, $\alpha \mapsto A_\alpha$, be a κ-filtered diagram of κ-compact objects with colimit A, and let $\gamma_\alpha\colon A_\alpha \to A$ be the canonical morphism. Then $\gamma_\alpha^*X \to \gamma_\alpha^*f^*Y$ has the right lifting property with respect to a set of generating trivial cofibrations in $\mathbf{E}_{A_\alpha}$, and hence this is a fibration in $\mathbf{E}_{A_\alpha}$. Since the right Quillen presheaf F is κ-accessible, this implies that $X \to f^*Y$ is a fibration in $\mathbf{E}_A$. This means $\overline{f}$ is a fibration in $\mathbf{E}$, so J is a set of generating trivial cofibrations.

Similarly, if $\overline{f}$ has the right lifting property with respect to the morphisms in I, then $X \to f^*Y$ is a trivial fibration in $\mathbf{E}_A$. To find a set of generating cofibrations we consider also the set I' of morphisms $\emptyset_\emptyset \to \emptyset_C$ and $\emptyset_{C\amalg C} \to \emptyset_C$ where C is a κ-compact object of $\mathbf{C}$ and $\emptyset_C$ denotes the initial object of $\mathbf{E}_C$. We claim that if $\overline{f}\colon X \to Y$ in $\mathbf{E}$, with image $f\colon A \to B$ in $\mathbf{C}$, has the right lifting property with respect to the morphisms in I', then f is an isomorphism in $\mathbf{C}$. To prove this it suffices to show that for every object $C \in \mathbf{C}$ the map $f_*\colon \mathrm{Hom}_{\mathbf{C}}(C,A') \to \mathrm{Hom}_{\mathbf{C}}(C,B')$ induced by composition with f is a bijection; since $\mathbf{C}$ is κ-presentable it is enough to prove this for C a κ-compact object. Since $\overline{f}$ has the right lifting property with respect to $\emptyset_\emptyset \to \emptyset_C$ and every morphism $C \to B$ induces a morphism $\emptyset_C \to Y$, there exists a lift in the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & B \end{array}$$

for every map $C \to B$; this shows that f_* is surjective. Moreover, given two morphisms $C \to A$ such that the composites $C \to B$ are equal, we get a lift in the diagram

$$\begin{array}{ccc} C \amalg C & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow f \\ C & \longrightarrow & B \end{array}$$

since $\overline{f}$ has the right lifting property with respect to $\emptyset_{C\amalg C} \to \emptyset_C$; thus the two morphisms $C \to A$ must be equal and so f_* is injective. It follows that if a morphism in $\mathbf{E}$ has the right lifting property with respect to the union $I \amalg I'$ then it is a trivial fibration, so $I \amalg I'$ is a set of generating cofibrations for $\mathbf{E}$. Hence $\mathbf{E}$ is a combinatorial model category. □

Remark 4.28 Let F be a right Quillen presheaf on a category $\mathbf{C}$, and let $\mathbf{E} \to \mathbf{C}$ be an opfibration associated to the underlying functor to categories. Write G for the associated "left Quillen presheaf" obtained by passing to left adjoints, and let $G^{\mathrm{cof}}\colon \mathbf{C} \to \mathrm{RelCat}$ be the functor to relative categories obtained by restricting to cofibrant objects. Then the full subcategory $\mathbf{E}^{\mathrm{cof}}$ of cofibrant objects in $\mathbf{E}$, with the model structure defined above, is the total space of the opfibration associated to G^{cof}, and the weak equivalences in $\mathbf{E}^{\mathrm{cof}}$ are precisely those considered above.

5 Rectifying Enriched ∞-Categories

Our goal in this section is to prove the main result of this paper: the homotopy theory of categories enriched in a nice monoidal model category $\mathbf{V}$ (with respect to the DK-equivalences) is equivalent to the homotopy theory of ∞-categories enriched in the monoidal ∞-category $\mathbf{V}[W^{-1}]$. We will do this in three steps:

(1) We first apply the results of § 3 to get an equivalence between the ∞-category obtained by inverting the weakly fully faithful morphisms in the category $\mathrm{Cat}_X(\mathbf{V})$ of $\mathbf{V}$-categories with a fixed set of objects X and the ∞-category $\mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$ of Δ_X^{op}-algebras.

(2) Next, using the results of § 4, we see that this induces an equivalence between the ∞-category obtained by inverting those morphisms in the category $\mathrm{Cat}(\mathbf{V})$ of small $\mathbf{V}$-categories that are weakly fully faithful and bijective on objects and the ∞-category $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$ of categorical algebras in $\mathbf{V}[W^{-1}]$ whose spaces of objects are sets.

(3) Finally, from this we deduce that the ∞-category obtained by inverting the DK-equivalences in $\mathrm{Cat}(\mathbf{V})$ is equivalent to the ∞-category $\mathrm{Cat}_\infty^{\mathbf{V}[W^{-1}]}$ of $\mathbf{V}[W^{-1}]$-∞-categories.

For the first step, the map we wish to prove is an equivalence is defined as follows:

Definition 5.1 Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom, and let X be a set. The map of generalized ∞-operads $\nu_X \colon \Delta_X^{\mathrm{op}} \to \mathbf{O}_X^{\otimes}$ defined in Proposition 2.10 gives an equivalence

$$\mathrm{Cat}_X(\mathbf{V}) \simeq \mathrm{Alg}_{\mathbf{O}_X}(\mathbf{V}) \xrightarrow{\sim} \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}).$$

As in Definition 3.16 the monoidal functor $\mathbf{V}^{\mathrm{cof}} \to \mathbf{V}[W^{-1}]$ induces, since the forgetful functor $\mathrm{Cat}_X(\mathbf{V}) \to \mathrm{Fun}(X \times X, \mathbf{V})$ preserves cofibrant objects by Corollary 3.15, a functor

$$\mathrm{Cat}_X(\mathbf{V})^{\mathrm{cof}} \to \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}]).$$

Let FF_X denote the class of morphisms in $\mathrm{Cat}_X(\mathbf{V})^{\mathrm{cof}}$ that are weakly fully faithful, i.e. given by weak equivalences on all morphism objects. It is clear that these are taken to equivalences in $\mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$ by this functor, and so there is an induced functor

$$\eta_X \colon \mathrm{Cat}_X(\mathbf{V})[\mathrm{FF}_X^{-1}] \to \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}]).$$

Moreover, it is clear that this is natural in X.

Proposition 5.2 *Suppose $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom, and let X be a set. The natural map*

$$\eta_X\colon \mathrm{Cat}_X(\mathbf{V})[\mathrm{FF}_X^{-1}] \to \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$$

is an equivalence.

Proof of Proposition 5.2 We apply [21, Corollary 4.7.4.16] as in the proof of [21, Theorem 4.1.4.4]: We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Cat}_X(\mathbf{V})[FF_X^{-1}] & \xrightarrow{\quad\eta_X\quad} & \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}]) \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle V} & \\ & \mathrm{Fun}(X \times X, \mathbf{V}[W^{-1}]), & \end{array}$$

where U_∞ is the functor of ∞-categories associated to the forgetful functor

$$U\colon \mathrm{Cat}_X(\mathbf{V}) \to \mathrm{Fun}(X \times X, \mathbf{V}),$$

which is a right Quillen functor, and V is given by restricting Δ_X^{op}-algebras to the fibre $(\Delta_X^{\mathrm{op}})_{[1]} \simeq X \times X$. Then we observe:

(a) The ∞-category $\mathrm{Cat}_X(\mathbf{V})[\mathrm{FF}_X^{-1}]$ is presentable by [21, Proposition 1.3.4.22], and the ∞-category $\mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$ is presentable by [13, Corollary B.5.7] since $\mathbf{V}[W^{-1}]$ is presentable by [21, Proposition 1.3.4.22] and the induced tensor product on $\mathbf{V}[W^{-1}]$ preserves colimits in each variable by [21, Lemma 4.1.4.8].

(b) The functor V admits a left adjoint G by [13, Theorem B.4.6].

(c) The functor U_∞ also admits a left adjoint F_∞ since it arises from a right Quillen functor.

(d) The functor V is conservative by [13, Lemma B.5.5] and preserves sifted colimits by [13, Corollary B.5.4].

(e) The functor U_∞ is conservative by the definition of the weak equivalences in $\mathrm{Alg}(\mathbf{V})$, and preserves sifted colimits by Lemma 3.20.

(f) The canonical map $V \circ G \to U_\infty \circ F_\infty$ is an equivalence since both induce, on the level of homotopy categories, the free $\mathbf{V}$-category monad

$$\Phi \mapsto \coprod_{n \geq 0} \coprod_{x_0, \dots, x_n \in X} \Phi(x_0, x_1) \otimes \cdots \otimes \Phi(x_{n-1}, x_n).$$

This is obvious for $U_\infty \circ F_\infty$ and for $V \circ G$ it follows by [13, Proposition B.4.9].

The hypotheses of [21, Corollary 4.7.4.16] thus hold, which implies that the morphism in question is an equivalence. □

For the second step, let us first define the class of maps in $\mathrm{Cat}(\mathbf{V})$ that we will invert:

Definition 5.3 We say that a functor $F\colon \mathbf{C} \to \mathbf{D}$ of $\mathbf{V}$-categories is *weakly fully faithful* if for all objects $X, Y \in \mathbf{C}$ the morphism $\mathbf{C}(X,Y) \to \mathbf{D}(FX,FY)$ is a weak equivalence in $\mathbf{V}$. We denote the class of morphisms in $\mathrm{Cat}(\mathbf{V})$ that are weakly fully faithful and given by bijections on sets of objects by FFB.

The map $\eta_X\colon \mathrm{Cat}_X(\mathbf{V})[\mathrm{FF}_X^{-1}] \to \mathrm{Alg}_{\Delta_X^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$ is natural in X, so it induces a natural transformation of functors $\mathrm{Set} \to \mathrm{Set}_\Delta^+$. Applying Corollary 4.22 we therefore get the required comparison of "pre-localized" homotopy theories:

Theorem 5.4 *The natural transformation η induces a functor*

$$\mathrm{Cat}(\mathbf{V})[\mathrm{FFB}^{-1}] \to \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$$

and this is an equivalence.

Remark 5.5 Using Proposition 4.25 we can combine the (fibrewise) model structures on $\mathrm{Cat}_X(\mathbf{V})$ to get a model structure on $\mathrm{Cat}(\mathbf{V})$. Explicitly, if $\mathbf{V}$ is a left proper tractable biclosed monoidal model category satisfying the monoid axiom, then there is a model structure on $\mathrm{Cat}(\mathbf{V})$ such that a morphism $F\colon \mathbf{C} \to \mathbf{D}$ is a weak equivalence if and only if F is weakly fully faithful and a bijection on objects, and a fibration if and only if $\mathbf{C}(x,y) \to \mathbf{D}(Fx,Fy)$ is a fibration in $\mathbf{V}$ for all $x, y \in \mathrm{ob}\,\mathbf{C}$. Thus $\mathrm{Cat}(\mathbf{V})[\mathrm{FFB}^{-1}]$ is the ∞-category associated to this model category.

The weakly fully faithful functors that are bijective on objects are clearly not the right weak equivalences between $\mathbf{V}$-categories — just as for ordinary categories the equivalences are the functors that are fully faithful and essentially surjective, here they should be the functors that are weakly fully faithful and essentially surjective up to homotopy, in the following sense:

Definition 5.6 Let $\mathbf{V}$ be a monoidal model category. Then the projection $\mathbf{V} \to \mathrm{h}\mathbf{V}$ to the homotopy category is a monoidal functor; this therefore induces a functor $\mathrm{Cat}(\mathbf{V}) \to \mathrm{Cat}(\mathrm{h}\mathbf{V})$. A functor of $\mathbf{V}$-categories is *homotopically essentially surjective* if its image in $\mathrm{Cat}(\mathrm{h}\mathbf{V})$ is essentially surjective, and a *DK-equivalence* if it is weakly fully faithful and homotopically essentially surjective (or equivalently if it induces an equivalence of $\mathrm{h}\mathbf{V}$-categories). We write DK for the class of DK-equivalences in $\mathrm{Cat}(\mathbf{V})$.

The DK-equivalences in $\mathrm{Cat}(\mathbf{V})$ clearly correspond to the fully faithful and essentially surjective functors in $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$, as defined in [13, § 5.2]. Theorem 5.4 therefore immediately implies the following:

Corollary 5.7 *Suppose* $\mathbf{V}$ *is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. Then* $\mathrm{Cat}(\mathbf{V})[\mathrm{DK}^{-1}]$ *is equivalent to the localization of* $\mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$ *with respect to the fully faithful and essentially surjective functors.*

Combining this with [13, Theorem 5.2.17] we get our main result:

Theorem 5.8 *Suppose* $\mathbf{V}$ *is a left proper tractable biclosed monoidal model category satisfying the monoid axiom. The functor* $\eta\colon \mathrm{Cat}(\mathbf{V})[\mathrm{FFB}^{-1}] \to \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$ *induces an equivalence*

$$\mathrm{Cat}(\mathbf{V})[\mathrm{DK}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathbf{V}[W^{-1}]}.$$

Proof By [13, Theorem 5.2.17], for any monoidal ∞-category $\mathcal{V}$ the localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})_{\mathrm{Set}}$ at the fully faithful and essentially surjective functors is equivalent to the corresponding localization of $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V})$, which is $\mathrm{Cat}_{\infty}^{\mathcal{V}}$ by [13, Theorem 5.5.6]. The result follows by combining this, in the case where $\mathcal{V}$ is $\mathbf{V}[W^{-1}]$, with Corollary 5.7. □

Remark 5.9 Under the hypotheses of Theorem 5.8 there is a model structure on the category $\mathrm{Cat}(\mathbf{V})$ whose weak equivalences are the DK-equivalences — the construction of Muro [24] requires slightly weaker hypotheses on $\mathbf{V}$ than our theorem. Thus we have shown that $\mathrm{Cat}_{\infty}^{\mathbf{V}[W^{-1}]}$ is the ∞-category associated to this model category. Other general constructions of model structures on enriched categories are given in [19, 6, 32] (see [6, § 1] for a historical discussion).

Example 5.10 The *stable model structure* on the category Sp^{Σ} of *symmetric spectra*, as described in [18], satisfies the hypotheses of Theorem 5.8. The associated monoidal ∞-category is the ∞-category of spectra with the smash product monoidal structure. Thus we have an equivalence

$$\mathrm{Cat}(\mathrm{Sp}^{\Sigma})[\mathrm{DK}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathrm{Sp}}$$

between spectral categories and spectral ∞-categories.

Example 5.11 The projective model structure on the category $\mathrm{Ch}^{\geq 0}(\mathrm{Mod}_R)$ of non-negatively graded chain complexes of modules over a commutative ring R, as described for example in [12], satisfies the hypotheses of Theorem 5.8. The same is true of the projective model structure on the category $\mathrm{Ch}(\mathrm{Mod}_R)$ of unbounded chain complexes of R-modules described in [17, §2.3]. The associated monoidal ∞-categories are the bounded and unbounded derived ∞-categories $\mathcal{D}^{\geq 0}_{\infty}(\mathrm{Mod}_R)$ and $\mathcal{D}_{\infty}(\mathrm{Mod}_R)$ of R-modules, as described in [21, §1.3.2]. (These are equivalent to the ∞-categories $\mathrm{Mod}^{\geq 0}_{\mathrm{H}R}$ and $\mathrm{Mod}_{\mathrm{H}R}$ of connective modules and all modules over the Eilenberg-MacLane ring spectrum $\mathrm{H}R$, respectively.) Thus we have equivalences

$$\mathrm{Cat}(\mathrm{Ch}^{\geq 0}(\mathrm{Mod}_R))[\mathrm{DK}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathcal{D}^{\geq 0}_{\infty}(\mathrm{Mod}_R)} \simeq \mathrm{Cat}_{\infty}^{\mathrm{Mod}^{\geq 0}_{\mathrm{H}R}},$$
$$\mathrm{Cat}(\mathrm{Ch}(\mathrm{Mod}_R))[\mathrm{DK}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathcal{D}_{\infty}(\mathrm{Mod}_R)} \simeq \mathrm{Cat}_{\infty}^{\mathrm{Mod}_{\mathrm{H}R}},$$

between ∞-categories of (two versions of) dg-categories and the appropriate corresponding enriched ∞-categories.

6 Comparison with Segal Categories

Segal categories are a model for the theory of $(\infty, 1)$-categories where composition is only associative up to coherent homotopy, inspired by Segal's model of A_{∞}-spaces. They first appeared in papers of Schwänzl and Vogt [28] and Dwyer, Kan, and Smith [11], though not with this name; they were later rediscovered by Hirschowitz and Simpson [16], who used them as a model for (∞, n)-categories. A generalization to Segal categories enriched in a Cartesian model category (i.e. a monoidal model category where the tensor product is the Cartesian product) was first given by Pellissier [25], further developed by Lurie [20], and finally extensively studied by Simpson [30]. In this section we will show that, for $\mathbf{V}$ a nice Cartesian model category with weak equivalences W, the homotopy theory of Segal categories enriched in $\mathbf{V}$ is equivalent to that of ∞-categories enriched in $\mathbf{V}[W^{-1}]$. We will first carry out the comparison in the case of a fixed set of objects, and then apply the results of §4 to prove the general comparison.

Definition 6.1 A model category is *Cartesian* if it is a monoidal model category with respect to the Cartesian product. If $\mathbf{V}$ is a Cartesian model category, a *$\mathbf{V}$-enriched Segal category* (or *Segal $\mathbf{V}$-category*) with set of objects S is a functor $\mathbf{C}\colon \Delta^{\mathrm{op}}_S \to \mathbf{V}$ such that for every object $(x_0, \ldots, x_n)$ of Δ^{op}_S the *Segal morphism* $\mathbf{C}(x_0, \ldots, x_n) \to \mathbf{C}(x_0, x_1) \times \cdots \mathbf{C}(x_{n-1}, x_n)$ induced by the projections $(x_0, \ldots, x_n) \to (x_i, x_{i+1})$ is a weak equivalence. We say the Segal category $\mathbf{C}$ is *fibrant* if the objects $\mathbf{C}(x_0, \ldots, x_n)$ in $\mathbf{V}$ are fibrant for all $x_0, \ldots, x_n \in S$, and *strictly unital* if the objects $\mathbf{C}(x)$ are final objects in $\mathbf{V}$ for all $x \in S$.

Remark 6.2 We can regard $\mathbf{V}$-categories as those Segal categories where the Segal morphisms are *isomorphisms*, rather than just weak equivalences.

We can describe fibrant Segal categories with a fixed set S of objects as the fibrant objects in a Bousfield localization of the projective model structure on $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$:

Definition 6.3 If X is an object of Δ_S^{op}, let $i_X \colon * \to \Delta_S^{\mathrm{op}}$ denote the functor with image X, write $i_X^* \colon \mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V}) \to \mathbf{V}$ for the functor given by composition with i_X, and let $i_{X,!} \colon \mathbf{V} \to \mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ be its left adjoint, given by left Kan extension along i_X. Then $i_{X,!}$ is a left Quillen functor with respect to the projective model structure on $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$. A functor $\mathbf{C} \colon \Delta_S^{\mathrm{op}} \to \mathbf{V}$ is a fibrant Segal category if and only if it is projectively fibrant and local with respect to the morphisms $i_{(x_0,x_1),!}A \coprod \cdots \coprod i_{(x_{n-1},x_n),!}A \to i_{(x_0,\ldots,x_n),!}A$ for all $x_0, \ldots, x_n$ in S and all A in a set of objects that generates $\mathbf{V}$ under colimits. If $\mathbf{V}$ is a left proper combinatorial Cartesian model category, then we can define a model structure whose fibrant objects are fibrant Segal categories by taking the left Bousfield localization of the projective model structure on $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ with respect to these morphisms — this exists under these hypotheses on $\mathbf{V}$ by a theorem of Smith (a proof can be found in [2, Theorem 4.7]). We refer to this model structure as the *Segal category model structure on functors* and write $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ for the category $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ equipped with this model structure.

To obtain a well-behaved model structure, it turns out to be better to consider only strictly unital Segal categories. This leads to considering the category of $\mathbf{V}$-*precategories*:

Definition 6.4 Let $\mathbf{V}$ be a left proper combinatorial Cartesian model category. A $\mathbf{V}$-*precategory* with set of objects S is a functor $\mathbf{C} \colon \Delta_S^{\mathrm{op}} \to \mathbf{V}$ such that $\mathbf{C}(x)$ is a final object for all $x \in S$. Write $\mathrm{Precat}_S(\mathbf{V})$ for the full subcategory of $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ spanned by the $\mathbf{V}$-precategories and $u^* \colon \mathrm{Precat}_S(\mathbf{V}) \to \mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ for the inclusion. Then u^* has a left adjoint, which we denote $u_!$.

There is a model structure on $\mathrm{Precat}_S(\mathbf{V})$ analogous to that for $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ we described above:

Proposition 6.5 (Simpson [30, Propostion 13.4.3]) *Suppose $\mathbf{V}$ is a left proper combinatorial Cartesian model category. There exists a (projective) model structure on $\mathrm{Precat}_S(\mathbf{V})$ where a morphism is a weak equivalence or fibration if it levelwise is one in $\mathbf{V}$. The functor $u^* \colon \mathrm{Precat}_S(\mathbf{V}) \to \mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$ is a right Quillen functor.*

Definition 6.6 Suppose $\mathbf{V}$ is a left proper combinatorial Cartesian model category. The *(projective) Segal category model structure on precategories* is the left Bousfield localization of this (projective) model structure on $\mathrm{Precat}_S(\mathbf{V})$ with respect to the morphisms $u_!(i_{(x_0,x_1),!}A \coprod \cdots \coprod i_{(x_{n-1},x_n),!}A) \to u_! i_{(x_0,\ldots,x_n),!}A$ for all $(x_0, \ldots, x_n)$ in S and all A in a set of objects that generates $\mathbf{V}$ under colimits. We write $\mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$ for the category $\mathrm{Precat}_S(\mathbf{V})$ equipped with this model structure.

Under mild hypotheses these two model categories in the fixed-objects case are equivalent:

Proposition 6.7 *Suppose* $\mathbf{V}$ *is a left proper combinatorial Cartesian model category where monomorphisms are cofibrations. Then the adjunction* $u_! \dashv u^*$ *gives a Quillen equivalence*

$$\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}} \rightleftarrows \mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}.$$

Proof It is obvious that u^* is a right Quillen functor, so this is a Quillen adjunction. Since u^* is fully faithful, the counit $u_! u^* F \to F$ is an isomorphism in $\mathrm{Precat}_S(\mathbf{V})$ for all F. By [30, Lemma 14.2.1] the functor $u_!$ only changes the values of a functor at the constant sequences $(x, \ldots, x)$ for $x \in S$, in which case $u_!F$ is given by forming the pushout

$$\begin{array}{ccc} F(x) & \longrightarrow & * \\ {\scriptstyle F(\sigma)}\downarrow & & \downarrow \\ F(x, \ldots, x) & \longrightarrow & u_!F(x, \ldots, x), \end{array}$$

where $\sigma\colon (x) \to (x, \ldots, x)$ is the map over the unique map $s\colon [0] \to [n]$ in Δ^{op}. If d is any map $[n] \to [0]$ in Δ^{op}, then $ds = \mathrm{id}$, hence $F(\sigma)$ is a monomorphism. By assumption it is therefore a cofibration, and so as $\mathbf{V}$ is left proper, the map $F(x, \ldots, x) \to u_!F(x, \ldots, x)$ is a weak equivalence if $F(x) \to *$ is a weak equivalence. Thus $F \to u^* u_! F$ is a levelwise weak equivalence if the map $F(x) \to *$ is a weak equivalence in $\mathbf{V}$ for every $x \in S$. Since every object of $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ is weakly equivalent to one for which this is true, it is clear that the Quillen adjunction $u_! \dashv u^*$ gives an equivalence of homotopy categories, and so is a Quillen equivalence. □

Next, we will compare the ∞-category associated to $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ to $\mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$. We know that the ∞-category associated to the projective model structure on $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})$

is equivalent to the ∞-categorical functor category $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V}[W^{-1}])$. The Bousfield-localized model category $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$ can therefore be identified with the full subcategory of $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V}[W^{-1}])$ spanned by the objects that are local with respect to certain maps. We can identify this with the ∞-category of Δ_S^{op}-monoids:

Definition 6.8 Recall that if $\mathcal{V}$ is an ∞-category with finite limits and $\mathcal{M}$ is a generalized non-symmetric ∞-operad, an $\mathcal{M}$-*monoid* in $\mathcal{V}$ is a functor $\mathcal{M} \to \mathcal{V}$ such that for every object $m \in \mathcal{M}_{[n]}$, if $m \to m_i$ ($i = 1, \dots, n$) are coCartesian morphisms corresponding to the inert maps $\rho_i \colon [1] \to [n]$ in Δ, then the induced morphism $F(m) \to F(m_1) \times \cdots \times F(m_n)$ is an equivalence. We write $\mathrm{Mon}_{\mathcal{M}}(\mathcal{V})$ for the full subcategory of $\mathrm{Fun}(\mathcal{M}, \mathcal{V})$ spanned by the monoids. There is a natural equivalence $\mathrm{Mon}_{\mathcal{M}}(\mathcal{V}) \simeq \mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$ (by [13, Proposition 3.5.3]).

Definition 6.9 Suppose $\mathcal{V}$ is a presentable ∞-category and $\mathcal{M}$ is a generalized non-symmetric ∞-operad. For $m \in \mathcal{M}$, write $i_m \colon * \to \mathcal{M}$ for the inclusion of this object, and let $i_{m,!}$ denote left Kan extension along i_m. Then for any functor $F \colon \mathcal{M} \to \mathcal{V}$ and $X \in \mathcal{V}$ we have $\mathrm{Map}(i_{m,!}c_X, F) \simeq \mathrm{Map}(c_X, i_m^* F) \simeq \mathrm{Map}_{\mathcal{V}}(X, F(m))$, where c_X is the functor $* \to \mathcal{V}$ with image X.

Lemma 6.10 *Suppose $\mathcal{V}$ is a presentable ∞-category such that the Cartesian product preserves colimits separately in each variable, and $\mathcal{M}$ is a small generalized non-symmetric ∞-operad. Then the ∞-category $\mathrm{Mon}_{\mathcal{M}}(\mathcal{V})$ is the localization of $\mathrm{Fun}(\mathcal{M}, \mathcal{V})$ with respect to the morphisms $i_{m_1,!}X \amalg \cdots \amalg i_{m_n,!}X \to i_{m,!}X$ for all $m \in \mathcal{M}$ with X ranging over a set of objects that generates $\mathcal{V}$ under colimits.*

Proof A functor $F \colon \mathcal{M} \to \mathcal{V}$ is a monoid if and only if it is local with respect to these morphisms. □

Since $\mathrm{Mon}_{\mathcal{M}}(\mathcal{V})$ is equivalent to $\mathrm{Alg}_{\mathcal{M}}(\mathcal{V})$, we have proved the following:

Proposition 6.11 *Suppose $\mathbf{V}$ is a left proper combinatorial Cartesian model category, and let $W_{\mathrm{Seg},S}$ denote the class of weak equivalences in $\mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})_{\mathrm{Seg}}$. Then the natural map $\alpha_S \colon \mathrm{Fun}(\Delta_S^{\mathrm{op}}, \mathbf{V})[W_{\mathrm{Seg},S}^{-1}] \to \mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$ is an equivalence. If moreover monomorphisms in $\mathbf{V}$ are cofibrations, then we also have a natural equivalence $\mathrm{Precat}_S(\mathbf{V})[W_{\mathrm{Pre},X}^{-1}] \to \mathrm{Alg}_{\Delta_S^{\mathrm{op}}}(\mathbf{V}[W^{-1}])$, where $W_{\mathrm{Pre},X}$ denotes the class of weak equivalences in $\mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$.*

Having dealt with the fixed-objects case, we will now allow the set of objects to vary:

Definition 6.12 Let $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ denote the total space of the right Quillen presheaf given by $S \mapsto \mathrm{Fun}(\Delta^{\mathrm{op}}_S, \mathbf{V})_{\mathrm{Seg}}$ and let $\mathrm{Precat}(\mathbf{V})$ denote the total space of the right Quillen presheaf given by $S \mapsto \mathrm{Precat}_S(\mathbf{V})_{\mathrm{Seg}}$. The adjunction $u_! \dashv u^*$ is natural and so gives a natural transformation between these right Quillen presheaves.

Proposition 6.13 *Let* $\mathbf{V}$ *be a left proper combinatorial Cartesian model category. There exist combinatorial model structures on the categories* $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ *and* $\mathrm{Precat}(\mathbf{V})$ *where a morphism* $F\colon \mathbf{C} \to \mathbf{D}$ *is a weak equivalence if and only if the induced morphism* f *on objects is a bijection and* $\mathbf{C} \to f^*\mathbf{D}$ *is a weak equivalence in* $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{ob}\,\mathbf{C}}, \mathbf{V})_{\mathrm{Seg}}$ *or* $\mathrm{Precat}_{\mathrm{ob}\,\mathbf{C}}(\mathbf{V})_{\mathrm{Seg}}$ *and a fibration if and only if* $\mathbf{C} \to f^*\mathbf{D}$ *is a fibration in* $\mathrm{Fun}(\Delta^{\mathrm{op}}_{\mathrm{ob}\,\mathbf{C}}, \mathbf{V})_{\mathrm{Seg}}$ *or* $\mathrm{Precat}_{\mathrm{ob}\,\mathbf{C}}(\mathbf{V})_{\mathrm{Seg}}$. *The adjunction*

$$u_! : \mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V}) \leftrightarrows \mathrm{Precat}(\mathbf{V}) : u^*$$

induced by the natural transformations $u_!$ *and* u^* *is a Quillen equivalence.*

Proof This is immediate from Proposition 4.25. □

Now combining Corollary 4.22 and Proposition 6.11 we get the following comparison of "algebraic" homotopy theories:

Theorem 6.14 *Suppose* $\mathbf{V}$ *is a left proper combinatorial Cartesian model category. The natural transformation* α *induces a functor* $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[W_{\mathrm{Fun}}^{-1}] \to \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$ *and this is an equivalence, where* W_{Fun} *denotes the weak equivalences in the model structure on* $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$. *If moreover monomorphisms in* $\mathbf{V}$ *are cofibrations, then we also have an equivalence* $\mathrm{Precat}(\mathbf{V})[W_{\mathrm{Precat}}^{-1}] \simeq \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}$.

The weak equivalences in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ are difficult to describe in general; however, a morphism $f\colon \mathbf{C} \to \mathbf{D}$ between fibrant objects, i.e. Segal categories, is a weak equivalence if and only if it is bijective on objects and a levelwise weak equivalence — in fact, given the Segal conditions, it suffices for f to give a weak equivalence $\mathbf{C}(x,y) \to \mathbf{D}(fx,fy)$ for all objects x,y in $\mathbf{C}$. To obtain the correct homotopy theory we clearly also need to invert the morphisms that are fully faithful and essentially surjective in the appropriate sense:

Definition 6.15 Composition with the projection $\mathbf{V} \to \mathrm{h}\mathbf{V}$ induces a functor

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V}) \to \mathrm{Seg}_{\mathrm{Fun}}(\mathrm{h}\mathbf{V}).$$

This takes Segal categories to categories enriched in $\mathrm{h}\mathbf{V}$. We say a morphism between Segal categories in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ is *weakly fully faithful and homotopically essentially surjective* if its image in $\mathrm{Seg}_{\mathrm{Fun}}(\mathrm{h}\mathbf{V})$ corresponds to a fully faithful and essentially surjective functor of $\mathrm{h}\mathbf{V}$-categories.

This definition extends to give a notion of weak equivalence in $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$, and similarly in $\mathrm{Precat}(\mathbf{V})$; we will refer to these as *Segal equivalences*, and denote the class of them as SE (in both $\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})$ and $\mathrm{Precat}(\mathbf{V})$). There are three model structures on $\mathrm{Precat}(\mathbf{V})$ with the Segal equivalences as weak equivalences, namely the *projective*, *injective*, and *Reedy* model structures, constructed in [30].

The Segal equivalences between Segal categories clearly correspond to the fully faithful and essentially surjective functors between categorical algebras, so we get the following:

Proposition 6.16 *Suppose* $\mathbf{V}$ *is a left proper combinatorial Cartesian model category. Then there is an equivalence*

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}[\mathrm{FFES}^{-1}].$$

If moreover monomorphisms in $\mathbf{V}$ *are cofibrations, then there is an equivalence*

$$\mathrm{Precat}(\mathbf{V})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Alg}_{\mathrm{cat}}(\mathbf{V}[W^{-1}])_{\mathrm{Set}}[\mathrm{FFES}^{-1}].$$

Combining this with [13, Theorem 5.2.17] gives our comparison result:

Theorem 6.17 *Suppose* $\mathbf{V}$ *is a left proper combinatorial Cartesian model category. There is an equivalence of* ∞*-categories*

$$\mathrm{Seg}_{\mathrm{Fun}}(\mathbf{V})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathbf{V}[W^{-1}]}.$$

If moreover monomorphisms in $\mathbf{V}$ *are cofibrations, then there is an equivalence*

$$\mathrm{Precat}(\mathbf{V})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{\infty}^{\mathbf{V}[W^{-1}]}.$$

Corollary 6.18 *Let* $\mathbf{V}$ *be a left proper tractable Cartesian model category that is a presheaf category such that the monomorphisms are the cofibrations. Then for all* $n \geq 0$ *there are equivalences of* ∞*-categories*

$$\mathrm{Precat}^{n}(\mathbf{V})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{(\infty,n)}^{\mathbf{V}[W^{-1}]}.$$

Proof We wish to apply Theorem 6.17 inductively. To do this we must check that if $\mathbf{V}$ satisfies the given hypotheses, then so does a suitable model structure on $\mathrm{Precat}(\mathbf{V})$. By [30, Theorem 21.3.2], if $\mathbf{V}$ is a left proper tractable Cartesian model category then the same is true of the *Reedy* model structure on $\mathrm{Precat}(\mathbf{V})$. Moreover, by [30, Proposition 15.7.2] if $\mathbf{V}$ is a presheaf category such that the monomorphisms are the cofibrations, then the injective and Reedy model structures on $\mathrm{Precat}(\mathbf{V})$ coincide, so the Reedy cofibrations are the monomorphisms, since these are clearly the injective cofibrations. Finally $\mathrm{Precat}(\mathbf{V})$ is also a presheaf category by [30, Proposition 12.7.6].

By induction it therefore follows that the Reedy model structure on $\mathrm{Precat}^n(\mathbf{V})$ satisfies the hypotheses of Theorem 6.17 for all n. Moreover, since the monoidal structures on both $\mathrm{Precat}(\mathbf{V})$ and $\mathrm{Cat}_\infty^{\mathbf{V}[W^{-1}]}$ are given by the Cartesian product, the equivalence between them is automatically an equivalence of symmetric monoidal ∞-categories, hence induces an equivalence $\mathrm{Cat}_\infty^{\mathrm{Precat}(\mathbf{V})} \xrightarrow{\sim} \mathrm{Cat}_{(\infty,2)}^{\mathbf{V}[W^{-1}]}$, etc. By induction we thus get a sequence of equivalences

$$\mathrm{Precat}^n(\mathbf{V})[\mathrm{SE}^{-1}] \simeq \mathrm{Cat}_\infty^{\mathrm{Precat}^{n-1}(\mathbf{V})[\mathrm{SE}^{-1}]} \simeq \mathrm{Cat}_{(\infty,2)}^{\mathrm{Precat}^{n-2}(\mathbf{V})[\mathrm{SE}^{-1}]} \simeq \cdots \simeq \mathrm{Cat}_{(\infty,n)}^{\mathbf{V}[W^{-1}]}.$$

□

Example 6.19 If we take $\mathbf{V}$ to be the category Set_Δ of simplicial sets, with the usual model structure, we get an equivalence

$$\mathrm{Precat}^n(\mathrm{Set}_\Delta)[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{(\infty,n)},$$

where the left-hand side is the ∞-category of the (∞,n)-categories of Pellissier-Hirschowitz-Simpson and the right-hand side is the ∞-category of (∞,n)-categories defined by iterated ∞-categorical enrichment.

Example 6.20 We would like to take $\mathbf{V}$ to be the category Set of sets, equipped with the trivial model structure, but of course this does not satisfy the hypothesis that cofibrations are monomorphisms. We therefore need to consider instead a model category $\mathbf{M}$, Quillen equivalent to Set, that does satisfy the hypotheses of the theorem. For example, following [30, § 22.1] we can let $\mathbf{M}$ be an appropriate localization of the Reedy model structure on $\mathrm{Precat}^2(*)$, or we can take $\mathbf{M}$ to be the Bousfield localization of the usual model structure on Set_Δ with respect to the morphisms $\partial\Delta^n \to \Delta^0$ for all $n \geq 2$. We then get an equivalence

$$\mathrm{Precat}^n(\mathrm{Set})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Precat}^n(\mathbf{M})[\mathrm{SE}^{-1}] \xrightarrow{\sim} \mathrm{Cat}_{(\infty,n)}^{\mathbf{M}[W^{-1}]} \xrightarrow{\sim} \mathrm{Cat}_n,$$

where the left-hand side is the ∞-category of Tamsamani's n-categories [33] and the right-hand side is the ∞-category of n-categories defined by iterated ∞-categorical enrichment.

7 Comparison with Iterated Segal Spaces

We saw in the previous section that the ∞-category $\mathrm{Cat}_{(\infty,n)}$ of (∞,n)-categories, obtained by iterated enrichment, is equivalent to that associated to the model category of n-fold Segal categories, which is another model for the homotopy theory of (∞, n)-categories. Since this model is known to satisfy the axioms of Barwick and Schommer-Pries [5], it follows that $\mathrm{Cat}_{(\infty,n)}$ is equivalent to all the usual models for (∞, n)-categories. However, this comparison was somewhat indirect. Our goal in this section is to give a more direct comparison between $\mathrm{Cat}_{(\infty,n)}$ and another established model of (∞, n)-categories, namely the iterated Segal spaces of Barwick [1].

We will deduce this comparison from a slightly more general result: we will prove that if $\mathcal{X}$ is an *absolute distributor*, in the sense of [20], then categorical algebras in $\mathcal{X}$ are equivalent to Segal spaces in $\mathcal{X}$, and complete categorical algebras are equivalent to complete Segal spaces. We begin with a brief review of the notion of distributor:

Definition 7.1 A *distributor* consists of an ∞-category $\mathcal{X}$ together with a full subcategory $\mathcal{Y}$ such that:

(1) The ∞-categories $\mathcal{X}$ and $\mathcal{Y}$ are presentable.

(2) The full subcategory $\mathcal{Y}$ is closed under small limits and colimits in $\mathcal{X}$.

(3) If $X \to Y$ is a morphism in $\mathcal{X}$ such that $Y \in \mathcal{Y}$, then the pullback functor $\mathcal{Y}_{/Y} \to \mathcal{X}_{/X}$ preserves colimits.

(4) Let $\mathcal{O}$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{X})$ spanned by those morphisms $f\colon X \to Y$ such that $Y \in \mathcal{Y}$, and let $\pi\colon \mathcal{O} \to \mathcal{Y}$ be the functor given by evaluation at $1 \in \Delta^1$. Since $\mathcal{X}$ admits pullbacks, the evaluation-at-1 functor $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ is a Cartesian fibration, hence so is π. Let $\chi\colon \mathcal{Y} \to \widehat{\mathrm{Cat}}_\infty^{\mathrm{op}}$ be a functor that classifies π. Then χ preserves small limits.

Definition 7.2 An *absolute distributor* is a presentable ∞-category $\mathcal{X}$ such that the unique colimit-preserving functor $\mathcal{S} \to \mathcal{X}$ that sends $*$ to the final object is fully faithful, and $\mathcal{S} \subseteq \mathcal{X}$ is a distributor.

Now we can recall the definition of a Segal space in an absolute distributor:

Definition 7.3 Suppose $\mathcal{C}$ is an ∞-category with finite limits. A *category object* in $\mathcal{C}$ is a simplicial object $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$ such that for each n the map

$$F_n \to F_1 \times_{F_0} \cdots \times_{F_0} F_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$ and $\{i\} \hookrightarrow [n]$ is an equivalence.

Definition 7.4 Let $\mathcal{X}$ be an absolute distributor. A *Segal space* in $\mathcal{X}$ is a category object $F\colon \Delta^{\mathrm{op}} \to \mathcal{X}$ such that $F([0])$ is in $\mathcal{S} \subseteq \mathcal{X}$.

Our goal is now to prove the following:

Theorem 7.5 *Suppose $\mathcal{X}$ is an absolute distributor. There is an equivalence*

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Seg}(\mathcal{X}),$$

given by sending a Δ^{op}_{S}-algebra $\mathcal{C}$ to the left Kan extension $\pi_!\mathcal{C}'$ of the composite

$$\mathcal{C}' \colon \Delta^{\mathrm{op}}_{S} \xrightarrow{\mathcal{C}} \mathcal{X}^{\times} \to \mathcal{X}$$

along $\pi\colon \Delta^{\mathrm{op}}_{S} \to \Delta^{\mathrm{op}}$, where the second map (which sends $(S_1, \dots, S_n) \in \mathcal{X}^{\times}_{[n]}$ to $S_1 \times \cdots \times S_n$) comes from a Cartesian structure in the sense of [21, Definition 2.4.1.1].

For the proof we need some more technical results:

Proposition 7.6 ([20, Corollary 1.2.5]) *Suppose $\mathcal{Y} \subseteq \mathcal{X}$ is a distributor. Let K be a small simplicial set, and let $\overline{\alpha}\colon \overline{p} \to \overline{q}$ be a natural transformation between functors $\overline{p}, \overline{q}\colon K^{\triangleright} \to \mathcal{X}$. If $\overline{q}$ is a colimit diagram in $\mathcal{Y}$ and $\alpha = \overline{\alpha}|_K$ is Cartesian, then $\overline{\alpha}$ is Cartesian if and only if $\overline{p}$ is a colimit diagram.*

Lemma 7.7 *Suppose $\mathcal{X}$ is an absolute distributor. Then for every space $X \in \mathcal{S}$, the map*

$$\gamma_X \colon \mathrm{Fun}(X, \mathcal{X}) \to \mathcal{X}_{/X}$$

that sends a functor $F\colon X \to \mathcal{X}$ to its colimit is an equivalence of ∞-categories.

Proof Let $\xi\colon X \to \mathcal{X}$ be the constant functor at the final object $* \in \mathcal{S} \subseteq \mathcal{X}$. Since X is a space, a functor $F\colon X \to \mathcal{X}$ sends every morphism in X to an equivalence in $\mathcal{X}$, and so the unique natural transformation $F \to \xi$ is Cartesian.

Write $\overline{\xi}\colon X^{\triangleright} \to \mathcal{X}$ for a colimit diagram extending ξ. Then γ_X factors as

$$\mathrm{Fun}(X, \mathcal{X}) \simeq \mathrm{Fun}(X, \mathcal{X})_{/\xi} \xrightarrow{\phi_1} \mathrm{Fun}(X^{\triangleright}, \mathcal{X})_{/\overline{\xi}} \xrightarrow{\phi_2} \mathcal{X}_{/X},$$

where ϕ_2 is given by evaluation at the cone point. The functor ϕ_1 gives an equivalence between $\mathrm{Fun}(X, \mathcal{X})_{/\xi}$ and the full subcategory $\mathcal{E}_1$ of $\mathrm{Fun}(X^{\triangleright}, \mathcal{X})_{/\overline{\xi}}$ spanned by the colimit diagrams. On the other hand, the restriction of ϕ_2 to the full subcategory $\mathcal{E}_2$ spanned by the Cartesian natural transformations to $\overline{\xi}$ is also clearly an equivalence. By Proposition 7.6 the subcategories $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide, and so the composite γ_X is indeed an equivalence. □

Proposition 7.8 *Let $\mathcal{O}$ be an ∞-category, and let $F\colon \mathcal{O} \to \mathcal{S}$ be a functor; write $\pi\colon \mathcal{O}_F \to \mathcal{O}$ for the left fibration associated to F. Suppose $\mathcal{X}$ is an absolute distributor. Then left Kan extension along π gives an equivalence*

$$\mathrm{Fun}(\mathcal{O}_F, \mathcal{X}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}, \mathcal{X})_{/F}.$$

Proof By [13, Proposition A.1.5] the ∞-category $\mathrm{Fun}(\mathcal{O}_F, \mathcal{X})$ is equivalent to the ∞-category of sections of the Cartesian fibration $\mathcal{E} \to \mathcal{O}$ whose fibre at $X \in \mathcal{O}$ is $\mathrm{Fun}(F(X), \mathcal{X})$. Since $\mathcal{X}$ is an absolute distributor, by Lemma 7.7 the ∞-category $\mathcal{E}$ is equivalent over $\mathcal{O}$ to the total space $\mathcal{E}'$ of the Cartesian fibration associated to the functor sending X to $\mathcal{X}_{/F(X)}$. Then $\mathcal{E}'$ is the pullback along F of the Cartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathcal{X}$ given by evaluation at 1, so we have an equivalence between the ∞-category $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{E}')$ of sections and the fibre of $\mathrm{Fun}(\mathcal{O} \times \Delta^1, \mathcal{X}) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{O}, \mathcal{X})) \to \mathrm{Fun}(\mathcal{O}, \mathcal{X})$ at F. This is clearly equivalent to $\mathrm{Fun}(\mathcal{O}, \mathcal{X})_{/F}$, which completes the proof. □

Proposition 7.9 *Let S be a space, and let $\pi\colon \Delta^{\mathrm{op}}_S \to \Delta^{\mathrm{op}}$ be the usual projection. Let $\pi_!\colon \mathrm{Fun}(\Delta^{\mathrm{op}}_S, \mathcal{X}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})$ be the functor given by left Kan extension along π. Then a functor $F\colon \Delta^{\mathrm{op}}_S \to \mathcal{X}$ is a Δ^{op}_S-monoid if and only if $\pi_! F$ is a Segal space.*

Proof It is clear that $\pi_! F([0])$ is equivalent to S. We must thus show that the Segal morphism

$$\pi_! F([n]) \to \pi_! F([1]) \times_S \cdots \times_S \pi_! F([1]) =: (\pi_! F)^{\mathrm{Seg}}_{[n]}$$

is an equivalence if and only if F is a Δ^{op}_S-monoid. Since π is a coCartesian fibration, we have an equivalence $\pi_! F([n]) \simeq \mathrm{colim}_{\xi \in S^{\times(n+1)}} F(\xi)$. It thus suffices to show that $(\pi_! F)^{\mathrm{Seg}}_{[n]}$ is also a colimit of this diagram if and only if F is a Δ^{op}_S-monoid. There is a natural transformation $(S^{\times(n+1)})^{\triangleright} \to \mathrm{Fun}(\Delta^1, \mathcal{X})$ that sends $\xi \in S^{\times(n+1)}$ to $F(\xi) \to \xi$ and ∞ to $(\pi_! F)^{\mathrm{Seg}}_{[n]} \to S^{\times(n+1)}$. Since $\mathcal{X}$ is an absolute distributor, by Proposition 7.6 the colimit is $(\pi_! F)^{\mathrm{Seg}}_{[n]}$ if and only if this natural transformation is Cartesian. Since $S^{\times(n+1)}$ is a space, this is equivalent to the square

$$\begin{array}{ccc} F(\xi) & \longrightarrow & (\pi_! F)^{\mathrm{Seg}}_{[n]} \\ \downarrow & & \downarrow \\ \xi & \longrightarrow & S^{\times(n+1)} \end{array}$$

being a pullback square for all ξ, so we are reduced to showing that the fibre of $(\pi_! F)^{\mathrm{Seg}}_{[n]} \to S^{\times(n+1)}$ at ξ is $F(\xi)$ if and only if F is a Δ^{op}_S-monoid. Since limits commute, if ξ is $(s_0, \ldots, s_n)$ this fibre is the iterated fibre product

$$(\pi_! F[1])_{(s_0,s_1)} \times_{(\pi_! F[0])_{(s_1)}} \cdots \times_{(\pi_! F[0])_{(s_{n-1})}} (\pi_! F[1])_{(s_{n-1},s_n)}.$$

But using Proposition 7.6 again it is clear that the natural maps $F(x,y) \to (\pi_! F[1])_{(x,y)}$ and $* \simeq F(x) \to (\pi_! F)_{(x)}$ are equivalences for all $x, y \in S$. Thus the map $F(\xi) \to (\pi_! F)^{\mathrm{Seg}}_{[n],\xi}$ is equivalent to the natural map

$$F(\xi) \to F(s_0, s_1) \times \cdots \times F(s_{n-1}, s_n).$$

By definition this is an equivalence for all $\xi \in \Delta^{\mathrm{op}}_S$ if and only if F is a Δ^{op}_S-monoid, which completes the proof. □

Definition 7.10 Let $i \colon * \to \Delta^{\mathrm{op}}$ denote the inclusion of the object $[0]$. Then composition with i gives a functor $i^* \colon \mathrm{Seg}(\mathcal{X}) \to \mathcal{S}$ with left and right adjoints $i_!$ and i_*, given respectively by left and right Kan extension. Observe that by definition $\Delta^{\mathrm{op}}_X \to \Delta^{\mathrm{op}}$ is the left fibration associated to $i_* X \in \mathrm{Seg}(\mathcal{S})$.

Corollary 7.11 *Let S be a space, and let $\pi \colon \Delta^{\mathrm{op}}_S \to \Delta^{\mathrm{op}}$ denote the canonical projection. By Proposition 7.8 the functor*

$$\pi_! \colon \mathrm{Fun}(\Delta^{\mathrm{op}}_S, \mathcal{X}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})_{/i_* S}$$

given by left Kan extension is an equivalence.

Under this equivalence, the full subcategory $\mathrm{Mon}_{\Delta^{\mathrm{op}}_S}(\mathcal{X})$ of Δ^{op}_S-monoids corresponds to the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})_{/i_* S}$ spanned by the Segal spaces $Y_\bullet$ such that $Y_0 \simeq S$ and the map $Y_\bullet \to i_* S$ is given by the adjunction unit $Y_\bullet \to i_* i^* Y_\bullet \simeq i_* S$.

Proof It is clear that $\pi_!$ takes $\mathrm{Mon}_{\Delta^{\mathrm{op}}_S}(\mathcal{X})$ into the full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})_{/i_* S}$ spanned by simplicial spaces $Y_\bullet$ with $Y_0 \simeq S$ and the map $Y_\bullet \to i_* S$ given by the adjunction unit $Y_\bullet \to i_* i^* Y \simeq i_* S$. The result therefore follows by Proposition 7.9. □

Corollary 7.12 *Let S be a space, and let $\pi \colon \Delta^{\mathrm{op}}_S \to \Delta^{\mathrm{op}}$ denote the canonical projection. The functor $\pi_! \colon \mathrm{Fun}(\Delta^{\mathrm{op}}_S, \mathcal{X}) \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{X})$ given by left Kan extension along π gives an equivalence of the full subcategory $\mathrm{Mon}_{\Delta^{\mathrm{op}}_S}(\mathcal{X})$ of Δ^{op}_S-monoids with the subcategory $\mathrm{Seg}(\mathcal{X})_S$ of Segal spaces with 0th space S and morphisms that are the identity on the 0th space.*

Proof of Theorem 7.5 If $\mathcal{V}$ is an ∞-category with finite products, pulling back the monoid fibration $\mathrm{Mon}(\mathcal{V}) \to \mathrm{Opd}^{\mathrm{ns}}_\infty$ of [13, Remark 3.6.3] along $\Delta^{\mathrm{op}}_{(\{\})}$ gives a Cartesian fibration $\mathrm{Mon}_{\mathrm{cat}}(\mathcal{V})$ with an equivalence

$$\mathrm{Alg}_{\mathrm{cat}}(\mathcal{V}) \xrightarrow{\sim} \mathrm{Mon}_{\mathrm{cat}}(\mathcal{V})$$

over $\mathcal{S}$. Taking left Kan extensions along the projections $\Delta^{\mathrm{op}}_{S} \to \Delta^{\mathrm{op}}$ for all $S \in \mathcal{S}$ we get (using Proposition 7.9) a commutative square

$$\begin{array}{ccc} \mathrm{Mon}_{\mathrm{cat}}(\mathcal{X}) & \xrightarrow{\Phi} & \mathrm{Seg}(\mathcal{X}) \\ & \searrow \quad \swarrow_{\mathrm{ev}_{[0]}} & \\ & \mathcal{S}. & \end{array}$$

By [13, Lemma A.1.6] it is clear that $\mathrm{ev}_{[0]}\colon \mathrm{Seg}(\mathcal{X}) \to \mathcal{S}$ is a Cartesian fibration, and the functor Φ preserves Cartesian morphisms by Proposition 7.6. It thus suffices to prove that for each $S \in \mathcal{S}$ the functor on fibres $\mathrm{Mon}_{\Delta^{\mathrm{op}}_{S}}(\mathcal{X}) \to \mathrm{Seg}(\mathcal{X})_S$ is an equivalence, which is the content of Corollary 7.12. □

Our goal is now to deduce that the equivalence of Theorem 7.5 induces an equivalence between complete categorical algebras and complete Segal spaces. We will first review the definition of the latter:

Definition 7.13 Write $\mathrm{Gpd}(\mathcal{S})$ for the full subcategory of $\mathrm{Seg}(\mathcal{S})$ spanned by the *groupoid objects*, i.e. the simplicial objects X such that for every partition $[n] = S \cup S'$ where $S \cap S'$ consists of a single element, the diagram

$$\begin{array}{ccc} X([n]) & \longrightarrow & X(S) \\ \downarrow & & \downarrow \\ X(S') & \longrightarrow & X(S \cap S') \end{array}$$

is a pullback square. Let $\mathcal{X}$ be an absolute distributor, and let $\Lambda\colon \mathcal{X} \to \mathcal{S}$ denote the right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathcal{X}$. The inclusion $\mathrm{Gpd}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{S}) \hookrightarrow \mathrm{Seg}(\mathcal{X})$ admits a right adjoint $\iota\colon \mathrm{Seg}(\mathcal{X}) \to \mathrm{Gpd}(\mathcal{S})$, which is the composite of the functor $\Lambda\colon \mathrm{Seg}(\mathcal{X}) \to \mathrm{Seg}(\mathcal{S})$ induced by Λ, and $\iota\colon \mathrm{Seg}(\mathcal{S}) \to \mathrm{Gpd}(\mathcal{S})$. We say a Segal space $F\colon \Delta^{\mathrm{op}} \to \mathcal{X}$ is *complete* if the groupoid object ιF is constant.

Remark 7.14 By [13, Lemma 5.1.14], a Segal space F is complete if and only if the map

$$\iota F(s^0)\colon \iota F[0] \to \iota F[1]$$

is an equivalence.

Definition 7.15 Let E^n denote the Segal space $i_*\{0,\ldots,n\}$. If $\mathcal{X}$ is an absolute distributor we also write E^n for E^n regarded as a Segal space in $\mathcal{X}$ via the inclusion $\mathcal{S} \hookrightarrow \mathcal{X}$.

Proposition 7.16 *Suppose $\mathcal{X}$ is an absolute distributor. Then a Segal space F in $\mathcal{X}$ is complete if and only if it is local with respect to the morphism $E^1 \to E^0$.*

Proof It is clear that F is local with respect to $E^1 \to E^0$, considered as a morphism in $\mathrm{Seg}(\mathcal{X})$, if and only if the Segal space ΛF in $\mathcal{S}$ is local with respect to $E^1 \to E^0$, considered as a morphism in $\mathrm{Seg}(\mathcal{S})$. On the other hand, F is complete if and only if ΛF is complete, so it suffices to prove this for Segal spaces in $\mathcal{S}$. This case is part of [26, Proposition 6.4]. □

Definition 7.17 Let $\mathrm{CSS}(\mathcal{X})$ denote the full subcategory of $\mathrm{Seg}(\mathcal{X})$ spanned by the complete Segal spaces; by Proposition 7.16 this is the localization of $\mathrm{Seg}(\mathcal{X})$ with respect to the morphism $E^1 \to E^0$.

Theorem 7.18 *Let $\mathcal{X}$ be an absolute distributor. The equivalence $\mathrm{Alg}_{\mathrm{cat}}(\mathcal{X}) \xrightarrow{\sim} \mathrm{Seg}(\mathcal{X})$ induces an equivalence $\mathrm{Cat}_\infty^{\mathcal{X}} \xrightarrow{\sim} \mathrm{CSS}(\mathcal{X})$.*

Proof It is clear that $E^n_{\mathcal{X}} \in \mathrm{Alg}_{\mathrm{cat}}(\mathcal{X})$ corresponds to $E^n \in \mathrm{Seg}(\mathcal{X})$ under this equivalence. Both sides are therefore the localization with respect to $E^1 \to E^0$. □

Definition 7.19 By [20, Corollary 1.3.4], if $\mathcal{X}$ is an absolute distributor, then $\mathrm{CSS}(\mathcal{X})$ is also an absolute distributor. We therefore have absolute distributors $\mathrm{CSS}^n(\mathcal{X})$ of *n-fold complete Segal spaces* in $\mathcal{X}$.

Applying Theorem 7.18 inductively, we get:

Corollary 7.20 *Let $\mathcal{X}$ be an absolute distributor. Then $\mathrm{Cat}^{\mathcal{X}}_{(\infty,n)} \simeq \mathrm{CSS}^n(\mathcal{X})$.*

In particular, taking $\mathcal{X}$ to be the ∞-category $\mathcal{S}$ of spaces, we obtain the desired comparison with iterated Segal spaces:

Corollary 7.21 *There is an equivalence $\mathrm{Cat}_{(\infty,n)} \simeq \mathrm{CSS}^n(\mathcal{S})$.*

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