STACKS OF UNIFORM CYCLIC COVERS OF CURVES AND THEIR PICARD GROUPS

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ABSTRACT. We study the stack $\mathcal{B}_{h,g,n}$ of uniform cyclic covers of degree n between smooth curves of genus h and g and, for $h \gg g$, present it as an open substack of a vector bundle over the universal Jacobian stack of $\mathcal{M}_g$. We use this description to compute the integral Picard group of $\mathcal{B}_{h,g,n}$, showing that it is generated by tautological classes of $\mathcal{B}_{h,g,n}$.

INTRODUCTION

Let k be a field and h, g, n be non negative integers with $n \geq 2$. We denote by $\mathcal{B}_{h,g,n}$ the stack over k of triples $(D \longrightarrow S, C \longrightarrow S, f)$ where $D \longrightarrow S$ is a smooth, geometrically connected genus h curve, $C \longrightarrow S$ is a smooth, geometrically connected genus g curve and $f\colon D \longrightarrow C$ is a uniform cyclic cover of degree n (see Section 4 for a definition of uniform cyclic covers). The aim of this paper is to describe the structure of $\mathcal{B}_{h,g,n}$ and compute its integral Picard group.

This work was inspired by the results in [AV04] and [BV12], where the authors compute the Picard group of similar moduli problems, namely the stack of uniform cyclic covers of projective spaces and of triple covers of curves of genus zero respectively. However the methods used here are different, as we do not use a presentation of $\mathcal{B}_{h,g,n}$ as a quotient stack. Another source of inspiration and, in fact, the starting point of our computation in genus one was the classical result of Mumford about the Picard group of the stack $\mathcal{M}_{1,1}$ of elliptic curves (see [Mum63] and also [FO10]).

This paper was born as a study of double covers of genus one curves, that is of the stacks $\mathcal{B}_{h,1,2}$ (which also explains the use of the letter $\mathcal{B}$ which stands for 'bielliptic'). The main obstacle in generalizing the results for $g \geq 2$ was the computation of the Picard group of the universal Jacobian of $\mathcal{M}_g$ (see below for a definition), since the methods we used for the same problem in genus one fail in higher genera. This last problem was solved in [MV14], allowing the generalization for higher genera.

Let us also remark that the case $n = 2$ is the most interesting from a "geometric" point of view, since (in characteristic different from 2) uniform cyclic covers of degree 2 are just covers of degree 2 and, therefore, $\mathcal{B}_{h,g,2}$ is the stack of double covers between smooth curves of genera h and g.

Let us also remark that the case $n = 2$ is the most interesting from a "geometric" point of view, since in this case (in characteristic different from 2) the "uniform cyclic" condition is automatic, and our stack coincides with the stack of double covers between smooth curves of genera h and g.

The Picard group of $\mathcal{B}_{h,0,n}$ was already computed in [AV04, Theorem 5.1]. Here the authors introduce moduli stacks of uniform cyclic covers of projective spaces, denoted by $\mathcal{H}_{\text{sm}}(r, n, d)$ for $r, n, d > 0$. In the one dimensional case $r = 1$ we have $\mathcal{H}_{\text{sm}}(1, n, d) = \mathcal{B}_{h,0,n}$, where d, n, h are related by the expression (1) below. In this paper we provide an alternative method for the computation of $\operatorname{Pic} \mathcal{B}_{h,0,n}$ which extends to higher genera.

In [Pag13] the author introduces moduli stacks of abelian covers of curves, which are related to our stacks $\mathcal{B}_{h,g,n}$ in the cyclic, totally ramified case. Let $\mathcal{Y}_{g,r,n}$ be the stack of tuples $(D, C, f, \sigma_1, \dots, \sigma_r)$ where $(C, \sigma_1, \dots, \sigma_r)$ is a r-pointed curve of genus g and $f\colon D \longrightarrow C$ is a uniform cyclic cover of degree n whose ramification locus is the union of the sections $\sigma_1, \dots, \sigma_r$. By forgetting the sections we obtain a functor $\mathcal{Y}_{g,nd,n} \longrightarrow \mathcal{B}_{h,g,n}$, where d, g, h, n are related by the expression (1) below, which is a S_{nd}-torsor. In [Pag13, Theorem 3] is proved that $\mathcal{Y}_{g,nd,n}$, which is denoted by $\mathcal{M}_{1,nd}(\mathrm{B}(\mathbb{Z}/n\mathbb{Z}), (1), \dots, (1))$, has trivial rational Picard group for $nd > 0$, which also implies the vanishing of the rational Picard group

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of $\mathcal{B}_{h,1,n}$. In this paper we recover this last result by explicitly describing the integral Picard group of $\mathcal{B}_{h,1,n}$, but we can not directly deduce the result in [Pag13, Theorem 3].

The main result of this paper is the following.

A. **Theorem.** *Let h, g, n be non negative integers with $n \geq 2$ and set*

$$d = 2\frac{h + n(1-g) - 1}{n(n-1)} \text{ so that } h = 1 + n(g-1) + \frac{n(n-1)}{2}d \tag{1}$$

The stack $\mathcal{B}_{h,g,n}$ is not empty if and only if $d \in \mathbb{N}$. Assume $d \in \mathbb{N}$. The stack $\mathcal{B}_{h,g,n}$ is algebraic and of finite type over k and, if $nd > 2g-2$ or $\operatorname{char} k \nmid n$, the forgetful functor $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{M}_g$ is smooth and surjective.

Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{B}_{h,g,n}$ be the universal genus g curve, $f\colon \mathcal{D} \longrightarrow \mathcal{C}$ be the universal uniform cyclic cover of degree n and $\mathcal{L}$ be the dual of the degree 1 part of the μ_n-equivariant sheaf $f_*\mathcal{O}_\mathcal{D}$. The sheaf $\mathcal{L}$ is invertible of degree d over $\mathcal{C}$ and we have the following.

(1) If $g = 0$ we have

$$\operatorname{Pic}\mathcal{B}_{h,0,n} \simeq \begin{cases} \mathbb{Z}/2n(nd-1)\mathbb{Z} & \text{generated by } \pi_*(\mathcal{L}\otimes\omega_\pi^{d/2}) \text{ if } d \text{ is even} \\ \mathbb{Z}/n(nd-1)\mathbb{Z} & \text{generated by } \det\pi_*(\mathcal{L}\otimes\omega_\pi^{(d-1)/2}) \text{ if } d \text{ is odd} \end{cases}$$

(2) If $g = 1$ then $\operatorname{Pic}\mathcal{B}_{h,1,n}$ is generated by $\pi_*\omega_\pi$ and $\det\pi_*\mathcal{L}$ with relations

$$\begin{aligned} &(\pi_*\omega_\pi)^4 \text{ and } \det\pi_*\mathcal{L} && \text{if } h=1,\ n=2,\ \operatorname{char} k \nmid 6 \\ &(\pi_*\omega_\pi)^6 \text{ and } (\pi_*\mathcal{L})^2\otimes(\pi_*\omega_\pi)^{-2} && \text{if } h=n=2,\ \operatorname{char} k \neq 2 \\ &(\pi_*\omega_\pi)^{12} \text{ and } (\det\pi_*\mathcal{L})^{2n^2}\otimes(\pi_*\omega_\pi)^{n(dn+d-2n)} && \text{if } nd>2,\ \operatorname{char} k \nmid nd \end{aligned}$$

As an abtract group we have

$$\operatorname{Pic}\mathcal{B}_{h,1,n} \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } h=1,\ n=2,\ \operatorname{char} k\nmid 3 \\ \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & \text{if } h=n=2 \\ \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2n^2\mathbb{Z} & \text{if } nd>2,\ \frac{n(dn+d-2n)}{2} \text{ is even},\ \operatorname{char} k\nmid nd \\ \mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4n^2\mathbb{Z} & \text{if } nd>2,\ \frac{n(dn+d-2n)}{2} \text{ is odd},\ \operatorname{char} k\nmid nd \end{cases}$$

(3) Assume that k is algebraically closed of characteristic 0 and either $nd > 2g-2$ and $g \geq 4$ or $nd > 2g-1$ and $g \geq 3$ or $nd > 2g$ and $g \geq 2$. Then $\operatorname{Pic}\mathcal{B}_{h,g,n}$ is generated by $\det\pi_*\omega_\pi$, $d_\pi(\mathcal{L})$ and $\det\pi_*(\mathcal{L}\otimes\omega_\pi)$ (see 2.4 for a defintion of $d_\pi(-)$) with the only relation

$$(\det\pi_*\omega_\pi)^{-2n^2}\otimes d_\pi(\mathcal{L})^{n(n-1)}\otimes(\det\pi_*(\mathcal{L}\otimes\omega_\pi))^{n(n+1)}$$

except for $g = 2$, for which we need to add the relation $(\det\pi_*\omega_\pi)^{10}$.

As an abstract group we have

$$\operatorname{Pic}\mathcal{B}_{h,g,n} \simeq \begin{cases} \mathbb{Z}/2n\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}\times\mathbb{Z} & \text{if } g=2 \text{ and } n \text{ is odd} \\ \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}\times\mathbb{Z} & \text{if } g=2 \text{ and } n \text{ is even} \\ \mathbb{Z}/2n\mathbb{Z}\times\mathbb{Z}^2 & \text{if } g>2 \text{ and } n \text{ is odd} \\ \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}^2 & \text{if } g>2 \text{ and } n \text{ is even} \end{cases}$$

When $h = n(g-1)+1$ (in particular $g \geq 1$), that is $d = 0$, uniform cyclic covers of degree n become μ_n-torsors. This case is not covered by the above theorem, except for $h = g = 1$ and $n = 2$ where $\operatorname{Pic}\mathcal{B}_{1,1,2}$ is computed by an ad hoc variation of the methods used in higher genera and degrees. When $d > 0$ uniform cyclic covers are never étale and the stacks $\mathcal{B}_{h,g,n}$ share a common description that we are now going to explain.

Fix $g \geq 0$, $d, n > 0$ and assume $\operatorname{char} k \nmid n$ or $g = 0$ and, if $g = 1$, $\operatorname{char} k \nmid d$. Set $\mathcal{U}_{d,g,n}$ for the stack of triples $(C \longrightarrow S, \mathcal{Q}, s)$ where C is a smooth, geometrically connected curve of genus g and $\mathcal{Q}$ is a degree d invertible sheaf with a section $s \in \mathcal{Q}^n$ that is not identically zero on any of the geometric fibers of $C \longrightarrow S$. The forgetful map $\mathcal{U}_{d,g,n} \longrightarrow \mathcal{M}_g$ defines the (universal) genus g curve $\pi\colon \mathcal{C} \longrightarrow \mathcal{U}_{d,g,n}$ together with an invertible sheaf $\mathcal{L}$ of degree d on $\mathcal{C}$ and a section $t \in \mathcal{L}^n$. The zero locus $\mathcal{H}_{d,g,n}$ of t in $\mathcal{C}$ is a degree nd cover of $\mathcal{U}_{d,g,n}$ and $\mathcal{B}_{h,g,n}$, where d, h, g, n are related by the expression (1) above, can

be identified with the étale locus of $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$ inside $\mathcal{U}_{d,g,n}$ (see 4.4). Since $\mathcal{U}_{d,g,n}$ is smooth and algebraic, the problem of computing $\operatorname{Pic}\mathcal{B}_{h,g,n}$ splits in two parts: compute $\operatorname{Pic}\mathcal{U}_{d,g,n}$ and describe the complement of $\mathcal{B}_{h,g,n}$ in $\mathcal{U}_{d,g,n}$.

Denote by $\mathcal{J}ac_{d,g}$ the stack of pairs $(C \longrightarrow S, \mathcal{Q})$ where C is a smooth, geometrically connected genus g curve and $\mathcal{Q}$ is a degree d invertible sheaf on C. This is the so called universal Jacobian of degree d on $\mathcal{M}_g$. Assume $nd > 2g-2$. The forgetful functor $\mathcal{U}_{d,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ makes $\mathcal{U}_{d,g,n}$ into the complement of the zero section of a vector bundle of rank $nd+1-g$ over $\mathcal{J}ac_{d,g}$ (this description is no longer true in general when $nd \leq 2g-2$ and this is why Theorem A does not cover this case). When $n \geq 2$ and since $nd+1-g \geq 2$ we can conclude that $\operatorname{Pic}\mathcal{U}_{d,g,n} \simeq \operatorname{Pic}\mathcal{J}ac_{d,g}$. When $g \geq 2$ the group $\operatorname{Pic}\mathcal{J}ac_{d,g}$ has been computed in [MV14]. If $g=0$ then $\operatorname{Pic}\mathcal{J}ac_{d,0} \simeq \mathbb{Z}$ (see 2.6). The case $g=1$ is harder than the case $g=0$ and our treatment differs from the methods used in [MV14] for $g \geq 2$. The result is that, if $g=1$, then $\operatorname{Pic}\mathcal{J}ac_{d,1} \simeq \mathbb{Z}/12 \times \mathbb{Z}$ (see 2.9) and it has been obtained by proving that the functor $\mathcal{J}ac_{d,1} \longrightarrow \mathcal{M}_{1,1}$ that maps $(E, \mathcal{Q})$ to $(\underline{\operatorname{Pic}}^0_E, [\mathcal{O}_E])$ is a trivial gerbe, that is $\mathcal{J}ac_{d,1} \simeq_{\mathcal{M}_{1,1}} \mathrm{B}_{\mathcal{M}_{1,1}} G_d$, where G_d is a smooth affine group over $\mathcal{M}_{1,1}$, and by computing the group of characters of G_d. The geometric fibers of $G_d \longrightarrow \mathcal{M}_{1,1}$ are particular cases of Theta groups, first defined by Mumford in his paper [Mum66] (see 2.16).

The last part in the computation of $\operatorname{Pic}\mathcal{B}_{h,g,n}$ is the description of $\mathcal{U}_{d,g,n} - \mathcal{B}_{h,g,n}$. Topologically this closed substack coincides with the discriminant locus $\mathcal{Z}_{d,g,n}$ of the cover $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$. By standard theory of covers the locus $\mathcal{Z}_{d,g,n}$ can be described as the zero locus of a section (the discriminant section) of an invertible sheaf of $\mathcal{U}_{d,g,n}$. The key point for the computation of $\operatorname{Pic}\mathcal{B}_{h,g,n}$ is that $\mathcal{Z}_{d,g,n}$ is reduced and, if $nd > 2g$ or $nd > 2g-1$ and $g \geq 3$ or $nd > 2g-2$ and $g \geq 4$, irreducible (see 3.2). In the special case $g=1$, $h=n=2$ (so that $d=1$ and $nd=2g$) ad hoc methods show that $\mathcal{Z}_{1,1,2}$ is a disjoint union of two irreducible components, allowing the computation of $\operatorname{Pic}\mathcal{B}_{2,1,2}$ (see 3.2). It is not clear whether $\mathcal{Z}_{d,g,n}$ is irreducible for the remaining values of h,g,n. The geometry of the loci $\mathcal{Z}_{d,g,n}$ is studied by reducing to the case $n=1$, showing that $\mathcal{U}_{d,g,1} \simeq \operatorname{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ (see section Notations) and that $\mathcal{H}_{d,g,1} \longrightarrow \operatorname{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ is the universal cover.

We remark that Theorem A is obtained by expressing $\operatorname{Pic}\mathcal{B}_{h,g,n}$ as quotient of $\operatorname{Pic}\mathcal{J}ac_{d,g}$ by a given relation and this description holds more generally than in the hypothesis of Theorem A (see 4.6 for a precise statement). For instance if $g \geq 2$ and $\operatorname{char} k \nmid n$ the knowledge of the integral (resp. rational) Picard group of $\mathcal{J}ac_{d,g}$ implies the knowledge of the integral (resp. rational) Picard group of $\mathcal{B}_{h,g,n}$. Unfortunately if $g \geq 2$ both the integral and rational Picard groups of $\mathcal{J}ac_{d,g}$ are known only in characteristic 0, although it seems reasonable to expect the same description for all but finitely many characteristics. See [MV14, Remark 1.4] for a discussion on the subject.

The paper is organized as follows. In Section 1 we collect useful remarks and lemmas, while in Section 2 we study the Picard group of the univeral Jacobian $\mathcal{J}ac_{d,g}$ over $\mathcal{M}_g$ for $g=0$ and $g=1$ and explain the results for $g \geq 2$ obtained in [MV14]. In Section 3 we introduce the canonical covers $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$ and describe their discriminant loci, while in Section 4 we introduce the stacks $\mathcal{B}_{h,g,n}$ and compute their Picard groups.

Notations. Given a base scheme S, by the words "scheme" or "stack" we always mean scheme or stack defined over this base scheme. Moreover by stacks we always mean a category fibered in groupoids which is a stack for the fppf topology. Let $\mathcal{S}$ be a stack.

A geometric point of $\mathcal{S}$ is a map $\operatorname{Spec} k \longrightarrow \mathcal{S}$, where k is an algebraically closed field.

Given a sheaf of groups $G\colon (\mathrm{Sch}/\mathcal{S})^{\mathrm{op}} \longrightarrow (\mathrm{grps})$ we denote by $G^\vee = \operatorname{Hom}_{\mathcal{S}}(G, \mathbb{G}_m)$ the group of characters of G over $\mathcal{S}$, i.e. group homomorphisms $G \longrightarrow \mathbb{G}_m$.

By a cover we mean an affine map $\mathcal{X} \xrightarrow{f} \mathcal{S}$ such that $f_*\mathcal{O}_\mathcal{X}$ is locally free of finite rank. Alternatively, a cover is a finite, flat and finitely presented map $\mathcal{X} \xrightarrow{f} \mathcal{S}$. The degree of f is the rank of $f_*\mathcal{O}_\mathcal{X}$. The discriminant section $s_f \in \det(f_*\mathcal{O}_\mathcal{X})^{-2}$ of f is the determinant of the map

$$f_*\mathcal{O}_\mathcal{X} \longrightarrow f_*\mathcal{O}_\mathcal{X}{}^\vee,\ x \longmapsto \operatorname{tr}_{f_*\mathcal{O}_\mathcal{X}}(x \cdot -)$$

where tr denotes the trace map. The discriminant locus of f is the zero locus in $\mathcal{S}$ of s_f. The discriminant section is stable under base change and the complement of the discriminant locus of f is the étale locus of f in $\mathcal{S}$.

A genus g curve over $\mathcal{S}$ is a representable (by algebraic spaces), flat, locally finitely presented and proper map $\mathcal{C} \longrightarrow \mathcal{S}$ of stacks whose geometric fibers are smooth and connected genus g curves. Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ be a genus g curve. We say that an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ has degree $d \in \mathbb{Z}$ if the pullback of $\mathcal{L}$ on every geometric fiber of $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ has degree d. We denote by ω_π the relative dualizing sheaf, which is an invertible sheaf on $\mathcal{C}$ of degree $2g-2$.

We denote by $\mathcal{M}_g$ the stack of genus g curves and by $\mathcal{M}_{g,1}$ the stack of genus g curves with a section. The forgetful functor $\mathcal{M}_{g,1} \longrightarrow \mathcal{M}_g$ is a genus g curve, called the universal genus g curve of $\mathcal{M}_g$. If it is given a map $\mathcal{S} \longrightarrow \mathcal{M}_g$, the universal genus g curve of $\mathcal{S}$ is the base change $\mathcal{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \longrightarrow \mathcal{S}$.

Let $\mathcal{X}$ be another stack and $f\colon \mathcal{X} \longrightarrow \mathcal{S}$ be a representable map. We denote by $\mathrm{Hilb}^n_{\mathcal{X}/\mathcal{S}}$ (or simply $\mathrm{Hilb}^n_{\mathcal{X}}$ whenever $\mathcal{S}$ is clear from the context) the stack over $\mathcal{S}$ whose objects over $S \longrightarrow \mathcal{S}$ are closed subschemes $Z \subseteq \mathcal{X} \times_{\mathcal{S}} S$ such that the projection $Z \longrightarrow S$ is a degree n cover. When $f\colon \mathcal{X} \longrightarrow \mathcal{S}$ is a projective map of schemes then $\mathrm{Hilb}^n_{\mathcal{X}/\mathcal{S}}$ is the usual Hilbert scheme of points.

We denote by $\mathrm{Hilb}^n_{\mathcal{X}/\mathcal{S}}$ (or simply $\mathrm{Hilb}^n_{\mathcal{X}}$ whenever $\mathcal{S}$ is clear from the context) the stack over $\mathcal{S}$ whose objects over $S \longrightarrow \mathcal{S}$ are closed subschemes $C \subseteq \mathcal{X} \times_{\mathcal{S}} S$ which are flat, proper and finitely presented over S, and such that for every geometric point $\overline{s} \to S$, the fiber $C_{\overline{s}} \subseteq \mathcal{X}_{\overline{s}}$ is 0-dimensional of lenght n.

We denote by $\underline{\mathrm{Pic}}_{\mathcal{X}/\mathcal{S}}$ (or simply $\underline{\mathrm{Pic}}_{\mathcal{X}}$ whenever $\mathcal{S}$ is clear from the context) the stack over $\mathcal{S}$ which is the fppf sheafification of the functor $(\mathrm{Sch}/\mathcal{S})^{\mathrm{op}} \longrightarrow (\mathrm{sets})$ that maps $S \longrightarrow \mathcal{S}$ to the set $\mathrm{Pic}(\mathcal{X} \times_{\mathcal{S}} S)$. If $f\colon \mathcal{X} \longrightarrow \mathcal{S}$ is a curve, we also denote by $\underline{\mathrm{Pic}}^d_{\mathcal{X}/\mathcal{S}}$ (or simply $\underline{\mathrm{Pic}}^d_{\mathcal{X}}$) the substack of $\underline{\mathrm{Pic}}_{\mathcal{X}/\mathcal{S}}$ of classes that are locally given by invertible sheaves having degree d on the geometric fibers of f. Given $n \in \mathbb{Z}$ we will denote by $[n]\colon \underline{\mathrm{Pic}}_{\mathcal{X}/\mathcal{S}} \longrightarrow \underline{\mathrm{Pic}}_{\mathcal{X}/\mathcal{S}}$ (or $[n]\colon \underline{\mathrm{Pic}}^d_{\mathcal{X}/\mathcal{S}} \longrightarrow \underline{\mathrm{Pic}}^{dn}_{\mathcal{X}/\mathcal{S}}$ if $\mathcal{X}$ is a curve over $\mathcal{S}$) the map induced by the multiplication by n.

The formation of $\mathrm{Hilb}^n_{\mathcal{X}/\mathcal{S}}$, $\underline{\mathrm{Pic}}_{\mathcal{X}/\mathcal{S}}$ and $\underline{\mathrm{Pic}}^d_{\mathcal{X}/\mathcal{S}}$ commute with arbitrary base change of the base $\mathcal{S}$. In particular, if $\mathcal{X} \longrightarrow \mathcal{S}$ is a curve, the stacks $\mathrm{Hilb}^d_{\mathcal{X}/\mathcal{S}}$ and $\underline{\mathrm{Pic}}^d_{\mathcal{X}/\mathcal{S}}$ for $d \in \mathbb{Z}$ are smooth over $\mathcal{S}$.

If $\mathcal{X}$ is an algebraic stack we will denote by $|\mathcal{X}|$ the topological space associated with $\mathcal{X}$.

Almost all the stacks $\mathcal{X}$ that we will introduce have a given map to $\mathcal{M}_g$, for some given g, and therefore have a given genus g curve over it, that, as remarked above, we will call the universal curve over (or of) $\mathcal{X}$. With abuse of notation we will usually denote them by the same symbol $\mathcal{C}$ for the total space and π for the structure map, that is $\pi\colon \mathcal{C} \longrightarrow \mathcal{X}$, but with the convention that this notation is fixed and remains coherent inside the statement of a lemma, proposition, theorem ... and its proof. The use of different symbols for such curves seemed to us not practical, while the use of subscripts would have encumbered the notation too much. Moreover this notation is supported by the idea that genus g curves can be seen as restriction of the universal curve over $\mathcal{M}_g$. Indeed, if $\pi\colon \mathcal{C} \longrightarrow \mathcal{M}_g$ is the universal curve and $q\colon \mathcal{X} \longrightarrow \mathcal{M}_g$ is a map, $\mathcal{C}$ can be seen as the functor $F\colon (\mathrm{Sch}/\mathcal{M}_g)^{\mathrm{op}} \longrightarrow (\mathrm{sets})$ which maps a genus g curve C over a scheme S to the set of sections $C(S)$, while q corresponds to a map $\mathrm{Sch}/\mathcal{X} \longrightarrow \mathrm{Sch}/\mathcal{M}_1$. The universal curve $\mathcal{C} \times_{\mathcal{M}_g} \mathcal{X} \longrightarrow \mathcal{X}$ of $\mathcal{X}$ then corresponds to the restriction of the functor F along the map $\mathrm{Sch}/\mathcal{X} \longrightarrow \mathrm{Sch}/\mathcal{M}_1$.

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1. Preliminaries

In this section we collect some general results and remarks that will be useful in the next sections. These results are well known, but for some of them we decided to include a proof for completeness and lack of exhaustive references. In this section we consider $\mathrm{Spec}\,\mathbb{Z}$ as the base scheme.

1.1. **Proposition.** *Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a locally finitely presented map of algebraic stacks and $\alpha\colon \mathcal{F} \longrightarrow \mathcal{H}$ be a map of finitely presented quasi-coherent sheaves on $\mathcal{X}$. If $\mathcal{H}$ is flat over $\mathcal{Y}$ then α is injective on*

the geometric fibers of f if and only if α is injective and $\operatorname{Coker}\alpha$ *is flat over $\mathcal{Y}$. In this case α remains injective after any base change $\mathcal{Y}' \longrightarrow \mathcal{Y}$ from an algebraic stack.*

Proof. It is easy to see that taking atlases of $\mathcal{Y}$ and $\mathcal{X}$ we can reduce the problem to the case of schemes, where it follows from [Gro66, Proposition 11.3.7]. □

1.2. REMARK. Let $\mathcal{Y}$ be an algebraic stack and G be a sheaf of groups over $\mathcal{Y}$. Then we have a natural isomorphism

$$\operatorname{Pic} \mathrm{B}_{\mathcal{Y}} G \simeq \operatorname{Pic} \mathcal{Y} \oplus \operatorname{Hom}(G, \mathbb{G}_m)$$

Indeed, by descent, an invertible sheaf over $\mathrm{B}_{\mathcal{Y}} G$ is given by a pair $(\mathcal{L}, \rho)$ where $\mathcal{L}$ is an invertible sheaf over $\mathcal{Y}$ and $\rho \in \operatorname{Hom}(G, \mathbb{G}_m)$, which defines an action of G on $\mathcal{L}$ given by $\rho\colon G \longrightarrow \mathbb{G}_m \simeq \underline{\operatorname{Aut}}(\mathcal{L})$.

In some proofs we will use dimension counting for algebraic stacks. We recall here some properties which are well known for schemes. We refer to [MBL99, Chapter 11] for definitions and basics about dimension theory for stacks.

1.3. REMARK. Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a locally of finite type map of algebraic stacks. If $\xi \in |\mathcal{X}|$, $\eta\colon \operatorname{Spec} k \longrightarrow \mathcal{Y}$, where k is a field, maps to $f(\xi)$ and $x \in |\mathcal{X} \times_{\mathcal{Y}} k|$ maps to ξ via the projection then the number

$$\dim_{\xi} f = \dim_x(\mathcal{X} \times_{\mathcal{Y}} k) \in \mathbb{Z}$$

does not depend on η and x. Indeed by standard arguments about fiber products one can reduce to the case $\mathcal{Y} = \operatorname{Spec} k$ and show that if L/k is a field extension and $\xi' \in |\mathcal{X} \times_k L|$ maps to $\xi \in |\mathcal{X}|$ then $\dim_{\xi'}(\mathcal{X} \times_k L) = \dim_{\xi} \mathcal{X}$. Using the definition of dimension for stacks one can assume that $\mathcal{X}$ is a scheme. In this case the result is standard (see for instance [SP014, Tag 02FW]).

1.4. **Definition.** Given $n \in \mathbb{Z}$, a locally of finite type map $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ of algebraic stacks has (pure) *relative dimension* n if all (the irreducible components of all) the fibers have dimension n.

1.5. REMARK. If $\mathcal{X}$ is a quasi-compact algebraic stack there exists $n \in \mathbb{N}$ and an atlas $X \longrightarrow \mathcal{X}$ of pure relative dimension n, where X is a quasi-compact scheme. Indeed if $P\colon X' \longrightarrow \mathcal{X}$ is an atlas from a quasi-compact scheme X', by [MBL99, Proposition 11.10] we have a decomposition $X' = \bigsqcup_{r=0}^{n} X_r$ such that, if $x \in X_r$, $r = \dim_x P(= \dim_x P_{|X_r})$. Taking into account [MBL99, Corollary 11.11] the map $Q\colon X = \bigsqcup_{r=0}^{n} \mathbb{A}^{n-r}_{X_r} \longrightarrow \mathcal{X}$ satisfies $\dim_x Q = n$ for all $x \in X$, i.e. it has pure relative dimension n.

1.6. **Proposition.** *Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a flat and locally of finite type map of locally noetherian algebraic stacks. Then*

$$\dim_{\xi} \mathcal{X} = \dim_{\xi} f + \dim_{f(\xi)} \mathcal{Y} \text{ for all } \xi \in |\mathcal{X}| \tag{2}$$

In particular if f has relative dimension $r \in \mathbb{Z}$ then

$$\dim \mathcal{X} = \dim \mathcal{Y} + r$$

Moreover if $\mathcal{Y}$ is locally of finite type over a field or $\mathbb{Z}$, f has pure relative dimension $r \in \mathbb{Z}$ and $\mathcal{Y}'$ is an irreducible component of $\mathcal{Y}$ then all irreducible components of $f^{-1}(\mathcal{Y}')$ dominate $\mathcal{Y}'$ and have dimension $\dim \mathcal{Y}' + r$.

Proof. We first prove (2) when $\mathcal{X}$ and $\mathcal{Y}$ are schemes. By [Gro66, Corollary 14.2.6] and since fibers have the subspace topology one can assume f surjective, of relative dimension $n \in \mathbb{N}$ and translate (2) in $\dim \mathcal{X} = \dim \mathcal{Y} + n$. By [Gro66, Theorem 14.2.1] we have $\dim(\mathcal{O}_{\mathcal{X},x}) = \dim(\mathcal{O}_{f^{-1}(f(x)),x}) + \dim(\mathcal{O}_{\mathcal{Y},f(x)})$ for all $x \in \mathcal{X}$. Since for all $y \in \mathcal{Y}$ we have $\max_{x \in f^{-1}(y)}\{\dim(\mathcal{O}_{f^{-1}(y),x})\} = n$ we get the desired expression.

When f is a smooth atlas (2) follows from definition of $\dim_{f(\xi)} \mathcal{Y}$. We show that

$$(2) \text{ for schemes} \implies (2) \text{ for algebraic spaces} \implies (2) \text{ for stacks}$$

Both implications follows from the same proof. One considers smooth atlases $X \longrightarrow \mathcal{X}$ and $Y \longrightarrow \mathcal{Y}$, choose a point $y \in Y$ over $f(\xi)$ and $x \in (X \times_{\mathcal{Y}} k(y))$ mapping to ξ. Writing a diagram of all possible fiber products the proof now consists in applying (2) several times on various projections of this diagram.

If f has relative dimension r the expression $\dim \mathcal{X} = \dim \mathcal{Y} + r$ follows from (2) and the fact that for all $\eta \in |\mathcal{Y}|$ we have $\max_{\xi \in f^{-1}(\eta)}\{\dim_\xi f\} = r$. We now consider the last claim in the statement. We can assume $\mathcal{Y}$ irreducible. Let $\mathcal{X}'$ be an irreducible component of $\mathcal{X}$ with generic point ξ. Notice that $\mathcal{X}'$ contains an open substack of $\mathcal{X}$. In particular $\dim_\xi \mathcal{X} = \dim_\xi \mathcal{X}'$. Moreover since f is open $f(\xi) = \eta$ is the generic point of $\mathcal{Y}$. By (2) we have $\dim_\xi \mathcal{X}' = \dim_\eta \mathcal{Y} + r$. Thus it suffices to show that if $\mathcal{Z}$ is an irreducible stack locally of finite type over a field or $\mathbb{Z}$ and ζ is its generic point then $\dim_\zeta \mathcal{Z} = \dim \mathcal{Z}$. First we can assume $\mathcal{Z}$ quasi-compact by taking an open substack of $\mathcal{Z}$ of the same dimension. Then by 1.5 there exists an atlas $P\colon Z \longrightarrow \mathcal{Z}$ of pure relative dimension r, so that $\dim Z = \dim \mathcal{Z} + r$. On the other hand we have seen that if z is a generic point of Z then $\dim_z Z = \dim_\zeta \mathcal{Z} + r$. This tells us that $\dim_z Z$ does not depend on the choice of the generic point and therefore, by [Gro66, Corollay 10.6.4, Example 10.7.1], $\dim_z Z = \dim \overline{\{z\}} = \dim Z$, which implies $\dim \mathcal{Z} = \dim_\zeta \mathcal{Z}$. □

1.7. **Corollary.** *Let $\mathcal{X}$ be an irreducible stack of finite type over a field and $\mathcal{Z}$ be the zero locus of a section of an invertible sheaf on $\mathcal{X}$. If $\emptyset \neq \mathcal{Z} \subsetneq \mathcal{X}$ then all irreducible components of $\mathcal{Z}$ have dimension $\dim \mathcal{X} - 1$.*

Proof. Let $P\colon X \longrightarrow \mathcal{X}$ be an atlas of pure relative dimension r (see 1.5), $\mathcal{Z}'$, Z' and X' be irreducible components of $\mathcal{Z}$, $P^{-1}(\mathcal{Z}')$ and X such that $Z' \subseteq X'$. Notice that $Z' \subsetneq X'$ because otherwise $P(Z')$ contains the generic point of $\mathcal{X}$. Since Z' is an irreducible component of a section of an invertible sheaf on X' we have $\dim Z' = \dim X' - 1$. On the other hand since P has pure relative dimension r we have $\dim Z' = \dim \mathcal{Z}' + r$ and $\dim X' = \dim \mathcal{X} + r$. □

1.8. **Corollary.** *Let $f\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a map of stacks locally of finite type over a field and assume that $\mathcal{X}$ is a Deligne-Mumford stack. Then $\dim \overline{f(\mathcal{X})} \leq \dim \mathcal{X}$, where $\overline{f(\mathcal{X})}$ is the reduced closed substack of $\mathcal{Y}$ whose topological space is $\overline{f(|\mathcal{X}|)}$.*

Proof. When $\mathcal{X}$ and $\mathcal{Y}$ are schemes the result is standard. We show how to reduce to this case. We can assume that f is dominant so that $\overline{f(\mathcal{X})} = \mathcal{Y}$. By taking an atlas of $\mathcal{Y}$ of pure relative dimension (see 1.5) we can assume that $\mathcal{Y}$ is a scheme. Moreover we can replace $\mathcal{X}$ by a scheme because étale atlases do not change dimension. □

1.9. **Proposition.** *Let $\mathcal{X}$ be a smooth and integral algebraic stack over a field.*

- *If $h\colon \mathcal{V} \longrightarrow \mathcal{X}$ is a vector bundle of finite rank then $h^*\colon \operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{V}$ is an isomorphism.*
- *If $\mathcal{Z}$ is a closed substack of $\mathcal{X}$ of codimension greater than 2 then the restriction map $\operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}(\mathcal{X} - \mathcal{Z})$ is an isomorphism.*
- *Given $\mathcal{L}_1, s_1, \ldots, \mathcal{L}_r, s_r$, where $\mathcal{L}_i$ is an invertible sheaf on $\mathcal{X}$ with a non zero global section s_i whose zero locus $Z(s_i)$ is integral, then the restriction map induces an isomorphism*

$$\operatorname{Pic}\mathcal{X}/\langle \mathcal{L}_1, \ldots, \mathcal{L}_r\rangle \simeq \operatorname{Pic}(\mathcal{X} - (Z(s_1) \cup \cdots \cup Z(S_r)))$$

Proof. Let $\sigma\colon \mathcal{X} \longrightarrow \mathcal{V}$ be the zero section. We must prove that $\operatorname{Pic}\mathcal{V} \xrightarrow{\sigma^*} \operatorname{Pic}\mathcal{X}$ is injective. Let $\mathcal{Q}$ be an invertible sheaf on $\mathcal{V}$ in the kernel and define the sheaf $F_\mathcal{Q}$ on the small smooth-étale site of $\mathcal{X}$ by

$$F_\mathcal{Q}(U) = \operatorname{Iso}_{h^{-1}(U)}(\mathcal{Q}_{|h^{-1}(U)}, \mathcal{O}_{h^{-1}(U)})$$

The map σ induces a map $F_\mathcal{Q} \longrightarrow \underline{\operatorname{Iso}}_\mathcal{X}(\sigma^*\mathcal{Q}, \mathcal{O}_\mathcal{X})$ and it suffices to prove that it is an isomorphism. This is a local question, so that we can assume $\mathcal{X} = \operatorname{Spec} D$ affine and $\mathcal{V}$ trivial. By standard intersection theory for schemes $\mathcal{Q}$ is trivial and therefore the previous map on the global sections is just $(D[x_1, \ldots, x_n])^* \simeq D^*$.

Let $\mathcal{U}$ be an open substack of $\mathcal{X}$. If $\mathcal{Q}$ is an invertible sheaf on $\mathcal{U}$ then by [MBL99, Corollary 15.5] there exists a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ such that $\mathcal{F}_{|\mathcal{U}} \simeq \mathcal{Q}$. Then the sheaf $\mathcal{L} = \mathcal{F}^{\vee\vee}$ is a reflexive sheaf of rank 1 and thus invertible and $\mathcal{L}_{|\mathcal{U}} \simeq \mathcal{Q}$. This shows that $\operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{U}$ is surjective. We now use the description of divisors given in [AV04, Proof of Lemma 5.2]. Let $\mathcal{L}$ be an invertible sheaf on $\mathcal{X}$ such that $\mathcal{L}_{|\mathcal{U}} \simeq \mathcal{O}_\mathcal{U}$. It follows that there is a divisor $\mathcal{D}$ on $\mathcal{X}$ such that $\mathcal{L} \simeq \mathcal{O}_\mathcal{X}(\mathcal{D})$ and the support of $\mathcal{D}$ is in $\mathcal{X} - \mathcal{U}$. In particular if $\mathcal{X} - \mathcal{U}$ has codimension greater than 2 then $\mathcal{D} = 0$. For the last point the sheaves $\mathcal{L}_i \simeq \mathcal{O}_\mathcal{X}(Z(s_i))$ restrict to $\mathcal{O}_\mathcal{U}$ on $\mathcal{U}$ and $\mathcal{L} \simeq \mathcal{L}_1^{m_1} \otimes \mathcal{L}_r^{m_r}$ where m_i is the multiplicity of $\mathcal{D}$ in $Z(s_i)$ (computed on an atlas). □

1.10. **Lemma.** *Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ be a genus g curve over an algebraic stack and $\mathcal{F}$ be a finitely presented quasi-coherent sheaf on $\mathcal{C}$, flat over $\mathcal{S}$. Then $\mathrm{R}^j \pi_* \mathcal{F}$ is locally free and satisfies base change for all $j \in \mathbb{N}$ in the following cases.*

(1) $\mathcal{F}$ is an invertible sheaf on $\mathcal{C}$ of degree 0 such that $[\mathcal{F}] = 0$ in $\underline{\mathrm{Pic}}^0_{\mathcal{C}/\mathcal{S}}$. In this case $\pi_*\mathcal{F}$ is an invertible sheaf and the canonical map

$$\pi^*\pi_*\mathcal{F} \longrightarrow \mathcal{F}$$

is an isomorphism.

(2) $\mathcal{F} = \omega_\pi$. The sheaf $\pi_*\omega_\pi$ has rank g and $\mathrm{R}^1\,\pi_*\omega_\pi \simeq \mathcal{O}_{\mathcal{S}}$. Moreover if $g = 1$ the map $\pi^*\pi_*\omega_\pi \simeq \omega_\pi$ is an isomorphism.

(3) $\mathcal{F}$ is an invertible sheaf on $\mathcal{C}$ of degree $d > 2g-2$ or $d < 0$. In this case $\operatorname{rk} \pi_*\mathcal{F} = \max\{d+1-g, 0\}$ and $\operatorname{rk} \mathrm{R}^1\,\pi_*\mathcal{F} = \max\{-d-1+g, 0\}$.

(4) $\mathcal{F}$ is supported on a closed substack of $\mathcal{C}$ which is quasi-finite over $\mathcal{S}$. In this case $\mathrm{R}^1\,\pi_*\mathcal{F} = 0$.

In all of the above cases but the last one we have an isomorphism

$$\mathrm{R}^1\,\pi_*\mathcal{F} \simeq \pi_*(\mathcal{F}^\vee \otimes \omega_\pi)^\vee$$

Proof. By [Kle80, II, Definition 10 and Theorem 21] there is a canonical map

$$\pi_* \underline{\mathrm{Hom}}(\mathcal{F}, \omega_\pi) \longrightarrow (\mathrm{R}^1\,\pi_*\mathcal{F})^\vee$$

which is an isomorphism if $\mathrm{R}^1\,\pi_*\mathcal{F}$ satisfies base change. In this case if $\mathcal{F}$ and $\mathrm{R}^1\,\pi_*\mathcal{F}$ are locally free we get the last formula in the statement by dualizing the above isomorphism.

All the other claims follow by standard semicontinuity theorems and Riemman-Roch. □

1.11. **Lemma.** *Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ be a genus g curve over an algebraic stack and $\mathcal{Q}$ be a degree d invertible sheaf on $\mathcal{C}$ with a section $s \in \mathcal{Q}$ which is non zero on the geometric fibers. Then the zero locus $\mathcal{Z}$ of s in $\mathcal{C}$ is a degree d cover of $\mathcal{S}$. When $d = 1$ this defines a section $\tau\colon \mathcal{S} \longrightarrow \mathcal{C}$ with an isomorphism $\mathcal{O}_{\mathcal{C}}(\tau) \simeq \mathcal{Q}$ sending 1 to s. If in addition $g = 1$ then the map $\mathcal{O}_{\mathcal{S}} \xrightarrow{s} \pi_*\mathcal{Q}$ is an isomorphism.*

Proof. By 1.1 the sequence

$$0 \longrightarrow \mathcal{Q}^{-1} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{Z}} \longrightarrow 0$$

is universally exact over $\mathcal{S}$ and $\mathcal{Z}$ is flat over $\mathcal{S}$. Moreover $\mathcal{Z} \longrightarrow \mathcal{S}$ is proper, finitely presented and, by looking at the geometric fibers, quasi-finite. By [Gro66, Theorem 8.11.1] we can conclude that $\mathcal{Z} \longrightarrow \mathcal{S}$ is a cover. By Riemman-Roch it has degree d.

Assume $d = 1$. The claim about the section τ follows from standard arguments. The last claim follows from the fact that $\pi_*\mathcal{Q}$ is invertible and satisfies base change by 1.10 and $s \in \pi_*\mathcal{Q}$ is nowhere vanishing by hypothesis. □

1.12. Remark. Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ be a genus g curve over an algebraic stack and $n \in \mathbb{Z}$. Then $[n]\colon \underline{\mathrm{Pic}}^r_{\mathcal{C}/\mathcal{S}} \longrightarrow \underline{\mathrm{Pic}}^{rn}_{\mathcal{C}/\mathcal{S}}$ is a cover of degree n^{2g} and it is étale if $n \in \mathcal{O}^*_{\mathcal{S}}$. Indeed, since the problem is local on $\mathcal{S}$, one can assume that $\mathcal{S}$ is a noetherian scheme and that $\mathcal{C} \longrightarrow \mathcal{S}$ has a section. This allows to reduce the problem to the case $r = 0$. Since $\underline{\mathrm{Pic}}^0_{\mathcal{C}/\mathcal{S}} \longrightarrow \mathcal{S}$ is flat and proper of relative dimension g, by the local flatness criterion [Gro66, Theorem 11.3.10] we can assume that $\mathcal{S}$ is the spectrum of an algebraically closed field. In this case the result follows from [Mil08, Proposition 7.1 and Theorem 7.2].

In particular all invertible sheaves on $\mathcal{C}$ of degree divisible by n are fppf locally (on $\mathcal{S}$) a n-th power of an invertible sheaf on $\mathcal{C}$. Moreover $(\underline{\mathrm{Pic}}^0_{\mathcal{C}/\mathcal{S}})[n]$ is a finite, flat and finitely presented group scheme over $\mathcal{S}$ of degree n^{2g} and it is étale if $n \in \mathcal{O}^*_{\mathcal{S}}$.

2. The universal Jacobian of degree d over $\mathcal{M}_g$ and its Picard group.

In this section we assume to work over a field of characteristic $p \geq 0$ and we fix a non negative integer g (the genus) and an integer d (the degree).

2.1. **Definition.** We denote by $\mathcal{J}ac_{d,g}$ the stack of pairs $(C, \mathcal{Q})$ where C is a curve of genus g and $\mathcal{Q}$ is an invertible sheaf over C of degree d. The stack $\mathcal{J}ac_{d,g}$ is called the *universal Jacobian* stack of degree d over $\mathcal{M}_g$.

The aim of this section is to describe the Picard group of $\mathcal{J}ac_{d,g}$. When $g \geq 2$ this has already been done in [MV14]. We will deal with the remaining cases, that is genus zero and one.

2.2. REMARK. The forgetful functor $\mathcal{J}ac_{d,g} \longrightarrow \mathcal{M}_g$ is the composition of a $\mathbb{G}_m$-gerbe $\mathcal{J}ac_{d,g} \longrightarrow \underline{\mathrm{Pic}}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ and the representable and smooth functor $\underline{\mathrm{Pic}}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g} \longrightarrow \mathcal{M}_g$. In particular $\mathcal{J}ac_{d,g}$ is a smooth and integral algebraic stack.

2.3. **Definition.** Let $\mathcal{C} \longrightarrow \mathcal{J}ac_{d,g}$ be the universal curve over $\mathcal{J}ac_{d,g}$. By construction, there exists an invertible sheaf $\mathcal{L}$ over $\mathcal{C}$ such that

$$\begin{array}{ccc} C & \xrightarrow{q} & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \xrightarrow{(C,\mathcal{Q})} & \mathcal{J}ac_{d,g} \end{array} \qquad\qquad q^*\mathcal{L} \simeq \mathcal{Q}$$

for all schemes T. We call the sheaf $\mathcal{L}$ the *universal invertible sheaf* over $\mathcal{C}$. Given a stack $\mathcal{Y}$ over $\mathcal{J}ac_{d,g}$ the universal invertible sheaf over the universal curve $\mathcal{C} \times_{\mathcal{J}ac_{d,g}} \mathcal{Y}$ of $\mathcal{Y}$ is the pull-back of $\mathcal{L}$ via the map $\mathcal{C} \times_{\mathcal{J}ac_{d,g}} \mathcal{Y} \longrightarrow \mathcal{C}$.

We now describe the result in [MV14] about $\mathrm{Pic}\, \mathcal{J}ac_{d,g}$ when $g \geq 2$.

2.4. REMARK. Let $\pi\colon \mathcal{C} \longrightarrow \mathcal{S}$ be a genus g curve. Given an invertible sheaf $\mathcal{T}$ on $\mathcal{C}$ one can define an invertible sheaf $d_\pi(\mathcal{T})$ on $\mathcal{S}$, called the determinant of cohomology of $\mathcal{T}$. When $\pi_*\mathcal{T}$ and $\mathrm{R}^1\pi_*\mathcal{T}$ are locally free one can simply set $d_\pi(\mathcal{T}) \simeq \det \pi_*\mathcal{T} \otimes (\det \mathrm{R}^1\pi_*\mathcal{T})^{-1}$. We refer to [MV14] for the general definition. In this paper we just use the fact that the formation of $d_\pi(\mathcal{T})$ commutes with arbitrary base changes.

Notice that from 1.10 it follows that $d_\pi(\omega_\pi) \simeq \det \pi_*\omega_\pi$ and that, if $\mathcal{T}$ is an invertible sheaf on $\mathcal{C}$ of positive degree, then $d_\pi(\mathcal{T} \otimes \omega_\pi) \simeq \det \pi_*(\mathcal{T} \otimes \omega_\pi)$.

2.5. **Theorem.** [MV14] *Assume that the ground field is algebraically closed of characteristic* 0 *and that* $g \geq 2$ *and* $d > 0$. *Let* $\pi\colon \mathcal{C} \longrightarrow \mathcal{J}ac_{d,g}$ *be the universal curve and* $\mathcal{L}$ *be the universal invertible sheaf over* $\mathcal{C}$. *Then* $\mathrm{Pic}\, \mathcal{J}ac_{d,g}$ *is freely generated by* $\det \pi_*\omega_\pi$, $d_\pi(\mathcal{L})$ *and* $\det \pi_*(\mathcal{L} \otimes \omega_\pi)$, *except for* $g = 2$, *in which case there is a single relation given by* $(\det \pi_*\omega_\pi)^{10}$. *Moreover for all* $n, k \geq 1$ *we have an isomorphism*

$$\det \pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq (\det \pi_*\omega_\pi)^{6k^2-6k-n^2+1} \otimes d_\pi(\mathcal{L})^{-nk+n(n+1)/2} \otimes (\det \pi_*(\mathcal{L} \otimes \omega_\pi))^{nk+n(n-1)/2}$$

Proof. Taking into account 2.4, everything follows from [MV14, Theorem A and 5.2, Notation 1.5, Remark 5.3]. □

2.1. **Genus zero case.** In this subsection we consider $g = 0$, while d is any integer. We will prove the following:

2.6. **Proposition.** *Let* $\pi\colon \mathcal{P} \longrightarrow \mathcal{J}ac_{d,0}$ *be the universal curve and* $\mathcal{L}$ *be the universal invertible sheaf over* $\mathcal{P}$. *If* d *is even then* $\mathrm{Pic}\, \mathcal{J}ac_{d,0}$ *is freely generated by* $\mathcal{L}_0 = \pi_*(\mathcal{L} \otimes \omega_\pi^{d/2})$ *and we have an isomorphism*

$$\det \pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq \mathcal{L}_0^{n\max\{nd-2k+1,0\}} \text{ for all } n, k \in \mathbb{Z}$$

If d *is odd then* $\mathrm{Pic}\, \mathcal{J}ac_{d,0}$ *is freely generated by* $\mathcal{L}_0 = \det \pi_*(\mathcal{L} \otimes \omega_\pi^{(d-1)/2})$ *and we have an isomorphism*

$$\det \pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq \mathcal{L}_0^{n\max\{nd-2k+1,0\}/2} \text{ for all } n, k \in \mathbb{Z}$$

We will need the following lemma, whose proof is standard and thus omitted.

2.7. **Lemma.** *The group* $(\mathrm{GL}_2)^\vee$ *is freely generated by* $\det\colon \mathrm{GL}_2 \longrightarrow \mathbb{G}_m$, *while* $(\mathrm{PGL}_2)^\vee = 0$

2.8. REMARK. Let $\pi\colon \mathcal{P} \longrightarrow \mathcal{S}$ be a curve of genus 0 over an algebraic stack and $\mathcal{Q}$ be an invertible sheaf on $\mathcal{P}$. If $\mathcal{Q}$ has degree 0, by 1.10 it follows that $\pi_*\mathcal{Q}$ is an invertible sheaf, it satisfies base change and the map $\pi^*\pi_*\mathcal{Q} \longrightarrow \mathcal{Q}$ is an isomorphism because it is so on the geometric fibers.

If $\mathcal{Q}$ has degree 1, by 1.10 it follows that $\pi_*\mathcal{Q}$ is a rank 2 locally free sheaf, it satisfies base change and the map $\pi^*\pi_*\mathcal{Q} \longrightarrow \mathcal{Q}$ is surjective because it is so on the geometric fibers. In particular we obtain an isomorphism $\mathcal{P} \longrightarrow \mathbb{P}(\pi_*\mathcal{Q})$: the pullback of $\mathcal{O}_{\mathbb{P}(\pi_*\mathcal{Q})}(1)$ is $\mathcal{Q}$ and therefore we get the Euler sequence

$$0 \longrightarrow \omega_\pi \otimes \mathcal{Q} \longrightarrow \pi^*\pi_*\mathcal{Q} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Proof of 2.6. Since $\mathcal{M}_0 \simeq \mathrm{B}\,\mathrm{PGL}_2$, by 1.2 and 2.7 we obtain $\operatorname{Pic}\mathcal{M}_0 = 0$. In particular $\det \pi_*(\omega_\pi^k)$ is trivial for all $k \in \mathbb{Z}$ in $\operatorname{Pic}\mathcal{J}ac_{d,0}$.

Assume d even. Tensoring by $\omega_\pi^{d/2}$ yields an isomorphism $\mathcal{J}ac_{d,0} \longrightarrow \mathcal{J}ac_{0,0}$ over $\mathcal{M}_0$. By 2.8 we see that the functors $\mathcal{J}ac_{0,0} \longrightarrow \mathrm{B}_{\mathcal{M}_0}\mathbb{G}_m$ mapping $(P \xrightarrow{q} S, \mathcal{Q})$ to $(P, q_*\mathcal{Q})$ and $\mathrm{B}_{\mathcal{M}_0}\mathbb{G}_m \longrightarrow \mathcal{J}ac_{0,0}$ mapping $(P \xrightarrow{q} S, \mathcal{T})$ to $(P, q^*\mathcal{T})$ are quasi-inverses of each other. Moreover by 1.2 we have $\operatorname{Pic}\mathrm{B}_{\mathcal{M}_0}\mathbb{G}_m \simeq \operatorname{Pic}\mathcal{M}_0 \oplus \mathbb{Z} \simeq \mathbb{Z}$, generated by the invertible sheaf given by the rule $(P, \mathcal{T}) \longmapsto \mathcal{T}$. The pullback of this sheaf via $\mathcal{J}ac_{d,0} \simeq \mathcal{J}ac_{0,0} \simeq \mathrm{B}_{\mathcal{M}_0}\mathbb{G}_m$ is isomorphic to $\mathcal{L}_0$, which therefore freely generates $\operatorname{Pic}\mathcal{J}ac_{d,0}$. By 2.8 we have $\mathcal{L} \simeq \pi^*\mathcal{L}_0 \otimes \omega_\pi^{-d/2}$ and, using projection formula,

$$\det \pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq \det(\mathcal{L}_0^n \otimes \pi_*(\omega_\pi^{k-nd/2})) \simeq \mathcal{L}_0^{n \operatorname{rk}\pi_*(\omega_\pi^{k-nd/2})}$$

Finally by 1.10 we see that $\operatorname{rk}\pi_*(\omega_\pi^{k-nd/2}) = \max\{nd - 2k + 1, 0\}$.

Assume now that d is odd. Tensoring by $\omega_\pi^{(d-1)/2}$ we get an isomorphism $\mathcal{J}ac_{d,0} \longrightarrow \mathcal{J}ac_{1,0}$. By 2.8 we see that the functors $\mathcal{J}ac_{1,0} \longrightarrow \mathrm{B}\,\mathrm{GL}_2$ mapping $(P \xrightarrow{q} S, \mathcal{Q})$ to $q_*\mathcal{Q}$ and $\mathrm{B}\,\mathrm{GL}_2 \longrightarrow \mathcal{J}ac_{1,0}$ mapping $\mathcal{E}$ to $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ are quasi-inverses of each other. Moreover by 1.2 we have $\operatorname{Pic}\mathrm{B}\,\mathrm{GL}_2 \simeq \mathbb{Z}$ generated by the invertible sheaf given by the rule $\mathcal{E} \longmapsto \det\mathcal{E}$. The pullback of this sheaf via $\mathcal{J}ac_{d,0} \simeq \mathcal{J}ac_{1,0} \simeq \mathrm{B}\,\mathrm{GL}_2$ is isomorphic to $\mathcal{L}_0$. Set $\mathcal{T} = \mathcal{L} \otimes \omega_\pi^{(d-1)/2}$, so that $\mathcal{L}_0 = \det\pi_*\mathcal{T}$. Applying the determinant to the Euler sequence in 2.8 associated with $\mathcal{T}$ we get an isomorphism

$$\omega_\pi \simeq \pi^*\mathcal{L}_0 \otimes \mathcal{T}^{-2}$$

Writing $\mathcal{L}^n \otimes \omega_\pi^k$ in terms of $\mathcal{T}$ and $\mathcal{L}_0$ and applying projection formula we obtain

$$\det \pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq \det(\mathcal{L}_0^{k-n(d-1)/2} \otimes \pi_*(\mathcal{T}^{nd-2k})) \simeq \mathcal{L}_0^{(k-n(d-1)/2)\operatorname{rk}\pi_*(\mathcal{T}^{nd-2k})} \otimes \det\pi_*(\mathcal{T}^{nd-2k})$$

By 1.10 we have that $\operatorname{rk}\pi_*(\mathcal{T}^{nd-2k}) = \max\{nd - 2k + 1, 0\}$. Thus it suffices to prove the expression $\det\pi_*(\mathcal{T}^q) \simeq \mathcal{L}_0^{q(q+1)/2}$ for $q \geq 0$. Considering the Euler sequence in 2.8 associated with $\mathcal{T}$, replacing ω_π by $\pi^*\mathcal{L}_0 \otimes \mathcal{T}^{-2}$ and tensoring by $\mathcal{T}^q$ we get an exact sequence

$$0 \longrightarrow \pi^*\mathcal{L}_0 \otimes \mathcal{T}^{q-1} \longrightarrow \pi^*\pi_*\mathcal{T} \otimes \mathcal{T}^q \longrightarrow \mathcal{T}^{q+1} \longrightarrow 0$$

The pushforward π_* of the above sequence for $q \geq 0$ is exact because $\mathrm{R}^1\pi_*(\pi^*\mathcal{L}_0 \otimes \mathcal{T}^{q-1}) = 0$ thanks to 1.10. Thus applying π_*, the determinant, the projection formula and using that $\operatorname{rk}\pi_*(\mathcal{T}^r) = r + 1$ for $r \geq -1$ we get an isomorphism

$$\det\pi_*(\mathcal{T}^{q+1}) \simeq \mathcal{L}_0 \otimes (\det\pi_*(\mathcal{T}^q))^2 \otimes (\det\pi_*(\mathcal{T}^{q-1}))^{-1}$$

It is now easy to check by induction that $\det\pi_*(\mathcal{T}^q) \simeq \mathcal{L}_0^{q(q+1)/2}$. □

2.2. **Genus one case.** In this subsection we consider $g = 1$ and $d > 0$. We will prove the following Theorem.

2.9. **Theorem.** *Let $\pi\colon \mathcal{E} \longrightarrow \mathcal{J}ac_{d,1}$ be the universal curve over $\mathcal{J}ac_{d,1}$, $\mathcal{L}$ be the universal invertible sheaf over $\mathcal{E}$ and assume $p \nmid d$. Then* $\operatorname{Pic}\mathcal{J}ac_{d,1}$ *is generated by $\pi_*\omega_\pi$ and $\det\pi_*\mathcal{L}$ with the only relation $(\pi_*\omega_\pi)^{12}$. Moreover we have an isomorphism*

$$\det\pi_*(\mathcal{L}^n \otimes \omega_\pi^k) \simeq (\det\pi_*\mathcal{L})^{n^2} \otimes (\pi_*\omega_\pi)^{dnk+(n-1)(dn-2n-2)/2} \text{ for } n > 0, k \in \mathbb{Z}$$

The starting point is the well known Theorem of Mumford, later generalized by Fulton and Olsson. (See [Mum63] and [FO10])

2.10. **Theorem** (Mumford, Fulton, Olsson)**.** *The Picard group of $\mathcal{M}_{1,1}$ is cyclic of order 12 and it is generated by $\pi_*\omega_\pi$, where $\pi\colon \mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ is the universal curve.*

We will proceed by showing that $\mathcal{J}ac_{d,1}$ is isomorphic to $\mathrm{B}_{\mathcal{M}_{1,1}} G_d$, for a certain group scheme G_d over $\mathcal{M}_{1,1}$. In particular we will conclude that $\operatorname{Pic}\mathcal{J}ac_{d,1} \simeq \operatorname{Pic}\mathcal{M}_{1,1} \oplus \operatorname{Hom}(G_d, \mathbb{G}_m)$ and we will conclude the section by computing the group of characters $\operatorname{Hom}(G_d, \mathbb{G}_m)$.

2.11. **Lemma.** *Let $\pi\colon \mathcal{E} \longrightarrow \mathcal{S}$ be a genus one curve over an algebraic stack. Then the functor*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\Omega\quad} & \underline{\operatorname{Pic}}^1_{\mathcal{E}/\mathcal{S}} \\ (\delta\colon T \longrightarrow \mathcal{E}\times_{\mathcal{S}} T) & \longmapsto & [\mathcal{O}_{\mathcal{E}\times_{\mathcal{S}} T}(\delta)] \end{array}$$

is an isomorphism. If $\sigma\colon \mathcal{S} \longrightarrow \mathcal{E}$ is a section, then also the functor

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \underline{\operatorname{Pic}}^0_{\mathcal{E}/\mathcal{S}} \\ (\delta\colon T \longrightarrow E\times_{\mathcal{S}} T) & \longmapsto & [\mathcal{O}_{\mathcal{E}\times_{\mathcal{S}} T}(\delta - \sigma\times_{\mathcal{S}} T)] \end{array}$$

is an isomorphism that sends σ to $[\mathcal{O}_{\mathcal{E}}]$

Proof. The last part of the statement follows from the first one. We start showing that Ω is an fppf epimorphism. Let $\chi \in \underline{\operatorname{Pic}}^1_{\mathcal{E}}(T)$, where T is an $\mathcal{S}$-scheme. We can replace T by $\mathcal{S}$ and assume the existence of an invertible sheaf $\mathcal{L}$ of degree 1 over $\mathcal{E}$ such that $\chi = [\mathcal{L}]$. By 1.10 the sheaf $\mathcal{Q} = \pi_*\mathcal{L}$ is invertible and we can assume it is trivial. Since $\pi_*\mathcal{L}$ satisfies base change, the section s of $\mathcal{L}$ corresponding to $1 \in \pi_*\mathcal{L}$ is always non-zero on the geometric fibers of π. Thus, by 1.11, we get a section $\tau\colon \mathcal{S} \longrightarrow \mathcal{E}$ with an isomorphism $\mathcal{O}_{\mathcal{E}}(\tau) \simeq \mathcal{L}$, so that $\Omega(\tau) = \chi$.

We now prove that Ω is injective over an $\mathcal{S}$-scheme T. Again we can replace T by $\mathcal{S}$, so that, in particular, $\mathcal{S}$ is a scheme. Let $\delta, \delta' \in \mathcal{E}(\mathcal{S})$ such that $[\mathcal{O}_{\mathcal{E}}(\delta')] = [\mathcal{O}_{\mathcal{E}}(\delta)]$. By 1.10 we have an isomorphism

$$\mathcal{O}_{\mathcal{E}}(\delta) \xrightarrow{\alpha} \mathcal{O}_{\mathcal{E}}(\delta') \otimes \pi^*\mathcal{Q}$$

for some invertible sheaf $\mathcal{Q}$ over $\mathcal{S}$. Since $\pi_*\mathcal{O}_{\mathcal{E}}(\delta)$ and $\pi_*\mathcal{O}_{\mathcal{E}}(\delta')$ are freely generated by the respective sections 1 by 1.11, applying π_* we get an element $x \in \mathcal{Q}$ such that $\alpha(1) = 1 \otimes \pi^* x$ and that freely generates $\mathcal{Q}$. Thus we obtain an isomorphism $\mathcal{O}_{\mathcal{E}}(\delta) \longrightarrow \mathcal{O}_{\mathcal{E}}(\delta')$ mapping 1 to 1 and therefore that $\delta = \delta'$. □

2.12. **Proposition.** *Let $\mathcal{E} \xrightarrow{\pi} \mathcal{S}$ be a genus one curve over an algebraic stack with a section σ. Then we have isomorphisms*

$$\det \pi_*\mathcal{O}_{\mathcal{E}}(n\sigma) \simeq \sigma^*\mathcal{O}_{\mathcal{E}}(\sigma)^{n(n+1)/2-1} \text{ for } n \geq 1$$

and

$$\sigma^*\mathcal{O}_{\mathcal{E}}(\sigma) \simeq (\pi_*\omega_\pi)^\vee$$

Moreover if $\mathcal{L}$ is an invertible sheaf over $\mathcal{E}$ of degree greater than zero and τ is another section (possibly equal to σ) we have an isomorphism

$$\det \pi_*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau)) \simeq \det \pi_*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau-\sigma)) \otimes \sigma^*\mathcal{L} \otimes \sigma^*\mathcal{O}_{\mathcal{E}}(\tau)$$

Proof. In what follows we will use 1.10 and, in particular, the isomorphism $\omega_\pi \simeq \pi^*\pi_*\omega_\pi$ without further comments. Notice that the first isomorphism in the statement follows from the last one with $\tau = \sigma$, induction and the isomorphism $\pi_*\mathcal{O}_{\mathcal{E}}(\sigma) \simeq \mathcal{O}_{\mathcal{S}}$ (see 1.11).

Consider an invertible sheaf $\mathcal{L}$ on $\mathcal{E}$ of degree greater or equal than zero and a section τ of $\mathcal{E}$. Tensoring by $\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau)$ the exact sequence on $\mathcal{E}$

$$0 \longrightarrow \mathcal{O}_{\mathcal{E}}(-\sigma) \longrightarrow \mathcal{O}_{\mathcal{E}} \longrightarrow \sigma_*\mathcal{O}_{\mathcal{S}} \longrightarrow 0$$

and applying π_* we get an exact sequence

$$0 \longrightarrow \pi_*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau-\sigma)) \longrightarrow \pi_*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau)) \longrightarrow \sigma^*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau)) \longrightarrow \mathrm{R}^1\pi_*(\mathcal{L}\otimes\mathcal{O}_{\mathcal{E}}(\tau-\sigma)) \longrightarrow 0$$

If $\mathcal{L} = \mathcal{O}_{\mathcal{E}}$ and $\tau = \sigma$, we obtain a surjection $\sigma^*\mathcal{O}_{\mathcal{E}}(\sigma) \longrightarrow \mathrm{R}^1\pi_*\mathcal{O}_{\mathcal{E}} \simeq (\pi_*\omega_\pi)^\vee$ which is therefore an isomorphism. If $\mathcal{L}$ has degree strictly greater than 0, the last term in the sequence is zero and taking the determinant we get the last isomorphism. □

2.13. **Proposition.** *The functor*

$$\begin{aligned} \mathcal{M}_{1,1} &\longrightarrow \mathcal{J}ac_{d,1} \\ (E,\sigma) &\longmapsto (E, \mathcal{O}_E(d\sigma)) \end{aligned}$$

is an epimorphism in the fppf topology and it is a section of the functor

$$\begin{aligned} \mathcal{J}ac_{d,1} &\longrightarrow \mathcal{M}_{1,1} \\ (E,\mathcal{L}) &\longmapsto (\underline{\mathrm{Pic}}^0_{E/S}, [\mathcal{O}_E]) \end{aligned}$$

Proof. The second part of the statement follows from 2.11. For the first one let $(E,\mathcal{L}) \in \mathcal{J}ac_{d,1}$. By 1.12 fppf locally we can write $\mathcal{L} \simeq \mathcal{T}^d$ for some degree 1 invertible sheaf $\mathcal{T}$ on E. By 2.11 E has a section σ such that $[\mathcal{T}] = [\mathcal{O}_E(\sigma)]$, which means that $\mathcal{T}$ (resp. $\mathcal{L}$) and $\mathcal{O}_E(\sigma)$ (resp. $\mathcal{O}_E(d\sigma)$) are fppf locally isomorphic. □

2.14. **Lemma.** *Let $p\colon \mathcal{X} \longrightarrow \mathcal{Y}$ be a map of stacks over a scheme S with a section $s\colon \mathcal{Y} \longrightarrow \mathcal{X}$ which is an fppf epimorphism. Then $\mathcal{X} \simeq \mathrm{B}_{\mathcal{Y}} G$ where G is the sheaf of groups over $\mathcal{Y}$ defined by*

$$G(T \xrightarrow{\xi} \mathcal{Y}) = \mathrm{Ker}(\mathrm{Aut}_{\mathcal{X}}(s(\xi)) \longrightarrow \mathrm{Aut}_{\mathcal{Y}}(ps(\xi)))$$

Given $\eta\colon T \longrightarrow \mathcal{X}$ the associated G-torsor over T is the inverse image of the identity section of $\underline{\mathrm{Aut}}_T(p(\eta))$ along the map

$$\underline{\mathrm{Iso}}_T(\eta, sp(\eta)) \xrightarrow{p} \underline{\mathrm{Iso}}_T(p(\eta), ps(p(\eta))) \simeq \underline{\mathrm{Aut}}_T(p(\eta))$$

Proof. Given $\xi\colon T \longrightarrow \mathcal{Y}$ denote by $\mathcal{X}_\xi$, G_ξ and $s_\xi\colon T \longrightarrow \mathcal{X}_\xi$ the base change of $\mathcal{X}$, G and $s\colon \mathcal{Y} \longrightarrow \mathcal{X}$ along ξ respectively. Given $\eta\colon T \longrightarrow \mathcal{X}$ denote by $F(\eta)$ the sheaf over Sch/T defined in the last part of the statement. A direct check shows that $F(\eta) \simeq \underline{\mathrm{Iso}}_{\mathcal{X}_{p(\eta)}}(\eta, s_{p(\eta)})$ and $G_\xi \simeq \underline{\mathrm{Aut}}_{\mathcal{X}_\xi}(s_\xi)$. Since η and $s_{p(\eta)}$ are fppf locally isomorphic, it follows that $F(\eta)$ is a $G_{p(\eta)}$-torsor. Thus we get a functor $\mathcal{X} \longrightarrow \mathrm{B}_{\mathcal{Y}} G$. Since the base change of this functor along any morphism $T \longrightarrow \mathcal{Y}$ is an equivalence by standard results, we obtain that it is globally an equivalence. □

2.15. **Definition.** We define the group functor G_d over $\mathcal{M}_{1,1}$ as the group G obtained as in 2.14 with respects to the maps defined in 2.13.

Let us describe the group G_d more concretely.

2.16. **Proposition.** *An element of $G_d(S \xrightarrow{(E,\sigma)} \mathcal{M}_{1,1})$ is a pair (f,λ) where $f\colon E \longrightarrow E$ is a translation by an element in $E[d]$ and $\lambda\colon \mathcal{O}_E(\sigma)^d \longrightarrow \mathcal{O}_E(f(\sigma))^d$ is an isomorphism. Moreover we have an exact sequence*

$$\begin{array}{ccccccccc} & & & & (f,\lambda) & \longmapsto & f(\sigma) & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G_d & \longrightarrow & \mathcal{E}[d] & \longrightarrow & 0 \\ & & \mu & \longmapsto & (\mathrm{id},\mu) & & & & \end{array}$$

in the Zariski topology of $\mathrm{Sch}/\mathcal{M}_{1,1}$, where $\mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ is the universal curve. In particular G_d is affine and of finite type over $\mathcal{M}_{1,1}$ and, if $p \nmid d$, it is smooth over $\mathcal{M}_{1,1}$.

Proof. By definition, an element of $G_d(S \xrightarrow{(E,\sigma)} \mathcal{M}_{1,1})$ is a pair (f,λ) where $f\colon E \longrightarrow E$ is an isomorphism such that $f_*\colon \underline{\mathrm{Pic}}^0_{E/S} \longrightarrow \underline{\mathrm{Pic}}^0_{E/S}$ is the identity and $\lambda\colon \mathcal{O}_E(\sigma)^d \longrightarrow \mathcal{O}_E(f(\sigma))^d$ is an isomorphism. Taking into account 2.11, the condition $f_* = \mathrm{id}$ means that, for all $\delta \in \mathcal{E}(T)$ and S-scheme T we have

$$[\mathcal{O}_{E_T}(f(\delta) - f(\sigma))] = [\mathcal{O}_{E_T}(\delta - \sigma)]$$

which implies that $f\colon E \longrightarrow E$ is a translation. The existence of λ also implies that $f(\sigma) \in E[d]$. In particular the sequence in the statement is well defined and, since $\pi_*\mathbb{G}_{m,E} \simeq \mathbb{G}_{m,S}$, it is exact in the first two terms.

It remains to prove that $G_d \longrightarrow \mathcal{E}[d]$ is a Zariski epimorphism. This will also imply that G_d is locally a product of $\mathbb{G}_m$ and $\mathcal{E}[d]$, proving the last sentence in the statement. Let $\delta \in \mathcal{E}[d]$ and t be the translation by δ, so that $t(\sigma) = \delta$. By 1.10, the equality $[\mathcal{O}_E(\delta - \sigma)^d] = 0$ in $\underline{\mathrm{Pic}}^0_{E/S}$ implies that $\mathcal{O}_E(\delta - \sigma)^d \simeq \pi^*\mathcal{Q}$, where $\mathcal{Q}$ is an invertible sheaf over S. In particular, where $\mathcal{Q}$ is trivial, we get an isomorphism $\lambda\colon \mathcal{O}_E(\sigma - t(\sigma))^d \longrightarrow \mathcal{O}_E$ and therefore a pair $(t, \lambda) \in G_d$ over δ. □

The groups obtained from G_d as base change along geometric points $\operatorname{Spec} k \xrightarrow{(E,p)} \mathcal{M}_{1,1}$ are particular cases of Theta groups, first defined by Mumford in his paper [Mum66]. With notation from this paper we have $G_d \times_{\mathcal{M}_{1,1}} k = \mathcal{G}(\mathcal{O}_E(dp))$.

2.17. **Corollary.** *We have an isomorphism $\mathcal{J}ac_{d,1} \simeq \mathrm{B}_{\mathcal{M}_{1,1}} G_d$.*

By 1.2 the last step in the computation of the Picard group of $\mathcal{J}ac_{d,1}$ is the study of the group of characters $\mathrm{Hom}(G_d, \mathbb{G}_m)$. For the remaining part of this subsection we assume $p \nmid d$.

2.18. **Lemma.** *The map*

$$\begin{aligned} G_d \times G_d &\xrightarrow{\hat{e}_d} \mathbb{G}_m \\ (x, y) &\longmapsto xyx^{-1}y^{-1} \end{aligned}$$

is bilinear and factors through a map $e_d\colon \mathcal{E}[d] \times \mathcal{E}[d] \longrightarrow \mathbb{G}_m$, where $\mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ is the universal curve. Moreover the map e_d induces an isomorphism

$$\mathcal{E}[d] \simeq \underline{\mathrm{Hom}}(\mathcal{E}[d], \mathbb{G}_m)$$

Proof. The map $\hat{e}_d$ is well defined because $\mathcal{E}[d]$ is abelian. Note that $\mathbb{G}_m \subseteq G_d$ is contained in the center. In particular the map e_d in the statement is well defined and we have

$$\hat{e}_d(xy, z) = xyzy^{-1}x^{-1}z^{-1} = x\hat{e}_d(y, z)zx^{-1}z^{-1} = \hat{e}_d(x, z)\hat{e}_d(y, z)$$

Finally $\hat{e}_d(x, y) = \hat{e}_d(y, x)^{-1}$ and $\hat{e}_d$ and e_d are therefore bilinear. Let ϕ be the induced map $\mathcal{E}[d] \longrightarrow \underline{\mathrm{Hom}}(\mathcal{E}[d], \mathbb{G}_m)$. This is a map between flat and finite group schemes and we can check that it is an isomorphism on the geometric points. So let E be an elliptic curve over an algebraically closed field. In this case $E[d] \simeq \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ and $\underline{\mathrm{Hom}}(E[d], \mathbb{G}_m) \simeq E[d]$ as abstract groups. The result then follows from the fact that e_d is non-degenerate thanks to [Mum66, §1, Theorem 1]. □

2.19. **Proposition.** *Let $\mathcal{E} \longrightarrow \mathcal{M}_{1,1}$ be the universal curve. Then*

$$\mathcal{E}[d]^\vee = \mathrm{Hom}(\mathcal{E}[d], \mathbb{G}_m) = 0$$

In particular the map $G_d^\vee \longrightarrow \mathbb{G}_m^\vee$ induced by the inclusion $\mathbb{G}_m \longrightarrow G_d$ is injective.

Proof. By 2.18 we have $\underline{\mathrm{Hom}}(\mathcal{E}[d], \mathbb{G}_m) \simeq \mathcal{E}[d]$. Therefore we have to prove that there are no sections $\mathcal{M}_{1,1} \longrightarrow \mathcal{E}[d]$ but the zero one. By contradiction, assume we have a non zero section $\delta\colon \mathcal{M}_{1,1} \longrightarrow \mathcal{E}[d]$. In particular, by base change, we have a section δ_E for all elliptic curves E. Since $\mathcal{E}[d]$ is étale and separated over $\mathcal{M}_{1,1}$, there are no elliptic curves E such that δ_E is zero. Moreover we can assume that d is prime. If $d = 2$, let k be a field having an irreducible and separable polynomial $g \in k[x]$ of degree 3 and consider the elliptic curve defined by the equation $y^2 = g(x)$. In this case $E[2](k) = 0$ by [Sil86, Group law algorithm 2.3] because g has no zeros in k and therefore $\delta_E = 0$, which is a contradiction.

So we can assume that d is an odd prime. Let k be a field having a degree 2 separable extension L and an elliptic curve E. We are going to prove that $\delta_E \in E(k)$ is invariant under the involution of E, that is that $\delta_E \in E[2](k)$. This will end the proof because $E[d](k) \cap E[2](k) = 0$.

We want to construct a new elliptic curve E' over k with the following construction. Let F be a sheaf of sets over a scheme S with an involution $i\colon F \longrightarrow F$, let $P \longrightarrow S$ be a $\mathbb{Z}/2\mathbb{Z}$-torsor and call $\sigma\colon P \longrightarrow P$ the induced order 2 automorphism. We define a new functor by

$$F_{P/S,i}\colon (\mathrm{Sch}/S)^{\mathrm{op}} \longrightarrow (\mathrm{sets}),\ F_{P/S,i}(T) = \{x \in F(T \times P) \mid \sigma^* x = i(x)\}$$

Since i and σ^* commute, it is easy to check that $F_{P/S,i}$ is a sheaf and that if $S' \longrightarrow S$ is any base change then $F_{P/S,i} \times S' = F_{P\times S'/S',i}$. Moreover, if $\Delta\colon T \longrightarrow P \times T$ is a section it is easy to check that

$$F_{P/S,i}(T) \longrightarrow F(T),\ z \longmapsto \Delta^* z$$

is a bijection. In particular for any torsor P we get an isomorphism $\tau\colon F_{P/S,i}\times P\longrightarrow F\times P$ induced by the diagonal section $P\longrightarrow P\times P$. We now claim that if there exist $x\in F_{P/S,i}(S)$ and $y\in F(S)$ such that $\tau(h^*x)=h^*y$, where $h\colon P\longrightarrow S$ is the structure morphism, then it follows that $i(y)=y$. Indeed by construction $\tau(h^*x)=x$, $\sigma^*h^*y=h^*y$ and, since $x\in F_{P/S,i}(S)$, $\sigma^*x=i(x)$. In particular $h^*i(y)=h^*y$ and by descent $i(y)=y$.

We apply the previous construction with $F=E$, i the involution of E and $P=\operatorname{Spec}L$. Set $E'=F_{P/k,i}$. The sheaf E' is a genus one curve and $0\in E'(k)\subseteq E(L)$. Thus E' is an elliptic curve and, by construction, $\tau\colon E'\times L\longrightarrow E\times L$ preserves the neutral element, that is it is an isomorphism of elliptic curves. Since δ_E and $\delta_{E'}$ come from a global section on $\mathcal{M}_{1,1}$ we should have $\tau(\delta_{E'\times L})=\delta_{E\times L}$, which implies that $i(\delta_E)=\delta_E$ as shown above. □

2.20. **Proposition.** *Consider the map $G_d^\vee\longrightarrow\mathbb{G}_m^\vee=\mathbb{Z}$ induced by the inclusion $\mathbb{G}_m\longrightarrow G_d$. The morphism*

$$\operatorname{Pic}\mathcal{J}ac_{d,1}\simeq\operatorname{Pic}\mathcal{M}_{1,1}\oplus G_d^\vee\longrightarrow\operatorname{Pic}\mathcal{M}_{1,1}\oplus\mathbb{Z}\simeq\mathbb{Z}/12\mathbb{Z}\oplus\mathbb{Z}$$

sends the invertible sheaf on $\mathcal{J}ac_{d,1}$ defined by $\mathcal{J}ac_{d,1}\ni(E\xrightarrow{\pi}S,\mathcal{L})\longmapsto\det\pi_*\mathcal{L}^n$ to the element $(1-nd(nd+1)/2,n^2d)$

Proof. Notice that $\mathrm{B}_{\mathcal{M}_{1,1}}\mathbb{G}_m$ is isomorphic to the stack $\mathcal{X}$ of triples $(E\xrightarrow{\pi}S,\sigma,\mathcal{Q})$ where $(E,\sigma)\in\mathcal{M}_{1,1}$ and $\mathcal{Q}$ is an invertible sheaf over S. Consider the functor

$$\begin{array}{ccc}\mathcal{X} & \xrightarrow{\quad\Omega\quad} & \mathcal{J}ac_{d,1}\\ (E\xrightarrow{\pi}S,\sigma,\mathcal{Q}) & \longmapsto & (E\xrightarrow{\pi}S,\mathcal{O}_E(d\sigma)\otimes\pi^*\mathcal{Q})\end{array}$$

We claim that the functor $\widetilde{\Omega}\colon\mathrm{B}_{\mathcal{M}_{1,1}}\mathbb{G}_m\simeq\mathcal{X}\longrightarrow\mathcal{J}ac_{d,1}\simeq\mathrm{B}_{\mathcal{M}_{1,1}}G_d$ is naturally equivalent to the map $\beta_*\colon\mathrm{B}_{\mathcal{M}_{1,1}}\mathbb{G}_m\longrightarrow\mathrm{B}_{\mathcal{M}_{1,1}}G_d$ induced by the inclusion $\mathbb{G}_m\longrightarrow G_d$. Given $(E\xrightarrow{\pi}S,\sigma,\mathcal{Q})\in\mathcal{X}$, the associated $\mathbb{G}_m$-torsor is $\underline{\mathrm{Iso}}_S(\mathcal{Q},\mathcal{O}_S)$, while $\widetilde{\Omega}(\underline{\mathrm{Iso}}_S(\mathcal{Q},\mathcal{O}_S))$ is the subsheaf of

$$\underline{\mathrm{Iso}}_{\mathcal{J}ac_{d,1}}((E,\mathcal{O}_E(d\sigma)\otimes\pi^*\mathcal{Q}),(E,\mathcal{O}_E(d\sigma)))$$

of isomorphism (f,λ) such that $f_*\colon\operatorname{Pic}^0_E\longrightarrow\operatorname{Pic}^0_E$ is the identity by 2.14. In particular there is a $\mathbb{G}_m$-equivariant map $\underline{\mathrm{Iso}}_S(\mathcal{Q},\mathcal{O}_S)\longrightarrow\widetilde{\Omega}(\underline{\mathrm{Iso}}_S(\mathcal{Q},\mathcal{O}_S))$ compatible with base changes and automorphisms of $\mathcal{Q}$. Recall that $\beta_*(P)=(P\times G_d)/\mathbb{G}_m$ for a $\mathbb{G}_m$-torsor P and that, if Q is a G_d-torsor, $\mathbb{G}_m$-equivariant maps $P\longrightarrow Q$ are in one to one corresponce with G_d-equivariant isomorphism $\beta_*(P)\longrightarrow Q$. Thus given P, the $\mathbb{G}_m$-equivariant map $P\longrightarrow\widetilde{\Omega}(P)$ induces an isomorphism $\beta_*(P)\longrightarrow\widetilde{\Omega}(P)$ of G_d-torsors and it is easy to check that this yields a natural isomorphism $\beta_*\longrightarrow\widetilde{\Omega}$.

We can conclude that the map in the statement can be seen as the pull-back along $\Omega\colon\mathcal{X}\longrightarrow\mathcal{J}ac_{d,1}$ on the Picard groups. The invertible sheaf defined in the statement is sent to

$$(E\xrightarrow{\pi}S,\sigma,\mathcal{Q})\longmapsto\det\pi_*(\mathcal{O}_E(nd\sigma)\otimes\pi^*\mathcal{Q}^n)\simeq(\det\pi_*\mathcal{O}_E(nd\sigma))\otimes\mathcal{Q}^{n^2d}$$

Since the invertible sheaf $(E,\sigma,\mathcal{Q})\longmapsto\mathcal{Q}$ is the generator of $\mathbb{G}_m^\vee$ in $\operatorname{Pic}\mathrm{B}_{\mathcal{M}_{1,1}}\mathbb{G}_m$, the result follows from 2.12. □

2.21. **Proposition.** *The image of the map $G_d^\vee\longrightarrow\mathbb{G}_m^\vee\simeq\mathbb{Z}$ induced by the inclusion $\mathbb{G}_m\longrightarrow G_d$ is $d\mathbb{Z}$.*

Proof. Call $\alpha\colon G_d^\vee\longrightarrow\mathbb{G}_m^\vee\simeq\mathbb{Z}$ the map in the statement. By 2.20 applied to $n=1$, we see that $d\mathbb{Z}\subseteq\operatorname{Im}\alpha$. Let E be an elliptic curve over an algebraically closed field k. By 2.18, it follows that there exist $x,y\in G_d(k)$ such that $\omega=e_d(x,y)=xyx^{-1}y^{-1}$ is a primitive d-root of unity. If $\psi\colon G_d\longrightarrow\mathbb{G}_m$ is a morphism and we set $r=\alpha(\psi)$, we have $1=\psi(\omega)=\omega^r$, which implies that $d\mid r$. □

Proof of theorem 2.9. By 2.17 we have $\mathcal{J}ac_{d,1}\simeq\mathrm{B}_{\mathcal{M}_{1,1}}G_d$ and by 1.2 we can conclude that $\operatorname{Pic}\mathcal{J}ac_{d,1}\simeq\operatorname{Pic}\mathcal{M}_{1,1}\oplus G_d^\vee$. By 2.19 and 2.21 we have $G_d^\vee=d\mathbb{G}_m^\vee\subseteq\mathbb{G}_m^\vee$ and by 2.20 that

$$\mathcal{T}=\det\pi_*\mathcal{L}\otimes(\pi_*\omega_\pi)^{d(d+1)/2-1}$$

freely generates $G_d^\vee$. Again by 2.20 we have

$$\det\pi_*(\mathcal{L}^n)\simeq\mathcal{T}^{n^2}\otimes(\pi_*\omega_\pi)^{1-nd(nd+1)/2}$$

Taking into account that $\omega_\pi \simeq \pi^*\pi_*\omega_\pi$ and using projection formula, a direct computation concludes the proof. □

3. Canonical covers and their discriminant loci.

In this section we work over a field of characteristic $p \geq 0$ and we fix non negative integers g and d and a positive integer n.

3.1. **Definition.** We denote by $\mathcal{V}_{d,g,n}$ (resp. $\mathcal{U}_{d,g,n}$) the stack of triples $(p\colon C \longrightarrow S, \mathcal{Q}, s)$ where $(p\colon C \longrightarrow S, \mathcal{Q}) \in \mathcal{J}ac_{d,g}$ and s is a section of $\mathcal{Q}^n$ (resp. that is not identically zero on any of the geometric fibers of $p\colon C \longrightarrow S$).

Let $\mathcal{C} \longrightarrow \mathcal{U}_{d,g,n}$ be the universal curve of $\mathcal{U}_{d,g,n}$ and $\mathcal{L}$ be its universal invertible sheaf. By definition, $\mathcal{L}^n$ carries a section and we will call it the *marked section* of $\mathcal{L}^n$.

If $d > 0$ we denote by $\mathcal{H}_{d,g,n}$ the zero locus of the marked section of $\mathcal{L}^n$. By 1.11 the map $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$ is a degree nd cover and we will call it the *canonical cover* of $\mathcal{U}_{d,g,n}$. Moreover the closed immersion $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{C}$ defines a functor $\rho_{d,g,n}\colon \mathcal{U}_{d,g,n} \longrightarrow \mathrm{Hilb}^{nd}_{\mathcal{M}_{g,1}/\mathcal{M}_g}$: an object $(C, \mathcal{Q}, s)$ is sent to the zero locus $Z(s) \subseteq C$ of $s \in \mathcal{Q}^n$. Finally we denote by $\mathcal{Z}_{d,g,n}$ the discriminant locus of the canonical cover $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$ of $\mathcal{U}_{d,g,n}$.

The aim of this section is to understand the geometry of the stacks $\mathcal{U}_{d,g,n}$ and $\mathcal{Z}_{d,g,n}$. We will prove the following.

3.2. **Theorem.** *Assume $d > 0$. Let $\mathcal{C} \xrightarrow{\pi} \mathcal{U}_{d,g,n}$ be the universal curve of $\mathcal{U}_{d,g,n}$ and $\mathcal{L}$ be the universal invertible sheaf over $\mathcal{C}$. Then $\mathcal{Z}_{d,g,n} \neq \emptyset$ if and only if $dn \neq 1$ and in this case $\mathcal{Z}_{d,g,n}$ is flat and surjective over $\mathcal{M}_g$ and we have:*

(1) $\mathcal{Z}_{d,g,n}$ is the zero locus in $\mathcal{U}_{d,g,n}$ of a section of the invertible sheaf
$$(\det \pi_*(\mathcal{L}^n \otimes \omega_\pi))^2 \otimes (\det \pi_*\omega_\pi)^{-2}$$
(2) if $p \nmid n$ or $g = 0$ then $\mathcal{Z}_{d,g,n}$ is geometrically reduced over $\mathcal{M}_g$ and, in particular, reduced;
(3) $\mathcal{Z}_{d,g,n}$ is irreducible in the following cases: $n = 1$; $dn > 2g$; $dn = 2g$ and $g \geq 3$; $dn = 2g - 1$ and $g \geq 4$;
(4) If $g = 1$ and $p \neq 2$ then $\mathcal{Z}_{1,1,2}$ is a disjoint union of two integral substacks of $\mathcal{U}_{1,1,2}$, one of which is the zero locus of a section of the invertible sheaf
$$\det(\pi_*\mathcal{L}^2) \otimes (\pi_*\mathcal{L})^{-2}$$

We start giving a more precise description of the stacks $\mathcal{U}_{d,g,n}$, proving in particular that they are algebraic and explaining the relation with the Hilbert scheme $\mathrm{Hilb}^{nd}_{\mathcal{M}_{g,1}/\mathcal{M}_g}$.

3.3. Remark. Assume $nd > 2g - 2$. In this case, by 1.10, the stack $\mathcal{V}_{d,g,n}$ is a vector bundle of rank $nd + 1 - g$ over $\mathcal{J}ac_{d,g}$ corresponding to the locally free sheaf $\pi_*(\mathcal{L}^n)$, where $\pi\colon \mathcal{C} \longrightarrow \mathcal{J}ac_{d,g}$ is the universal curve and $\mathcal{L}$ is the universal invertible sheaf over $\mathcal{C}$. Moreover $\mathcal{U}_{d,g,n}$ is the complement of the zero section of $\mathcal{V}_{d,g,n}$.

3.4. **Proposition.** *If $d > 0$ the functor $\rho_{d,g,1}\colon \mathcal{U}_{d,g,1} \longrightarrow \mathrm{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ is an equivalence. A quasi-inverse is given by*
$$\begin{aligned} \mathrm{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g} &\longrightarrow \mathcal{U}_{d,g,1} \\ (C, Z \subseteq C) &\longmapsto (C, \mathcal{I}_Z^\vee, 1) \end{aligned}$$
where $\mathcal{I}_Z$ is the sheaf of ideals defining Z.

Proof. Given an object $Z \subseteq C \in \mathrm{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}(S)$, we have to prove that $\mathcal{I}_Z$ is invertible over C. Since $\mathrm{Hilb}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ is smooth over $\mathcal{M}_g$, we can assume that S and therefore C are smooth over the base field. By flatness of Z, I_Z is invertible on the fibers of $C \longrightarrow S$ and therefore $\dim_{k(p)} \mathcal{I}_Z \otimes k(p) = 1$ for all $p \in C$, which implies that I_Z is invertible over C. The above discussion shows that the functor in the statement is well defined. The fact that the functors are quasi-inverses of each other is standard. □

3.5. **Proposition.** *If $(C, \mathcal{Q}, s) \in \mathcal{U}_{0,g,n}$ then $s\colon \mathcal{O}_C \longrightarrow \mathcal{Q}^n$ is an isomorphism. Moreover the functor $\mathcal{U}_{0,g,n} \longrightarrow (\underline{\mathrm{Pic}}^0_{\mathcal{M}_{g,1}/\mathcal{M}_g})[n]$ is a μ_n-gerbe.*

Proof. The first claim follows from the fact that a non zero section of a degree 0 invertible sheaf on a proper curve over an algebraically closed field always generates it. Denote by F the functor in the statement. Given $\xi\colon T \longrightarrow (\underline{\mathrm{Pic}}^0_{\mathcal{M}_{g,1}/\mathcal{M}_g})[n]$ denote by $\mathcal{Y} \longrightarrow T$ the base change of F along ξ. In order to prove that $\mathcal{Y}$ is a μ_n-gerbe, we can assume that ξ is given by a curve $p\colon C \longrightarrow T$ and the class of some invertible sheaf $\mathcal{L}$ on C. Since $n[\mathcal{L}] = 0$ in $\underline{\mathrm{Pic}}^0_C$ the sheaf $p_*(\mathcal{L}^n)$ is invertible by 1.10 and we can assume that it is trivial. An object of $\mathcal{Y}(T)$ is a pair $(\mathcal{Q}, s)$ where $\mathcal{Q}$ is an invertible sheaf over C, $s\colon \mathcal{O}_C \longrightarrow \mathcal{Q}^n$ is an isomorphism and $[\mathcal{Q}] = [\mathcal{L}]$. Those data define an invertible sheaf $\mathcal{T} = p_*(\mathcal{L} \otimes \mathcal{Q}^{-1})$ with an isomorphism $\mathcal{O}_T \longrightarrow \mathcal{T}^n$. Since T is arbitrary we get a functor $\mathcal{Y} \longrightarrow \mathrm{B}_T\,\mu_n$ which is easily seen to be an equivalence. □

3.6. **Proposition.** *The map*

$$\begin{array}{ccc} \mathcal{J}ac_{d,g} & \xrightarrow{\ n\ } & \mathcal{J}ac_{dn,g} \\ (C, \mathcal{Q}) & \longmapsto & (C, \mathcal{Q}^n) \end{array}$$

is the composition of a μ_n-gerbe followed by a surjective cover and, if $p \nmid n$ or $g = 0$, it is smooth.

Proof. Consider the diagram

$$\begin{array}{ccccc} \mathcal{J}ac_{d,g} & & \xrightarrow{\ \Omega\ } & & \mathcal{J}ac_{dn,g} \\ & \searrow^{h} & & & \downarrow \\ & & \mathcal{F} & \longrightarrow & \mathcal{J}ac_{dn,g} \\ & & \downarrow & & \downarrow \\ & & \underline{\mathrm{Pic}}^d_{\mathcal{M}_{g,1}/\mathcal{M}_g} & \xrightarrow{\ [n]\ } & \underline{\mathrm{Pic}}^{dn}_{\mathcal{M}_{g,1}/\mathcal{M}_g} \end{array}$$

where the square diagram is Cartesian. By 1.12 the map $[n]$ and therefore the map $\mathcal{F} \longrightarrow \mathcal{J}ac_{dn,g}$ are covers of degree n^{2g} and they are étale if $p \nmid n$ or $g = 0$. It remains to prove that $h\colon \mathcal{J}ac_{d,g} \longrightarrow \mathcal{F}$ is a μ_n gerbe. An object of $\mathcal{F}$ over a scheme S is a triple $(C, \chi, \mathcal{Q})$ where $(C, \mathcal{Q}) \in \mathcal{J}ac_{dn,g}(S)$, $\chi \in \underline{\mathrm{Pic}}^d_C(S)$ satisfying $[n]\chi = [\mathcal{Q}]$. The map h sends $(C, \mathcal{T})$ to $(C, [\mathcal{T}], \mathcal{T}^n)$. Let $\mathcal{Y} \longrightarrow T$ be the base change of h along a map $T \longrightarrow \mathcal{F}$ given by the data $(q\colon C \longrightarrow T, \chi, \mathcal{Q})$. The objects in $\mathcal{Y}(T)$ are pairs $(\mathcal{T}, \lambda)$ where $\mathcal{T}$ is a degree d invertible sheaf on C with $\chi = [\mathcal{T}]$ and $\lambda\colon \mathcal{T}^n \longrightarrow \mathcal{Q}$ is an isomorphism. We can assume $\chi = [\mathcal{T}_0]$ and, since $[n]\chi = [\mathcal{Q}]$ means $[\mathcal{T}_0^n] = [\mathcal{Q}]$ so that $\mathcal{T}_0^n \simeq \mathcal{Q} \otimes q^*\mathcal{R}$ for some invertible sheaf $\mathcal{R}$ on T by 1.10, that $\mathcal{Y}(T) \neq \emptyset$. In this case $\mathcal{Y}(T)$ is isomorphic to the category of pairs $(\overline{\mathcal{T}}, \mu)$ where $\overline{\mathcal{T}}$ is a degree 0 invertible sheaf with $[\overline{\mathcal{T}}] = 0$ in $\underline{\mathrm{Pic}}^0_C$ and $\mu\colon \overline{\mathcal{T}}^n \longrightarrow \mathcal{O}_C$ is an isomorphism. It follows that $\mathcal{Y} \longrightarrow T$ is the base change of the map $\mathcal{U}_{0,g,n} \longrightarrow (\underline{\mathrm{Pic}}^0_{\mathcal{M}_{g,1}/\mathcal{M}_g})[n]$ along the map $T \longrightarrow (\underline{\mathrm{Pic}}^0_{\mathcal{M}_{g,1}/\mathcal{M}_g})[n]$ given by $(C, [\mathcal{O}_C])$ and it is therefore a μ_n-gerbe (see 3.5). □

3.7. **Proposition.** *The stack $\mathcal{U}_{d,g,n}$ is algebraic, flat and of finite type over $\mathcal{M}_g$. Moreover the map*

$$\begin{array}{ccc} \mathcal{U}_{d,g,n} & \xrightarrow{\ \Omega\ } & \mathcal{U}_{dn,g,1} \\ (C, \mathcal{Q}, s) & \longmapsto & (C, \mathcal{Q}^n, s) \end{array}$$

is flat, surjective and of finite type. If $d > 0$ then $\rho_{dn,g,1} \circ \Omega = \rho_{d,g,n}$ (see 3.1 for the notation) and, in particular, $\Omega^{-1}(\mathcal{Z}_{dn,g,1}) = \mathcal{Z}_{d,g,n}$. If $p \nmid n$ or $g = 0$ then Ω is smooth and $\mathcal{U}_{d,g,n}$ is smooth over $\mathcal{M}_g$.

Proof. The equality $\rho_{dn,g,1} \circ \Omega = \rho_{d,g,n}$ follows from the fact that the base change of $\mathcal{H}_{dn,g,1} \longrightarrow \mathcal{U}_{dn,g,1}$ along Ω is $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$. Since the map Ω is the base change of $\mathcal{J}ac_{d,g} \longrightarrow \mathcal{J}ac_{dn,g}$ (see 3.6) along the map $\mathcal{U}_{dn,g,1} \longrightarrow \mathcal{J}ac_{dn,g}$, by 3.6 we can reduce the problem to the case $n = 1$, where all the claims follow from 3.4 and 3.5. □

The remaining part of the section is dedicated to the proof of Theorem 3.2. In particular in what follows we assume $d > 0$.

Proof of theorem 3.2, (1). Set $\mathcal{H}_{d,g,n} = \operatorname{Spec} \mathscr{A}$. By definition $\mathcal{Z}_{d,g,n}$ is the zero locus of a section of $(\det \mathscr{A})^{-2}$. We have an exact sequence on $\mathcal{C}$

$$0 \longrightarrow \mathcal{L}^{-n} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow \mathcal{O}_{\mathcal{H}_{d,g,n}} \longrightarrow 0$$

Applying π_* we get an exact sequence on $\mathcal{U}_{d,g,n}$

$$\pi_*\mathcal{L}^{-n} = 0 \longrightarrow \mathcal{O}_{\mathcal{U}_{d,g,n}} \longrightarrow \mathscr{A} \longrightarrow \mathrm{R}^1\,\pi_*(\mathcal{L}^{-n}) \longrightarrow \mathrm{R}^1\,\pi_*\mathcal{O}_{\mathcal{C}} \longrightarrow 0 = \mathrm{R}^1\,\pi_*\mathcal{O}_{\mathcal{H}_{d,g,n}}$$

where the first and last vanishing follows from 1.10. We also have

$$\mathrm{R}^1\,\pi_*(\mathcal{L}^{-n}) \simeq \pi_*(\mathcal{L}^n \otimes \omega_\pi)^\vee \text{ and } \mathrm{R}^1\,\pi_*\mathcal{O}_{\mathcal{C}} \simeq (\pi_*\omega_\pi)^\vee$$

again by 1.10. We can now deduce the following formula for the determinant of $\mathscr{A}$

$$\det \mathscr{A} \simeq (\det \pi_*(\mathcal{L}^n \otimes \omega_\pi))^{-1} \otimes \det \pi_*\omega_\pi$$

so that $\det \mathscr{A}^{-2}$ is isomorphic to the invertible sheaf in the statement. □

3.8. REMARK. If R is a local ring, $R \longrightarrow S$ is a flat map of rings and M is an R-module then

$$\mathrm{l}_S(M \otimes_R S) = \mathrm{l}_S(S/m_R S)\mathrm{l}_R(M)$$

where l denotes the length function.

The following Theorem is one of the crucial points of the whole paper.

3.9. **Theorem.** *Assume $n \geq 2$. The discriminant locus of the universal degree n cover of $\operatorname{Hilb}^n_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ is flat, surjective and geometrically integral over $\mathcal{M}_g$. In particular it is integral.*

Proof. The problem is local on $\mathcal{M}_g$, therefore we can replace $\mathcal{M}_g$ by a noetherian scheme Y and $\mathcal{M}_{g,1}$ by a genus g curve C over Y. Set Z for the discriminant locus. By definition Z is the zero locus of a section of an invertible sheaf over $\operatorname{Hilb}^n_{C/Y}$. Moreover this section is always non zero over a geometric point of Y because curves over an algebraically closed field always have n distinct rational points. By 1.1 we can conclude that Z is flat over Y. For the remaining part of the statement we can assume $Y = \operatorname{Spec} k$, where k is an algebraically closed field.

Let C' be a non-empty open subset of a projective, integral and generically smooth curve over k (we will reduce to a calculation on a plane curve, that might have singularities). Given indices $i \neq j \leq n$ we denote by $\Delta_{i,j}(C'^n)$ the effective Cartier divisor of C'^n given by

$$\Delta_{i,j}(C'^n) = \{(p_1, \ldots, p_n) \in C'^n \mid p_i = p_j\}$$

Let $\mathcal{H}(C'^{n+1}) \subseteq C'^{n+1}$ be the zero locus of $1 \in \mathcal{O}_{C'^{n+1}}(\Delta_{1,n+1}(C'^{n+1}) + \cdots + \Delta_{n,n+1}(C'^{n+1}))$. Notice that if $C'' \subseteq C'$ is a non-empty open subset then $\mathcal{H}(C''^{n+1}) \longrightarrow C''^n$ is the restriction of $\mathcal{H}(C'^{n+1}) \longrightarrow C'^n$. Indeed by definition we have

$$\forall i \quad \Delta_{i,n+1}(C'^{n+1}) \cap (C''^n \times C') = \Delta_{i,n+1}(C''^{n+1}) \implies \mathcal{H}(C'^{n+1}) \cap (C''^n \times C') = \mathcal{H}(C''^{n+1})$$

We claim that $\mathcal{H}(C'^{n+1}) \longrightarrow C'^n$ is a degree n cover. Indeed, by discussion above, we can assume that C' is projective: in this case the map $\mathcal{H}(C'^{n+1}) \longrightarrow C'^n$ is flat thanks to 1.1, proper, quasi-finite and generically étale of degree n. Set Z' for the discriminant locus of $\mathcal{H}(C'^{n+1})$ in C'^n and $P \in C'^n$ for the generic point of $\Delta_{1,2}(C'^n)$, which lies in Z'. We will show that $\mathrm{l}(\mathcal{O}_{Z',P}) = 2$. We first show how to conclude the proof from this fact. When $C' = C$ the cover $\mathcal{H}(C^{n+1})$ induces a map

$$f \colon C^n \longrightarrow \operatorname{Hilb}^n_{C/k}$$

which factors through an isomorphism $S^nC \simeq \operatorname{Hilb}^n_{C/k}$. In particular f is a cover and $Z' = f^{-1}(Z)$. Moreover, topologically, $Z = f(\Delta_{1,2}(C^n))$ and it coincides with the ramification locus of f. In particular Z is non empty and irreducible. Since $\operatorname{Hilb}^n_{C/k}$ is smooth, to see that Z is reduced we have to prove that $\mathrm{l}(\mathcal{O}_{Z,f(P)}) = 1$. By 3.8 and our assumption we have

$$2 = \mathrm{l}(\mathcal{O}_{Z',P}) = \mathrm{l}(\mathcal{O}_{Z',P}/m_{f(P)}\mathcal{O}_{Z',P})\mathrm{l}(\mathcal{O}_{Z,f(P)}) \geq \mathrm{l}(\mathcal{O}_{Z,f(P)})$$

and we can exclude the equality because otherwise f will be unramified in P, and by transitivity of S_n, also over $f(P)$, which is not the case.

We have to prove that $\mathrm{l}(\mathcal{O}_{Z',P}) = 2$. Thanks to the above discussion we see that the ring $\mathcal{O}_{Z',P}$ (and therefore the number $\mathrm{l}(\mathcal{O}_{Z',P})$) does not change is we take an open subset of C'. In particular we can assume that C' is a closed subscheme of $\mathbb{A}^2_k$:

$$C' = \operatorname{Spec} A \text{ where } A = k[x,y]/(f)$$

We have $C'^n = \operatorname{Spec} B$, where

$$B = A^{\otimes n} = k[x_1, y_1, \dots, x_n, y_n]/(f(x_1,y_1), \dots, f(x_n,y_n))$$

and $P = (x_2 - x_1, y_2 - y_1)$. By definition $\mathcal{H}(C'^{n+1})$ is the spectrum of the B-algebra

$$D = B[\alpha,\beta]/(f(\alpha,\beta), \prod_j I_j) \text{ where } I_j = (\alpha - x_j, \beta - y_j)$$

We have to compute the discriminant locus of the cover $D_P = D \otimes B_P$ over B_P and show that it has length 2. Since C' is generically smooth, B_P is a DVR. In particular we can assume that P is generated by $x_1 - x_2$ in B_P, so that $x_1 - x_2 \mid y_1 - y_2$ in B_P. Notice that

$$(\alpha - x_1)\cdots(\alpha - x_l), (\alpha - x_{l+1}) \in I_1 \cdots I_l + I_{l+1} \implies (x_{l+1} - x_1)\cdots(x_{l+1} - x_l) \in I_1 \cdots I_l + I_{l+1}$$

in $B[\alpha,\beta]/(f(\alpha,\beta))$. Looking at the quotient $B/P = A^{\otimes(n-1)}$ we see that $(x_{l+1} - x_1)\cdots(x_{l+1} - x_l) \notin P$ for $l > 1$. So $I_1 \cdots I_l + I_{l+1}$ is the trivial ideal in D_P and applying the Chinese remainder theorem inductively it follows that

$$D_P = (D_P/I_1I_2) \times D_P/I_3 \times \cdots \times D_P/I_n$$

Since $D_P/I_l \simeq B_P$, the discriminant locus of D_P over B_P coincides with the discriminant locus of $E = (D_P/I_1I_2)$ over B_P, which is a cover of degree 2, since D_P is a cover of degree n of B_P. From $I_1I_2 = 0$ in E we get relations

$$\alpha^2 = (x_1 + x_2)\alpha - x_1x_2 \text{ and } (x_1 - x_2)\beta = (y_1 - y_2)\alpha + x_1y_2 - y_1x_2$$

Since $x_1 - x_2$ divides $y_1 - y_2$ in B_P, E is generated by $1, \alpha$ as B_P-module. Moreover, since E is a free B_P-module of rank 2, $1, \alpha$ is also a B_P-basis of E. Finally a direct computation shows that

$$\operatorname{tr}(\alpha) = x_1 + x_2,\ \operatorname{tr}(\alpha^2) = (x_1 + x_2)^2 - 2x_1x_2,\ \det \begin{pmatrix} \operatorname{tr}(1) & \operatorname{tr}(\alpha) \\ \operatorname{tr}(\alpha) & \operatorname{tr}(\alpha^2) \end{pmatrix} = (x_1 - x_2)^2$$

where $\operatorname{tr} = \operatorname{tr}_{E/B_P}$. The last determinant is the discriminant section of the cover E/B_P and therefore its discriminant locus has length 2, as claimed. □

Proof of theorem 3.2, first sentence and (2). By 3.7 we have $\mathcal{Z}_{d,g,n} = \emptyset \iff \mathcal{Z}_{dn,g,1} = \emptyset$ and, by 3.9, this happens if and only if $dn = 1$. So assume $dn > 1$. Again by 3.7 we can assume $n = 1$. The result then follows from 3.4 and 3.9. □

We now deal with the problem of reducibility of the stacks $\mathcal{Z}_{d,g,n}$.

3.10. **Lemma.** *Let k be an algebraically closed field and C be a genus g curve over k. If $\mathcal{Q}$ is an invertible sheaf of degree d with $d > 2g - 2$ and $q \in C(k)$ then the map*

$$\mathrm{H}^0(\mathcal{Q}) \longrightarrow \mathcal{Q} \otimes (\mathcal{O}_{C,q}/m_q^2\mathcal{O}_{C,q}) \tag{3}$$

where m_q is the maximal ideal of $\mathcal{O}_{C,q}$, has cokernel $\mathrm{H}^0(\mathcal{Q}^{-1} \otimes \mathcal{O}_C(2q) \otimes \omega_C)$ and it is surjective if $d > 2g$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{Q} \otimes \mathcal{O}_C(-2q) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{Q} \otimes (\mathcal{O}_{C,q}/m_q^2\mathcal{O}_{C,q}) \longrightarrow 0$$

Since $\mathrm{H}^1(\mathcal{Q}) \simeq \mathrm{H}^0(\mathcal{Q}^{-1} \otimes \omega_C) = 0$ by degree reasons, applying H^0 we get an exact sequence

$$\mathrm{H}^0(\mathcal{Q}) \longrightarrow \mathcal{Q} \otimes (\mathcal{O}_{C,q}/m_q^2\mathcal{O}_{C,q}) \longrightarrow \mathrm{H}^1(\mathcal{Q} \otimes \mathcal{O}_C(-2q)) \simeq \mathrm{H}^0(\mathcal{Q}^{-1} \otimes \mathcal{O}_C(2q) \otimes \omega_C) \longrightarrow 0$$

Finally, if $d > 2g$ then $\mathrm{H}^0(\mathcal{Q}^{-1} \otimes \mathcal{O}_C(2q) \otimes \omega_C) = 0$, again by degree reasons. □

Proof of theorem 3.2, (3). The case $n = 1$ follows from 3.4 and 3.9. So we focus on the case $nd > 2g-2$. We remark that the proof is a bit simpler if we have the stronger inequality $nd > 2g$, and the intermediate cases require a finer inspection.

Set $\mathcal{V} = \mathcal{V}_{d,g,n}$ and $\mathcal{U} = \mathcal{U}_{d,g,n}$. By 3.3 $\mathcal{V}$ is a vector bundle over $\mathcal{J}ac_{d,g}$ and $\mathcal{U}$ is the complement of the zero section. Consider the diagram

$$\begin{array}{ccc} \mathcal{V} \times_{\mathcal{J}ac_{d,g}} \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \pi \\ \mathcal{V} & \longrightarrow & \mathcal{J}ac_{d,g} \end{array}$$

where $\mathcal{C}$ is the universal curve over $\mathcal{J}ac_{d,g}$. Denote by $\mathcal{L}$ the universal invertible sheaf over $\mathcal{C}$, so that $\mathcal{V}$ corresponds to $\pi_*\mathcal{L}^n$. From [Gro67, Section 16.7] there exists a locally free sheaf $\mathcal{F}$ on $\mathcal{C}$, the 2-th bundle of principal parts of $\mathcal{L}^n$, and a map $\alpha\colon \pi^*\pi_*(\mathcal{L}^n) \longrightarrow \mathcal{F}$ such that for all algebraically closed fields k and triples $(C, \mathcal{Q}, q) \in \mathcal{C}(k)$, where $q \in C(k)$, we have $\mathcal{F} \otimes k \simeq \mathcal{Q}^n \otimes \mathcal{O}_{C,q}/m_q^2$, where m_q is the maximal ideal of $\mathcal{O}_{C,q}$ and

$$\alpha \otimes k\colon \mathrm{H}^0(\mathcal{Q}^n) \simeq \pi^*\pi_*(\mathcal{L}^n) \otimes k \longrightarrow \mathcal{F} \otimes k \simeq \mathcal{Q}^n \otimes \mathcal{O}_{C,q}/m_q^2$$

is the restriction. If $nd > 2g$, by 3.10 we can conclude that α is surjective. In this case in what follows set $\mathcal{W} = \mathcal{J}ac_{d,g}$ and $\mathcal{Y} = \emptyset$. If $nd \leq 2g$ we want to find an open substack $\mathcal{W}$ of $\mathcal{J}ac_{d,g}$ over which α is surjective. If $dn = 2g$ and $g \geq 3$ consider the map

$$\beta\colon \mathcal{M}_{g,1} \longrightarrow \mathcal{J}ac_{dn,g},\ (C \longrightarrow S, \sigma) \longmapsto (C, \mathcal{O}_C(2\sigma) \otimes \omega_{C/S})$$

while if $dn = 2g-1$ and $g \geq 4$ the map

$$\beta\colon \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \longrightarrow \mathcal{J}ac_{dn,g},\ (C \longrightarrow S, \sigma, \tau) \longmapsto (C, \mathcal{O}_C(2\sigma - \tau) \otimes \omega_{C/S})$$

In both cases denote by $\mathcal{Y}'$ the closed substack of $\mathcal{J}ac_{dn,g}$ whose topological space is the closure of the image of β, $\mathcal{Y} = n^{-1}(\mathcal{Y}')$ where $n\colon \mathcal{J}ac_{d,g} \longrightarrow \mathcal{J}ac_{dn,g}$ is the elevation to the n-th power and $\mathcal{W} = \mathcal{J}ac_{d,g} - \mathcal{Y}$.

We will denote by $-_{\mathcal{W}}$ the restriction to $\mathcal{W}$. We want to prove that $\alpha_{\mathcal{W}}$ is surjective and we can check this on the geometric points of $\mathcal{C}_{\mathcal{W}}$. Given $(C, \mathcal{Q}, q) \in \mathcal{C}_{\mathcal{W}}(k)$, where k is an algebraically closed field, by 3.10 the cokernel of $\alpha \otimes k$ is $\mathrm{H}^0(\mathcal{T})$ where $\mathcal{T} = \mathcal{Q}^{-n} \otimes \mathcal{O}_C(2q) \otimes \omega_C$. Notice that $\mathcal{T}$ has degree $-dn + 2g$. Assume by contradiction $\mathrm{H}^0(\mathcal{T}) \neq 0$. If $dn = 2g$ then $\mathcal{T} \simeq \mathcal{O}_C$ and therefore $n(C, \mathcal{Q}) \simeq \beta(C, q)$ which is not the case by construction of $\mathcal{W}$. Finally if $dn = 2g-1$ then by 1.11 there exists $q' \in C$ such that $\mathcal{T} \simeq \mathcal{O}_C(q')$, which means $n(C, \mathcal{Q}) \simeq \beta(C, q, q')$ and it is again not possible by construction.

We want to prove that $(\mathcal{Z}_{d,g,n})_{\mathcal{W}}$ is irreducible. Since $\mathcal{J}ac_{d,g}$ and $\mathcal{C}$ are integral and the vector bundle associated with $\pi^*\pi_*(\mathcal{L}^n)$ is $\mathcal{V} \times_{\mathcal{J}ac_{d,g}} \mathcal{C}$, the kernel $\mathcal{Z}$ of $\alpha_{\mathcal{W}}$ is an integral closed substack of $(\mathcal{V} \times_{\mathcal{J}ac_{d,g}} \mathcal{C})_{\mathcal{W}}$. Let $\widehat{\mathcal{Z}}$ be the image of $\mathcal{Z}$ via the projection $(\mathcal{V} \times_{\mathcal{J}ac_{d,g}} \mathcal{C})_{\mathcal{W}} \longrightarrow \mathcal{V}_{\mathcal{W}}$. We want to prove that $(\mathcal{Z}_{d,g,n})_{\mathcal{W}} = \widehat{\mathcal{Z}} \cap \mathcal{U}$ topologically. This will imply the irriducibility of $(\mathcal{Z}_{d,g,n})_{\mathcal{W}}$.

In what follows k will be an arbitrary algebraically closed field. The objects of $\mathcal{Z}(k)$ are tuples $(C, \mathcal{Q}, q, s) \in (\mathcal{V} \times_{\mathcal{J}ac_{d,g}} \mathcal{C})_{\mathcal{W}}(k)$, where $q \in C(k)$ and $s \in \mathcal{Q}^n$, such that $v_q(s) \geq 2$, where v_q denotes the valuation in q. Thus the objects of $(\widehat{\mathcal{Z}} \cap \mathcal{U})(k)$ are triples $(C, \mathcal{Q}, s) \in \mathcal{U}_{\mathcal{W}}(k)$ for which there exists $q \in C(k)$ such that $v_q(s) \geq 2$. The result then follows from the following remark. If $(C, Q, s) \in \mathcal{U}(k)$ then the zero locus $Z(s) \subseteq C$ of s is étale over k if and only if for all $q \in C(k)$ the ring $\mathcal{O}_{C,q}/s_q (\simeq \mathcal{O}_{Z(s),q})$ is either 0 or k, that is if and only if for all $q \in C(k)$ we have $v_q(s) < 2$.

If $dn > 2g$, so that $\mathcal{W} = \mathcal{J}ac_{d,g}$, then $(\mathcal{Z}_{d,g,n})_{\mathcal{W}} = \mathcal{Z}_{d,g,n}$ is irreducible as required. So assume $2g - 2 < dn \leq 2g$ and $g \geq 2$. Denote by $f : \mathcal{U} \longrightarrow \mathcal{J}ac_{d,g}$ the structure map. Topologically we have

$$\mathcal{Z}_{d,g,n} \subseteq \overline{(\mathcal{Z}_{d,g,n})_{\mathcal{W}}} \cup f^{-1}(\mathcal{Y})$$

where the closure is taken inside $\mathcal{U}$. If $\mathcal{Z}_{d,g,n} = \mathcal{U}$ (which a posteriori will not be the case) there is nothing to prove. Otherwise, since $\mathcal{Z}_{d,g,n} \neq \emptyset$, by 1.7 the equality $\mathcal{Z}_{d,g,n} = \overline{(\mathcal{Z}_{d,g,n})_{\mathcal{W}}}$ follows from

$\dim \mathcal{U} - \dim f^{-1}(\mathcal{Y}) \geq 2$, which we are going to prove. Thanks to 3.6 the map $\mathcal{U} \xrightarrow{f} \mathcal{J}ac_{d,g} \xrightarrow{n} \mathcal{J}ac_{dn,g}$ has constant relative dimension. Moreover $n^{-1}(\mathcal{Y}') = \mathcal{Y}$ and, by 1.6, we get

$$\dim \mathcal{U} - \dim f^{-1}(\mathcal{Y}) = \dim \mathcal{J}ac_{dn,g} - \dim \mathcal{Y}'$$

If $dn = 2g$ set $\mathcal{M} = \mathcal{M}_{g,1}$, while if $dn = 2g-1$ set $\mathcal{M} = \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$. Let us also write $\delta\colon \mathcal{J}ac_{dn,g} \longrightarrow \underline{\mathrm{Pic}}^{dn}_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ for the natural map, and $\gamma = \delta \circ \beta$. Since $\mathcal{M}$ and $\underline{\mathrm{Pic}}^{dn}_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ are proper over $\mathcal{M}_g$ and they are Deligne-Mumford stacks we can conclude that $\gamma(\mathcal{M})$ is closed and, by 1.8, that

$$\dim \gamma(\mathcal{M}) \leq \dim \mathcal{M}$$

Since $\delta\colon \mathcal{J}ac_{dn,g} \longrightarrow \underline{\mathrm{Pic}}^{dn}_{\mathcal{M}_{g,1}/\mathcal{M}_g}$ is a $\mathbb{G}_m$-gerbe we also have $\delta^{-1}(\gamma(\mathcal{M})) = \mathcal{Y}'$, so that, by 1.6,

$$\dim \mathcal{J}ac_{dn,g} - \dim \mathcal{Y}' = \dim \underline{\mathrm{Pic}}^{dn}_{\mathcal{M}_{g,1}/\mathcal{M}_g} - \dim \gamma(\mathcal{M}) \geq g + \dim \mathcal{M}_g - \dim \mathcal{M}$$

Since $\dim \mathcal{M} = 1 + \dim \mathcal{M}_g$ if $dn = 2g$ and $\dim \mathcal{M} = 2 + \dim \mathcal{M}_g$ if $dn = 2g-1$ we get the desired formula. □

When $dn \leq 2g-2$ it is not clear whether $\mathcal{Z}_{d,g,n}$ is irreducible or not. The main technical issue here is that $\mathcal{V}_{d,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ is no longer a vector bundle and $\mathcal{U}_{d,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ may not be surjective.

When $dn > 2g-2$ we have shown above that the stack $\mathcal{Z}_{d,g,n}$ is irreducible except in the following cases: $g = 1$ and $(d,n) = (1,2)$; $g = 2$ and $(d,n) \in \{(1,4),(2,2),(1,3)\}$; $g = 3$ and $(d,n) = (1,5)$. In the remaining part of the section we work out the case $g = 1$, $d = 1$ and $n = 2$ (so that $dn = 2g$). As claimed in 3.2, (4) we will see that $\mathcal{Z}_{1,1,2}$ is reducible. In the other cases just listed, it is again not clear whether $\mathcal{Z}_{d,g,n}$ is irreducible or not.

3.11. **Lemma.** *Assume $p \neq 2$. Let $\mathcal{C} \longrightarrow \mathcal{S}$ be a curve over an algebraic stack with a section τ and denote by $\mathcal{W}$ the zero locus of $1 \in \mathcal{O}_{\mathcal{C}}(2\tau)$ in $\mathcal{C}$. Then $\rho\colon \mathcal{W} \longrightarrow \mathcal{S}$ is a degree 2 cover. Moreover τ factors trough $\mathcal{W}$ and the induced map $\rho_*\mathcal{O}_{\mathcal{W}} \longrightarrow \mathcal{O}_{\mathcal{S}}$ is $\mathrm{tr}_{\rho_*\mathcal{O}_{\mathcal{W}}}/2$, where* tr *denotes the trace map, and its kernel is a square zero ideal.*

Proof. The map $\rho\colon \mathcal{W} \longrightarrow \mathcal{S}$ is a degree 2 cover thanks to 1.11 and the section $\mathcal{S} \longrightarrow \mathcal{C}$ factors through $\mathcal{W}$ by definition of this last space. Denote by $\psi\colon \rho_*\mathcal{O}_{\mathcal{W}} \longrightarrow \mathcal{O}_{\mathcal{S}}$ the induced map. Since $\mathcal{M}_{g,1}$ is reduced, we can assume $\mathcal{S} = \operatorname{Spec} k$, where k is an algebraically closed field. In this case the result follows because $\rho_*\mathcal{O}_{\mathcal{W}} = k[x]/(x^2)$. □

Proof of theorem 3.2, (4). If we denote by $\overline{\mathcal{E}}$ the universal curve over $\mathcal{M}_{1,1}$, we are going to show that there is a μ_2-gerbe $\mathcal{Z}_{1,1,2} \longrightarrow \overline{\mathcal{E}}[2]$. Since this last group is a disjoint union of two irreducible components and gerbes are geometrically irreducible, we will conclude that $\mathcal{Z}_{1,1,2}$ is also a disjoint union of two irreducible components. We will then study the component over the zero section of $\overline{\mathcal{E}}[2]$ and represent it as zero locus of a section of an invertible sheaf.

Set $\mathcal{E} = \mathcal{C}$ for the universal curve of $\mathcal{U}_{1,1,2}$. The canonical cover $\mathcal{H}_{1,1,2} \longrightarrow \mathcal{U}_{1,1,2}$ of $\mathcal{U}_{1,1,2}$ has degree 2. By standard theory of double covers it is given by an invertible sheaf $\mathcal{T}$ over $\mathcal{U}_{1,1,2}$ and a section $r \in \mathcal{T}^2$, so that $\mathcal{H}_{1,1,2} = \operatorname{Spec} \mathscr{A}$ where $\mathscr{A} = \mathcal{O}_{\mathcal{U}_{1,1,2}} \oplus \mathcal{T}^{-1}$. By an easy local computation, the discriminant section of the canonical cover coincides, up to an invertible element, with r and therefore $\mathcal{Z}_{1,1,2}$ is the zero locus of r. Set $\mathcal{Z} = \mathcal{Z}_{1,1,2}$. We will use the symbol $-_{\mathcal{Z}}$ for base changes along $\mathcal{Z} \longrightarrow \mathcal{U}_{1,1,2}$. For instance $\mathcal{E}_{\mathcal{Z}}$ is the universal curve over $\mathcal{Z}$ with universal invertible sheaf $\mathcal{L}_{\mathcal{Z}}$. Since $r_{\mathcal{Z}} = 0$ we have $(\mathcal{T}_{\mathcal{Z}}^{-1})^2 = 0$ in $\mathscr{A}_{\mathcal{Z}}$. Therefore the projection $\mathscr{A}_{\mathcal{Z}} \longrightarrow \mathcal{O}_{\mathcal{Z}}$ is a ring homomorphism and thus induces a section $\mathcal{Z} \longrightarrow \mathcal{H}_{\mathcal{Z}}$ and therefore a section $\tau\colon \mathcal{Z} \longrightarrow \mathcal{E}_{\mathcal{Z}}$. This yields a unique map $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(\tau) \longrightarrow \mathcal{L}_{\mathcal{Z}}^2$ that sends 1 to s and therefore a section $s' \in \mathcal{N} = \mathcal{L}_{\mathcal{Z}}^2 \otimes \mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(-\tau)$. Since s' is non zero on the geometric fibers of $\pi_{\mathcal{Z}}\colon \mathcal{E}_{\mathcal{Z}} \longrightarrow \mathcal{Z}$, by 1.11 there exists another section $\tau'\colon \mathcal{Z} \longrightarrow \mathcal{E}_{\mathcal{Z}}$ and an isomorphism $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(\tau') \simeq \mathcal{N}$ sending 1 to s'. Since the cover $(\mathcal{H}_{1,1,2})_{\mathcal{Z}} \longrightarrow \mathcal{Z}$ is topologically an homeomorphism, the sections τ and τ' coincide on the geometric fibers of $\pi_{\mathcal{Z}}$. Since $\mathcal{Z}$ is reduced thanks to 3.2, (2), we can conclude that $\tau = \tau'$. Moreover the induced isomorphism $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(2\tau) \longrightarrow \mathcal{L}_{\mathcal{Z}}^2$ sends 1 to s.

Define $\mathcal{Z}'$ as the stack of tuples $(E, \mathcal{G}, \tilde{\tau}, \lambda)$ where E is a genus one curve over S, $\mathcal{G}$ is a degree 0 invertible sheaf over E, $\tilde{\tau}$ is a section of E and $\lambda\colon \mathcal{G}^2 \longrightarrow \mathcal{O}_E$ is an isomorphism. Discussion above

shows that we have a map $\mathcal{Z} \longrightarrow \mathcal{Z}'$ which sends $(E, \mathcal{Q}, s)$ to $(E, \mathcal{G}, \tilde{\tau}, \lambda)$ where $\tilde{\tau}$ is induced by the section $\tau\colon \mathcal{Z} \longrightarrow \mathcal{E}_{\mathcal{Z}}$, $\mathcal{G} = \mathcal{Q} \otimes \mathcal{O}_E(-\tilde{\tau})$ and the isomorphism λ is the base change of the isomorphism $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(2\tau) \longrightarrow \mathcal{L}^2_{\mathcal{Z}}$. Conversely we can define a map $\mathcal{Z}' \longrightarrow \mathcal{Z}$ by sending $(E, \mathcal{G}, \tilde{\tau}, \lambda)$ to $(E, \mathcal{Q}, s)$ where $\mathcal{Q} = \mathcal{G} \otimes \mathcal{O}_E(\tilde{\tau})$ and s is the image of 1 under the isomorphism

$$\mathcal{O}_E(2\tilde{\tau}) \simeq \mathcal{G}^2 \otimes \mathcal{O}_E(2\tilde{\tau}) \simeq \mathcal{Q}^2$$

By 3.11 we see that the last functor is well defined and that the composition $\mathcal{Z}' \longrightarrow \mathcal{Z} \longrightarrow \mathcal{Z}'$ is equivalent to the identity. Conversely the composition $\mathcal{Z} \longrightarrow \mathcal{Z}' \longrightarrow \mathcal{Z}$ is equivalent to the identity because the map $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(2\tau) \longrightarrow \mathcal{L}^2_{\mathcal{Z}}$ sends 1 to s.

We define the map

$$\begin{array}{ccc} \mathcal{Z}' & \xrightarrow{\quad p \quad} & \overline{\mathcal{E}}[2] \\ (E, \mathcal{G}, \tilde{\tau}, \lambda) & \longmapsto & (E, \tilde{\tau}, [\mathcal{G}]) \end{array}$$

where we identify $\overline{\mathcal{E}}$ with $\underline{\mathrm{Pic}}^0_{\overline{\mathcal{E}}/\mathcal{M}_{1,1}}$ (see 2.11), which is easily seen to be a μ_2-gerbe.

Now we prove that $\overline{\mathcal{E}}[2]$ is a disjoint union of two irreducible components, one being the zero section $\mathcal{M}_{1,1} \longrightarrow \overline{\mathcal{E}}[2]$. First of all, since $\overline{\mathcal{E}}[2]$ is étale, the zero section is a connected component of $\overline{\mathcal{E}}[2]$. So we need to prove that the complement H is irreducible as well. But $H \longrightarrow \mathcal{M}_{1,1}$ is an étale degree 3 cover and thus, if H is not connected (and therefore irreducible being smooth), it should have a section, which is not the case thanks to 2.18 and 2.19.

Since $p\colon \mathcal{Z}' \longrightarrow \overline{\mathcal{E}}[2]$ is a μ_2-gerbe and thus has irreducible fibers, we can conclude that $\mathcal{Z}' \simeq \mathcal{Z}$ is a disjoint union of two irreducible substack, one of which is $\mathcal{Z}_0 = p^{-1}(\mathcal{M}_{1,1})$. We identify $\mathcal{Z}'$ with $\mathcal{Z}$ and we are going to write $\mathcal{Z}_0$ as the zero locus of a section of the invertible sheaf in the statement. So with an object $(E, \mathcal{Q}, s) \in \mathcal{Z}$ are associated a section $\tilde{\tau}$ of E, base change of $\tau\colon \mathcal{Z} \longrightarrow \mathcal{E}_{\mathcal{Z}}$ and an isomorphism $\mathcal{O}_E(2\tilde{\tau}) \longrightarrow \mathcal{Q}^2$ sending 1 to s, base change of the isomorphism $\mathcal{O}_{\mathcal{E}_{\mathcal{Z}}}(2\tau) \longrightarrow \mathcal{L}^2_{\mathcal{Z}}$. The objects of $\mathcal{Z}_0$ are the triples $(E, \mathcal{Q}, s) \in \mathcal{Z}(S)$ such that $\mathcal{Q}$ and $\mathcal{O}_E(\tilde{\tau})$ differ by an invertible sheaf from the base S, that is $[\mathcal{Q}] = [\mathcal{O}_E(\tilde{\tau})]$ in $\underline{\mathrm{Pic}}^1_{E/S}$. Since $\mathcal{L}$ is an invertible sheaf of degree 1 on $\mathcal{E}$, $\mathcal{R} = \pi_*\mathcal{L}$ is an invertible sheaf by 1.10 and there exists a unique section $\sigma\colon \mathcal{U}_{1,1,2} \longrightarrow \mathcal{E}$ with an isomorphism $\mathcal{L} \simeq \mathcal{O}_{\mathcal{E}}(\sigma) \otimes \pi^*\mathcal{R}$ by 2.11.

Let $\mathcal{W}$ be the zero locus in $\mathcal{E}$ of the section $1 \in \mathcal{O}_{\mathcal{E}}(2\sigma)$. The induced map $\mathcal{W} \longrightarrow \mathcal{U}_{1,1,2}$ is a degree 2 cover by 3.11. Tensoring the exact sequence defining $\mathcal{O}_{\mathcal{W}}$ by $\mathcal{L}^2$, we get an exact sequence

$$0 \longrightarrow \pi^*\mathcal{R}^2 \xrightarrow{\alpha} \pi^*\mathcal{R}^2 \otimes \mathcal{O}_E(2\sigma) \simeq \mathcal{L}^2 \longrightarrow \mathcal{O}_W \otimes \mathcal{L}^2 \longrightarrow 0$$

where $\alpha(x) = x \otimes 1$. Applying π_* and taking into account 1.10 we get an exact sequence

$$0 \longrightarrow \mathcal{R}^2 \xrightarrow{\pi_*\alpha} \pi_*\mathcal{L}^2 \longrightarrow \pi_*(\mathcal{O}_W \otimes \mathcal{L}^2) \longrightarrow \mathrm{R}^1\pi_*(\pi^*\mathcal{R}^2) \longrightarrow 0$$

of locally free sheaves on $\mathcal{U}_{1,1,2}$. Note that the exact sequence on $\mathcal{E}$ satisfies base change for π_*. Set $\mathcal{N} = \mathrm{Coker}(\pi_*\alpha)$. This is an invertible sheaf and applying the determinant we see that it coincides with the invertible sheaf in the statement. The section $s \in \pi_*\mathcal{L}^2$ induces a section $t \in \mathcal{N}$ and we claim that its zero locus is exactly $\mathcal{Z}_0$. This will conclude the proof. Let $\chi = (E \xrightarrow{f} S, \mathcal{Q}, s) \in \mathcal{U}_{1,1,2}$. We will denote by $-_\chi$ the base change along the corresponding map $S \longrightarrow \mathcal{U}_{1,1,2}$. For instance $\mathcal{L}_\chi = \mathcal{Q}$, $f = \pi_\chi$ and, with abuse of notation, $s_\chi = s$. We have that $t_\chi = 0$ if and only if $s_\chi \in \mathrm{Im}(f_*\alpha_\chi) \subseteq f_*\mathcal{Q}^2$ if and only if $s \in \mathrm{Im}(\alpha_\chi) \subseteq \mathcal{Q}^2$. So $t_\chi = 0$ if and only if the square of the isomorphism $\mathcal{Q} \simeq f^*\mathcal{R}_\chi \otimes \mathcal{O}_E(\sigma_\chi)$ sends s to a section of the form $x \otimes 1$. We want to show that those are exactly the objects of $\mathcal{Z}_0$, that is $t_\chi = 0$ if and only if $\chi \in \mathcal{Z}_0$.

If $\chi \in \mathcal{Z}_0$, we have $[\mathcal{Q}] = [\mathcal{O}_E(\sigma_\chi)] = [\mathcal{O}_E(\tilde{\tau})]$ in $\underline{\mathrm{Pic}}^1_{E/S}$, which implies $\sigma_\chi = \tilde{\tau}$ by 2.11. Moreover we have an isomorphism $\mathcal{O}_E(2\tilde{\tau}) \simeq \mathcal{Q}^2$ sending 1 to s. We can conclude observing that all the isomorphisms $\mathcal{O}_E(2\tilde{\tau}) \longrightarrow \mathcal{O}_E(2\tilde{\tau}) \otimes f^*\mathcal{R}^2$ send 1 to a section of the form $1 \otimes x$.

Assume now $t_\chi = 0$, so that $s \in \mathcal{Q}^2$ corresponds to a section $x \otimes 1 \in f^*\mathcal{R}^2_\chi \otimes \mathcal{O}_E(2\sigma_\chi)$. If $x \in f^*\mathcal{R}^2_\chi$ does not generate this sheaf, then the zero locus of $s \in \mathcal{Q}^2$ inside E cannot be a cover of S, because it will have non zero dimensional fibers, contradicting the fact that $\chi \in \mathcal{U}_{1,1,2}$. So $f^*\mathcal{R}^2_\chi \simeq \mathcal{O}_E$, and the zero locus of s in E is the base change of $\mathcal{W} \subseteq \mathcal{E}$, the zero locus of $1 \in \mathcal{O}_{\mathcal{E}}(2\sigma)$. Taking into account

3.11, this shows that $\chi \in \mathcal{Z}$. It also implies that $\tilde{\tau} = \sigma$, so that $[\mathcal{Q} \otimes \mathcal{O}_E(-\tilde{\tau})] = 0$ in $\underline{\mathrm{Pic}}^0_{E/S}$. This exactly means that $\chi \in \mathcal{Z}_0$, as required. □

4. Stacks of uniform cyclic covers and their Picard groups.

In this section we work over a field of characteristic $p \geq 0$ and we fix a non negative integer g and a positive integer n with $n \geq 2$.

4.1. **Definition.** Let Y be a scheme. A uniform cyclic cover of degree n of Y is a map $f\colon X \longrightarrow Y$ together with an action of μ_n on X such that for all $q \in Y$ there exists an affine open neighborhood $U = \operatorname{Spec} R$ of q, an element $h \in R$ and a μ_n-equivariant isomorphism of U-schemes $f^{-1}(U) \simeq \operatorname{Spec} R[x]/(x^n - h)$, where the right hand side is given the action for which $\deg x = 1$.

Unifor cyclic covers of degree n form a stack that we denote by $\mathcal{UC}_n$.

Notice that uniform cyclic covers of degree n are covers of degree n and can be seen as a generalization of double covers when $p \neq 2$. The definition of uniform cyclic covers in [AV04] is slightly different from our, because in 4.1 we do not require that h is a non zero divisor. The reason is that this is automatic for uniform cyclic covers between schemes smooth on a common base and that, avoiding this restriction, uniform cyclic covers are stable by base change.

4.2. **Definition.** Let h be a natural number. We denote by $\mathcal{B}_{h,g,n}$ the stack of triples (D, C, f) where $D \longrightarrow S$ is a genus h curve, $C \longrightarrow S$ is a genus g curve and $f\colon D \longrightarrow C$ is a uniform cyclic cover of degree n.

We define the number $d(h,g,n) = 2\frac{h+n(1-g)-1}{n(n-1)}$, so that $h = 1 + n(g-1) + \frac{n(n-1)}{2}d(h,g,n)$.

The aim of this section is to describe $\mathcal{B}_{h,g,n}$ and compute its Picard group, at least for $h \gg g$. We start by describing explicitly uniform cyclic covers.

4.3. Remark. Let $\mathcal{Y}_n$ be the stack parametrizing pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and $s \in \mathcal{L}^n$. There is an equivalence $\mathcal{Y}_n \longrightarrow \mathcal{UC}_n$ that maps $(\mathcal{L}, s) \in \mathcal{Y}_n(S)$ to

$$X = \operatorname{Spec} \mathscr{A} \longrightarrow S \text{ where } \mathscr{A} = \mathcal{O}_S \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-(n-1)}$$

where μ_n acts on $\mathscr{A}$ via the given grading and the equivariant algebra structure on $\mathscr{A}$ is obtained as follows: given $0 \leq u, v, z < n$ such that $z \equiv u + n \bmod (n)$ the multiplication is

$$(\mathcal{L}^{-u} \otimes \mathcal{L}^{-v} \longrightarrow \mathcal{L}^{-z}) \simeq \begin{cases} \mathcal{L}^{-u-v} \xrightarrow{\mathrm{id}} \mathcal{L}^{-z} & \text{if } u+v < n \\ \mathcal{L}^{-u-v} \simeq \mathcal{L}^{-z} \otimes \mathcal{L}^{-n} \xrightarrow{\mathrm{id}\otimes s} \mathcal{L}^{-z} & \text{if } u+v \geq n \end{cases}$$

A quasi inverse $\Lambda\colon \mathcal{UC}_n \longrightarrow \mathcal{Y}_n$ is obtained as follows. Given a uniform cyclic cover $f\colon X \longrightarrow S$ of degree n, the group μ_n acts on $f_*\mathcal{O}_X$. The degree 1 part of $f_*\mathcal{O}_X$ is an invertible sheaf on S and we set $\mathcal{L}$ for its dual. Since the multiplication $f_*\mathcal{O}_X \otimes f_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X$ is μ_n-equivariant, we get a map from $\mathcal{L}^{-n}$ to the degree 0 part of $f_*\mathcal{O}_X$, which is $\mathcal{O}_S$. This yields a section $s \in \mathcal{L}^n$.

4.4. **Proposition.** *Let h be a natural number and set $d = d(h,g,n)$. If $d \notin \mathbb{N}$ then $\mathcal{B}_{h,g,n} = \emptyset$. If $d \in \mathbb{N}$ the functor (see 4.3 for the notation)*

$$\begin{array}{ccc} \mathcal{B}_{h,g,n} & \xrightarrow{\psi_{h,g,n}} & \mathcal{U}_{d,g,n} \\ (D, C, f) & \longmapsto & (C, \Lambda_C(f)) \end{array}$$

is well defined and an open immersion. If $h > n(g-1)+1$ the image of $\psi_{h,g,n}$ is the complement of $\mathcal{Z}_{d,g,n}$ in $\mathcal{U}_{d,g,n}$, which is the étale locus of the canonical cover $\mathcal{H}_{d,g,n} \longrightarrow \mathcal{U}_{d,g,n}$. If $h = n(g-1)+1$ and $g \geq 1$ the image of $\psi_{h,g,n}$ is the substack of $\mathcal{U}_{0,g,n}$ of triples $(C \longrightarrow S, \mathcal{Q}, s)$ such that $\mathcal{Q}, \ldots, \mathcal{Q}^{n-1}$ are not trivial on the geometric fibers of $C \longrightarrow S$.

Proof. Let $(D, C, f) \in \mathcal{B}_{h,g,n}(k)$, where k is an algebrically closed field, and set $(\mathcal{Q}, s) = \Lambda_C(f)$, so that

$$f_*\mathcal{O}_D \simeq \mathcal{O}_C \oplus \mathcal{Q}^{-1} \oplus \cdots \oplus \mathcal{Q}^{-(n-1)}$$

Since D is integral and connected we get $\dim_k \mathrm{H}^0(\mathcal{Q}^{-i}) = 0$ for $i = 1, \dots, n-1$. By Riemman-Roch it follows that $\dim_k \mathrm{H}^1(\mathcal{Q}^{-i}) = i \deg \mathcal{Q} + g - 1$ and therefore

$$\dim_k \mathrm{H}^1(f_*\mathcal{O}_D) = h = n(g-1) + 1 + (\deg \mathcal{Q})n(n-1)/2$$

In particular $\deg \mathcal{Q} =\in \mathbb{Z}$. On the other hand s cannot be zero since D is smooth and therefore $\deg \mathcal{Q} \geq 0$. In conclusion we see that $\mathcal{B}_{h,g,n} = \emptyset$ if $d \notin \mathbb{N}$, and that $\psi_{h,g,n}$ is well defined if $d \in \mathbb{N}$.

From now on we assume $d \in \mathbb{N}$. From 4.3 it follows that $\psi_{h,g,n}$ is fully faithful.

Given $(C \longrightarrow S, \mathcal{Q}, s) \in \mathcal{U}_{d,g,n}$ we have to check under what conditions the total space of the unifor cyclic cover $D \longrightarrow C$ associated with $(\mathcal{Q}, s)$ (see 4.3) is a smooth curve of genus h over S. It is easy to see that everything follows from the case $S = \operatorname{Spec} k$, where k is an algebraically closed field. Assume $d > 0$. We have $\mathrm{H}^0(\mathcal{Q}^{-i}) = 0$ for $i > 0$, which tells us that D is connected and, by definition of d, that $\dim_k \mathrm{H}^1(\mathcal{O}_D) = h$. The result then follows because the scheme D is regular if and only if the zero locus of $s \in \mathcal{Q}^n$ is étale over k. This can be checked locally using that if (R, m_R) is a DVR and $h \in R$ then $R[x]/(x^n - h)$ is regular if and only if $h \notin m_R^2$.

Now assume $d = 0$. By 3.5 the map $s\colon \mathcal{O}_C \longrightarrow \mathcal{Q}^n$ is an isomorphism. In particular $D \longrightarrow C$ is a μ_n-torsor and therefore D is smooth. Moreover D is connected if and only if $\mathrm{H}^0(\mathcal{Q}^{-i}) = 0$ for $i = 1, \dots, n-1$, in which case has exactly genus h by definition of d. Since $\mathcal{Q}$ has degree 0 we have that $\mathrm{H}^0(\mathcal{Q}^{-i}) \neq 0$ if and only if $\mathcal{Q}^i \simeq \mathcal{O}_C$, which concludes the proof. □

4.5. **Proposition.** *Let h be a natural number with $d = d(h,g,n) \in \mathbb{N}$. Then $\mathcal{B}_{h,g,n}$ is a non empty algebraic stack of finite type and if $nd > 2g-2$ or $p \nmid n$ then $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{M}_g$ is smooth and surjective.*

Proof. Let k be an algebraically closed field. We are going to prove that $\mathcal{B}_{h,g,n} \neq \emptyset$ and, if $dn > 2g-2$ or $p \nmid n$, that $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{M}_g$ is surjective. All the other claims follow from 3.3, 3.7 and 4.4

Assume $d > 0$. By 4.4 we have $\mathcal{B}_{h,g,n} = \mathcal{U}_{d,g,n} - \mathcal{Z}_{d,g,n}$. Moreover by 3.7 there is a surjective map $\mathcal{U}_{d,g,n} - \mathcal{Z}_{d,g,n} \longrightarrow \mathcal{U}_{nd,g,1} - \mathcal{Z}_{nd,g,1}$. We can conclude that $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{M}_g$ is surjective because if C is a genus g curve over k and $p_1, \dots, p_{nd}$ are distinct rational points then $(C, \mathcal{O}_C(p_1 + \cdots + p_{nd}), 1) \in (\mathcal{U}_{nd,g,1} - \mathcal{Z}_{nd,g,1})(k)$.

Assume $d = 0$ and let C be a genus g curve over k. We have $g \geq 1$ because if $g = 0$ then $h = 1 - n < 0$. By 4.4 the fiber of $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{M}_g$ over $C \in \mathcal{M}_g(k)$ is not empty if and only if $\operatorname{Pic} C$ has an element of order n. If $p \nmid n$ this is always the case thanks to 1.12. If $p \mid n$ we have to show that this holds when C is general. We can assume $n = q^l$ for some prime q. If $q \neq p$ then $\operatorname{Pic} C$ has an element of order n by 1.12. Assume $p = q$. By [FvdG04, Theorem 2.3] when C is general there exists an invertible sheaf $\mathcal{T}$ on $\mathcal{C}$ of order p. Since $[p^{l-1}]\colon \underline{\operatorname{Pic}}^0_C \longrightarrow \underline{\operatorname{Pic}}^0_C$ is surjective by 1.12, there exists $\mathcal{Q} \in \operatorname{Pic} C$ such that $\mathcal{Q}^{p^{l-1}} \simeq \mathcal{T}$. It is easy to check that $\mathcal{Q}$ has order exactly $p^l = n$. □

The following result explains the relation between $\operatorname{Pic} \mathcal{B}_{h,g,n}$ and $\operatorname{Pic} \mathcal{J}ac_{d(h,g,n),g}$.

4.6. **Proposition.** *Let h be a natural number such that $d = d(h,g,n) \in \mathbb{Z}$ and $nd > 2g-2$ and let $\pi\colon \mathcal{C} \longrightarrow \mathcal{J}ac_{d,g}$ be the universal curve and $\mathcal{L}$ be the universal invertible sheaf on $\mathcal{C}$. Set also $\mathcal{T} = (\det \pi_*(\mathcal{L}^n \otimes \omega_\pi))^2 \otimes (\det \pi_* \omega_\pi)^{-2}$. Then the map $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ induces a surjective morphism*

$$\gamma\colon (\operatorname{Pic} \mathcal{J}ac_{d,g})/\langle \mathcal{T} \rangle \longrightarrow \operatorname{Pic} \mathcal{B}_{h,g,n}$$

If $\mathcal{Z}_{d,g,n}$ is integral then γ is an isomorphism. If $h = n = 2$, $g = 1$ (so that $d = 1$) and $p \neq 2$ then the kernel of γ is generated by $(\pi_*\mathcal{L})^2 \otimes (\pi_*\omega_\pi)^{-2}$.

If $g = 0$, with notations from 2.6, then $\mathcal{T} \simeq \mathcal{L}_0^{2n(nd-1)}$ if d is even, $\mathcal{T} \simeq \mathcal{L}_0^{n(nd-1)}$ if d is odd. If $g = 1$ and $p \nmid d$ then $\mathcal{T} \simeq (\det \pi_* \mathcal{L})^{2n^2} \otimes (\pi_*\omega_\pi)^{n(dn+d-2n)}$. If $g \geq 2$ and $p = 0$ then $\mathcal{T} \simeq (\det \pi_* \omega_\pi)^{-2n^2} \otimes d_\pi(\mathcal{L})^{n(n-1)} \otimes (\det \pi_*(\mathcal{L} \otimes \omega_\pi))^{n(n+1)}$.

Proof. Notice that we must have $d \geq 1$, otherwise $g = 0$, $d = 0$ and $h = 1 - n \leq -1$. The stack $\mathcal{J}ac_{d,g}$ is smooth and irreducible by 2.2. Moreover $\mathcal{V}_{d,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ is a vector bundle of rank $nd + 1 - g$ and $\mathcal{U}_{d,g,n}$ is the complement of the zero section in $\mathcal{V}_{d,g,n}$ by 3.3. Notice that $\operatorname{rk} \mathcal{V}_{d,g,n} = nd + 1 - g \geq 2$ using that $nd \geq \max\{2, 2g-1\}$. Thus $\mathcal{U}_{d,g,n}$ is smooth and integral and, by 1.9, we can conclude that the map $\mathcal{U}_{d,g,n} \longrightarrow \mathcal{J}ac_{d,g}$ induces an isomorphism on Picard groups. By 4.4 and 4.5 we have $\mathcal{B}_{h,g,n} = \mathcal{U}_{d,g,n} - \mathcal{Z}_{d,g,n} \neq \emptyset$, while by 1.9 and 3.2 the map γ is well defined, surjective and, if $\mathcal{Z}_{d,g,n}$ is

integral, an isomorphism. The claim about the case $h = n = 2$, $g = 1$ and $p \neq 2$ follows again by 1.9 and 3.2.

The expressions for $\mathcal{T}$ in the last part of the statement follow by a direct computation from 2.6, 2.9 and 2.5 respectively. □

(Proof of Theorem A, except the case of $\mathcal{B}_{1,1,2}$). The first part of the statement follows from 4.4 and 4.5.

By construction $\mathcal{L}$ is the universal invertible sheaf on $\mathcal{C}$ with respect to the map $\mathcal{B}_{h,g,n} \longrightarrow \mathcal{J}ac_{d,g}$. We first consider the case $g = 1$ and $h = n = 2$. By 2.9 and 4.6 the group $\operatorname{Pic}\mathcal{B}_{2,1,2}$ is generated by $\alpha = \pi_*\mathcal{L}$ and $\beta = \pi_*\omega_\pi$ with relations $8\alpha = 2\beta$, $2\alpha = 2\beta$ and $12\beta = 0$. Those relations are equivalent to $6\beta = 0$ and $2\alpha = 2\beta$, which yields $\operatorname{Pic}\mathcal{B}_{2,1,2} \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In all the other cases we have that $nd > 2g-2$ and that $\mathcal{Z}_{d,g,n}$ is integral by 3.2. In particular the map γ defined in 4.6 is an isomorphism. Using 4.6 the description of $\operatorname{Pic}\mathcal{B}_{h,g,n}$ with generators and relations follows from 2.6 for $g = 0$, 2.9 for $g = 1$ and 2.5 for $g \geq 2$.

We now deal with the description as abstract groups. For $g = 0$ the result is clear.

Consider now the case (2), that is $g = 1$ and $nd > 2$. Set $A = n(dn + d - 2n)$ and notice that $2 \mid A$. The group $H = \operatorname{Pic}\mathcal{B}_{h,1}$ is isomorphic to $\mathbb{Z}^2/\langle(0, 12), (2n^2, A)\rangle$. The element $(0, 4)$ has order 3 in H. A direct check shows that the map $\phi\colon H \longrightarrow \mathbb{Z}/3\mathbb{Z}$ given by $\phi(0, 1) = 1$ and $\phi(1, 0) = x$, where $x = 0$ if $3 \mid n$ and $x = A/n^2$ otherwise, is well defined because $3 \mid n$ implies $3 \mid A$. Since $\phi(0, 4) = 1$ we obtain $H \simeq \mathbb{Z}/3\mathbb{Z} \times G$ where $G = H/\langle(0, 4)\rangle$. We have $G = \mathbb{Z}^2/\langle(0, 4), (2n^2, A)\rangle$. If $4 \mid A$ then $G \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2n^2\mathbb{Z}$. So assume $A \equiv 2 \bmod (4)$. The map $\psi\colon G \longrightarrow \mathbb{Z}/2\mathbb{Z}$ given by $\psi(u, v) = v$ is well defined. Moreover $(n^2, 1)$ has order 2 in G and $\psi(n^2, 1) = 1$. We obtain $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^2/\langle(0, 4), (n^2, 1)\rangle$. It is now easy to check that the last factor is cyclic of order $4n^2$.

Consider now the case (3) and set $x = (1, 0, 0) \in \mathbb{Z}^3$. Then $\operatorname{Pic}\mathcal{B}_{h,g,n}$ is isomorphic to the group H quotient of $\mathbb{Z}^3$ by the relations $(-2n^2, n(n + 1), n(n - 1))$ and, if $g = 2$, $10x$. Set $l = 10$ if $g = 2$ and $l = 0$ otherwise. It is easy to see that $\langle x\rangle \simeq \mathbb{Z}/l\mathbb{Z}$. A direct computation shows that the map $\psi\colon H \longrightarrow \mathbb{Z}/l\mathbb{Z}$ given by $\psi(u, v, z) = u + v + z$ is well defined. Since $\psi(x) = 1$ we can conclude that $H \simeq \mathbb{Z}/l\mathbb{Z} \times G$, where $G = H/x \simeq \mathbb{Z}^2/\langle(n(n + 1), n(n - 1))\rangle$. Set m for the great common divisor of $n(n + 1)$ and $n(n - 1)$. An easy computation shows that $m = n$ if n is even and $m = 2n$ is n is odd. Let $\alpha, \beta \in \mathbb{Z}$ such that $\alpha n(n + 1) + \beta n(n - 1) = m$. Consider the map

$$\phi = \begin{pmatrix} \alpha & \beta \\ -\frac{n(n-1)}{m} & \frac{n(n+1)}{m} \end{pmatrix} : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$$

By construction ϕ is an isomorphism because $\det\phi = 1$. Moreover $\phi(n(n+1), n(n-1)) = (m, 0)$ and therefore $G \simeq \mathbb{Z}^2/\langle(m, 0)\rangle \simeq \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}$ as required. □

In the remaining part of this section we will deal with the case of $\mathcal{B}_{1,1,2}$. As pointed out at the beginning, this case is peculiar and needs a variation of the methods used for higher genera. Nevertheless, the steps in the computation of $\operatorname{Pic}\mathcal{B}_{1,1,2}$ are very similar to the ones in the computation of $\operatorname{Pic}\mathcal{B}_{h,1,n}$, for $h \gg 0$.

In what follows we consider $g = 1$ and assume that $p \nmid 6$. We denote by $\widetilde{\mathcal{M}}_{1,2}$ the universal curve over $\mathcal{M}_{1,1}$, which is the moduli stack of triples (E, σ_1, σ_2) where E is a genus one curve and σ_1, σ_2 are sections. The map $\widetilde{\mathcal{M}}_{1,2} \longrightarrow \mathcal{M}_{1,1}$ is the functor that forgets the second section.

4.7. **Proposition.** *The functor*

$$\begin{aligned} \widetilde{\mathcal{M}}_{1,2} &\longrightarrow \mathcal{J}ac_{0,1} \\ (E, \sigma_1, \sigma_2) &\longmapsto (E, \mathcal{O}_E(\sigma_2 - \sigma_1)) \end{aligned}$$

is an epimorphism in the fppf topology and it is a section of the functor

$$\begin{aligned} \mathcal{J}ac_{0,1} &\longrightarrow \widetilde{\mathcal{M}}_{1,2} \\ (E, \mathcal{Q}) &\longmapsto (\underline{\operatorname{Pic}}^0_{E/S}, [\mathcal{O}_E], [\mathcal{Q}]) \end{aligned}$$

Proof. The second part of the statement follows from 2.11. For the first one let $(E, \mathcal{Q}) \in \mathcal{J}ac_{0,1}$. We can assume that E has a section σ_1. Again by 2.11, $[\mathcal{Q}] = [\mathcal{O}_E(\sigma_2 - \sigma_1)]$, for some section σ_2 of E, which means that $\mathcal{Q}$ and $\mathcal{O}_E(\sigma_2 - \sigma_1)$ are fppf locally isomorphic. □

4.8. **Definition.** We define the group functor G_0 over $\widetilde{\mathcal{M}}_{1,2}$ as the group G obtained as in 2.14 with respects to the maps defined in 4.7.

4.9. **Proposition.** *An element of $G_0(S \xrightarrow{(E,\sigma_1,\sigma_2)} \widetilde{\mathcal{M}}_{1,2})$ is a pair (f, λ) where $f\colon (E, \sigma_1) \longrightarrow (E, \sigma_1)$ is a translation and $\lambda\colon \mathcal{O}_E(\sigma_2 - \sigma_1) \longrightarrow \mathcal{O}_E(f(\sigma_2) - f(\sigma_1))$ is an isomorphism. Moreover we have an exact sequence*

$$\begin{array}{ccccccccc} & & & & (f,\lambda) & \mapsto & f(\sigma_1) & & \\ 0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & G_0 & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \mu & \longmapsto & (\mathrm{id}, \mu) & & & & \end{array}$$

in the Zariski topology of $\mathrm{Sch}/\widetilde{\mathcal{M}}_{1,2}$*, where $\mathcal{E} \longrightarrow \widetilde{\mathcal{M}}_{1,2}$ is the universal curve. In particular G_0 is smooth over $\widetilde{\mathcal{M}}_{1,2}$.*

Proof. By definition, an element of $G_0(S \xrightarrow{(E,\sigma_1,\sigma_2)} \widetilde{\mathcal{M}}_{1,2})$ is a pair (f, λ) where $f\colon E \longrightarrow E$ is an isomorphism such that $f_*\colon \underline{\mathrm{Pic}}^0_{E/S} \longrightarrow \underline{\mathrm{Pic}}^0_{E/S}$ is the identity and $\lambda\colon \mathcal{O}_E(\sigma_2 - \sigma_1) \longrightarrow \mathcal{O}_E(f(\sigma_2) - f(\sigma_1))$ is an isomorphism. As in the proof of 2.16, $f_* = \mathrm{id}$ means that f is a translation. In particular the sequence in the statement is well defined and, since $\pi_*\mathbb{G}_{m,E} \simeq \mathbb{G}_m$, it is exact in the first two terms.

It remains to prove that $G_0 \longrightarrow \mathcal{E}$ is a Zariski epimorphism. This will also imply that G_0 is locally a product of $\mathbb{G}_m$ and $\mathcal{E}$ and therefore smooth. Let $(E, \sigma_1, \sigma_2, \delta) \in \mathcal{E}$ and t be the translation by δ, so that $t(\sigma_1) = \delta$. Since t is a translation we have

$$[\mathcal{O}_E(\sigma_2 - \sigma_1)] = [\mathcal{O}_E(t(\sigma_2) - t(\sigma_1))] \text{ in } \underline{\mathrm{Pic}}^0_{E/S}$$

which means that the sheaves differ from an invertible sheaf coming from the base thanks to 1.10. So Zariski locally we get an isomorphism $\lambda\colon \mathcal{O}_E(\sigma_2 - \sigma_1) \longrightarrow \mathcal{O}_E(t(\sigma_2) - t(\sigma_1))$ and therefore $(t, \lambda) \in G_0(E, \sigma_1.\sigma_2))$ is mapped to $(E, \sigma_1, \sigma_2, \delta)$. □

4.10. **Lemma.** *Let $\mathcal{X}$ be a smooth algebraic stack, $\mathcal{Y} \xrightarrow{p} \mathcal{X}$ be a $\mathbb{G}_m$-torsor and $\mathcal{L}$ be the invertible sheaf over $\mathcal{X}$ corresponding to it. Then we have an exact sequence*

$$\mathbb{Z} \xrightarrow{\mathcal{L}} \mathrm{Pic}\,\mathcal{X} \xrightarrow{p^*} \mathrm{Pic}\,\mathcal{Y} \longrightarrow 0$$

Proof. The stack $\mathcal{Y}$ and the line bundle $\mathcal{Y}' \longrightarrow \mathcal{X}$ corresponding to $\mathcal{L}$ can be seen as the relative sheaves

$$\underline{\mathrm{Iso}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \mathcal{L}) \text{ and } \underline{\mathrm{Hom}}_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}, \mathcal{L})$$

on $\mathrm{Sch}/\mathcal{X}$ respectively. Denote by $q\colon \mathcal{Y}' \longrightarrow \mathcal{X}$ the structural morphism. The stack $\mathcal{Y}$ is the open substack of $\mathcal{Y}'$ whose complement $\mathcal{Z}$ is the zero section of $\mathcal{Y}' \longrightarrow \mathcal{X}$. The stack $\mathcal{Z}$ is integral since $\mathcal{X}$ is so and it is the zero locus of the universal section of $q^*\mathcal{L}$. The result then follows from 1.9. □

In what follows we denote by $\mathcal{F}$ the complement of the zero section $\mathcal{M}_{1,1} \longrightarrow \widetilde{\mathcal{M}}_{1,2}$ in $\widetilde{\mathcal{M}}_{1,2}[2]$.

4.11. **Proposition.** *The composition $\mathcal{B}_{1,1,2} \longrightarrow \mathcal{J}ac_{0,1} \longrightarrow \widetilde{\mathcal{M}}_{1,2}$ has image in $\mathcal{F}$. The induced map $\mathcal{B}_{1,1,2} \longrightarrow \mathcal{F}$ yields an isomorphism on Picard groups and factors as a $\mathbb{G}_m$-torsor $\mathcal{B}_{1,1,2} \longrightarrow \mathrm{B}_{\mathcal{F}} G_0$ followed by the projection $\mathrm{B}_{\mathcal{F}} G_0 \longrightarrow \mathcal{F}$.*

Proof. Set $\mathcal{X} = \mathrm{B}_{\mathcal{F}} G_0$. By 2.14 and 4.7, we see that $\mathcal{J}ac_{0,1} \simeq \mathrm{B}_{\widetilde{\mathcal{M}}_{1,2}} G_0$. In particular $\mathcal{X}$ can be seen as the closed substack of $\mathcal{J}ac_{0,1}$ of pairs $(E, \mathcal{Q})$ such that $(\underline{\mathrm{Pic}}^0_E, [\mathcal{O}_E], [\mathcal{Q}]) \in \mathcal{F}$. By 4.4 we see that the forgetful map $\mathcal{B}_{1,1,2} \longrightarrow \mathcal{X}$ is a $\mathbb{G}_m$-torsor corresponding to the invertible sheaf $\pi_*(\mathcal{L}^2)$, where $\pi\colon \mathcal{E} \longrightarrow \mathcal{X}$ is the universal curve and $\mathcal{L}$ is the universal invertible sheaf over it. Notice that $\mathcal{X}$ is smooth because it is an open substack of $\mathcal{J}ac_{0,1}$, which is smooth thanks to 2.2. In particular from 4.10 the pull-back of $\mathcal{B}_{1,1,2} \longrightarrow \mathcal{X}$ induces an isomorphism

$$\mathrm{Pic}\,\mathcal{B}_{1,1,2} \simeq \mathrm{Pic}\,\mathcal{X}/\langle \pi_*(\mathcal{L}^2) \rangle$$

Moreover from 1.2, we have $\operatorname{Pic}\mathcal{X} = \operatorname{Pic}\mathcal{F} \oplus G_0^\vee$. We are going to show that $G_0^\vee \simeq \mathbb{Z}$ and that the component of $\pi_*(\mathcal{L}^2)$ in $\operatorname{Pic}\mathcal{X}$ with respect to $G_0^\vee$ generates this last group. This will imply that the composition of pull-backs $\operatorname{Pic}\mathcal{F} \longrightarrow \operatorname{Pic}\mathcal{X} \longrightarrow \operatorname{Pic}\mathcal{B}_{1,1,2}$ is an isomorphism.

Taking into account 4.9, the inclusion $\mathbb{G}_m \longrightarrow G_0$ yields a map $\alpha\colon G_0^\vee \longrightarrow \mathbb{G}_m^\vee \simeq \mathbb{Z}$ whose kernel is the group of characters of the universal curve $\widetilde{\mathcal{E}}$ of $\mathcal{F}$. If $\phi\colon \widetilde{\mathcal{E}} \longrightarrow \mathbb{G}_m$ is such a homomorphism, $\phi^{-1}(1)$ is a closed substack of $\widetilde{\mathcal{E}}$ and, by checking on the geometric fibers, we see that they are topologically equal. Since $\widetilde{\mathcal{E}}$ is reduced we can conclude that ϕ is trivial and therefore that α is injective. As in the proof of 2.20, considering the functor

$$\begin{aligned} \mathrm{B}_{\mathcal{F}}\mathbb{G}_m &\longrightarrow \mathcal{X} \\ (E \xrightarrow{\pi} S, \sigma_1, \sigma_2, \mathcal{Q}) &\longmapsto (E, \mathcal{O}_E(\sigma_2 - \sigma_1) \otimes \pi^*\mathcal{Q}) \end{aligned}$$

and the expression

$$\pi_*[(\mathcal{O}_E(\sigma_2 - \sigma_1) \otimes \pi^*\mathcal{Q})^2] \simeq \pi_*(\mathcal{O}_E(2\sigma_2 - 2\sigma_1)) \otimes \mathcal{Q}^2$$

we see that $\pi_*(\mathcal{L}^2)$ is sent to 2 by the map $\operatorname{Pic}\mathcal{X} \simeq \operatorname{Pic}\mathcal{F} \oplus G_0^\vee \xrightarrow{\mathrm{id}\oplus\alpha} \operatorname{Pic}\mathcal{F} \oplus \mathbb{G}_m^\vee \longrightarrow \mathbb{G}_m^\vee = \mathbb{Z}$. In particular $2\mathbb{Z} \subseteq \operatorname{Im}\alpha$ and we need to prove that those groups are equal, or, equivalently, that α is not an isomorphism. Assume by contradiction that α is an isomorphism. This exactly means that the map $\mathbb{G}_m \longrightarrow G_0$ has a section. Thus also the map $G_0 \longrightarrow \widetilde{\mathcal{E}}$ has a section. Since this last map is a $\mathbb{G}_m$-torsor, we can rephrase this saying that the invertible sheaf over $\widetilde{\mathcal{E}}$ corresponding to G_0 is trivial. We are going to compute this sheaf and prove that it is not trivial. Given $(E \xrightarrow{\pi} S, \sigma_1, \sigma_2, \sigma_3) \in \widetilde{\mathcal{E}}$ and denoted by $t\colon (E, \sigma_1) \longrightarrow (E, \sigma_1)$ the translation by σ_3, so that $t(\sigma_1) = \sigma_3$, the invertible sheaf $\mathcal{K}$ over $\widetilde{\mathcal{E}}$ corresponding to G_0 is given by the following calculation

$$\begin{aligned} \underline{\mathrm{Iso}}_E(\mathcal{O}_E(\sigma_2 - \sigma_1), \mathcal{O}_E(t(\sigma_2) - t(\sigma_1)) &\simeq \underline{\mathrm{Iso}}_E(\mathcal{O}_E(\sigma_2 + \sigma_3 - \sigma_1), \mathcal{O}_E(t(\sigma_2))) \\ &\simeq \underline{\mathrm{Iso}}_S(\pi_*\mathcal{O}_E(\sigma_2 + \sigma_3 - \sigma_1), \mathcal{O}_S) \simeq \pi_*\mathcal{O}_E(\sigma_2 + \sigma_3 - \sigma_1)^\vee \end{aligned}$$

where we have used that $\pi_*\mathcal{O}_E(t(\sigma_2)) \simeq \mathcal{O}_S$ by 1.11. Using 2.12 twice we also have

$$\pi_*\mathcal{O}_E(\sigma_2 + \sigma_3 - \sigma_1)^\vee \simeq \sigma_1^*\mathcal{O}_E(\sigma_2) \otimes \sigma_1^*\mathcal{O}_E(\sigma_3) \otimes \sigma_3^*\mathcal{O}_E(-\sigma_2) \otimes \pi_*\omega_\pi$$

Given an elliptic curve E over an algebraically closed field with origin p_1 and $p_2 \in E[2] - \{p_1\}$, we consider the object $\chi = (E \times E \xrightarrow{\mathrm{pr}_2} E, \sigma_1, \sigma_2, \Delta) \in \widetilde{\mathcal{E}}(E)$ where $\sigma_i = p_i \times \mathrm{id}_E\colon E \longrightarrow E \times E$ for $i = 1, 2$ and $\Delta\colon E \longrightarrow E \times E$ is the diagonal. Using isomorphism above, the pull-back of $\mathcal{K}$ to E is given by

$$\sigma_1^*\mathcal{O}_{E\times E}(\sigma_2) \otimes \sigma_1^*\mathcal{O}_{E\times E}(\Delta) \otimes \Delta^*\mathcal{O}_{E\times E}(-\sigma_2) \otimes \mathrm{pr}_{2*}\,\omega_{\mathrm{pr}_2} \simeq \mathcal{O}_E(p_1 - p_2)$$

which is not trivial. □

4.12. **Proposition.** *We have* $\operatorname{Pic}\mathcal{F} \simeq \mathbb{Z}/4\mathbb{Z}$*, generated by the invertible sheaf* $\pi_*\omega_\pi$*, where* $\pi\colon \mathcal{E} \longrightarrow \mathcal{F}$ *is the universal curve over* $\mathcal{F}$*.*

Proof. Let k be the base field and set $U = \operatorname{Spec} R$, where $R = k[a, b]_\Delta$ with $\Delta = 4a^3 + 27b^2$ and $\rho\colon \mathcal{F} \longrightarrow \mathcal{M}_{1,1}$ for the structure map, which is an étale degree 3 cover. Since $\operatorname{char} k \nmid 6$, the map $U \longrightarrow \mathcal{M}_{1,1}$ given by the general Weierstrass curve

$$W = \operatorname{Proj}(R[x, y, z]/(f)) \longrightarrow U \text{ where } f = y^2z - x^3 - axz^2 - bz^3$$

is a $\mathbb{G}_m$-torsor corresponding to a generator $\mathcal{K}$ of $\operatorname{Pic}\mathcal{M}_{1,1}$, either $p_*\omega_p$ or its dual, where $p\colon \widetilde{\mathcal{E}} \longrightarrow \mathcal{M}_{1,1}$ is the universal curve. In particular the base change $V = U \times_{\mathcal{M}_{1,1}} \mathcal{F} \longrightarrow \mathcal{F}$ is the $\mathbb{G}_m$-torsor corresponding to the pull-back $\rho^*\mathcal{K}$ and it coincides with $W[2]$ minus the zero section. By [Sil86, Group law algorithm 2.3], 2-torsion points are obtained modding out by y and therefore we get

$$V \simeq \operatorname{Spec} k[a, b, x]_\Delta/(x^3 + ax + b)$$

In particular V is an open subscheme of $\mathbb{A}^2 = \operatorname{Spec} k[a, x]$ and therefore $\operatorname{Pic} V = 0$. By 4.10 we can conclude that $\operatorname{Pic}\mathcal{F}$ is generated by $\omega = \pi_*\omega_\pi$. This is because, if $\omega' = p_*\omega_p$, then $\rho^*\omega' \simeq \omega$.

We want to prove that the order r of ω in $\operatorname{Pic}\mathcal{F}$ is exactly 4. Since ω' has order 12 in $\operatorname{Pic}\mathcal{M}_{1,1}$ by 2.10, we have

$$\omega^r \simeq \rho^*\omega'^r \simeq \mathcal{O}_{\mathcal{F}} \implies \rho_*\mathcal{O}_{\mathcal{F}} \simeq \omega'^r \otimes \rho_*\mathcal{O}_{\mathcal{F}} \overset{\det}{\implies} \omega'^{3r} \simeq \mathcal{O}_{\mathcal{M}_{1,1}} \implies 12 \mid 3r \implies 4 \mid r$$

Consider now the invertible sheaf $\mathcal{T}$ on $\mathcal{F}$ given by

$$\mathcal{T}\colon (E \xrightarrow{q} S, \sigma_1, \sigma_2) \longmapsto \sigma_2^*\mathcal{O}_E(\sigma_2 - \sigma_1) \otimes \sigma_1^*\mathcal{O}_E(\sigma_1 - \sigma_2)$$

Since $\mathcal{O}_E(\sigma_2 - \sigma_1)^2 \simeq q^*q_*(\mathcal{O}_E(\sigma_2 - \sigma_1)^2)$ by definition of $\mathcal{F}$ and 1.10, we see that $\mathcal{T}^2 \simeq \mathcal{O}_{\mathcal{F}}$. On the other hand, since σ_1 and σ_2 are disjoint, we have $\sigma_1^*\mathcal{O}_E(\sigma_2) \simeq \sigma_2^*\mathcal{O}_E(\sigma_1) \simeq \mathcal{O}_E$ and therefore $\mathcal{T} \simeq \omega^2$ thanks to 2.12. In conclusion $\mathcal{O}_{\mathcal{F}} \simeq \mathcal{T}^2 \simeq \omega^4$ and therefore $r = 4$. □

(Proof of Theorem A, the case of $\mathcal{B}_{1,1,2}$). By 4.4 $\mathcal{L}$ is a degree 0 invertible sheaf on $\mathcal{C}$ which is never trivial on the geometric fibers of $\mathcal{C} \longrightarrow \mathcal{B}_{1,1,2}$. By Grauert we can conclude that $\pi_*\mathcal{L} = 0$, so that $\det \pi_*\mathcal{L}$ is trivial. The result then follows from 4.11 and 4.12. □

REFERENCES

[AV04] Alessandro Arsie and Angelo Vistoli. Stacks of cyclic covers of projective spaces. *Compositio Mathematica*, 140(3):647–666, May 2004.

[BV12] Michele Bolognesi and Angelo Vistoli. Stacks of trigonal curves. *Transactions of the American Mathematical Society*, 364:3365–3393, 2012.

[FO10] William Fulton and Martin Olsson. The Picard group of $M_{1,1}$. *Algebra and Number Theory*, 4:87–104, April 2010.

[FvdG04] Carel Faber and Gerard van der Geer. Complete subvarieties of moduli spaces and the Prym map. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2004(573), January 2004.

[Gro66] Alexander Grothendieck. *EGAIV-3 - Étude locale des schémas et des morphismes de schémas (Troisiém partie) - Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné)*. Inst. Hautes Études Sci. Publ. Math. 28, 1966.

[Gro67] Alexander Grothendieck. *EGAIV-4 - Étude locale des schémas et des morphismes de schémas (Quatrième partie) - Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné)*. Inst. Hautes Études Sci. Publ. Math. 32, 1967.

[Kle80] Steven L. Kleiman. Relative Duality for Quasi-Coherent Sheaves. *Compositio Mathematica*, 41:39 – 60, 1980.

[MBL99] Laurent Moret-Bailly and Gerard Laumon. *Champs algébriques*. Springer, first edition, 1999.

[Mil08] James S. Milne. *Abelian Varieties*. Online lecture notes, 2008.

[Mum63] David Mumford. Picard groups of moduli problems. *Arithmetical Algebraic Geometry*, pages 33–81, 1963.

[Mum66] David Mumford. On the equations defining abelian varieties. I. *Inventiones Mathematicae*, 1(4):287–354, December 1966.

[MV14] Margarida Melo and Filippo Viviani. The Picard group of the compactified universal Jacobian. *Documenta Mathematica*, 19:457–506, July 2014.

[Pag13] Nicola Pagani. Moduli of abelian covers of elliptic curves. *arxiv id: 1303.2991*, page 24, March 2013.

[Sil86] Joseph H. Silverman. *The Arithmetic of Elliptic Curves*. 1986.

[SP014] Stacks Project, 2014.

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