# GORENSTEIN DIMENSIONS MODULO A REGULAR ELEMENT

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**Abstract**

Let $R$ be a commutative ring. In this paper we study the behavior of Gorenstein homological dimensions of a homologically bounded $R$-complex under special base changes to the rings $R_x$ and $R/xR$, where $x$ is a regular element in $R$. Our main results refine some known formulae for the classical homological dimensions. In particular, we provide the Gorenstein counterpart of a criterion for projectivity of finitely generated modules, due to Vasconcelos.

*Keywords and phrases*: Gorenstein homological dimension, derived category.

## 1. Introduction

Throughout this paper, $R$ is a non-trivial commutative ring with a unit element, and $x$ is an element of $R$ that is neither a zero-divisor nor invertible. In [8] we investigated the relation between homological behavior of ring $R$ and those of rings $R_x$ and $R/xR$. See, for instance, [8, 3.4, 3.7]. It is also proved that for a complex of $R$-modules $M$, following equalities hold

$$\operatorname{fd}_R M = \max\{\operatorname{fd}_{R/xR}(R/xR \otimes^{\mathbf{L}}_R M), \operatorname{fd}_{R_x} M_x\}. \tag{1}$$

$$\operatorname{id}_R M = \max\{\operatorname{id}_{R/xR}(R/xR \otimes^{\mathbf{L}}_R M), \operatorname{id}_{R_x} M_x\}. \tag{2}$$

See [8, 3.2, 4.2] for detailed statements.
In this paper we prove the Gorenstein counterparts of (1) and (2). More precisely, We prove that for every homologically bounded complex $M$ over coherent ring $R$, the following equality holds

$$\operatorname{Gfd}_R M = \max\{\operatorname{Gfd}_{\overline{R}} \overline{M}, \operatorname{Gfd}_{R_x} M_x\} \tag{3}$$

where $\overline{R}$ denotes the factor ring $R/(x)$ and $\overline{M}$ is a complex of $\overline{R}$-module, see section 3 for details. A similar formula holds for Gorenstein injective dimensions when $R$ is noetherian with dualizing complex, see (3.3).

In [9], Vasconcolos proved a criterion for projectivity of finitely generated modules, i.e. it's proved that (see [9, Theorem 1.6]) when $M$ is a finitely generated module over $R$ and $x$ is a non-zero divisor on both $R$ and $M$, $M$ is projective over $R$ if and only if $M/xM$ is projective over $R/x$ and $M_x$ is projective over $R_x$. It is therefore natural to ask whether the same criterion for Gorenstein projectivity is true. In fact, when $M$ is finitely generated and $R$ is noetherian, a very special case of the equation (3) gives affirmative answer to this question, see (3).

## 2. Prerequisites

In this short section, we fix our notation and prove some easy lemmas that will be used later. Throughout, $R$ is a non-trivial commutative ring with a unit element, and $x$ is an element of $R$ that is neither a zero-divisor nor invertible. We sometimes write '$R$-complex' in place of 'a complex of $R$-modules'. Complexes are graded homologically. Thus, an $R$-complex $M$ has the form

$$\cdots \to M_{\ell+1} \xrightarrow{\partial^M_{\ell+1}} M_\ell \xrightarrow{\partial^M_\ell} M_{\ell-1} \to \cdots .$$

Modules are considered to be complexes concentrated in degree zero. We write $\Sigma M$ for complex with

$$(\Sigma M)_n = M_{n-1} \quad \text{and} \quad \partial^{\Sigma M} = -\partial^M.$$

The *supremum* and *infimum* of $M$ are defined as follows:

$$\begin{aligned} \sup(M) &= \sup\{\ell \in \mathbb{Z} | \operatorname{H}_\ell(M) \neq 0\} \\ \inf(M) &= \inf\{\ell \in \mathbb{Z} | \operatorname{H}_\ell(M) \neq 0\}, \end{aligned}$$

with the usual conventions that one sets $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

The derived category is written $\mathcal{D}(R)$ and subscript "$\square$" signifies the homological boundedness condition. Thus $\mathcal{D}_\square(R)$ denotes the full subcategory of $\mathcal{D}(R)$ of homologically bounded complexes.

For each $R$- complex $M$, we set

$$\overline{M} = R/xR \otimes^{\mathbf{L}}_R M.$$

Note that $\overline{M}$ is an $R/xR$-complex and as an $R$-complex, it is quasi-isomorphic to the mapping cone of the homothety morphism $x_M : M \to M$. We also denote by $\overline{R}$ the ring $R/xR$.

Remark. Let $M$ be a complex of $R$-modules. It follows from [8, 2.1] that

(i) $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are bounded if and only if $\mathrm{H}(M)$ is bounded.

(ii) $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are trivial if and only if $\mathrm{H}(W)$ is trivial.

**2.1. Auslander and Bass Classes** Recall that a complex $C \in \mathcal{D}^f_\Box(R)$ is said to be *semidualizing* provided that the canonical map $R \to \mathbf{R}\mathrm{Hom}_R(C,C)$ is an isomorphism.

The *Auslander category* (see [1]) with respect to a semidualizing complex $C$ is the full subcategory $\mathcal{A}_R(C)$ of $\mathcal{D}_\Box(R)$ consisting of all $R$-complexes $M$ such that $C\otimes^{\mathbf{L}}_R M \in \mathcal{D}_\Box(R)$ and the canonical morphism $M \to \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R M)$ is an isomorphism.

Dually, the *Bass category* with respect to a semidualizing complex $C$ is the full subcategory $\mathcal{B}_R(C)$ of $\mathcal{D}_\Box(R)$ consisting of all $R$-complexes $N$ such that $\mathbf{R}\mathrm{Hom}_R(C,N) \in \mathcal{D}_\Box(R)$ and the canonical morphism $C\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,N) \to N$ is an isomorphism.

Let $C$ be a semidualizing complex. For $R$-complexes $M$ and $N$ set

$$\triangle_C(M) = \mathrm{Cone}(M \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C\otimes^{\mathbf{L}}_R M))$$

and

$$\Lambda_C(N) = \mathrm{Cone}(C\otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,N) \longrightarrow N).$$

It is clear that the following biimplications hold when $M$ and $N$ are homologically bounded:

$$M \in \mathcal{A}_R(C) \iff C\otimes^{\mathbf{L}}_R M \in \mathcal{D}_\Box(R) \wedge \mathrm{H}(\triangle_C(M)) = 0$$

and

$$N \in \mathcal{B}_R(C) \iff \mathbf{R}\mathrm{Hom}_R(C,N) \in \mathcal{D}_\Box(R) \wedge \mathrm{H}(\Lambda_C(N)) = 0.$$

Note also that for any semidualizing complex $C$,

$$\mathrm{fd}_R\, M \leq \infty \Rightarrow M \in \mathcal{A}_R(C)$$

and

$$\mathrm{id}_R\, N \leq \infty \Rightarrow N \in \mathcal{B}_R(C)$$

(see [1, 4.4]).

LEMMA 2.1. *Let $C$ be a semidualizing complex for $R$ and $\varphi : R \to S$ a ring homomorphism such that $S$ belongs to $\mathcal{A}_R(C)$. Then $C\otimes^{\mathbf{L}}_R S$ is a semidualizing complex for $S$.*

PROOF. See [2, 5.1] □

COROLLARY 2.2. *If $C$ be a semidualizing complex for $R$, then $\overline{C}$ and $C_x$ are semidualizing for $\overline{R}$ and $R_x$, respectively.* □

PROPOSITION 2.3. *Let $C$ be a semidualizing complex for $R$ and suppose that $M$ is a homologically bounded complex of $R$-modules. Then*

*(i)* $M \in \mathcal{A}_R(C) \Longleftrightarrow \overline{M} \in \mathcal{A}_{\overline{R}}(\overline{C}) \wedge M_x \in \mathcal{A}_{R_x}(C_x)$.

*(ii)* $M \in \mathcal{B}_R(C) \Longleftrightarrow \overline{M} \in \mathcal{B}_{\overline{R}}(\overline{C}) \wedge M_x \in \mathcal{B}_{R_x}(C_x)$.

PROOF. We only prove (i). A dual argument proves (ii).

"$\Leftarrow$" Since $\mathrm{H}(\overline{M})$ and $\mathrm{H}(M_x)$ are bounded, (2) shows that $\mathrm{H}(M)$ is bounded. Similarly, $H(C \otimes^{\mathbf{L}}_R M)$ is also bounded. On the other hand, $\mathrm{H}(\Delta_{\overline{C}}(\overline{M})) = H(\overline{\Delta_C(M)})$ and $\mathrm{H}(\Delta_{C_x} M_x) = H((\Delta_C M)_x)$ are both trivial and so is $H(\Delta_C M)$ by (2). Therefore $M \in \mathcal{A}_R(C)$.

"$\Rightarrow$" Follows from [1, 5.8]. $\square$

REMARK. Recall that a *dualizing complex* is a semidulazing complex with finite injective dimension. It follows from (2.2) and [8, 4.2] that if $D$ is a dualizing complex for $R$, then $D_x$ and $\overline{D}$ are dualizing for $R_x$ and $\overline{R}$, respectively.

## 3. Main Results

It is well established that Gorenstein homological dimensions refines many results in classical homological theory of modules. It is also believed that any result in the classical theory has a Gorenstein counterpart. Our main results are the Gorenstein counterpart of some formulae given in [8].

The literature on Gorenstein homological algebra are rich and extensive. Thus, recollecting the basic definitions and facts (in this short paper) seemed to us out of place. We quote here just what we need in establishing our formulae. The reader is referred to the literature for more information. See, for example [1], [4] and [7].

In [4] Christensen, Frankild and Holm showed that the Gorenstein injective dimension $\mathrm{Gid}_R M$ of a homologically bounded complex $M$ can be computed by the following formula:

$$\mathrm{Gid}_R M = \sup\{-\sup \mathbf{R}\mathrm{Hom}_R(U, M) - \sup(U) \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\},$$

and when $R$ is coherent,

$$\mathrm{Gfd}_R M = \sup\{\sup(U \otimes^{\mathbf{L}}_R M) - \sup U \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\},$$

where $\mathcal{I}(R)$ denotes the class of all $R$-complexes with finite injective dimension.

LEMMA 3.1. *Let $\varphi : R \to S$ be a ring homomorphism of coherent rings with $\mathrm{fd}_R\, S < \infty$. Then, for any homologically bounded $R$-complex $M$,*

$$\mathrm{Gfd}_S(S \otimes^{\mathbf{L}}_R M) \leq \mathrm{Gfd}_R\, M.$$

PROOF. By assumption, the forgetful functor $\mathcal{D}(S) \to \mathcal{D}(R)$ gives an embedding $\mathcal{I}(S) \to \mathcal{I}(R)$. For each $S$-complex $U$ with finite injective dimension, we have

$$\sup(U \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R M)) - \sup(U) = \sup(U \otimes^{\mathbf{L}}_R M) - \sup(U).$$

Thus

$$\begin{aligned}
\mathrm{Gfd}_S(S \otimes^{\mathbf{L}}_R M) &= \sup\{\sup(U \otimes^{\mathbf{L}}_S (S \otimes^{\mathbf{L}}_R M)) - \sup(U) \mid U \in \mathcal{I}(S) \wedge \mathrm{H}(U) \neq 0\} \\
&= \sup\{\sup(U \otimes^{\mathbf{L}}_R M) - \sup(U) \mid U \in \mathcal{I}(S) \wedge \mathrm{H}(U) \neq 0\} \\
&\leq \sup\{\sup(U \otimes^{\mathbf{L}}_R M) - \sup(U) \mid U \in \mathcal{I}(R) \wedge \mathrm{H}(U) \neq 0\} \\
&= \mathrm{Gfd}_R\, M.
\end{aligned}$$

Where the inequality holds because $U$ ranges over two different classes, one of which is larger that the other. □

THEOREM 3.2. *Let $R$ be a coherent ring and $M$ a homologically bounded complex. Then*

$$\mathrm{Gfd}_R\, M = \max\{\mathrm{Gfd}_{\overline{R}}\, \overline{M}, \mathrm{Gfd}_{R_x}\, M_x\}.$$

*If every flat $R$-module has finite projective dimension and $M$ has with finitely presented homologies, then*

$$\mathrm{Gpd}_R\, M = \max\{\mathrm{Gpd}_{\overline{R}}\, \overline{M}, \mathrm{Gpd}_{R_x}\, M_x\}.$$

PROOF. "$\leq$" For each $R$-complex $U$ we have

$$\mathrm{H}(\overline{U \otimes^{\mathbf{L}}_R M}) \cong \mathrm{H}(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M})$$

and

$$\mathrm{H}(U \otimes^{\mathbf{L}}_R M)_x \cong \mathrm{H}(U_x \otimes^{\mathbf{L}}_{R_x} M_x).$$

Thus, it follows from [8, 2.2] that

$$\begin{aligned}
\sup(U \otimes^{\mathbf{L}}_R M) - \sup(U) &= \max\{\sup(\overline{U \otimes^{\mathbf{L}}_R M}) - 1, \sup(U \otimes^{\mathbf{L}}_R M)_x\} - \sup U \\
&= \max\{\sup(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M}) - 1 - \sup U, \sup(U_x) \otimes^{\mathbf{L}}_{R_x} M_x - \sup U\} \\
&\leq \max\{\sup(\overline{U} \otimes^{\mathbf{L}}_{\overline{R}} \overline{M}) - \sup \overline{U}, \sup(U_x \otimes^{\mathbf{L}}_R M_x) - \sup U_x\} \\
&\leq \max\{\mathrm{Gfd}_{\overline{R}}\, \overline{M}, \mathrm{Gfd}_{R_x}\, M_x\}.
\end{aligned}$$

The other inequality "$\geq$" follows from (3.1).

For the last assertion, see [4, 3.8. (b)]. □

REMARK. Let $R$ be a coherent ring and $M$ be an $R$-module. When $x$ is a non-zero divisor on $M$, $\overline{M}$ and $M/xM$ are indistinguishable in the derived category $\mathcal{D}(R)$. Thus the previous theorem shows that $M$ is a Gorenstein flat $R$-module if and only of $M/xM$ and $M_x$ are Gorenstein flat modules over the rings $\overline{R}$ and $R_x$ respectively. Similarly one has a criterion for Gorenstein projectivity of finitely presented modules. In particular, if $M$ is finitely generated and $R$ is noetherian, then $M$ is Gorenstein projective if and only if $M/xM$ and $M_x$ are Gorenstein projective over $\overline{R}$ and $R_x$, respectively. This is the Gorenstein counterpart of a result of Vasconcelos [9, 1.6] where he proved the criterion for projectivity without assuming that $R$ is noetherian. It is therefore natural to ask about the validity of the criterion (resp. the formula for Gfd and Gpd) without any condition on the commutative ring $R$.

The following properties of Gorenstein injective dimension are used in proof of the next theorem.

REMARK. For a homologically bounded complex $M$ over a noetherian ring $R$ with dualizing complex $D$, $\operatorname{Gid}_R M$ is finite if and only if $M$ belongs to $\mathcal{B}_R(D)$. Also, if $\operatorname{Gid}_R M$ happens to be finite, then

$$\operatorname{Gid}_R M = \sup\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \operatorname{Spec}(R)\},$$

See [4, Theorem 4.4] and [5, Theorem 2.2].

THEOREM 3.3. *Let $R$ be a noetherian ring with dualizing complex $D$. The following equality holds for any homologically bounded complex $M$:*

$$\operatorname{Gid}_R M = \max\{\operatorname{Gid}_{\overline{R}} \overline{M} + 1, \operatorname{Gid}_{R_x} M_x\}.$$

PROOF. By (3), (2.1) and (2.3) we may assume that both side are finite. To prove "$\leq$", choose a prime ideal $\mathfrak{p}$ such that

$$\operatorname{Gid}_R M = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}.$$

We divide into two cases.

Case I : If $x$ does not belong to $\mathfrak{p}$, set $\mathfrak{q} = \mathfrak{p}_x \in \operatorname{Spec}(R_x)$. Then

$$\begin{aligned} \operatorname{Gid}_R M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &= \operatorname{depth}(R_x)_{\mathfrak{q}} - \operatorname{width}_{(R_x)_{\mathfrak{q}}} (M_x)_{\mathfrak{q}} \\ &\leq \operatorname{Gid}_{R_x} M_x. \end{aligned}$$

Therefore, in this case, the desired inequality holds.

Case II : If $x \in \mathfrak{p}$, then $\mathfrak{q} = \mathfrak{p}/(x) \in \operatorname{Spec}(\overline{R})$ and we have

$$\begin{aligned} \operatorname{Gid}_R M &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &= \operatorname{depth} \overline{R_{\mathfrak{q}}} + 1 - \operatorname{width}_{\overline{R_{\mathfrak{q}}}} (\overline{M})_{\mathfrak{q}} \\ &\leq \operatorname{Gid}_{\overline{R}} \overline{M} + 1. \end{aligned}$$

For the other inequality "$\geq$", choose $\mathfrak{q} \in \operatorname{Spec}(\overline{R})$ such that $\operatorname{Gid}_{\overline{R}} \overline{M} = \operatorname{depth}_R(\overline{R})_{\mathfrak{q}} - \operatorname{width}_{(\overline{R})_{\mathfrak{q}}}(\overline{M})_{\mathfrak{q}}$. There exists $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}/(x) = \mathfrak{q}$ and we have

$$\begin{aligned} \operatorname{Gid}_{\overline{R}} \overline{M} &= \operatorname{depth}_R(\overline{R})_{\mathfrak{q}} - \operatorname{width}_{(\overline{R})_{\mathfrak{q}}}(\overline{M})_{\mathfrak{q}} \\ &= \operatorname{depth} R_{\mathfrak{p}} - 1 - \operatorname{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \operatorname{Gid}_R M - 1. \end{aligned}$$

It remains to prove $\operatorname{Gid}_R M \geq \operatorname{Gid}_{R_x} M_x$. This can be proved in the exact same manner as in [1, 6.2.13]. □

REMARK. Restricted homological dimensions for complexes are defined in [3]. Exactly the same argument as given in the proof of (3.2) shows that the equality

$$\mathsf{R}\operatorname{fd}_R M = \max\{\mathsf{R}\operatorname{fd}_{\overline{R}} \overline{M}, \mathsf{R}\operatorname{fd}_{R_x} M_x\}$$

hold for each homologically bounded complex $M$. Turning to the restricted injective dimensions, it is natural to ask if they satisfy the same equality as one given in (3.3). The method used in our proof of (3.3) needs a Chouinard-type formula. But, to the best of authors' knowledge, this formula only holds under some restricting hypotheses (see [3, 5.12]).

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## References

[1] L. W. Christensen, *Gorenstein dimensions* Lecture Notes in Mathematics, **1747**, Springer-Verlag, Berlin, 2000.

[2] L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353**(5), 1839-1883 (2001)

[3] L. W. Christensen; H. B. Foxby; A. Frankild, *Restricted homological dimensions and Cohen-Macaulayness*, J. Algebra **251** (2002), 479–502.

[4] L. W. Christensen; A. Frankild; H. Holm, *On Gorenstein projective, injective and flat dimensions—a functorial description with applications*, J. Algebra **302** (2006), 231–279.

[5] L. W. Christensen and S. Sather-Wagstaff, *Transfer of Gorenstein dimensions along ring homomorphisms*, J. Pure Appl. Algebra 214 (2010), 982-989.

[6] E. E. Enochs; O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, **30**, Walter de Gruyter & Co., Berlin, 2000.

[7] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), 167–193.

[8] S. Rajabi, B. Tortecillas, S. Yassemi, *Homological Dimensions and Special Base Changes*, to apper in J. Pure Appl. Algebra.

[9] W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. **138** (1969), 505–512.

[10] S. Yassemi, *Width of complexes of modules*, Acta Math. Vietnam, **23** (1998), 161–169.

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