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H-Harmonic Maaß-Jacobi forms of degree 1

The analytic theory of some indefinite theta series

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Abstract

It was shown in previous work that the one-variable $\widehat{\mu}$ -function defined by Zwegers (and Zagier) and his indefinite theta series attached to lattices of signature $(r+1, 1)$ are both Heisenberg harmonic Maaß-Jacobi forms. We extend the concept of Heisenberg harmonicity to Maaß-Jacobi forms of arbitrary many elliptic variables, and produce indefinite theta series of “product type” for non-degenerate lattices of signature $(r+s, s)$. We thus obtain a clean generalization of $\widehat{\mu}$ to these negative definite lattices. From restrictions to torsion points of Heisenberg harmonic Maaß-Jacobi forms, we obtain harmonic weak Maaß forms of higher depth in the sense of Zagier and Zwegers. In particular, we explain the modular completion of some, so-called degenerate indefinite theta series in the context of higher depth mixed mock modular forms. The structure theory for Heisenberg harmonic Maaß-Jacobi forms developed in this paper also explains a curious splitting of Zwegers’s two-variable $\widehat{\mu}$ -function into the sum of a meromorphic Jacobi form and a one-variable Maaß-Jacobi form.

Keywords: Real-analytic Jacobi forms, Generalized $\widehat{\mu}$ -functions, Mixed mock modular forms

Mathematics Subject Classification: Primary 11F50; Secondary 11F27

Background

In his celebrated thesis [34], Zwegers employed the so-called $\widehat{\mu}$ -function to provide an automorphic completion of the until then mysterious mock theta functions. The $\widehat{\mu}$ -function is a real-analytic Jacobi form of one modular and two elliptic variables. A remarkable fact was commented on by Zagier [31]: The “two-variable” $\widehat{\mu}$ -function can be written as the sum of a meromorphic Jacobi form and a real-analytic Jacobi form that only depends on the difference of the two elliptic variables.^a

$$\widehat{\mu}(\tau, u, v) = \frac{\zeta(\tau, u) - \zeta(\tau, v) - \zeta(\tau, u - v)}{\theta(\tau, u - v)} + \widehat{\mu}(\tau, u - v),$$

where ζ is the Weierstrass ζ -function and θ is the Jacobi θ -function. The second term is the “one-variable” $\widehat{\mu}$ -function, which we denote, abusing notation, by the same letter as the original $\widehat{\mu}$ -function. One outcome of the present paper is a natural explanation for this behavior of $\widehat{\mu}$. The construction of $\widehat{\mu}$ can be naturally phrased in terms of indefinite theta series. We extend this construction to more general lattices.

Classical Jacobi forms were defined in Ref. [14] and have been applied in many contexts since then. In some cases, the generating functions of interesting arithmetic quantities turn out to be Jacobi forms [15, 32]; in other cases, classical Jacobi forms and their generalizations have been used to understand the structure of modular forms of different types. For example, Jacobi forms occur as Fourier–Jacobi coefficients of holomorphic and non-holomorphic Siegel modular forms—see [7, 18] for an explanation of how Fourier–Jacobi coefficients can be obtained from the latter. Jacobi forms also serve as a tool to better understand elliptic modular forms, quasimodular forms [33], and mock modular forms. Quasimodular forms, for example, occur as Taylor coefficients of classical Jacobi forms. A more recent accomplishment that is based on Jacobi forms, and which is closely connected to the subject of this paper, is the findings by Zwegers [34]. He provided three different ways, all based on non-classical Jacobi forms, to understand mock modular forms—see [25] for details on the latter. First, he defined the $\widehat{\mu}$ -function, a real-analytic Jacobi form which specializes at certain torsion points to automorphic completions of mock theta functions. Second, he defined indefinite theta series for lattices of signature $(r - 1, 1)$, which are also real-analytic Jacobi forms. They can be employed in a similar way as the $\widehat{\mu}$ -function to understand mock theta functions. Third, Zwegers analyzed Fourier coefficients of meromorphic Jacobi forms, to obtain mock modular forms.

Motivated by this success of real-analytic Jacobi forms (defined in an ad hoc way), several attempts were made to give a precise definition of real-analytic Jacobi forms and, more specifically, harmonic weak Maaß-Jacobi forms. In the past few years, several such definitions, all based on the Casimir operator for the extended real Jacobi group, were suggested by Berndt and Schmidt, Pitale, Bringmann and Richter, Conley and the author, and Bringmann, Richter and the author [5, 7, 10, 11, 24]. To discuss these definitions, recall that Jacobi forms are functions $\phi : \mathbb{H} \times \mathbb{C}^l \rightarrow \mathbb{C}$, depending on a modular variable $\tau \in \mathbb{H} \subset \mathbb{C}$ in the Poincaré upper half plane and elliptic variables $z \in \mathbb{C}^l$. The index of a Jacobi form is an $l \times l$ matrix. A Jacobi form is semi-holomorphic if it is holomorphic as a function of z . The Casimir operator is a certain invariant, central differential operator that annihilates constant functions.

Berndt and Schmidt, and Pitale gave definitions of real-analytic Jacobi forms that were motivated by representation theoretic ideas, therefore restricting themselves to functions that satisfy a polynomial growth condition with respect to the modular variable. By their definition, a real-analytic Jacobi form is an eigenfunction of the Casimir operator. In addition, Berndt and Schmidt require a real-analytic Jacobi form to be an eigenfunction of another differential operator which is invariant, but not central, and which is similar to Δ^H defined in Ref. [7]. This is elaborated on in more detail in “Preliminaries”. Pitale then showed that it suffices to consider semi-holomorphic forms to study smooth vectors in automorphic representations for the extended Jacobi group. This led him to require that Maaß-Jacobi forms be semi-holomorphic eigenfunctions of the Casimir operator.

The work by Bringmann and Richter introduced a new idea. Restricting to functions that are annihilated by the Casimir operator, they relaxed the growth condition, requiring at most exponential growth, and called the Jacobi forms that arise this way harmonic Maaß-Jacobi forms. To distinguish them from the real analytic Jacobi forms mentioned

so far, we will call them harmonic weak Maaß-Jacobi forms. It is important to notice that functions that only satisfy a weak growth condition currently cannot be incorporated into a satisfactory representation theoretic framework. However, they gain importance by the tremendous amount of applications in which harmonic weak Maaß forms [2] and harmonic weak Maaß-Jacobi forms show up—see, for example, [8, 12, 13]. Note that, even though Bringmann and Richter formally did not impose any further condition in a formal way, their work treats only the semi-holomorphic case.

The weak growth condition was used in Ref. [11] to give a definition of semi-holomorphic harmonic weak Maaß-Jacobi forms of the lattice index. In the context of Ref. [11], this type of Jacobi form is relevant because of its connection with Siegel modular forms of higher genus [26]. In Section 3 of [11], it was shown that the vanishing conditions with respect to analogs of the above Δ^H , in this setting, lead to semi-holomorphic functions, if the Jacobi index is not scalar. This is an important observation, which is treated in more detail in Proposition 3.6 of the present paper.

The later work [7] focused on scalar Jacobi indices. When allowing certain kinds of singularities, the class of harmonic weak Maaß-Jacobi forms that are annihilated by Δ^H is strictly larger than the class of semi-holomorphic harmonic weak Maaß-Jacobi forms. A complete structure theory of Heisenberg-harmonic (H-harmonic) Maaß-Jacobi forms with scalar Jacobi indices was built up. We restrict ourselves to reminding the reader that the one-variable $\widehat{\mu}$ -function is an H-harmonic Maaß-Jacobi form in the very sense of Ref. [6].

The definition of H-harmonic Maaß-Jacobi forms

As discussed above, the results on real-analytic Jacobi forms, so far, either restrict to the semi-holomorphic case or to the case of scalar Jacobi indices. The two-variable $\widehat{\mu}$ -function is neither semi-holomorphic, nor has a scalar Jacobi index. To study it as a real-analytic Jacobi form, we develop the theory of H-harmonic Maaß-Jacobi forms of arbitrary indices L that are lattices. To explain the definition, we recall in more detail the main result of Section 3 in [11]. Let L^{JH} and R^{JH} denote the lowering and raising operators with respect to the elliptic variables—explicit expressions are given in “Preliminaries”. These are maps that assign to every $b \in L \otimes \mathbb{R}$ a differential operator, and we write $L^{\text{JH}[b]}$ and $R^{\text{JH}[b]}$ for the images under b . We call them the lowering and raising operator in the direction of b . In Ref. [11], it is shown that, if the rank of L exceeds 1 and L is non-degenerate, then any smooth function from the Jacobi upper half space $\mathbb{H} \times (L \otimes \mathbb{C})$ to \mathbb{C} that is annihilated by $R^{\text{JH}[b]} L^{\text{JH}[b]}$ for all $b \in L \otimes \mathbb{R}$ is holomorphic in z . In other words, a coordinate independent definition of H-harmonic Maaß-Jacobi forms leads to semi-holomorphic forms whenever the Jacobi index is not scalar. From the perspective taken in Ref. [11], coordinate independence is a reasonable assumption, which holds automatically for Maaß-Jacobi forms that are obtained from real-analytic Siegel modular forms and orthogonal modular forms. The two-variable $\widehat{\mu}$ -function, however, is not semi-holomorphic. One is thus led to consider coordinate dependent H-harmonicity. Fixing a basis B of linear independent vectors in $L \otimes \mathbb{R}$, we consider Heisenberg harmonic functions with H-harmonicity B that, by definition, are annihilated by $R^{\text{JH}[b]} L^{\text{JH}[b]}$ for all $b \in B$ —see Definition 3.1. The basic theory of H-harmonic Maaß-Jacobi forms is developed in “H-harmonic Maaß-Jacobi forms”.

Indefinite theta series

It is natural to ask for examples of H-harmonic Maaß-Jacobi forms beyond Zwegers’s $\widehat{\mu}$ -function. We construct theta series for some indefinite lattices, following closely the definition proposed in Ref. [34] in the case of signature $(l_+, 1)$.

Given a lattice L with bilinear form $\langle \cdot, \cdot \rangle_L$ of signature $(l_+, 1)$, Zwegers’s indefinite theta functions depend on a pair (c_1, c_2) of non-positive vectors in $L \otimes \mathbb{R}$. They provide modular completions of

$$\begin{aligned} & \sum_{v \in L} (\operatorname{sgn}\langle c_1, v \rangle_L - \operatorname{sgn}\langle c_2, v \rangle_L) \exp(L[v]\tau + \langle v, b \rangle_L) \\ &= \widetilde{\sum}_{\substack{v \in L \\ \operatorname{sgn}\langle c_1, v \rangle_L = -\operatorname{sgn}\langle c_2, v \rangle_L}} \exp(L[v]\tau + \langle v, b \rangle_L), \end{aligned}$$

where for technical reasons we introduce some suitable $b \in L \otimes \mathbb{R}$ such that the sum converges. The tilde decorating the sum refers to signs and boundary terms that we suppress to present a clearer picture of the construction. The condition $\langle c_1, v \rangle_L$ and $\langle c_2, v \rangle_L$ on the right hand side can be interpreted as restricting summation to a cone in $L \otimes \mathbb{R}$.

For arbitrary lattices, there are several possibilities to restrict summation to achieve convergence. Utilizing the intersection of cones that are defined by two vectors, each is the most straightforward one. Given a set C of pairs of vectors (c_1, c_2) that span mutually orthogonal spaces of $L \otimes \mathbb{R}$, we set

$$\begin{aligned} \operatorname{cone}_L(C) &= \{v \in L : \forall (c_1, c_2) \in C : \operatorname{sgn}\langle c_1, v \rangle_L = -\operatorname{sgn}\langle c_2, v \rangle_L\} \\ &= \bigcap_{(c_1, c_2) \in C} \{v \in L : \operatorname{sgn}\langle c_1, v \rangle_L = -\operatorname{sgn}\langle c_2, v \rangle_L\}. \end{aligned}$$

If $\#C = l_-$, where (l_+, l_-) is the signature of L and if all c_1, c_2 are negative, then

$$\widetilde{\theta}_L^C(\tau, 0) = \widetilde{\sum}_{v \in \operatorname{cone}_L(C)} \exp(L[v]\tau + \langle v, b \rangle_L) \tag{1.1}$$

converges. In “[Indefinite theta series](#)”, we provide their modular completion. As a special case, if we choose negative vectors c_1 and isotropic vectors c_2 , we obtain H-harmonic Maaß-Jacobi forms for $B = \{c_1 : (c_1, c_2) \in C\}$. Their image under the Heisenberg ξ -operator, discussed in Proposition 3.4, is a skew theta series. It is attached to the majorant of L that is defined by taking the negative of span B . In particular, there are plenty of preimages of the same skew theta series, which are distinguished by different choices of bases B of span B .

In case that for all $c_1, c_2 \in L \otimes \mathbb{Q}$, the intersection of $L \otimes \mathbb{Q}$ with $\operatorname{span}\{c_1, c_2\}$ is two-dimensional, the theta series (1.1) can be written as a sum of products of theta series for lattices of signature $(1, 1)$ and one for a positive definite lattice. For this reason, theta series of the kind that we treat are occasionally called theta series of the product type. Note that under these specific assumptions on C , modular completions of (1.1) can be furnished by employing Zwegers’s indefinite theta series. We will elaborate on this in “[Reduction to smaller lattices](#)”.

From a purely philosophical standpoint, one expects that indefinite theta series of product type should show up rather frequently—many counting problems feature degenerate cases. We give two examples, which one of the referees suggested mentioning. In their work on Torus knots, Hikami and Lovejoy encountered an indefinite theta series for which they could not provide a modular completion [17]. A concrete expression can be found in Theorem 5.6 of loc. cit. where the last sum runs over three variables, each restricted with respect to its sign. That is, the cone that appears in a corresponding indefinite theta series has three walls. If this theta series is degenerate, it should be possible to express that cone as a union or suitable intersection of cones defined by two pairs of walls as in (1.1). Specifically, we expect that Hikami’s and Lovejoy’s theta series is, after multiplication with a definite theta series, of the form

$$\sum_{v \in \text{cone}} \exp(L[v]\tau), \quad \text{where } \text{cone} = \{v \in L : \text{sgn}\langle c_1, v \rangle_L = \text{sgn}\langle c_2, v \rangle_L = \text{sgn}\langle c_3, v \rangle_L\}$$

with suitable c_1, c_2, c_3 . The question is whether $\text{cone} = \bigcup_C \text{cone}_L(C)$ for a suitable collection of C s as above.

Another example of possibly degenerate theta series can be found in the work of Lau and Zhou on open Gromov–Witten potentials [21]. Formulas (4.10) and (4.11) therein both feature sums over cones with three walls.

Theta-like decompositions

Another foundation to our understanding of H-harmonic Maaß-Jacobi forms and indefinite theta series are theta-like decompositions as in Ref. [6]. The theta-like decomposition introduced for Maaß-Jacobi forms of scalar Jacobi index in Ref. [6] provides a more flexible way to construct examples. We prove Theorem 5.2, which extends the theta-like decomposition studied so far to the case of arbitrary Jacobi indices. Recall the statement in the case of scalar indices. An H-harmonic Maaß-Jacobi form ϕ of index $-m < 0$ can be written as

$$\sum_{l \pmod{2m}} h_l(\tau) \widehat{\mu}_{m,l}(\tau, z) + \psi(\tau, z),$$

where h_l are the components of a vector-valued elliptic modular form, $\widehat{\mu}_{m,l}$ are functions depending only on m and l , and ψ is a meromorphic Jacobi form. In our setting, such a decomposition result must incorporate additional meromorphic terms. In that matter, it is interesting to note that Bringmann, Creutzig, Rolin, and Zagier recently showed that the meromorphic term ψ also admits a decomposition, namely by means of partial theta functions, if the Jacobi index m is negative [1, 9].

A prototypical decomposition of a Jacobi form of index $\begin{pmatrix} 1 & 0 \\ 0 & -m \end{pmatrix}$, which can still be phrased in terms of Zagier’s $\widehat{\mu}$ -function, is

$$\sum_{l \pmod{2m}} \psi_{m,l}(\tau, z_1) \widehat{\mu}_{m,l}(\tau, z_2) + \psi(\tau, z),$$

where $\psi_{m,l}$ are meromorphic Jacobi forms that depend on ϕ .

The $\widehat{\mu}$ -function by Zagier has been one of the most prominent players in the field of real-analytic modular forms. It also appears in the theta-like decomposition, since its image under the Heisenberg ξ -operator, as shown in Ref. [6], is a unary theta series. We

generalize $\widehat{\mu}$ by employing the previously constructed indefinite theta series. For a negative definite lattice L and an orthogonal basis B of $L \otimes \mathbb{R}$, the images of our functions $\widehat{\mu}_L^B$ under the Heisenberg ξ -operator are anti-holomorphic theta series attached to L .

Splittings of H-harmonic Maaß-Jacobi forms

Zwegers’s $\widehat{\mu}$ -function falls under Definition 3.1, and its splitting can be explained in the setting of H-harmonic Maaß-Jacobi forms. In “Non-trivial H-harmonicities are orthogonal”, we prove that if L is degenerate, then any H-harmonic Maaß-Jacobi form must be a sum of meromorphic and anti-meromorphic functions on $L_0 \otimes \mathbb{R}$, the totally isotropic part of $L \otimes \mathbb{R}$. For example, in the case of Jacobi index $L = \begin{pmatrix} 0 & 0 \\ 0 & -m \end{pmatrix}$, Proposition 3.9 and Theorem 5.2 imply that an H-harmonic Maaß-Jacobi form ϕ can be written as

$$\phi(\tau, (z_1, z_2)) = \psi(\tau, (z_1, z_2)) + \psi_{\overline{\text{hol}}}(\tau, z_1)\phi_{\overline{\text{hol}}}(\tau, z_2) + \psi_{\text{hol}}(\tau, z_1)\phi_{\text{hol}}(\tau, z_2)$$

for a meromorphic function ψ , anti-meromorphic and H-harmonic functions $\psi_{\overline{\text{hol}}}$ and $\phi_{\overline{\text{hol}}}$, and meromorphic and H-harmonic functions ψ_{hol} and ϕ_{hol} , respectively.

Restrictions to torsion points

In the spirit of, for example, [34], it is interesting to study the restrictions of H-harmonic Maaß-Jacobi forms to torsion points. In fact, most of the contemporary theory on mock theta functions is formulated in terms of such restrictions. In “Theta-like decompositions”, we describe their analytic properties. In particular, we connect H-harmonic Maaß-Jacobi forms with Zagier’s and Zwegers’s notion of harmonic weak Maaß forms of higher depth, or equivalently mixed mock modular forms of higher depth—see “Restrictions to torsion points” for a definition.^b Let $\mathbb{M}_k^{[d]}$ denote the space of depth d harmonic weak Maaß forms. Then, for example, any H-harmonic Maaß-Jacobi form ϕ that is not singular at $z = 0$ gives

$$\phi(\tau, 0) \in \mathbb{M}_k^{[d]}.$$

By restricting H-harmonic Maaß-Jacobi forms to torsion points, one obtains sums of products of harmonic weak Maaß forms. Such products cannot be characterized by differential operators, a paucity which let emerge the approach of mixed mock modular forms. Our results reconcile in parts with the approach taken by geometers, who tend to focus on harmonic modular forms, and physicists, who often encounter products of harmonic weak Maaß forms and holomorphic modular forms as generating series. For example, characters of Kac-Moody Lie superalgebras [19, 20] are typically not mock theta functions, but mixed mock modular forms, depending on the signature of the Lie superalgebra’s Cartan matrix.

We suggest studying which of the mixed mock modular forms that have been encountered so far can be obtained as “holomorphic parts” of restrictions of H-harmonic Maaß-Jacobi forms to torsion points. Specifically, if there is, say, a counting problem with coefficients $c(n)$ such that

$$\sum_n c(n) \exp(2\pi i n \tau)$$

is a mixed mock modular form of higher depth, then we suggest trying to refine it in a natural way $c(n) = \sum_r c(n, r)$ such that

$$\sum_{n,r} c(n, r) \exp(2\pi i (n\tau + r(z)))$$

is a “mock Jacobi form”. This has already been carried out in Ref. [4] for Kac–Wakimoto characters that were studied in Ref. [19]. An approach in the same spirit has also been helpful to investigate moments of partition counting functions [3], where Taylor expansions of mock Jacobi forms occurred naturally.

Preliminaries

Lattices

A lattice is a free \mathbb{Z} -module L together with an \mathbb{R} -valued quadratic form $L[\cdot]$ on L . The rank of L will be denoted by l . We say that L is integral if $L[\cdot]$ takes values in \mathbb{Z} . The associated rational, real, and complex spaces are denoted by $L_{\mathbb{Q}} = L \otimes \mathbb{Q}$, $L_{\mathbb{R}} = L \otimes \mathbb{R}$, and $L_{\mathbb{C}} = L \otimes \mathbb{C}$, respectively. There is a bilinear form $\langle z, z' \rangle_L = L[z + z'] - L[z] - L[z']$ canonically attached to L . It extends to a linear form on $L_{\mathbb{C}}$ (i.e., a form which is complex linear in both the first and second component), which we also denote by $\langle \cdot, \cdot \rangle_L$.

A vector $v \in L$ is called isotropic if $L[v] = 0$. The maximal totally isotropic subspace $\{v \in L : \forall v' \in L : \langle v, v' \rangle_L = 0\}$ will be denoted by L_0 , while its dimension is denoted by l_0 . We call L non-degenerate if $l_0 = 0$. Writing (l_+, l_-) for the signature of L , where l_+ and l_- are the dimensions of maximal positive and negative definite subspaces, we therefore have $l = l_+ + l_0 + l_-$. The abbreviation $l_{\pm} = l_+ + l_-$ will appear frequently. It is standard to call L positive or negative semi-definite if $l_- = 0$ or $l_+ = 0$, respectively. We say that L is positive or negative definite, if $l_+ = l$ or $l_- = l$. A totally isotropic lattice satisfies definition $l_0 = l$.

Fixing an ordered basis for L , we can identify L with a Gram matrix $2m = 2m_L \in \text{Mat}_l^T(\mathbb{R})$. The determinant of $2m$ is independent of any choice. We let the reduced covolume $|L|$ be the determinant of the matrix $2m_{L/L_0}$ that is associated with the non-degenerate lattice L/L_0 .

The real dual $L_{\mathbb{R}}^{\vee}$ is defined for arbitrary L and consists of all linear functions on $L_{\mathbb{R}}$. We call $L^{\vee} = \{v^{\vee} \in L_{\mathbb{R}}^{\vee} : \forall v \in L : v^{\vee}(v) \in \mathbb{Z}\}$ the dual of L . It can be identified with $\{v \in L \otimes \mathbb{Q} : \forall v' \in L : \langle v, v' \rangle \in \mathbb{Z}\}$, if L is non-degenerate. In this case, we write $\text{disc}(L)$ for the discriminant module L^{\vee}/L . If L is non-degenerate, then $L^{\vee} \subseteq L_{\mathbb{R}}^{\vee}$ via $v^{\vee}(v) = \langle v^{\vee}, v \rangle_L \in \mathbb{Z}$ for $v^{\vee} \in L^{\vee}$ and $v \in L$. We define a scalar product $\langle \cdot, \cdot \rangle_L$ on $L_{\mathbb{R}}^{\vee}$ as follows. On $(L_0)_{\mathbb{R}}^{\vee}$, it is zero. A complement to $(L_0)_{\mathbb{R}}^{\vee}$ is given by the inclusion $(L/L_0)_{\mathbb{R}} \hookrightarrow L_{\mathbb{R}}^{\vee}$, on which we set $L[\langle v, \cdot \rangle_L] = 4L[v]$ for $v \in L_{\mathbb{R}}$. This quadratic form is the same that arises in the case of $l_0 = 0$ from the inclusion $L^{\vee} \supseteq L$.

The Weil representation associated with a non-degenerate lattice L is a representation of the metaplectic cover $\text{Mp}_2(\mathbb{Z})$ of $\text{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[\text{disc}(L)]$ (see [28]). A natural basis for $\mathbb{C}[\text{disc}(L)]$ is given in terms of e_v , where v runs through $\text{disc}(L)$. By abuse of the notation, we write S and T for the generators of the metaplectic cover of $\text{SL}_2(\mathbb{Z})$ that project to the corresponding generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\text{SL}_2(\mathbb{Z})$. In terms of our basis of $\mathbb{C}[\text{disc}(L)]$ and these generators of $\text{Mp}_2(\mathbb{Z})$, the Weil representation is defined as:

$$\rho_m(T) \epsilon_v := e(L[v]) \epsilon_v, \quad \rho_m(S) \epsilon_v := \frac{1}{\sigma_L \sqrt{2|L|}} \sum_{v' \in \text{disc}(L)} e(-\langle v, v' \rangle_L) \epsilon_{v'}, \tag{2.1}$$

where $\sigma_L = \sqrt{2|L|}^{-1} \sum_{v \in \text{disc}(L)} e(-L[x])$ is the signum of L . Here, we use the shorthand notation $e(x) = \exp(2\pi i x)$ for $x \in \mathbb{C}$ that will appear later again. Throughout the paper, we will pass from $\text{SL}_2(\mathbb{Z})$ to its metaplectic cover whenever necessary and without further mentioning it.

Given any set of vectors v_1, \dots, v_n , we denote their span by $\text{span}\{v_1, \dots, v_n\}$. It will be clear by the context, whether we mean the span over \mathbb{Z}, \mathbb{R} , or \mathbb{C} . The orthogonal complement of v_1, \dots, v_n will be denoted by $\{v_1, \dots, v_n\}^\perp$, or in the case of $n = 1$ by v_1^\perp . For a subset B of $L_{\mathbb{Q}}$, the span of B is denoted by $L_B = L \cap \text{span}_{\mathbb{Q}} B$.

Given a subset B of $L \otimes \mathbb{R}$, we define

$$B_+ = \{b \in B : L[b] > 0\}, \quad B_- = \{b \in B : L[b] < 0\}, \quad B_0 = \{b \in B : L[b] = 0\}. \tag{2.2}$$

Jacobi forms

The Poincaré upper half plane and the Jacobi upper half space attached to a lattice L are

$$\mathbb{H} = \{\tau = x + iy : y > 0\} \subset \mathbb{C}, \quad \text{and} \quad \mathbb{H}^{(L)} = \mathbb{H} \times (L \otimes \mathbb{C}^2).$$

The latter is isomorphic to $\mathbb{H}^{(1)} = \mathbb{H} \times \mathbb{C}^l$ in a non-canonical way. Typically, elements of $\mathbb{H}^{(L)}$ are written as pairs (τ, z) , where $z = u + iv$ with $u, v \in L_{\mathbb{R}}$. Recall the notation $e(x) = \exp(2\pi i x)$ for $x \in \mathbb{C}$. The variable q stands for $e(\tau)$. Given $r \in L_{\mathbb{R}}^\vee$, we set $\zeta^r = e(r(z))$.

Multiplication in the real Jacobi group attached to L

$$G^{(L)} := \text{SL}_2(\mathbb{R}) \ltimes (L \otimes \mathbb{R}^2)$$

is given by

$$(\gamma, \lambda, \mu) \cdot (\gamma', \lambda', \mu') = (\gamma\gamma', (\lambda, \mu)\gamma' + (\lambda', \mu')).$$

Here, a typical element of $G^{(L)}$ is denoted by $g^l = (\gamma, \lambda, \mu)$, where $\lambda, \mu \in L \otimes \mathbb{R}$. The pair (λ, μ) is viewed as an element of $L \otimes \mathbb{R}^2$, on which γ' acts trivial on the first component and by its standard representation from the right on the second component. Note that $G^{(L)}$ is independent of the quadratic form q_L . The discrete subgroup $\Gamma^{(L)} = \text{SL}_2(\mathbb{Z}) \ltimes (L \otimes \mathbb{Z}^2) \subset G^{(L)}$ is called the full Jacobi group.

The action of $\text{SL}_2(\mathbb{R})$ on \mathbb{H} is given by

$$\gamma\tau = \frac{a\tau + b}{c\tau + d},$$

where here and throughout the element γ of $\text{SL}_2(\mathbb{R})$ is written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The real Jacobi group acts on the Jacobi upper half plane via

$$(\gamma, \lambda, \mu)(\tau, z) = \left(\gamma\tau, \frac{z + \lambda\tau + \mu}{c\tau + d} \right).$$

Fix an integer k and a lattice L as above. The action of $G^{(L)}$ on $\mathbb{H}^{(L)}$ admits a cocycle

$$\alpha_L^J(g^J, z) = e\left(\frac{-cL[z]}{c\tau + d} + \langle z, \lambda \rangle_L + L[\lambda]\tau\right), \tag{2.3}$$

which leads to the following slash action on $C^\infty(\mathbb{H}^{(JL)})$.

$$\left(\phi|_{k,L}(\gamma, \lambda, \mu)\right)(\tau, z) = (c\tau + d)^{-k} \alpha_L^J(g^J, z) \phi((\gamma, \lambda, \mu)(\tau, z)). \tag{2.4}$$

This is the usual Jacobi slash action.

We say that a function $\phi : \mathbb{H}^{(JL)} \rightarrow \mathbb{C}$ has non-moving singularities, if there are finitely many linear maps $\lambda^\vee : L_Q Q \rightarrow \mathbb{Q}$ and finitely many $\alpha, \beta \in \mathbb{Q}$, such that the singularities of ϕ are located at $\{z : \lambda(z) = \alpha + \beta\tau\} + L + \tau L$. We say that such a singularity at (τ_0, z_0) has +(i) meromorphic type, (ii) almost meromorphic type, or (iii) real-analytic quotient type, if there is a neighborhood $U \subset \mathbb{H}$ of τ_0 , a function $z_0 : \mathbb{H} \rightarrow L \otimes \mathbb{C}$ with $z_0(\tau_0) = z_0$, and a function $\psi : U \setminus \{(\tau, z_0(\tau)) : \tau \in U\}$ which +(i) is meromorphic, (ii) is the quotient of a real analytic by a holomorphic function, or (iii) is the quotient of two real-analytic functions such that $\phi - \psi$ can be continued to a real-analytic function in the neighborhood of (τ_0, z_0) . A complex valued function on $\mathbb{H}^{(JL)}$ that is meromorphic as a function of the second component z will be called semi-meromorphic. Any such function is called semi-holomorphic if it has no singularities.

Remark 2.1 It appears that the term “non-moving poles” was coined by Zagier. The intuition behind it is that torsion points $\alpha\tau + \beta$, $\alpha, \beta \in L_Q$ should be viewed as fixed while varying τ . A typical example of a modular form with non-moving singularities of meromorphic type is the inverse theta series.

A meromorphic Jacobi form of weight k and index L with non-moving singularities is a meromorphic function $\phi : \mathbb{H}^{(JL)} \rightarrow \mathbb{C}$, such that $\phi|_{k,L} \gamma^J = \phi$ for all $\gamma^J \in \Gamma^{(JL)}$. The space of such forms is denoted by $\mathcal{M}_{k,L}^J$, where \mathcal{M} refers to meromorphicity.

Finally, it is clear that the concept of vector-valued elliptic modular forms extends to Jacobi forms. The Weil representation, previously described as a representation of $SL_2(\mathbb{Z})$ or its metaplectic double cover, can be viewed as representation of the full Jacobi group by defining $\rho_L(\lambda) = \rho_L(\mu) = \text{id}$, the identity.

Differential operators

We discuss the theory of covariant and invariant differential operators for the extended real Jacobi group. The results coincide with the ones in Ref. [11], if L is non-degenerate. Even if L is degenerate, we still employ the results of [11] in a crucial way. The reader is referred to this work for details on the Lie theoretic background. Note that our notation significantly differs from the one that appeared in a previous work. It is close to the nowadays common notation for the classical differential operators for Maaß forms—see page 177 of [22]. This notation seems thus better suited for usage outside the context of Lie groups, but inside the scope of modular forms and Jacobi forms.

Let $D^\infty(\mathbb{H}^{(JL)})$ be the algebra of differential operators on $\mathbb{H}^{(JL)}$. The subalgebra of $G^{(JL)}$ -invariant differential operators for the slash action defined in (2.4) is denoted by $\mathbb{D}(k, L)$. We may regard the imaginary part v of $z \in L_C$ as an element $\langle v, \cdot \rangle_L$ of $L_{\mathbb{R}}^\vee \otimes D^\infty(\mathbb{H}^{(JL)})$,

where $D^\infty(\mathbb{H}^{(JL)})$ is the space of smooth differential operators on $\mathbb{H}^{(JL)}$. We view ∂_z and $\partial_{\bar{z}}$ as elements of $L_{\mathbb{R}}^\vee \otimes D^\infty(\mathbb{H}^{(JL)})$, so that, in particular, $\langle \nu, \partial_z \rangle_L \in D^\infty(\mathbb{H}^{(JL)})$.

$$\begin{aligned} L_{k,L}^J &:= -2iy(y\partial_{\bar{z}} + \langle \nu, \partial_{\bar{z}} \rangle_L), & R_{k,L}^J &:= 2i\left(\partial_z + y^{-1}\langle \nu, \partial_z \rangle_L + 2\pi i y^{-2}L[\nu]\right) + ky^{-1}, \\ L_{k,L}^{JH} &:= -iy\partial_{\bar{z}}, & R_{k,L}^{JH} &:= i\partial_z - 2\pi y^{-1}\langle \nu, \cdot \rangle_L. \end{aligned} \tag{2.5}$$

As is common by now, we suppress the subscripts k, L , if they are clear from the context. We write $L^{JH[\nu]}$ and $R^{JH[\nu]}$ for the evaluation of L^{JH} and R^{JH} , respectively, at $\nu \in L_{\mathbb{R}}$.

Remark 2.2 The superscript J of the raising and lowering operator should remind the reader of Jacobi forms, on which they act. The superscript H refers to the Heisenberg subgroup of $G^{(JL)}$. The lowering and raising operators L^{JH} and R^{JH} both act only on the elliptic variable z , which originates in the Heisenberg subgroup of $G^{(JL)}$.

The commutation relations of lowering and raising operators are

$$\begin{aligned} [L^J, R^J] &= -k, & [L^J, L^{JH}] &= 0, & [L^J, R^{JH}] &= -L^{JH}, \\ [R^J, R^{JH}] &= 0, & [L^{JH}, R^J] &= R^{JH}, & [L^{JH}, R^{JH}] &= -\pi \langle \cdot, \cdot \rangle_L \in L_{\mathbb{R}}^\vee \otimes L_{\mathbb{R}}^\vee. \end{aligned} \tag{2.6}$$

They can be verified readily by means of Helgason’s theory of differential operators [16] as displayed in Ref. [11], and a computation of the images of y or ν .

The commutation relations in (2.6) show that we can view the raising and lowering operators in conjunction with one further element k as generators of an abstract algebra. The additional element k acts on Jacobi forms by multiplication with their weight. The commutation relations of k are

$$[L^J, k] = 2L^J, \quad [R^J, k] = -2R^J, \quad [L^{JH}, k] = L^{JH}, \quad [R^{JH}, k] = -R^{JH}.$$

We write D^J for this algebra, and \mathbb{D}^J for the k centralizer subalgebra of D^J . It consists of elements that act on Jacobi forms as invariant differential operators. To emphasize the dependence on L , we occasionally add the subscript L to D^J and \mathbb{D}^J .

We define a Casimir operator $\mathcal{C}_{k,L}^J \in \mathbb{D}^\infty(\mathbb{H}^{(JL)})$, extending the expression in Ref. [11] to the case of degenerate L . It is given by

$$\begin{aligned} &-2R^JL^J + i\left(R^J\langle L^{JH}, L^{JH} \rangle_L - L^J\langle R^{JH}, R^{JH} \rangle_L\right) - \frac{1}{2}\left(L[R^{JH}]L[L^{JH}] - \langle R^{JH}\langle R^{JH}, L^{JH} \rangle_L, L^{JH} \rangle_L\right) \\ &- \frac{i}{2}(2k - l - 3)\langle R^{JH}, L^{JH} \rangle_L, \end{aligned} \tag{2.7}$$

which equals

$$\begin{aligned} &-2\Delta_{k-l_{\pm}/2} + \frac{y^2}{\pi i}(\partial_{\bar{z}}L[\partial_z] + \partial_zL[\partial_{\bar{z}}]) - 8y\partial_z\langle \nu, \partial_{\bar{z}} \rangle_L \\ &+ \frac{y^2}{32\pi^2}\left(4L[\partial_{\bar{z}}]L[\partial_z] - \langle \partial_z, \partial_{\bar{z}} \rangle_L^2\right) + \frac{y}{2\pi i}\langle \nu, \partial_{\bar{z}} \rangle_L\langle \partial_z, \partial_u \rangle_L - \frac{(2k-l+1)y}{8\pi}\langle \partial_{\bar{z}}, \partial_u \rangle_L \\ &+ 2\langle \nu, \partial_{\bar{z}} \rangle_L\langle \partial_z, \partial_{\bar{z}} \rangle_L + (2k - l_+ - l_- - 1)i\langle \nu, \partial_{\bar{z}} \rangle_L. \end{aligned} \tag{2.8}$$

We easily check that $C^J_{k,L}$ commutes with all invariant differential operators by using that this is the case for non-degenerate L and that C^J depends continuously on $L[\cdot]$.

Suppose that L is degenerate, then differentials with respect to $z \in L_0 \otimes \mathbb{C} \subseteq L_{\mathbb{C}}$ do not occur in C^J . Instead, additional operators arise from the totally isotropic part of L . From the formal element k , we obtain an \mathfrak{sl}_2 -triple (L^J, k, R^J) . As stated before, the kernel of the commutator $[k \cdot, \cdot]$ consists of invariant differential operators. Let

$$D^{JH_0} = \mathbb{C} \left[L^{JH[v]}, R^{JH[v]} : v \in L_0 \otimes \mathbb{R} \right]$$

be the subalgebra of D^J attached to the totally isotropic part of $L_{\mathbb{R}}$.

Proposition 2.3 *The center of D^J_L is generated by C^J , viewed as an element of D^J_L by (2.7), and the \mathfrak{sl}_2 -invariants $H^0(\text{span}\{L^J, k, R^J\}, D^{JH_0})$.*

Remark 2.4 Note that we refer, in the above proposition, to the center of D^J , and not the one of \mathbb{D}^J . The latter might be larger, but we believe that they agree.

Proof We have already asserted that C^J is central in \mathbb{D}^J . From results in Ref. [11], it spans the center of D^J_{L/L_0} . Since also D^{JH_0} is the kernel of $D^J_L \rightarrow D^J_{L/L_0}$, the result follows. \square

Let us fix the notation for the following extra differential operators, which lie in the part of the center of \mathbb{D}^J that arises from $L_0 \otimes \mathbb{R}$.

$$\Delta_L^{JH[v,v']} := L^{JH[v]}R^{JH[v']} - L^{JH[v']}R^{JH[v]}, \quad v, v' \in L_0 \otimes \mathbb{R}, v \neq v'. \tag{2.9}$$

The classical weight k Laplace operator on \mathbb{H} arises from C^J when we consider the lattice L of rank 0, which we provisionally denote by \emptyset .

$$\Delta_k := R^J_{k,\emptyset} L^J_{k,\emptyset} = 4y^2 \partial_{\tau} \partial_{\bar{\tau}} - 2kiy \partial_{\bar{\tau}}. \tag{2.10}$$

It factors as a product $\Delta_k = \xi_{2-k} \xi_k$, where

$$\xi_k f = y^{k-2} \overline{L^J_{k,\emptyset} f} \tag{2.11}$$

is the classical ξ operator that first appeared in Ref. [2].

The Heisenberg Laplace operator is similar to Δ_k , but acts merely on the z -variable of functions on $\mathbb{H}^{(JL)}$. Given $v \in L_{\mathbb{R}}$, we set

$$\Delta_L^{JH[v]} := R^J_{k-1,m} L^J_{k,m} = y \partial_z(v) \partial_{\bar{z}}(v) + 4\pi i \langle v, v \rangle_L \partial_{\bar{z}}(v). \tag{2.12}$$

Harmonic weak Maaß forms

We revisit briefly the theory of harmonic weak Maaß forms. A nice exposition can be found in Ref. [8]. Vector valued elliptic modular forms are invariant under the $|_{k,\rho}$ slash action, which is associated with some weight $k \in \frac{1}{2}\mathbb{Z}$ and a type, i.e., a finite dimensional, complex representation ρ of $SL_2(\mathbb{Z})$ or $Mp_2(\mathbb{Z})$. The slash action is defined by

$$(f|_{k,\rho} \gamma)(\tau) = (c\tau + d)^{-k} \rho(\gamma)^{-1} f(\gamma\tau),$$

where $\gamma \tau$ denotes the action of $SL_2(\mathbb{R})$ on \mathbb{H} . Recall that we pass to the metaplectic cover in case that $k \notin \mathbb{Z}$.

The space of vector-valued modular forms of weight k and type ρ is the space of holomorphic functions $f : \mathbb{H} \rightarrow V$ that satisfy (i) $f|_{k,\rho} \gamma = f$ for all $\gamma \in \Gamma$ and (ii) $f(\tau) = O(1)$ as $y \rightarrow \infty$. We denote this space by $M_k(\rho)$. The space of weakly holomorphic elliptic modular forms is the space of holomorphic functions $f : \mathbb{H} \rightarrow V$ where the second condition is weakened: $f(\tau) = O(e^{ay})$ for some $a > 0$ as $y \rightarrow \infty$. This space is denoted by $M_k^1(\rho)$.

The space of harmonic weak Maaß forms of weight k and type ρ , which is denoted by $\mathbb{M}_k(\rho)$, consists of real-analytic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ satisfying (i) $f|_{k,\rho} \gamma = f$ for all $\gamma \in \Gamma$, (ii) $\Delta_k f = 0$, and (iii) $f(\tau) = O(e^{ay})$ as $y \rightarrow \infty$ for some $a > 0$.

Skew Jacobi forms

We close the preliminaries with the definition of skew(-holomorphic) Jacobi forms of the matrix index. The original definition of skew Jacobi forms of scalar indices m was given in Ref. [29]. It was formulated referring to a slash action:

$$|c\tau + d|^{-1} \overline{(c\tau + d)}^{k-1} e\left(\frac{-m\tau^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right).$$

This allowed choosing positive Jacobi indices m . However, in the case of general Jacobi indices L , such a choice is slightly impractical, when used in conjunction with the differential operator L^{JH} , as we will see later. More importantly, it hides an interesting structural analogy that occurs when passing from real-analytic Jacobi forms to skew Jacobi forms. For this reason, we choose a different equivalent slash action to define skew Jacobi forms.

Given L , fix a maximal negative definite subspace $L_{\mathbb{R}}^{\text{sk}} \subseteq L_{\mathbb{R}}$. Since $L_{\mathbb{R}}^{\text{sk}}$ is non-degenerate, it allows for an orthogonal decomposition of z into z_1 and $z_2 \in L_{\mathbb{R}}^{\text{sk}} \otimes \mathbb{C}$. We write $\lambda = \lambda_1 + \lambda_2$, $\mu = \mu_1 + \mu_2$, and $g_1^{\text{J}} = (g, \lambda_1, \mu_1)$, $g_2^{\text{J}} = (g, \lambda_2, \mu_2)$. For functions $\phi : \mathbb{H}^{(L)} \rightarrow \mathbb{C}$, the skew slash action attached to $L_{\mathbb{R}}^{\text{sk}}$ is defined by

$$\phi|_{k,L}^{\text{sk}[L_{\mathbb{R}}^{\text{sk}}]} g^{\text{J}} = |c\tau + d|^{-l} \overline{(c\tau + d)}^{l-2-k} \alpha_L^{\text{J}}(\gamma_1^{\text{J}}, z_1) \overline{\alpha_L^{\text{J}}(\gamma_2^{\text{J}}, z_2)} (\phi \circ (\gamma, \lambda, \mu)). \tag{2.13}$$

The cocycle α^{J} can be found in (2.3).

Set $L^{\text{sk}} = L \cap L_{\mathbb{R}}^{\text{sk}}$, which is by means of the restriction of $(\cdot, \cdot)_L$ a negative definite lattice. The skew heat operator attached to L is the complex conjugate of the usual one:

$$\mathbb{L}_L^{\text{sk}} = \partial_{\bar{\tau}} + \frac{1}{8\pi i} L[\partial_{\bar{z}}]. \tag{2.14}$$

It annihilates anti-holomorphic theta series

$$\theta_{L^{\text{sk}}, \bar{v}}^{\text{sk}} = \sum_{v \in \bar{v} + L^{\text{sk}}} \bar{q}^{L[v]} \bar{\zeta}^{(z,v)_L} \tag{2.15}$$

that can be defined for $\bar{v} \in (L^{\text{sk}} \otimes \mathbb{Q})/L^{\text{sk}}$:

$$\mathbb{I}_L^{\text{sk}} \theta_{L^{\text{sk}}, \bar{v}}^{\text{sk}}(\tau, z) = \sum_{v \in \bar{v} + L^{\text{sk}}} \left(-2\pi i L[v] + \frac{2}{8\pi i} (2\pi i)^2 L[\langle v, \cdot \rangle_L] \right) \bar{q}^{L[v]} \bar{\zeta}^{\langle z, v \rangle_L},$$

where we have explicitly written the inclusion of $v \in L_{\mathbb{R}}$ into $L_{\mathbb{R}}^{\vee}$. Recall that by definition, we have $L[\langle v, \cdot \rangle_L] = 4L[v]$, which implies that $\mathbb{I}_L^{\text{sk}} \theta_{L^{\text{sk}}, \bar{v}}^{\text{sk}} = 0$.

Definition 2.5 Let L be a lattice, and let $L_{\mathbb{R}}^{\text{sk}}$ and $z = z_1 + z_2$ be as above. A function $\phi : \mathbb{H}^{(JL)} \rightarrow \mathbb{C}$ that is real-analytic, except for non-moving singularities of real-analytic quotient type, is called a skew Jacobi form of weight k , index L , and skewness $L_{\mathbb{R}}^{\text{sk}}$ if it satisfies the following conditions:+

- (i) For all $\gamma^J \in \Gamma^{(JL)}$, we have $\phi|_{k,L}^{\text{sk}[L_{\mathbb{R}}^{\text{sk}}]} \gamma^J = \phi$.
- (ii) The function ϕ is meromorphic in z_1 and antimeromorphic in z_2 .
- (iii) We have $\mathbb{I}_L^{\text{sk}} \phi = 0$.
- (iv) We have $\phi(\tau, z) = O(1)$ as $y \rightarrow \infty$, if $z = \alpha\tau + \beta$, $\alpha, \beta \in L_{\mathbb{R}}$ is a non-singular point of ϕ .

We will denote the space of such skew Jacobi forms by $\mathcal{M}_{k,L}^{\text{sk}[L_{\mathbb{R}}^{\text{sk}}]}$.

Singularities of skew Jacobi forms cannot be located in arbitrary position. The first instance of a related discussion appears in Ref. [6], which we extend to the case of lattice indices.

Proposition 2.6 Let $\phi \in \mathcal{M}_{k,L}^{\text{sk}[L_{\mathbb{R}}^{\text{sk}}]}$. If ϕ is singular along $\{(\tau, z) : v^{\vee}(z) = \alpha\tau + \beta\}$ for $v^{\vee} \in L_{\mathbb{Q}}^{\vee}$ and $\alpha, \beta \in \mathbb{Q}$, then v^{\vee} lies in the orthogonal complement of $\langle L_{\mathbb{R}}^{\text{sk}}, \cdot \rangle_L \subseteq L_{\mathbb{R}}^{\vee}$.

Proof We prove the proposition by contradiction. Suppose that there is $v^{\text{sk}} \in L_{\mathbb{R}}^{\text{sk}}$ with $v^{\vee}(v^{\text{sk}}) \neq 0$ along which some skew Jacobi form ϕ has singularities.

We employ coordinates z_1, \dots, z_l with respect to which L has the form $L[z] = \sum_i s_i z_i^2$ with $s_i \in \{-1, 0, 1\}$. Further, we may assume that $L_{\mathbb{R}}^{\text{sk}} = \text{span}\{z_1, \dots, z_{l^{\text{sk}}}\}$ for $l^{\text{sk}} = \dim L_{\mathbb{R}}^{\text{sk}}$. Since ϕ is anti-meromorphic with respect to z_i , $1 \leq i \leq l^{\text{sk}}$, we have a local Laurent expansion

$$\sum_{n=(n_1, \dots, n_l) \in \mathbb{Z}^l} c_n(\tau) \prod_{i=1}^{l^{\text{sk}}} \overline{(z_i - z_{i,0}(\tau))}^{n_i} \prod_{i=l^{\text{sk}}+1}^l (z_i - z_{i,0}(\tau))^{n_i},$$

where $z_{i,0}$ is linear in τ , and $c_n = 0$ for small enough n_i . By definition, ϕ is annihilated by the skew heat operator, which implies that

$$\sum_n \prod_{i=1}^{l^{\text{sk}}} \overline{(z_i - z_{i,0}(\tau))}^{n_i} \prod_{i=l^{\text{sk}}+1}^l (z_i - z_{i,0}(\tau))^{n_i} \times \left(\partial_{\tau} c_n(\tau) + \sum_{i=1}^{l^{\text{sk}}} \frac{-\partial_{\tau} \overline{z_{i,0}(\tau)}}{z_i - z_{i,0}(\tau)} + \sum_{\substack{i,j=1 \\ i \neq j}}^{l^{\text{sk}}} \frac{n_i n_j}{(z_i - z_{i,0}(\tau))(z_j - z_{j,0}(\tau))} + \sum_{i=1}^{l^{\text{sk}}} \frac{n_i(n_i + 1)}{(z_i - z_{i,0}(\tau))^2} \right) \tag{2.16}$$

vanishes. We show in three steps that $n_i \geq 0$ if $c_n \neq 0$ and $1 \leq i \leq l^{\text{sk}}$. Exploiting symmetry, it suffices to consider the case $i = 1$. Fix n with $c_n \neq 0$ that is minimal with respect to the ordering

$$n' < n \Leftrightarrow \exists i_0 : n_{i_0} < n'_{i_0} \wedge \forall i < i_0 n_i = n'_i. \tag{2.17}$$

First, consider the last term in (2.16), which must vanish if $n_1 < 0$, since the exponent of $z_i - z_{i,0}(\tau)$ that arises from it is minimal with respect to (2.17). From this, we see that $n_1 \geq -1$. Second, if $n_1 = -1$, we inspect the term next to the last term in (2.16). It shows that $n_i = 0$ for $2 \leq i \leq l^{\text{sk}}$. Third, we inspect the second term. If $z_{1,0}$ is constant, it is zero, so that the first term forces $n_1 \geq 0$. Otherwise, it yields the lowest term of (2.16), and hence again $n_1 \geq 0$. \square

The previous proposition is essential, since it allows us to obtain theta decompositions with respect to $L^{\text{sk}} \subseteq L$ for all skew Jacobi forms, as opposed to holomorphic Jacobi forms, for which singularities can occur.

Proposition 2.7 Fix L and L^{sk} as above, and suppose that $L^{\text{sk}} \subseteq L$ splits off as a direct summand: $L = L' \oplus L^{\text{sk}}$ for some lattice L' . Recall the decomposition of z into z_1 and z_2 . For any $\phi \in \mathcal{M}_{k,L}^{\text{sk}[L^{\text{sk}}]}$, we have a theta decomposition

$$\phi(\tau, z) = \sum_{\nu_0 \in \text{disc}(L^{\text{sk}})} \psi_{\nu_0}(\tau, z_1) \theta_{L^{\text{sk}}, \nu_0}^{\text{sk}}(\tau, z_2), \tag{2.18}$$

for a vector-valued meromorphic Jacobi form

$$(\psi_{\nu_0})_{\nu_0 \in \text{disc}(L^{\text{sk}})} \in \mathcal{M}_{k-L-/2, L'}(\rho_{L^{\text{sk}}}).$$

Proof We only provide a sketch of the proof and leave the verification of the details to the reader. Fixing a skew Jacobi form ϕ as in the statement, we have a local Fourier expansion at $(\tau_0, z_0) \in \mathbb{H}^{(lL)}$ that are of the form

$$\sum_{\substack{n \in \mathbb{Z} \\ r \in L^\vee}} c_{(\tau_0, z_0)}(n, r; y, \nu) q^n \zeta^r.$$

Proposition 2.6 says that given $z \in L_{\mathbb{C}}$, the function $z_2 \in L_{\mathbb{C}}^{\text{sk}} \mapsto \phi(\tau, z + z_2)$ has no singularities. In conjunction with invariance of ϕ under the action of $L^{\text{sk}} \otimes \mathbb{Z}^2 \subset \Gamma^{(lL)}$, we find that

$$c_{(\tau_0, z_0)}(n, r; y, \nu) = c_{(\tau_0, z_0)}(n', r + 2r^{\text{sk}}; y, \nu)$$

for all $r^{\text{sk}} \in L^{\text{sk}}$ and with $4n + L[r] = 4n' + L[r + 2r^{\text{sk}}]$. From this, one concludes directly that there is a theta decomposition of the asserted form. \square

Remark 2.8 We assume that L^{sk} splits off from L , but this restriction is not essential. We can always pass to sublattices of L that splits into orthogonal sums and employ vector-valued Jacobi forms. This procedure requires the extension of the full Jacobi group defined in Ref. [11].

H-harmonic Maaß-Jacobi forms

In Ref. [11], Conley and the author provided a definition of H-harmonic Maaß-Jacobi forms for non-degenerate lattice indices. We chose a coordinate-independent definition and were able to show that if the rank of L exceeds 1, then all instances of H-harmonic Maaß-Jacobi forms are semi-meromorphic. In other words, for such lattices, H-harmonic Maaß-Jacobi forms in the sense of [11] fall under the scope of previous considerations.

We now provide a definition of H-harmonic Maaß-Jacobi forms that depends on the choice of a subset B of $\mathbb{P}(L_{\mathbb{R}})$. Elements of B will be frequently identified with preimages in $L_{\mathbb{R}}$ to simplify notation. This section contains the study of basic properties of H-harmonic Maaß-Jacobi forms. The definition is formulated for arbitrary B , but we will see in Propositions 3.6 and 3.11 and Corollary 3.7 that it suffices to consider a finite set of mutually orthogonal, negative vectors. This explains our choice of notation: the reader should think of B as a basis for a negative definite subspace of $L_{\mathbb{R}}$ (very much related to $L_{\mathbb{R}}^{\text{sk}}$ in the previous section).

Definition 3.1 Let L be a lattice, and $B \subseteq \mathbb{P}(L_{\mathbb{R}})$ with $\text{span } B = L_{\mathbb{R}}$. Let $\phi : \mathbb{H}^{(JL)} \rightarrow \mathbb{C}$ be real-analytic except for non-moving singularities of real-analytic quotient type. We say that ϕ is an H-harmonic Maaß-Jacobi form of weight k , index L , and H-harmonicity B if it satisfies the following conditions: +

- (i) For all $\gamma^J \in \Gamma^{(JL)}$, we have $\phi|_{k,L} \gamma^J = \phi$.
- (ii) The function ϕ is annihilated by the center of $\mathbb{D}_{k,L}^J$. In particular, we have $\mathcal{C}_{k,L}^J \phi = 0$.
- (iii) We have $\Delta^{\text{H}[b]} \phi = 0$ for all $b \in B$.
- (iv) The growth condition $\phi(\tau, \alpha\tau + \beta) = O(e^{ay})$ as $y \rightarrow \infty$ is satisfied for some $a > 0$ provided that $z = \alpha\tau + \beta, \alpha, \beta \in L_{\mathbb{R}}$ is a non-singular point of ϕ .

We denote the space of all such H-harmonic Maaß-Jacobi forms by $\mathcal{M}_{k,L}^{\Delta, \text{H}[B]}$. The notation for spaces of vector-valued Jacobi forms is analogous to the one for elliptic modular forms, introduced below. We refer to a representation ρ of $\Gamma^{(JL)}$ in parenthesis.

Remark 3.2

- (1) We assume in Definition 3.1 that H-harmonic Maaß-Jacobi forms have singularities of real-analytic quotient type. However, it follows from the theory that we develop in this paper that they automatically have almost meromorphic singularities. This is a consequence of Theorem 5.2, which in this regard is based on Proposition 2.7.
- (2) The definition extends to half-integral weights and indices, and to complex representations of $\Gamma^{(JL)}$. We do not treat these cases explicitly, as they lead to additional technical difficulties and do not yield further insight. Zwegers’s $\hat{\mu}$ -function, however, strictly speaking does not fall under Definition 3.1. Confer [6] for more details on the latter.
- (3) Equally well, one can define H-harmonic Maaß-Jacobi forms for any $\Gamma = \Gamma' \times L^2$ where $\Gamma' \subset \text{SL}_2(\mathbb{Z})$ has finite index.
- (4) An analogous definition can be made for the skew Jacobi slash action, which subsumes the image of $\xi^{\text{H}[b]}$ defined below.

Various spaces of H-harmonic Maaß-Jacobi forms

In analogy with the ideas that were presented in Ref. [6], we define several subspaces of $\mathcal{M}_{k,L}^{\Delta,H[B]}$, and remark on how they might be studied. It should be clear that not all of them appear in the present work, but it seems advantageous to suggest uniform notation for sequels to this paper.

We will very soon restrict our attention to $\mathcal{M}_{k,L}^{\mathbb{H}[B]}$, which is defined by

$$\mathcal{M}_{k,L}^{\mathbb{H}[B]} = \mathcal{M}_{k,L}^{\mathbb{J},\mathbb{H}[B]} := \mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{H}[B]} \cap \ker L^{\mathbb{J}}.$$

This space is very much analogous to the one primarily studied in Ref. [6]. We will deduce a theta-like decomposition for the forms in it in “Theta-like decompositions”. The notation defined here for general H-harmonic Maaß-Jacobi forms transfers directly to those annihilated by $L^{\mathbb{J}}$.

If L is non-degenerate, and $B \subseteq L_{\mathbb{R}}$ does not span $L_{\mathbb{R}}$, then we set

$$\mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{H}[B]} := \mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{H}[B+B^{\perp}]} \cap \ker_{b \in B^{\perp}} L^{\mathbb{H}[b]}.$$

We refrain from extending this definition to degenerate lattices, since $L_0 \otimes \mathbb{R} \subset B^{\perp}$ for every B , so that the notation would become ambiguous. Instead, we set for $B_{\text{hol}} \subseteq \mathbb{P}(L_{\mathbb{R}})$ with $B_{\text{hol}} + \text{span } B = L_{\mathbb{R}}$

$$\mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{h}[B_{\text{hol}}]\mathbb{H}[B]} := \mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{H}[B+B_{\text{hol}}]} \cap \ker_{b \in B_{\text{hol}}} L^{\mathbb{H}[b]}.$$

In accordance with the notation in Ref. [6], we write $\mathcal{M}_{k,L}^{\mathbb{J},\Delta,h}$ for $\mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{h}[L_{\mathbb{R}}]} = \mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{h}[L_{\mathbb{R}}]\mathbb{H}[\emptyset]}$.

Remark 3.3

- (1) The theory of Maaß-Jacobi forms in $\mathcal{M}_{k,L}^{\mathbb{J},\Delta,h}$ was developed in Ref. [11]. It resembles strongly the theory of harmonic weak Maaß forms.
- (2) A theory similar to the one developed in Ref. [6] could be developed for $\mathcal{M}_{k,L}^{\mathbb{J},\Delta,\mathbb{H}[B]}$ by considering the restrictions.
- (3) If L is non-degenerate, then $\mathcal{M}_{k,L}^{\mathbb{H}} = \mathcal{M}_{k,L}^{\mathbb{J}}$ is the space of meromorphic Jacobi forms with non-moving singularities of meromorphic type.

The Heisenberg ξ -operators

As in the case of $l = 1$, dealt with in Ref. [6], there is a Heisenberg ξ -operator, which we define now. Given $v \in L_{\mathbb{R}}$, we set

$$\xi_L^{\mathbb{H}[b]} := \frac{\text{sgn}(L[b] \langle v, b \rangle_L)}{\sqrt{|L[b]y|}} \exp\left(\frac{-\pi \langle v, b \rangle_L^2}{L[b]y}\right) L_L^{\mathbb{H}[b]}. \tag{3.1}$$

If B is a set of orthogonal vectors, none of which is isotropic, we set $\xi^{\mathbb{H}[B]} = \prod_{b \in B} \xi^{\mathbb{H}[b]}$.

Proposition 3.4 *Fix a subspace $L_{\mathbb{R}}^{\text{sk}} \subseteq L_{\mathbb{R}}$ that contains no totally isotropic vector. Given a set $B \subset \mathbb{P}(L_{\mathbb{R}})$ of mutually orthogonal vectors that are also orthogonal to $L_{\mathbb{R}}^{\text{sk}}$, and none of which is isotropic, we let $L_{\mathbb{R}}^{\text{sk}} + B$ be the span of $L_{\mathbb{R}}^{\text{sk}}$ and B . Then for any smooth function ϕ on $\mathbb{H}^{(L)}$ and any $g^{\mathbb{J}} \in G^{(L)}$, we have*

$$\xi_L^{\text{H}[B]} (\phi_{k,L}^{\text{sk}[L^{\text{sk}}]} g^J) = (\xi_L^{\text{H}[B]} \phi)_{k,L}^{\text{sk}[L^{\text{sk}}+B]} g^J.$$

Proof It suffices to treat the case $B = \{b\}$. We write $z_b = b\langle z, b \rangle_L / 2L[b]$ and $\lambda_b = b\langle \lambda, b \rangle_L / 2L[b]$. Using the known covariance of L^{H} and $y^{\frac{-1}{2}}$, we only need to establish that

$$\begin{aligned} & \exp\left(\frac{-\pi \langle v, b \rangle_L^2}{L[b]y}\right) e\left(\frac{-cL[z_b + \lambda_b\tau]}{c\tau + d} + \langle z_b, b \rangle_L + L[\lambda_b]\tau\right) \\ &= e\left(\frac{-cL[\bar{z}_b + \lambda_b\bar{\tau}]}{c\bar{\tau} + d} + \langle \bar{z}_b, b \rangle_L + L[\lambda_b]\bar{\tau}\right) \left(\exp\left(\frac{-\pi \langle v, b \rangle_L^2}{L[b]y}\right)\right)_{\tau \mapsto \frac{-1}{\bar{\tau}}, z \mapsto \frac{z}{\bar{\tau}}}. \end{aligned}$$

Checking this for $g^J = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0, 0\right)$ and for $g^J = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \lambda, 0\right)$ separately establishes the statement. \square

Remark 3.5 The image of $\xi^{\text{H}[B']}$ with $B' \subseteq B$ applied to $\mathcal{M}_{k,L}^{\Delta, \text{H}[B]}$, in general, is an instance of skew H-harmonic Maaß-Jacobi forms, referred to in Remark 3.2(4). If $B' = B$, we obtain proper skew Jacobi forms as in Definition 2.5. This is a consequence of Proposition 3.15.

Non-trivial H-harmonicities are orthogonal

Definition 3.1 of H-harmonic Maaß-Jacobi forms refers to a set $B \subseteq \mathbb{P}(L_{\mathbb{R}})$. However, it suffices to study sets of mutually orthogonal vectors. The argument that we employ to show this is similar to the one in Ref. [11]. But the degenerate case does not allow for a quite straightforward generalization.

Proposition 3.6 *For any $B \subseteq \mathbb{P}(L_{\mathbb{R}})$, we have*

$$\mathcal{M}_{k,L}^{\text{H}[B]} \subseteq \mathcal{M}_{k,L}^{\text{h}[B \neq 0] \text{H}[B=0]}, \quad \text{where} \quad \begin{aligned} B_{=0} &= \{b' \in B : \forall b \in B, b \neq b' : \langle b, b' \rangle_L = 0\}, \\ B_{\neq 0} &= \{b' \in B : \exists b \in B, b \neq b' : \langle b, b' \rangle_L \neq 0\}. \end{aligned} \tag{3.2}$$

Proof Fix $b, b' \in B$ with $b \neq b'$ and $\langle b, b' \rangle_L \neq 0$. We lift both to elements of $L_{\mathbb{R}}$, denoting these again by b and b' . In case that $b - b' \in L_0 \otimes \mathbb{R}$, then replace, without loss of restriction, b' by $2b'$. In particular, we may assume that $b - b'$ is not totally isotropic.

We have to show that any $\phi \in C^\infty(\mathbb{H}^{(J,L)})$ that is annihilated by $\Delta^{\text{H}[b]}$ and $\Delta^{\text{H}[b']}$ is already annihilated $L^{\text{H}[b]}$ and $L^{\text{H}[b']}$. Now, the remainder of the proof is the same as the proof of Theorem 3.4 of [11]. For convenience of the reader, we reproduce a variation of it. It suffices to consider functions $a(y, v)$ that are annihilated by $\Delta^{\text{H}[b]}$ and $\Delta^{\text{H}[b']}$. The commutator

$$[\Delta^{\text{H}[b]}, \Delta^{\text{H}[b']}] = \pi \langle b, b' \rangle_L (R^{\text{H}[b']L^{\text{H}[b]}} - R^{\text{H}[b]L^{\text{H}[b]'}})$$

also annihilates $a(y, v)$, which implies that

$$\partial_\nu(b') a(y, \nu) = \frac{\langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L} \partial_\nu(b) a(y, \nu).$$

Because

$$\partial_\nu(b') \frac{\langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L} = \frac{\langle b', b' \rangle_L \langle \nu, b \rangle_L - \langle b, b' \rangle_L \langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L^2},$$

we find that $\Delta^{\mathbb{H}[b']}$ acts on $a(y, \nu)$ as

$$\frac{\langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L} \partial_\nu(b)^2 + \left(\frac{-2\pi \langle \nu, b' \rangle_L^2}{y \langle \nu, b \rangle_L} + \frac{\langle b', b' \rangle_L \langle \nu, b \rangle_L - \langle b, b' \rangle_L \langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L^2} \right) \partial_\nu(b).$$

We now consider the action of $\Delta^{\mathbb{H}[b']} - \frac{\langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L} \Delta^{\mathbb{H}[b]}$, which is

$$\left(\frac{-2\pi \langle \nu, b' \rangle_L^2}{y \langle \nu, b \rangle_L} + \frac{\langle b', b' \rangle_L \langle \nu, b \rangle_L - \langle b, b' \rangle_L \langle \nu, b' \rangle_L}{\langle \nu, b \rangle_L^2} + \frac{2\pi \langle \nu, b' \rangle_L \langle \nu, b' \rangle_L^2}{y \langle \nu, b \rangle_L \langle \nu, b \rangle_L} \right) \partial_\nu(b).$$

Therefore $a(y, \nu)$ is constant in direction of b if the first factor is not zero. Since this factor is a non-trivial polynomial in $\langle \nu, b \rangle_L$ and $\langle \nu, b' \rangle_L$, this implies the statement. \square

Corollary 3.7 *Let $B \subseteq \mathbb{P}(L \otimes \mathbb{R})$ be a set of mutually orthogonal vectors that spans a maximal non-degenerate subspace of $L_{\mathbb{R}}$. Then*

$$\mathcal{M}_{\mathbb{J}_{k,L}^{\mathbb{H}[B]}} = \mathcal{M}_{\mathbb{J}_{k,L}^{\mathbb{h}[B_0]\mathbb{H}[B_{\pm}]}}$$

where $B_0 = \{b \in B : L[b] = 0\}$ and $B_{\pm} = \{b \in B : L[b] \neq 0\}$ are the subsets of isotropic and non-isotropic vectors in B .

Proof By the assumption that span B is maximal non-degenerate, for each $b \in B_0$ there is $b' \in B$ with $\langle b, b' \rangle_L \neq 0$. Proposition 3.6 therefore implies the corollary. \square

The totally isotropic subspace of $L_{\mathbb{R}}$ would pose particular technical problems, if only vanishing with respect to the Casimir operator is imposed. However, Definition 3.1 also features the degenerate central invariant differential operators $\Delta^{\mathbb{H}[\nu, \nu']}$ given in (2.9). They enforce reasonable analytic behavior on $\mathbb{H} \times (L_0 \otimes \mathbb{C}) \subseteq \mathbb{H}^{(I)L}$.

Lemma 3.8 *Any orthogonal set $B \subseteq \mathbb{P}(L_{\mathbb{R}})$ with span $B = L_{\mathbb{R}}$ contains a generating set of $L_0 \otimes \mathbb{R}$.*

Proof Passing to the quotient L/L_0 , observe that span $B = (L/L_0) \otimes \mathbb{R}$. Further, vectors in $L/L_0 \otimes \mathbb{R}$ are orthogonal to each other if and only if they are so in $L_{\mathbb{R}}$. Thus, we see that B contains exactly $l_+ + l_-$ vectors that are not totally isotropic. The remaining vectors must span $L_0 \otimes \mathbb{R}$, since B spans $L_{\mathbb{R}}$. \square

Proposition 3.9 *Any $\phi \in \mathcal{M}_{\mathbb{J}_{k,L}^{\mathbb{H}[B]}}$ is either holomorphic or anti-holomorphic with respect to $L_0 \otimes \mathbb{C} \subseteq L_{\mathbb{C}}$.*

Proof Lemma 3.8 tells us that B_0 generates $L_0 \otimes \mathbb{R}$. By assumption, ϕ is annihilated by $\Delta^{\text{H}[b]}$ for $b \in B_0$. If ϕ is constant with respect to every $b \in B_0$, then we are done. Assuming this is not the case, we can fix one b so that either $L^{\text{H}[b]} \phi \neq 0$ or $R^{\text{H}[b]} \phi \neq 0$. Further, fix $v \in L_0 \otimes \mathbb{R}$ which is not a multiple of b .

Since both b and v are totally isotropic, $L^{\text{H}[v]} = -iy\partial_{\bar{z}}(v)$, $R^{\text{H}[v]} = i\partial_z(v)$, $L^{\text{H}[b]} = -iy\partial_{\bar{z}}(b)$, and $R^{\text{H}[b]} = i\partial_z(b)$ commute. Therefore, ϕ vanishing under $\Delta^{\text{H}[v]}$ and $\Delta^{\text{H}[b]}$ is equivalent to ϕ being locally a sum of a holomorphic and an anti-holomorphic function in the direction of v and b each.

Consider the equation $\Delta^{\text{H}[b,v]} \phi = 0$ which is implied by Condition (ii) of Definition 3.1 in conjunction with (2.9). It means that

$$L^{\text{H}[v]} R^{\text{H}[b]} \phi = R^{\text{H}[v]} L^{\text{H}[b]} \phi.$$

The left hand side is holomorphic with respect to b , and the right hand side is anti-holomorphic with respect to b . Since ϕ is a sum of a holomorphic and an anti-holomorphic function in b , this implies that both the left and the right hand side are zero. Therefore, if $L^{\text{H}[v]} \phi \neq 0$, then $R^{\text{H}[b]} \phi = 0$, and if $R^{\text{H}[v]} \phi \neq 0$, then $L^{\text{H}[b]} \phi = 0$. This proves the claim. \square

For reference, we fix the notation for spaces that are meromorphic and anti-meromorphic on $L_0 \otimes \mathbb{R}$

$$\begin{aligned} \mathcal{M}_{\mathbb{J}^{\text{h}[B_{\text{hol}}] \text{h}_0 \text{H}[B]}} &= \mathcal{M}_{\mathbb{J}^{\text{h}[B_{\text{hol}}] \text{H}[B+L_0 \otimes \mathbb{R}]}} \cap \ker_{b \in L_0 \otimes \mathbb{R}} L^{\text{H}[b]}, \\ \mathcal{M}_{\mathbb{J}^{\text{h}[B_{\text{hol}}] \bar{\text{h}}_0 \text{H}[B]}} &= \mathcal{M}_{\mathbb{J}^{\text{h}[B_{\text{hol}}] \text{H}[B+L_0 \otimes \mathbb{R}]}} \cap \ker_{b \in L_0 \otimes \mathbb{R}} R^{\text{H}[b]}. \end{aligned} \tag{3.3}$$

Example 3.10 Recall that $\zeta(\tau, z)$ is the Weierstass ζ function, and $\theta(\tau, z)$ is the Jacobi theta function.

$$\begin{aligned} \theta(\tau, z) &= \sum_{n \in \frac{1}{2} + \mathbb{Z}} (-1)^{n-\frac{1}{2}} q^{n^2} \zeta^n = q^{\frac{1}{8}} \zeta^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1} q^{n-1}) \quad \text{and} \\ \zeta(\tau, z) &= \frac{\partial_z \theta(\tau, z)}{\theta(\tau, z)} = \frac{1}{z} + \sum_{\substack{z_0 \in \mathbb{Z} + \tau\mathbb{Z} \\ z_0 \neq 0}} \left(\frac{1}{z - z_0} + \frac{1}{z_0} + \frac{z}{z_0^2} \right). \end{aligned}$$

From this representation of θ and ζ , we directly read off their zeros and poles.

The splitting of Zwegers’s two-variable $\widehat{\mu}$ -function is well known, but in [31] Zagier only remarked on it in a footnote. Zwegers’s definition of $\widehat{\mu}$ is

$$\begin{aligned} \widehat{\mu}(\tau, z_1, z_2) &= \frac{e^{\pi i z_1}}{\theta(\tau, z_2)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n e^{\left(\frac{n^2+n}{2} \tau + n z_2\right)}}{1 - e^{(n\tau + z_1)}} \\ &\quad - \frac{i}{2\sqrt{\pi}} \sum_{n \in \frac{1}{2} + \mathbb{Z}} \left(-\text{sgn}(n) - H^{\text{H}(-1,1)}(y, u_1 - u_2; n) \right) (-1)^{n+\frac{1}{2}} e^{\left(\frac{-n^2}{2} \tau + n(z_1 - z_2)\right)}, \end{aligned} \tag{3.4}$$

which we obtain after unraveling Formula (1.8) in Ref. [34]. Its index, Theorem 1.11 of [34] states, is the matrix $m = \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Note that here we fix a basis for the corresponding lattice with Gram matrix m .

The one-variable $\widehat{\mu}$ -function was never defined by Zwegers in a formal way, and Zagier [31] simply asserted the existence of a one-variable $\widehat{\mu}$ -function. So, we have to prove that

$$\widehat{\mu}(\tau, z_1, z_2) = \frac{\zeta(\tau, z_1) - \zeta(\tau, z_2) + \zeta(\tau, z_1 - z_2)}{\theta(\tau, z_1 - z_2)} \tag{3.5}$$

transforms like a Jacobi form and only depends on $z_1 - z_2$.

The two-variable $\widehat{\mu}$ function has simple poles at $z_2 \in \mathbb{Z} + \tau\mathbb{Z}$ [from the prefactor of the first term of (3.4)] and at $z_1 \in \mathbb{Z} + \tau\mathbb{Z}$ [from the denominator inside the sum of the first term of (3.4)]. The residue at $z_1 = 0$ is easily seen to be $\theta(\tau, z_2)^{-1}$, and the symmetry of $\widehat{\mu}(\tau, z_1, z_2)$ stated in Theorem 1.11 (3) of [34] shows that the residue at $z_2 = 0$ is $\theta(\tau, z_1)^{-1}$. The poles of $\zeta(\tau, z)$ lie at $z \in \mathbb{Z} + \tau\mathbb{Z}$. One thus quickly checks that the function in (3.5) has poles at $z_1 - z_2 \in \mathbb{Z} + \tau\mathbb{Z}$.

We conveniently inspect the modular properties of $\zeta(\tau, z_1) - \zeta(\tau, z_2) + \zeta(\tau, z_1 - z_2)$ by means of [27]—also refer to [23]—which implies it is a Jacobi form of index 0. The quotient by $\theta(\tau, z_1 - z_2)$ yields a meromorphic Jacobi form of index m , the same as the one of $\widehat{\mu}(\tau, z_1, z_2)$. Therefore, (3.5) displays a real analytic Jacobi form of index m whose poles are supported on $z_1 - z_2 \in \mathbb{Z} + \tau\mathbb{Z}$.

The image of $\widehat{\mu}(\tau, z_1, z_2)$ under $\xi^{\mathbb{H}[b_2]}$ vanishes, from which we infer that $\widehat{\mu}$ is meromorphic in the direction of $b_2 = (1, 1)$, and so is (3.5). From our above considerations of poles and residues, we further find that it is holomorphic in the direction of b_2 . Fixing $z_1 - z_2 \in \mathbb{C} \setminus (\mathbb{Z} + \tau\mathbb{Z})$ and letting $z_1 + z_2$ vary, we obtain a holomorphic Jacobi form of index $m[(1, 1)] = 0$. By the classical theory of Jacobi forms—see, for example, Theorem 1.2 of [14]—it is constant. This establishes the splitting of the two-variable $\widehat{\mu}$ -function.

Fourier expansions

Fourier expansions are crucial to understand further properties of H-harmonic Maaß-Jacobi forms, even more so, since they allow us to reason about restrictions. Any term of the Fourier expansion is indexed by $n \in \mathbb{R}$ and $r \in L_{\mathbb{R}}^{\vee}$. The overall situation is similar to the case of scalar Jacobi indices: the space of Fourier coefficients $c(n, r; y, \nu)q^n \zeta^r$ that are annihilated by \mathcal{C}^1 and $\Delta^{\mathbb{H}[b]}$ for $b \in B, B \subseteq \mathbb{P}(L_{\mathbb{R}})$ is finite dimensional, if $\text{span } B = L_{\mathbb{R}}$.

Maaß-Jacobi forms with non-moving singularities admit a local Fourier expansion as explained in Ref. [6]. For $\phi \in \mathcal{M}_{k,L}^{\mathbb{H}[B]}$, we denote it by

$$\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r \in L^{\vee}} c(\phi; n, r; y, \nu) q^n \zeta^r. \tag{3.6}$$

Note that, if L is degenerate, then L^{\vee} cannot be canonically identified with a subspace of $L_{\mathbb{Q}}$, as is common otherwise.

Propositions 3.6 and 3.9 allow us to focus on $B \subseteq \mathbb{P}(L_{\mathbb{R}})$ that consists of mutually orthogonal, negative vectors, and $B_{\text{hol}} \subseteq \mathbb{P}(L_{\mathbb{R}})$ which is orthogonal to B and contains no isotropic vectors. Our goal is to describe (3.6) explicitly

$$\phi \in \mathcal{M}_{\mathbb{J}_{k,L}^{\text{h}[B_{\text{hol}}] \text{h}_0 \text{H}[B]}} \quad \text{and for} \quad \phi \in \mathcal{M}_{\mathbb{J}_{k,L}^{\text{h}[B_{\text{hol}}] \bar{\text{h}}_0 \text{H}[B]}}.$$

We split $z \in L_{\mathbb{C}}$ into $z_{\text{hol}} \in \text{span } B_{\text{hol}}$, $z_0 \in L_0 \otimes \mathbb{C}$, and $z_b = \langle z, b \rangle_L b/2L[b] \in \mathbb{C}b$ for $b \in B$, such that $z = z_{\text{hol}} + z_0 + \sum_b z_b$.

Besides the holomorphic terms, two building blocks suffice to explicitly describe Fourier expansions. The function H , up to slight modifications, appeared in Ref. [2] first. It is usually employed to describe Fourier expansions of harmonic weak Maaß forms. The function $H^{\text{H}[b]}$, adopted from Ref. [6], is adjusted to the setting of lattice indices. It features the lower incomplete gamma function $\gamma(s, y) = \int_0^y t^{s-1} e^{-t} dt$. We set

$$H(y; D) := e^{-y} \int_{-2y}^{\infty} e^{-t} t^{-k+l_{\pm}/2} dt \Big|_{y=\pi Dy/2|L|}, \quad \text{if } D \neq 0, |L| \neq 0; \tag{3.7}$$

$$H(y; D) := y^{-k+l_{\pm}/2}, \quad \text{if } D = 0; \text{ and} \tag{3.8}$$

$$H^{\text{H}[b]}(y, v; r) := \text{sgn}(\langle v + ry, b \rangle_L) \gamma\left(\frac{1}{2}, -\pi \frac{\langle v + ry, b \rangle_L^2}{L[b]y}\right), \quad \text{if } L[b] < 0. \tag{3.9}$$

If the integral representation for H does not converge, we mean its analytic continuation with respect to k . If $D < 0$, one of the special functions above simplifies, and we find that

$$H(y; D) = \exp(-\pi Dy/2|L|) \Gamma\left(1 + \frac{l_{\pm}}{2} - k, -Dy/|L|\right),$$

where $\Gamma(s, y) = \int_y^{\infty} t^{s-1} e^{-t} dt$ is the upper incomplete gamma function.

Given $n \in \mathbb{R}$ and $r \in L_{\mathbb{R}}^{\vee}$, we call

$$D = D_L(n, r) := |L|(4n - L[r])$$

the discriminant of (n, r) . We suppress the dependence of $D_L(n, r)$ on L, n , and r , which will be clear from the context.

As an important side product to our study of Fourier expansion, we obtain further restrictions on non-trivial H-harmonic Maaß Jacobi forms.

Proposition 3.11 *Given an orthogonal set $B \subseteq \mathbb{P}(L_{\mathbb{R}})$ of non-isotropic vectors, then we have*

$$\mathcal{M}_{\mathbb{J}_{k,L}^{\text{H}[B]}} = \mathcal{M}_{\mathbb{J}_{k,L}^{\text{h}[B_+] \text{H}[B_-]}}$$

where $B_{\pm} = \{b \in B : \text{sgn}(L[b]) = \pm 1\}$.

Proposition 3.12 *Let $\phi \in \mathcal{M}_{\mathbb{J}_{k,L}^{\Delta \text{h}[B_{\text{hol}}] \text{h}_0 \text{H}[B]}}$ be an H-harmonic Maaß-Jacobi with local Fourier expansion as in (3.6). With the same assumptions on B_{hol} and B as above, for fixed n and $r, c(\phi; n, r; y, v)$ it is an element of the complex vector space spanned by*

$$\prod_{b \in B'} H^{\text{H}[b]}(y, v; r), \quad \text{and} \quad H(y; D) \prod_{b \in B'_-} H^{\text{H}[b]}(y, v; r), \quad \text{for some } B' \subseteq B. \tag{3.10}$$

Analogously, local Fourier expansions of $\phi \in \mathcal{M}_{\mathbb{J}_{k,L}}^{\Delta \text{h}[B_{\text{hol}}] \bar{\text{h}}_0 \text{H}[B]}$ lie in the space spanned by

$$e(2r(iv_0)) \prod_{b \in B'} H^{\text{H}[b]}(y, v; r), \quad \text{and} \quad H(y; D) e(2r(iv_0)) \prod_{b \in B'_-} H^{\text{H}[b]}(y, v; r), \quad \text{for some } B' \subseteq B. \tag{3.11}$$

The proof of Proposition 3.12 is a complicated calculation, the parts of which we split up into three separate propositions.

Proposition 3.13 Given $b \in L_{\mathbb{R}}$ that is not isotropic, and a smooth function a , we have

$$\Delta^{\text{H}[b]} a \left(\frac{\langle v, b \rangle_L^2}{4|L[b]|y} \right) = \frac{|L[b]| \Delta_{\frac{1}{2}} a(y) e(\text{sgn}(L[b])\tau)}{y e(\text{sgn}(L[b])\tau)} \Big|_{y = \frac{\langle v, b \rangle_L^2}{4|L[b]|y}},$$

where $\Delta_{1/2}$ is the classical weight $\frac{1}{2}$ Laplace operator defined in (2.10). Every solution to $\Delta^{\text{H}[b]}$ can be written as a function depending on L, b, y , and v only by means of $\langle v, b \rangle_L^2 / 4|L[b]|y$.

More specifically, in the case of $L[b] < 0$ solutions to $\Delta^{\text{H}[b]} a(y, v; r) \zeta^r$ for some $r \in L_{\mathbb{R}}^{\vee}$ are linear combinations of the constant function and

$$a(y, v; r) = \text{sgn}(\langle v + ry, b \rangle_L) \gamma \left(\frac{1}{2}, -\pi \frac{\langle v + ry, b \rangle_L^2}{L[b]y} \right).$$

Proposition 3.14 Let B be as above, and fix $n \in \mathbb{R}$ and $r \in L_{\mathbb{R}}^{\vee}$. Given solutions $a_b(y, \langle v, b \rangle_L)$ to the differential equations

$$\Delta^{\text{H}[b]} a_b(y, \langle v, b \rangle_L) \zeta^r = 0, \quad b \in B,$$

set $a_B(y, v) = \prod_{b \in B} a_b(y, \langle v, b \rangle_L)$. Then the differential equations

$$C_{k,L}^{\downarrow} a(y) a_B(y, v) q^n \zeta^r = 0 \quad \text{and} \quad C_{k,L}^{\downarrow} a(y) e(2r(iv_0)) a_B(y, v) q^n \zeta^r = 0$$

are both equivalent to

$$\Delta_{k-L_{\pm}/2} a(y) q^{D/4|L|} = 0.$$

Proposition 3.15 Given $b \in L_{\mathbb{R}}$ that is not isotropic, let $a(y, v)$ be a real smooth function that is annihilated by $\Delta^{\text{H}[b]}$. Then, $\xi_L^{\text{H}[b]} a(y, v)$ is constant, and $\xi_{k,L}^{\downarrow} a(y, v) = 0$. More specifically, if $L[b] < 0$ then

$$\xi_L^{\text{H}[b]} H^{\text{H}[b]}(v, y; 0) = 2\sqrt{\pi}.$$

Proofs of Proposition 3.13, 3.14, and 3.15 can be found below.

Proof of Proposition 3.11 Assume that there is some $\phi \in \mathcal{M}_{\mathbb{J}_{k,L}}^{\text{H}[B]}$ and some $b \in B_+$ such that $\xi_L^{\text{H}[b]} \phi \neq 0$. We choose a maximal subset \tilde{B} of B that contains b and satisfies

$$\xi_L^{\text{H}[\bar{B}]} \phi \neq 0.$$

By Propositions 3.4 and 3.15, this is a non-trivial skew Jacobi form. Further, by Proposition 2.6, it cannot have any singularity in direction of b , which by a standard argument contradicts $L[b] > 0$. For details, the reader is referred to Theorem 1.2 in [14]. It should be kept in mind that we use opposite signs for Jacobi indices. \square

Proof of Proposition 3.12 Proposition 3.11 allows us to focus on the case $B = B_-$. Applying Proposition 3.13 to the case of negative $L[b]$ shows that any Fourier expansion is of the form

$$a(y) \prod_{b \in B_-} H^{\text{H}[b]}(y, \nu; r).$$

Then Proposition 3.14 establishes the statement. \square

Proof of Proposition 3.13 Recall that $\Delta^{\text{H}[b]} = -iy(i\partial_z(b) - 2\pi y^{-1} \langle \cdot, \nu \rangle_L) \partial_{\bar{z}}(b)$. A straightforward computation yields

$$\Delta^{\text{H}[b]} a \left(\frac{\langle \nu, b \rangle_L^2}{4L[b]y} \right) = \frac{\langle \nu, b \rangle_L^2}{4y} a'' \left(\frac{\langle \nu, b \rangle_L^2}{4L[b]y} \right) + \left(\frac{|L[b]|}{2} - \frac{\pi \langle \nu, b \rangle_L^2}{\text{sgn}(L[b])y} \right) a' \left(\frac{\langle \nu, b \rangle_L^2}{4L[b]y} \right).$$

On the other hand, $\Delta_{\frac{1}{2}} = 4y^2 \partial_\tau \partial_{\bar{\tau}} - iy \partial_{\bar{\tau}}$, from which we get

$$\Delta_{\frac{1}{2}} a(y) e(\text{sgn}(L[b])\tau) = \left(y^2 a''(y) + \left(\frac{y}{2} - 4\pi \text{sgn}(L[b])y^2 \right) a'(y) \right) e(\text{sgn}(L[b])\tau).$$

This completes the proof of the first part. The second part follows by combining the covariance of $\Delta^{\text{H}[b]}$ with respect to the action of $G^{(L)}$ and

$$\Delta_{\frac{1}{2}} \gamma \left(\frac{1}{2}, 4\pi y \right) e(-\tau) = 0.$$

\square

Proof of Proposition 3.14 Invariance of $\mathcal{C}_{k,L}^j$ and $\Delta^{\text{H}[b]}$ under the action of the real Jacobi group implies that we can restrict ourselves to the case $r = 0$. Further, it suffices to treat the case of non-degenerate lattices L , since derivatives with respect to $L_0 \otimes \mathbb{R}$ do not occur in \mathcal{C}^j . Then, $D/4|L| = n$.

We first deduce three relations of differential operators, which hold under the mere assumption that $L[b] \neq 0$ for all $b \in B$. Sums and products in these formulas, if not indicated differently, run over elements in B . Since $L[b] \neq 0$, Proposition 3.13 implies that $a_b(y, \nu) = \tilde{a}_b \left(\frac{\langle \nu, b \rangle_L^2}{2|L[b]|y} \right)$ for some suitable function \tilde{a}_b . We deduce that $\partial_y a_b(y, \nu)$ equals

$$\frac{-\langle \nu, b \rangle_L^2}{4|L[b]|y^2} \tilde{a}'_b \left(\frac{\langle \nu, b \rangle_L^2}{4|L[b]|y} \right) = \frac{-\langle \nu, b \rangle_L}{4L[b]y} \frac{\langle \nu, b \rangle_L}{\text{sgn}(L[b])y} \tilde{a}'_b \left(\frac{\langle \nu, b \rangle_L^2}{4|L[b]|y} \right) = \frac{-\langle \nu, b \rangle_L}{4L[b]y} \partial_\nu(b) a_b(y, \nu). \tag{3.12}$$

Applying Relation (3.12) to $\prod a_b(y, \nu)$ and using orthogonality of B , we find that

$$\partial_y \prod_{b \in B} a_b(y, \nu) = \sum_{b \in B} \frac{-\langle \nu, b \rangle_L \partial_\nu(b)}{4L[b]y} \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu) \tag{3.13}$$

$$\partial_y^2 \prod_{b \in B} a_b(y, \nu) = \left(\sum_{b, b' \in B} \frac{\langle \nu, b \rangle_L \langle \nu, b' \rangle_L \partial_\nu(b) \partial_\nu(b')}{16L[b]L[b']y^2} + \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{4L[b]y^2} \right) \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu). \tag{3.14}$$

In addition, using the equation $\Delta^{\text{H}[b]} a_b(y, \nu) = 0$, we infer that

$$\partial_\nu(b)^2 a_\nu(y, \nu) = \frac{4\pi \langle \nu, b \rangle_L \partial_\nu(b)}{y} a_b(y, \nu). \tag{3.15}$$

The actual proof of Proposition 3.14 is a computation, which we start by expanding Equation (2.14) for \mathcal{C}^J with respect to derivatives ∂_τ and $\partial_{\bar{\tau}}$ that are applied to the factor $a(y)q^n$.

$$\begin{aligned} \mathcal{C}_{k,L}^J a(y)q^n \prod_b a_b(y, \nu) &= -2(\Delta_{k-l_{\pm}/2} a(y)q^n) \prod_b a_b(y, \nu) \\ &\quad - 8y^2 (\partial_\tau a(y)q^n) \partial_{\bar{\tau}} \prod_b a_b(y, \nu) \end{aligned} \tag{3.16}$$

$$- 8y^2 (\partial_{\bar{\tau}} a(y)q^n) \partial_\tau \prod_b a_b(y, \nu) \tag{3.17}$$

$$+ \frac{y^2}{\pi i} \left((\partial_\tau a(y)q^n) L[\frac{i}{2} \partial_\nu] + (\partial_{\bar{\tau}} a(y)q^n) L[\frac{-i}{2} \partial_\nu] \right) \prod_b a_b(y, \nu) \tag{3.18}$$

$$- 8y (\partial_\tau a(y)q^n) \langle \nu, \partial_{\bar{z}} \rangle_L \prod_b a_b(y, \nu) \tag{3.19}$$

$$+ a(y)q^n \mathcal{C}_{k,L}^J \prod_b a_b(y, \nu). \tag{3.20}$$

To establish the proposition, it suffices to show that the sum of the last five terms (3.16) through (3.20) vanishes.

The sum of (3.16) and (3.17) equals

$$-2y^2 (2a'(y)q^n - 4\pi n a(y)q^n) \partial_y \prod_{b \in B} a_b(y, \nu) = y(a'(y) - 2\pi n a(y))q^n \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{L[b]} \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu)$$

by Relation (3.13). In the term (3.18), terms involving $a'(y)$ cancel. To treat the derivative $L[\partial_\nu]$ applied to $\prod a_b(y, \nu)$, we recall one elementary fact. Since B consists of mutually orthogonal vectors, we have

$$\partial_\nu \prod_{b \in B} a_b(y, \nu) = \sum_{b \in B} \frac{\langle b, \cdot \rangle_L}{2L[b]} \partial_\nu(b) \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu). \tag{3.21}$$

Since $L[\langle b, \cdot \rangle_L] = 4L[b]$, we find that (3.18) yields

$$\begin{aligned} \frac{-ny^2}{2} a(y) q^n L[\partial_\nu] \prod_{b \in B} a_b(y, \nu) &= \frac{-ny^2}{2} a(y) q^n \sum_{b \in B} \frac{4L[b]}{4L[b]^2} \partial_\nu(b)^2 \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu) \\ &= -2\pi n y a(y) q^n \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{L[b]} \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu) \end{aligned}$$

by Relation (3.15). Finally, the term (3.19) can be expanded in a straightforward way:

$$-y(a'(y) - 4\pi n a(y)) q^n \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{L[b]} \prod_{\tilde{b} \in B} a_{\tilde{b}}(y, \nu).$$

By what we have computed so far, we see that (3.16) + (3.17) + (3.18) + (3.19) = 0.

To prove the proposition, we are therefore reduced to showing that (3.20) vanishes. This, in turn, is the same as showing that $C_{k,L}^J \prod_b a_b(y, \nu) = 0$. Considering expression (2.8) for $C_{k,L}^J$, we see that no terms involving ∂_u contributes. The second term in (2.8) does not contribute either, because ∂_τ and $\partial_{\bar{\tau}}$ cancel each other. The order four term in (2.8) also does not contribute, as is easily seen by means of (3.21).

We are left with the following expression for $C_{k,L}^J \prod_b a_b(y, \nu)$, which originates in the first, third, seventh, and eighth term of (2.8).

$$\left(-2y^2 \partial_y^2 - (2k - l_\pm) y \partial_y - 2y \partial_y \langle \nu, \partial_\nu \rangle_L - \frac{1}{2} \langle \nu, \langle \nu, \partial_\nu \rangle_L, \partial_\nu \rangle_L - \frac{2k - l_\pm - 1}{2} \langle \nu, \partial_\nu \rangle_\nu \right) \prod_{b \in B} a_b(y, \nu).$$

We simplify it by employing (3.13), (3.14), and (3.21). It suffices to show vanishing of:

$$\begin{aligned} & - \sum_{b,b'} \frac{\langle \nu, b \rangle_L \langle \nu, b' \rangle_L \partial_\nu(b) \partial_\nu(b')}{8L[b]L[b']} - \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{2L[b]} + (2k - l_\pm) \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{4L[b]} \\ & + \sum_{b,b' \in B} \frac{\langle b', \nu \rangle_L \partial_\nu(b') \langle b, \nu \rangle_L \partial_\nu(b)}{2L[b']} - \frac{1}{2} \sum_{b,b' \in B} \frac{\langle b, \nu \rangle_L \langle b', \nu \rangle_L \partial_\nu(b) \partial_\nu(b')}{4L[b]L[b']} - \frac{2k - l_\pm - 1}{2} \sum_{b \in B} \frac{\langle \nu, b \rangle_L \partial_\nu(b)}{2L[b]}. \end{aligned}$$

The first, the order 2 contribution of the fourth, and the fifth term cancel. The order 1 contribution of the fourth term is $\sum_b \langle b, \nu \rangle_L \partial_\nu(b) / 2L[b]$, so that the remaining terms also cancel. This completes the proof. \square

Proof of Proposition 3.15 The statement on ξ^J is a consequence of (3.21) in the proof of Proposition 3.14.

Our proof of the first statement on $\xi^{JH[b]}$ uses Proposition 3.13. Let $a(y)$ be a solution to the differential equation $\Delta_{\frac{1}{2}} a(y) e(\text{sgn}(L[b])\tau)$. If $a(y) \neq 1$, we can choose a scalar multiple of $a(y)$ in such a way that

$$\xi_{\frac{1}{2}} a(y) e(\text{sgn}(L[b])\tau) = e(-\text{sgn}(L[b])\tau), \quad \text{therefore } a'(y) = y^{-\frac{1}{2}} \exp(4\pi \text{sgn}(b)y).$$

The proposition then follows from a straightforward computation. \square

Indefinite theta series

In this section, we slightly generalize the results of Chapter 2 of [34]. Zwegers defined indefinite theta series for lattices of signature $(r + 1, 1)$, $r \geq 0$. These theta series are H-harmonic, as can be deduced from Proposition 3.12. Zwegers also remarked that his construction generalizes to arbitrary, non-degenerate lattices. Here, we make a small step toward the most general type of indefinite theta series by considering the ones of product type (this terminology is explained in the introduction). In the previous section, we have seen that in our setting, we may only expect real analytic contributions from the negative definite part of L . Further, such real analytic terms in the H-harmonic setting must be associated with mutually orthogonal directions in L . One possibility for achieving this is by considering contribution from orthogonal subspaces of $L_{\mathbb{R}}$ that have signature $(1, 1)$. Many cases that we consider here could be alternatively constructed using products of Zwegers’s indefinite theta series. However, in “Reduction to smaller lattices”, we show that this is not always true. On the other hand, we would like to emphasize that the intention behind this section is not to construct all indefinite theta series. In fact, even when restricting to H-harmonic indefinite theta series, the presented construction does not exhaust all cases. Rather, in the present section, we provide some non-trivial examples to bring the previously developed theory to life.

Throughout this section, we assume that L is non-degenerate. We repeat Zwegers’s construction on orthogonal pieces of signature $(1, 1)$, given by a set

$$C = \{(c_1, c_2) \in \mathbb{P}(L_{\mathbb{R}})^2 : \text{sgn}(\text{span}\{c_1, c_2\}) = (1, 1), \langle c_1, c_2 \rangle_L < 0\}$$

of size L_- . For any distinct pairs (c_1, c_2) and (c'_1, c'_2) in C , we assume that $\text{span}\{c_1, c_2\}$ is orthogonal to $\text{span}\{c'_1, c'_2\}$. A set C with this property will be called a $(1, 1)$ -decomposition of L . As an obvious restriction on the existence of $(1, 1)$ -decompositions, we have $L_+ \geq L_-$. Indeed, $\text{span } C \subseteq L_{\mathbb{R}}$ has signature (L_-, L_-) .

To define the indefinite theta series, we need an analogue of Zwegers’s ρ function for $c \in \mathbb{P}(L_{\mathbb{R}})$ with $L[c] \leq 0$:

$$\begin{aligned} \rho^c(\tau, z; \nu) &= \text{sgn}(\langle c, \nu \rangle_L), & \text{if } L[c] = 0; \text{ and} \\ \rho^c(\tau, z; \nu) &= H^{\text{H}[c]}(y, \nu; \nu), & \text{if } L[c] < 0. \end{aligned} \tag{4.1}$$

Definition 4.1 Let C be a $(1, 1)$ -decomposition of L . The vector-valued (indefinite) theta series of L attached to C is defined by

$$\theta_L^C(\tau, z) = \sum_{\bar{v} \in \text{disc } L} \epsilon_{\bar{v}} \sum_{\nu \in \bar{v} + L} e(L[\nu]\tau + \langle z, \nu \rangle_L) \prod_{(c_1, c_2) \in C} (\rho^{c_1} - \rho^{c_2})(\tau, z; \nu). \tag{4.2}$$

Example 4.2 (Zwegers’s indefinite theta series for lattices of signature $(r - 1, 1)$) Zwegers analyzed indefinite theta series for Lorentzian lattices. For the time being, we adopt the notation from Chapter 2 of [34]. Starting with a quadratic space V of signature $(r - 1, 1)$, $r \geq 2$, we choose isotropic or negative vectors $c_1, c_2 \in \mathbb{R}^r$. For any such choice, one obtains an indefinite theta series

$$\theta^{c_1, c_2}(\tau, z) = \sum_{\nu \in \mathbb{Z}^r} \rho^{c_1, c_2}(\nu; \tau) e(L[\nu]\tau + \langle z, \nu \rangle_L),$$

where, in the present paper's notation, we have

$$\rho^{c_1, c_2}(v; \tau) = \rho^{c_1}(\tau, z; v) - \rho^{c_2}(\tau, z; v).$$

We can recover θ^{c_1, c_2} as the ϵ_0 -component of θ_L^C with $C = \{(c_1, c_2)\}$, as long as c_1 is not a multiple of c_2 . If $c_1 = c_2$ as elements of $\mathbb{P}(V_{\mathbb{R}})$, then $\theta^{c_1, c_2} = 0$.

Indefinite theta series have singularities, whose location is determined by the geometry of C . We set

$$D(C) := \{(\tau, z) \in \mathbb{H}^{(1, L)} : \langle b_1, v/y \rangle_L, \langle b_2, v/y \rangle_L \notin \mathbb{Z} \text{ for all } (b_1, b_2) \in C\}.$$

Proposition 4.3 *Given a $(1, 1)$ -decomposition of L , the theta series θ_L^C converges absolutely and locally uniformly for any $z \in D(C)$. It can be analytically continued to $\mathbb{H}^{(1, L)}$, except for non-moving singularities of almost meromorphic type.*

The proof of Proposition 4.3 relies on a reduction to rational c_i s and c_j s. The next lemma allows us, in addition, to replace isotropic vectors by negative ones.

Lemma 4.4 *Let C be a $(1, 1)$ -decomposition of L . Fix $(c_1, c_2) \in C$ such that for $i = 1$ and $i = 2$, we have $\mathbb{R}c_i \cap L_{\mathbb{Q}} = \emptyset$ or $L[c_i] = 0$. Then there is $\tilde{c} \in \mathbb{P}(L_{\mathbb{Q}})$ with $L[\tilde{c}] < 0$, such that both*

$$(\tilde{c}, c_2) \cup C \setminus \{(c_1, c_2)\} \quad \text{and} \quad (c_1, \tilde{c}) \cup C \setminus \{(c_1, c_2)\}$$

are $(1, 1)$ -decompositions of L .

Proof We analyze the following Grassmannian of negative definite, rational subspaces

$$H = \text{Gr}_- \{v \in L_{\mathbb{Q}} : \forall (c'_1, c'_2) \in C, (c'_1, c'_2) \neq (c_1, c_2) : \langle v, c'_i \rangle_L = 0\}.$$

Since $\text{span}\{c_1, c_2\}$ has signature $(1, 1)$, $(C \setminus \{(c_1, c_2)\})^\perp$ has signature $(l_+ - l_- + 1, 1)$. Therefore, H is a manifold of dimension at least 1. By our assumption on c_1 and c_2 , we find that $\mathbb{R}c_1, \mathbb{R}c_2 \notin H$. It thus suffices to choose any $\mathbb{Q}\tilde{c} \in H$. □

The next proof is essentially due to Zwegers [34]. A little care must be taken when splitting off $\text{span}\{c_1, c_2\}$ for $(c_1, c_2) \in C$, and for this reason we give some details. For our purpose, it is also important that we prove that θ_L^C has non-moving singularities of almost meromorphic type. This fact was not mentioned in Ref. [34], even though it is immediate from the treatment given there.

Proof of Proposition 4.3 Using Lemma 4.4, we can successively write θ_L^C as the sum of $\theta_{\tilde{C}}^C$ s where for each $(c_1, c_2) \in \tilde{C}$ at least one of the c_i s is rational and negative. After swapping c_1 and c_2 if necessary, we can thus assume without loss of generality that $L[c_1] < 0$ and $c_1 \in L_{\mathbb{Q}}$ for all $(c_1, c_2) \in C$.

It also suffices to show convergence for $\bar{v} = 0$. Indeed, given $\bar{v} \in \text{disc}(L)$, we can replace z by $z + v$ for a representative v of \bar{v} . The set $D(C)$ remains unchanged under this substitution, since $v \in L^\vee$.

Under these hypotheses, we can proceed in a similar way as Zwegers. It will be convenient to write z as $a + \tau b$ for $a, b \in L_{\mathbb{R}}$. In particular, we have $v/y = a$. Let $\beta(y) = \int_y^\infty t^{-\frac{1}{2}} e^{-\pi t} dt$ be the beta function. We have

$$H^{\text{H}[c]}(y, v; v) = \sqrt{\pi} \operatorname{sgn}(\langle a + v, c \rangle_L) \left(1 - \beta(-y \langle a + v, c \rangle_L^2 / L[c]) \right).$$

So, up to scalar multiples of $\sqrt{\pi}$, we can write $\rho^{c_1} - \rho^{c_2}$ as a sum or difference of terms

$$\begin{aligned} & \operatorname{sgn}(\langle a + v, c_1 \rangle_L) \beta(-y \langle a + v, c_1 \rangle_L^2 / L[c_1]), \\ & \operatorname{sgn}(\langle a + v, c_2 \rangle_L) \beta(-y \langle a + v, c_2 \rangle_L^2 / L[c_2]), \quad \text{and} \\ & \operatorname{sgn}(\langle a + v, c_1 \rangle_L) - \operatorname{sgn}(\langle a + v, c_2 \rangle_L). \end{aligned} \tag{4.3}$$

We have to estimate the product $\prod_{(c_1, c_2)} (\rho^{c_1} - \rho^{c_2})$. The orthogonality relation imposed on elements of C allows us to focus on each term individually, so that Zwegers’s estimates apply word by word for negative c_1 or c_2 .

We can thus focus on the case that $c_1 \in L_{\mathbb{Q}}$ is negative and c_2 is isotropic. For simplicity, write $C_0 \subseteq C$ for the set (c_1, c_2) with isotropic c_2 . We discuss the contribution of the third term in (4.3) in more detail, since we need it to understand the singularities of θ_L^C . We are free to replace c_1 by a scalar multiple of itself without changing θ_L^C . In particular, we may assume, by Lemma 4.4, that $c_1 \in L$.

Given $(c_1, c_2) \in C_0$, we decompose $a + v = \mu + nc_1$ with $n \in \mathbb{Z}$ and $0 \leq \langle c_2, \mu \rangle_L < \langle c_2, c_1 \rangle_L$. As in the proof of Proposition 2.4 in [34], we can use the following equality, since $z \in D(C)$.

$$\begin{aligned} & \sum_{\substack{n \in \mathbb{Z} \\ (c_1, c_2) \in C}} \left(\operatorname{sgn}(\langle \mu, c_2 \rangle_L) - \operatorname{sgn}\left(n + \frac{\langle \mu, c_1 \rangle_L}{\langle c_1, c_2 \rangle_L}\right) \right) e(\langle \mu, c_2 \rangle n \tau + \langle b, c_2 \rangle n) \\ & = \frac{2}{1 - e(\langle \mu, c_2 \rangle_L \tau + \langle b, c_2 \rangle_L)} - \delta(\langle \mu, c_1 \rangle_L), \end{aligned} \tag{4.4}$$

where $\delta(0) = 1$ and $\delta(t) = 0$ for $t \neq 0$.

Because $(c_1, c_2) \in C$ are mutually orthogonal, we can in fact decompose $a + v = \mu + \sum_{(c_1, c_2) \in C_0} n_{c_1}$. By passing to the corresponding quotient of L , we see that

$$\begin{aligned} \mathcal{M}(C_0) &= \left\{ \mu \in a + L : \forall (c_1, c_2) \in C_0 : 0 \leq \langle c_2, \mu \rangle_L < \langle c_2, c_1 \rangle_L \right\} / \\ & \quad \left(\operatorname{span}\{c_1 : (c_1, c_2) \in C_0\} \right)^\perp \end{aligned}$$

is finite. Up to sign, the contribution to θ_L^C under consideration thus equals

$$\begin{aligned} & \sum_{\substack{\mu \in \mathcal{M}(C_0) \\ v \in (\operatorname{span}\{c_1 : (c_1, c_2) \in C_0\})^\perp}} e(L[\mu + v]\tau + \langle z, \mu + v \rangle_L) \prod_{(c_1, c_2) \in C \setminus C_0} (\rho^{c_1} - \rho^{c_2})(\tau, z, \mu + v) \\ & \times \prod_{(c_1, c_2) \in C_0} \left(\frac{2}{1 - e(\langle \mu + v, c_2 \rangle_L \tau + \langle b, c_2 \rangle_L)} - \delta(\langle \mu, c_1 \rangle_L) \right). \end{aligned} \tag{4.5}$$

This reduces to convergence of $\theta_{L'}^{C \setminus C_0}$ with $L' = (\operatorname{span}\{c_1 : (c_1, c_2) \in C_0\})^\perp$, for which we have already referred the reader to Ref. [34]. From this point on, it is clear how to establish the convergence of θ_L^C as in Ref. [34].

We are left with proving that θ_L^C has non-moving singularities of almost meromorphic type. Observe that (4.4) can be continued to all except, finally, many $b \pmod L$. Therefore, singularities of (4.5) arise from products of meromorphic functions with real-analytic ones. This completes the proof. \square

We can go one step beyond Proposition 4.3, by establishing uniform convergence. The next lemma will help us to prove Theorem 4.6.

Lemma 4.5 *Fix a $(1, 1)$ -decomposition C of L , and one pair $(c_1, c_2) \in C$. Let \tilde{c} be a continuous function $\tilde{c} : [0, \infty) \rightarrow \text{span}\{c_1, c_2\}$. Assume that $L[\tilde{c}(t)] < 0$, $L[c_2] = 0$, and $\tilde{c}(0) = c_2$, and that $\text{span}\{c_1\tilde{c}(t)\}$ has signature $(1, 1)$ for all $t > 0$.*

Set $C(t) = \{(\tilde{c}(t), c_2)\} \cup C \setminus \{(c_1, c_2)\}$. Then, $\lim_{t \rightarrow 0} \theta_L^{C(t)} = \theta_L^C$.

Proof The proof of Proposition 2.7 (4) in [34] applies almost word by word. We give some details for convenience.

In analogy to the proof of Proposition 4.3, we can assume that $\bar{v} = 0$. Zwegers’s argument that we can restrict to the situation where $c_1 = c_2$ and $\text{span}\{c_1, \tilde{c}(t)\}$ are constant for all $t > 0$ $\tilde{c}(t)$ can be transferred to our setting without difficulty. We thus have to show that

$$\sum_{v \in \bar{v} + L} e(L[v]\tau + \langle z, v \rangle_L) \prod_{(c'_1, c'_2) \in C(t)} (\rho^{c'_1} - \rho^{c'_2})(\tau, z; v) \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

Writing $\rho^{c_1} - \rho^{\tilde{c}(t)}$ as a sum of

$$\begin{aligned} & \text{sgn}(\langle a + v, \tilde{c}(t) \rangle_L) \beta(-y(a + v, \tilde{c}(t))_L^2 / L[\tilde{c}(t)]) \quad \text{and} \\ & \text{sgn}(\langle a + v, c_1 \rangle_L) - \text{sgn}(\langle a + v, \tilde{c}(t) \rangle_L), \end{aligned}$$

we are reduced to two separate cases. It suffices to apply the inequalities

$$\left| \text{sgn}(\langle a + v, c_1 \rangle_L) - \text{sgn}(\langle a + v, c_2 \rangle_L) \right| \leq \left| \text{sgn}(\langle a + v, c_1 \rangle_L) - \text{sgn}(\langle a + v, \tilde{c}(t) \rangle_L) \right|,$$

and

$$\left| \beta(-y(a + v, \tilde{c}(t))_L^2 / L[\tilde{c}(t)]) e(L[a + v]\tau + \langle a + v, z \rangle_L) \right| \leq e(\tilde{L}[a + v]\tau + \langle a + v, z \rangle_{\tilde{L}})$$

for a suitable quadratic form \tilde{L} , which depends on $\langle c_1, a + v \rangle_L$ and $\langle c_2, a + v \rangle_L$. Both estimates were literally established by Zwegers. We close by revisiting the construction of \tilde{L} .

Zwegers splits up the set L into three smaller ones, which he calls P_1 , P_2 , and P_3 . With P_1 and P_2 , one quickly obtains a majorant; on P_3 however, one needs to use additional vectors $\tilde{c}_1 = \frac{\langle c_1, c_2 \rangle_L}{2L[c_1]} c_1 - c_2$ and $\tilde{c}_2 = -c_2$ and a quadratic form

$$\tilde{L}[v] = L[v^\perp] - \frac{\langle c_2, v \rangle_L (\langle c_1, c_2 \rangle_L \langle c_1, v \rangle_L - L[c_1] \langle c_2, v \rangle_L)}{\langle c_1, c_2 \rangle_L^2}.$$

The concrete estimate in the last case is involved, but we nevertheless urge the reader to consult [34] on this matter. \square

If L is an integral lattice, then we immediately see that θ_L^C is invariant under the action of the Heisenberg group in $\Gamma^{(l)}$. The correct transformation behavior with respect to $SL_2(\mathbb{Z})$ can be inferred, again, in the same way as in Ref. [34]. Zwegers, however, reminded the author of Vignéras’s results [30], which we wrote in a particularly neat way.

Theorem 4.6 *For every L and every C , the theta series θ_L^C transforms like a Jacobi form of weight $l/2$, index L , and type ρ_L^l with respect to the action of $SL_2(\mathbb{Z}) \subset \Gamma^{(l)}$.*

Corollary 4.7 *Suppose that L is an even lattice, then the theta series θ_L^C transforms like a Jacobi form of weight $l/2$, index L , and type ρ_L^l . If $L[c_2] = 0$ for all $(c_1, c_2) \in C$, then*

$$\theta_L^C \in \mathcal{M}_{\frac{l}{2}, L}^{\text{H}[B]}(\rho_L), \quad \text{where } B = \{c_1 : (c_1, c_2) \in C\}.$$

Proof of Theorem 4.6 In light of Lemma 4.5, we can reduce ourselves to the case that c_1 and c_2 are negative for all $(c_1, c_2) \in C$. The estimates in Proposition 4.3, i.e., in Proposition 2.4 of [34], show that

$$e(L[v]\tau + \langle z, v \rangle_L) \prod_{(c_1, c_2) \in C} (\rho^{c_1} - \rho^{c_2})(\tau, z; v)$$

is of exponential decay. In particular, it is integrable and square integrable with respect to v , and so are its differentials and products with arbitrary polynomials in v . Vignéras states that θ_L^C is modular, if the above term at $y = 1$ is annihilated by $\frac{1}{4\pi} \Delta_L - E_L$, where Δ_L is the Laplace operator attached to L and E_L is the corresponding Euler operator. One verifies that this, up to scalar constants, equals $\langle R^{\text{H}}, L^{\text{H}} \rangle_L$ restricted to $y = 1$.

It suffices to verify the differential equation after expanding the product over C . So, fix $i_j \in \{1, 2\}$ for $1 \leq j \leq l_-$ and set $B = \{c_{i_j} : (c_1, c_2) \in C\}$. Then,

$$\langle R^{\text{H}}, L^{\text{H}} \rangle_L = \langle R^{\text{H}}, L^{\text{H}} \rangle_{(\text{span } B)^\perp} + \sum_{b \in B} \Delta^{\text{H}[b]}.$$

By construction, $\prod_{b \in B} \rho^b$ has H- harmonicity B , and this completes the proof. \square

Reduction to smaller lattices

The relation between indefinite theta series of product type and products of indefinite theta series is not quite obvious. We give a sufficient criterion that asserts splitting into more than one factor. It is crucial to note that in the assumptions of the next proposition, we intersect the span of $C_j \subset C$ with a rational subspace of L .

Proposition 4.8 *Let L and C be as above. Suppose that C can be written as a disjoint union of C_j , with $1 \leq j \leq l_C$. Suppose that further $L_{\mathbb{Q}} \supset \bigoplus_{j=1}^{l_C} L_j$ for lattices L_j such that $l = \sum_j l_j$. If $\text{span } C_j \cap L_{j, \mathbb{Q}} = L_{j, \mathbb{Q}}$, then*

$$\pi_L \left(\bigoplus \theta_{L_j}^{C_j} \right) = \theta_L^C,$$

where π_L is the natural projection of $\bigotimes_j \rho_{L_j}$ onto ρ_L that is induced by the inclusion $\text{disc}(L) \subseteq \bigotimes \text{disc}(L_j)$.

Proof It is clear how to reduce ourselves to the case that $L = \bigoplus_j L_j$ by means of the projection π_L . Then it suffices to inspect the definition (4.2) of θ_L^C , to prove the proposition. \square

We give two examples of indefinite theta series, which illustrate extreme cases that can occur. Both are attached to the same lattice of signature (2, 2), but we use different C . The first one is a product of two indefinite theta series, and the second one cannot possibly split.

Example 4.9 Let L_0 be the lattice with Gram matrix $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, and set $L = L_0 \oplus L_0$. To simplify the notation, we further fix the corresponding bases and coordinates z_1, z_2 and z_1, z_2, z_3, z_4 for z , respectively. +

- (1) Set $c_{0,1} = (-1, 1)$, $c_{0,2} = (0, 1)$, and $C_0 = \{(c_{0,1}, c_{0,2})\}$. We consider the product of two of Zwegers’s indefinite theta series attached to L_0 and C_0 . It yields an H-harmonic Maaß-Jacobi form:

$$\theta_L^C(\tau, z) = \theta_{L_0}^{C_0}(\tau, z_1, z_2) \cdot \theta_{L_0}^{C_0}(\tau, z_3, z_4) \quad \text{with}$$

$$C = \{((-1, 1, 0, 0), (0, 1, 0, 0)), ((0, 0, -1, 1), (0, 0, 0, 1))\}.$$

Its H- harmonicity is $B = \{(0, 1, 0, 0), (0, 0, 0, 1)\}$, and it is a product of two indefinite theta series.

- (2) The same lattice L allows for an indefinite theta series that is not a product (but per terminology is of product type).

$$\theta_L^C(\tau, z) \quad \text{with} \quad C = \{((1, 1, \pi^2, \pi^2), (1, \pi^2, 1, \pi)), ((\pi^2, \pi^2, 1, 1), (1, \pi, 1, \pi^2))\}.$$

This theta series is H-harmonic, because $(1, 1, \pi^2, \pi^2)$ and $(\pi^2, \pi^2, 1, 1)$ are isotropic. To check that it cannot split as a product, it suffices to observe that

$$\text{span}_{\mathbb{R}} \{(1, 1, \pi^2, \pi^2), (1, \pi^2, 1, \pi)\} \cap L_{\mathbb{Q}}$$

has dimension 0, while the vectors in C span $L_{\mathbb{R}}$.

Remark 4.10 The preceding example makes clear that one has to carefully distinguish between the product of the indefinite theta series and the indefinite theta series of the product type. The latter terminology can be explained by alluding to a yet to be developed adelic theory of mock theta series. At ∞ , such a theory should not distinguish between the two concepts, and the difference that was showcased in Example 4.9 should originate in the finite places. This also explains why there is such a huge overlap between the proofs here and those in Ref. [34]. Convergence and even modularity of theta series is mostly an issue that can be handled “at the infinite places”.

Theta-like decompositions

We conclude this work with a theta-like decomposition for H-harmonic Maaß-Jacobi forms. Generalized $\widehat{\mu}$ functions play the role of θ functions.

Generalized $\widehat{\mu}$ -functions

Indefinite theta series can be used to obtain preimages of skew theta series under $\xi_L^{\text{H}[B]}$. This construction, in the case of $L_- = 1$, should be somewhat known to experts. We write $\langle \cdot, \epsilon_{\bar{v}} \rangle_{\epsilon}$ for the \bar{v} -th coordinate of a vector-valued function. Given a negative definite, even lattice L and an orthogonal basis B of L , we define

$$\widehat{\mu}_L^B(\tau, z) = \frac{\sum_{\bar{v} \in \text{disc } L} \epsilon_{\bar{v}} \langle \theta_{L \oplus L(-1)}^C(\tau, (z, 0)), \epsilon_{\bar{v}} \epsilon_0 \rangle_{\epsilon}}{\langle \theta_{L(-1)}(\tau, 0), \epsilon_0 \rangle_{\epsilon}}, \tag{5.1}$$

where $C = \{((b, 0), (b, b)) : b \in B\}$. Clearly, $(0, b)$ is negative definite, while (b, b) is isotropic. Singularities of $\theta_{L \oplus L(-1)}^C(\tau, (z, 0))$ lie outside of $D(B) = \{(\tau, z) \in \mathbb{H}^{(JL)}; \langle b, v/y \rangle_L \notin \mathbb{Z} \text{ for all } b \in B\}$.

Proposition 5.1 *Given L and B as above, we have*

$$\xi_L^{\text{H}[B]} \widehat{\mu}_L^B = 2\sqrt{\pi} \theta_L^{\text{span } B}.$$

Proof It suffices to determine the image of ρ^c in (4.1) under $\xi_L^{\text{H}[B]}$. This was done in Proposition 3.15. □

Theorem 5.2 *Suppose that $L = L' \oplus \bigoplus_{b \in B} \mathbb{Z}b$. Then we have*

$$\mathcal{M}_{\mathbb{J}_{k,L}^{\text{H}[B]}} = \sum_{B' \subseteq B} \left\langle \widehat{\mu}_L^{B'}, \mathcal{M}_{\mathbb{J}_{k-\#B'/2, (\text{span } B')^\perp}(\rho_{\text{span } B'})} \right\rangle. \tag{5.2}$$

Remark 5.3

- (1) The condition that L be $L' \oplus \bigoplus \mathbb{Z}b$ might seem surprisingly restrictive. However, for every lattice L and every $B \subseteq \mathbb{P}(L_{\mathbb{Q}})$ we can pass to such a situation by considering suitable suplattices $L_{\perp} \supseteq L$ in conjunction with the theory of vector-valued Jacobi forms. On the other hand, overlapping singularities of meromorphic Jacobi forms in decomposition (5.2) hinder us from passing back from L_{\perp} to L .
- (2) A similar decomposition can be found for $\mathcal{M}_{\mathbb{J}_{k,L}^{\text{h}_0\text{H}[B]}}$ and $\mathcal{M}_{\mathbb{J}_{k,L}^{\text{h}_0\text{H}[B]}}$, in the case of degenerate L .

Proof of Theorem 5.2 The theorem follows by induction: by applying $\xi^{\text{H}[B]}$ to $\phi \in \mathcal{M}_{\mathbb{J}_{k,L}^{\text{H}[B]}}$ to obtain a vector-valued meromorphic Jacobi form by means of the partial theta decomposition Proposition 2.7. Using $\widehat{\mu}_L^B$, we can find an H-harmonic Maaß-Jacobi form ψ that has the same image under $\xi^{\text{H}[B]}$ as ϕ . Thus, the difference $\phi - \psi$ is a sum of H-harmonic Maaß-Jacobi forms with H-harmonicities B' , $\#B' < \#B$. This completes the proof. □

Restriction to torsion points

To formulate the final corollary, we have to define harmonic weak Maaß forms of higher depth. It does not seem adequate to employ the generally preferable language of vector-valued modular forms. Technicalities would require an extra exposition. Therefore, given

$\Gamma \subseteq \text{SL}_2(\mathbb{Z})$, we set $\mathbb{M}_k^{[0]}(\Gamma) = M_k(\Gamma)$. For depth $d \geq 1$, we let $\mathbb{M}_k^{[d]}(\Gamma)$ be the space of real-analytic functions $f : \mathbb{H} \rightarrow \mathbb{C}$ that

- (i) are invariant under the weight k slash action of Γ ;
- (ii) whose image under ξ_k lies in $\sum_l \overline{M_l(\Gamma)} \otimes \mathbb{M}_{k-l}^{[d-1]}(\Gamma)$;
- (iii) satisfy the growth condition $f(iy) = O(e^{ay})$ as $y \rightarrow \infty$ for some $a \in \mathbb{R}$.

The definition of higher depth harmonic weak Maaß forms goes back to Zagier and Zwegers, but currently there is no literature on it.

Corollary 5.4 *Suppose that $(\tau, \alpha\tau + \beta)$ for $\alpha, \beta \in \frac{1}{N}L$ is not a singularity of $\phi \in \mathcal{M}_{k,L}^{\text{H}[B]}$. Then we have*

$$\left(\phi|_{k,L} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha, \beta \right)\right)(\tau, 0) \in \mathbb{M}_k^{[\#B]}(\Gamma(N)),$$

where $\Gamma(N)$ is the principal congruence subgroup of level N .

Remark 5.5 The statement could be refined even more by specifying that each time we apply ξ to the restriction of an H-harmonic Maaß-Jacobi form, a unary theta series splits off.

Proof The transformation behavior is clear. The analytic properties follow from restricting to torsion points the Fourier expansions in Proposition 3.6. □

Endnotes

^aZagier’s formula has a sign mistake, as one referee pointed out. Following that referee’s suggestion, we sketch a proof in Example 3.10.

^bThere is no definition of higher depth harmonic weak Maaß forms in the literature, but it has been communicated by Zagier and Zwegers in some talks.

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Competing interests

The author declares that they have no competing interests.

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