

BAD REDUCTION OF GENUS THREE CURVES WITH COMPLEX MULTIPLICATION

IRENE BOUW, JENNY COOLEY, KRISTIN LAUTER, ELISA LORENZO GARCIA, MICHELLE MANES, RACHEL NEWTON, EKIN OZMAN

ABSTRACT. Let C be a smooth, absolutely irreducible genus 3 curve over a number field M . Suppose that the Jacobian of C has complex multiplication by a sextic CM-field K . Suppose further that K contains no imaginary quadratic subfield. We give a bound on the primes \mathfrak{p} of M such that the stable reduction of C at \mathfrak{p} contains three irreducible components of genus 1.

1. INTRODUCTION

In [GL07], Goren and Lauter study genus 2 curves whose Jacobians are absolutely simple and have complex multiplication (CM) by the ring of integers \mathcal{O}_K of a quartic CM-field K , and they show that if such a curve has bad reduction to characteristic p then there is a solution to the embedding problem, formulated as follows [GL07]:

Let K be a quartic CM-field which does not contain a proper CM-subfield, and let p be a prime. The embedding problem concerns finding a ring embedding $\iota : \mathcal{O}_K \hookrightarrow \text{End}(E_1 \times E_2)$, such that the Rosati involution coming from the product polarization induces complex conjugation on \mathcal{O}_K , and E_1, E_2 are supersingular elliptic curves over $\overline{\mathbb{F}}_p$.

In this paper, we consider genus 3 curves whose Jacobians have CM by a sextic CM-field that does not contain a proper CM-subfield. By analogy with [GL07], we formulate an embedding problem for the genus 3 case as follows.

Problem 6.3 (The embedding problem) Let \mathcal{O} be an order in a sextic CM-field K , and let p be a prime number. The *embedding problem* for \mathcal{O} and p is the problem of finding elliptic curves E_1, E_2, E_3 defined over $\overline{\mathbb{F}}_p$, and a ring embedding

$$i : \mathcal{O} \hookrightarrow \text{End}(E_1 \times E_2 \times E_3)$$

such that the Rosati involution on $\text{End}(E_1 \times E_2 \times E_3)$ induces complex conjugation on \mathcal{O} . We call such a ring embedding a *solution to the embedding problem* for \mathcal{O} and p .

In this paper, we prove the following result on solutions to the embedding problem. We refer to Section 6.3 for the precise statement.

Theorem 6.9 *Let K be a sextic CM-field such that K does not contain a proper CM-subfield. Let \mathcal{O} be an order in K . There exists an explicit bound on the rational primes p for which the embedding problem has a solution, and this bound depends only on the order \mathcal{O} .*

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As in the genus 2 case, Theorem 6.9 yields a bound on certain primes of bad reduction for the curve C . However, the result is not as strong as in the genus 2 case, since there are more possibilities for the reduction of C . We discuss the statement of the result.

Let C be a smooth, absolutely irreducible genus 3 curve over a number field M whose Jacobian has CM by an order \mathcal{O} in a sextic CM-field K . We say that C has bad reduction at a rational prime p if there exists a prime \mathfrak{p} of M above p at which C has bad reduction. In Corollary 4.3, we observe that if C has bad reduction at a prime \mathfrak{p} , there are two possibilities for the stable reduction $\overline{C}_{\mathfrak{p}}$ of C at \mathfrak{p} . Either $\overline{C}_{\mathfrak{p}}$ contains three irreducible components of genus 1, or $\overline{C}_{\mathfrak{p}}$ contains one irreducible component of genus 1 and one of genus 2.

In this paper, we restrict our attention to the first of these two possibilities. In Proposition 6.5, we show that if C has bad reduction at a prime \mathfrak{p} above p and the stable reduction contains three genus 1 curves, then the embedding problem for \mathcal{O} and p has a solution. Theorem 6.9 therefore yields the following result on the primes of bad reduction of C .

Theorem 6.8 *Let C be a genus 3 curve whose Jacobian has CM by an order \mathcal{O} in a sextic CM-field K that does not contain a proper CM-subfield. There exists an explicit bound on the primes p where the stable reduction contains three irreducible components of genus 1.*

We do not consider all primes of bad reduction of C in Theorem 6.8 for the following reason. If the stable reduction of C at \mathfrak{p} contains three irreducible components of genus 1, then the reduction $\overline{J}_{\mathfrak{p}}$ of the Jacobian J of C is isomorphic to the product $E_1 \times E_2 \times E_3$ of elliptic curves as polarized abelian varieties (Proposition 4.2). This yields a ring embedding

$$\iota : \mathcal{O} = \text{End}(J) \hookrightarrow \text{End}(\overline{J}_{\mathfrak{p}}) = \text{End}(E_1 \times E_2 \times E_3),$$

which has the property that the Rosati involution on $\text{End}(E_1 \times E_2 \times E_3)$ restricts to complex conjugation on the image of \mathcal{O} (Section 4.3). This is precisely the statement that ι is a solution to the embedding problem for \mathcal{O} and p .

Consider a prime \mathfrak{p} where the curve C has bad reduction, but the stable reduction $\overline{C}_{\mathfrak{p}}$ contains an irreducible component E of genus 1 and an irreducible component D of genus 2 (Corollary 4.3). In this case — an example of which is described in Section 5.2 — the reduction $\overline{J}_{\mathfrak{p}}$ of the Jacobian of C is the product of E with the Jacobian of D as polarized abelian varieties. The abelian variety $\overline{J}_{\mathfrak{p}}$ is still isogenous to a product of elliptic curves (Theorem 4.5), but $\overline{J}_{\mathfrak{p}}$ is not isomorphic to a product of elliptic curves as polarized abelian varieties. This suggests that a different formulation of the embedding problem would be needed to draw conclusions for such primes \mathfrak{p} . We do not discuss the correct formulation of the embedding problem for this case in the present paper, but leave it as a direction for future work.

The assumption that the CM-field K does not contain a proper CM-field is also present in the genus 2 case in [GL07]. However, in the genus 2 case, this assumption is equivalent to the assumption that the CM-type of the Jacobian J is primitive. We refer to Section 3.4 for more details. In characteristic zero, the condition that the CM-type corresponding to J is primitive is equivalent to the assumption that J is absolutely simple (Theorem 3.2).

In the genus 3 case, the assumption that the CM-field K does not contain a proper CM-subfield still implies that the CM-type of the Jacobian J is primitive. However, the converse does not hold. Even in the case that the sextic CM-field K contains a proper CM-subfield there exist primitive CM-types (Section 3). In Section 6.4, we discuss why the embedding

problem needs to be formulated differently for such CM-fields. We show that, in the case where K contains a proper CM-subfield, the embedding problem as we have formulated it has solutions for any prime p and some order \mathcal{O} of K .

Finally, we have not included the condition that the elliptic curves E_i are supersingular in the formulation of the embedding problem, in contrast to the formulation in genus 2, because for a set of Dirichlet density $1/2$, the elliptic curves E_i are ordinary.

1.1. Relation to a result of Gross and Zagier. One of the motivations of Goren and Lauter for studying solutions of the embedding problem in genus 2 was generalizing a result of Gross and Zagier on singular moduli of elliptic curves [GZ85]. Recall that singular moduli are values $j(\tau)$ of the modular function j at imaginary quadratic numbers τ . Gross and Zagier define the product

$$J(d_1, d_2) = \left(\prod_{[\tau_1], [\tau_2]} (j(\tau_1) - j(\tau_2)) \right)^{4/w_1 w_2},$$

where the product runs over equivalence classes of imaginary quadratic numbers τ_i with discriminants d_i , where the d_i are assumed to be relatively prime. Here w_i denotes the number of units in $\mathbb{Q}(\tau_i)$. The function J is closely related to the value of the Hilbert class polynomial of an imaginary quadratic field at a point τ corresponding to a different imaginary quadratic field.

Under some assumptions, Gross and Zagier show that $J(d_1, d_2)$ is an integer, and their main result gives a formula for the factorization of this integer. The result of Gross and Zagier may be reinterpreted as a formula for the number of isomorphisms between the reductions of the elliptic curves E_i corresponding to the τ_i at all rational primes p . This problem is equivalent to counting embeddings of $\text{End}(E_2)$ into the endomorphism ring of the reduction of E_1 at p .

Goren and Lauter ([GL07], Corollary 5.1.3) prove a generalization of the result of Gross and Zagier. They consider curves of genus 2 with CM by a quartic CM-field. In their result, the function J is replaced by suitable Siegel modular functions f/Θ^k . Here f is a Siegel modular form of weight $10k$ with values in a number field and Θ is a concrete Siegel modular form of weight 10. The modular function f/Θ^k has the property that for any τ in the Siegel upper half plane the genus 2 curve corresponding to τ has bad reduction at the primes dividing the denominator of $(f/\Theta^k)(\tau)$. (See [GL07], Corollary 5.1.2 for the precise statement.)

The Igusa class polynomials are an analog of the Hilbert class polynomials for quartic CM-fields, where the j -invariant is replaced by the absolute Igusa invariants. Goren and Lauter and collaborators (see for example [GL07], [GL13], [LV12]) deduce results on the denominators of the coefficients of the Igusa class polynomials from results on the embedding problem for quartic CM-fields.

The embedding problem for curves of genus 3 studied in this paper does not immediately yield a statement analogous to that of Gross and Zagier. One of the ingredients that is missing is finding good coordinates for the moduli space of curves of genus 3, analogous to the absolute Igusa invariants in genus 2.

In this paper, we discuss several differences between the reduction of CM-curves in genus 2 and in genus 3. The embedding problem in the formulation of Problem 6.3 does not cover all types of bad reduction. Also, in the case that the sextic CM-field K contains a proper

CM-subfield the embedding problem should be adapted. It would be interesting to study the implication of these differences for a possible analog of the Igusa class polynomials for sextic CM-fields.

1.2. Outline. The structure of this paper is as follows. Section 2 gives the possibilities for the Galois group of the Galois closure of a sextic CM-field, following work of Dodson in [Dod84]. Section 3 describes the possible CM-types for a sextic CM-field. We note which of the CM-types are primitive, meaning that they can arise as the CM-type of a simple abelian variety. In Section 4, we describe the possibilities for the reduction of a genus 3 curve and its Jacobian to characteristic $p > 0$. We also give some properties of the Rosati involution attached to a polarized abelian variety, which will be used in Section 6. In Section 5, we give various examples of genus 3 curves with CM; we calculate their CM-types and the reductions of the curves and their Jacobians to characteristic $p > 0$. In Section 6, we consider a genus 3 curve C over a number field M such that its Jacobian has CM by a sextic CM-field K with no proper CM-subfield. We prove a bound on primes such that there exists a solution to the embedding problem, and we use that to give a bound on the primes p such that the stable reduction of C at p contains three elliptic curves. We show that if we drop the assumption that K has no proper CM-subfield, then the embedding problem as stated cannot be used to give a bound on the primes p as above.

We include as an appendix a collection of conditions that a solution to the embedding problem must satisfy, written as equations in the entries of certain matrices in the image of the embedding. These equations may be useful for future work. A refinement of the embedding problem (for example, a version which includes conditions pertaining to the CM-type) would result in extra equations in addition to those in the appendix. It is to be hoped that studying this larger set of equations would yield an explicit bound on the primes for which they have a solution. This would give a bound on the primes p such that the stable reduction of C at p contains three curves of genus 1, even in the case where the CM-field K contains a proper CM-subfield.

1.3. Notation and conventions. We set the following notation, to be used throughout.

- \mathbb{F}_p is the finite field with p elements.
- ζ_N is a primitive N th root of unity.
- For a field k , \bar{k} is an algebraic closure.
- K is a sextic CM-field, i.e., K is a totally imaginary extension of K^+ , where K^+ is a totally real cubic extension of \mathbb{Q} .
- \mathcal{O} is an order of K .
- F and L are Galois closures of K/\mathbb{Q} and K^+/\mathbb{Q} respectively, with $G = \text{Gal}(F/\mathbb{Q})$ and $G^+ = \text{Gal}(L/\mathbb{Q})$.
- ψ is a complex embedding $K \hookrightarrow \mathbb{C}$, and ρ is complex conjugation. Hence $\{\psi, \rho \circ \psi\}$ is a conjugate pair of embeddings.
- (K, φ) is a CM-type, i.e., a choice of one embedding from each pair of complex conjugate embeddings.
- A is an abelian variety, $\text{End}(A)$ is the endomorphism ring of A , and $\text{End}^0(A)$ is $\text{End}(A) \otimes \mathbb{Q}$.
- For $f \in \text{End}(A)$, $f^\vee \in \text{End}(A^\vee)$ is the dual isogeny. The Rosati involution associated with a fixed polarization is denoted by $f \mapsto f^*$, $\text{End}^0(A) \rightarrow \text{End}^0(A)$.

- E is an elliptic curve, $j(E)$ is the j -invariant of E .
- We denote an isomorphism between two abelian varieties over an algebraic closure of the field of definition by \simeq .
- We denote an isogeny between two abelian varieties over an algebraic closure of the field of definition by \sim .
- M is a number field, ν (or \mathfrak{p}) is a finite place of M , \mathcal{O}_ν is the valuation ring of ν , and k_ν is the residue field.
- C is a curve over a number field with Jacobian J and genus $g = g(C)$. A curve C is always assumed to be smooth, projective and absolutely irreducible, unless explicitly mentioned otherwise.
- $B_{p,\infty}$ is the quaternion algebra ramified at p and ∞ , and R is a maximal order of $B_{p,\infty}$.
- For a matrix T , $\mathbf{Tr}(T)$ denotes the sum of its diagonal entries, the trace.
- \mathbf{Tr}_{K/K_1} denotes the trace of a field extension K/K_1 .
- For an element of a central simple algebra, \mathbf{Nrd} denotes the reduced norm.
- \mathbf{Nm}_{K/K_1} denotes the norm of a field extension K/K_1 ; we use \mathbf{Nm} when the extension is clear.

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2. THE GALOIS GROUP OF THE GALOIS CLOSURE OF A SEXTIC CM-FIELD

Let K be a sextic CM-field, i.e., K is a totally imaginary quadratic extension of a totally real field K^+ with $[K^+ : \mathbb{Q}] = 3$. We denote the Galois closure of K^+/\mathbb{Q} by L and the Galois closure of K/\mathbb{Q} by F . We write $G = \text{Gal}(F/\mathbb{Q})$ and $G^+ = \text{Gal}(L/\mathbb{Q})$. The following proposition lists the possibilities for G .

Proposition 2.1. *Let K be a sextic CM-field, and let G be the Galois group of the Galois closure of K/\mathbb{Q} . Then G is one of the following groups:*

- (1) $C_2 \times C_3 \simeq C_6$,
- (2) $C_2 \times S_3 \simeq D_{12}$,
- (3) $(C_2)^3 \rtimes G^+$ with $G^+ \in \{C_3, S_3\}$ acting by permutations on the three copies of C_2 .

In particular, if K/\mathbb{Q} is Galois, then the Galois group $G = \text{Gal}(K/\mathbb{Q}) \simeq C_6$ is cyclic.

Proof. This is proved in Section 5.1.1 of [Dod84], for example. □

In the rest of this section, we sketch the proof of Proposition 2.1, following Dodson. Since we restrict to the case of sextic CM-fields, the presentation can be simplified. In the course of the proof, we also give more details on the structure of the extensions F/\mathbb{Q} and K^+/\mathbb{Q} in the different cases. In particular, we show that Case 3 is precisely the case where K does not contain an imaginary quadratic subfield.

Galois theory implies that we have the following exact sequence of groups:

$$1 \rightarrow \text{Gal}(F/L) \rightarrow G \rightarrow G^+ \rightarrow 1.$$

Lemma 2.2. *We have*

$$\text{Gal}(F/L) \simeq (C_2)^v, \quad 1 \leq v \leq 3$$

and

$$G^+ \in \{C_3, S_3\}.$$

Proof. This lemma is a special case of the proposition in Section 1.1 of [Dod84]. We give the proof here for convenience.

We first remark that $K = K^+(\sqrt{-\delta})$ for some totally positive square-free $\delta \in K^+$. We write $\delta_1 := \delta, \delta_2, \dots, \delta_r$ for the G^+ -conjugates of δ . It follows that

$$F = L(\sqrt{-\delta_1}, \dots, \sqrt{-\delta_r}).$$

Every element $h \in \text{Gal}(F/L)$ sends $\sqrt{-\delta_i}$ to $\pm\sqrt{-\delta_i}$. Moreover, h is determined by its action on these elements. It follows that $\text{Gal}(F/L) \simeq (C_2)^v$ is an elementary abelian 2-group.

Since $\delta \in K^+$ it follows that $[\mathbb{Q}(\delta) : \mathbb{Q}]$ divides 3. We conclude that the number of G^+ -conjugates of δ is at most 3.

The statement on G^+ immediately follows from the fact that $[K^+ : \mathbb{Q}] = 3$. This proves the lemma. \square

Proof of Proposition 2.1. We start the classification. Note that $\text{Gal}(K/K^+)$ is generated by complex conjugation. It follows that complex conjugation is also an element of G . This element, which we denote by ρ , is an element of the center of G .

Case I: K/\mathbb{Q} Galois.

Since K/\mathbb{Q} is Galois, $G = \text{Gal}(K/\mathbb{Q})$ is a group of order 6, hence either cyclic or S_3 . Since the Galois closure L of K^+/\mathbb{Q} is a totally real subfield of K , it follows that $K^+ = L$. This implies that $\text{Gal}(K/K^+)$ is a normal subgroup of G which has order 2. It follows that $G \simeq C_6$ is cyclic. Note that K contains the imaginary quadratic subfield $K_1 := K^{C_3}$ and $K = K_1K^+$. This corresponds to Case 1 of Proposition 2.1.

Case II: K/\mathbb{Q} is not Galois and K contains an imaginary quadratic field K_1 .

Since K contains an imaginary quadratic field K_1 , we have $F = LK_1$ and $G \simeq C_2 \times G^+$. If $G^+ \simeq C_3$, then $L = K^+$ and K/\mathbb{Q} is Galois, which contradicts our assumption. It follows that $G^+ \simeq S_3$ and $G \simeq C_2 \times S_3$. This is Case 2 of Proposition 2.1. We obtain the field diagram in Figure 1.

Case III: K/\mathbb{Q} is not Galois and K does not contain an imaginary quadratic subfield.

This case corresponds to Case 3 of Proposition 2.1. In this case the integer v from Lemma 2.2 is not equal to 1, i.e., we have $v = 2$ or 3. The following claim completes the proof of Proposition 2.1.

Claim: The case $v = 2$ does not occur. This claim is a special case of the second proposition in Section 5.1.1 of [Dod84]. We give the proof here for completeness.

Recall that $\rho \in \text{Gal}(F/L)$ denotes complex conjugation and is contained in the center of G . Let $\sigma \in G^+$ be an element of order 3. Then σ acts on $\text{Gal}(F/L) = (C_2)^v$ by conjugation. This action has two orbits of length 1, corresponding to the identity element and ρ . All other orbits have length 3. It follows that $3 \mid (2^v - 2)$. The claim follows. \square

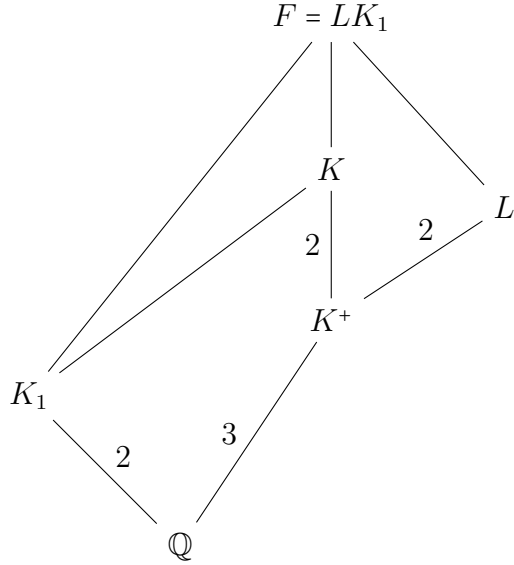


FIGURE 1. Field extensions in Case 2

Of primary interest to us in the rest of this paper is Case 3 of Proposition 2.1, in which K does not contain an imaginary quadratic subfield. We have seen that $G \simeq (C_2)^3 \rtimes G^+$ with $G^+ \in \{C_3, S_3\}$. The following diagram describes the field extensions in Case 3.

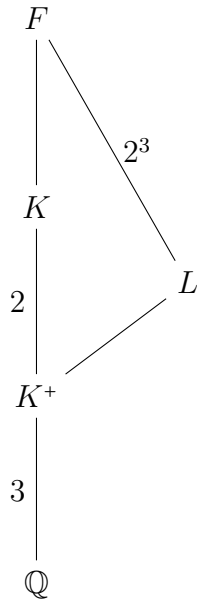


FIGURE 2. Field extensions in Case 3

3. PRIMITIVE CM-TYPES

Let K be a sextic CM-field. As in Section 2, we write K^+ for the totally real cubic subfield of K . The complex embeddings $K \hookrightarrow \mathbb{C}$ come in pairs $\{\psi, \rho \circ \psi\}$, where ρ denotes complex

conjugation. Recall that a CM-type (K, φ) is a choice of one embedding from each of these pairs. The goal of this section is to determine the primitive CM-types. We start by recalling the definition from [Mil06], Section 1.1. For examples we refer to Section 5.

Definition 3.1. *Let (K, φ) and (K_1, φ_1) be CM-types. We say that (K, φ) is induced from (K_1, φ_1) if K_1 is a subfield of K and the restriction of φ to K_1 coincides with φ_1 . A CM-type is called primitive if it is not induced from a CM-type on any proper CM-subfield of K .*

Let A be an abelian variety and let K be a CM-field with $[K : \mathbb{Q}] = 2 \dim(A)$. We say that A has *complex multiplication (CM) by K* if the endomorphism algebra $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$ contains K . We say that a curve C has *CM by K* if its Jacobian has CM by K . We say that A (or C) has CM if there exists a CM-field K such that A (or C) has CM by K . If $\text{End}(A)$ is an order \mathcal{O} in a CM-field K with $[K : \mathbb{Q}] = 2 \dim(A)$, we say that A has *CM by \mathcal{O}* .

The following theorem gives a geometric interpretation of what it means for the CM-type of a CM-abelian variety to be primitive in characteristic zero. For convenience, we say that an abelian variety A defined over a field M is simple if it is absolutely simple, meaning that $A \otimes_M \overline{M}$ is not isogenous to a product of abelian varieties of lower dimension. Similarly, we say that two abelian varieties A_1, A_2 defined over M are isogenous if there exists an isogeny $\varphi : A_1 \rightarrow A_2$ defined over the algebraic closure of M .

Theorem 3.2. *Let A be an abelian variety defined over a field of characteristic zero. Suppose that A has CM with CM-type (K, φ) . Then the CM-type (K, φ) is primitive if and only if the abelian variety A is simple.*

Proof. This is proved in Theorem 3.5 of Chapter 1 of [Lan83]. See also Remark 1.5.4.2 of [CCO14]. \square

We refer to Section 1.5.5 of [CCO14] for an explanation of why we need to assume that A is defined over a field of characteristic zero in Theorem 3.2.

The following result gives a useful criterion for determining whether a given CM-type is primitive. For a proof, we refer to Theorem 3.6 of Chapter 1 of [Lan83]. For a CM-type (K, φ) and $h \in \text{Aut}(K)$, we write

$$\varphi h = \{\varphi_i \circ h \mid \varphi_i \in \varphi\}.$$

Proposition 3.3. *Let (K, φ) be a CM-type. We write (F, Φ) for the induced CM-type of the Galois closure of K/\mathbb{Q} . Let*

$$H_\Phi = \{h \in G = \text{Gal}(F/\mathbb{Q}) \mid \Phi h = \Phi\}.$$

Then (K, φ) is primitive if and only if

$$K = F^{H_\Phi}.$$

We now determine the primitive sextic CM-types in each of the cases of Proposition 2.1. We first consider Case 3. Recall that in the proof of Proposition 2.1 we showed that Case 3 is precisely the case where K does not contain an imaginary quadratic subfield.

Corollary 3.4. *Suppose that we are in Case 3 of Proposition 2.1, i.e., K does not contain an imaginary quadratic field. Then every CM-type (K, φ) is primitive.*

Proof. Suppose that (K, φ) is not primitive. Then K contains a proper CM-subfield K_1 . Since K is sextic, K_1 is an imaginary quadratic field. This yields a contradiction. \square

3.1. Primitive types in Case 1. We now consider Case 1 from Proposition 2.1. This is the case in which K/\mathbb{Q} is Galois, with Galois group $G \simeq C_6$. We choose a generator σ of G . Note that complex conjugation corresponds to σ^3 . Up to replacing φ by its complex conjugate, every CM-type (K, φ) may be written as

$$\varphi_{a,b} = \{1, \sigma^a, \sigma^b\}, \quad 0 < a, b < 6, \quad a \equiv 1 \pmod{3}, \quad b \equiv 2 \pmod{3}.$$

We find 4 cases:

$$\{a, b\} \in \{\{1, 2\}, \{1, 5\}, \{4, 2\}, \{4, 5\}\}.$$

Note that changing the generator σ of G to σ^{-1} changes $\{4, 5\}$ to $\{1, 2\}$, therefore we do not have to consider the choice $\{4, 5\}$.

We write $H_{a,b}$ for the subgroup fixing the CM-type as in Proposition 3.3. Then $H_{1,2} = H_{1,5} = \{1\}$ and $H_{4,2} = \langle \sigma^2 \rangle \simeq C_3$. Note that $K_1 := K^{H_{4,2}}$ is the imaginary quadratic subfield of K , which is a CM-field. We conclude that $\varphi_{4,2}$ is induced from K_1 , and hence imprimitive. The other CM-types are primitive.

3.2. Primitive types in Case 2. We now consider Case 2 from Proposition 2.1. We refer to Section 2 for a description of the fields involved. Recall that $K = K_1 K^+$. Therefore, an embedding $\psi : K \hookrightarrow \mathbb{C}$ corresponds to an ordered pair (ψ_1, ψ^+) , where $\psi_1 : K_1 \hookrightarrow \mathbb{C}$ is an embedding of K_1 and $\psi^+ : K^+ \hookrightarrow \mathbb{C}$ is an embedding of K^+ . Since K^+ is totally real, the image of ψ^+ is contained in \mathbb{R} . We denote the three possible complex embeddings of K^+ by χ_i for $i = 1, 2, 3$. We fix a complex embedding of K_1 and denote it by 1. We denote the other complex embedding of K_1 by -1 .

A CM-type (K, φ) consists of a triple of these ordered pairs in which no two of the pairs are complex conjugates. Since $\text{Gal}(K_1/\mathbb{Q})$ is generated by complex conjugation, we simply choose one of the two complex embeddings of K_1 for each embedding χ_i of K^+ . This means that we may write

$$\varphi = \{(\epsilon_i, \chi_i) \mid i = 1, 2, 3\}, \quad \epsilon_i \in \{\pm 1\}.$$

Identifying φ with its complex conjugate yields four different CM-types.

We determine the imprimitive types. The only CM-field properly contained in K is the imaginary quadratic field K_1 . The restriction of the embedding (ϵ_i, χ_i) to K_1 is just ϵ_i . Therefore, the CM-type $\varphi = \{(\epsilon_i, \chi_i)\}$ is imprimitive if and only if ϵ_i is independent of i . We conclude that there is a unique imprimitive CM-type. The other three are primitive.

3.3. Examples of CM-types. We give examples of CM-types illustrating each of the three cases of Proposition 2.1.

Example 3.5 (K/\mathbb{Q} is Galois with Galois group $G \simeq C_6$.) Let K be $\mathbb{Q}(\zeta_7)$ where ζ_7 is a primitive seventh root of unity. The maximal totally real subfield of K is $K^+ = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, which has degree three over \mathbb{Q} (the minimal polynomial of $\zeta_7 + \zeta_7^{-1}$ over \mathbb{Q} is $x^3 + x^2 - 2x - 1$). The field K is a totally imaginary quadratic extension of K^+ .

The automorphism σ which maps ζ_7 to ζ_7^5 generates $\text{Gal}(K/\mathbb{Q})$. The fixed field of $\langle \sigma^2 \rangle$ is $\mathbb{Q}(\zeta_7^4 + \zeta_7^2 + \zeta_7) = \mathbb{Q}(\sqrt{-7})$. This is the unique imaginary quadratic extension of \mathbb{Q} contained in $\mathbb{Q}(\zeta_7)$. Therefore, the only imprimitive CM-type admitted by K is $\varphi_{2,4} = \{1, \sigma^2, \sigma^4\}$; the CM-types $\varphi_{a,b} = \{1, \sigma^a, \sigma^b\}$ for $\{a, b\} \neq \{4, 2\}$ with $a \equiv 1 \pmod{3}$, $b \equiv 2 \pmod{3}$ are all primitive.

The following examples have been taken from the database of Klüners and Malle ([KM]).

Example 3.6 (The Galois closure of K/\mathbb{Q} is D_{12}). Let K be the sextic field obtained by adjoining a root of the irreducible polynomial $f(x) = x^6 - 3x^5 + x^4 + 10x^2 - 9x + 3$. Then K is a totally imaginary quadratic extension of the totally real cubic field $K^+ = \mathbb{Q}(\alpha)$ where the minimal polynomial of α is $g(x) = x^3 - 7x^2 + 12x - 3$. The Galois closure F of K/\mathbb{Q} is the compositum of the Galois closure of K^+ with the unique imaginary quadratic subfield K_1 of K , given by the minimal polynomial $x^2 + 3x + 3$. The Galois group of F is isomorphic to $S_3 \times C_2 \simeq D_{12}$. Denote the roots of $g(x)$ by $\alpha_1 := \alpha, \alpha_2, \alpha_3$.

Let $\chi_i : \alpha_1 \mapsto \alpha_i$ denote the three real embeddings of K^+ and ± 1 denote the two complex embeddings of K_1 . Then the CM-type $\varphi = \{(1, \chi_1), (1, \chi_2), (1, \chi_3)\}$ of K is imprimitive, since its restriction to the quadratic imaginary subfield K_1 is also a CM-type. The remaining three CM-types of K are primitive. For clarity, the primitive CM-types are as follows: $\{(1, \chi_1), (-1, \chi_2), (-1, \chi_3)\}, \{(1, \chi_1), (1, \chi_2), (-1, \chi_3)\}, \{(1, \chi_1), (-1, \chi_2), (1, \chi_3)\}$.

Example 3.7 (The Galois closure of K/\mathbb{Q} is $(C_2)^3 \rtimes C_3$). Let $K = \mathbb{Q}(\beta)$ be the degree 6 extension of \mathbb{Q} where the minimal polynomial of β is $f(x) = x^6 - 2x^5 + 5x^4 - 7x^3 + 10x^2 - 8x + 8$. Let F be the Galois closure of K . Then $\text{Gal}(F/\mathbb{Q})$ is $(C_2)^3 \rtimes C_3$. Moreover, K is a CM-field since K is a totally imaginary quadratic extension of $K^+ = \mathbb{Q}(\alpha)$ where the minimal polynomial of α over \mathbb{Q} is $g(x) = x^3 - 7x^2 + 14x - 7$. Note that K contains no quadratic subfield, hence every CM-type is primitive.

3.4. Comparison with the genus 2 case. The following proposition characterizes primitive CM-types for quartic CM-fields.

Proposition 3.8. *Let K be a quartic CM-field. The following are equivalent.*

- (1) *The CM-type is primitive.*
- (2) *The CM-field K does not contain an imaginary quadratic subfield.*

Proof. We recall the argument from Example 8.4.(2) of [Shi98] in which we find a classification of the possible Galois groups of quartic CM-fields K together with the possible CM-types. It follows from this classification that if K contains a proper CM-subfield $K_1 \neq \mathbb{Q}$ then K/\mathbb{Q} is Galois with Galois group $G \simeq C_2 \times C_2$. Moreover, in this case all CM-types are imprimitive. Namely, denoting again complex conjugation by ρ , we may write $G = \{1, \rho, \sigma, \rho\sigma\}$. Then the possible CM-types are $\{1, \sigma\}$ and $\{1, \rho\sigma\}$, which are fixed by $\langle \sigma \rangle$ and $\langle \rho\sigma \rangle$, respectively. Therefore, the statement follows from Proposition 3.3. \square

Proposition 3.8 explains why Goren and Lauter ([GL06], [GL07]) restrict to the case where the quartic CM-field does not contain an imaginary quadratic subfield. For quartic CM-fields, this is equivalent to requiring that the CM-type is primitive. However, as we have seen in our discussion of the primitive types in Cases 1 and 2 of Proposition 2.1, these two properties are not equivalent for sextic CM-fields.

We give two concrete examples of genus 2 curves with CM to illustrate Proposition 3.8. These are similar to the genus 3 examples given in Section 5.1. We consider two smooth projective curves defined by the following affine equations

$$\begin{aligned} D_1 : \quad & y^5 = x(x-1), \\ D_2 : \quad & y^8 = x(x-1)^4. \end{aligned}$$

One easily verifies that both curves have genus 2.

The curve D_1 has CM by $K_1 := \mathbb{Q}(\zeta_5)$ with CM-type $(1, 2)$ in the notation of Section 5.1. The Galois group of K_1/\mathbb{Q} is cyclic of order 4, hence its unique subgroup of order 2 is generated by complex conjugation, which cannot fix the CM-type. Indeed, the Jacobian of D_1 is simple. In the genus 2 case, all CM-types of a cyclic CM-field are primitive. We have already seen that this does not hold in general for genus $g \geq 3$.

The curve D_2 has CM by $K_2 := \mathbb{Q}(\zeta_8)$. The corresponding Galois group is isomorphic to $C_2 \times C_2$, hence the CM-type is imprimitive. Indeed, the CM-type is $(1, 3)$ which is fixed by $\langle 3 \rangle \subset (\mathbb{Z}/8\mathbb{Z})^*$. The CM-type $(1, 3)$ is induced from the CM-type of the elliptic curve $E := D_2/\langle \tau \rangle$, where $\tau(x, y) = (1/x, y^3/x(x-1))$ is an automorphism of order 4.

4. REDUCTION OF CM-CURVES AND THEIR JACOBIANS

Our main result (Theorem 6.8) deals with curves C of genus 3 defined over some number field whose Jacobians have CM by a sextic CM-field K . In this section, we describe the possibilities for the reduction of these curves and their Jacobians to characteristic $p > 0$.

4.1. The theorem of Serre–Tate. Let C be a curve of genus $g \geq 2$ defined over a number field M , and let $J := \text{Jac}(C)$ be its Jacobian. In the course of our arguments, we allow ourselves to replace M by a finite extension, which we still denote by M . Let ν be a finite place of M . We write \mathcal{O}_ν for the valuation ring of ν and k_ν for its residue field. We write \overline{k}_ν for an algebraic closure of k_ν .

Recall that the abelian variety J has *good reduction* at ν if there exists an abelian scheme \mathcal{J} over \mathcal{O}_ν with $\mathcal{J} \otimes_{\mathcal{O}_\nu} M \simeq J$. This implies that the reduction $\overline{J} := \mathcal{J} \otimes_{\mathcal{O}_\nu} \overline{k}_\nu$ is an abelian variety. We say that J has *potentially good reduction* at ν if there exists a finite extension M'/M and an extension ν' of ν such that $J \otimes_M M'$ has good reduction at ν' .

The following theorem is Theorem 6 of [ST68].

Theorem 4.1. (*Serre–Tate*) *Let J be an abelian variety with CM defined over a number field M . Let ν be a finite place of M . Then J has potentially good reduction at ν .*

Since there are at most finitely places where J does not have good reduction, there exists a finite extension of M over which J has good reduction everywhere.

4.2. Reduction of genus 3 curves with CM. We now describe the restrictions imposed by Theorem 4.1 on the reduction of the curve C .

Recall that C is a curve of genus $g(C) \geq 2$ defined over a number field M . We say that C has *good reduction* at a finite place ν of M if there exists a model \mathcal{C} over \mathcal{O}_ν with $\mathcal{C} \otimes_{\mathcal{O}_\nu} M \simeq C$ such that the reduction $\overline{C} := \mathcal{C} \otimes_{\mathcal{O}_\nu} \overline{k}_\nu$ is smooth. Similarly, C has *potentially good reduction* at ν if it has good reduction over a finite extension of M .

We say that C has *semistable reduction* at ν if there exists a model \mathcal{C} over \mathcal{O}_ν with $\mathcal{C} \otimes_{\mathcal{O}_\nu} M \simeq C$ such that the reduction \overline{C} is semistable. This means that \overline{C} is reduced and has at most ordinary double points as singularities. The corresponding model $\mathcal{C} = \mathcal{C}_\nu$ is called a *semistable model* of C at ν . The Stable Reduction Theorem ([DM69], Corollary 2.7) states that every curve C admits a semistable model at ν after replacing M by a finite extension. Since we assume that $g(C) \geq 2$, there exists a unique minimal semistable model, which is called the *stable model* at ν . Its special fiber \overline{C} is called the *stable reduction* of C at ν . The minimality of the stable model implies that C has potentially good reduction if and only if the stable reduction \overline{C} is smooth. If the finite place ν is fixed, we usually omit it.

We now turn to our situation of interest, namely that of a genus 3 curve whose Jacobian has CM by a sextic CM-field. The following proposition is a consequence of Theorem 4.1.

We say that C has *bad reduction* at ν if it does not have potentially good reduction at ν . This is equivalent to the stable reduction \overline{C} having singularities. We say that the reduction \overline{C} of C is *tree-like* if the intersection graph of the irreducible components of \overline{C} is a tree. Note that we always consider the reduction \overline{J} (resp. \overline{C}) as an abelian variety (resp. curve) defined over the algebraically closed field $\overline{k_\nu}$ for convenience.

Proposition 4.2. *Let C be a curve of genus 3 defined over a number field M such that its Jacobian $J = \text{Jac}(C)$ has CM. Let ν be place of M where C has bad reduction. Then*

- (a) *the stable reduction reduction \overline{C} of C is tree-like,*
- and*
- (b) *the reduction \overline{J} of J is the product of the Jacobians of the irreducible components of \overline{C} (as polarized abelian varieties).*

Proof. Let ν be a finite place of M . After replacing M by a finite extension and choosing an extension of ν , we may assume that C has stable reduction at ν . Let \mathcal{C} be the stable model of C . Set $S = \text{Spec}(\mathcal{O}_\nu)$, and define $\text{Pic}^0(\mathcal{C}/S)$ to be the identity component of the Picard variety. Since the stable reduction \overline{C} of C is reduced, Theorem 1 in Section 9.5 of [BLR90] states that $\text{Pic}^0(\mathcal{C}/S)$ is a Néron model of J .

Theorem 4.1 implies that J has potentially good reduction, i.e., there exists an abelian variety \mathcal{J} over S with generic fiber J . Proposition 8 of Section 1.2 in [BLR90] shows that \mathcal{J}/S is a Néron model. Since two different Néron models are canonically isomorphic, it follows that $\text{Pic}^0(\mathcal{C}/S) \simeq_S \mathcal{J}$. In particular, it follows that the special fiber $\text{Pic}^0(\mathcal{C}/S) \otimes_{\mathcal{O}_\nu} \overline{k_\nu} \simeq \text{Pic}^0(\overline{C})$ is an abelian variety.

Example 8 of Section 9.2 in [BLR90] shows that $\text{Pic}^0(\overline{C})$ is given by an exact sequence

$$1 \rightarrow T \rightarrow \text{Pic}^0(\overline{C}) \rightarrow B := \prod_i \text{Jac}(\tilde{C}_i) \rightarrow 1, \quad (4.1)$$

where B is an abelian variety and T is a torus. The product on the right-hand side is taken over the irreducible components of \overline{C} . We denote the normalization of an irreducible component C_i of \overline{C} by \tilde{C}_i . The torus T satisfies

$$T \simeq \mathbb{G}_{m, \overline{k_\nu}}^t$$

for some $t \geq 0$. The torus \mathbb{G}_m is not compact, and hence not an abelian variety. Since $\text{Pic}^0(\overline{C})$ is an abelian variety, the exact sequence (4.1) implies that $t = 0$, i.e., $\text{Pic}^0(\overline{C})$ contains no torus. By Corollary 12.b of [BLR90], this means that the intersection graph of the irreducible components of \overline{C} is a tree. Both statements of the proposition follow from this. \square

The corollary below follows immediately from Proposition 4.2. In Section 5, we give examples of each of the cases.

Corollary 4.3. *Let C be a genus 3 curve with CM defined over a number field M , and let ν be a finite place of M . One of the following three possibilities holds for the irreducible components of \overline{C} of positive genus:*

- (i) *(good reduction) \overline{C} is a smooth curve of genus 3,*
- (ii) *\overline{C} has three irreducible components of genus 1,*

(iii) \overline{C} has an irreducible component of genus 1 and one of genus 2.

Note that the stable reduction \overline{C} may contain irreducible components of genus 0. This happens for the stable reduction \overline{C}_1 to characteristic 3 of the curve C_1 from Lemma 5.3, for example. One may show that \overline{C}_1 has four irreducible components: one of genus 0 and three of genus 1. The three elliptic curves each intersect the genus 0 curve in one point but do not intersect each other. Since the irreducible components of genus 0 do not contribute to the Jacobian, we have not listed them in Corollary 4.3.

Remark 4.4. Let C be a curve of genus 3 with CM, defined over a number field M . Suppose that C has bad reduction at a finite place ν of M . In Case (ii) of Corollary 4.3, the reduction \overline{C} of C contains three irreducible components E_i of genus 1. Proposition 4.2 implies that

$$\overline{J} \simeq E_1 \times E_2 \times E_3$$

as polarized abelian varieties, i.e., the polarization on \overline{J} is the product polarization.

In Case (iii) of Corollary 4.3, \overline{C} contains an irreducible component E of genus 1 and an irreducible component D of genus 2. In this case, we have

$$\overline{J} \simeq E \times \text{Jac}(D)$$

and the polarization on \overline{J} is induced by $E \times \{0\} + \{0\} \times D \hookrightarrow \overline{J}$. We show below that in this case \overline{J} is still isogenous to a product of elliptic curves (Theorem 4.5). However, it is **not** true that the polarization of \overline{J} is induced by polarization on the three elliptic curves as we had in Case (ii).

Even in the case where C has good reduction (Case (i) of Corollary 4.3), the reduction \overline{J} of the Jacobian need not be simple even if J is. In this case, the polarization of \overline{J} is induced by the embedding of \overline{C} in its Jacobian and hence is not a product polarization.

The following theorem is a generalization of Theorem 3.2 to positive characteristic.

Theorem 4.5. *Let J be an abelian variety of dimension 3 with CM, defined over a number field M . Suppose that the reduction \overline{J} of J at a finite place of M is not simple. Then \overline{J} is isogenous to the product of three copies of the same elliptic curve E .*

Proof. The result is essentially a special case of Theorem 1.3.1.1 of [CCO14]. For the convenience of the reader we sketch a direct proof in our situation.

By assumption, J has CM by the sextic CM-field K . This implies that we have an embedding

$$K \hookrightarrow \text{End}^0(\overline{J}).$$

Decompose \overline{J} into isotypic components: $\overline{J} \sim \prod_i A_i^{n_i}$ where the A_i are simple and $A_i \not\sim A_j$ for $i \neq j$. Since \overline{J} is not simple by assumption, for dimension reasons there exists j such that $A_j = E$ is an elliptic curve. We have $K \hookrightarrow \text{End}^0(\overline{J}) = \prod_i M_{n_i}(\text{End}^0(A_i))$. Projecting this ring homomorphism on the j th factor gives an injection $K \hookrightarrow M_{n_j}(\text{End}^0(E))$. A dimension argument shows that $n_j = 3$ and therefore $\overline{J} \sim E^3$. \square

Proposition 4.6. *Let C be a genus 3 curve with CM, defined over a number field M . Suppose that C has bad reduction at a finite place ν of M . Then the reduction \overline{J} of the Jacobian J of C is supersingular or K contains an imaginary quadratic field K_1 .*

Proof. Let C and \bar{J} be as in the statement of the proposition. Since C has bad reduction at ν , Corollary 4.3 shows that \bar{C} has an irreducible component E_1 of genus 1. It follows that we may regard E_1 as abelian subvariety of \bar{J} . (This is slightly weaker than the statement in Remark 4.4.) In particular, \bar{J} is not simple. Theorem 4.5 implies therefore that \bar{J} is isogenous to the product of three copies of an elliptic curve E . Note that \bar{J} is supersingular if and only if E is.

We assume that E is ordinary. Since E may be defined over a finite field, it has CM and $K_1 := \text{End}^0(E)$ is an imaginary quadratic field contained in the center of $\text{End}^0(E^3) = M_3(K_1)$. Since \bar{J} is isogenous to E^3 , we obtain an embedding

$$K = \text{End}^0(J) \hookrightarrow \text{End}^0(\bar{J}) \simeq \text{End}^0(E^3) = M_3(K_1).$$

Theorem 1.3.1.1 of [CCO14] states that K is its own centralizer in $M_3(K_1)$. Since the center of $M_3(K_1)$ is K_1 , we conclude that K_1 is contained in K and the result follows. \square

The following corollary summarizes the results so far in the case that the CM-field K does not contain an imaginary quadratic subfield K_1 .

Corollary 4.7. *Let C be a genus 3 curve with CM by K , defined over a number field M . Suppose that K does not contain an imaginary quadratic subfield. Then the following hold:*

- (a) *the CM-type (K, φ) of J is primitive, and J is absolutely simple,*
- (b) *if C has bad reduction at a finite place ν , then the reduction of J at ν is supersingular.*

Proof. Part (a) follows from Corollary 3.4 and Theorem 3.2. Part (b) follows from Proposition 4.6. \square

4.3. Polarizations and the Rosati involution. In the rest of this section, we recall some results on the Rosati involution following Sections 20 and 21 of [Mum70] and Section 17 of [Mil08]. For precise definitions and more details, we refer to these sources. Let A be an abelian variety and $\lambda : A \rightarrow A^\vee$ be a polarization associated with an ample line bundle \mathcal{L} on A . The polarization λ is an isogeny and therefore has an inverse $\frac{1}{\deg \lambda} \lambda^\vee = \lambda^{-1} \in \text{Hom}(A^\vee, A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

The Rosati involution on $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$ is defined by

$$f \mapsto f^* = \lambda^{-1} \circ f^\vee \circ \lambda.$$

It satisfies

$$(f + g)^* = f^* + g^*, \quad (fg)^* = g^* f^*, \quad a^* = a$$

for $f, g \in \text{End}^0(A)$ and $a \in \mathbb{Q}$. In the case where λ is a principal polarization, i.e., $\deg(\lambda) = 1$, the Rosati involution acts as an involution on $\text{End}(A)$. This is because λ^{-1} is in $\text{Hom}(A^\vee, A)$ and not just in $\text{Hom}(A^\vee, A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The natural polarization on a Jacobian is a principal polarization.

The Rosati involution is a positive involution (Theorem 1 of Section 21 in [Mum70]). This means that

$$(f, g) \mapsto \mathbf{Tr}(f \cdot g^*), \quad \text{End}^0(A) \rightarrow \mathbb{Q}$$

defines a positive definite quadratic form on $\text{End}^0(A)$. (We refer to Section 21 of [Mum70] for the precise definition of the trace.) In the case that $A = E$ is an elliptic curve, we choose the polarization λ defined as

$$\lambda : E \rightarrow \text{Pic}^0(E), \quad P \mapsto [P] - [O].$$

The corresponding Rosati involution sends an isogeny f to its dual isogeny f^\vee and $\mathbf{Tr}(f \cdot f^\vee)$ is $\deg(f)$, the degree of the endomorphism f .

Proposition 4.8. *Let A be a simple abelian variety defined over a field of characteristic zero with principal polarization λ . Assume that A has CM by a field K . Then the Rosati involution associated with λ induces complex conjugation on the CM-field K .*

Proof. Since A is simple, the endomorphism algebra $\text{End}^0(A)$ equals K and the proposition is proved, for example, in Lemma 1.3.5.4 of [CCO14]. \square

Remark 4.9. Let A be a simple abelian variety with $\text{End}^0(A) = K$ as in the statement of Proposition 4.8. Let M be a number field over which A can be defined, and let \mathfrak{p} be a prime of M at which A has good reduction. Write \overline{A} for the reduction. We obtain an embedding

$$K \hookrightarrow \text{End}^0(\overline{A}).$$

The Rosati involution on $\text{End}^0(\overline{A})$ is an extension of the Rosati involution on $\text{End}^0(A) = K$, which is complex conjugation by Proposition 4.8.

The following proposition was used in the proofs of [GL07] but not stated there explicitly.

Proposition 4.10. *Suppose that $A = E^n$ is a product of elliptic curves as polarized abelian varieties. Then the Rosati involution acts as*

$$M_n(\text{End}(E)) \rightarrow M_n(\text{End}(E)), \quad (f_{i,j}) \mapsto (f_{j,i}^\vee).$$

Proof. The result is well known to the experts. We sketch the argument. The proof we present here is a variant of the proof of Proposition 11.28 (ii) of [GM].

Let $A = E^n$ be as in the statement of the lemma, and write $p_i : A \rightarrow E$ for the projection on the i th coordinate. Then any line bundle \mathcal{L} on A satisfies $\mathcal{L} = p_1^* \mathcal{L}_1 \otimes \cdots \otimes p_n^* \mathcal{L}_n$ for suitable line bundles \mathcal{L}_i on E .

Consider the natural map $A^\vee = \text{Pic}^0(A) \rightarrow (\text{Pic}^0(E))^n = (E^\vee)^n$ which sends a line bundle $\mathcal{L} \in \text{Pic}^0(A)$ to the n -tuple $(\mathcal{L}|_{E_i})_i \in (\text{Pic}^0(E))^n$ of the restrictions of \mathcal{L} to the i th copy $E_i := (\cdots, 0, E, 0, \cdots)$ of E . One shows that this map is an isomorphism (Exercise 6.2 of [GM]). The product polarization $\lambda_A : A \rightarrow A^\vee = (E^\vee)^n$ is induced by the natural polarization $\lambda : E \rightarrow \text{Pic}^0(E)$ on E . In particular, it is also a principal polarization.

Using this identification, it suffices to prove the proposition in the case that $f \in \text{End}(A)$ corresponds to a $n \times n$ matrix with an endomorphism $\alpha \in \text{End}(E)$ as (j, i) th component and zeros everywhere else. The endomorphism $f^\vee : A^\vee \rightarrow A^\vee$ induced by f sends a line bundle \mathcal{L} on A to $p_i^*(\alpha^* \mathcal{L}_i)$. We conclude that the dual isogeny $f^* : A \rightarrow A = E^n$ corresponds to the matrix with the dual isogeny α^\vee in the (i, j) th coordinate and zeros elsewhere. This proves the proposition. \square

5. EXAMPLES

In this section we discuss some examples of genus 3 curves with CM.

5.1. Cyclic covers. The first type of examples we consider are N -cyclic covers of the projective line branched at exactly three points, see also Sections 1.6 and 1.7 of Chapter 1 of [Lan83]. More precisely, let C be a smooth projective curve defined over a field of characteristic zero which admits a Galois cover $\pi : C \rightarrow \mathbb{P}^1$ whose Galois group is cyclic of order N

such that π is branched exactly at three points. We may assume the three branch points to be $0, 1, \infty \in \mathbb{P}^1$.

Kummer theory implies the existence of integers $0 < a_1, a_2 < N$ with $\gcd(N, a_1, a_2) = 1$ such that the extension of function field corresponding to π is

$$\mathbb{Q}(x) \subset \mathbb{Q}(x)[y]/(y^N - x^{a_1}(x-1)^{a_2}).$$

The Galois group of π is generated by $\alpha(x, y) = (x, \zeta_N y)$, where ζ_N is a primitive N th root of unity.

Define $0 < a_3 < N$ by $a_1 + a_2 + a_3 \equiv 0 \pmod{N}$. Then a chart at ∞ may be given by

$$w^N = z^{a_3}(z-1)^{a_2},$$

where $z = 1/x$. The condition that π is branched at ∞ is therefore equivalent to $a_3 \equiv -(a_1 + a_2) \not\equiv 0 \pmod{N}$. The Riemann–Hurwitz formula shows that

$$2g(C) - 2 = -2N + \sum_{i=1}^3 (N - \gcd(N, a_i)).$$

In Lemma 5.1 below, we show that the endomorphism ring of $\text{Jac}(C)$ contains $\mathbb{Q}(\zeta_N)$. Therefore $\text{Jac}(C)$ has CM by $\mathbb{Q}(\zeta_N)$ if and only if $2g(C) = \varphi(N)$, where φ denotes Euler's totient function. For example, this condition is satisfied if N is an odd prime. This case is discussed by Lang (Section 1.7 of Chapter 1 of [Lan83]).

The condition $2g(C) = \varphi(N)$ is satisfied for exactly three curves C_i , up to isomorphism. Kummer theory implies that two tuples (N, a_1, a_2, a_3) and (M, b_1, b_2, b_3) define isomorphic curves if and only if $N = M$ and there exists an integer c with $\gcd(c, N) = 1$ and a permutation $\sigma \in S_3$ such that $b_i \equiv ca_{\sigma(i)} \pmod{N}$ for all i . This is similar to the argument in Section 1.7 of Chapter 1 of [Lan83].

The three curves satisfying this property are:

$$\begin{aligned} C_1 : \quad & y^9 = x(x-1)^3, \\ C_2 : \quad & y^7 = x(x-1)^2, \\ C_3 : \quad & y^7 = x(x-1). \end{aligned}$$

An alternative equation for C_1 is

$$y^3 = z^4 - z, \quad \text{where } z^3 = x. \tag{5.1}$$

We put $K_{N_i} = \mathbb{Q}(\zeta_{N_i})$ and $G_{N_i} = (\mathbb{Z}/N_i\mathbb{Z})^*$. In the three cases we consider in Lemma 5.1, we have $G_{N_i} \simeq C_6$. For $j \in (\mathbb{Z}/N_i\mathbb{Z})^*$, we denote the corresponding element of $\text{Gal}(K_{N_i}/\mathbb{Q})$ by

$$\sigma_j : \zeta_{N_i} \mapsto \zeta_{N_i}^j,$$

or also by j when no confusion can arise.

The following lemma summarizes the properties of the curves C_i .

Lemma 5.1. (a) *The curve C_1 has CM by $\mathbb{Q}(\zeta_9)$. The CM-type is $(1, 2, 4)$. This type is primitive.*

(b) *The curve C_2 has CM by $\mathbb{Q}(\zeta_7)$ and CM-type $(1, 2, 4)$. This type is imprimitive.*

(c) *The curve C_3 has CM by $\mathbb{Q}(\zeta_7)$ and CM-type $(1, 2, 3)$. This type is primitive.*

Proof. It is easy to check that the automorphism α of C_i has a fixed point. Using this point to embed the curve C_i in its Jacobian, we see that α induces an endomorphism $\alpha \in \text{End}(\text{Jac}(C_i))$ of multiplicative order N_i .

We may regard $\alpha \in \text{End}(\text{Jac}(C_i))$ as a primitive N_i th root of unity. In all three cases, we have $2g(C_i) = 6 = \varphi(N_i) = [\mathbb{Q}(\zeta_{N_i}) : \mathbb{Q}]$. It follows that C_i has CM by K_{N_i} .

To calculate the CM-type of C_i we follow the strategy of Section 1.7 of Chapter 1 of [Lan83], and identify the cohomology group $H^0(C_i, \Omega)$ of holomorphic differentials with the tangent space of $\text{Jac}(C)$. It suffices to find a basis of $H^0(C_i, \Omega)$ consisting of eigenvectors of α^* , the map induced by α on $H^0(C_i, \Omega)$. Such a basis is computed in Section 1.7 of Chapter 1 of [Lan83]. The statement on the CM-type easily follows from this. (The fact that the action of $\langle \alpha \rangle$ on $H^0(C_i, \Omega)$ does not factor through the action of a quotient group provides a second proof that α defines an endomorphism of order N_i of $\text{Jac}(C_i)$.)

We explain what happens for C_1 . We use a slightly different notation from Theorem 1.7.1 of Chapter 1 of [Lan83]. A basis of $H^0(C_1, \Omega)$ is given by

$$\omega_1 = \frac{y dx}{x(x-1)}, \quad \omega_2 = \frac{y^2 dx}{x(x-1)}, \quad \omega_4 = \frac{y^4 dx}{x(x-1)^2}.$$

Note that $\alpha^* \omega_i = \zeta_9^i \omega_i$. The statement on the CM-type of $\text{Jac}(C_1)$ follows. Primitivity is shown in Section 3.1.

In Example 3.5 we have determined all primitive CM-types for $\mathbb{Q}(\zeta_7)$. The statements on the (im)primitivity of the CM-types of C_2 and C_3 follow from this. \square

Remark 5.2. Lemma 5.1.(b) implies that $\text{Jac}(C_2)$ is not simple. We may also check this directly. The curve C_2 admits an automorphism

$$\beta(x, y) = \left(\frac{1}{1-x}, \frac{y^2}{1-x} \right).$$

The curve $E := C_2 / \langle \beta \rangle$ has genus 1. This curve has CM by the field $K_1 = \mathbb{Q}(\zeta_7)^{\langle \sigma_2 \rangle} = \mathbb{Q}(\sqrt{-7})$.

One checks that $\langle \alpha, \beta \rangle \simeq \mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/3\mathbb{Z}$ is a non-abelian group. Using the method of Kani–Rosen ([KR89] or [Pau08]), one may also deduce from this that

$$\text{Jac}(C_2) \sim E^3.$$

Our next goal is to describe the reduction behavior of the curves C_1 and C_3 .

Lemma 5.3. (a) *The curve C_1 has bad reduction at $p = 3$ and good reduction at all other primes.*

(b) *The reduction $\overline{J}_{1,p}$ of the Jacobian J_1 of C_1 to characteristic p is ordinary if and only if $p \equiv 1 \pmod{9}$ and supersingular if and only if $p = 3$ or $p \equiv 2 \pmod{3}$.*

(c) *If $p \equiv 4, 7 \pmod{9}$, then the abelian variety $\overline{J}_{1,p}$ is simple.*

Proof. It is easy to see that C_1 has good reduction to characteristic $p \neq 3$. Indeed, (5.1) still defines a smooth projective curve in characteristic $p \neq 3$. We consider the reduction at $p = 3$. In this case, the extension of function fields

$$\mathbb{F}_3(z) \subset \mathbb{F}_3(z)[y]/(y^3 - z(z^3 - 1))$$

defines a purely inseparable field extension. This implies that $\mathbb{F}_3(z)[y]/(y^3 - z(z^3 - 1))$ is the function field of a curve of genus 0. This does not imply that C_1 has bad reduction to characteristic 3, since there could be a different model.

We claim that there does not exist a curve of genus 3 in characteristic 3 with an automorphism of order 9. This claim implies that C has bad reduction to characteristic 3. Indeed, if C has potentially good reduction, then the automorphism group $\text{Aut}(\overline{C})$ of the reduction \overline{C} of C contains $\text{Aut}(C)$. Hence, in particular, $\text{Aut}(\overline{C})$ contains an automorphism of order 9.

To obtain a contradiction, we assume that X is a curve of genus 3 in characteristic 3 with an automorphism γ of order 9. We consider the Galois cover

$$X \rightarrow X/\langle \gamma \rangle.$$

This cover is wildly ramified of order 9 above at least one point. We apply the Riemann–Hurwitz formula to this cover. It follows from Theorem 1.1 of [OP10] that the contribution of a wild ramification point with ramification index 9 to $2g(X) - 2$ in the Riemann–Hurwitz formula is at least $2 \cdot (9 - 1) + 5 \cdot (3 - 1) = 26$, which contradicts the assumption that X has genus 3. This proves (a).

We have shown that C_1 has bad reduction to characteristic 3. Let $\overline{C}_{1,3}$ be the stable reduction of C_1 to characteristic 3. Then $\overline{C}_{1,3}$ contains at least 2 irreducible components of positive genus (Corollary 4.3). Furthermore, there is an automorphism of order 9 acting on $\overline{C}_{1,3}$. The only way this is possible is if $\overline{C}_{1,3}$ contains three irreducible components of positive genus, which are then elliptic curves, each with an automorphism of order 3. The automorphism of order 9 permutes these components. There is a unique elliptic curve with an automorphism of order 3, namely the elliptic curve with $j = 0$. In characteristic 3 this curve may be given by

$$w^3 - w = v^2. \tag{5.2}$$

This curve is supersingular by the Deuring–Shafarevich formula ([Cre84]). We conclude that the reduction $\overline{J}_{1,3}$ of the Jacobian of C_1 to characteristic 3 is supersingular. Proposition 4.2.(b) implies that $\overline{J}_{1,3}$ is in fact superspecial: the Jacobian $\overline{J}_{1,3}$ is isomorphic to three copies of the supersingular elliptic curve (5.2) as a polarized abelian variety.

The rest of (b) may be deduced from [Yui80]. For $p \equiv 4, 7 \pmod{9}$, Yui’s results [Yui80] imply that \overline{J} is neither ordinary nor supersingular. In fact, her results imply that \overline{J} has p -rank zero, but is not supersingular. Theorem 4.5 therefore implies that \overline{J} is simple. \square

The situation for C_3 is similar but somewhat easier.

- Lemma 5.4.** (a) *The curve C_3 has good reduction at $p \neq 7$ and potentially good reduction at $p = 7$.*
(b) *The reduction $\overline{J}_{3,p}$ of the Jacobian J_3 of C_3 to characteristic p is ordinary if and only if $p \equiv 1 \pmod{7}$ and supersingular if and only if $p = 7$ or $p \equiv -1, 3, 5 \pmod{7}$.*

Proof. The fact that C_3 has good reduction to characteristic $p \neq 7$ follows as in the proof of Lemma 5.3. The curve C_3 has potentially good reduction to characteristic 7 as well, see Example 3.8 of [BW12]. The curve C_3 does not have good reduction over \mathbb{Q}_7 but acquires good reduction over the extension $\mathbb{Q}_7(\zeta_7)$ of \mathbb{Q}_7 .

Statement (b) for $p \neq 7$ follows from [Yui80]. We consider the reduction $\overline{C}_{3,7}$ of C to characteristic 7. In characteristic 7, the reduction $\overline{C}_{3,7}$ is given by

$$w^7 - w = v^2,$$

by Example 3.8 of [BW12]. By the Deuring–Shafarevich formula, it follows that the Jacobian $\overline{J}_{3,7}$ of $\overline{C}_{3,7}$ has p -rank 0. To show that it is supersingular, it suffices to find an elliptic quotient of the curve $\overline{C}_{3,7}$.

The curve $\overline{C}_{3,7}$ admits an extra automorphism of order 3 given by

$$\beta(v, w) = (\zeta_3 v, \zeta_3^2 w),$$

where $\zeta_3 \in \mathbb{F}_7^\times$ is an element of order three. The automorphism β has exactly two fixed points, namely the points with $w = 0, \infty$. It follows that $E_{3,7} := \overline{C}_{3,7}/\langle \beta \rangle$ is an elliptic curve. This shows that $\overline{J}_{3,7}$ is supersingular. \square

5.2. A Picard curve example. We end this section by considering Example 3 from Section 5 of [KW05], wherein Koike and Weng study Picard curves with CM. We show that the curve in the aforementioned example has bad reduction to characteristic $p = 5$, and that the stable reduction consists of an elliptic curve and a curve of genus 2. We will show that the Jacobian has superspecial reduction in this case. This is an example where the reduction \overline{J} of the Jacobian is isomorphic to E^3 , but the polarization is neither that of a smooth curve nor the product polarization $E \times \{0\} \times \{0\} + \{0\} \times E \times \{0\} + \{0\} \times \{0\} \times E$.

A Picard curve is a curve of genus 3 given by an equation

$$y^3 = f(x),$$

where $f(x) \in \mathbb{C}[x]$ is a polynomial of degree 4 with simple roots. Every Picard curve admits an automorphism $\alpha(x, y) = (x, \zeta_3 y)$. Therefore, the endomorphism ring of the Jacobian contains $\mathbb{Q}(\zeta_3)$.

Let C_4 be the smooth projective curve defined by

$$y^3 = f(x) := x^4 - 13 \cdot 2 \cdot 7^2 \cdot x^2 + 2^3 \cdot 13 \cdot 5 \cdot 47 \cdot x - 5^2 \cdot 31 \cdot 13^2.$$

Koike and Weng show that the Jacobian of C_4 has CM by the field $K = K^+ K_1$ with $K_1 = \mathbb{Q}(\zeta_3)$ and $K^+ = \mathbb{Q}[t]/(t^3 - t^2 - 4t - 1)$. The CM-field K is Galois over \mathbb{Q} , hence we are in Case 1 of Proposition 2.1. One may show that the corresponding CM-type is primitive. For example, one may check using [Bou01] that the reduction $\overline{J}_{4,7}$ of the Jacobian J_4 of C_4 to characteristic 7 has p -rank 1, and hence is neither ordinary nor supersingular. It follows from this that the Jacobian J_4 is simple. The primitivity of the CM-type follows from this, by Theorem 3.2.

We now consider the reduction of C_4 . The discriminant of f is $2^{12} \cdot 5^6 \cdot 13^4$ which shows that C_4 has good reduction for $p \neq 2, 3, 5, 13$. One may check that C_4 also has good reduction at $p = 2, 13$. We do not consider what happens for $p = 3$.

We determine the reduction at $p = 5$. Note that

$$f(x) \equiv x^2(x+2)(x-2) = x^4 + x^2 \pmod{5}. \quad (5.3)$$

Therefore, the stable reduction of C_4 contains an irreducible component \overline{D} of genus 2 given by the equation

$$\overline{y}^3 = \overline{x}^2(\overline{x}^2 + 1). \quad (5.4)$$

The reason that this curve has genus 2 rather than 3 is that the 3-cyclic cover $(\overline{x}, \overline{y}) \mapsto \overline{x}$ has only 4 branch points in characteristic 5, and not 5 branch points as it had in characteristic zero. It follows that the curve C_4 has bad reduction to characteristic 5, and the reduction of C_4 consists of the curve \overline{D} of genus 2 intersecting with an elliptic curve. (We do not actually

have to compute the elliptic component to conclude this.) The reduction \overline{J}_4 of the Jacobian of C_4 is therefore isogenous to the product of an elliptic curve and the abelian surface $\text{Jac}(\overline{D})$. To determine the reduction type of \overline{J}_4 , we first consider the Jacobian $\text{Jac}(\overline{D})$ of the curve \overline{D} given by the equation (5.4).

One may show by computing the Hasse–Witt matrix of \overline{D} that the Jacobian $J(\overline{D})$ is supersingular. This is a similar calculation to the one we did in Section 5.1. However, since \overline{D} has genus 2, it suffices to compute the p -rank. In fact, the Hasse–Witt matrix is identically zero, which shows that $J(\overline{D})$ is superspecial, i.e., isomorphic to the product of two supersingular elliptic curves.

Alternatively, we may note that \overline{D} has additional automorphisms given by

$$\tau(\bar{x}, \bar{y}) = (-\bar{x}, \bar{y}), \quad \rho(\bar{x}, \bar{y}) = \left(-\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}^2}\right), \quad \tau \circ \rho(\bar{x}, \bar{y}) = \left(\frac{1}{\bar{x}}, \frac{\bar{y}}{\bar{x}^2}\right).$$

Note that τ fixes the two points with $\bar{x} = 0, \infty$ and ρ fixes the two points with $\bar{x}^2 = -1$. The quotients $\overline{C}_4/\langle\tau\rangle$ and $\overline{C}_4/\langle\rho\rangle$ are elliptic curves, each with an automorphism of order 3. In particular, these elliptic curves have $j = 0$. Since $p = 5 \equiv 2 \pmod{3}$, they are supersingular. Theorem 4.5 implies that \overline{J} is isogenous to E_0^3 , where E_0 denotes the supersingular elliptic curve over $\overline{\mathbb{F}}_5$ with $j = 0$.

Remark 5.5. The examples we discussed in this section all have the property that the CM-field K contains a CM-subfield K_1 with $\mathbb{Q} \subsetneq K_1 \subsetneq K$. In Section 6.4, we will show that this implies that the embedding problem, which we formulate in Section 6, has degenerate solutions for every prime. This explains why we exclude this case in Theorem 6.8.

Remark 5.6. Let K be a sextic CM-field. It is known how to construct genus 3 curves C in characteristic 0 with CM by K ([Shi98], Sections 6.2 and 14.3). We sketch the construction.

We fix a CM-type (K, φ) . Let $\delta_{K/\mathbb{Q}}$ be the different. For any ideal \mathfrak{a} of \mathcal{O}_K we consider the lattice $\varphi(\mathfrak{a}) = (\varphi_1(\mathfrak{a}), \varphi_2(\mathfrak{a}), \varphi_3(\mathfrak{a}))$. Then

$$A := \mathbb{C}^3/\varphi(\mathfrak{a})$$

is an abelian variety with CM by (K, φ) . Shimura (Theorem 3 of Section 6.2 in [Shi98]) shows that all CM abelian varieties occur in this way.

In Section 14.3 of [Shi98], Shimura also describes all Riemann forms defining principal polarizations on A . Such a Riemann form exists if the following two conditions are satisfied.

- The ideal $\delta_{K/\mathbb{Q}}\mathfrak{a}\bar{\mathfrak{a}} = (a)$ is principal.
- There exists a unit $u \in \mathcal{O}_K$ such that ua is totally imaginary and the imaginary part of $\varphi_i(ua)$ is negative for all i .

Every principally polarized abelian variety of dimension 3 is isomorphic to the Jacobian of a (possibly singular) genus 3 curve C by Theorem 4 of [OU73]. More precisely, Oort and Ueno show that the curve C is of compact type, meaning that A is isomorphic to the product of the Jacobians of the irreducible components of positive genus of C . (This notion is essentially the same as the notion “tree-like” that we used in Section 4.2.) In our situation, the abelian variety A is simple, and it follows that the curve C is smooth.

6. EMBEDDING PROBLEM

6.1. Formulation of the embedding problem. Let C be a genus 3 curve defined over some number field M . We assume that the Jacobian $J = \text{Jac}(C)$ has CM by a sextic CM-field K . After replacing M by a finite extension if necessary, we may assume that J has good reduction (Theorem 4.1) and that C has stable reduction at all finite places of M .

In this section, we make the following important assumption.

Assumption 6.1. We assume that K does not contain an imaginary quadratic subfield.

Recall that Assumption 6.1 implies that the CM-type of C is primitive (Corollary 4.7). The reason for making this assumption is discussed in Section 6.4.

Let \mathfrak{p} be a finite prime of M where the curve C has bad reduction. We write \bar{k} for the algebraic closure of the residue field at \mathfrak{p} and let p denote the residue characteristic. We want to bound these primes p . (See Theorem 6.8 for the precise statement of our result.) Recall from Corollary 4.3 that there are two possibilities for the reduction \bar{C} of C . In this section, we only deal with the case where \bar{C} has three irreducible components of genus 1 and postpone the other case for future work. To summarize, we make the following assumption on the prime \mathfrak{p} .

Assumption 6.2. Let \mathfrak{p} be a finite prime of M , such that the stable reduction $\bar{C} = \bar{C}_{\mathfrak{p}}$ of C at \mathfrak{p} contains three elliptic curves as irreducible components (Case (ii) of Corollary 4.3).

Let \mathfrak{p} be as in Assumption 6.2. We write E_1, E_2, E_3 for the three elliptic curves that are the irreducible components of \bar{C} . We write \bar{J} for the reduction of J at \mathfrak{p} . Recall from Remark 4.4 that we have an isomorphism

$$\bar{J} \simeq E_1 \times E_2 \times E_3$$

as polarized abelian varieties, i.e., the polarization on \bar{J} is the product polarization. Corollary 4.7 implies that the E_i are supersingular. In particular, they are isogenous. (This also follows from Theorem 4.5).

Let $\text{End}(J) = \mathcal{O} \subset \mathcal{O}_K$. Reduction at the prime \mathfrak{p} gives an injective ring homomorphism

$$\mathcal{O} \hookrightarrow \text{End}(\bar{J}) \simeq \text{End}(E_1 \times E_2 \times E_3).$$

Problem 6.3 (The embedding problem). Let \mathcal{O} be an order in a sextic CM-field K , and let p be a prime number. The *embedding problem* for \mathcal{O} and p is the problem of finding elliptic curves E_1, E_2, E_3 defined over a field of characteristic p , and a ring embedding

$$i : \mathcal{O} \hookrightarrow \text{End}(E_1 \times E_2 \times E_3)$$

such that the Rosati involution on $\text{End}(E_1 \times E_2 \times E_3)$ induces complex conjugation on \mathcal{O} . We call such a ring embedding a *solution to the embedding problem* for \mathcal{O} and p .

The following result states that if we have a solution to the embedding problem then the elliptic curves E_i are automatically isogenous. The proof we give here works directly with the abelian variety $E_1 \times E_2 \times E_3$ without considering it as the reduction of an abelian variety in characteristic zero. However the proof is essentially the same as the proofs of Theorem 4.5 and Proposition 4.6.

Lemma 6.4. *Let K be a sextic CM-field. Suppose that there exist elliptic curves E_1, E_2, E_3 defined over a field of characteristic $p > 0$ and an injective \mathbb{Q} -algebra homomorphism*

$$i : K \hookrightarrow \text{End}^0(E_1 \times E_2 \times E_3).$$

Then the elliptic curves E_1 , E_2 and E_3 are all isogenous. Furthermore, if K contains no imaginary quadratic subfield then the E_i are supersingular.

Proof. First suppose that no two of the elliptic curves E_1 , E_2 , E_3 are isogenous. Then

$$i : K \hookrightarrow \text{End}^0(E_1 \times E_2 \times E_3) = \begin{pmatrix} \text{End}^0 E_1 & 0 & 0 \\ 0 & \text{End}^0 E_2 & 0 \\ 0 & 0 & \text{End}^0 E_3 \end{pmatrix} = \text{End}^0 E_1 \times \text{End}^0 E_2 \times \text{End}^0 E_3.$$

Projecting on the factor $\text{End}^0 E_i$ gives a ring homomorphism $K \hookrightarrow \text{End}^0 E_i$. Since K is a field, this ring homomorphism must be injective. But $\text{End}^0 E_i$ is either an imaginary quadratic field or a quaternion algebra, neither of which can contain a sextic field.

Now suppose that exactly two of the elliptic curves are isogenous. Without loss of generality, we may assume that $E_1 \sim E_2$ and $E_1 \not\sim E_3$. Then

$$i : K \hookrightarrow \text{End}^0(E_1 \times E_2 \times E_3) = \begin{pmatrix} \text{End}^0 E_1 & \text{End}^0 E_1 & 0 \\ \text{End}^0 E_1 & \text{End}^0 E_1 & 0 \\ 0 & 0 & \text{End}^0 E_3 \end{pmatrix} = M_2(\text{End}^0 E_1) \times \text{End}^0 E_3.$$

Again, projecting on the factor $\text{End}^0 E_3$, we see that $K \hookrightarrow \text{End}^0 E_3$. This is impossible for dimension reasons. Thus, we have proved that all three elliptic curves are isogenous.

Now suppose that K contains no imaginary quadratic subfield and that the elliptic curves E_i are ordinary. Then $\text{End}^0 E_1 = K_1$ for some imaginary quadratic field K_1 and

$$i : K \hookrightarrow \text{End}^0(E_1 \times E_2 \times E_3) = M_3(K_1).$$

Let β be a generator for K over \mathbb{Q} and let f be its minimal polynomial, which has degree 6. The matrix $i(\beta) \in M_3(K_1)$ has a minimal polynomial of degree at most 3 over K_1 . Since i is an injective \mathbb{Q} -algebra homomorphism, this means that f splits over K_1 . Since K_1 is quadratic, this implies that $K_1 \hookrightarrow K$, contradicting the assumption that K contains no imaginary quadratic subfield. \square

Proposition 6.5. *Let C be a genus 3 curve such that $\mathcal{O} := \text{End}(\text{Jac}(C))$ is an order in a sextic CM-field K satisfying Assumption 6.1. Let M be a number field over which C is defined, and let \mathfrak{p} be a prime of bad reduction of C such that Assumption 6.2 is satisfied. Write p for the residue characteristic of \mathfrak{p} . Then there exists a solution to the embedding problem for \mathcal{O} and p . Moreover, in this situation the three elliptic curves are supersingular.*

Proof. Let C be as in the statement of the proposition. Then the CM-type of its Jacobian J is primitive (Corollary 4.7.(a)). Therefore the Rosati involution acts as complex conjugation on $\text{End}^0(J) = K$ by Proposition 4.8. The canonical polarization on the Jacobian J is a principal polarization, therefore the Rosati involution also acts on $\text{End}(J) = \mathcal{O}$.

Assumption 6.2 implies that the reduction \overline{J} of the Jacobian at \mathfrak{p} is isomorphic to a product of three elliptic curves E_i as polarized abelian varieties. These elliptic curves are supersingular (Corollary 4.7.(b)). Remark 4.9 shows that we obtain a solution to the embedding problem. \square

6.2. Endomorphisms of \overline{J} as 3×3 matrices. In this section we describe the ring $\text{End}(E_1 \times E_2 \times E_3)$ from the embedding problem (Problem 6.3). Recall that we may assume that the E_i are isogenous (Lemma 6.4). We recall from Proposition 4.10 the description of the Rosati involution corresponding to the product polarization on $E_1 \times E_2 \times E_3$.

We can view an element $f \in \text{End}(E_1 \times E_2 \times E_3)$ as a matrix

$$f = \begin{pmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{pmatrix},$$

where $f_{i,j} \in \text{Hom}(E_j, E_i)$. Given two endomorphisms f, g the composition $f \circ g$ corresponds to multiplication of matrices. Since the polarization on $\bar{J} = E_1 \times E_2 \times E_3$ is the product polarization, the Rosati involution $f \mapsto f^*$ sends f to

$$\begin{pmatrix} f_{1,1}^\vee & f_{2,1}^\vee & f_{3,1}^\vee \\ f_{1,2}^\vee & f_{2,2}^\vee & f_{3,2}^\vee \\ f_{1,3}^\vee & f_{2,3}^\vee & f_{3,3}^\vee \end{pmatrix}$$

where $f_{i,j}^\vee$ denotes the dual isogeny of $f_{i,j}$.

For $i = 2, 3$, let $\psi_i : E_1 \rightarrow E_i$ be an isogeny of degree δ_i . Let $f \in \text{End}(E_1 \times E_2 \times E_3)$. Then the composition

$$E_1 \times E_1 \times E_1 \xrightarrow{(1, \psi_2, \psi_3)} E_1 \times E_2 \times E_3 \xrightarrow{(1, \delta_2^{-1} \psi_2^\vee, \delta_3^{-1} \psi_3^\vee)} E_1 \times E_1 \times E_1$$

induces an injective \mathbb{Q} -algebra homomorphism

$$\text{End}^0(E_1 \times E_2 \times E_3) \hookrightarrow \text{End}^0(E_1 \times E_1 \times E_1) = M_3(\text{End}^0 E_1). \quad (6.1)$$

Let Φ denote the composite map

$$\Phi : K \hookrightarrow \text{End}^0(E_1 \times E_2 \times E_3) \hookrightarrow M_3(\text{End}^0 E_1).$$

It is easily seen that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} \Phi(\mathcal{O}) \subset M_3(\text{End} E_1).$$

Under the assumptions made in Section 6.1, we may assume that the elliptic curves E_i in the formulation of the embedding problem are supersingular (Proposition 6.5). We therefore recall some well-known facts on the endomorphism ring of a supersingular elliptic curve.

Let $p \in \mathbb{Z}_{>0}$ be the rational prime lying below \mathfrak{p} .

Proposition 6.6. *Let E be a supersingular elliptic curve defined over a field of characteristic p . Then $\text{End}^0 E$ is a quaternion algebra over \mathbb{Q} ramified at precisely the places $\{p, \infty\}$. This quaternion algebra is non-canonically isomorphic to the algebra $B_{p,\infty}$, where $B_{p,\infty} = \left(\frac{-1, -1}{\mathbb{Q}}\right)$ if $p = 2$ and if p is odd, $B_{p,\infty} = \left(\frac{-\varepsilon, -p}{\mathbb{Q}}\right)$ where*

$$\varepsilon = \begin{cases} 1 & \text{if } p \equiv 3 \pmod{4}, \\ 2 & \text{if } p \equiv 5 \pmod{8}, \\ \ell & \text{if } p \equiv 1 \pmod{8}. \end{cases}$$

In the case that $p \equiv 1 \pmod{8}$, $\ell \in \mathbb{Z}_{>0}$ is a prime such that $\ell \equiv 3 \pmod{4}$ and ℓ is not a square modulo p . Any isomorphism sends $\text{End} E$ to an order of $B_{p,\infty}$ and the involution given by taking the dual isogeny corresponds to the canonical involution on $B_{p,\infty}$.

Proof. The fact that the endomorphism algebra $\text{End}^0(E)$ of a supersingular elliptic curve is a quaternion algebra over \mathbb{Q} ramified precisely at $\{p, \infty\}$ is proved, for example, in Section 21 of [Mum70]. The statement on the Rosati involution is also proved in loc. cit. The uniqueness of the quaternion algebra is proved, for example, in Theorem III.3.1 of [Vig80].

For $p = 2$, let $Q = \left(\frac{-1, -1}{\mathbb{Q}}\right)$. For every odd prime p , let ε be as in the statement of the proposition and let $Q = \left(\frac{-\varepsilon, -p}{\mathbb{Q}}\right)$ be the corresponding quaternion algebra. The statement that Q is exactly ramified at the places $\{p, \infty\}$ follows easily from the properties of the Hilbert symbol (page 37 of [Vig80]). \square

For $b \in B_{p, \infty}$, we write $\text{Nrd}(b) = bb^*$ (where b^* represents the involution on the quaternion algebra) for the reduced norm of b . The reduced norm corresponds to the degree of an endomorphism under the identification in Proposition 6.6.

Lemma 6.7 (Elements of small norm commute). [GL07, Corollary 2.1.2] *Let R be a maximal order of $B_{p, \infty}$. If $k_1, k_2 \in R$ and $\text{Nrd}(k_1), \text{Nrd}(k_2) < \sqrt{p}/2$ then $k_1k_2 = k_2k_1$.*

6.3. Bounding the primes of bad reduction for C . Recall that $J = \text{Jac}(C)$ is the Jacobian of a genus 3 curve C which has complex multiplication by an order \mathcal{O} in a sextic CM-field K which does not contain an imaginary quadratic field (Assumption 6.1). Let K^+ denote the totally real cubic subfield of K . The main result of this section is Theorem 6.8 which gives an upper bound on the primes of bad reduction for C satisfying Assumption 6.2.

Theorem 6.8. *Suppose that K does not contain an imaginary quadratic subfield. Let $\mathfrak{p} \mid p$ be a prime of bad reduction for C satisfying Assumption 6.2. Write $K = \mathbb{Q}(\sqrt{\alpha})$ for some totally negative element $\alpha \in K^+ \setminus \mathbb{Z}$ with $\sqrt{\alpha} \in \mathcal{O} = \text{End}(J)$. Then $p \leq 4 \text{Tr}_{K^+/\mathbb{Q}}(\alpha)^6/3^6$.*

The existence of such α is guaranteed because the sextic CM-field K contains no imaginary quadratic subfield. By Proposition 6.5, the following result implies Theorem 6.8.

Theorem 6.9. *Suppose that K does not contain an imaginary quadratic subfield. Let p be a prime such that there exists a solution to the embedding problem (Problem 6.3) for some order \mathcal{O} of K . Write $K = \mathbb{Q}(\sqrt{\alpha})$ for some totally negative element $\alpha \in K^+ \setminus \mathbb{Z}$ with $\sqrt{\alpha} \in \mathcal{O}$. Then $p \leq 4 \text{Tr}_{K^+/\mathbb{Q}}(\alpha)^6/3^6$.*

We break down the proof of Theorem 6.9 into several lemmas.

Let

$$Q = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}$$

be the image of $\sqrt{\alpha}$ in $\text{End}(E_1 \times E_2 \times E_3)$. By Proposition 4.8, the Rosati involution corresponds to complex conjugation on K , so we have

$$\begin{pmatrix} r^\vee & u^\vee & x^\vee \\ s^\vee & v^\vee & y^\vee \\ t^\vee & w^\vee & z^\vee \end{pmatrix} = \begin{pmatrix} -r & -s & -t \\ -u & -v & -w \\ -x & -y & -z \end{pmatrix}. \quad (6.2)$$

Lemma 6.10. *We may assume that the homomorphisms $s : E_2 \rightarrow E_1$ and $t : E_3 \rightarrow E_1$ are both nonzero.*

Proof. Suppose for contradiction that both s and t are zero. Then the image of α in $\text{End}(E_1 \times E_2 \times E_3)$ is

$$Q^2 = \begin{pmatrix} -rr^\vee & 0 & 0 \\ 0 & -vv^\vee - ww^\vee & vw + wz \\ 0 & -w^\vee v - zw^\vee & -w^\vee w - zz^\vee \end{pmatrix}.$$

For $i = 2, 3$, let $\psi_i : E_1 \rightarrow E_i$ be an isogeny of degree δ_i . As seen in (6.1), the ψ_i induce an injective \mathbb{Q} -algebra homomorphism $\text{End}^0(E_1 \times E_2 \times E_3) \rightarrow \text{End}^0(E_1 \times E_1 \times E_1) = M_3(\text{End}^0 E_1)$ sending Q^2 to

$$S = \begin{pmatrix} -rr^\vee & 0 & 0 \\ 0 & -vv^\vee - ww^\vee & \delta_2^{-1}\psi_2^\vee(vw + wz)\psi_3 \\ 0 & \delta_3^{-1}\psi_3^\vee(-w^\vee v - zw^\vee)\psi_2 & -w^\vee w - zz^\vee \end{pmatrix}.$$

Since $(vw + wz)^\vee = -w^\vee v - zw^\vee$, the entries of S commute and therefore form a subfield L of $\text{End}^0 E_1$. Since S is the image of α under an injective \mathbb{Q} -algebra homomorphism, the minimal polynomial of S over L divides the minimal polynomial of α over \mathbb{Q} . Recall that $rr^\vee \in \mathbb{Z}$ is the degree of r . Now $-rr^\vee$ is an eigenvalue of S and therefore a root of its minimal polynomial. But this means that the minimal polynomial of α over \mathbb{Q} has a root in \mathbb{Z} , contradicting its irreducibility.

Therefore, at least one of s, t is nonzero. Using E_2 in place of E_1 , we see that at least one of s, w is nonzero. Using E_3 in place of E_1 , we see that at least one of t, w is nonzero. Putting all these conditions together and reordering the elliptic curves E_1, E_2, E_3 if necessary, we may assume that s and t are both nonzero. \square

Henceforth, we assume that s and t are nonzero. Therefore, we can use s^\vee and t^\vee to give an injective \mathbb{Q} -homomorphism $\text{End}^0(E_1 \times E_2 \times E_3) \hookrightarrow \text{End}^0(E_1 \times E_1 \times E_1)$ as in (6.1). The image of $\sqrt{\alpha}$ in $M_3(\text{End}^0 E_1)$ is

$$T = \begin{pmatrix} r & \delta_2 & \delta_3 \\ -1 & svs^\vee/\delta_2 & swt^\vee/\delta_2 \\ -1 & -tw^\vee s^\vee/\delta_3 & tzt^\vee/\delta_3 \end{pmatrix}, \quad (6.3)$$

where $\delta_2 = \deg(s)$ and $\delta_3 = \deg(t)$.

Since K contains no imaginary quadratic subfield, Lemma 6.4 shows that the elliptic curves E_1, E_2 and E_3 are supersingular. By Proposition 6.6, we may choose an isomorphism $\text{End}^0 E_1 \rightarrow B_{p,\infty}$. The isomorphism sends $\text{End} E_1$ to a maximal order of $B_{p,\infty}$ and the Rosati involution on $\text{End} E_1$ corresponds to the usual involution on $B_{p,\infty}$. We abuse notation slightly by continuing to write T for the image of $\sqrt{\alpha}$ in $M_3(B_{p,\infty})$.

Lemma 6.11. *Suppose that K contains no imaginary quadratic subfield. Let T denote the image of $\sqrt{\alpha}$ in $M_3(B_{p,\infty})$. Then the entries of the matrix T do not all commute with each other.*

Proof. Suppose for contradiction that the entries of T commute. Let K_1 denote the subfield of $B_{p,\infty}$ generated by the entries of T . A subfield of $B_{p,\infty}$ is either \mathbb{Q} or a quadratic subfield which splits $B_{p,\infty}$. But $B_{p,\infty}$ is ramified at the infinite place, so it is not split by any real field. Thus, K_1 is either \mathbb{Q} or an imaginary quadratic field. By assumption, K contains no imaginary quadratic subfield. Thus, the minimal polynomial of $\sqrt{\alpha}$ over \mathbb{Q} remains irreducible over K_1 .

Let g denote the minimal polynomial of T over K_1 . The degree of g is at most 3. Since T is the image of $\sqrt{\alpha}$ under an injective \mathbb{Q} -algebra homomorphism, g divides the minimal polynomial of $\sqrt{\alpha}$ over \mathbb{Q} , which has degree 6. Thus, the minimal polynomial of $\sqrt{\alpha}$ over \mathbb{Q} factorizes over K_1 , giving the required contradiction. \square

We restrict to the case where p is odd; the case $p = 2$ is very similar. By Proposition 6.6, $B_{p,\infty}$ has a \mathbb{Q} -basis $1, i, j, k$ where $i^2 = -\varepsilon$, $j^2 = -p$, $ij = k$, $ji = -ij$ and ε is as in Proposition 6.6. We embed $B_{p,\infty}$ into $M_4(\mathbb{Q})$ via

$$1 \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, i \mapsto \begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 0 & 0 & 1 & 0 \end{pmatrix}, j \mapsto \begin{pmatrix} 0 & 0 & -p & 0 \\ 0 & 0 & 0 & p \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, k \mapsto \begin{pmatrix} 0 & 0 & 0 & -\varepsilon p \\ 0 & 0 & -p & 0 \\ 0 & \varepsilon & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This induces an embedding $M_3(B_{p,\infty}) \hookrightarrow M_{12}(\mathbb{Q})$. Let U denote the image of α in $M_{12}(\mathbb{Q})$. Write $\mathbf{Tr}(T^2)$ for the sum of the elements on the diagonal of T^2 . Define $\mathbf{Tr}(Q^2)$ in the same way. It is easily checked that $\mathbf{Tr}(T^2) = \mathbf{Tr}(Q^2)$. By the construction of the embedding $B_{p,\infty} \hookrightarrow M_4(\mathbb{Q})$, we have

$$\mathbf{Tr}(U) = 4 \mathbf{Tr}(T^2). \quad (6.4)$$

Lemma 6.12. *Let T denote the image of $\sqrt{\alpha}$ in $M_3(B_{p,\infty})$. Then $\mathbf{Tr}(T^2) = \mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha)$.*

Proof. Let $\alpha = \alpha_1, \alpha_2, \alpha_3$ denote the conjugates of α . The characteristic polynomial of U is $(X - \alpha_1)^{m_1}(X - \alpha_2)^{m_2}(X - \alpha_3)^{m_3}$ for some $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$ with $m_1 + m_2 + m_3 = 12$. The trace of U is $m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 \in \mathbb{Q}$. If we can show that $m_1 = m_2 = m_3 = 4$, then equation (6.4) gives

$$4 \mathbf{Tr}(T^2) = \mathbf{Tr}(U) = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 = 4(\alpha_1 + \alpha_2 + \alpha_3) = 4 \mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha). \quad (6.5)$$

Therefore, it is enough to show that $m_1 = m_2 = m_3$. Since $\alpha \in \mathcal{O}_{K^+}$, we have $\alpha_1 + \alpha_2 + \alpha_3 \in \mathbb{Z}$ and therefore $(m_2 - m_1)\alpha_2 + (m_3 - m_1)\alpha_3 \in \mathbb{Q}$. Suppose for contradiction that we are not in the case $m_1 = m_2 = m_3$. Then, without loss of generality, $(m_2 - m_1) \neq 0$ and since $\alpha_2 \notin \mathbb{Q}$ it follows that $(m_3 - m_1) \neq 0$. Therefore, $\alpha_3 = \lambda\alpha_2$ for some $\lambda \in \mathbb{Q}$. But α_3 is a Galois conjugate of α_2 and the Galois group of the Galois closure of K^+/\mathbb{Q} is either C_3 or S_3 . Therefore, the automorphism sending α_2 to α_3 has order dividing 6 and hence λ is a sixth root of unity in \mathbb{Q} . Therefore, $\lambda = -1$ and $\alpha_3 = -\alpha_2$. But this gives $\mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha) = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1$. So $\alpha = \alpha_1 = \mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha) \in \mathbb{Q}$, which is a contradiction. \square

Proof of Theorem 6.9. Suppose for contradiction that $p > 4 \mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha)^6/3^6$. We will show that the entries of the matrix T commute, contradicting Lemma 6.11. The key ingredients will be Lemma 6.7 (which states that elements of a maximal order whose reduced norms are smaller than $\sqrt{p}/2$ commute) and equation (6.7) below.

Recall that

$$T = \begin{pmatrix} r & \delta_2 & \delta_3 \\ -1 & sv s^\vee / \delta_2 & sw t^\vee / \delta_2 \\ -1 & -tw^\vee s^\vee / \delta_3 & tzt^\vee / \delta_3 \end{pmatrix} \quad (6.6)$$

where $\delta_2 = \deg(s)$ and $\delta_3 = \deg(t)$. We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} T \in M_3(\text{End } E_1).$$

We have chosen an isomorphism $\text{End}^0 E_1 \rightarrow B_{p,\infty}$, sending $\text{End} E_1$ to a maximal order of $B_{p,\infty}$. The dual on $\text{End} E_1$ corresponds to the usual involution on $B_{p,\infty}$. We identify $\text{End}^0 E_1$ with $B_{p,\infty}$ and write $\text{Nrd}(f) = \deg(f) = ff^\vee$ for $f \in \text{End} E_1$.

By Lemma 6.12, we have $\mathbf{Tr}(T^2) = \mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha)$. Writing out the entries on the diagonal of T^2 gives

$$0 < \deg(r) + 2\deg(s) + 2\deg(t) + \deg(v) + 2\deg(w) + \deg(z) = -\mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha) < 3\sqrt[6]{p/4}. \quad (6.7)$$

Note that the sum of degrees is a sum of non-negative integers. We want to use (6.7) to

bound the reduced norms of the non-scalar entries of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} T$. Recall that, in light of

Lemma 6.10, we are assuming that s and t are nonzero. Therefore, $\deg(s), \deg(t) \geq 1$ and (6.7) gives

- i) $\text{Nrd}(r) = \deg(r) < 3\sqrt[6]{p/4} - 4 < \sqrt{p}/2$,
- ii) $2\deg(s) + \deg(v) < 3\sqrt[6]{p/4}$,
- iii) $2(\deg(s) + \deg(t) + \deg(w)) < 3\sqrt[6]{p/4}$,
- iv) $2\deg(t) + \deg(z) < 3\sqrt[6]{p/4}$.

Observe that $\text{Nrd}(swt^\vee) = \deg(s)\deg(w)\deg(t) = \text{Nrd}(-tw^\vee s^\vee)$. So it remains to bound the reduced norms of svs^\vee , swt^\vee and tzt^\vee . Let $a \in \mathbb{R}_{>0}$. The maximum of the function $f(x) = x^2(a - 2x)$ for $x \geq 0$ is achieved at $x = a/3$ and we have $f(a/3) = (a/3)^3$. Applying this to ii) with $a = 3\sqrt[6]{p/4}$, we see that

$$\text{Nrd}(svs^\vee) = \deg(s)^2 \deg(v) < (\sqrt[6]{p/4})^3 = \sqrt{p}/2.$$

Similarly, using iv) we get

$$\text{Nrd}(tzt^\vee) = \deg(t)^2 \deg(z) < (\sqrt[6]{p/4})^3 = \sqrt{p}/2.$$

Using iii), we get

$$\text{Nrd}(swt^\vee) = \deg(s)\deg(w)\deg(t) \leq (\deg(s) + \deg(w))^2 \deg(t) < (\sqrt[6]{p/4})^3 = \sqrt{p}/2.$$

Therefore, by Lemma 6.7, the entries of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} T$ commute. Since the entries of

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{pmatrix} T$ are just scalar multiples of the entries of T , this means that the entries of T commute. But this contradicts Lemma 6.11. Therefore, the assumption $p > 4\mathbf{Tr}_{K^+/\mathbb{Q}}(\alpha)^6/3^6$ does not hold. \square

6.4. Solutions to the embedding problem in the case that K contains an imaginary quadratic subfield. In this section, we consider the case where the sextic CM-field K contains an imaginary quadratic subfield K_1 . We show that the embedding problem 6.3 has solutions for every prime p (Corollary 6.15). The solutions are constructed via the reduction at p of a CM-abelian variety $A = E^3$ in characteristic zero, where E is an elliptic curve. In particular, the CM-type of A is imprimitive (Theorem 3.2). The solutions we construct may therefore be called *degenerate solutions* to the embedding problem.

The point is that if K is a CM-field which contains an imaginary quadratic subfield then there always exist imprimitive CM-types for K . This is what allows for the existence of degenerate solutions to the embedding problem. Recall from Corollary 3.4 that there do not exist imprimitive CM-types (K, φ) for CM-fields that do not contain a proper CM-subfield.

The proof of Theorem 6.8 relied on showing non-existence of solutions of the embedding problem for sufficiently large primes (Theorem 6.9) in the case where the sextic CM-field contains no proper CM-subfield. In contrast, if C is a curve whose Jacobian has CM by a sextic CM-field K which contains an imaginary quadratic field, then this strategy breaks down because there the embedding problem has degenerate solutions for all primes p (Corollary 6.15). The embedding problem, as formulated in Problem 6.3, does not take the CM-type into consideration. It may be possible to prove an analogous result to Theorem 6.8, in the case that K contains a proper CM-subfield, using a more refined formulation of the embedding problem that includes the CM-type as part of the data.

Proposition 6.13. *Let K be a sextic CM-field containing a proper CM subfield K_1 . Let E be an elliptic curve over an arbitrary field and suppose that there exists an embedding $K_1 \hookrightarrow \text{End}^0(E)$. Then there exists an order \mathcal{O} of K and a ring embedding*

$$\mathcal{O} \hookrightarrow \text{End}(E^3) = M_3(\text{End}(E))$$

such that the Rosati involution on $\text{End}(E^3)$ corresponding to the product polarization on $A = E^3$ induces complex conjugation on \mathcal{O} .

Proof. It suffices to give an injective \mathbb{Q} -algebra homomorphism

$$K \hookrightarrow \text{End}^0(E^3) = M_3(\text{End}^0(E)). \quad (6.8)$$

This can be achieved as follows. Write $K = K^+K_1$ where K^+/\mathbb{Q} is a totally real field with $[K^+ : \mathbb{Q}] = 3$. Choose a primitive element α of K^+/\mathbb{Q} , so $K^+ = \mathbb{Q}(\alpha)$. Embed K_1 diagonally via the fixed embedding of K_1 into $\text{End}^0(E)$. Map α to a symmetric matrix $Q \in M_3(\mathbb{Q})$ which has the same minimal polynomial as α . Since all the conjugates of α are real, the existence of the matrix Q is proved in Theorem 4 of [Ben68]. Extend to a \mathbb{Q} -algebra homomorphism. \square

In Remark 5.6, we reviewed the construction in characteristic 0 of genus 3 curves with CM by a sextic CM-field K . Similarly, when K_1 is an imaginary quadratic field, elliptic curves with CM by K_1 exist in characteristic zero. For example, we may take $E = \mathbb{C}/\mathcal{O}_{K_1}$, where we consider the maximal order \mathcal{O}_{K_1} of K_1 as lattice in \mathbb{C} ([Sil94], Remark II.4.1.1). Then $\text{End}(E) = \mathcal{O}_{K_1}$. Moreover, $j(E)$ is an algebraic integer ([Sil94], Theorem II.6.1). (This can be deduced from Theorem 4.1 which states that E has potentially good reduction.) In particular, E can be defined over the number field $M := \mathbb{Q}(j(E))$.

We now show the existence of elliptic curves with CM by K_1 in positive characteristic. As above, E/M is an elliptic curve defined over the number field M with $\text{End}(E) = \mathcal{O}_{K_1}$. We choose a rational prime p , and let \mathfrak{p} be a prime of M above \underline{p} . After extending M if necessary, we may assume that E has good reduction at \mathfrak{p} . Write $\overline{E}_{\mathfrak{p}}$ for the reduction of E at \mathfrak{p} . We obtain an embedding

$$\mathcal{O}_{K_1} = \text{End}(E) \hookrightarrow \text{End}(\overline{E}_{\mathfrak{p}}).$$

This proves the following lemma.

Lemma 6.14. *Let p be a prime. Then there exists an elliptic curve $\overline{E}_{\mathfrak{p}}$ in characteristic p with $\mathcal{O}_{K_1} \hookrightarrow \text{End}(\overline{E}_{\mathfrak{p}})$.*

The following result follows immediately from Lemma 6.14 and Proposition 6.13.

Corollary 6.15. *Let K be a sextic CM-field containing an imaginary quadratic field K_1 . Then there exists an order \mathcal{O} of K for which there exists a solution to the embedding problem for \mathcal{O} and p for every prime number p .*

Corollary 6.15 does not specify whether the elliptic curve \overline{E}_p from Lemma 6.14 is ordinary or supersingular. The following proposition answers this question. Note that it follows that the set of primes where the elliptic curve \overline{E}_p is supersingular has Dirichlet density $1/2$.

Proposition 6.16. *(Deuring's Theorem) Let E/M be an elliptic curve with CM by \mathcal{O}_{K_1} . Let p be a rational prime and \mathfrak{p} be a prime of M above p such that E has good reduction at \mathfrak{p} . Then the reduction $\overline{E}_{\mathfrak{p}}$ of E at \mathfrak{p} is supersingular if and only if p is inert or ramified in K_1 .*

Proposition 6.16 is well known, but hard to find explicitly in the literature. The statement can be proved using Theorem 10 of Section 10.4 of [Lan87]. We give the idea of the proof of the proposition. Let $\overline{E}/\mathbb{F}_q$ be an elliptic curve. Write π for its q -Frobenius endomorphism. Then \overline{E} is supersingular if and only if there exists integers n, m such that $\pi^n = [p]^m$, where $[p]$ denotes multiplication by p . (See for example the proof of the Theorem of Deuring in Section 22 of [Mum70]). The theorem from [Lan87] shows that this happens if and only if p is inert or ramified in K_1 .

APPENDIX A. EQUATIONS

In this section, we list the equations obtained from a possible solution to the embedding problem. We start by setting some notation.

Let K^+ be the maximal real subfield of the sextic CM-field $K = K^+(\eta)$. Take an integral basis of \mathcal{O}_{K^+} , so $\mathcal{O}_{K^+} = \alpha_1\mathbb{Z} \oplus \alpha_2\mathbb{Z} \oplus \alpha_3\mathbb{Z}$. We may assume that $K^+ = \mathbb{Q}(\alpha_1)$. We fix the following notation:

- $\mathbf{Tr}_{K/K^+}(\eta) = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$
- $\mathbf{Nm}_{K/K^+}(\eta) = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3$
- $f_i(x) = x^3 + m_ix^2 + n_ix + s_i$ is the characteristic polynomial of α_i over \mathbb{Q} for $i = 1, 2, 3$.

A solution to the embedding problem (Problem 6.3) gives us three elliptic curves E_1, E_2, E_3 and an embedding of $\iota : \mathcal{O}_K \hookrightarrow \text{End}(E_1 \times E_2 \times E_3)$ such that Rosati involution on $E_1 \times E_2 \times E_3$ restricts to complex conjugation in the image of \mathcal{O}_K . This gives the following conditions on $\iota(\alpha_i)$ and $\iota(\eta)$:

- (1) Commutativity:
 - (a) $\iota(\alpha_i)\iota(\eta) = \iota(\eta)\iota(\alpha_i)$ for all $i = 1, 2, 3$.
 - (b) $\iota(\alpha_i)\iota(\alpha_j) = \iota(\alpha_j)\iota(\alpha_i)$ for all $i \neq j \in \{1, 2, 3\}$.
- (2) Characteristic polynomial: $f_i(\iota(\alpha_i)) = 0$ for all $i = 1, 2, 3$.
- (3) Norm: $\iota(\eta)\iota(\eta)^\dagger = b_1\iota(\alpha_1) + b_2\iota(\alpha_2) + b_3\iota(\alpha_3)$, where \dagger denotes the conjugate transpose.
- (4) Trace: $\iota(\eta) + \iota(\eta)^\dagger = a_1\iota(\alpha_1) + a_2\iota(\alpha_2) + a_3\iota(\alpha_3)$.
- (5) Duality/Complex conjugation: $\iota(\alpha_i) = \iota(\alpha_i)^\dagger$ for all $i = 1, 2, 3$. Since we are interested in the case that Rosati involution induces complex multiplication and since η can be chosen so that $\eta^2 \in K^+$ is totally negative, we have $\iota(\eta)^\dagger = -\iota(\eta)$.

In the rest of this appendix, we will only write the conditions for $i = 1$ which is enough if we have a power basis. In any case, the other relations for $i = 2, 3$ are similar. We now write

the conditions above in terms of matrix coefficients. We are using the conventions and maps introduced in Section 6.2.

$$\text{Let } M = \iota(\alpha_1) \text{ be the matrix } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} \text{ and } N = \iota(\eta) \text{ be the matrix } \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & y \end{pmatrix}.$$

A.0.1. *Equations for duality/complex conjugation condition.* The relation $\iota(\eta)^\dagger = -\iota(\eta)$ trans-

lates into $M = M^\vee$ i.e., $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & \ell \end{pmatrix} = \begin{pmatrix} a^\vee & d^\vee & g^\vee \\ b^\vee & e^\vee & h^\vee \\ c^\vee & f^\vee & \ell^\vee \end{pmatrix}.$

This gives us the following relations.

Remark A.1. Note that we name the relations with respect to the variables we intend to use later on. Our aim is to simplify the equations and write everything in terms of the upper triangular entries of our matrices which are a, b, c, e, f, ℓ in the case of M and p, q, r, t, u, y in the case of N .

(b-d) $d = b^\vee$

(c-g) $g = c^\vee$

(f-h) $h = f^\vee$

(int) a, e, ℓ are integral and in \mathbb{Q} , hence they are integers.

The relation $\iota(\eta)^\vee = -\iota(\eta)$ translates into: $\begin{pmatrix} p & q & r \\ s & t & u \\ v & w & y \end{pmatrix} = \begin{pmatrix} -p^\vee & -s^\vee & -v^\vee \\ -q^\vee & -t^\vee & -w^\vee \\ -r^\vee & -u^\vee & -y^\vee \end{pmatrix}.$

This gives us the following relations:

(q-s) $s = -q^\vee$

(r-v) $v = -r^\vee$

(u-w) $w = -u^\vee$

(trace) $p = -p^\vee, t = -t^\vee,$ and $y = -y^\vee$

i.e., $p, t,$ and y have trace zero in $\text{End}(E_1), \text{End}(E_2),$ and $\text{End}(E_3)$ respectively.

A.0.2. *Equations for commutativity condition.* Using M and N as above, the condition means $MN = NM$ which translates into the following equations:

(i-i) $ap + bs + cv = pa + qd + rg.$ (By equation (int) in Section A.0.1, a is an integer. Hence $ap = pa$ and $bs + cv = qd + rg.$)

(i-ii) $aq + bt + cw = pb + qe + rh$

(i-iii) $ar + bu + cy = pc + qf + rl$

(ii-i) $dp + es + fv = sa + td + ug$

(ii-ii) $dq + et + fw = sb + te + uh$ (By equation (int) in Section A.0.1, e is an integer. Hence $et = te$ and $dq + fw = sb + uh.$)

(ii-iii) $dr + eu + fy = sc + tf + ul$

(iii-i) $gp + hs + lv = va + wd + yg$

(iii-ii) $gq + ht + lw = vb + we + yh$

(iii-iii) $gr + hu + ly = vc + wf + yl$ (By equation (int) in Section A.0.1, ℓ is an integer. Hence $ly = yl$ and $gr + hu = vc + wf.$)

A.0.3. *Combining duality and commutativity conditions.* Now we will plug in the equations we obtained in Section A.0.1 into the equations we obtained in Section A.0.2. Note that our aim is to simplify the equations and write everything in terms of the upper triangular entries of our matrices which are a, b, c, e, f, ℓ in the case of M and p, q, r, t, u, y in the case of N .

Relation	Obtained using:
$bq^\vee + cr^\vee + rc^\vee + qb^\vee = 0$	(i-i), (c-g), (q-s), (v-r)
$pb + qe + rf^\vee - aq - bt + cu^\vee = 0$	(i-ii), (u-w), (f-h)
$ar + bu + cy - pc - qf - r\ell = 0$	(i-iii)
$b^\vee p - eq^\vee - fr^\vee + q^\vee a - tb^\vee - uc^\vee = 0$	(ii-i), (b-d), (q-s), (r-v), (q-s)
$b^\vee q - fu^\vee + q^\vee b - uf^\vee = 0$	(ii-ii), (b-d), (u-w), (q-s), (f-h)
$dr + eu + fy + q^\vee c - tf - ul = 0$	(ii-iii), (q-s)
$c^\vee p - f^\vee q^\vee + (a - \ell)r^\vee + u^\vee b - yc^\vee = 0$	(iii-i), (c-g), (f-h), (s-q), (r-v), (u-w), (b-d), (int)
$c^\vee q + f^\vee t + (e - \ell)u^\vee + r^\vee b - yf^\vee = 0$	(iii-i), (c-g), (f-h), (u-w), (r-v), (int)
$c^\vee r + f^\vee u + r^\vee c + u^\vee f = 0$	(iii-i), (f-h), (u-w), (r-v)

A.0.4. *Equations for characteristic polynomial condition.* The characteristic polynomial condition for $i = 1$ translates into $0 = M^3 + m_1 M^2 + n_1 M + s_1$. Combining this equality with Equation (int) of Section A.0.2 gives the following equations. For instance, for the top left corner of the matrix sum we get

$$0 = a^3 + abd + acg + bda + bed + bfg + cga + chd + clg + m_1(a^2 + bd + cg) + n_1 a + s_1.$$

If we apply Condition (int) this turns into

$$(2a + e + m_1)bd + (2a + \ell + m_1)cg + bfg + chd + a^3 + m_1 a^2 + n_1 a + s_1 = 0.$$

The following is the list of equations coming from all nine entries.

- (i) $(2a + e + m_1)bd + (2a + \ell + m_1)cg + bfg + chd + a^3 + m_1 a^2 + n_1 a + s_1 = 0$
- (ii) $(a^2 + ae + e^2 + m_1 a + m_1 e + n_1)b + (e + \ell + m_1 + a)ch + bdb + bfh + cgb = 0$
- (iii) $(a^2 + al + \ell^2 + m_1 a + m_1 \ell + n_1)c + (a + e + \ell + m_1)bf + bdc + cgc + chf = 0$
- (iv) $(a^2 + ea + e^2 + m_1 a + m_1 e + n_1)d + (e + a + \ell + m_1)fg + dbd + dcg + fhd = 0$
- (v) $(a + 2e + m_1)db + (2e + \ell + m_1)fh + dch + fgb + e^3 + m_1 e^2 + n_1 + s_1 = 0$
- (vi) $(a + \ell + e + m_1)dc + (e^2 + e\ell + \ell^2 + m_1 e + m_1 \ell + n_1)f + dbf + fgc + fhf = 0$
- (vii) $(a^2 + la + \ell^2 + m_1 a + m_1 \ell + n_1)g + (a + e + \ell + m_1)hd + gbd + gcg + hfg = 0$
- (viii) $(e^2 + le + \ell^2 + m_1 e + m_1 \ell + n_1)h + (a + e + \ell + m_1)gb + gch + hdb + hfh = 0$
- (ix) $(a + 2\ell + m_1)gc + (e + 2\ell + m_1)hf + gbf + hdc + \ell^3 + m_1 \ell^2 + n_1 \ell + s_1 = 0$

A.0.5. *Combining duality and characteristic polynomial conditions.* Now we will plug in the equations we obtained in Section A.0.1 into the equations we obtained in Section A.0.4. Note that our aim is to simplify the equations and write everything in terms of the upper triangular entries of our matrices which are a, b, c, e, f, ℓ in the case of M and p, q, r, t, u, y in the case of N . Note that $\text{Nrd}(x) = xx^\vee$, $\mathbf{Tr}(x) = x + x^\vee$ denote the reduced norm and trace of an element. Since the norm and trace are scalars, they commute with everything else.

We start with the relations coming from M :

- (I) $(2a + e + m_1)\text{Nrd}(b) + (2a + \ell + m_1)\text{Nrd}(c) + \mathbf{Tr}(bfc^\vee) + a^3 + m_1 a^2 + n_1 a + s_1 = 0$
- (II) $(a^2 + ae + e^2 + m_1 a + m_1 e + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))b + (a + e + \ell + m_1)cf^\vee = 0$

- (III) $(a^2 + al + \ell^2 + m_1a + m_1\ell + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))c + (a + e + \ell + m_1)bf = 0$
- (IV) $(a^2 + ae + e^2 + m_1a + m_1e + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))b^\vee + (a + e + \ell + m_1)fc^\vee = 0$
- (V) $(a + 2e + m_1) \text{Nrd}(b) + (2e + \ell + m_1) \text{Nrd}(f) + \mathbf{Tr}(b^\vee c f^\vee) + e^3 + m_1e^2 + n_1e + s_1 = 0$
- (VI) $(e^2 + e\ell + \ell^2 + m_1e + m_1\ell + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))f + (a + \ell + e + m_1)b^\vee c = 0$
- (VII) $(a^2 + al + \ell^2 + m_1a + m_1\ell + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))c^\vee + (a + e + \ell + m_1)f^\vee b^\vee = 0$
- (VIII) $(e^2 + e\ell + \ell^2 + m_1e + m_1\ell + n_1 + \text{Nrd}(b) + \text{Nrd}(c) + \text{Nrd}(f))f^\vee + (a + e + \ell + m_1)c^\vee b = 0$
- (IX) $(a + 2\ell + m_1) \text{Nrd}(c) + (e + 2\ell + m_1) \text{Nrd}(f) + \mathbf{Tr}(c^\vee b f) + \ell^3 + m_1\ell^2 + n_1\ell + s_1 = 0$

Write $\mathbf{Tr}(X)$ for the sum of the entries on the main diagonal of a matrix X . Notice that if we take $\eta = \sqrt{\alpha_1}$ like in Section 6.3, then

$$-m_1 = \mathbf{Tr}(\alpha_1) = \mathbf{Tr}(N^2) = \mathbf{Tr}(M) = a + e + \ell,$$

where the first equality follows by definition, the second equality is Lemma 6.12, the third equality holds because we took $\eta = \sqrt{\alpha_1}$, and the final equality is the definition of $\mathbf{Tr}(M)$. This implies that Equation (II) = Equation (IV), Equation (III) = Equation (VII) and Equation (VI) = Equation (VIII).

Combining $-m_1 = a + e + \ell$ with relations (I)-(IX), we deduce the following relations on the coefficients m_1, n_1, s_1 of the characteristic polynomial of α_1 .

- (1) $m_1 = -(a + e + \ell)$
- (2) $n_1 = ae + e\ell + al - \text{Nrd}(b) - \text{Nrd}(c) - \text{Nrd}(f)$ (using Equation (1) together with Equations (II), (III) and (VI).)
- (3) $s_1 = a \text{Nrd}(f) + e \text{Nrd}(c) + \ell \text{Nrd}(b) - ae\ell - \mathbf{Tr}(bfc^\vee)$ (using Equation (1) together with Equations (I), (V) and (IX).)

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