# BIPERMUTAHEDRON AND BIASSOCIAHEDRON

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Abstract. We give a simple description of the face poset of a version of the biassociahedra that generalizes, in a straightforward manner, the description of the faces of the Stasheff's associahedra via planar trees. We believe that our description will substantially simplify the notation of [8] making it, as well as the related papers, more accessible.



Contents



## History and pitfalls

In this introductory section we recall the history and indicate the pitfalls of the 'quest for the biassociahedron,' hoping to elucidate the rôle of the present paper in this struggle.

**History.** Let us start by reviewing the precursor of the biassociahedron. J. Stasheff in his seminal paper [9] introduced $A_\infty$-spaces (resp. $A_\infty$-algebras, called also strongly homotopy or sh associative algebras) as spaces (resp. algebras) with a multiplication associative up to a coherent system of homotopies. The central object of his approach was a cellular operad $K = \{K_m\}_{m\geq 2}$ whose $m$th piece $K_m$ was a convex $(m-2)$-dimensional polytope called the Stasheff associahedron. $A_\infty$-space was then defined as a topological space on which the operad $K$ acted, while $A_\infty$-algebras were algebras over the operad $C_*(K)$ of cellular chains on $K$. Let us briefly recall the basic features of the construction of [9], emphasizing the algebraic side. More details can be found for instance in [7, II.1.6] or in the original source [9].

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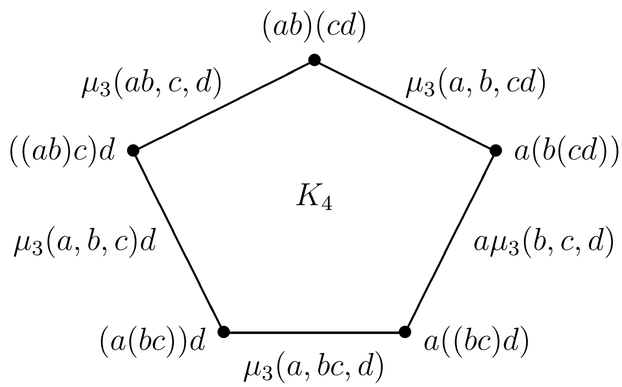
Consider a dg-vector space $V$ with a homotopy associative multiplication $\mu : V^{\otimes 2} \to V$. This means that there is a chain homotopy $\mu_3 : V^{\otimes 3} \to V$ between $\mu(\mu \otimes 1\!\!1)$ and $\mu(1\!\!1 \otimes \mu)$, where $1\!\!1$ denotes the identity endomorphism $1\!\!1 : V \to V$. The homotopy $\mu_3$ will be symbolized by the interval

$$K_3 := \ (ab)c \ \overset{\mu_3(a,b,c)}{\bullet\!\!\longrightarrow\!\!\!\!\!\!\longrightarrow\!\!\bullet} \ a(bc)$$

connecting the two possible products, $(ab)c$ and $a(bc)$, of three elements $a, b, c \in V$. We abbreviate, as usual, $(ab)c := \mu\big(\mu(a,b),c\big) = \mu(\mu \otimes 1\!\!1)(a,b,c)$, &c. As the next step, consider all possible products of four elements and organize them into the vertices of the pentagon:




The products labeling adjacent vertices are homotopic and we labelled the edges by the corresponding homotopies. Observe that all these homotopies are constructed using $\mu_3$ and the multiplication $\mu_2$.

Next, we require the homotopy for the associativity to be *coherent*, by which we mean that the pentagon $K_4$ can be 'filled' with a higher homotopy $\mu_4 : V^{\otimes 4} \to V$ whose differential equals the sum (with appropriate signs) of the homotopies labelling the edges. This process can be continued, giving rise to a sequence $K = \{K_m\}_{m \geq 2}$ of the *Stasheff associahedra*. It turns out that $K$ is a polyhedral operad. An $A_\infty$-algebra is then an algebra over the operad $C_*(K)$ of cellular chains on $K$.

Much later there appeared another, purely algebraic, way to introduce $A_\infty$-algebras. As proved in [5], the operad $\mathcal{A}ss$ for associative algebras admits a unique, up to isomorphism, minimal cofibrant model $\mathcal{A}_\infty$ which turns out to be isomorphic to the operad $C_*(K)$. We may thus as well say that $A_\infty$-algebras are algebras over the minimal model of $\mathcal{A}ss$. Finally, one can describe $A_\infty$-algebras explicitly, as a structure with operations $\mu_m : V^{\otimes m} \to V$, $m \geq 2$, satisfying a very explicit infinite set of axioms, see [9, page 294]. In the case of $A_\infty$-algebras thus *topology*, represented by the associahedron, preceded *algebra*.

There were similar attempts to find a suitable notion of $A_\infty$-bialgebras,[1] that is, structures whose multiplication and comultiplication are compatible and (co)associative up to a system of coherent homotopies. The motivation for such a quest was, besides restless nature of human mind, homotopy invariance and the related transfer properties which these structures should posses. For instance, given a (strict) bialgebra $H$, each dg-vector space quasi-isomorphic to the underlying dg-vector space of $H$ ought to have an induced $A_\infty$-bialgebra structure.

[1]Other possible names are $B_\infty$-algebras or strongly homotopy bialgebras.

Here *algebra* by far preceded *topology*. The existence of a minimal model $\mathcal{B}_\infty$ for the PROP $B$ governing bialgebras[2] was proved in [3]. According to general philosophy [4], $A_\infty$-bialgebras defined as algebras over $\mathcal{B}_\infty$ are homotopy invariant concepts. Moreover, it follows from the description of $\mathcal{B}_\infty$ given in [3] that an $A_\infty$-bialgebra defined in this way has operations $\mu^n_m : V^{\otimes m} \to V^{\otimes n}$, $m, n \in \mathbb{N}$, $(m, n) \neq (1, 1)$, but axioms as explicit as the ones for $A_\infty$-algebras were given only for $m + n \leq 6$.

**Pitfalls.** It is clearly desirable to have some polyhedral PROP $KK = \{KK^n_m\}$ playing the same rôle for $A_\infty$-bialgebras as the Stasheff's operad plays for $A_\infty$-algebras. By this we mean that $\mathcal{B}_\infty$ should be isomorphic to the PROP of cellular chains of $KK$, so the differential in $\mathcal{B}_\infty$ and therefore also the axioms of $A_\infty$-bialgebras would be encoded in the combinatorics of $KK$. To see where the pitfalls are hidden, we try to mimic the inductive construction of the associahedra in the context of bialgebras.

The first step is obvious. Assume we have a dg-vector space $V$ with a multiplication $\mu : V^{\otimes 2} \to V$ and a comultiplication $\Delta : V \to V^{\otimes 2}$ such that $\mu$ is associative up to a homotopy $\mu^1_3 : V^{\otimes 3} \to V$ symbolized by the interval

$$K^1_3 := \ (ab)c \ \bullet\!\!\xrightarrow{\quad\mu^2_3(a,b,c)\quad}\!\!\bullet \ a(bc),$$

$\mu$ and $\Delta$ are compatible up to a homotopy $\mu^2_2 : V^{\otimes 2} \to V^{\otimes 2}$ symbolized by

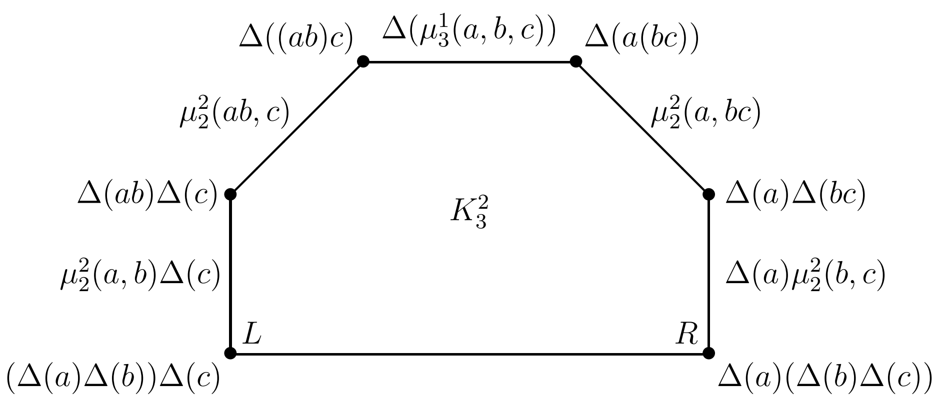
$$K^2_2 := \quad \Delta(ab) \ \bullet\!\!\xrightarrow{\quad\mu^2_2(a,b)\quad}\!\!\bullet \ \Delta(a)\Delta(b),$$

and $\Delta$ is coassociative up to a homotopy $\mu^3_1 : V \to V^{\otimes 3}$ depicted as

$$K^3_1 := (\Delta \otimes 1\!\!1)\Delta(a) \ \bullet\!\!\xrightarrow{\quad\mu^3_1(a)\quad}\!\!\bullet \ (1\!\!1 \otimes \Delta)\Delta(a).$$

Let us take all elements of $V^{\otimes 2}$ constructed out of three elements of $V$ using $\Delta$ and the multiplication on the tensor powers of $V$ induced in the standard manner by $\mu$. Let us call such elements *algebraic*. There are six of them, labelling the vertices of a hexagon:

(1)




All products labelling adjacent vertices except the two bottom ones are homotopic via an 'algebraic' homotopy, i.e. a homotopy constructed using $\Delta$, $\mu^1_3$, $\mu^2_2$, and the multiplication induced by $\mu$ on the powers of $V$.

Let us inspect the vertices $L$ and $R$. The 'obvious' candidate $\mu^1_3\big(\Delta(a), \Delta(b), \Delta(c)\big)$ for the connecting homotopy does not have any meaning. The labels of these vertices are,

[2]PROPs generalize operads. We briefly recall them in the Appendix.

however, still homotopic but in an unexpected manner. For $a, b, c \in V$ define $X(a, b, c) \in V^{\otimes 2}$ by

$$X(a, b, c) := \big(\mu(1\!\!1 \otimes \mu) \otimes \mu(\mu \otimes 1\!\!1)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big)$$

where $\sigma\big(\tbinom{3}{2}\big) : V^{\otimes 6} \to V^{\otimes 6}$ is the permutation acting on $v_1, \ldots, v_6 \in V$ as

$$\sigma\big(\tbinom{3}{2}\big)(v_1 \otimes v_2 \otimes v_3 \otimes v_4 \otimes v_5 \otimes v_6) := (v_1 \otimes v_3 \otimes v_5 \otimes v_2 \otimes v_4 \otimes v_6).$$

Similarly, put

$$Y(a, b, c) := \big(\mu(\mu \otimes 1\!\!1) \otimes \mu(1\!\!1 \otimes \mu)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big).$$

Define furthermore the homotopies $H_l, H_r, G_l, G_r : V^{\otimes 3} \to V^{\otimes 2}$ by the formulas

$$\begin{aligned}
H_l(a, b, c) &: = \big(\mu_3^1 \otimes \mu(\mu \otimes 1\!\!1)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big),\\
H_r(a, b, c) &: = \big(\mu(1\!\!1 \otimes \mu) \otimes \mu_3^1\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big),\\
G_l(a, b, c) &: = \big(\mu(\mu \otimes 1\!\!1) \otimes \mu_3^1\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big), \quad \text{and}\\
G_r(a, b, c) &: = \big(\mu_3^1 \otimes \mu(1\!\!1 \otimes \mu)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big).
\end{aligned}$$

Observing that

$$\begin{aligned}
\big(\Delta(a)\Delta(b)\big)\Delta(c) &= \big(\mu(\mu \otimes 1\!\!1) \otimes \mu(\mu \otimes 1\!\!1)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big), \text{ and}\\
\Delta(a)\big(\Delta(b)\Delta(c)\big) &= \big(\mu(1\!\!1 \otimes \mu) \otimes \mu(1\!\!1 \otimes \mu)\big)\sigma\big(\tbinom{3}{2}\big)\big(\Delta(a) \otimes \Delta(b) \otimes \Delta(c)\big),
\end{aligned}$$

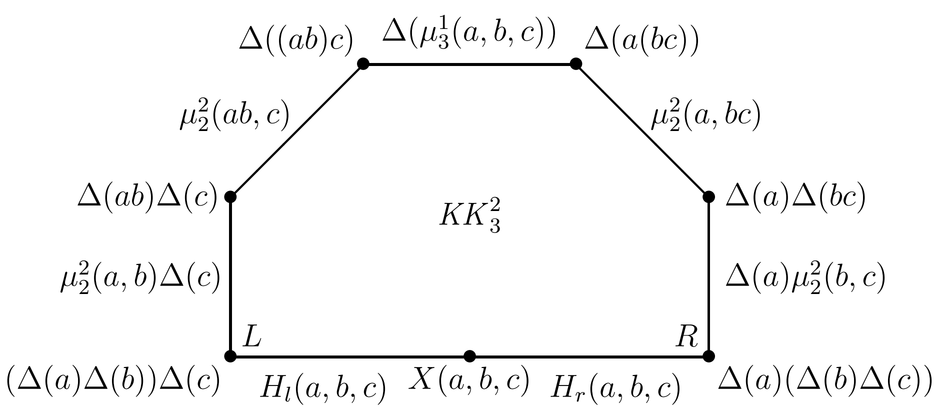
we see the following composite chain of homotopies

$$\begin{tikzcd}
(\Delta(a)\Delta(b))\Delta(c) \arrow[r, dash, "H_l(a,b,c)"] & \bullet^{X(a,b,c)} \arrow[r, dash, "H_r(a,b,c)"] & \Delta(a)(\Delta(b)\Delta(c))
\end{tikzcd}$$

and also

$$\begin{tikzcd}
(\Delta(a)\Delta(b))\Delta(c) \arrow[r, dash, "G_l(a,b,c)"] & \bullet^{Y(a,b,c)} \arrow[r, dash, "G_r(a,b,c)"] & \Delta(a)(\Delta(b)\Delta(c)).
\end{tikzcd}$$

To proceed as in the case of the associahedron, we need to subdivide the bottom edge of the hexagon $K_3^2$ in (1) and consider the heptagon $KK_3^2$

(2)



Observe that the subdivision and therefore also $KK_3^2$ is *not unique*, we could as well take $Y, G_l, G_r$ instead of $X, H_l, H_r$. Notice also that *neither* the expressions $X, Y$ *nor* the homotopies $H_l, H_r, G_l, G_r$ are *algebraic*.

**Two types of biassociahedra.** We can already glimpse the following pattern. There naturally appear polytopes $K_m^n$, $m, n \in \mathbb{N}$, such that $K_m^1$ and $K_1^m$ are isomorphic to Stasheff's associahedron $K_m$. We will also see that $K_m^2$ is isomorphic to the multiplihedron $J_m$. We call these polytopes the *step-one biassociahedra*. In this paper we give a simple and clean description of their face posets.

To continue as in the case of $A_\infty$-algebras, one however needs to subdivide some faces of $K^n_m$; and example of such a subdivision is the heptagon $KK^2_3$ in (2) subdividing the hexagon $K^2_3$. The subdivisions must be compatible so that the result will be a cellular PROP $KK = \{KK^n_m\}$. Its associated cellular chain complex is moreover required to be isomorphic to the minimal model $\mathcal{B}_\infty$ of the bialgebra PROP. We call these polyhedra *step-two biassociahedra*.

The polytopes $KK^n_m$ were, for $m + n \leq 6$, constructed in [3]. In higher dimensions, the issue of the compatibility of the subdivisions arises. In [8], a construction of the step-two biassociahedra was proposed, but we admit that we were not able to verify it. By our opinion, a reasonably simple construction of the polyhedral PROP $KK$ or at least a convincing proof of its existence still remains a challenge.

We think that a necessary starting point to address the above problems is a suitable notation. In this paper we give a simple description of the face poset of the step-one biassociahedron $K^n_m$ that generalize the classical description of the Stasheff associahedron in terms of planar directed trees. We will also give a 'coordinate-free' characterization of $K^n_m$ which shows that it is not a human invention but has existed since the beginning of time. As a by-product of our approach, it will be obvious that $K^2_m$ is isomorphic to the multiplihedron $J_m$, for each $m \geq 2$.

**Notation and terminology.** Some low-dimensional examples of the step-two biassociahedra appeared for the first time, without explicit name, in [3]; they were denoted $B^n_m$ there. The word *biassociahedron* was used by S. Saneblidze and R. Umble, see e.g. [8], referring to what we called above the step-two biassociahedron; they denoted it $KK_{m,n}$. Step-one biassociahedra can also be, without explicit name, found in [8]; they were denoted $K_{m,n}$ there. Whenever we mention the biassociahedron in this paper, we always mean the *step-one* biassociahedron which we denote by $K^n_m$.

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## Main results

Let us recall some standard facts [7, 11]. The *permutahedron*[3] $P_{m-1}$ is, for $m \geq 2$, a convex polytope whose poset of faces $\mathcal{P}_{m-1}$ is isomorphic to the set $lT_m$ of planar directed trees with levels and $m$ leaves, with the partial order generated by identifying adjacent levels. The permutahedron $P_{m-1}$ can be realized as the convex hull of the vectors obtained by permuting the coordinates of $(1, \ldots, m-1) \in \mathbb{R}^{m-1}$, its vertices correspond to elements of the symmetric group $\Sigma_{m-1}$. The face poset $\mathcal{K}_m$ of the Stasheff's *associahedron* $K_m$ is the set of directed planar trees (no levels) $T_m$ with $m$ leaves; the partial order is given by contracting the internal edges. The obvious epimorphism $\varpi_m : lT_m \twoheadrightarrow T_m$ erasing the levels induces the *Tonks projection* $Ton : \mathcal{P}_{m-1} \twoheadrightarrow \mathcal{K}_m$ of the face posets, see [11] for details.

[3]Sometimes also spelled permut<u>o</u>hedron.

There is a conceptual explanation of Tonks' projection that uses a natural map

$$\varpi_m : lT_m \to \mathsf{F}(\xi_2, \xi_3, \ldots)(m) \tag{3}$$

to the arity $m$ piece of the free non-$\Sigma$ operad [6, Section 4] generated by the operations $\xi_2, \xi_3, \ldots$ of arities $2, 3, \ldots$, respectively. The map $\varpi_m$, roughly speaking, replaces the vertices of a tree $T \in lT_m$ with the generators of $\mathsf{F}(\xi_2, \xi_3, \ldots)$ whose arities equal the number of inputs of the corresponding vertex, and then composes these generators using $T$ as the composition scheme, see §2.1.

It is almost evident that the set $T_m$ is isomorphic to the image of $\varpi_m$. In other words, the face poset $\mathcal{K}_m$ of the associahedron $K_m$ can be *defined* as the quotient of $lT_m$ modulo the equivalence that identifies elements having the same image under $\varpi_m$, with the induced partial order. Tonks' projection then appears as the epimorphism in the factorization

$$\begin{tikzcd} lT_m \arrow[r, two heads, "Ton"] \arrow[rr, bend right, "\varpi_m"'] & T_m \arrow[r, hook] & \mathsf{F}(\xi_2, \xi_3, \ldots). \end{tikzcd}$$

The aim of this note is to define in the same manner the poset $\mathcal{K}_m^n$ of faces of the step-one *biassociahedron* $K_m^n$ constructed in [8, §9.5].[4] To this end, we introduce in 1.2, for each $m, n \geq 1$, the set $lT_m^n$ of *complementary pairs* of directed planar trees with levels. The set $lT_m^n$ has a partial order $<$ similar to that of $lT_m$. The poset $\mathcal{P}_m^n = (lT_m^n, <)$ provides a natural indexing of the face poset of the *bipermutahedron* $P_m^n$ of [8].[5] It turns out that the posets $\mathcal{P}_m^n$ and $\mathcal{P}_{m+n-1}$ are isomorphic; we give a simple proof of this fact in Section 1. Comparing it with the proof of the analogous [8, Proposition 6] convincingly demonstrates the naturality of our language of complementary pairs.

As the next step, we describe, for each $m, n \geq 1$, a natural map

$$\varpi_m^n : lT_m^n \to \mathsf{F}\big(\xi_a^b \,|\, a, b \geq 1,\ (a,b) \neq (1,1)\big).$$

The object in the right hand side is the free PROP [6, Section 8] generated by operations $\xi_a^b$ of biarity $(b, a)$, i.e. with $b$ outputs and $a$ inputs. We then *define* the face poset $\mathcal{K}_m^n$ of the biassociahedron as $lT_m^n$ modulo the relation that identifies the complementary pairs of trees having the same image under $\varpi_m^n$, with the induced partial order. We prove that $\mathcal{K}_m^n$ is isomorphic to the poset of complementary pairs of planar directed trees $zT_m^n$ *with zones*, see Definition A on page 14. It will be obvious that this is the simplest possible description of the poset of faces of the Saneblidze-Umble biassociahedron that generalizes the standard description of the face poset of the Stasheff's associahedron.

In the last section, we analyze in detail the special case of $\mathcal{K}_m^2$ when complementary pairs of trees with zones can equivalently be described as trees with a *diaphragm*. Using this description we prove that $\mathcal{K}_m^2$ is isomorphic to the face poset of the multiplihedron $J_m$. Necessary facts about PROPs and calculus of fractions are recalled in the Appendix.

The main definitions are **Definition A** on page 14 and **Definition B** on page 15. The main result is **Theorem C** on page 18 and the main application is **Proposition D** on page 23.

[4] In [8] it was denoted $K_{m,n}$.

[5] Denoted $P_{m-1,n-1}$ in [8].

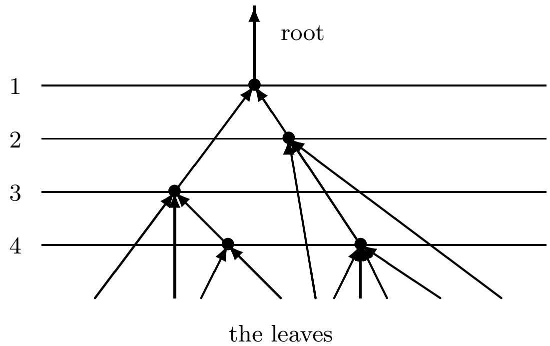



FIGURE 1. An up-rooted tree with 10 leaves and 5 vertices aligned at 4 levels, none of them 'dummy.' The edges are oriented towards the root.

## 1. Trees with levels and (bi)permutahedra

1.1. **Up- and down-rooted trees.** Let us start by recalling some classical material from [7, II.1.5]. A planar directed (also called rooted) tree is a planar tree with a specified leg called the *root*. The remaining legs are the *leaves*. We will tacitly assume that all vertices have at least three adjacent edges.

We will distinguish between *up-rooted* trees whose all edges different from the root are oriented *towards* the root while, in *down-rooted* trees, we orient the edges to point *away* from the root. The set of vertices $\mathit{Vert}(T)$ of an up- or down-rooted tree $T$ is partially ordered by requiring that $u < v$ if and only if there exist an oriented edge path starting at $u$ and ending at $v$.

An up-rooted planar tree with $h$ *levels*, $h \geq 1$, is an up-rooted planar tree $U$ with vertices placed at $h$ horizontal lines numbered $1, \ldots, h$ from the top down. More formally, an up-rooted tree with $h$ levels is an up-rooted planar tree $U$ together with a strictly order-preserving *level function* $\ell : \mathit{Vert}(U) \to \{1, \ldots, h\}$.[6] We tacitly assume that the level function is an epimorphism (no 'dummy' levels with no vertices); if this is not the case, we say that $\ell$ is *degenerate*. We believe that Figure 1 clarifies these notions. Since we numbered the level lines from the top down, saying that vertex $v'$ lies *above* $v''$ means $\ell(v') < \ell(v'')$.

Let us denote by $lT_m$ the set of up-rooted trees with levels and $m$ leaves. It forms a category whose morphisms $(U', \ell') \to (U'', \ell'')$ are couples $(\phi, \hat{\phi})$ consisting of a map[7] of up-rooted planar trees $\phi : U' \to U''$ and of an order-preserving map $\hat{\phi} : \{1, \ldots, h'\} \to \{1, \ldots, h''\}$ forming the commutative diagram

$$\begin{tikzcd}
\mathit{Vert}(U') \arrow[r, "\phi"] \arrow[d, "\ell'"'] & \mathit{Vert}(U'') \arrow[d, "\ell''"] \\
\{1, \ldots, h'\} \arrow[r, "\hat{\phi}"] & \{1, \ldots, h''\}
\end{tikzcd}$$

in which we denote $\phi : U' \to U''$ and the induced map $\mathit{Vert}(U) \to \mathit{Vert}(U'')$ by the same symbol.

[6] Strictly order-preserving means that $v' < v''$ implies $\ell(v') < \ell(v'')$.

[7] By a map of trees we understand a sequence of contractions of internal edges. In particular, the root and leaves are fixed.

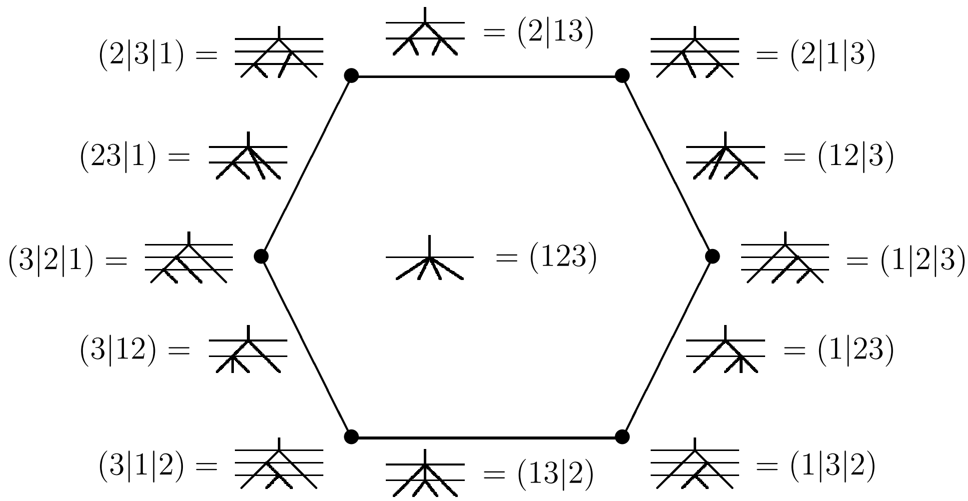



FIGURE 2. The faces of the permutahedron $P_3$ indexed by the set $lT_4$. The ordered partitions of $\{1, 2, 3\}$ corresponding to the faces under the correspondence described in §1.3 are also shown.

We say that $(U', \ell') < (U'', \ell'')$ if there exists a morphism $(U', \ell') \to (U'', \ell'')$. Since all endomorphisms in $lT_m$ are the identities, the relation $<$ is a partial order. The set $lT_m$ with this partial order is isomorphic to the face poset $\mathcal{P}_{m-1}$ of the permutahedron $P_{m-1}$. This result is so classical that we will not give full details here, see [11].[8] For $m = 4$, this isomorphism is illustrated in Figure 2.

The definition of the poset $(lT^n, <)$ of down-rooted trees with levels and $n$ leaves is similar. We will also need the *exceptional tree* $|$ with one edge and no vertices. We define $lT_1 = lT^1 := \{|\}$.

1.2. **Complementary pairs.** For $m, n \geq 1$ we denote by $lT^n_m$ the set of all triples $(U, D, \ell)$ consisting of an up-rooted planar tree $U$ with $m$ leaves and a down-rooted planar tree $D$ with $n$ leaves, equipped with a strictly order-preserving level function

$$\ell : \mathit{Vert}(U) \cup \mathit{Vert}(D) \twoheadrightarrow \{1, \ldots, h\}.$$

Observe that if we denote $\ell_u := \ell|_{\mathit{Vert}(U)}$ (resp. $\ell_d := \ell|_{\mathit{Vert}(D)}$), then $(U, \ell_u)$ (resp. $(D, \ell_d)$) is an up-rooted (resp. down-rooted) rooted tree with possibly degenerate level function.

We call the objects $(U, D, \ell)$ the *complementary pairs* of trees with levels. Figure 3 explains the terminology, concrete examples can be found in Figure 6. When the level function is clear from the context, we drop it from the notation. The set $lT^n_m$ forms a category in the same way as $lT_m$. A morphism

$$\phi : (U', D', \ell') \to (U'', D'', \ell'')$$

is a triple $(\phi_u, \phi_d, \hat{\phi})$ consisting of a morphism $\phi_u : U' \to U''$ (resp. $\phi_d : D' \to D''$) of up-rooted (resp. down-rooted) planar trees and of an order-preserving map $\hat{\phi} : \{1, \ldots, h'\} \to \{1, \ldots, h''\}$

[8] We however recall the correspondence between trees with levels and ordered partitions in §1.3.

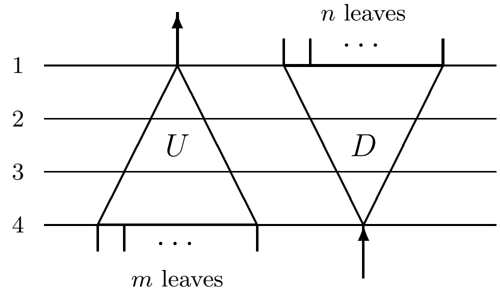



FIGURE 3. A schematic picture of a couple $(U, D) \in lT_m^n$ of complementary trees with 4 levels.

such that the diagram

$$\begin{tikzcd} Vert(U') \cup Vert(D') \arrow[r, "\phi_u \cup \phi_d"] \arrow[d, "\ell'"'] & Vert(U'') \cup Vert(D'') \arrow[d, "\ell''"] \\ \{1, \ldots, h'\} \arrow[r, "\hat{\phi}"] & \{1, \ldots, h''\} \end{tikzcd} \tag{4}$$

commutes. The partial order of $lT_m^n$, analogous to that of $lT_m$, is given by the existence of a morphism in the above category.

**Theorem 1.2.1.** *The posets $\mathcal{P}_{m+n-2} = (lT_{m+n-1}, <)$ and $\mathcal{P}_m^n = (lT_m^n, <)$ are naturally isomorphic for each $m, n \geq 1$.*

*Proof.* Let us describe an isomorphisms of the underlying sets. Since clearly $lT_u^1 \cong lT_1^u \cong lT_u$ for each $u \geq 1$, it is enough to construct, for each $n \geq 2$, $m \geq 1$, an isomorphism

$$lT_m^n \cong lT_{m+1}^{n-1}. \tag{5}$$

Let $X = (U, D') \in lT_m^n$. Denote by $v$ the initial vertex of the leftmost leaf of $D'$ and $L$ its level line. Amputation of this leftmost leaf at $v$ gives a down-rooted tree $D$ with $n-1$ leaves. Now extend the up-rooted tree $U$ by grafting an up-going leaf at the rightmost point $w$ in which the level line $L$ intersects $U$. If $w$ is a vertex, we graft the leaf at this vertex, if $w$ is a point of an edge, we introduce a new vertex with two input edges.

Let $U'$ denotes this extended tree. The isomorphism (5) is given by the correspondence $(U, D') \mapsto (U', D)$, with the pair $(U', D)$ equipped with the level function induced, in the obvious manner, by the level function of $(U, D')$. We believe that Figure 4 makes isomorphism (5) obvious. It would, of course, be possible to define it using the formal language of trees with level functions, but we consider the above informal, intuitive definition more satisfactory. It is simple to verify that (5) preserves the partial orders, giving rise to a poset isomorphism $\mathcal{P}_m^n \cong \mathcal{P}_{m+1}^{n-1}$, for each $n \geq 2$, $m \geq 1$. □

**Example.** The isomorphism of Theorem 1.2.1 is, for $m + n = 4$, presented in Figure 5.

**Example.** Isomorphism (5) of complementary pairs is, for $m + n = 5$, shown in the table of Figure 6. Comparing the entries in the leftmost column with the corresponding entries of the 4th one, we see a nontrivial isomorphism between the posets of up- and down-rooted trees. We suggest, as an exercise, to decorate the faces of the permutahedron $P_4$ in Figure 2 by the corresponding entries of the table in Figure 6 on page 11 to verify in this particular case that the partial orders are indeed preserved.

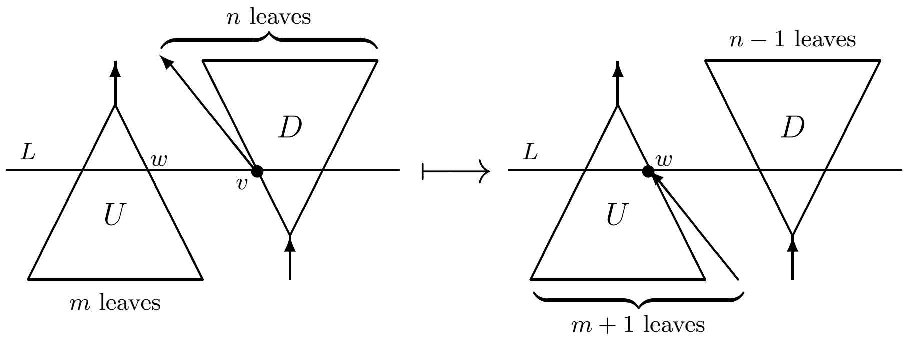



FIGURE 4. Isomorphism (5).

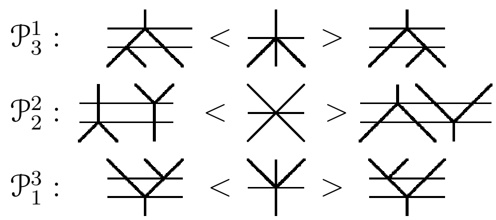



FIGURE 5. The isomorphic posets $\mathcal{P}_2 = \mathcal{P}_3^1$, $\mathcal{P}_2^2$ and $\mathcal{P}_1^3$.

1.3. **Relation to the standard permutahedron.** In this subsection we recall the well-known isomorphism between the poset $\mathcal{P}_m = (lT_m, <)$ of rooted planar trees with levels and the poset of ordered decompositions of the set $\{1, \ldots, m-1\}$ which we denote by $\mathcal{SP}_m = (Dec_m, <)$ (the $\mathcal{S}$ in front of $\mathcal{P}$ abbreviating the *standard* permutahedron). This isomorphism, which forms a necessary link to [8] and related papers, extends to an isomorphism between $(\mathcal{P}_m^n, <)$ and the poset of ordered *bi*partitions $\mathcal{SP}_m^n = (Dec_m^n, <)$. As the gadgets described here will not be used later in our note, this subsections can be safely skipped.

We start by drawing a tree $U \in lT_m$, as always in this note, with the root up, and labelling the intervals between its leaves, from the left to the right, by $1, \ldots, m-1$. We then replace the labels by party balloons, release them and let them lift to the highest possible level.[9] The first set of the corresponding partition is formed by the balloons that lifted to the root (level 1), the second by the balloons that lifted to level 2, &c. For instance, to the tree $U \in lT_8$ in Figure 7 one associates the ordered decomposition $(4|57|12|36) \in Dec_8$. Another instance of the above correspondence is shown in Figure 2.

Let us denote the resulting isomorphism by $\gamma : \mathcal{P}_m \cong \mathcal{SP}_m$. Combined with the isomorphisms of Theorem 1.2.1, it leads to an isomorphism (denoted by the same symbol) $\gamma : \mathcal{P}_m^n \cong \mathcal{SP}_{m+m-2}$ for each $m, n \geq 1$.

**Example.** A particular instance of the above isomorphism is $\gamma : \mathcal{P}^m \cong \mathcal{SP}_m$. On the other hand the posets $\mathcal{P}_m = (lT_m, <)$ and $\mathcal{P}^m = (lT^m, <)$ are in fact the *same*, it is only that we draw the trees in $\mathcal{P}_m$ with the root up, and those in $\mathcal{P}^m$ with the root down. One can prove that the composite

$$\tau : \mathcal{SP}_m \overset{\gamma^{-1}}{\cong} \mathcal{P}_m = \mathcal{P}^m \overset{\gamma}{\cong} \mathcal{SP}_m \tag{6}$$

[9] Alternatively, replace the labels by balls and change the direction of gravity.

$\boxed{\text{Le banquet céleste}}$

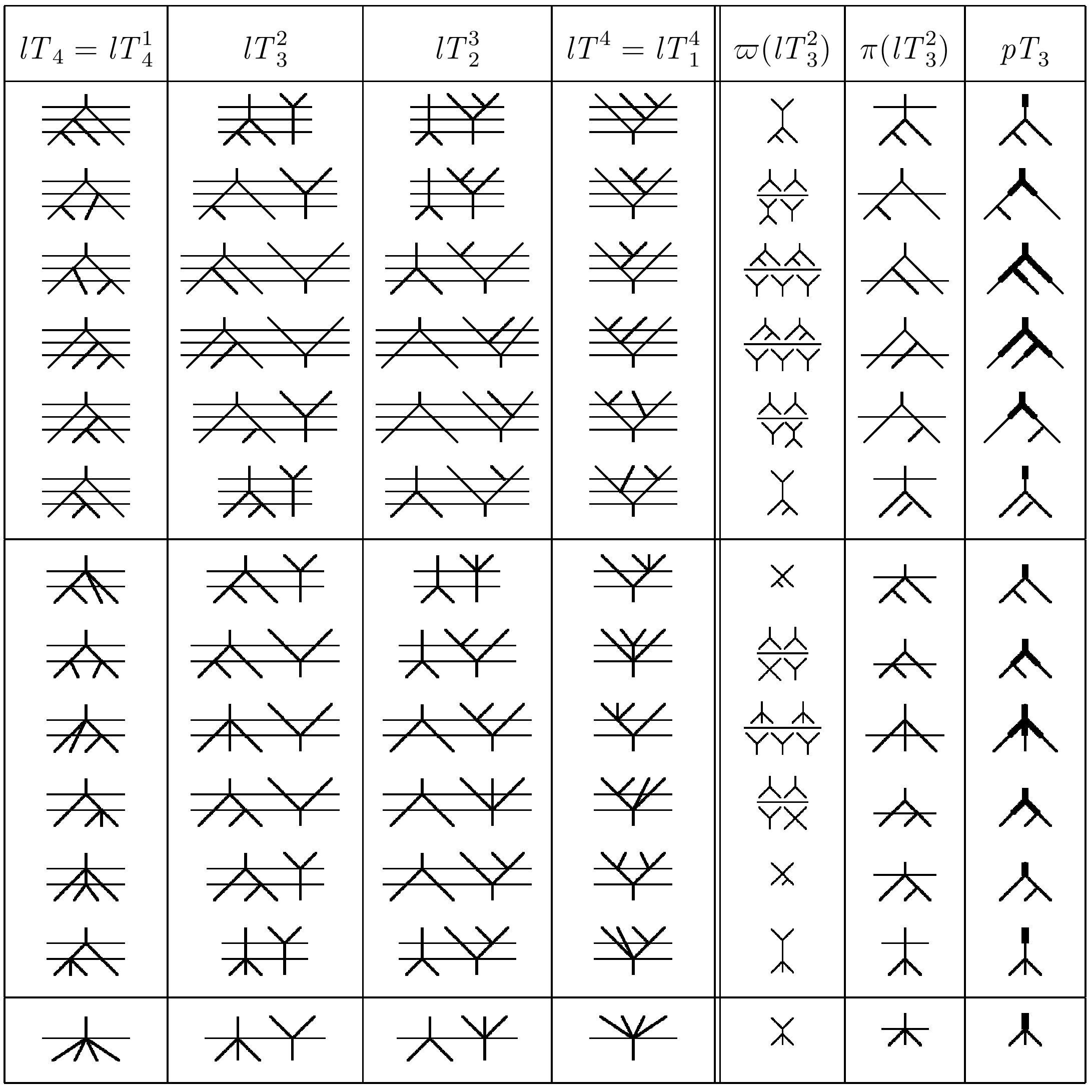



FIGURE 6. The isomorphic sets $lT_m^n$ with $m+n=5$, $\varpi(lT_3^2)$, $zT_3^2$ and $pT_3$. The upper section of the left part corresponds to the vertices, the middle to the edges, and the bottom row to the 2-cell of the bipermutahedron $P_3 = P_4^1 \cong P_3^2 \cong P_2^3 \cong P_1^4$.

is given by reversing the order of the members of the decomposition. For instance,

$$\tau(4|57|12|36) = (36|12|57|4).$$

Let us extend the above isomorphism to complementary pairs of trees. For $m, n \geq 1$, denote by $\mathcal{SP}_m^n = (Dec_m^n, <)$ the poset of ordered *bipartitions* of the sets $\{1, \ldots, m-1\}$, $\{m, \ldots, m+n-2\}$, by which we mean arrays

$$\left( \left| \begin{array}{c} D_\ell \\ U_1 \end{array} \right| \left| \begin{array}{c} D_{\ell-1} \\ U_2 \end{array} \right| \cdots \left| \begin{array}{c} D_2 \\ U_{\ell-1} \end{array} \right| \left| \begin{array}{c} D_1 \\ U_\ell \end{array} \right| \right),$$

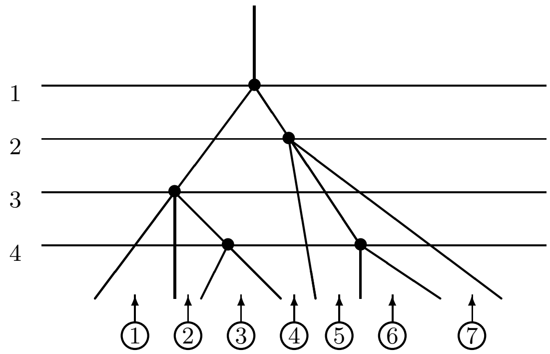



FIGURE 7. An up-rooted tree $U \in lT_8$ with party balloons ready to take off.

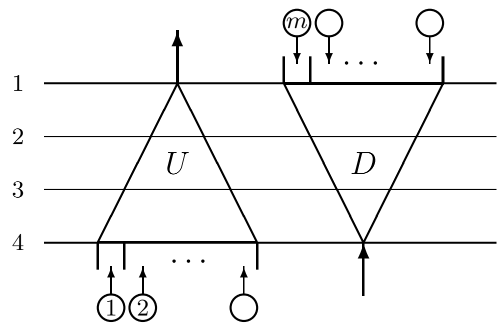



FIGURE 8. A complementary pair decorated by $m+n-2$ balloons and balls. The labels $m-1$, $m+1$ and $n+m-2$ are not shown since the balloons resp. balls are too small to hold them.

where $U_1, U_2, \ldots, U_\ell$ (resp. $D_1, D_2, \ldots, D_\ell$) is an ordered partition of $1, \ldots, m-1$ (resp. an ordered partition of $m, \ldots, m+n-2$). Here we allow some of the sets $U_j$ (resp. $D_j$) to be empty, but we require $U_j \cup D_j \neq \emptyset$ for each $1 \leq j \leq \ell$.

We associate to a complementary pair $X = (U, D) \in lT_m^n$ a bipartition in $Dec_m^n$ as follows. We attach to the intervals between the leaves of the up-rooted tree $U$ balloons labeled $1, \ldots, m-1$, and to the intervals between the leaves of the down-rooted tree $D$ balls labelled $m, \ldots, m+n-2$ as indicated in Figure 8. Then $U_j$ (resp. $D_j$) is the set of balloons (resp. balls) that lift (resp. fall) to level $j$, $1 \leq j \leq \ell$. Observe that the reversing map (6) is already built in the above assignment.

**Example.** The following bipartitions correspond to the entries of the second line of the first part of the table in Figure 6:

$$\left(\begin{array}{|c||c||c|} & & \\ 2 & 3 & 1 \end{array}\right), \left(\begin{array}{|c||c||c|} & 3 & \\ 2 & & 1 \end{array}\right), \left(\begin{array}{|c||c||c|} & 3 & 2 \\ & & 1 \end{array}\right) \text{ and } \left(\begin{array}{|c||c||c|} 1 & 3 & 2 \\ & & \end{array}\right),$$

while the bipartitions corresponding to the second line of the second part are

$$\left(\begin{array}{|c||c|} & \\ 2 & 13 \end{array}\right), \left(\begin{array}{|c||c|} 3 & \\ 2 & 1 \end{array}\right), \left(\begin{array}{|c||c|} 3 & 2 \\ & 1 \end{array}\right), \text{ and } \left(\begin{array}{|c||c|} 13 & 2 \\ & \end{array}\right).$$

As an exercise, we recommend describing the isomorphism of Theorem 1.2.1 in terms of bipartitions. The above example serves as a clue how to do so.

[March 6, 2013]

## 2. Trees with zones and the biassociahedron $\mathcal{K}^n_m$

In this section we present our definition of the face poset $\mathcal{K}^n_m$ of the biassociahedron. Let us recall the classical associahedron first.

2.1. **The associahedron $K_m$.** As in (3), denote by $\mathsf{F}(\xi_2, \xi_3, \ldots)$ the free non-$\Sigma$ operad in the monoidal category of sets, generated by the operations of $\xi_2, \xi_3, \ldots$ of arities $2, 3, \ldots$, respectively. Its component of arity $n$ consists of (up-rooted) planar rooted trees with vertices having at least 2 inputs [6, Section 4]. We can therefore define the map (3) simply by forgetting the level functions. We however give a more formal, inductive definition which exhibits some features of other constructions used later in this note.

Let $(U, \ell) \in lT_m$. Since our description of the map (3) will not depend on the level function, we drop it from the notation. If $U$ is the up-rooted $n$-corolla $c_m$, $m \geq 2$, i.e. the tree with one vertex and $m$ leaves, we put

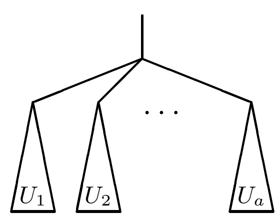
$$\varpi(c_m) := \xi_m \in \mathsf{F}(\xi_2, \xi_3, \ldots)(m).$$

Agreeing that $c_1$ denotes the exceptional tree $|$, we extend the above formula for $m = 1$ by

$$\varpi_1(c_1) := e \in \mathsf{F}(\xi_2, \xi_3, \ldots)(1),$$

where $e$ is the operad unit. Let us proceed by induction on the number of vertices. An arbitrary $U \in lT_m$, $m \geq 2$, is of the form

(7)




with some up-rooted, possibly exceptional, trees $U_1, \ldots, U_a$, $a \geq 2$, each having strictly fewer vertices than $U$. We then put

$$\varpi(U) := \xi_a\big(\varpi(U_1), \ldots, \varpi(U_a)\big) \in \mathsf{F}(\xi_2, \xi_3, \ldots)(m),$$

where $-(-, \ldots, -)$ in the right hand side denotes the operad composition. Notice that we simplified the notation by dropping the subscripts of $\varpi$.

For $(U', \ell), (U'', \ell'') \in lT_m$ let $(U', \ell') \sim (U'', \ell'')$ if $\varpi(U', \ell') = \varpi(U'', \ell'')$. Since obviously the latter equality holds if and only if $U' = U''$, the levels disappear and the quotient $lT_m/\sim$ is isomorphic to the set $T_m$ of up-rooted trees with $m$ leaves. The partial order of $lT_m$ induces the standard partial order[10] on $T_m$, so we have the isomorphism

$$\mathcal{K}_m \cong \mathcal{P}_{m-1}/\sim .$$

We can take the above equation as a *definition* of the face poset of the associahedron $K_m$. The discrepancy between the indices ($m$ versus $m - 1$) is of historical origin.

[10]The one such that $T' < T''$ if and only if there exists a morphism of planar up-rooted trees $T' \to T''$.

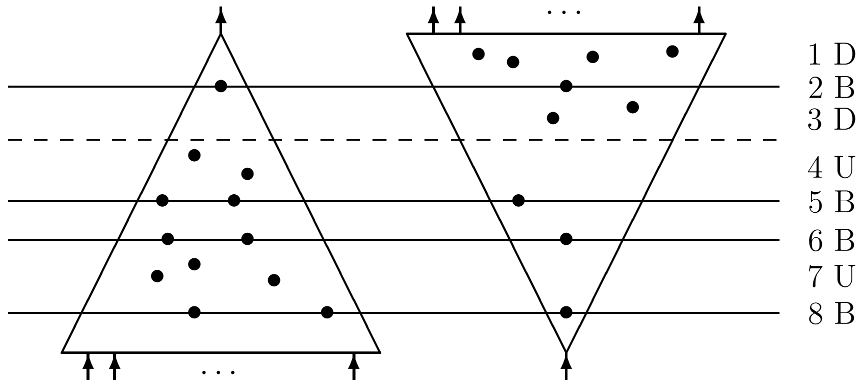



FIGURE 9. An element of $zT^n_m$.

2.2. **Complementary pairs with zones.** For $m, n \geq 1$, consider a triple $(U, D, z)$ consisting of an up-rooted planar tree $U$ with $m$ leaves, a down-rooted planar tree $D$ with $n$ leaves, and an order preserving epimorphism

$$z : \mathit{Vert}(U) \cup \mathit{Vert}(D) \twoheadrightarrow \{1, \ldots, l\}. \tag{8}$$

We call $i \in \{1, \ldots, l\}$ such that $z^{-1}(i)$ contains both a vertex of $U$ and a vertex of $D$ a *barrier*. The remaining $i$'s are the *zones* of $z$. If $z^{-1}(i) \subset \mathit{Vert}(U)$ (resp. $z^{-1}(i) \subset \mathit{Vert}(D)$), we call $i$ an *up-zone* (resp. *down-zone*).

**Definition A.** We call (8) a *zone function* if

(i) $z$ is strictly order-preserving on barriers and
(ii) there are no adjacent zones of the same type.

We denote by $zT^n_m$ the set of all triples $(U, D, z)$ consisting of a planar up-rooted tree $U$ with $m$ leaves, a planar down-rooted tree with $n$ leaves, and a zone function (8).

Condition (i) means the following. If $u', u'' \in \mathit{Vert}(U)$ and $v \in \mathit{Vert}(D)$ are such that $z(u') = z(u'') = z(v)$, then $u'$ and $u''$ are unrelated.[11] Dually, if $v', v'' \in \mathit{Vert}(D)$ and $u \in \mathit{Vert}(U)$ are such that $z(u) = z(v') = z(v'')$, then $v'$ and $v''$ are unrelated. Condition (ii) can be rephrased as follows. For $i \in (1, \ldots, l)$ let

$$t_z(i) := \begin{cases} \text{U} & \text{if } i \text{ is an up-zone,} \\ \text{D} & \text{if } i \text{ is an down-zone, and} \\ \text{B} & \text{if } i \text{ is a barrier.} \end{cases}$$

Condition (ii) then says that the sequence $\big(t_z(1) \cdots t_z(l)\big)$ does not contains subsequences UU or DD. We call $\big(t_z(1) \cdots t_z(l)\big)$ the *type* of $z$.

The notion of complementary pairs with zones is illustrated in Figure 9. In the picture, the values 1,3,4,7 are zones, the values 2,5,6 are barriers. The type of the zone function is (DBDUBBUB).

**Example.** Let us look at case $n = 2$ of Definition A. For $X = (U, \mathsf{Y}, z) \in zT^2_m$, only the following three cases may happen.

*Case $l = 1$.* 1 is a barrier if $m \geq 2$ and 1 is a down-zone if $m = 1$.

[11]By this we mean that neither $u' < u''$ nor $u' > u''$.

*Case* $l = 2$. One has four possibilities for the type of $z$, namely (DU), (UD), (BU) or (UB). In the (DU) and (UD) cases $m \geq 2$, in the remaining two cases $m \geq 3$.

*Case* $l = 3$. $m \geq 3$ and the only possibility for the type is Alfred Jarry's (UBU).

For $i \in \{1, \ldots, h\}$ define its *closure* $\overline{i} \subset \{1, \ldots, h\}$ by $\overline{i} := \{i\}$ if $i$ is a barrier, and $\overline{i}$ to be the set consisting of $i$ and its adjacent barriers in the opposite case. For the complementary pair in Figure 9,

$$\begin{array}{llll} \overline{1} = \{1,2\}, & \overline{2} = \{2\}, & \overline{3} = \{2,3\}, & \overline{4} = \{4,5\}, \\ \overline{5} = \{5\}, & \overline{6} = \{6\}, & \overline{7} = \{6,7,8\}, & \overline{8} = \{8\}. \end{array}$$

A *morphism* $(U', D', z') \to (U'', D'', z'')$ is a triple $(\phi_u, \phi_d, \hat{\phi})$ consisting of morphisms $\phi_u : U' \to U''$, $\phi_d : D'' \to D''$ of trees and of an order-preserving map $\hat{\phi} : \{1, \ldots, l'\} \to \{1, \ldots, l''\}$ such that the closures are preserved, that is

$$z''\big(\phi_u(u)\big) \in \overline{\hat{\phi}\big(z'(u)\big)} \ \text{ and } z''\big(\phi_d(v)\big) \in \overline{\hat{\phi}\big(z'(v)\big)},$$

for $u \in \mathit{Vert}(U)$ and $v \in \mathit{Vert}(D)$. We may also say that the obvious analog of (4), i.e.

$$\begin{tikzcd} \mathit{Vert}(U') \cup \mathit{Vert}(D') \arrow[r, "\phi_u \cup \phi_d"] \arrow[d, "z'_u \cup z'_d"'] & \mathit{Vert}(U'') \cup \mathit{Vert}(D'') \arrow[d, "z''_u \cup z''_d"] \\ \{1, \ldots, h'\} \arrow[r, "\hat{\phi}"] & \{1, \ldots, h''\}. \end{tikzcd} \tag{9}$$

commutes up to the closures. The notion of a morphism induces a partial order $<$.

**Definition B.** The *face poset of the (step-one) biassociahedron* is the poset $\mathcal{K}^n_m := (zT^n_m, <)$ of complementary pairs of trees with zones, with the above partial order.

Let $(U, D, \ell) \in lT^n_m$ be a pair with the level function $\ell : \mathit{Vert}(U) \cup \mathit{Vert}(D) \to \{1, \ldots, h\}$. We call $i \in \{1, \ldots, h\}$ an *up-level* (resp. *down-level*) if $\ell^{-1}(i) \subset \mathit{Vert}(U)$ (resp. $\ell^{-1}(i) \subset \mathit{Vert}(D)$).

**Definition 2.2.1.** Let $(U, D, \ell) \in lT^n_m$ be as above and $(1, \ldots, l)$ the quotient cardinal obtained from $(1, \ldots, h)$ by identifying the adjacent up-levels and the adjacent down-levels. Denote by $p : (1, \ldots, h) \twoheadrightarrow (1, \ldots, l)$ the projection and define

$$\pi(U, D, \ell) := (U, D, z) \in zT^n_m, \quad \text{with } z := p \circ \ell.$$

We call the map $\pi : lT^n_m \to zT^n_m$ defined in this way the *canonical projection* and $z = p \circ \ell$ the *induced zone function*.

It is easy to show that the map $\pi$ preserves the partial orders, giving rise to the projection $\mathcal{P}^n_m \twoheadrightarrow \mathcal{K}^n_m$ of posets. We finish this subsection by two statements needed in the proof of Theorem C.

**Proposition 2.2.2.** *Let $(U, D, z'), (U, D, z'') \in zT^n_m$. If, for each vertices $u \in \mathit{Vert}(U)$ and $v \in \mathit{Vert}(D)$,*

$$z'(u) < z'(v) \ (\textit{resp. } z'(u) = z'(v), \ \textit{resp. } z'(u) > z'(v)) \tag{10a}$$

*is equivalent to*

$$z''(u) < z''(v) \ (\textit{resp. } z''(u) = z''(v), \ \textit{resp. } z''(u) > z''(v)), \tag{10b}$$

*then $z' = z''$.*

*Proof.* Let

$$z' : \mathit{Vert}(U) \cup \mathit{Vert}(D) \to \{1, \ldots, h'\} \ \text{ and } \ z'' : \mathit{Vert}(U) \cup \mathit{Vert}(D) \to \{1, \ldots, h''\}$$

be zone functions as in the proposition. Let us show that, for $x, y \in \mathit{Vert}(U) \cup \mathit{Vert}(D)$,

$$\text{if } z'(x) = z'(y) \text{ then } z''(x) = z''(y). \tag{11}$$

The above implication immediately follows from the assumptions if $x \in \mathit{Vert}(U)$ and $y \in \mathit{Vert}(D)$, or if $x \in \mathit{Vert}(D)$ and $y \in \mathit{Vert}(U)$. So assume $x, y \in \mathit{Vert}(U)$, $z'(x) = z'(y)$ and, say, $z''(x) > z''(y)$. Since, by definition, $z''$ does not have two adjacent zones of type U, there must exist $v \in \mathit{Vert}(D)$ such that $z''(x) \geq z''(v) \geq z''(y)$, where at least one relation is sharp. Assuming the equivalence between (10a) and (10b), we get $z'(x) \geq z'(v) \geq z'(y)$ where again at least one relation is sharp, so $z'(x) \neq z'(y)$, which is a contradiction. The case $x, y \in \mathit{Vert}(D)$ can be discussed in the same way, thus (11) is established.

For $i \in \{1, \ldots, h'\}$ (resp. $j \in \{1, \ldots, h''\}$) denote $S'_i := z'^{-1}(i)$ (resp. $S''_j := z''^{-1}(j)$). By (11), for each $i$ there exists a unique $j$ such that $S'_i \subset S''_j$. Exchanging the rôles of $z'$ and $z''$, we see that, vice versa, for each $j$ there exists a unique $i$ such that $S''_j \subset S'_i$. This obviously means that there exists an automorphism $\varphi : \{1, \ldots, h'\} \to \{1, \ldots, h''\}$ such that $z' = \varphi \circ z''$. As both $z'$ and $z''$ are order-preserving epimorphisms, $\varphi$ must be the identity. □

The following proposition in conjunction with Proposition 2.2.2 shows that the induced zone function remembers the relative heights of vertices of $U$ and $D$ but nothing more.

**Proposition 2.2.3.** *Let $(U, D, \ell) \in lT^n_m$ and $z$ be the zone function induced by $\ell$. Then, for each $u \in \mathit{Vert}(U)$ and $v \in \mathit{Vert}(D)$,*

$$\ell(u) < \ell(v) \ (\textit{resp. } \ell(u) = \ell(v), \ \textit{resp. } \ell(u) > \ell(v))$$

*if and only if*

$$z(u) < z(v) \ (\textit{resp. } z(u) = z(v), \ \textit{resp. } z(u) > z(v)).$$

*Proof.* The proof is a simple application of the definition of the induced zone function. □

2.3. **The map $\varpi : lT^n_m \to \mathsf{F}(\Xi)$.** This subsection relies on the notation and terminology recalled in the Appendix. For $(U, D, \ell) \in lT^n_m$ and subtrees $\overline{U} \subset U$, $\overline{D} \subset D$ with, say, $\overline{m}$ and resp. $\overline{n}$ leaves, one has a natural *restriction*

$$r_{\overline{U}, \overline{D}}(U, D, \ell) = (\overline{U}, \overline{D}, \overline{\ell}) \in lT^{\overline{n}}_{\overline{m}} \tag{12}$$

with the level function $\overline{\ell} : \mathit{Vert}(\overline{U}) \cup \mathit{Vert}(\overline{D}) \to (1, \ldots, \overline{h})$ defined as follows. The image of the restriction of $\ell$ to $\mathit{Vert}(\overline{U}) \cup \mathit{Vert}(\overline{D})$ is a sub-cardinal of $(1, \ldots, h)$, canonically isomorphic to $(1, \ldots, \overline{h})$ for some $\overline{h} \leq h$. The level function $\overline{\ell}$ is then the composition of the restriction of $\ell$ with this canonical isomorphism. In other words, $\overline{\ell}$ is the epimorphism in the factorization

$$\mathit{Vert}(\overline{U}) \cup \mathit{Vert}(\overline{D}) \stackrel{\overline{\ell}}{\twoheadrightarrow} (1, \ldots, \overline{h}) \hookrightarrow (1, \ldots, h)$$

of the restriction of $\ell$.

Let $\mathsf{F}(\Xi)$ be the free PROP in the category of sets generated by

$$\Xi := \big\{ \xi^n_m \mid m, n \geq 1, \ (m, n) \neq (1, 1) \big\},$$

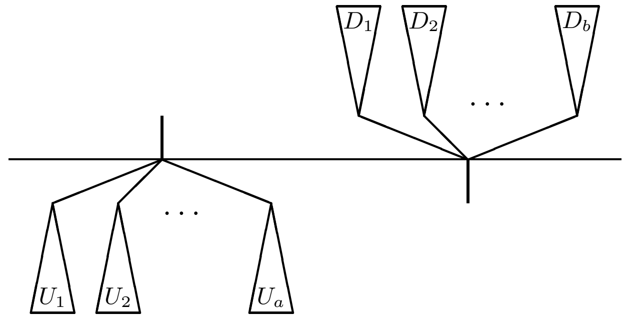
$X =$ 


FIGURE 10. The decomposition of $U$ and $D$ in the 2nd case.

where $\xi^n_m$ is the generator of biarity $(n,m)$ ($m$ inputs and $n$ outputs). Observe that one has, for each $m, n \geq 1$, the inclusions

$$\iota_U : \mathsf{F}(\xi_2, \xi_3, \ldots)(m) \hookrightarrow \mathsf{F}(\Xi)\binom{1}{m} \ \text{ and } \ \iota_D : \mathsf{F}(\xi_2, \xi_3, \ldots)(n) \hookrightarrow \mathsf{F}(\Xi)\binom{n}{1}, \tag{13}$$

given by

$$\iota_U(\xi_a) := \xi^1_a \ \text{ resp. } \ \iota_D(\xi_a) := \xi^a_1,\ a \geq 2.$$

We will use $\iota_U$ to identify $\mathsf{F}(\xi_2, \xi_3, \ldots)(m)$ with a subset of $\mathsf{F}(\Xi)\binom{1}{m}$.

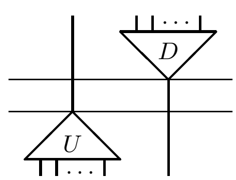
Let us start the actual construction of the map $\varpi : lT^n_m \to \mathsf{F}(\Xi)\binom{n}{m}$. First of all, for a down-rooted tree $D$, i.e. an element of $lT^n_1$, define

$$\varpi(D) := \iota_D\big(\varpi(D')\big) \in \mathsf{F}(\Xi)\binom{n}{1},$$

where $D'$ is $D$ turned upside down (i.e. an up-rooted tree), $\varpi$ is as in 2.1 and $\iota_D$ the second inclusion of (13).

Each element $X \in lT^n_m$ is a triple $X = (U, D, \ell)$, where $U$ is an up-rooted tree with $m$ leaves and $D$ a down-rooted tree with $n$ leaves. We construct $\varpi(X)$ by induction on the number of vertices of $D$. We distinguish three cases.

*Case 1.* The root vertex of $D$ is above the root vertex of $U$,[12] schematically




Then we put

$$\varpi(X) := \frac{\varpi(D)}{\varpi(U)} = \varpi(D) \circ \varpi(U) \in \mathsf{F}(\Xi)\binom{n}{m}. \tag{14}$$

*Case 2.* The vertex of $D$ is at the same level as the root vertex of $U$. We decompose $U$ as in (7) and $D$ in the obvious dual manner, the result is portrayed in Figure 10. We then define

$$\varpi(X) := \frac{\varpi(D_1) \cdots \varpi(D_b)}{\dfrac{\xi^b_a}{\varpi(U_1) \cdots \varpi(U_a)}} = \big(\varpi(D_1) \boxtimes \cdots \boxtimes \varpi(D_b)\big) \circ \xi^b_a \circ \big(\varpi(U_1) \boxtimes \cdots \boxtimes \varpi(U_a)\big). \tag{15}$$

[12]That is the vertex adjacent to the root.

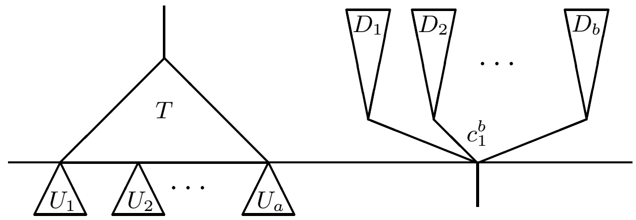
$X =$ 


FIGURE 11. The decomposition of $U$ and $D$ in the 3rd case.

*Case 3.* The root vertex of $D$ is below the root vertex of $U$. In this case we decompose $U$ and $D$ as in Figure 11 in which $T$ is the maximal up-rooted tree containing all vertices above the level of the root vertex of $D$ and the up-rooted trees $U_1, \ldots, U_a$ contain all the remaining vertices of $U$. The decomposition of $D$ is the same as in Case 2. Using the restriction (12), we denote

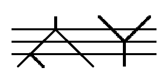
$$X_i := r_{U_i, c_1^b}(X),\ 1 \leq i \leq a, \quad \text{and } Y_j := r_{T, D_j}(X),\ 1 \leq j \leq b. \tag{16}$$

Clearly $\varpi(X_i)$'s fall into the previous two cases. Since $D_j$ has strictly less vertices than $D$, $\varpi(Y_j)$'s have been defined by induction. We put

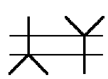
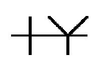
$$\varpi(X) = \frac{\varpi(Y_1) \cdots \varpi(Y_b)}{\varpi(X_1) \cdots \varpi(X_a)} \in \mathsf{F}(\Xi)\binom{n}{m}. \tag{17}$$

**Remark.** In the above construction of the map $\varpi$, the root vertex of $D$ plays a different rôle than the root vertex of $U$. One can exchange the rôles of $U$ and $D$, arriving at a formally different formula for $\varpi(X)$. Due to the associativity of fractions [3, Section 6], both formulas give the same element of $\mathsf{F}(\Xi)\binom{n}{m}$. It is also possible to write a non-inductive formula for $\varpi(X)$ based on the technique of *block transversal matrices* developed in [8].

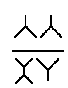
**Example.** If

$X =$  ,

we are in Case 3, with $T = U_1$ the up-rooted 2-corollas , and $U_2$ the exceptional tree . We have

$X_1 =$  and $X_2 =$  .

Both $X_1$ and $X_2$ fall into Case 1, and $\varpi(X_1) = \xi_1^2 \circ \xi_2^1$ while $\varpi(X_2) = \xi_1^2$. Formula (17) gives

$$\varpi(X) = \frac{\xi_2^1\ \ \xi_2^1}{\frac{\xi_1^2}{\xi_2^1}\ \ \xi_1^2} =$$  .

The rightmost term is obtained by depicting $\xi_m^n$ as an oriented corolla with $m$ inputs and $n$ outputs. The same notation is used in the 5th column of the table of Figure 6 which lists elements in the image $\varpi(lT_3^2)$.

Let us formulate the main result of this note which relates the canonical projection of Definition 2.2.1 with the map $\varpi$.

**Theorem C.** *Let $X', X'' \in lT_m^n$ be two complementary pairs of trees. Then*

$$\varpi(X') = \varpi(X'') \ \textit{(equality in } \mathsf{F}(\Xi)\tbinom{n}{m}\textit{)}$$

[March 6, 2013]

*if and only if*

$$\pi(X') = \pi(X'') \text{ (equality in } zT^n_m\text{)},$$

*so the image of* $\varpi : lT^n_m \to \mathsf{F}(\Xi)$ *is isomorphic to* $zT^n_m$.

Our *proof* of this theorem occupies the rest of this section.

**Proof that $\pi(X') = \pi(X'')$ implies $\varpi(X') = \varpi(X'')$.** Parallel to $r_{\overline{U},\overline{D}}(U, D, \ell) \in lT^{\overline{n}}_{\overline{m}}$ of (12) there is a similar restriction $s_{\overline{U},\overline{D}}(U, D, z) \in zT^{\overline{n}}_{\overline{m}}$ defined for each $(U, D, z) \in zT^n_m$ and subtrees $\overline{U} \subset U$ and $\overline{D} \subset D$. They commute with the canonical projection in the sense that, for $X = (U, D, \ell) \in lT^n_m$,

$$\pi\big(r_{\overline{U},\overline{D}}(X)\big) = s_{\overline{U},\overline{D}}\big(\pi(X)\big). \tag{18}$$

The restriction $s_{\overline{U},\overline{D}}$ can be defined along similar lines as $r_{\overline{U},\overline{D}}$. The only subtlety is that the zone function restricted to $\mathit{Vert}(\overline{U}) \cup \mathit{Vert}(\overline{D})$ need not satisfy (ii) of Definition A, so we need to identify, in its image, adjacent zones of the same type. We leave the details to the reader.

The construction of $\varpi(X)$ given in §2.3 was divided into three cases, determined by the relative positions of the root vertices of $U$ and $D$. This information is, by Proposition 2.2.3, retained by the induced zone function of $\pi(X)$. Therefore the case into which $X$ falls depends only on the projection $\pi(X)$.

Let $X', X'' \in lT^n_m$ be such $\pi(X') = \pi(X'')$. Then $X'$ and $X''$ may differ only by the level functions, i.e. $X' = (U, D, \ell')$ and $X'' = (U, D, \ell'')$. Let us proceed by induction on the number of vertices of $D$.

If one (hence both) of $X'$, $X''$ falls into Case 1 or Case 2 of our construction of $\varpi$, then clearly $\varpi(X'') = \varpi(X'')$ since in these cases $\varpi$ manifestly depends only on the trees $U$ and $D$ not on the level function. The induced zone function of $\pi(X') = \pi(X'')$ must be of type (DU) in Case 1 and (DBU) in Case 2.

In Case 3 we observe first that the exact form of the decomposition in Figure 11 depends only on the relative positions of the root vertex of $D$ and the vertices of $U$. It is therefore, by Proposition 2.2.3, determined by the induced zone function, so it is the same for both $X'$ and $X''$. Now we invoke the commutativity (18) to check that

$$\pi(X'_i) = \pi\big(r_{U_i,c^b_1}(X')\big) = s_{U_i,c^b_1}\big(\pi(X')\big) = s_{U_i,c^b_1}\big(\pi(X'')\big) = \pi\big(r_{U_i,c^b_1}(X'')\big) = \pi(X''_i)$$

for each $1 \leq i \leq a$. Similarly we verify that $\pi(Y'_j) = \pi(Y''_j)$ for each $1 \leq j \leq b$. By the induction assumption,

$$\varpi(X'_i) = \varpi(X''_i) \ \text{ and } \ \varpi(Y'_j) = \varpi(Y''_j) \ \text{ for } \ 1 \leq i \leq a,\ 1 \leq j \leq b,$$

therefore

$$\varpi(X') = \frac{\varpi(Y'_1) \cdots \varpi(Y'_b)}{\varpi(X'_1) \cdots \varpi(X'_a)} = \frac{\varpi(Y''_1) \cdots \varpi(Y''_b)}{\varpi(X''_1) \cdots \varpi(X''_a)} = \varpi(X'').$$

This finishes our proof of the implication $\pi(X') = \pi(X'') \Longrightarrow \varpi(X') = \varpi(X'')$.

[March 6, 2013]

**Proof that $\varpi(X') = \varpi(X'')$ implies $\pi(X') = \pi(X'')$.** Let us show that $\pi(X)$ is uniquely determined by $\varpi(X) \in \mathsf{F}(\Xi)\binom{n}{m}$. As we already remarked, elements of $\mathsf{F}(\Xi)$ are represented by directed graphs $G$ whose vertices are corollas $c_a^b$ with $a$ inputs and $b$ outputs, where $a, b \geq 1$, $(a,b) \neq (1,1)$. Let $e$ be an internal edge of $G$, connecting an output of $c_r^s$ with an input of $c_u^v$. We say that $e$ is *special* if either $s = 1$ or $u = 1$. The graph $G$ is *special* if all its internal edges are special. Finally, and element of $\mathsf{F}(\Xi)$ is *special* if it is represented by a special graph. We have the following simple lemma whose proof immediately follows from the definition of the fraction.

**Lemma 2.3.1.** *Let $A_1, \dots A_l, B_1, \dots B_k \in \mathsf{F}(\Xi)$ be as in Definition A.0.1. The fraction*

$$\frac{A_1 \cdots A_l}{B_1 \cdots B_k}$$

*is special if and only if all $A_1, \dots A_l, B_1, \dots B_k$ are special and if $k = 1$ or $l = 1$.*

Lemma 2.3.1 implies that $\varpi(X)$ is special if and only if $X$ falls into Case 1 or Case 2 of §2.3. It is also clear that $X$ falls into Case 2 if and only if $\varpi(X)$ is special and the graph representing $\varpi(X)$ has a (unique) vertex $c_a^b$ with $a, b \geq 2$. Therefore $\varpi(X)$ bears the information to which case of its construction $X = (U, D, \ell) \in lT_m^n$ falls.

Suppose that $X$ falls to Case 1 of our definition of $\varpi(X)$. Clearly, formula (14) uniquely determines the planar up-rooted tree $U$ and a down-rooted tree $D$ such that $X = (U, D, \ell)$. The only possible zone function $z$ for $\pi(X) = (U, D, z)$ is of type (DU) with

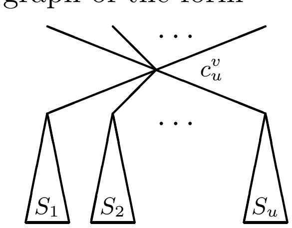
$$z\big(\mathit{Vert}(D)\big) = \{1\},\ z\big(\mathit{Vert}(U)\big) = \{2\}.$$

If $X = (U, D, \ell)$ falls into Case 2 of our construction of $\varpi(X)$, we argue as in the previous paragraph. Formula (15) uniquely determines $\varpi(U_1), \dots, \varpi(U_a)$ and $\varpi(Y_1), \dots, \varpi(Y_b)$ and therefore also the trees $U_1, \dots, U_a, Y_1, \dots, Y_b$ in the decomposition in Figure 10. Therefore also $U$ and $D$ are uniquely determined, and clearly the only possible zone function $z$ for $\pi(X) = (U, D, z)$ is of type $(\mathrm{D}, \mathrm{B}, \mathrm{U})$ with

$$z\big(\mathit{Vert}(D_j)\big) = \{1\},\ z\big(\mathit{Vert}(U_i)\big) = \{3\},$$
$$z(\text{root vertex of } U) = z(\text{root vertex of } D) = \{2\},$$

$1 \leq i \leq a$, $1 \leq j \leq b$.

Assume that $X$ falls into Case 3 of our construction. Let us call, only for the purposes of this proof, a directed graph a *generalized tree*, if it is obtained by grafting directed up-rooted trees $S_1, , , \dots, S_u$ into the inputs of the directed corolla $c_u^v$, with some $u, v \geq 1$, $(u,v) \neq (1,1)$. So a generalized tree is a directed graph of the form




where we keep our convention that all edges are oriented to point upwards. Since $X$ falls into Case 3, we know that $\varpi(X)$ is as in (17), for some $X_i$, $Y_j$, $1 \leq i \leq a$, $1 \leq j \leq b$. We moreover know that the complementary pairs $X_i$ fall into Case 1 or Case 2 of the construction of $\varpi(X_i)$.

Now let $G_1, \ldots, G_r$ be the maximal generalized trees containing the inputs of the graph representing $\varpi(X)$, numbered from left to right. It is clear from the definition of the fraction that $r = a$ and that $G_i$ is, for each $1 \leq i \leq a$, the graph representing $\varpi(X_i)$. So all $\varpi(X_1), \ldots, \varpi(X_a)$ are determined by $\varpi(X)$. A simple argument shows that if

$$\frac{A'_1 \cdots A'_l}{B_1 \cdots B_k} = \frac{A''_1 \cdots A''_l}{B_1 \cdots B_k}$$

for some $A'_1, \ldots, A'_l, A''_1, \ldots, A''_l \in \mathsf{F}(\Xi)\binom{*}{k}$ and $B_1, \ldots, B_k \in \mathsf{F}(\Xi)\binom{l}{*}$, then $A'_i = A''_i$ for each $1 \leq i \leq l$. We therefore see that also $\varpi(Y_1), \ldots, \varpi(Y_b)$ are determined by $\varpi(X)$.

Let us summarize what we have. We know each $\varpi(X_i)$. Since the construction of $\varpi(X_i)$ falls into Case 1 or Case 2, we know, as we have already proved, the trees $U_1, \ldots, U_a$ in Figure 11, and also the relative positions of vertices of $U_1, \ldots, U_a$ and the root vertex of $D$.

Now we preform a similar analysis of $\varpi(Y_1), \ldots, \varpi(Y_b)$ and repeat this process until we get trivial trees. It is clear that, during this process, we fully reconstruct the trees $U, D$ in $X = (U, D, \ell)$ and the relative positions of their vertices. By Proposition 2.2.3, this uniquely determines the zone function in $\pi(X) = (U, D, z)$. This finishes our proof of the second implication.

## 3. The particular case $\mathcal{K}^2_m$

In this section we analyze in detail the poset $\mathcal{K}^2_m$ for which the notion of complementary pairs with zones takes a particularly simple form.

3.1. **Trees with a diaphragm.** Let us consider the ordinal $\big\{(-\infty, 1) < 1 < (1, +\infty)\big\}$. A *diaphragm* of an up-rooted tree $U$ is an order-preserving map

$$\zeta : \mathit{Vert}(U) \to \big\{(-\infty, 1), 1, (1, +\infty)\big\} \tag{19}$$

which is strictly order-preserving at 1. By this we mean that, if $\zeta(v') = \zeta(v'') = 1$ then neither $v' < v''$ nor $v' > v''$. We will denote $dT^2_m$ the set of all pairs $(U, \zeta)$, where $U$ is an up-rooted tree with $m$ leaves and $\zeta$ a diaphragm.

One may imagine a tree with a diaphragm as a planar up-rooted tree crossed by a horizontal line, i.e. a diaphragm, see the rightmost column of the table in Figure 6 for examples. It is convenient to introduce the following subsets of $\mathit{Vert}(U)$:

$$\mathit{Vert}_{<1}(U) := \zeta^{-1}(-\infty, 1),\ \ \mathit{Vert}_1(U) := \zeta^{-1}(1),\ \ \mathit{Vert}_{>1}(U) := \zeta^{-1}(1, +\infty)$$

and the 'closures'

$$\mathit{Vert}_{\leq 1}(U) := \mathit{Vert}_{<1}(U) \cup \mathit{Vert}_1(U) \ \text{ and } \ \mathit{Vert}_{\geq 1}(U) := \mathit{Vert}_{>1}(U) \cup \mathit{Vert}_1(U).$$

A *morphism* $\phi : (U', \zeta') \to (U'', \zeta'')$ of trees with a diaphragm is a morphism $\phi : U' \to U''$ of planar up-rooted trees which preserves the closures, i.e.

$$\phi\big(\mathit{Vert}_1(U')\big) \subset \mathit{Vert}_1(U''),\ \phi\big(\mathit{Vert}_{<1}(U')\big) \subset \mathit{Vert}_{\leq 1}(U''),\ \phi\big(\mathit{Vert}_{>1}(U')\big) \subset \mathit{Vert}_{\geq 1}(U'').$$

We say that $(U', \zeta') < (U'', \zeta'')$ if there exists a morphism $(U', \zeta') \to (U'', \zeta'')$.

**Proposition 3.1.1.** *The posets $(zT^2_m, <)$ and $(dT^2_m, <)$ are, for each $m \geq 1$, naturally isomorphic.*

[March 6, 2013]

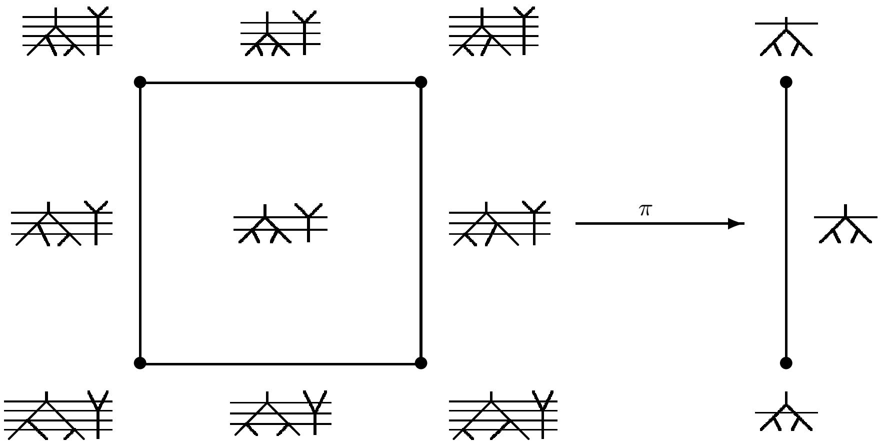



FIGURE 12. The projection of a face of $P_4^2$ to $K_4^2$. The faces of the interval in $K_4^2$ (right) are indexed by trees with a diaphragm.

*Proof.* For $X = (U, \mathsf{Y}, z) \in zT_m^2$, denote $L$ the value of $z$ on the vertex of $\mathsf{Y}$. Define $\zeta$ by

$$\zeta(v) := \begin{cases} (-\infty, 0) & \text{if } z(v) < L, \\ 1 & \text{if } z(v) = L, \text{ and} \\ (1, +\infty) & \text{if } z(v) > L. \end{cases}$$

It is easy to see that $(U, \zeta)$ is a tree with a diaphragm, that the correspondence $(U, z) \mapsto (U, \zeta)$ is one-to-one and that it preserves the partial orders. $\square$

The natural projection $\pi : lT_m^2 \to zT_m^2$ can be, in terms of trees with a diaphragm, described as follows. Let $X = (U, \mathsf{Y}, \ell) \in lT_m^2$ and assume that the vertex of $\mathsf{Y}$ is placed at level $L$. Then $\pi(X) := (U, \zeta)$, with the diaphragm

$$\zeta(v) := \begin{cases} (-\infty, 0) & \text{if } \ell(v) < L, \\ 1 & \text{if } \ell(v) = L, \text{ and} \\ (1, +\infty) & \text{if } \ell(v) > L. \end{cases}$$

**Example.** The $\pi$-images of complementary pairs in $lT_3^2$ are listed in the rightmost column of the table in Figure 6.

**Example.** Figure 12 illustrates the projection $P_4^2 \to K_4^2$. It shows the face poset of a square face of the 3-dimensional $P_4^2$ together with the corresponding complementary pairs in $lT_4^2$ and its projection, which is in this case the poset of the interval indexed by the corresponding trees with a diaphragm.

As an exercise, we recommend describing the map $\varpi : zT_m^2 \to \mathsf{F}(\Xi)\binom{2}{m}$ in terms of trees with a diaphragm. One may generalize the above description of the poset $zT_m^n$ also to $n > 2$. In this case, the tree $U$ corresponding to $(U, D, z) \in zT_m^n$ may have *several* diaphragms, depending on the relative positions of the vertices of $D$. The combinatorics of this kind of description becomes, however, unmanageably complicated with growing $n$.

3.2. **Relation to the multiplihedron.** Multiplihedra appeared in the study of homotopy multiplicative maps between $A_\infty$-spaces [10]. The $m$-th multiplihedron $J_m$ is a convex polytope of dimension $m - 1$ whose vertices correspond to ways of bracketings $m$ variables and

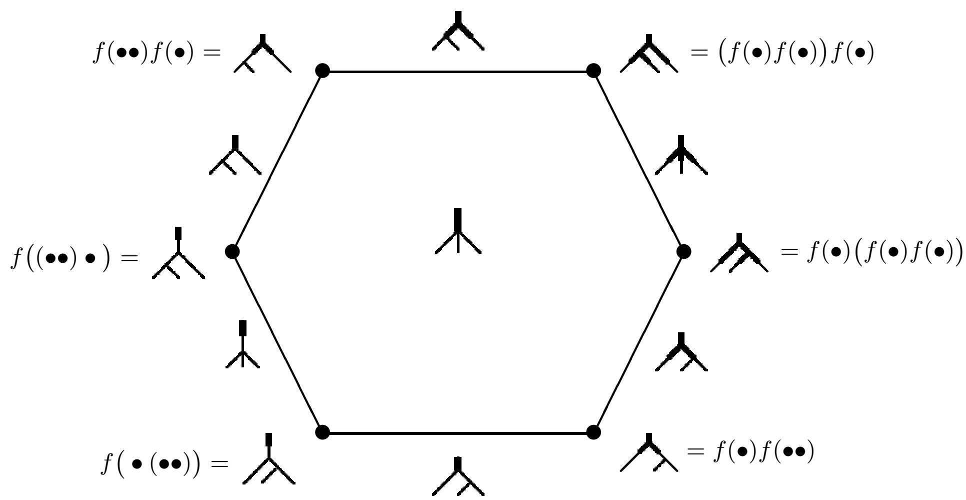



FIGURE 13. The faces of the multiplihedron $J_3$ indexed by the set $pT_3$ of painted 3-trees. The labels of vertices in terms of bracketings of 3 variables and an operation $f$ are also shown.

applying an operation. As explained in [1], the faces of $J_m$ are indexed by *painted $m$-trees* which are, by definition, directed (rooted) planar trees with $m$ leaves, two types of edges – black and white – and vertices of the following two types:

(i) vertices with at least two inputs whose all adjacent edges are of the same color, or
(ii) vertices whose all inputs are white and whose output is black.

The set $pT_m$ of all painted $m$-trees has a partial order $<$ induced by contracting the edges. The poset $\mathcal{J}_m := (pT_m, <)$ is then the poset of faces of the $m$-th multiplihedron $J_m$. We believe that Figure 13 makes the above definitions clear.

**Proposition D.** *The face poset $\mathcal{K}^2_m$ of the biassociahedron $K^2_m$ is isomorphic to the face poset $\mathcal{J}_m$ of the multiplihedron $J_m$, for each $m \geq 2$.*

*Proof.* By Proposition 3.1.1, it suffices to prove that the posets $(dT_m, <)$ and $(pT_m, <)$ are isomorphic. It is very simple. Having a tree $U$ with a diaphragm, we paint everything that lies above[13] the diaphragm black, and everything below white. If the diaphragm intersects an edge of $U$, we introduce at the intersection a new vertex of type (ii) with one input edge. The result will obviously be a painted tree belonging to $pT_m$. The isomorphism we have thus described clearly preserves the partial orders. □

The correspondence of Proposition D is, for $m = 3$, illustrated by the two rightmost columns of the table in Figure 6.

**Remark.** S. Forcey in [1] constructed an explicit realization of the poset $\mathcal{J}_m$ by the face poset of a convex polyhedron. This, combined with Proposition D proves, independently of [8], that $\mathcal{K}^2_m$ is the face poset of a convex polyhedron, too.

[13] We keep our convention that all edges are oriented to point upwards.

## Appendix A. Calculus of fractions

A PROP in the monoidal category of sets is a sequence of sets $\mathsf{P} = \left\{\mathsf{P}\binom{n}{m}\right\}_{m,n\geq 1}$ with compatible left $\Sigma_m$- right $\Sigma_n$-actions and two types of equivariant compositions, vertical:

$$\circ : \mathsf{P}\binom{n}{u} \times_{\Sigma_u} \mathsf{P}\binom{u}{m} \to \mathsf{P}\binom{n}{m},\ m,n,u \geq 1,$$

and horizontal:

$$\boxtimes : \mathsf{P}\binom{n_1}{m_1} \times \mathsf{P}\binom{n_2}{m_2} \to \mathsf{P}\binom{n_1+n_2}{m_1+m_2},\ m_1,m_2,n_1,n_2 \geq 1,$$

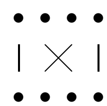
together with an identity $e \in \mathsf{P}\binom{1}{1}$, satisfying appropriate axioms [2, 6]. One can imagine elements of $\mathsf{P}\binom{n}{m}$ as 'abstract' operations with $m$ inputs and $n$ outputs. We say that $X$ has *biarity* $\binom{n}{m}$ if $X \in \mathsf{P}\binom{n}{m}$.

Calculus of fractions was devised in [3, Section 4] to handle particular types of compositions in PROPs. For $k,l \geq 1$ and $1 \leq i \leq kl$, let $\sigma\binom{l}{k} \in \Sigma_{kl}$ be the permutation given by

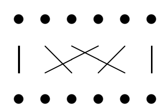
$$\sigma\binom{l}{k}(i) := l(i-1-(s-1)k)+s,$$

where $s$ is such that $(s-1)k < i \leq sk$.

**Example.** We have

$$\sigma\binom{2}{2} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} =$$  $$\in \Sigma_4.$$

Similarly

$$\sigma\binom{3}{2} := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix} =$$  $$\in \Sigma_6.$$

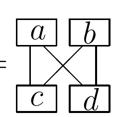
**Definition A.0.1.** Let $\mathsf{P}$ be an arbitrary PROP. Let $k,l \geq 1$, $a_1,\ldots,a_l,b_1,\ldots,b_k \geq 1$, $A_1,\ldots,A_l \in \mathsf{P}\binom{k}{a_j}$ and $B_1,\ldots,B_k \in \mathsf{P}\binom{b_i}{l}$. Then define the *fraction*

$$\frac{B_1 \cdots B_k}{A_1 \cdots A_l} := (B_1 \boxtimes \cdots \boxtimes B_k) \circ \sigma\binom{l}{k} \circ (A_1 \boxtimes \cdots \boxtimes A_l) \in \mathsf{P}\binom{b_1+\cdots+b_k}{a_1+\cdots+a_l}.$$

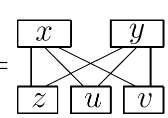
**Example.** If $k=1$ or $l=1$, the fractions give the 'operadic' and 'cooperadic' compositions:

$$\frac{B_1}{A_1 \cdots A_l} = B_1 \circ (A_1 \boxtimes \cdots \boxtimes A_l) \ \text{ and } \ \frac{B_1 \cdots B_k}{A_1} = (B_1 \boxtimes \cdots \boxtimes B_k) \circ A_1.$$

**Example.** For $\boxed{a}, \boxed{b} \in \mathsf{P}\binom{*}{2}$ and $\boxed{c}, \boxed{d} \in \mathsf{P}\binom{2}{*}$,

$$\frac{\boxed{a}\ \boxed{b}}{\boxed{c}\ \boxed{d}} = (\boxed{a} \boxtimes \boxed{b}) \circ \sigma\binom{2}{2} \circ (\boxed{c} \boxtimes \boxed{d}) =$$  .

Similarly, for $\boxed{x}, \boxed{y} \in \mathsf{P}\binom{*}{3}$ and $\boxed{z}, \boxed{u}, \boxed{v} \in \mathsf{P}\binom{2}{*}$,

$$\frac{\boxed{x}\ \ \boxed{y}}{\boxed{z}\ \boxed{u}\ \boxed{v}} = (\boxed{x} \boxtimes \boxed{y}) \circ \sigma\binom{3}{2} \circ (\boxed{z} \boxtimes \boxed{u} \boxtimes \boxed{v}) =$$  .

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