

Quantum hypermultiplet moduli spaces in $\mathcal{N} = 2$ string vacua: a review

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ABSTRACT. The hypermultiplet moduli space \mathcal{M}_H in type II string theories compactified on a Calabi-Yau threefold X is largely constrained by supersymmetry (which demands quaternion-Kählerity), S-duality (which requires an isometric action of $SL(2, \mathbb{Z})$) and regularity. Mathematically, \mathcal{M}_H ought to encode all generalized Donaldson-Thomas invariants on X consistently with wall-crossing, modularity and homological mirror symmetry. We review recent progress towards computing the exact metric on \mathcal{M}_H , or rather the exact complex contact structure on its twistor space.

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1. Introduction

String vacua with $N = 2$ supersymmetry in four dimensions offer a unique opportunity to investigate non-perturbative aspects of the low energy effective action and of the spectrum of black hole bound states. Unlike in vacua with higher supersymmetry, the two-derivative effective action in general receives non-trivial quantum corrections, while degeneracies of BPS black holes depend non trivially on the value of the moduli at spatial infinity. Both issues are in fact related, since BPS black holes in 4 dimensions yield BPS instantons upon reduction on a circle, and the resulting instanton corrections to the three-dimensional effective action can sometimes (after T-duality along the circle) lift back to 4 dimensions.

For ungauged $\mathcal{N} = 2$ vacua, the complete two-derivative effective action is encoded in the Riemannian metric on the moduli space, which famously factorizes as the product $\mathcal{M}_V \times \mathcal{M}_H$ of the vector multiplet (VM) and hypermultiplet (HM) moduli spaces, respectively [1, 2]. A complete understanding of the former was achieved in the 90s, leading to deep connections with algebraic geometry, most notably the discovery of classical mirror symmetry. By contrast, our understanding of the latter has long remained rudimentary, mainly due to the difficulty of parametrizing quaternion-Kähler (QK) metrics on \mathcal{M}_H , as required by supersymmetry. The situation has considerably improved in recent years, as twistorial techniques [3, 4, 5, 6, 7, 8, 9] were used to reformulate this problem analytically, in terms of the complex contact structure on the twistor space \mathcal{Z} (a \mathbb{P}^1 -bundle over \mathcal{M}_H), and a suitable set of complex Darboux coordinates on \mathcal{Z} .

The purpose of this contribution is to give a survey of recent progress towards determining the exact hypermultiplet moduli space metric (see [10] for a review with different emphasis). We focus on type II strings compactified on Calabi-Yau (CY) threefolds, although other dual formulations of the same vacua (see [11, 12, 13] for recent progress on $K3 \times T^2$ heterotic vacua) may eventually be useful for achieving the stated goal. In §2, we summarize the structure of the perturbative hypermultiplet moduli space in type IIA and type IIB vacua, emphasizing its twistorial description. In §3, we discuss instanton corrections from Euclidean D-branes wrapped on supersymmetric cycles inside the CY threefold X , and provide a twistorial construction of these corrections parametrized by the Donaldson-Thomas invariants of X . We explain the consistency of this construction with wall-crossing using the so called QK/HK correspondence which allows to reformulate the resulting corrections to the QK metric on \mathcal{M}_H in terms of corrections to an auxiliary, or “dual”, hyperkähler (HK) space \mathcal{M}'_H . In §4, we discuss the implications of the modular symmetry of type IIB strings (known as S-duality) for these D-instanton corrections, with particular emphasis on D3-brane instantons (corresponding to divisors in X). We show that in the large volume, one-instanton approximation, S-duality holds thanks to special modular properties of the DT invariants for divisors, and of the indefinite theta series which sum up D1-D(-1) instanton effects at fixed D3-brane charge. In §5, we use the same duality to obtain Neveu-Schwarz (NS) fivebrane instantons from D5-instantons, and relate these contributions to the topological string amplitude on X . We also discuss some conjectural relations between NS5-branes and quantum integrable systems.

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2. Perturbative moduli space

In this section we discuss the one-loop corrected hypermultiplet moduli space in type IIA and type IIB string theories compactified on CY threefolds X and \hat{X} , respectively. Higher loop corrections are expected to vanish after suitable field redefinitions [14, 15, 16]. If (X, \hat{X}) is a mirror pair, then the moduli spaces are isometric as a consequence of classical mirror symmetry [17].

2.1. Type IIA.

2.1.1. *Topology.* Type IIA string theory associates to each compact CY threefold X a real $4(h_{2,1}(X) + 1)$ -dimensional quaternion-Kähler space $\mathcal{M}_H = \mathcal{M}_H(X)$. Topologically, \mathcal{M}_H is a \mathbb{C}^\times -bundle

$$(2.1) \quad \mathbb{C}^\times \longrightarrow \mathcal{M}_H(X) \longrightarrow \mathcal{J}_W(X) \longrightarrow \mathcal{M}_C(X)$$

over the Weil intermediate Jacobian $\mathcal{J}_W(X)$. The latter is a torus bundle over the complex structure moduli space $\mathcal{M}_C(X)$, with generic fiber $\mathcal{T} = H^3(X, \mathbb{R})/H^3(X, \mathbb{Z})$, endowed with the Weil complex structure where $H^{3,0} \oplus H^{1,2}$ generate the holomorphic tangent space.

To see how this arises from physics, consider the \mathbb{C}^\times -bundle $\mathcal{L}_X \rightarrow \mathcal{M}_C(X)$ with fibre the space of nowhere vanishing holomorphic 3-forms $\Omega^{3,0}$ on X . Fixing a symplectic basis (A^Λ, B_Λ) , $\Lambda = 0, \dots, h_{2,1}(X)$ of $\Gamma = H_3(X, \mathbb{Z})$, the period integrals

$$(2.2) \quad X^\Lambda = \int_{A^\Lambda} \Omega^{3,0}, \quad F_\Lambda = \int_{B_\Lambda} \Omega^{3,0}$$

realize \mathcal{L}_X as a complex Lagrangian cone in $H^3(X, \mathbb{C})$. Locally, the B -periods F_Λ can be expressed in terms of the A -periods X^Λ as derivatives of a holomorphic prepotential $F(X^\Lambda)$ homogeneous of degree 2. The ratios $z^a = X^a/X^0$, $a = 1, \dots, h_{2,1}$, parametrize the moduli space of complex structures $\mathcal{M}_C(X)$, and describe the scalar degrees of freedom in type IIA/ X originating from the metric in 10 dimensions. The periods (X^Λ, F_Λ) are valued in the Hodge bundle \mathcal{L}_X (times a symplectic vector bundle associated to changes of the symplectic basis).

In addition, the periods of the ten-dimensional Ramond-Ramond (RR) three-form C

$$(2.3) \quad \zeta^\Lambda = \int_{A^\Lambda} C, \quad \tilde{\zeta}_\Lambda = \int_{B_\Lambda} C$$

yield scalar moduli valued in $H^3(X, \mathbb{R})$. Invariance under large gauge transformations $C \rightarrow C + H$ with $H \in \Gamma$ imply that $(\zeta^\Lambda, \tilde{\zeta}_\Lambda)$ are periodic with integer periods, hence live in the torus \mathcal{T} . Sometimes we abuse notation and write $\Omega^{3,0} = (X^\Lambda, F_\Lambda)$, $C = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$. Just like $\Omega^{3,0}$, the vector C transforms by a symplectic rotation under monodromies in $\mathcal{M}_C(X)$, implying that \mathcal{T} is non-trivially fibered over \mathcal{M}_X . The total space of this bundle is the intermediate Jacobian $\mathcal{J}_W(X)$.

Finally, the four-dimensional dilaton e^ϕ and the Poincaré dual σ to the B -field in four dimensions provide an additional complex scalar degree of freedom in four dimensions, corresponding to the \mathbb{C}^\times fiber in (2.1). Large gauge transformations of the B -field identify $\sigma \mapsto \sigma + 2\kappa$ with κ

integer (for a suitable normalization), while the afore-mentioned large gauge transformations also act on the axion σ by a shift [18, 19],

$$(2.4) \quad (C, \sigma) \mapsto (C + H, \sigma + 2\kappa + \langle C, H \rangle + 2c(H)).$$

Here $c(H)$ provides the quadratic refinement $\lambda(H) \equiv (-1)^{2c(H)}$ of the intersection form $\langle \cdot, \cdot \rangle$ on Γ satisfying

$$(2.5) \quad \lambda(H + H') = (-1)^{\langle H, H' \rangle} \lambda(H) \lambda(H').$$

Given a choice of symplectic basis of $H^3(X, \mathbb{Z})$, any quadratic refinement can be parametrized as

$$(2.6) \quad \lambda(H) = (-1)^{2c(H)} = e^{-i\pi m_\Lambda n^\Lambda + 2\pi i(m_\Lambda \theta^\Lambda - n^\Lambda \phi_\Lambda)},$$

where $H = (n^\Lambda, m_\Lambda)$ are the components of H along (A^Λ, B_Λ) and $\Theta \equiv (\theta^\Lambda, \phi_\Lambda)$ are a choice of characteristics in \mathcal{T} . Note that (2.6) defines $c(H)$ only modulo integers, but the corresponding ambiguity in (2.4) can be absorbed in κ . The extra shift $2c(H)$ of σ in (2.4) is needed to ensure the closure of the group action. Altogether, the large gauge transformations of the B and C fields define a discrete Heisenberg group action $\mathbb{H}(\mathbb{Z})$ which will play a central role in the discussion of NS5-brane instanton effects in §5. Eq. (2.4) completely specifies the restriction of the \mathbb{C}^\times -bundle (or rather, its unit circle bundle \mathcal{C}_σ , with S^1 -fiber parametrized by the axion σ) in (2.1) to the torus \mathcal{T} . The topology of the bundle over the full intermediate Jacobian $\mathcal{J}_W(X)$ will be discussed in the next paragraph after discussing the one-loop corrected metric and in more detail in §5.1.

2.1.2. Perturbative metric. At tree-level, the metric on \mathcal{M}_H belongs to the class of ‘semi-flat’ QK metrics discovered in [20, 21]. In particular, its restriction to $\mathcal{M}_C(X)$ is the special Kähler metric $g_{\mathcal{M}_C(X)}$ deduced from the prepotential $F(X^\Lambda)$, with Kähler potential [22]

$$(2.7) \quad \mathcal{K} = -\log \left[i \int_X \Omega^{3,0} \wedge \overline{\Omega^{3,0}} \right] = -\log [i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)],$$

while along $\mathcal{T} \times S^1$ it has continuous isometries. As shown in [14, 23, 15, 24], the one-loop correction takes the metric outside the above class, still preserving flatness along $\mathcal{T} \times S^1$. The resulting metric on \mathcal{M}_H can be written as

$$(2.8) \quad g_{\text{pert}} = \frac{r + 2c}{r^2(r + c)} dr^2 + \frac{4(r + c)}{r} g_{\mathcal{M}_C(X)} + \frac{1}{r} g_{\mathcal{T}}(c) + \frac{r + c}{16r^2(r + 2c)} (d\sigma + \mathcal{A}(c))^2,$$

where $r = e^\phi$ and the parameter $c = -\chi(X)/(192\pi)$ encodes the one-loop correction, governed solely by the Euler number of X . Here $g_{\mathcal{T}}(c)$ denotes a deformation of the standard Weil metric on the torus \mathcal{T} which can be found in [24]. Most importantly, the connection $\mathcal{A}(c)$ on the circle bundle \mathcal{C}_σ is given by

$$(2.9) \quad \mathcal{A}(c) = \tilde{\zeta}_\Lambda d\zeta^\Lambda - \zeta^\Lambda d\tilde{\zeta}_\Lambda + 8c\mathcal{A}_K, \quad \mathcal{A}_K = \frac{i}{2}(\partial_{z^a}\mathcal{K}dz^a - \partial_{\bar{z}^{\bar{a}}}\mathcal{K}d\bar{z}^{\bar{a}}).$$

Here \mathcal{A}_K is the Kähler connection on the Hodge bundle \mathcal{L}_X with Kähler potential (2.7). The second term in (2.9) follows by reducing the topological coupling $B \wedge I_8$ in the ten-dimensional type IIA action and dualizing B into σ (see [19] for details). The tree-level metric is recovered by setting $c = 0$.

The connection (2.9) implies that the circle bundle \mathcal{C}_σ has non-trivial curvature both along the torus \mathcal{T} , in accordance with (2.4), but also along the base $\mathcal{M}_C(X)$ of the intermediate Jacobian $\mathcal{J}_W(X)$; the first Chern class is given by

$$(2.10) \quad c_1(\mathcal{C}_\sigma) = \omega_{\mathcal{T}} + \frac{\chi(X)}{24} \omega_C, \quad \omega_{\mathcal{T}} \equiv d\tilde{\zeta}_\Lambda \wedge d\zeta^\Lambda, \quad \omega_C \equiv -\frac{1}{2\pi} d\mathcal{A}_K.$$

We return to the topology of the axion circle bundle in relation to NS5-instantons in §5. For now, notice that putative higher loop corrections would in general induce corrections to $c_1(\mathcal{C}_\sigma)$ suppressed by inverse powers of r , contradicting the requirement that $c_1(\mathcal{C}_\sigma) \in H^2(\mathcal{M}_H, \mathbb{Z})$. Note that for $\chi(X) > 0$, the metric (2.8) has a curvature singularity at $r = -2c$ (while $r = 0$, $r = -c$ are coordinate singularities). This singularity is expected to be resolved once the full set of non-perturbative corrections is included [25].

2.1.3. Twistor space description. The most convenient way of describing a QK manifold \mathcal{M} is via its twistor space \mathcal{Z} [3]. Recall that a quaternion-Kähler manifold of real dimension $4n$ has holonomy group contained in $USp(n) \times SU(2) \subset SO(4n)$. In particular, it has a triplet of almost complex structures \vec{J} (defined locally up to $SU(2)$ rotations) satisfying the quaternion algebra, corresponding two-forms \vec{w} and a globally defined closed 4-form $\vec{w} \wedge \vec{w}$. The J_i 's are not integrable unless the scalar curvature of \mathcal{M} vanishes, in which case \mathcal{M} is hyperkähler. Nevertheless, it is possible to encode the geometry of \mathcal{M} complex analytically, by passing to its twistor space \mathcal{Z} , the total space of a canonical \mathbb{P}^1 -bundle over \mathcal{M} . \mathcal{Z} carries a canonical complex contact structure, given by the kernel of the $\mathcal{O}(2)$ -twisted, $(1,0)$ -form

$$(2.11) \quad Dt = dt + p_+ - ip_3 t + p_- t^2,$$

where t is a stereographic coordinate on \mathbb{P}^1 and $(p_\pm = -\frac{1}{2}(p_1 \mp ip_2), p_3)$ denotes the $SU(2)$ -part of the Levi-Civita connection on \mathcal{M} .

Locally on an open patch $\mathcal{U}_i \subset \mathcal{Z}$ there exists a function $\Phi_{[i]}$, the ‘contact potential’, which is holomorphic along the twistor lines (i.e. the fibers of $\mathcal{Z} \rightarrow \mathcal{M}$) and such that the product

$$(2.12) \quad \mathcal{X}^{[i]} = -4i e^{\Phi_{[i]}} Dt/t$$

is a holomorphic (i.e. $\bar{\partial}$ -closed) one-form. The nowhere vanishing holomorphic top-form $\mathcal{X} \wedge (d\mathcal{X})^n$ defines the complex contact structure on \mathcal{Z} . Locally, by a complex-contact analogue of the Darboux theorem, one can always choose complex coordinates $(\xi_{[i]}^\Lambda, \tilde{\xi}_\Lambda^{[i]}, \alpha^{[i]})$ in \mathcal{U}_i such that the contact one-form (2.12) takes the canonical form [26, 27]

$$(2.13) \quad \mathcal{X}^{[i]} = d\alpha^{[i]} + \xi_{[i]}^\Lambda d\tilde{\xi}_\Lambda^{[i]}.$$

In what follows it will often be convenient to combine ξ^Λ and $\tilde{\xi}_\Lambda$ into a symplectic vector $\Xi = (\xi^\Lambda, \tilde{\xi}_\Lambda)$, and to define a variant $\tilde{\alpha} = -2\alpha - \tilde{\xi}_\Lambda \xi^\Lambda$ of the coordinate α such that

$$(2.14) \quad \mathcal{X}^{[i]} = -\frac{1}{2} \left(d\tilde{\alpha}^{[i]} + \tilde{\xi}_\Lambda^{[i]} d\xi_{[i]}^\Lambda - \xi_{[i]}^\Lambda d\tilde{\xi}_\Lambda^{[i]} \right) = -\frac{1}{2} \left(d\tilde{\alpha}^{[i]} + \langle \Xi^{[i]}, d\Xi^{[i]} \rangle \right).$$

The global complex contact structure on \mathcal{Z} is then encoded into the set of complex contact transformations between overlapping Darboux coordinate systems on $\mathcal{U}_i \cap \mathcal{U}_j$. A convenient way of specifying these contact transformations is via a set of Hamilton generating functions $H^{[ij]}(\xi, \tilde{\xi}, \alpha) \in H^1(\mathcal{Z}, \mathcal{O}(2))$, as explained in detail in [27, 16]. The QK metric can be reconstructed by (i) parametrizing the twistor lines, i.e. expressing the complex Darboux coordinates (Ξ, α) in terms of the local coordinates (t, x^μ) on $\mathbb{P}^1 \times \mathcal{M}$; (ii) evaluating the contact one-form (2.13) and matching the result with (2.11) so as to extract the $SU(2)$ connection \vec{p} ; (iii) computing the quaternionic 2-forms \vec{w} via $d\vec{p} + \frac{1}{2}\vec{p} \wedge \vec{p} = \frac{\nu}{2}\vec{w}$ (ν is the constant curvature of \mathcal{M}_H); (iv) constructing the space of $(1,0)$ -forms with respect to J_3 , by expanding the differentials $(d\Xi, d\alpha)$ around $t = 0$, and finally, (v) contracting the Kähler form w_3 with the complex structure J_3 .

In this framework, the perturbative metric (2.8) is captured by the following Darboux coordinates in the patch $\mathcal{U}_0 = \mathbb{P}^1 \setminus \{0, \infty\}$ [27] (building on earlier work [28, 26, 24]):

$$(2.15) \quad \begin{aligned} \xi^\Lambda &= \zeta^\Lambda + 2\sqrt{r+c}e^{\mathcal{K}/2}(t^{-1}X^\Lambda - t\bar{X}^\Lambda), \\ \tilde{\xi}_\Lambda &= \tilde{\zeta}_\Lambda + 2\sqrt{r+c}e^{\mathcal{K}/2}(t^{-1}F_\Lambda - t\bar{F}_\Lambda), \\ \tilde{\alpha} &= \sigma + 2\sqrt{r+c}e^{\mathcal{K}/2}(t^{-1}W - t\bar{W}) - 8c \log t, \end{aligned}$$

where $W \equiv F_\Lambda \zeta^\Lambda - X^\Lambda \tilde{\zeta}_\Lambda$, whereas the contact potential coincides with the dilaton, $\Phi = \phi$. The last term in the expression for $\tilde{\alpha}$ is the sole effect of the one-loop correction in this framework (except for a field redefinition $r \rightarrow r+c$). Under a holomorphic rescaling $\Omega^{3,0} \rightarrow e^f \Omega^{3,0}$, the Kähler potential \mathcal{K} and coordinates t, σ vary according to $\mathcal{K} \rightarrow \mathcal{K} - f - \bar{f}$, $t \mapsto e^{i \operatorname{Im} f} t$, $\sigma \mapsto \sigma - 8c \operatorname{Im} f$, leaving (2.15) invariant.

For our purposes, it is important to note two key properties of the twistorial approach. First, quaternionic isometries of \mathcal{M} (i.e. preserving the 4-form $\vec{w} \wedge \vec{w}$) are classified by the Čech cohomology group $H^0(\mathcal{Z}, \mathcal{O}(2))$ via the moment map construction, and therefore lift to holomorphic actions on \mathcal{Z} [29]. In particular, the action of the Heisenberg group (2.4) on \mathcal{M}_H lifts to a holomorphic action on the Darboux coordinates (2.15) as

$$(2.16) \quad (\Xi, \tilde{\alpha}) \mapsto (\Xi + H, \tilde{\alpha} + 2\kappa + \langle \Xi, H \rangle + 2c(H)).$$

The second property is that linear deformations of a QK space \mathcal{M} are classified by sections of $H^1(\mathcal{Z}, \mathcal{O}(2))$ [30].

2.2. Type IIB. We now turn to the perturbative HM moduli space $\hat{\mathcal{M}}_H(\hat{X})$ in type IIB string theory compactified on a CY threefold \hat{X} . Mirror symmetry requires that it should be isometric to the previously discussed type IIA HM moduli space $\mathcal{M}_H(X)$ whenever (X, \hat{X}) form a dual pair. The two spaces however come with different natural coordinates, and it is important to determine the ‘mirror map’ between the two sides.

On the type IIB side, the HM moduli space has a similar fibration structure as in (2.1),

$$(2.17) \quad \mathbb{C}^\times \longrightarrow \hat{\mathcal{M}}_H(\hat{X}) \longrightarrow \mathcal{J}_K(\hat{X}) \longrightarrow \mathcal{M}_K(\hat{X}),$$

where the ‘even Jacobian’ $\mathcal{J}_K(\hat{X})$ is a torus bundle over $\mathcal{M}_K(\hat{X})$, the moduli space of complexified Kähler structures on \hat{X} , with fiber $\hat{\mathcal{T}} = H^{\text{even}}(\hat{X}, \mathbb{R})/\hat{\Gamma}$ where $\hat{\Gamma}$ is a lattice which will be specified below. Physically, the \mathbb{C}^\times -fiber is parametrized by the type IIB dilaton $\tau_2 = 1/g_s$ and the NS-axion ψ , while $\hat{\mathcal{T}}$ corresponds to the periods of the ten-dimensional RR form $C^{\text{even}} = C^{(0)} + C^{(2)} + C^{(4)} + C^{(6)} = H^{\text{even}}(\hat{X}, \mathbb{R})$. A convenient set of coordinates is given by [31, 16]

$$(2.18) \quad \begin{aligned} c^0 &= C^{(0)}, & c^a &= \int_{\gamma^a} C^{(2)}, & \tilde{c}_a &= - \int_{\gamma_a} \left(C^{(4)} - \frac{1}{2} B \wedge C^{(2)} \right), \\ \tilde{c}_0 &= - \int_{\hat{X}} \left(C^{(6)} - B \wedge C^{(4)} + \frac{1}{3} B \wedge B \wedge C^{(2)} \right), & b^a + it^a &= \int_{\gamma^a} (B + iJ). \end{aligned}$$

where $\gamma_a, a = 1, \dots, h^{1,1}$, is a basis of 4-cycles in $H_4(\hat{X}, \mathbb{R})$, and γ^a is the dual basis of 2-cycles in $H_2(\hat{X}, \mathbb{R})$.

By classical mirror symmetry, $\mathcal{M}_K(\hat{X}) = \mathcal{M}_C(X)$, for a suitable map between the complex structure moduli $z^a = X^a/X^0$ and the Kähler moduli (b^a, t^a) . In the large volume limit on the type IIB side, the prepotential on $\mathcal{M}_K(\hat{X})$ is given by

$$(2.19) \quad F^{\text{cl}} = -\kappa_{abc} \frac{X^a X^b X^c}{6X^0} + \frac{1}{2} A_{\Lambda\Sigma} X^\Lambda X^\Sigma,$$

where κ_{abc} is the triple intersection product on $H^2(\hat{X}, \mathbb{Z})$ and $A_{\Lambda\Sigma}$ is a real symmetric matrix which does not affect the Kähler potential, but which is important for consistency with charge quantization [19]. In this limit, the mirror map reduces to $z^a = b^a + it^a$ for a suitable choice of symplectic basis on the type IIA side adapted to the point of maximal unipotent monodromy. The classical metric on $\hat{\mathcal{M}}_H(\hat{X})$ then takes the semi-flat form (2.8) with $c = 0$ and the prepotential (2.19), provided $\mathcal{M}_K(\hat{X})$ is identified with $\mathcal{M}_C(X)$ and the natural coordinates on the type IIB side are related to the coordinates (2.3) on the type IIA side by [32]:

$$(2.20) \quad \begin{aligned} r &= \frac{\tau_2^2}{2} \mathcal{V}, & X^a/X^0 &= z^a = b^a + it^a, & \zeta^0 &= \tau_1, & \zeta^a &= -(c^a - \tau_1 b^a), \\ \tilde{\zeta}'_a &= \tilde{c}_a + \frac{1}{2} \kappa_{abc} b^b (c^c - \tau_1 b^c), & \tilde{\zeta}'_0 &= \tilde{c}_0 - \frac{1}{6} \kappa_{abc} b^a b^b (c^c - \tau_1 b^c), \\ \sigma &= -2(\psi + \frac{1}{2} \tau_1 \tilde{c}_0) + \tilde{c}_a (c^a - \tau_1 b^a) - \frac{1}{6} \kappa_{abc} b^a c^b (c^c - \tau_1 b^c), \end{aligned}$$

where $\mathcal{V} = \frac{1}{6} \kappa_{abc} t^a t^b t^c$ denotes the volume of \hat{X} and the prime denotes fields obtained by the symplectic transformation removing the quadratic term in (2.19), namely,

$$(2.21) \quad \tilde{\zeta}'_\Lambda = \tilde{\zeta}_\Lambda - A_{\Lambda\Sigma} \zeta^\Sigma.$$

By mirror symmetry, the lattice¹ $\hat{\Gamma} \subset H^{\text{even}}(\hat{X}, \mathbb{R})$ must be (indeed, is) the image of the lattice $\Gamma \subset H^3(X, \mathbb{R})$ under the map (2.20) between the type IIA RR fields $\zeta^\Lambda, \tilde{\zeta}_\Lambda$ and the type IIB RR fields $c^0, c^a, \tilde{c}_a, \tilde{c}_0$. Beyond the large volume limit, the prepotential (2.19) and mirror map (2.20) acquire worldsheet instanton corrections, governed by the genus zero Gopakumar-Vafa invariants $n_{q_a}^{(0)}$ of \hat{X} [33, 16, 34, 35]. Together with the one-loop correction proportional to $\chi(\hat{X})$, this produces the same metric (2.8) as on the type IIA side.

Besides establishing mirror symmetry, the mirror map (2.20) has another virtue: it exposes the invariance of the HM moduli space $\mathcal{M}_H(X) = \hat{\mathcal{M}}_H(\hat{X})$, in the large volume/weak coupling limit where it holds, under the action of $SL(2, \mathbb{R})$, corresponding to the continuous S-duality symmetry of ten-dimensional type IIB supergravity. Of course, this continuous symmetry is broken by quantum corrections, but there is overwhelming evidence that a discrete $SL(2, \mathbb{Z})$ subgroup remains unbroken, providing a strong constraint on possible non-perturbative effects [36]. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ is simplest in type IIB variables [14, 32]:

$$(2.22) \quad \begin{aligned} \tau &\mapsto \frac{a\tau + b}{c\tau + d}, & t^a &\mapsto t^a |c\tau + d|, & \tilde{c}_a &\mapsto \tilde{c}_a - c_{2,a} \varepsilon(g), \\ \begin{pmatrix} c^a \\ b^a \end{pmatrix} &\mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c^a \\ b^a \end{pmatrix}, & \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix} &\mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{c}_0 \\ \psi \end{pmatrix}. \end{aligned}$$

In this action, we have included a shift of the RR coordinate \tilde{c}_a , overlooked in early studies but crucial for maintaining S-duality invariance under D3 and D5-NS5 instanton corrections [19, 37], as we shall see in §4 and §5. Here, $c_{2,a}$ and $\varepsilon(g) \in \mathbb{Q}$ are defined by

$$(2.23) \quad c_{2,a} \equiv \int_{\gamma_a} c_2(\hat{X}), \quad \eta \left(\frac{a\tau + b}{c\tau + d} \right) / \eta(\tau) = \mathbf{E}(\varepsilon(g)) (c\tau + d)^{-1/2},$$

where $\eta(\tau)$ is the Dedekind eta-function and $\mathbf{E}(x) = \exp(2\pi i x)$. We stress that as defined so far, the metric on $\hat{\mathcal{M}}_H(\hat{X})$ is only invariant under $SL(2, \mathbb{Z})$ in the strict infinite volume, zero string coupling limit, where it is actually enhanced to $SL(2, \mathbb{R})$. Both worldsheet instanton corrections to the classical prepotential (2.19) and the one-loop correction break this symmetry, and it is

¹More precisely, Γ and $\hat{\Gamma}$ are local system of lattices over $\mathcal{M}_C(X)$ and $\mathcal{M}_K(\hat{X})$, due to monodromies.

necessary to include non-perturbative effects in order to recover it. The holomorphic action of S-duality in twistor space will be described in §4.1.

3. D-instantons, wall-crossing and the QK/HK correspondence

The perturbative metric (2.8), while being valid to all orders at small string coupling g_s , is expected to receive non-perturbative corrections of order e^{-1/g_s} , due to Euclidean D-branes wrapping supersymmetric cycles in X (or \hat{X}). In this section we discuss some general aspects of D-instantons and their relation with Donaldson-Thomas (DT) invariants, describe how they modify the twistorial description of §2.1.3, and how they result in a smooth quantum corrected metric, despite discontinuities of the DT invariants across certain walls in complex structure (or Kähler) moduli space. For this purpose, a new duality between quaternion-Kähler and hyperkähler manifolds will turn out to be useful.

3.1. D-instantons, Donaldson-Thomas invariants and wall-crossing.

3.1.1. *Derived category of D-instantons.* On the type IIA side, the leading corrections to the perturbative hypermultiplet metric g_{pert} described in §2.1.2 come from Euclidean D2-branes wrapping Lagrangian 3-cycles (sLags) in X endowed with a flat $U(1)$ connection. On the type IIB side, they correspond to superpositions of D(-1)-D1-D3-D5 instantons wrapping complex even-dimensional cycles, or more generally coherent sheaves (holomorphic vector bundles supported on (singular) submanifolds). Most generally, D-instantons are objects in a bounded derived Fukaya category $\text{D}^b\text{Fuk}(X)$ on the type IIA side, or the derived category of coherent sheaves $\text{D}^b\text{Coh}(\hat{X})$ on the type IIB side [38, 39]. Each of them is graded by the Grothendieck group, an extension of the lattice $\Gamma = H_3(X, \mathbb{Z})$ or $\hat{\Gamma} \subset H_{\text{even}}(\hat{X}, \mathbb{Z})$ of electromagnetic charges. The fact that the same categories also govern the spectrum of BPS states in type IIB on X and type IIA on \hat{X} , respectively, can be understood by compactifying on a circle down to 3 space-time dimensions: T-duality along the circle exchanges 4-dimensional D-instantons with 4-dimensional BPS states whose worldline winds around the circle [16]. Kontsevich's homological mirror symmetry conjecture [40], the mathematical counterpart of non-perturbative mirror symmetry [41], asserts that these two categories are isomorphic when (X, \hat{X}) is a dual pair, in particular $\Gamma \simeq \hat{\Gamma}$.

3.1.2. *Stability and DT invariants.* Among all the objects in the derived category, those which correspond to supersymmetric, elementary D-instantons (or dually, one-particle BPS states) are the semi-stable ones [39]. Stability can be assessed using the central charge Z , a homomorphism $Z : \Gamma \rightarrow \mathbb{C}$ which varies holomorphically over \mathcal{B} ($= \mathcal{M}_C(X)$ or $\mathcal{M}_K(\hat{X})$) and which determines the classical action (or dually, the mass) of the instanton. In type IIA we have

$$(3.1) \quad Z_\gamma(z) = e^{\mathcal{K}/2} \int_\gamma \Omega^{3,0},$$

while in the large-volume limit in type IIB

$$(3.2) \quad Z_\gamma(z) = \int_{\hat{X}} e^{B+iJ} \text{ch}(\mathcal{E}) \sqrt{\text{Td}(\hat{X})},$$

where \mathcal{E} is a coherent sheaf, whose Mukai vector $\text{ch}(\mathcal{E}) \sqrt{\text{Td}(\hat{X})} \in H^{\text{even}}(\hat{X}, \mathbb{Q})$ is identified with the charge vector γ of E . Semi-stability is most easily defined for Abelian categories as follows. An object F with charge γ is called semi-stable if for every subobject $F' \subset F$ with charge γ' , $\varphi(\gamma') \leq \varphi(\gamma)$, where $\varphi(\gamma)$ is the argument of the central charge $Z_\gamma(z)$. F is called stable if the inequality is strict for strict subobjects. This notion can be extended to derived categories, by

considering an Abelian subcategory of the derived category, the “heart of the t -structure” [42, 43]. On the IIA side, semi-stable objects of $D^b\text{Fuk}(X)$ are *special* (or calibrated) Lagrangian cycles L , i.e. such that the phase of $\Omega|_L/dV_L$ is constant, where dV_L is the volume form on L [44]. On the IIB side in the infinite volume limit, semi-stable objects are the semi-stable coherent sheaves in the classical sense of Gieseker stability.

An important property of semi-stable objects is that their space of deformations is finite-dimensional, although it can be singular. The generalized DT invariant $\Omega(\gamma; z)$ is defined as the (weighted) Euler number of this moduli space. It is the mathematical counterpart of the BPS index, which counts BPS black holes or instantons of charge γ . It is a locally constant function of the moduli z (through the central charge Z_γ), away from certain walls of marginal stability described below. It is also monodromy invariant, in the sense that $\Omega(M \cdot \gamma; M \cdot z) = \Omega(\gamma; z)$, where $M \in Sp(m; \mathbb{Z})$ is the symplectic rotation induced by a monodromy along a loop in \mathcal{B} . Homological mirror symmetry implies that the DT invariants $\Omega(\gamma; z)$ associated to $D^b\text{Fuk}(X)$ and $D^b\text{Coh}(\hat{X})$ are the same, provided the charges γ and moduli z are related according to the classical mirror map. As we shall see, this guarantees that the D-instanton corrected HM moduli spaces $\mathcal{M}_H(X)$ and $\hat{\mathcal{M}}_H(\hat{X})$ are isometric.

3.1.3. Wall-crossing. Physically, the jump of the DT invariants $\Omega(\gamma, z)$ across codimension one walls in \mathcal{B} corresponds to the decay of bound states into more elementary stable constituents. For any pair of charge vectors (γ_1, γ_2) , the decay of a D-brane of charge $\gamma = M\gamma_1 + N\gamma_2$ into constituents of charges $M_i\gamma_1 + N_i\gamma_2$ with $\sum(M_i, N_i) = (M, N)$ is energetically possible only if the phases of the central charges align, i.e. $\varphi(\gamma_1) = \varphi(\gamma_2)$, which defines the wall of marginal stability $W(\gamma_1, \gamma_2) \subset \mathcal{B}$. Let $z_\pm \in \mathcal{B}$ denote two points infinitesimally displaced on either side of such a wall. We can always choose the basis γ_1, γ_2 of the two-dimensional lattice $\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$ such that only the first and third quadrants are populated on either side of the wall, $\Omega^\pm(M\gamma_1 + N\gamma_2) = 0$ if $MN \leq 0$ [45].

Several formulae exist in the mathematics and physics literature for how to compute the jump of $\Omega(M\gamma_1 + N\gamma_2; z)$ across the wall $W(\gamma_1, \gamma_2)$ (see, e.g., [46] for a review). The one relevant here is the Kontsevich-Soibelman (KS) formula [47], which has a clear geometric interpretation. To write their formula, one introduces the Lie algebra of (twisted) infinitesimal symplectomorphisms of the complex torus $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}^\times$ generated by vector fields $(e_\gamma)_{\gamma \in \Gamma}$ satisfying

$$(3.3) \quad [e_\gamma, e_{\gamma'}] = (-1)^{\langle \gamma, \gamma' \rangle} \langle \gamma, \gamma' \rangle e_{\gamma + \gamma'}.$$

For any $\gamma \in \Gamma$, $z \in \mathcal{B}$ we also define the group element

$$(3.4) \quad U_\gamma(z) = \exp \left(\Omega(\gamma; z) \sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right).$$

The KS wall-crossing formula then asserts the following equality between oppositely ordered infinite products of symplectomorphisms [47]

$$(3.5) \quad \prod_{\substack{\gamma = m\gamma_1 + n\gamma_2 \\ m \geq 0, n \geq 0}}^{\widehat{\prod}} U_\gamma(z_+) = \prod_{\substack{\gamma = m\gamma_1 + n\gamma_2 \\ m \geq 0, n \geq 0}}^{\widehat{\prod}} U_\gamma(z_-).$$

By projecting this equality on finite dimensional quotients, one can determine $\Delta\Omega = \Omega(\gamma; z_+) - \Omega(\gamma; z_-)$ for any $\gamma = M\gamma_1 + N\gamma_2$. This formula was interpreted physically in the context of $\mathcal{N} = 2$ gauge theory à la Seiberg-Witten in [48], as ensuring the smoothness of the hyperkähler metric on the Coulomb branch of the gauge theory on $\mathbb{R}^3 \times S^1$. From this point of view, the U_γ 's are complex

symplectomorphisms relating different Darboux coordinate systems on the twistor space. Below we will show that an extension of (3.5), where the U_γ 's now are complex contact transformations, also ensures the smoothness of the HM moduli space \mathcal{M}_H in $\mathcal{N} = 2$ string vacua across walls in \mathcal{B} .

3.2. D-instantons in twistor space. Away from the zero-coupling limit, the perturbative hypermultiplet metric (2.8) receives non-perturbative corrections due to D-brane instantons. In the one-instanton approximation these corrections take the schematic form [16]

$$(3.6) \quad g_D \sim \sum_{\gamma \in \Gamma} \lambda_D(\gamma) \bar{\Omega}(\gamma; z) \exp\left(-8\pi\sqrt{r}|Z_\gamma| - 2\pi i \langle \gamma, C \rangle\right),$$

where the exponential is the classical action of the D-instanton, with $\sqrt{r} \sim 1/g_s$. The prefactor in principle originates from integrating the fluctuation determinant around the classical solution over collective coordinates. It is natural to expect that it is proportional to the DT invariant $\Omega(\gamma, z)$ introduced in §3.1.2, however consistency with wall-crossing will dictate the less obvious product of a quadratic refinement $\lambda_D(\gamma)$, analogous to the one in (2.5), with the ‘rational DT invariant’ [47, 49, 50]

$$(3.7) \quad \bar{\Omega}(\gamma; z) = \sum_{d|\gamma} \frac{\Omega(\gamma/d; z)}{d^2}.$$

When γ is a primitive charge vector, $\bar{\Omega}(\gamma; z) = \Omega(\gamma; z)$.

In order to incorporate these corrections to the metric while maintaining its quaternion-Kähler structure, it is best to do this at the level of the twistor space \mathcal{Z} . As explained in [16, 51, 52], in close analogy with the field theory construction in [48], the instanton corrections modify the contact structure on \mathcal{Z} , presented in §2.1.3, by replacing the patch \mathcal{U}_0 around the equator of \mathbb{P}^1 with an infinite set of angular sectors separated by so called ‘‘BPS-rays’’

$$(3.8) \quad \ell_\gamma = \{t \in \mathbb{P}^1 : Z_\gamma(z)t^{-1} \in i\mathbb{R}_-\},$$

where $Z_\gamma(z)$ is the central charge function (3.1). Across ℓ_γ the Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \tilde{\alpha})$ must jump by a complex contact transformation. We postulate that the jump of the holomorphic Fourier modes on the torus $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}^\times$,

$$(3.9) \quad \mathcal{X}_\gamma = \mathbf{E}(-\langle \gamma, \Xi \rangle) = e^{-2\pi i(q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda)},$$

is the standard KS symplectomorphism

$$(3.10) \quad U_\gamma : \mathcal{X}_{\gamma'} \mapsto \mathcal{X}_{\gamma'} (1 - \lambda_D(\gamma) \mathcal{X}_\gamma)^{\Omega(\gamma) \langle \gamma, \gamma' \rangle}.$$

Requiring that the contact one-form (2.13) is preserved determines the discontinuity in the remaining Darboux coordinate $\tilde{\alpha}$. As a result, the full contact transformation is given by

$$(3.11) \quad V_\gamma : (\mathcal{X}_{\gamma'}, \tilde{\alpha}) \mapsto \left(\mathcal{X}_{\gamma'} (1 - \lambda_D(\gamma) \mathcal{X}_\gamma)^{\Omega(\gamma) \langle \gamma, \gamma' \rangle}, \tilde{\alpha} + \frac{\Omega(\gamma)}{2\pi^2} L_{\lambda_D(\gamma)}(\lambda_D(\gamma) \mathcal{X}_\gamma) \right),$$

where $L_\epsilon(x)$ is a variant of the Rogers dilogarithm,

$$(3.12) \quad L_\epsilon(x) \equiv \text{Li}_2(x) + \frac{1}{2} \log(\epsilon^{-1}x) \log(1-x).$$

Requiring further that the Darboux coordinates reduce to the uncorrected ones (2.15) near $t = 0$ and $t = \infty$, one may recast the gluing conditions (3.11) across the BPS rays ℓ_γ as a system

of integral equations for the Fourier modes \mathcal{X}_γ [48, 51],

$$(3.13) \quad \mathcal{X}_\gamma(t) = \mathcal{X}_\gamma^{\text{sf}}(t) \exp \left[\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma') \langle \gamma, \gamma' \rangle \int_{\ell_{\gamma'}} \frac{dt' t + t'}{t' t - t'} \log(1 - \lambda_{\text{D}}(\gamma') \mathcal{X}_{\gamma'}(t')) \right],$$

where $\mathcal{X}_\gamma^{\text{sf}}$ are the ‘semi-flat’ Fourier modes obtained from (2.15)

$$(3.14) \quad \mathcal{X}_\gamma^{\text{sf}}(t) = \exp \left[-2\pi i \left(\langle \gamma, C \rangle + \frac{\tau_2}{2} e^{-\kappa/2} (t^{-1} Z_\gamma - t \bar{Z}_\gamma) \right) \right].$$

The remaining coordinate $\tilde{\alpha}$ and the contact potential, which in this case is globally defined, independent of t and can be identified with the 4-dimensional dilaton $\Phi = \phi$, are then obtained from the solutions of (3.13) via [51, 52]

$$(3.15) \quad \tilde{\alpha} = \sigma + t^{-1} \mathcal{W} - t \bar{\mathcal{W}} + \frac{i\chi(X)}{24\pi} \log t + \frac{i}{8\pi^3} \sum_{\gamma} \Omega(\gamma) \int_{\ell_\gamma} \frac{dt' t + t'}{t' t - t'} L_{\lambda_{\text{D}}(\gamma)}(\lambda_{\text{D}}(\gamma) \mathcal{X}_\gamma),$$

$$(3.16) \quad e^\phi = \frac{\tau_2^2}{16} e^{-\kappa} + \frac{\chi(X)}{192\pi} - \frac{i\tau_2}{64\pi^2} e^{-\kappa/2} \sum_{\gamma} \Omega(\gamma) \int_{\ell_\gamma} \frac{dt}{t} (t^{-1} Z_\gamma - t \bar{Z}_\gamma) \log(1 - \lambda_{\text{D}}(\gamma) \mathcal{X}_\gamma),$$

where the expression for \mathcal{W} can be found in [52]. While (3.13) cannot be solved exactly in general, it can be solved approximately by first plugging in $\mathcal{X}_\gamma = \mathcal{X}_\gamma^{\text{sf}}$ on the r.h.s., computing the l.h.s. and iterating. This generates a formal infinite series of terms labelled by decorated rooted trees, interpreted as multi-instanton corrections [48, 53]. The first term in this expansion, known as the one-instanton approximation, is an integral governed by a saddle point at $t = i\sqrt{Z_\gamma/\bar{Z}_\gamma}$, which produces a result of the expected form (3.6). Ref. [54] argued that the third term on the right hand side in (3.16) corresponds to the contribution of multi-particle states to the Witten index (whereas $\Omega(\gamma)$ counts single-particle BPS states).

It should be stressed that the gluing conditions (3.11) apply only in an open set on \mathcal{Z} away from any wall of marginal stability. Across such a wall, the DT invariants $\Omega(\gamma)$ will jump, but so will the ordering of the BPS rays ℓ_γ . By the same reasoning as in [48], the consistency of the construction and the smoothness of the instanton corrected metric (including all multi-instanton corrections) requires an analogue of the KS wall-crossing formula (3.5), where the U_γ ’s are replaced by V_γ ’s, and equality holds modulo the axion periodicity $\tilde{\alpha} \rightarrow \tilde{\alpha} + 2\kappa$, $\kappa \in \mathbb{Z}$. On the other hand, unlike the HK situation in [48], the twistor space \mathcal{Z} is not a trivial fibration $\mathcal{M}_H \rightarrow \mathcal{Z} \rightarrow \mathbb{P}^1$, but rather an opposite, non-trivial fibration $\mathbb{P}^1 \rightarrow \mathcal{Z} \rightarrow \mathcal{M}_H$, and the above construction does not address the global structure of the twistor space. To circumvent this problem, it will be convenient to relate the twistor space of the D-instanton corrected QK manifold to the twistor space of a ‘dual’ HK manifold, using a general correspondence between QK and HK manifolds with isometries, which we now explain.

3.3. The QK/HK correspondence. Let us first recall the notion of *hyperholomorphic line bundle* on a HK manifold \mathcal{M}' : A line bundle $\mathcal{L} \rightarrow \mathcal{M}'$ is hyperholomorphic if its first Chern class $c_1(\mathcal{L})$ is of type (1, 1) with respect to the whole S^2 of complex structures on \mathcal{M}' [55, 56, 57]. In real dimension 4 this reduces to the notion of self-dual curvature. A hyperholomorphic connection is a one-form λ whose curvature $d\lambda$ satisfies the same condition. It is the curvature of a hyperholomorphic line bundle if and only if $d\lambda \in H^2(\mathcal{M}', \mathbb{Z})$.

3.3.1. *Theorem* [58, 52, 59]. Given a quaternion-Kähler manifold \mathcal{M} with a quaternionic circle action generated by a Killing vector κ , there exists a ‘dual’ hyperkähler manifold \mathcal{M}' of the same dimension, equipped with a circle action generated by the Killing vector κ' , that fixes one of the complex structures, J'_3 , and rotates J'_1, J'_2 . Choosing coordinates θ, θ' adapted to the circle actions on both sides, such that $\kappa = \partial_\theta, \kappa' = \partial_{\theta'}$, the QK metric on \mathcal{M} and HK metric on \mathcal{M}' are related by

$$(3.17) \quad ds_{\mathcal{M}}^2 = \tau (d\theta + \Theta)^2 + ds_{\mathcal{M}/\partial_\theta}^2, \quad ds_{\mathcal{M}'}^2 = (\nu + \rho) (d\theta' + \Theta')^2 + ds_{\mathcal{M}'/\partial_{\theta'}}^2$$

with

$$(3.18) \quad ds_{\mathcal{M}'/\partial_{\theta'}}^2 = \frac{d\rho^2}{\rho} + 4\rho |p_+|^2 - 2\rho ds_{\mathcal{M}/\partial_\theta}^2.$$

Here, ρ is a function on $\mathcal{M}'/\partial_{\theta'}$ defined as the moment map of κ' with respect to J'_3 ; it is identified with the function $1/(2|\vec{\mu}|)$ on $\mathcal{M}/\partial_\theta$, where $\vec{\mu}$ is quaternionic moment map of κ on \mathcal{M} . τ and ν are functions on $\mathcal{M}/\partial_\theta$ and $\mathcal{M}'/\partial_{\theta'}$, respectively, related by $\tau = \frac{\nu + \rho}{2\rho^2\nu}$. Finally, p_+ is the $+$ component of the $SU(2)$ connection on \mathcal{M} , related to the same component of the $SU(2)$ connection on \mathcal{M}' via $p'_+ = \rho e^{i\theta'} p_+$, while the one-forms Θ and Θ' appear in the decomposition of the third component, $p_3 = -\frac{1}{\rho}(d\theta + \Theta) + \Theta', p'_3 = \rho(d\theta' + \Theta') - \Theta$.

Under this correspondence, the HK manifold \mathcal{M}' is naturally endowed with a hyperholomorphic connection²

$$(3.19) \quad \lambda = \nu (d\theta' + \Theta') + \Theta, \quad d\lambda = 2i\partial\bar{\partial}\rho - w'_3 \in H^2(\mathcal{M}', \mathbb{Z}),$$

where w'_3 is the Kähler form on \mathcal{M}' associated to J'_3 and ∂ is the Dolbeault derivative in the same complex structure. Conversely, given a HK manifold with a hyperholomorphic line bundle, there exists a one-parameter family of QK metrics given by the same formula. The one-parameter ambiguity stems from the fact that the moment map ρ of κ' is defined up to an additive constant, which can be absorbed in a shift of ν . This affects the hyperholomorphic connection λ but not its curvature. Examples of such dual pairs are provided by the rigid and (one-loop deformed) local c -map spaces associated to the same prepotential $F(X)$ [52, 60].

3.3.2. *Twistor space realization.* The QK/HK correspondence is most easily understood by using Swann’s relation between QK manifolds and hyperkähler cones [7]. Indeed, the total space \mathcal{S} of the $\mathcal{O}(-2)$ line bundle over the twistor space \mathcal{Z} of a QK manifold \mathcal{M} carries a canonical HK cone metric. Any quaternionic circle action of \mathcal{M} lifts to a triholomorphic circle action on \mathcal{S} . Taking the hyperkähler quotient of \mathcal{S} with respect to this circle action then produces the dual hyperkähler manifold \mathcal{M}' , equipped with a natural hyperholomorphic circle bundle [55, 56].

For our purposes it will be more useful to realize the QK/HK correspondence directly at the level of the twistor spaces \mathcal{Z} and \mathcal{Z}' , without invoking the Swann bundle. For this purpose, choose local contact Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$ on the QK side, such that the Killing vector κ globally lifts to ∂_α (the Reeb vector for the contact one-form (2.12)). This implies that contact transformations between different patches must reduce to symplectomorphisms of $(\xi^\Lambda, \tilde{\xi}_\Lambda)$, supplemented by a suitable, Ξ -dependent shift of α . On the HK side, we choose local (symplectic) Darboux coordinates $(\eta^\Lambda, \mu_\Lambda)$ on \mathcal{Z}' such that the holomorphic symplectic form on \mathcal{Z}' is $d\eta^\Lambda \wedge d\mu_\Lambda$. We further choose coordinates x^μ on $\mathcal{M}'/\partial_{\theta'}$, ζ on the \mathbb{P}^1 base on the HK side and t on the \mathbb{P}^1 fiber on the QK side, such that $\zeta = 0, \infty$ correspond to the complex structure J'_3 preserved by κ' . The fact that κ' rotates J'_1 into J'_2 means that the Darboux coordinates $(\eta^\Lambda, \mu_\Lambda)$ on \mathcal{Z}' depend

²Hyperholomorphicity is guaranteed by the second equation in (3.19).

only on x^μ and $\zeta e^{-i\theta'}$. Moreover, it implies that transition functions between different patches must be complex symplectomorphisms of $(\eta^\Lambda, \mu_\Lambda)$, independent of ζ . Then the correspondence shows that the Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda)$ on \mathcal{Z} , as functions of (x^μ, θ, t) , can be identified with $(\eta^\Lambda, \mu_\Lambda)$ for $t = \zeta e^{-i\theta'}$. In particular, the complex contact structure on \mathcal{Z} and symplectic structure on \mathcal{Z}' can be described globally by the same symplectomorphisms, supplemented on the QK side by a suitable shift of α . In fact, the Darboux coordinate α provides, on the dual HK side, a holomorphic section

$$(3.20) \quad \Upsilon^{[i]} \equiv e^{2\pi i \alpha^{[i]}}$$

of a line bundle $\mathcal{L}'_{\mathcal{Z}}$ over \mathcal{Z}' , with holomorphic connection given by the contact one-form (2.13). This section is non-zero along each twistor line, and hence by the Atiyah-Ward twistor correspondence (see [61]) yields a hyperholomorphic line bundle \mathcal{L} over \mathcal{M}' with connection

$$(3.21) \quad \lambda = \frac{1}{4} \left(\bar{\partial}^{(\zeta)} \alpha + \partial^{(\zeta)} \bar{\alpha} \right),$$

which can be shown to agree with (3.19). It is also worth noting that the contact potential e^ϕ on the QK side is identified with the moment map ρ on the HK side.

3.4. Wall-crossing revisited. After this digression on the QK/HK correspondence, we now return to the problem of wall-crossing on the D-instanton corrected HM moduli space \mathcal{M}_H . Since D-instanton corrections are independent of the NS-axion σ , they preserve the quaternionic Killing vector ∂_σ . Therefore, by the QK/HK correspondence, we can trade the construction of the QK space \mathcal{M}_H with that of a HK space \mathcal{M}'_H equipped with a hyperholomorphic connection λ . Twistorially, this is equivalent to constructing the twistor space \mathcal{Z}' and complex line bundle $\mathcal{L}_{\mathcal{Z}'}$. The space \mathcal{Z}' , parametrized by the Darboux coordinates $(\xi^\Lambda, \tilde{\xi}_\Lambda)$, is defined by the same gluing conditions (3.10) as in [48], specialized to the case where the prepotential is homogeneous. To construct³ $\mathcal{L}_{\mathcal{Z}'}$, one must lift the symplectomorphisms U_γ to complex gauge transformations V_γ preserving the holomorphic connection (2.13), which we identify with the contact transformations (3.11). The consistency of these gluing conditions across walls of marginal stability require that the KS formula (3.5) lifts to

$$(3.22) \quad \prod_{\substack{\gamma = m\gamma_1 + n\gamma_2 \\ m \geq 0, n \geq 0}}^{\widehat{\quad}} V_\gamma(z_+) = \prod_{\substack{\gamma = m\gamma_1 + n\gamma_2 \\ m \geq 0, n \geq 0}}^{\widehat{\quad}} V_\gamma(z_-).$$

Clearly, assuming (3.5) is satisfied, (3.22) could fail at most by a translation $\tilde{\alpha} \rightarrow \tilde{\alpha} + \Delta\tilde{\alpha}$ along the \mathbb{C}^\times -fiber. The global existence of $\mathcal{L}_{\mathcal{Z}'}$ requires $\Delta\tilde{\alpha} \in 2\mathbb{Z}$, the natural ambiguity in the coordinate $\tilde{\alpha}$. To see why this is so, let us rewrite (3.22) as an identity $\prod_s V_{\gamma_s}^{\epsilon_s} = 1$ by assembling all operators on one side. Here, ϵ_s is a sign that changes from $+1$ on the right side of the product (corresponding to the r.h.s. of (3.22)) to -1 on the left side (corresponding to the inverse of the l.h.s. of (3.22)). The total shift of $\tilde{\alpha}$ so obtained can be written as

$$(3.23) \quad \Delta\tilde{\alpha} = \frac{1}{2\pi^2} \sum_s \epsilon_s \Omega(\gamma_s) L_{\lambda_D(\gamma_s)}(\mathcal{X}_{\gamma_s}(s)),$$

where $\mathcal{X}_{\gamma_s}(s) = U_{\gamma_{s-1}} \circ U_{\gamma_{s-2}} \circ \dots \circ U_{\gamma_1} \cdot \mathcal{X}_{\gamma_s}$. In [52] it was shown that the quantization property $\Delta\tilde{\alpha} \in 2\mathbb{Z}$ follows from the motivic wall-crossing formula of Kontsevich and Soibelman [47] in an

³The construction of a canonical hyperholomorphic connection on the Coulomb branch of $\mathcal{N} = 2$ gauge theories was independently given in [62].

appropriate classical limit. This formula generalizes various known and conjectural identities for the Rogers dilogarithm associated with cluster algebras of Dynkin quivers (see, e.g., [63, 64]).

Let us end this discussion with an important remark. Although the above construction formally gives a satisfactory solution to the wall-crossing problem in the hypermultiplet sector of type IIA string theory on a CY threefold X , it ignores a crucial problem, namely the exponential growth $\Omega(\gamma) \sim e^{S(\gamma)}$ of the DT invariants for large charges, where $S(\gamma)$ is the entropy of a 4D BPS black hole of charge γ . Since $S(\gamma)$ scales quadratically in γ for large classical black holes, any generating function of the form $\sum_{\gamma} \Omega(\gamma) q^{\gamma}$ is divergent, hence making the integral equation (3.13) ill-defined. It was observed in [65] that the ambiguity in such divergent sums is of the same order e^{-1/g_s^2} as NS5-instanton effects, which might therefore cure this problem. We return to NS5-instantons in §5, after discussing S-duality in the presence of D3-instantons.

4. S-duality, D3-instantons and mock theta series

The invariance of ten-dimensional type IIB string theory under S-duality is a well supported fact (see e.g. [36] and much subsequent work). It is important to test whether it continues to hold in vacua with less supersymmetry, in particular in type IIB compactified on a CY threefold \hat{X} . Assuming that it does, one may e.g. deduce D1-D(-1) instanton corrections from worldsheet instantons [33, 34], obtaining the first hint of the form of D-instanton corrections to the HM metric, or NS5-instantons from D5 [19], as we discuss in §5. Here, we focus on the intermediate case of D3-instantons, which are singlets of $SL(2, \mathbb{Z})$ and should therefore preserve S-duality by themselves [35, 37]. We shall show that this is indeed the case thanks to special modular properties of D3-D1-D(-1) Donaldson-Thomas invariants and of certain indefinite theta series.

4.1. S-duality in twistor space. We start by discussing general constraints imposed by S-duality on the twistor space construction. By a suitable choice of coordinates, we can assume that, even after the inclusion of instanton corrections, $SL(2, \mathbb{Z})$ acts on $\hat{\mathcal{M}}_H(\hat{X})$ by (2.22). This action must lift to a holomorphic action on the twistor space \mathcal{Z} . At the classical level, using the Darboux coordinates (2.15) with $F = F^{\text{cl}}$ (2.19), $r = \frac{r_0^2}{16} e^{-\mathcal{K}}$ and $c = 0$, one can check that (2.22) lifts to the complex contact transformation [16]

$$(4.1) \quad \begin{aligned} \xi^0 &\mapsto \frac{a\xi^0 + b}{c\xi^0 + d}, & \xi^a &\mapsto \frac{\xi^a}{c\xi^0 + d}, & \tilde{\xi}'_a &\mapsto \tilde{\xi}'_a + \frac{c}{2(c\xi^0 + d)} \kappa_{abc} \xi^b \xi^c - c_{2,a} \varepsilon(g), \\ \begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} &\mapsto \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} \tilde{\xi}'_0 \\ \alpha' \end{pmatrix} + \frac{1}{6} \kappa_{abc} \xi^a \xi^b \xi^c \begin{pmatrix} c^2 / (c\xi^0 + d) \\ -[c^2(a\xi^0 + b) + 2c] / (c\xi^0 + d)^2 \end{pmatrix}, \end{aligned}$$

provided the action (2.22) on \mathcal{M}_H is supplemented by a suitable action on the \mathbb{P}^1 fiber, e.g. in the gauge $X^0 = 1$,

$$(4.2) \quad z \mapsto \frac{c\bar{\tau} + d}{|c\tau + d|} z, \quad z \equiv \frac{t + i}{t - i}.$$

Beyond the classical limit, the Darboux coordinates are no longer given by the simple formulae (2.15), however, they should still transform as in (4.1) up to local contact transformations, if S-duality is to remain unbroken. This in turn constrains the transformations of the transition functions on overlapping patches. The constraint can be easily formulated by considering a covering by an infinite set of open patches $\mathcal{U}_{m,n}$, which are mapped to each other by S-duality, including a S-duality invariant patch $\mathcal{U}_0 \equiv \mathcal{U}_{0,0}$ [34, 35]. Then at the linearized level S-duality requires that

the generating functions $H_{m,n}$ of the contact transformations from \mathcal{U}_0 to $\mathcal{U}_{m,n}$ transform as

$$(4.3) \quad H_{m,n} \mapsto \frac{H_{m',n'}}{c\xi^0 + d} + \text{reg.}, \quad \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Heuristically, ignoring the fact that the functions $H_{m,n}$ are attached to different (pairs of) patches, the constraint (4.3) says that the formal sum $\sum H_{m,n}$ should transform as a holomorphic modular form of weight -1 . In [35] this constraint was promoted to the full non-linear level under the assumption that the QK space has two commuting continuous isometries, which holds for the D3-instanton corrected HM moduli space having two continuous isometries along fivebrane axions. How to drop this assumption and describe an arbitrary QK space carrying an isometric action of $SL(2, \mathbb{Z})$ was understood in [66, 67].

4.2. Modularity of DT invariants. Before discussing the S-duality invariance of the D3-instanton corrected metric, let us first recall the modular properties of DT invariants associated to dimension 2 sheaves. The same invariants control D4-D2-D0 and M5-brane black holes, which have been the subject of much research [68, 69, 70, 71, 72, 73]. As discussed in §3.1.2, given a coherent sheaf \mathcal{E} on \hat{X} , D-brane charges are components of the generalized Mukai charge vector γ :

$$(4.4) \quad \gamma = \text{ch}(\mathcal{E}) \sqrt{\text{Td} \hat{X}} = p^0 + p^a \omega_a - q'_a \omega^a + q'_0 \omega_{\hat{X}},$$

where $\{\omega_a\}$, $\{\omega^a\}$ and $\omega_{\hat{X}}$ are respectively a basis of 2-forms, 4-forms and the volume form of \hat{X} , and $q'_\Lambda = q_\Lambda - A_{\Lambda\Sigma} p^\Sigma$ are in general non-integral [19, 37]. We shall denote the corresponding DT invariants by $\Omega(p^0, p^a, q'_a, q'_0; z)$. Dimension-one or zero sheaves ($p^a = p^0 = 0$) correspond to D1-D(-1) instantons. As shown in [33], the D1-D(-1)-instanton corrected metric is invariant under $SL(2, \mathbb{Z})$ provided

$$(4.5) \quad \Omega(0, 0, 0, q_0) = -\chi(\hat{X}), \quad \Omega(0, 0, q_a, q_0) = n_{q_a}^{(0)},$$

where $n_{q_a}^{(0)}$ are the genus 0 Gopakumar-Vafa invariants governing the worldsheet instanton corrections. D3-brane instantons correspond to dimension-two sheaves ($p^0 = 0, p^a \neq 0$), supported on a divisor $\mathcal{D} \subset \hat{X}$. Dimension-three sheaves will be discussed in §5.2.1.

We assume that \mathcal{D} is an ample divisor, i.e. that $[\mathcal{D}]$ belongs to the Kähler cone. The intersection matrix of 2-cycles of an ample divisor provides a natural quadratic form $\kappa_{abc} p^c$ on $\Lambda = H_4(\hat{X}, \mathbb{Z})$, with signature $(1, b_2(\hat{X}) - 1)$. This also provides a quadratic form $\kappa^{ab} = (\kappa_{abc} p^c)^{-1}$ on Λ^* . In the following, we shall use this quadratic form to identify Λ as a sublattice of Λ^* , $k^a \mapsto k_a \equiv \kappa_{abc} p^b k^c$. In the following we shall denote the vector $(k^1, \dots, k^{b_2(\hat{X})})$ as \mathbf{k} . For a general $\mathbf{k} \in \Lambda$, the vectors $\mathbf{k}_\pm \in \Lambda \otimes \mathbb{R}$ are projections of \mathbf{k} onto the positive and negative definite subspaces of $\Lambda \otimes \mathbb{R}$ defined by the magnetic charge vector \mathbf{p} and the Kähler moduli \mathbf{t} :

$$(4.6) \quad \mathbf{k}_+ = \frac{\mathbf{k} \cdot \mathbf{t}}{\mathbf{p} \cdot \mathbf{t}^2} \mathbf{t}, \quad \mathbf{k}_- = \mathbf{k} - \mathbf{k}_+, \quad \mathbf{k}^2 = \mathbf{k}_+^2 + \mathbf{k}_-^2,$$

which satisfy $\mathbf{k}_+^2 > 0$, $\mathbf{k}_-^2 < 0$ for all $\mathbf{k} \neq \mathbf{0}$. We also use the notation k_+ to denote the modulus of the vector \mathbf{k}_+ .

As explained in §3.1.2, the DT invariants $\Omega(\gamma; \mathbf{z})$ are piecewise constant in Kähler moduli, but can be discontinuous across walls of marginal stability. This moduli dependence persists in the large volume limit, and complicates the analysis of the modular properties of $\Omega(\gamma; \mathbf{z})$ [74, 50]. To deal with this problem, we express the DT invariants $\Omega(\gamma; \mathbf{z})$ in terms of the ‘MSW invariants’

$\Omega_{\mathbf{p}}(\mathbf{q}', q'_0)$. The latter coincide with the DT invariants at the so-called ‘large volume attractor point’ [75]⁴

$$(4.7) \quad \Omega_{\mathbf{p}}(\mathbf{q}', q'_0) = \Omega(0, \mathbf{p}, \mathbf{q}', q'_0; \mathbf{z}_\infty(\gamma)), \quad \mathbf{z}_\infty(\gamma) = \lim_{\lambda \rightarrow +\infty} (\mathbf{b}(\gamma) + i\lambda \mathbf{t}(\gamma)),$$

where $\mathbf{z}(\gamma) = \mathbf{b}(\gamma) + i\mathbf{t}(\gamma)$ is the standard attractor point. Away from the large volume attractor point $\mathbf{z}_\infty(\gamma)$ (but still at large volume), the DT invariant $\Omega(0, \mathbf{p}, \mathbf{q}', q'_0; \mathbf{z})$ differs from the MSW invariant $\Omega_{\mathbf{p}}(\mathbf{q}', q'_0)$ by terms of higher order in the MSW invariants [74, 76]. The higher order terms can be thought of as describing bound states of the MSW constituents, which exist away from the large volume attractor point $\mathbf{z}_\infty(\gamma)$. This decomposition is analogous to the decomposition of the index in terms of the multi-centered black hole bound states. As shown in [74], the ‘two-centered’ contribution leads to a modular invariant partition function with the same modular properties as the elliptic genus, and it is expected that modularity persists to all orders in the MSW invariants. We stress that the expansion in MSW invariants is not a Taylor expansion in a small parameter, rather it is a finite sum in any chamber separated by a finite number of walls from the large volume attractor chamber. In the remainder of this work, we shall be concerned with only the first term in this expansion, which we call the ‘one-instanton approximation’ or ‘dilute instanton approximation’.

As the name suggests, the MSW invariants are the BPS indices of the MSW (0, 4) superconformal field theory (SCFT) describing D4-brane or M5-branes wrapped on the divisor \mathcal{D} [68]. They are unchanged by the ‘spectral flow’ transformations of the charges:

$$(4.8) \quad \mathbf{p} \mapsto \mathbf{p}, \quad \mathbf{q}' \mapsto \mathbf{q}' - \epsilon, \quad q'_0 \mapsto q'_0 - \epsilon \cdot \mathbf{q}' + \frac{1}{2} \mathbf{p} \cdot \epsilon^2,$$

which are induced by monodromies around the large volume point of $\mathcal{M}_H(\hat{X})$, $\mathbf{z} \rightarrow \mathbf{z} + \epsilon$, and leave

$$(4.9) \quad \hat{q}_0 \equiv q'_0 - \frac{1}{2} \mathbf{q}'^2$$

invariant. Decomposing

$$(4.10) \quad \mathbf{q}' = \boldsymbol{\mu} + \epsilon + \frac{1}{2} \mathbf{p},$$

where $\boldsymbol{\mu} \in \Lambda^*/\Lambda$ is the residue class of $\mathbf{q}' - \frac{1}{2}\mathbf{p}$ modulo ϵ , it follows that the MSW invariant $\Omega_{\mathbf{p}}(\mathbf{q}', q'_0) \equiv \Omega_{\mathbf{p}, \boldsymbol{\mu}}(\hat{q}_0)$ depends only on \mathbf{p} , $\boldsymbol{\mu}$ and \hat{q}_0 . The partition function of MSW invariants for fixed divisor \mathcal{D} is the elliptic genus of the SCFT,

$$(4.11) \quad \mathcal{Z}_{\mathbf{p}}(\tau, \mathbf{y}) = \sum_{\mathbf{q}', q'_0} (-1)^{\mathbf{p} \cdot \mathbf{q}'} \bar{\Omega}_{\mathbf{p}}(\mathbf{q}', q'_0) \mathbf{E} \left(-\hat{q}_0 \tau - \frac{1}{2} \mathbf{q}'^2 \tau - \frac{1}{2} \mathbf{q}'^2 \bar{\tau} + \mathbf{q}' \cdot \mathbf{y} \right)$$

with $\mathbf{y} \in \Lambda \otimes \mathbb{C}$ and $\bar{\Omega}_{\mathbf{p}}(\mathbf{q}', q'_0)$ the rational MSW invariant defined analogously to (3.7). When γ is primitive, it follows from general properties of the MSW SCFT that the elliptic genus (4.11) is a multi-variable Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$ under $SL(2, \mathbb{Z})$, with multiplier system $M_{\mathcal{Z}} = \mathbf{E}(\varepsilon(g) \mathbf{c}_2 \cdot \mathbf{p})$, where $\varepsilon(g)$ is as in (2.23). If \mathbf{p} is not a primitive vector, the BPS indices might not be related to a proper CFT due to states at threshold stability. However, wall-crossing arguments [50] and explicit calculations [77] suggest that nevertheless the generating function of $\bar{\Omega}_{\mathbf{p}}(\mathbf{q}', q'_0)$ exhibits good modular properties under $SL(2, \mathbb{Z})$.

⁴Note that for CY threefolds with $b_2(\hat{X}) = 1$, the walls of marginal stability for D3-instantons do not extend to large volume regime, hence the MSW and DT invariants coincide.

The invariance of $\overline{\Omega}_{\mathbf{p}}(\mathbf{q}', q'_0)$ under the spectral flow (4.8) implies that the elliptic genus has a theta function decomposition:

$$(4.12) \quad \mathcal{Z}_{\mathbf{p}}(\tau, \mathbf{y}, \mathbf{t}) = \sum_{\mu \in \Lambda^*/\Lambda} h_{\mathbf{p}, \mu}(\tau) \overline{\theta_{\mathbf{p}, \mu}(\tau, \mathbf{y}, \mathbf{t})},$$

where $\theta_{\mathbf{p}, \mu}$ is the Siegel-Narain theta series associated to the lattice Λ equipped with the quadratic form κ_{ab} of signature $(1, b_2(\hat{X}) - 1)$,

$$(4.13) \quad \theta_{\mathbf{p}, \mu}(\tau, \mathbf{y}, \mathbf{t}) = \sum_{\mathbf{k} \in \Lambda + \mu + \frac{1}{2}\mathbf{p}} (-1)^{\mathbf{k} \cdot \mathbf{p}} \mathbf{E} \left(\frac{1}{2} \mathbf{k}_+^2 \tau + \frac{1}{2} \mathbf{k}_-^2 \bar{\tau} + \mathbf{k} \cdot \mathbf{y} \right),$$

and the \mathbf{y} -independent coefficients are given by

$$(4.14) \quad h_{\mathbf{p}, \mu}(\tau) = \sum_{\hat{q}_0 \leq \tau \chi(\mathcal{D})/24} \overline{\Omega}_{\mathbf{p}, \mu}(\hat{q}_0) \mathbf{E}(-\hat{q}_0 \tau).$$

Since the theta series (4.13) is a vector valued Jacobi form of modular weight $(\frac{1}{2}, \frac{b_2(\hat{X})-1}{2})$ and multiplier system M_{θ} under $SL(2, \mathbb{Z})$, it follows that $h_{\mathbf{p}, \mu}$ must transform as a vector-valued holomorphic⁵ modular form of negative weight $(-\frac{b_2(\hat{X})}{2} - 1, 0)$ and multiplier system $M(g) = M_{\mathcal{Z}} \times \overline{M_{\theta}}^{-1}$ under the full modular group $SL(2, \mathbb{Z})$. The latter is equivalent to $M(g) = M_{\mathcal{Z}} \times M_{\theta}$ since M_{θ} is unitary.

4.3. Dilute D3-instantons. Let us now explain why D3-D1-D(-1) instanton corrections are consistent with S-duality, at least in the dilute instanton approximation. To this end it is sufficient to consider the contribution of a single homology class \mathbf{p} , but to include the sum over embedded classes \mathbf{q} and q_0 . This avoids the complications arising from the divergent sum over $\mathbf{p} \in H^4(\hat{X}, \mathbb{Z})$, as mentioned at the end of §3.2.

4.3.1. *Transition functions.* First, we observe that the twistor description of D3-instantons satisfies the requirement of §4.1, namely that the formal sum of transition functions transforms as a holomorphic modular form of weight -1 . The transition functions generating the contact transformations (3.11) are

$$(4.15) \quad H_{\gamma} = \frac{\Omega(\gamma)}{(2\pi)^2} \text{Li}_2(\lambda_{\text{D}}(\gamma) \mathcal{X}_{\gamma}) + \dots,$$

where $\gamma = (0, \mathbf{p}, \mathbf{q}, q_0)$ and \dots denote terms non-linear in the DT invariants whose explicit form can be found in [51]. In the leading term one can replace $\text{Li}_2(x)$ by x at the cost of replacing the integer DT invariants by their rational counterparts (3.7). Furthermore, in the dilute instanton approximation the latter can be replaced by the MSW invariants $\overline{\Omega}_{\mathbf{p}}(\mathbf{q}', q'_0)$ introduced in the previous subsection. Choosing $\lambda_{\text{D}}(\gamma) = (-1)^{\mathbf{p} \cdot \mathbf{q}}$, we arrive at the formal sum

$$(4.16) \quad H_{\mathbf{p}} = \sum_{\mathbf{q}', q'_0} \overline{H}_{\gamma}, \quad \overline{H}_{\gamma} = \frac{(-1)^{\mathbf{p} \cdot \mathbf{q}'}}{(2\pi)^2} \overline{\Omega}_{\mathbf{p}, \mu}(\hat{q}_0) \mathbf{E}(\mathbf{p} \cdot \tilde{\xi}' - \mathbf{q}' \cdot \xi - q'_0 \xi^0).$$

Similarly to (4.12), using monodromy invariance the sum can be rewritten as

$$(4.17) \quad H_{\mathbf{p}} = \frac{1}{(2\pi)^2} \mathbf{E}(\mathbf{p} \cdot \tilde{\xi}') \sum_{\mu \in \Lambda^*/\Lambda} h_{\mathbf{p}, \mu}(\xi^0) \Xi_{\mathbf{p}, \mu}(\xi^0, \xi),$$

⁵Non-compact directions in the target space of the CFT could potentially lead to mock modular forms and thus holomorphic anomalies [78]. For local CY manifolds, e.g. $\mathcal{O}(-K_{\mathbb{P}^2}) \rightarrow \mathbb{P}^2$, it is known that the holomorphic generating series of DT-invariants for sheaves with rank > 1 requires a non-holomorphic addition in order to transform as a modular form [79]. We assume that this issue does not arise here.

where $h_{\mathbf{p},\mu}(\xi^0)$ is the modular function defined in (4.14), now evaluated at $\tau = \xi^0$, and $\Xi_{\mathbf{p},\mu}(\xi^0, \boldsymbol{\xi})$ is a holomorphic theta series defined by the quadratic form $-\kappa_{ab}$ which transforms formally as a holomorphic Jacobi form of weight $\frac{b_2}{2}$, multiplier system M_θ^{-1} and index $m_{ab} = -\frac{1}{2}\kappa_{ab}$ [37]. Finally, thanks to the term proportional to $c_{2,a}$ in the transformation of $\tilde{\boldsymbol{\xi}}'$, the exponential prefactor in (4.17) transforms as the automorphy factor of a multi-variable holomorphic Jacobi theta series with the index $m_{ab} = \frac{1}{2}\kappa_{ab}$ and multiplier system M_θ^{-1} . Combining the transformation properties of all factors, we conclude that, under the action (4.1), the formal sum (4.16) indeed transforms as a holomorphic Jacobi form of weight -1 and trivial multiplier system, as required by S-duality.

However, this analysis overlooks the important fact that the quadratic form κ_{ab} has indefinite signature $(1, b_2 - 1)$, and therefore the theta series $\Xi_{\mathbf{p},\mu}(\xi^0, \boldsymbol{\xi})$ is divergent. Fortunately, it never actually arises as such in the computation of the metric, rather each of the terms in (4.16) must be integrated along a different contour, which renders the resulting series convergent. To put the above heuristic argument on solid ground, one should analyze the transformation properties of the quantities which enter the computation of the metric, namely the Darboux coordinates and the contact potential. Below we restrict to the contact potential and Darboux coordinate $\boldsymbol{\xi}$, which exhibit the key mechanism, referring the reader to [37] for a complete discussion.

4.3.2. Contact potential. The contact potential e^ϕ is given in (3.16). The first term on the right hand side is the classical term, with $\frac{1}{8}e^{-\mathcal{K}(z,\bar{z})}$ equal to the volume $\mathcal{V} = \frac{1}{6}\mathbf{t}^3$ of \hat{X} . Using the transformation properties of τ_2 and \mathbf{t} , one easily checks that the classical part transforms with weight $(-\frac{1}{2}, -\frac{1}{2})$. The instanton corrections involve the Darboux coordinates \mathcal{X}_γ , which are solutions to the integral equations (3.13). In order to relate these corrections to the modular functions of the previous section, we start by writing the 3rd term on the right hand side of (3.16) as:

$$(4.18) \quad \delta_{\mathbf{p}} e^\phi = -\frac{i\tau_2}{16} \sum_{q_\Lambda} \int_{\ell_\gamma} \frac{dt}{t} (t^{-1}Z_\gamma - t\bar{Z}_\gamma) \overline{H}_\gamma(\xi^0, \boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}),$$

where \overline{H}_γ is given in (4.16). In the dilute instanton approximation we can replace the Darboux coordinates appearing in the arguments of \overline{H}_γ by their classical expressions (2.15) with the pre-potential (2.19). Furthermore, we keep only the leading terms in the limit where $\mathbf{t} \rightarrow \infty$ and the product $z\mathbf{t}$ remains constant (where z is the Cayley-rotated coordinate on \mathbb{P}^1 , see (4.2)). This is motivated by the fact that the saddle points of the integral lie at $z_\gamma = -i(k+b)_+/\sqrt{\mathbf{p}\cdot\mathbf{t}^2}$ for $\mathbf{p}\cdot\mathbf{t}^2 > 0$. As a result, $\overline{H}_\gamma(\xi^0, \boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})$ simplifies to the following form [37]:

$$(4.19) \quad \overline{H}_\gamma = \frac{(-1)^{p\cdot q}}{(2\pi)^2} \overline{\Omega}_{\mathbf{p},\mu}(\hat{q}_0) \mathbf{E} \left(iS_{\text{cl}} - \frac{1}{2}(\mathbf{q} + \mathbf{b})_+^2 \bar{\tau} - \left(\hat{q}_0 + \frac{1}{2}(\mathbf{q} + \mathbf{b})_-^2 \right) \tau + \mathbf{c} \cdot (\mathbf{q} + \frac{1}{2}\mathbf{b}) + iQ_\gamma(z) \right),$$

where S_{cl} is the leading part of the Euclidean D3-instanton action in the large volume limit, and $Q_\gamma(z)$ is the only part which depends on the fiber coordinate z ,

$$(4.20) \quad S_{\text{cl}} = \frac{\tau_2}{2} \mathbf{p} \cdot \mathbf{t}^2 - i\tilde{\mathbf{c}} \cdot \mathbf{p}, \quad Q_\gamma(z) = \tau_2 \mathbf{p} \cdot \mathbf{t}^2 \left(z + i \frac{(k+b)_+}{\sqrt{\mathbf{p}\cdot\mathbf{t}^2}} \right)^2.$$

Keeping the leading contributions to the remaining terms in (4.18) leads to

$$(4.21) \quad \delta_{\mathbf{p}} e^\phi = -\frac{\tau_2}{4} \sum_{q_\Lambda} \int_{\ell_\gamma} dz \left[\hat{q}_0 + \frac{1}{2}(\mathbf{k} + \mathbf{b} - iz\mathbf{t}) \cdot (\mathbf{k} + \mathbf{b} - 3iz\mathbf{t}) \right] H_\gamma + \text{c.c.}$$

The integral over z is now Gaussian, leading to

$$(4.22) \quad \delta_{\mathbf{p}} e^{\phi} = \frac{\tau_2 e^{-2\pi S_{cl}}}{16\pi^2 \sqrt{2\tau_2} \mathbf{p} \cdot \mathbf{t}^2} \mathcal{D}_{-\frac{3}{2}} \sum_{\mu \in \Lambda^*/\Lambda} h_{\mathbf{p},\mu}(\tau) \overline{\theta_{\mathbf{p},\mu}(\tau, \mathbf{t}, \mathbf{b}, \mathbf{c})} + \text{c.c.},$$

where $\theta_{\mathbf{p},\mu}(\tau, \mathbf{t}, \mathbf{b}, \mathbf{c})$ is a theta function similar to (4.13), which transforms as a vector-valued modular form with weight $\frac{1}{2}(1, b_2(\hat{X}) - 1)$ and multiplier system M_θ , see [37, Eq. (A.6)]. The action of $\mathcal{D}_{-\frac{3}{2}} = \frac{1}{2\pi i} \left(\partial_\tau - \frac{3/2}{2i\tau_2} \right)$ raises the modular weight from $(-\frac{3}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{1}{2})$, while the overall factor of τ_2 reduces this to $(-\frac{1}{2}, -\frac{1}{2})$. The transformation property of \tilde{c}_a (2.22) is now seen to cancel the non-trivial phases due to the multiplier system $M_{\mathcal{Z}}$ of $\mathcal{Z}_{\mathbf{p}}(\tau, \mathbf{y}, \mathbf{t})$, establishing that the instanton correction $\delta_{\mathbf{p}} e^{\phi}$ transforms correctly under S-duality.

4.3.3. *Darboux coordinates and Eichler integrals.* Our next example is the coordinate ξ on twistor space, which has the general form:

$$(4.23) \quad \xi = \zeta + \frac{\tau_2}{2} \left(\frac{z}{t} - \bar{z}t \right) + \frac{1}{8\pi^2} \sum_{\gamma} \Omega(\gamma; z) \mathbf{p} \int_{\ell_\gamma} \frac{dt' t + t'}{t' t - t'} \log[1 - \sigma_\gamma \mathcal{X}_\gamma].$$

As explained in [35], the S-duality transformations of such expressions can be greatly simplified by modifying the integration kernel by a t -independent term, at the cost of correcting the mirror map (2.20). After this change and in the large volume limit, $\delta_{\mathbf{p}} \xi$ becomes

$$(4.24) \quad \delta_{\mathbf{p}} \xi = \mathbf{p} \sum_{q_\Lambda} \int_{\ell_\gamma} \frac{dz'}{z' - z} \overline{H}_\gamma.$$

Unlike (4.21), the integral is no longer Gaussian but can be expressed as an Eichler integral

$$(4.25) \quad \delta_{\mathbf{p}} \xi = -\frac{e^{-2\pi S_{cl}}}{4\pi} \mathbf{p} \sum_{\mu \in \Lambda^*/\Lambda} h_{\mathbf{p},\mu}(\tau) \int_{\bar{\tau}}^{-i\infty} \frac{\overline{\Upsilon_{\mu}(w, \bar{\tau}; \bar{z})} d\bar{w}}{\sqrt{i(\bar{w} - \tau)}},$$

where $\Upsilon_{\mu}(w, \bar{\tau}; \bar{z})$ (given [37, Eq. (4.24)]) is a modular form of weight $(\frac{3}{2}, \frac{b_2-1}{2})$. Such integrals transform inhomogeneously under $SL(2, \mathbb{Z})$,

$$(4.26) \quad \delta_{\mathbf{p}} \xi \mapsto (c\tau + d)^{-1} \left(\delta_{\mathbf{p}} \xi + \frac{e^{-2\pi S_{cl}}}{4\pi} \mathbf{p} \sum_{\mu \in \Lambda^*/\Lambda} h_{\mathbf{p},\mu}(\tau) \int_{-d/c}^{i\infty} \frac{\overline{\Upsilon_{\mu}(w, \bar{\tau}; \bar{z})} d\bar{w}}{[i(\bar{w} - \tau)]^{1/2}} \right).$$

The overall weight $(-1, 0)$ is in agreement with the classical transformation of ξ (4.1), however the period integral $\int_{-d/c}^{i\infty} d\bar{w}$ makes this transformation anomalous. Remarkably, using standard techniques for mock theta series [80] one can show that

$$(4.27) \quad \delta_{\mathbf{p}} \widehat{\xi} = \delta_{\mathbf{p}} \xi - 2\pi i \mathbf{p} H_{\text{anom}}$$

does transform as a modular form of weight $(-1, 0)$. Here H_{anom} is an indefinite theta function:

$$(4.28) \quad H_{\text{anom}} = \frac{1}{2} \sum_{q_\Lambda} [\text{sgn}((\mathbf{k} + \mathbf{b}) \cdot \mathbf{t}) - \text{sgn}((\mathbf{k} + \mathbf{b}) \cdot \mathbf{t}')] \overline{H}_\gamma,$$

with \mathbf{t}' lying on the boundary of the Kähler cone. H_{anom} transforms also by a period integral, precisely cancelling the one of $\delta_{\mathbf{p}} \xi$. More generally, H_{anom} generates a contact transformation which precisely cancels the modular anomaly in all Darboux coordinates, thus establishing the S-duality invariance of the D3-instanton corrected HM metric. It would be very interesting to derive H_{anom} and the choice of \mathbf{t}' from first principles.

5. Toward NS5-instanton effects

In addition to D-instanton effects, the metric on the HM moduli space \mathcal{M}_H receives instanton corrections of order e^{-1/g_s^2} from Euclidean NS5-branes wrapping the whole threefold X [81]. On the type IIB side, it is clear that such corrections are necessary to restore S-duality, since D5 and NS5 branes transform as a doublet under $SL(2, \mathbb{Z})$. For k NS5-branes the expected correction is, schematically,

$$(5.1) \quad g_{\text{NS5}} \sim e^{-4\pi|k|r - \pi i k \sigma} \mathcal{Z}_k,$$

where \mathcal{Z}_k is the partition function for the degrees of freedom localized on the NS5-branes and $r \sim \mathcal{V}/g_s^2$. In particular, unlike D-instanton effects, these corrections break the continuous translational symmetry of the NS-axion to a discrete subgroup $\sigma \mapsto \sigma + 2\kappa$, $\kappa \in \mathbb{Z}$, as anticipated in equation (2.4). Our goal in this section is to infer the form of \mathcal{Z}_k (more precisely, the corresponding correction to the twistor space \mathcal{Z}) from the known D5-instanton corrections, at least in a linearized approximation. Before doing this, we discuss constraints on \mathcal{Z}_k coming from the topology of the axion circle bundle \mathcal{C}_σ . We close with some speculative comments on relations between NS5-instantons and quantum integrable systems.

5.1. Topology of the axion circle bundle.

5.1.1. *NS5-partition function and theta series.* The fivebrane partition function \mathcal{Z}_k is in general a function of the dilaton ϕ , complex structure moduli z^a and RR moduli C (in type IIA variables). Its dependence on C is strongly constrained by the fact that the corrected metric must stay invariant under the large gauge transformations (2.4). In view of (5.1), this implies that, under an integer shift $C \mapsto C + H$ with $H \in H^3(X, \mathbb{Z})$,

$$(5.2) \quad \mathcal{Z}_k(C + H) = (\lambda(H))^k \mathbf{E}\left(\frac{k}{2}\langle C, H \rangle\right) \mathcal{Z}_k(C).$$

In words, \mathcal{Z}_k must be a section of the theta line bundle $(\mathcal{L}_\Theta)^{\otimes k}$ over \mathcal{T} , with first Chern class $c_1(\mathcal{L}_\Theta) = \omega_{\mathcal{T}}$ (see (2.10)). In the Weil complex structure, $(\mathcal{L}_\Theta)^{\otimes k}$ is known to admit $|k|^{b_3(X)/2}$ holomorphic sections [82], corresponding to the Siegel theta series

$$(5.3) \quad \vartheta_{k,\mu}(C) = \sum_{n \in \Gamma_m + \mu + \theta} \mathbf{E}\left(\frac{k}{2}(\zeta^\Lambda - n^\Lambda) \bar{\mathcal{N}}_{\Lambda\Sigma} (\zeta^\Sigma - n^\Sigma) + k(\tilde{\zeta}_\Lambda - \phi_\Lambda) n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)\right)$$

labelled by vectors $\mu \in \Gamma_m/|k|\Gamma_m$, where Γ_m is a Lagrangian sublattice of $H^3(X, \mathbb{Z})$, here spanned by A-cycles. Physically, the sum over n^Λ labels the topological sectors of the (imaginary) self-dual 3-form field strength \mathcal{H} living on k fivebranes. Indeed, the Siegel theta series can be obtained by holomorphic factorization of the (non-holomorphic) partition function $\mathcal{Z}_{k,3\text{-form}}$ of a Gaussian 3-form on X [82, 83, 84, 85, 86]:

$$\mathcal{Z}_{k,3\text{-form}} \sim \sum_{\mu \in \Gamma_m/|k|\Gamma_m} \vartheta_{k,\mu}(C) \overline{\vartheta_{k,\mu}(C)}.$$

In general however, the chiral fivebrane worldvolume theory is non-Gaussian, and the only conclusion that can be drawn from (5.2) is that \mathcal{Z}_k must be a linear combination of non-Gaussian theta series

$$(5.4) \quad \mathcal{Z}_k(C) = \sum_{\mu \in \Gamma_m/|k|\Gamma_m} \sum_{n \in \Gamma_m + \mu + \theta} \Psi_{k,\mu}(\zeta^\Lambda - n^\Lambda) \mathbf{E}\left(k(\tilde{\zeta}_\Lambda - \phi_\Lambda) n^\Lambda + \frac{k}{2}(\theta^\Lambda \phi_\Lambda - \zeta^\Lambda \tilde{\zeta}_\Lambda)\right),$$

where we only displayed the dependence on the C -field. The $\Psi_{k,\mu}$'s can be interpreted as wave-functions in the real polarization corresponding to the Lagrangian subspace of $H_3(X, \mathbb{R})$ spanned by periods along the A-cycles.

5.1.2. *Metric dependence.* More generally, for consistency of the correction (5.1), the partition function \mathcal{Z}_k should be a (not necessarily holomorphic) section of \mathcal{C}_σ^k , where \mathcal{C}_σ is the circle bundle where the NS axion $e^{i\pi\sigma}$ is valued [19]. As indicated in (2.10), this circle bundle has curvature both over the fiber \mathcal{T} and base $\mathcal{M}_C(X)$ of the intermediate Jacobian $\mathcal{J}_W(X)$. The curvature over \mathcal{T} reflects the non-trivial behavior (2.4), (5.2) under large gauge transformations, while the curvature over $\mathcal{M}_C(X)$ shows that under a monodromy M in $\mathcal{M}_C(X)$, under which the holomorphic 3-form transforms as $\Omega_{3,0} \mapsto e^f \Omega_{3,0}$, the NS-axion and NS5 partition function by

$$(5.5) \quad \sigma \mapsto \sigma + \frac{\chi(X)}{24\pi} \text{Im } f + 2\kappa(M), \quad \mathcal{Z}_k \mapsto e^{ik\frac{\chi(X)}{24}} \text{Im } f + 2\pi i k \kappa(M) \mathcal{Z}_k,$$

where $\kappa(M)$ is the logarithm of a character of the monodromy group (see [19] for more details).

The cancellation of phases between $e^{-i\pi k\sigma}$ and \mathcal{Z}_k is expected by the general inflow mechanism for local anomalies [87]. A complication however is that the first Chern class (2.10) is not an integer cohomology class, rather $2c_1(\mathcal{C}_\sigma) \in H_2(\mathcal{M}_H, \mathbb{Z})$, since $\frac{1}{12}\chi(X)\omega_C$ can be identified as the first Chern class of the determinant line bundle of the Dirac operator for chiral spinors on X [88, 89], also known as the BCOV line bundle [90]. This means that σ is only defined up to a half-period, and \mathcal{Z}_k has a sign ambiguity. This sign ambiguity corresponds to a choice of character $\kappa(M)$ in (5.5), and also appears to be related to the 'orientation data' in the theory of generalized Donaldson-Thomas invariants [47, 91]. We expect that global anomaly cancellation will ensure that these two ambiguities cancel.

5.2. Fivebrane corrections to the contact structure.

5.2.1. *NS5-branes from D5-branes.* Since (NS5,D5)-branes transform as a doublet under S-duality, the corrections to the complex contact structure on \mathcal{Z} induced by NS5-instantons can in principle be inferred from the D5-instanton corrections described in §3.2. This procedure was implemented at the linearized level in [19] as follows. Start from the transition function

$$(5.6) \quad \bar{H}_\gamma = \frac{\bar{\Omega}(\gamma)}{(2\pi)^2} \lambda_D(\gamma) \mathbf{E} \left(p^\Lambda \tilde{\xi}'_\Lambda - q'_\Lambda \xi^\Lambda \right)$$

generating the contact transformation across a BPS ray associated to an instanton with non-vanishing D5-brane charge p^0 , and act on it by an $SL(2, \mathbb{Z})$ transformation $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, choosing $(c, d) = (-k/p^0, p/p^0)$ such that $p^0 = \text{gcd}(k, p)$. Such a transformation maps a D5-brane into a (k, p) -fivebrane, where k labels the NS5-brane charge. Using (4.1) and (4.3), one finds that the transition function associated to such a fivebrane takes the form

$$(5.7) \quad H_{k,p,\hat{\gamma}} = -\frac{\bar{\Omega}(\gamma)}{(2\pi)^2} \frac{k}{p^0} (\xi^0 - n^0) \lambda_D(\gamma) \mathbf{E} \left(k S_\alpha + \frac{p^0 (k \hat{q}_a (\xi^a - n^a) + p^0 \hat{q}_0)}{k^2 (\xi^0 - n^0)} + a \frac{p^0 q'_0}{k} - c_{2,a} p^a \epsilon(g) \right),$$

where $\hat{\gamma}$ denotes the remaining D3-D1-D(-1)-charges and

$$(5.8) \quad S_\alpha = \alpha + n^\Lambda \tilde{\xi}'_\Lambda + F^{\text{cl}}(\xi - n) - \frac{1}{2} A_{\Lambda\Sigma} n^\Lambda n^\Sigma, \quad (n^0, n^a) = \left(\frac{p}{k}, \frac{p^a}{k} \right) \in \mathbb{Z}/k.$$

This transition function generates a discontinuity in the Darboux coordinates across the image of the BPS ray ℓ_γ under the same $SL(2, \mathbb{Z})$ transformation, namely a meridian $\ell_{k,p,\hat{\gamma}} \subset \mathbb{P}^1$ joining the two roots of the equation $\xi^0(t) = n^0$ where (5.7) has essential singularities.

One can perform two consistency checks on the above result. First, by evaluating the Penrose transform of (5.7)

$$(5.9) \quad \int_{\ell_{k,p,\hat{\gamma}}} \frac{dt}{t} H_{k,p,\hat{\gamma}}(\xi(t), \tilde{\xi}(t), \alpha(t))$$

in the small string coupling limit, which justifies the use of the saddle point approximation, it was checked [19] that (5.7) produces corrections with the correct semi-classical action known from the analysis of instanton solutions in $N = 2$ supergravity [92]. The second check is to verify that (5.7) is consistent with large gauge transformations (2.4) and monodromy around the large volume point. This requires that the set of transition functions should be mapped to itself under the corresponding actions on twistor space. This is indeed the case, as shown in [19] and further clarified in [93]. The problem of going beyond the linearized regime and elevating the infinitesimal contact transformations generated by (5.7) to their finite counterparts was addressed in [67, 93].

5.2.2. Relation to topological strings. A general prediction of S-duality is that the partition function of a single NS5-brane ($k = 1$) on a CY threefold X should be governed by the (ordinary) DT invariants of X with $p^0 = 1$, which are in turn related to higher genus Gromov-Witten invariants [94], and therefore to topological strings [84, 95, 96]. A precise relation of this sort follows immediately from (5.7). Indeed, the formal sum over all charges

$$(5.10) \quad H_{\text{NS5}}^{(1)}(\xi, \tilde{\xi}, \alpha) = \sum_{p,p^a,q_\Lambda} H_{1,p,\hat{\gamma}}(\xi, \tilde{\xi}, \alpha)$$

is invariant under Heisenberg transformations (2.16). Thus it can be cast in the form (5.4) of a non-Gaussian theta series [19]

$$(5.11) \quad H_{\text{NS5}}^{(1)}(\xi, \tilde{\xi}, \alpha) = \frac{1}{4\pi^2} \sum_{n^\Lambda} \mathcal{H}_{\text{NS5}}^{(1)}(\xi^\Lambda - n^\Lambda) \mathbf{E}\left(\alpha + n^\Lambda(\tilde{\xi}_\Lambda - \phi_\Lambda)\right).$$

Remarkably, the wave-function $\mathcal{H}_{\text{NS5}}^{(1)}$ turns to be proportional (up to ξ^0 -dependent factors involving the Mac-Mahon function) to the topological A-model string amplitude on \hat{X} in the real polarization,

$$(5.12) \quad \mathcal{H}_{\text{NS5}}^{(1)}(\xi^\Lambda) \sim \Psi_{\mathbb{R}}^{\text{top}}(\xi^\Lambda).$$

This relation indicates that the proper habitat for the topological string amplitude is the space $H^1(\mathcal{Z}_D, O(2))$ parametrizing deformations of the D-instanton corrected twistor space.

5.3. NS5-branes and quantization of cluster varieties. We conclude our discussion of quantum corrected HM moduli spaces by collecting several indications pointing to deep relations with quantum integrable systems. The general relation to integrable systems is of course built in from the fact that \mathcal{M}'_H , the HK manifold dual to the perturbative \mathcal{M}_H by the QK/HK correspondence of §3.3, is a HK manifold fibered by algebraic tori, and therefore provides an example of a complex integrable system.⁶

More interestingly, this relation seems to extend in the presence of D-instanton corrections, in view of the fact [48, 98] that the integral equations (3.13) governing the D-instanton corrections to the metric on \mathcal{M}_H coincide with equations of Thermodynamic Bethe Ansatz (TBA) typically describing the spectrum of two-dimensional integrable models [99]. Moreover, the D-instanton

⁶See [97] for a recent analysis of the relation between these integrable systems and wall-crossing in Donaldson-Thomas theory.

corrected contact potential (3.16) is identified with the free energy of the system associated to this TBA, and the corresponding S-matrix can be shown to satisfy all axioms of integrability [98].

The TBA equations can often be rewritten as a system of discrete equations, known as Y-system, corresponding to gluing conditions (3.11) of our framework. In particular, the same dilogarithm identities which ensure consistency with wall-crossing are well-known to arise as a consequence of periodicity of Y-systems [100]. Their mathematical underpinning is Fomin and Zelevinsky's theory of *cluster algebras*⁷ [102], and its generalization to *cluster varieties*, developed by Fock and Goncharov [103]. In particular, a structure very similar to NS5-instanton corrections arises when quantizing *cluster \mathcal{A} -varieties* [104]. An N -dimensional cluster \mathcal{A} -variety is a collection of complex tori $(\mathbb{C}^\times)^N$ glued together into a symplectic algebraic variety using cluster transformations. Geometric quantization produces a pre-quantum vector bundle $\mathcal{V}_{\hbar} \rightarrow \mathcal{A}$, depending on a rational parameter $\hbar = s/r$, where s is the first Chern class and r encodes the rank of the bundle. For $\hbar = 1$, the corresponding line bundle \mathcal{V}_1 is described by exactly the same gluing conditions (3.11) as for the holomorphic bundle $\mathcal{L}_{\mathcal{Z}}$ entering the QK/HK-correspondence. Thus we may view the hyperholomorphic bundle \mathcal{L} as a pre-quantum line bundle for the geometric quantization of \mathcal{M}' (as also observed in [59]).

To explain the relation with NS5-branes, let us for simplicity restrict to the case of a two-dimensional cluster \mathcal{A} -variety, with local coordinates $(a, b) \in \mathbb{C}^\times \times \mathbb{C}^\times$. Sections of \mathcal{V}_1 are multivalued functions $F(a, b)$ on $\mathbb{C}^\times \times \mathbb{C}^\times$, and by a choice of polarization one may restrict to holomorphic 'wave functions' $\Psi(a)$ on \mathbb{C}^\times . Fock and Goncharov implement this restriction explicitly via Fourier expansion in one of the variables [104]:

$$(5.13) \quad F(a, b) = \sum_{n \in \mathbb{Z}} \Psi(\log a - 2\pi i n) \exp \left[-\frac{\log a \log b}{4\pi i} \right] b^n.$$

Identifying $a = e^{2\pi i \xi}$ and $b = e^{2\pi i \tilde{\xi}}$, up to the factor $e^{-i\pi \tilde{\alpha}}$, this takes the same form as the NS5-partition function (5.11). Thus the twistor space partition function of a single NS5-brane ($k = 1$) may be thought of as a section of a pre-quantum line bundle over an \mathcal{A} -cluster variety. More generally, for $\hbar \neq 1$, sections of the vector bundle \mathcal{V}_{\hbar} can be represented by vector-valued functions $\Psi_{\hbar^{-1}, \ell}(\xi)$ via a generalization of the expansion (5.13). In the case $s = 1$ and $r = 1/\hbar \in \mathbb{Z}_+$, this becomes

$$(5.14) \quad F_{\hbar}(\xi, \tilde{\xi}) = \sum_{\ell \in \mathbb{Z}/(\mathbb{Z}/\hbar)^{-1}} \sum_{n \in \mathbb{Z} + \ell/\hbar} \Psi_{\hbar^{-1}, \ell}(\xi - n) \mathbf{E} \left((2n\tilde{\xi} - \xi\tilde{\xi})/2\hbar \right),$$

which is recognized as the non-abelian Fourier expansion corresponding to k NS5-branes [105, 19], provided we identify $\hbar = 1/k$ and set the characteristics (θ, ϕ) to zero. This identification of the NS5-brane charge k with the inverse of a quantization parameter \hbar will be confirmed from a different perspective in §5.4 below.

Moreover, as explained in [104], the gluing conditions for wave functions $\Psi_{\hbar^{-1}, \ell}$ involve a convolution with the quantum dilogarithm. Since the latter also governs the wall-crossing behavior of motivic DT invariants $\Omega_{\hbar^{-1}}(\gamma)$ [47], this suggests that NS5-instantons should be controlled by the same. On the other hand, S-duality implies that (p, k) 5-brane instantons are determined by the ordinary DT invariants $\Omega(\gamma)$ suggesting an intriguing relation between ordinary and motivic DT invariants which it would be very interesting to spell out. Finally, while the analogy with quantum

⁷The relation between wall-crossing and cluster transformations was noted already in [47], and further analyzed in the context of $\mathcal{N} = 2$ gauge theories [101] and $\mathcal{N} = 2$ supergravity [52].

cluster varieties seems to open the way to the construction of a natural section of $H^1(\mathcal{Z}, O(2))$, hence a linear correction to the D-instanton corrected metric on \mathcal{M}_H , it is as yet unclear how this could be lifted to a full non-linear deformation, consistent with S-duality and regularity.

5.4. Universal hypermultiplet and free fermions. Another interesting relation to quantum integrable systems can be seen in the special case of type IIA string theory compactified on a rigid CY threefold with $h^{2,1}(X) = 0$. The corresponding HM moduli space is a 4-dimensional QK space, often known as ‘the universal hypermultiplet’ (although it is hardly universal).

In the absence of NS5-brane corrections, \mathcal{M}_H has a continuous isometry corresponding to shifts of the NS-axion σ . Four-dimensional QK spaces with an isometry are known to be described by solutions of the Toda equation [106]

$$(5.15) \quad \partial_z \partial_{\bar{z}} T + \partial_\rho^2 e^T = 0,$$

which appears as the lowest equation of the dispersionless (i.e. classical) limit of Toda integrable hierarchy. In fact, one can show that the twistor framework is equivalent to the Lax formalism for this hierarchy [107]. In particular, the Darboux coordinates ξ and $\tilde{\xi}$ coincide with the two Lax operators, whereas the gluing conditions are identified with the so called string equations. To understand the role of the Darboux coordinate α , consider the perturbative HM moduli space (2.8) with $F = -\frac{i}{4} X^2$. In this case the corresponding solution of (5.15) takes particularly simple form [108, 109]

$$(5.16) \quad T = \log(\rho + c).$$

In [110] it was shown that the Darboux coordinate α given in (2.15) is related in a simple way to the WKB phase of the quasiclassical Baker-Akhiezer function Ψ associated with the solution (5.16). This, together with the known wave-function property of the Baker-Akhiezer function, suggests that Ψ might be related to NS5-brane effects, which typically have an exponential dependence on α , see (5.7).

Indeed, in [111] it was suggested that, for compactifications on rigid CY, NS5-brane instanton corrections to the contact structure on the twistor space are generated by the following holomorphic function

$$(5.17) \quad H_k^{\text{NS5}} \sim (\xi - 2i\tilde{\xi})^{8\pi ck} e^{-\pi i k \tilde{\alpha} - \pi k (\frac{1}{4}\xi^2 + \tilde{\xi}^2)},$$

which is the type IIA counterpart of the type IIB description based on (5.7). One can easily verify that H_k^{NS5} is proportional to the Baker-Akhiezer function Ψ provided one identifies the NS5-brane charge with the inverse quantization parameter of the integrable hierarchy [110]

$$(5.18) \quad \hbar^{-1} = 8\pi k.$$

This nicely agrees with the relation found below (5.14), up to normalization.

Furthermore, this 4-dimensional example hints for a possible relation to free fermions. Indeed, the solution of Toda hierarchy based on (5.16) is known to describe non-critical $c = 1$ string theory in a non-trivial tachyon background compactified on a circle of the self-dual radius [112]. In turn, this theory is described by Matrix Quantum Mechanics, where integrability is manifest, and this matrix model is known to reduce to a system of free fermions. In this description, the Baker-Akhiezer function, shown above to encode NS5-brane effects, is just the one-fermion wave function. It would be exciting to use this idea to compute the exact quantum HM moduli

space for rigid CY threefolds, and see whether S-duality or one of its extensions is indeed realized [105, 113, 114, 115].

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