

Towards large genus asymptotics of intersection numbers on moduli spaces of curves

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Abstract

We explicitly compute the diverging factor in the large genus asymptotics of the Weil-Petersson volumes of the moduli spaces of n -pointed complex algebraic curves. Modulo a universal multiplicative constant we prove the existence of a complete asymptotic expansion of the Weil-Petersson volumes in the inverse powers of the genus with coefficients that are polynomials in n . This is done by analyzing various recursions for the more general intersection numbers of tautological classes, whose large genus asymptotic behavior is also extensively studied.

1 Introduction and statement of results

In this note, we study the asymptotic behavior of the Weil-Petersson volumes $V_{g,n}$ of the moduli spaces $\mathcal{M}_{g,n}$ of n -pointed complex algebraic curves of genus g as $g \rightarrow \infty$. Here

$$V_{g,n} = \frac{1}{(3g-3+n)!} \int_{\mathcal{M}_{g,n}} \omega_{g,n}^{3g-3+n},$$

where $\omega_{g,n}$ is the Weil-Petersson symplectic form on $\mathcal{M}_{g,n}$.

The following conjecture was made in [Z] on the basis of numerical data:

Conjecture 1.1. *For any fixed $n \geq 0$*

$$V_{g,n} = (2g-3+n)!(4\pi^2)^{2g-3+n} \frac{1}{\sqrt{g\pi}} \left(1 + \frac{c_n}{g} + O\left(\frac{1}{g^2}\right) \right)$$

as $g \rightarrow \infty$.

The objective of this paper is to prove the statements formulated below.

Theorem 1.2. *There exists a universal constant $C \in (0, \infty)$ such that for any given $k \geq 1, n \geq 0$,*

$$V_{g,n} = C \frac{(2g-3+n)!(4\pi^2)^{2g-3+n}}{\sqrt{g}} \left(1 + \frac{c_n^{(1)}}{g} + \dots + \frac{c_n^{(k)}}{g^k} + O\left(\frac{1}{g^{k+1}}\right) \right),$$

as $g \rightarrow \infty$. Each term $c_n^{(i)}$ in the asymptotic expansion is a polynomial in n of degree $2i$ with coefficients in $\mathbb{Q}[\pi^{-2}, \pi^2]$ that are effectively computable. Moreover, the leading term of $c_n^{(i)}$ is equal to $\frac{(-1)^i}{i!(2\pi^2)^i} n^{2i}$.

Remark 1.3. Note that Conjecture 1.1 claims that $C = \frac{1}{\sqrt{\pi}}$. Numerical data suggest that the coefficients of $c_n^{(i)}$ actually belong to $\mathbb{Q}[\pi^{-2}]$. For example,

$$c_n^{(1)} = -\frac{n^2}{2\pi^2} - \left(\frac{1}{4} - \frac{5}{2\pi^2}\right)n + \frac{7}{12} - \frac{17}{6\pi^2}.$$

Our method also implies that given $k \geq 1, n \geq 0$ we have

$$\frac{V_{g,n+1}}{8\pi^2 g V_{g,n}} = 1 + \frac{a_n^{(1)}}{g} + \dots + \frac{a_n^{(k)}}{g^k} + O\left(\frac{1}{g^{k+1}}\right), \quad (1.1)$$

$$\frac{V_{g-1,n+2}}{V_{g,n}} = 1 + \frac{b_n^{(1)}}{g} + \dots + \frac{b_n^{(k)}}{g^k} + O\left(\frac{1}{g^{k+1}}\right), \quad (1.2)$$

as $g \rightarrow \infty$, where the coefficients $a_n^{(i)}$ and $b_n^{(i)}$ can be explicitly computed. However, here we do this only for $a_n^{(1)}$ and $b_n^{(1)}$:

Theorem 1.4. For any fixed $n \geq 0$:

$$\frac{V_{g,n+1}}{8\pi^2 g V_{g,n}} = 1 + \left(\left(\frac{1}{2} - \frac{1}{\pi^2}\right)n - \frac{5}{4} + \frac{2}{\pi^2}\right) \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right), \quad (1.3)$$

$$\frac{V_{g-1,n+2}}{V_{g,n}} = 1 + \frac{3-2n}{\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right). \quad (1.4)$$

With the help of the identity $\frac{V_{g+1,n}}{V_{g,n}} = \frac{V_{g+1,n}}{V_{g,n+2}} \cdot \frac{V_{g,n+2}}{V_{g,n+1}} \cdot \frac{V_{g,n+1}}{V_{g,n}}$ this theorem immediately yields

Corollary 1.5. Let $n \geq 0$ be fixed, then

$$\frac{V_{g+1,n}}{V_{g,n}} = (4\pi^2)^2 (2g+n-1)(2g+n-2) \left(1 - \frac{1}{2g}\right) \cdot \left(1 + \frac{r_n^{(2)}}{g^2} + O\left(\frac{1}{g^3}\right)\right),$$

as $g \rightarrow \infty$.

Note that $r_n^{(2)} = 3/8 - c_n^{(1)}$ by Theorem 1.2, so that

$$r_n^{(2)} = \frac{n^2}{2\pi^2} + \left(\frac{1}{4} - \frac{5}{2\pi^2}\right)n - \frac{5}{24} + \frac{17}{6\pi^2}.$$

Remark 1.6. Since $\prod_{g=1}^{\infty} (1 + a_g)$ converges when $a_g = O(1/g^2)$, Corollary 1.5 easily implies that there exists

$$\lim_{g \rightarrow \infty} \frac{V_{g,n} \sqrt{g}}{(2g-3+n)! (4\pi^2)^{2g-3+n}} = C \in (0, \infty).$$

The proof of Theorem 1.2 is based on (1.1) and (1.2) (see Theorems 4.1, 4.2) and follows similar lines as well. The method we use also allows to calculate the error terms explicitly: even though we only do the calculation for the coefficient of $1/g$, the error terms of order of $1/g^s$ can be written in terms of values of the intersections of ψ classes on surfaces of genus at most s . However, this method does not provide any information about the exact value of C .

Analyzing the signs of the error terms of order $1/g^2$ in Theorem 1.4, we get

Corollary 1.7. *Given $n \geq 2$, there exists g_0 such that the sequence*

$$\left\{ \frac{V_{g-1, n+2}}{V_{g, n}} \right\}_{g \geq g_0}$$

is increasing. Similarly, for $n \geq 3$ there exists g_0 such that the sequence

$$\left\{ \frac{8\pi^2 g V_{g, n}}{V_{g, n+1}} \right\}_{g \geq g_0}$$

is increasing.

We also obtain somewhat weaker results when n varies as $g \rightarrow \infty$:

Theorem 1.8. *For any sequence $\{n(g)\}_{g=1}^\infty$ of non-negative integers with*

$$\lim_{g \rightarrow \infty} \frac{n(g)^2}{g} = 0,$$

we have

$$V_{g, n(g)} = \frac{C}{\sqrt{g}} (2g - 3 + n(g))! (4\pi^2)^{2g-3+n(g)} \left(1 + O\left(\frac{1 + n(g)^2}{g}\right) \right)$$

as $g \rightarrow \infty$.

In fact, we prove that

$$\frac{\sqrt{g} V_{g, n(g)}}{C (2g - 3 + n(g))! (4\pi^2)^{2g-3+n(g)}} = O\left(\frac{(g+n)!}{g! g^n}\right).$$

Notes and remarks.

- It may be instructive to compare Theorem 1.2 to the asymptotic formula for the Weil-Petersson volumes for fixed $g \geq 0$ and $n \rightarrow \infty$ (cf. [MZ], Theorem 6.1):

Theorem 1.9. *For any fixed $g \geq 0$*

$$V_{g, n} = n! C^n n^{(5g-7)/2} \left(c_g^{(0)} + \frac{c_g^{(1)}}{n} + \dots \right), \quad (1.5)$$

as $n \rightarrow \infty$, where $C = -\frac{2}{x_0 J_0'(x_0)}$, x_0 is the first positive zero of the Bessel function J_0 , and the coefficients $c_g^{(0)}, c_g^{(1)}, \dots$ are effectively computable.

- Penner [Pe] developed a different method for computing Weil-Petersson volumes by integrating the Weil-Petersson volume form over simplices in the cellular decomposition of the moduli space. In [Gr], this method of integration was used to prove that for a fixed $n > 0$ there are constants $C_1, C_2 > 0$ such that

$$C_1^g \cdot (2g)! < V_{g,n} < C_2^g \cdot (2g)!.$$

(This result was extended to the case of $n = 0$ by an algebro-geometric argument in [ST].) Note that these estimates do not give much information about the growth of $V_{g-1,n+2}/V_{g,n}$ and $V_{g,n}/V_{g,n+1}$ when $g \rightarrow \infty$.

- The estimates from [M3] imply that given $n \geq 0$ there exists $m > 0$ such that

$$g^{-m} \leq \frac{V_{g,n}}{(2g-3+n)!(4\pi^2)^{2g-3+n}} \leq g^m. \quad (1.6)$$

In general,

$$\lim_{g+n \rightarrow \infty} \frac{\log(V_{g,n})}{(2g+n)\log(2g+n)} = 1,$$

but understanding the asymptotics of $V_{g,n}$ for arbitrary g, n seems to be more complicated.

2 Relations between intersection numbers

To begin with, let us recall some well-known facts about tautological classes on $\overline{\mathcal{M}}_{g,n}$ and their intersections. For $n > 0$, there are n tautological line bundles \mathcal{L}_i on $\overline{\mathcal{M}}_{g,n}$ whose fiber at the point $(C, x_1, \dots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line to C at x_i , and we put $\psi_i = c_1(\mathcal{L}_i) \in H_2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ (cf. e.g. [HM] or [AC]). **Notation.** For $d = (d_1, \dots, d_n)$ with $d_i \in \mathbb{Z}_{\geq 0}$ put $|d| = d_1 + \dots + d_n$ and, assuming $|d| \leq 3g - 3 + n$, put $d_0 = 3g - 3 + n - |d|$. Define

$$\begin{aligned} [\tau_{d_1} \dots \tau_{d_n}]_{g,n} &= \left[\prod_{i=1}^n \tau_{d_i} \right]_{g,n} = \frac{\prod_{i=1}^n 2^{2d_i} (2d_i + 1)!!}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \omega_{g,n}^{d_0} \\ &= \prod_{i=1}^n 2^{2d_i} (2d_i + 1)!! \frac{(2\pi^2)^{d_0}}{d_0!} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n} \kappa_1^{d_0}, \end{aligned}$$

where $\kappa_1 = \frac{[\omega_{g,n}]}{2\pi^2}$ is the first Mumford class on $\overline{\mathcal{M}}_{g,n}$. According to [M1], for $L = (L_1, \dots, L_n)$ the Weil-Petersson volume of the moduli space of hyperbolic surfaces of genus g with $n > 0$ geodesic boundary components of lengths $2L_1, \dots, 2L_n$ can be written as

$$V_{g,n}(2L) = \sum_{\substack{d_1, \dots, d_n \\ |d| \leq 3g-3+n}} \left[\prod_{i=1}^n \tau_{d_i} \right]_{g,n} \frac{L_1^{2d_1}}{(2d_1 + 1)!} \dots \frac{L_n^{2d_n}}{(2d_n + 1)!}. \quad (2.1)$$

Recursive formulas. The following recursions for the intersection numbers $[\tau_{d_1} \dots \tau_{d_n}]_{g,n}$ hold:

$$(Ia) \quad \left[\tau_0 \tau_1 \prod_{i=1}^n \tau_{d_i} \right]_{g,n+2} = \left[\tau_0^4 \prod_{i=1}^n \tau_{d_i} \right]_{g-1,n+4} + \\ + 6 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1, \dots, n\}}} \left[\tau_0^2 \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+2} \cdot \left[\tau_0^2 \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+2} .$$

$$(Ib) \quad \left[\tau_0^2 \tau_{l+1} \prod_{i=1}^n \tau_{d_i} \right]_{g,n+3} = \left[\tau_0^4 \tau_l \prod_{i=1}^n \tau_{d_i} \right]_{g-1,n+5} + \\ + 8 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1, \dots, n\}}} \left[\tau_0^2 \tau_l \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+3} \cdot \left[\tau_0^2 \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+2} + \\ + 4 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1, \dots, n\}}} \left[\tau_0 \tau_l \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+2} \cdot \left[\tau_0^3 \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+3} .$$

$$(II) \quad (2g-2+n) \left[\prod_{i=1}^n \tau_{d_i} \right]_{g,n} = \frac{1}{2} \sum_{l=1}^{3g-2+n} \frac{(-1)^{l-1} l \pi^{2l-2}}{(2l+1)!} \left[\tau_l \prod_{i=1}^n \tau_{d_i} \right]_{g,n+1} .$$

(III) Put $a_i = (1 - 2^{1-2i}) \zeta(2i)$, where ζ is the Riemann zeta function and $i \in \mathbb{Z}_{\geq 0}$. Then

$$[\tau_{d_1} \dots \tau_{d_n}]_{g,n} = A_d + B_d + C_d,$$

where

$$A_d = 8 \sum_{j=2}^n \sum_{l=0}^{d_0} (2d_j + 1) a_l \left[\tau_{d_1+d_j+l-1} \prod_{i \neq 1, j} \tau_{d_i} \right]_{g,n-1}, \quad (2.2)$$

$$B_d = 16 \sum_{l=0}^{d_0} \sum_{\substack{k_1+k_2= \\ =l+d_1-2}} a_l \left[\tau_{k_1} \tau_{k_2} \prod_{i \neq 1} \tau_{d_i} \right]_{g-1,n+1}, \quad (2.3)$$

$$C_d = 16 \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{1, \dots, n\}}} \sum_{l=0}^{d_0} \sum_{\substack{k_1+k_2= \\ =l+d_1-2}} a_l \left[\tau_{k_1} \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+1} \cdot \left[\tau_{k_2} \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+1} . \quad (2.4)$$

Basic properties of the sequence $\{a_i = (1 - 2^{1-2i}) \zeta(2i)\}$. It is easy to check that for $i \geq 1$

$$a_i = \frac{1}{(2i-1)!} \int_0^\infty \frac{t^{2i-1}}{1+e^t} dt$$

and

$$a_{i+1} - a_i = \int_0^\infty \frac{1}{(1+e^t)^2} \left(\frac{t^{2i+1}}{(2i+1)!} + \frac{t^{2i}}{2i!} \right) dt. \quad (2.5)$$

Lemma 2.1. *The sequence $\{a_i\}_{i=1}^\infty$ is increasing. Moreover,*

(i)

$$\lim_{i \rightarrow \infty} a_i = 1, \quad \sum_{i=0}^\infty (a_{i+1} - a_i) = \frac{1}{2},$$

(ii) $a_{i+1} - a_i$ has the order of $1/2^{2i}$, i.e., there exist $C > 0$ such that

$$\frac{1}{C \cdot 2^{2i}} < a_{i+1} - a_i < \frac{C}{2^{2i}}, \quad (2.6)$$

(iii)

$$\sum_{i=0}^\infty i(a_{i+1} - a_i) = \frac{1}{4}, \quad (2.7)$$

(iv) for $j \in \mathbb{Z}, j \geq 2$, the sum

$$\sum_{i=0}^\infty i^j (a_{i+1} - a_i)$$

is a polynomial in π^2 of degree $[j/2]$ with rational coefficients.

Proof. Both (i) and (ii) easily follow from the definition of a_i and (2.5). As for (iii), let

$$S_1 = \sum_{i=0}^\infty \int_0^\infty \frac{1}{(1+e^t)^2} \cdot \frac{t^{2i+1}}{(2i+1)!} dt$$

and

$$S_2 = \int_0^\infty \frac{t e^t}{(1+e^t)^2} dt.$$

We have

$$S_1 = \int_0^\infty \frac{(e^t - e^{-t}) dt}{2(1+e^t)^2} = -\frac{1}{2} + \log 2, \quad S_2 = \log 2,$$

so that from (2.5),

$$2 \sum_{i=1}^\infty i(a_{i+1} - a_i) + S_1 = S_2,$$

which implies (2.7).

We will prove (iv) by induction in j . The base case $j = 1$ being checked in (iii), we observe that

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{i^j t^{2i+1}}{(2i+1)!} &= (tDt^{-1})^j (\sinh t) = \sum_{l=0}^j t^l (a_{1,j}^{(l)} \cosh t + b_{1,j}^{(l)} \sinh t), \\ \sum_{i=0}^{\infty} \frac{i^j t^{2i}}{(2i)!} &= D^j (\cosh t) = \sum_{l=0}^j t^l (a_{2,j}^{(l)} \cosh t + b_{2,j}^{(l)} \sinh t),\end{aligned}$$

where $D = \frac{t}{2} \cdot \frac{d}{dt}$, and the coefficients $a_{*,j}^{(l)}, b_{*,j}^{(l)}$ are rational numbers. A standard computation shows that

$$\begin{aligned}\int_0^{\infty} \frac{t^l e^{-t}}{(1+e^t)^2} dt &= l!(1 - 2(1 - 2^{-l})\zeta(l+1) + (1 - 2^{1-l})\zeta(l)), \\ \int_0^{\infty} \frac{t^l e^t}{(1+e^t)^2} dt &= l!(1 - 2^{1-l})\zeta(l).\end{aligned}$$

From here we see that the values of the zeta function at odd $l = 2k + 1$ do not contribute to the sum in (iv) if and only if

$$(2k+1) \cdot (a_{1,j}^{(2k+1)} + a_{2,j}^{(2k+1)}) - a_{1,j}^{(2k)} - a_{2,j}^{(2k)} + b_{1,j}^{(2k)} + b_{2,j}^{(2k)} = 0. \quad (2.8)$$

The condition (2.8) is not hard to verify by induction from j to $j + 1$. Thus, only the values of the zeta function at even l contribute to the sum. \square

References.

- The relationship between the Weil-Petersson volumes and the intersection numbers of ψ -classes on $\mathcal{M}_{g,n}$ is discussed in [W] and [AC]. An explicit formula for the volumes in terms of the intersections of ψ -classes was given in [KMZ], cf. also [MZ].
- Recursion **(Ia)** is a special case of Proposition 3.3 in [LX1]. Similarly, recursion **(Ib)** is a simple corollary of Propositions 3.3 and 3.4 in [LX1].
- For different proofs of **(II)** see [DN] and [LX1]. In terms of the volume polynomial $V_{g,n}(L)$, recursion **(II)** can be written as follows ([DN]):

$$\frac{\partial V_{g,n+1}}{\partial L_{n+1}}(L_1, \dots, L_n, 2\pi\sqrt{-1}) = 2\pi\sqrt{-1}(2g - 2 + n)V_{g,n}(L_1, \dots, L_n).$$

When $n = 0$,

$$V_{g,1}(2\pi\sqrt{-1}) = 0,$$

and

$$\frac{\partial V_{g,1}}{\partial L}(2\pi\sqrt{-1}) = 2\pi\sqrt{-1}(2g - 2)V_{g,0}. \quad (2.9)$$

- For a proof of **(III)** see [M2]; note that **(III)** applies only when $n > 0$ (in case of $n = 0$, formula (2.9) gives the necessary estimates on the growth of $V_{g,0}$). In fact, **(III)** can be interpreted as a recursive formula for the volumes of moduli spaces $\mathcal{M}_{g,n}(L)$ of hyperbolic surfaces of genus g with n geodesic boundary components of lengths L_1, \dots, L_n that describes a removal of a pair of pants on a surface containing at least one of its boundary components. Although **(III)** is written here in purely combinatorial terms, it is related to the topology of different pant decompositions of a surface, cf. also [Mc] and [LX2].
- When $d_1 + \dots + d_n = 3g - 3 + n$, recursion **(III)** reduces to the Virasoro constraints for the intersection numbers of ψ_i -classes predicted by Witten [W], cf. also [MS]. For different proofs and discussions of these relations see, [Ko], [OP], [M1], [KL], and [EO].
- In this paper, we are mainly interested in the intersection numbers of κ_1 and ψ_i classes. For generalizations of **(III)** to the case of intersection numbers involving higher Mumford's κ -classes see [LX1], [E] and [Ka].

3 Asymptotics of intersection numbers when n is fixed

In this section, we prove Theorem 1.4. This theorem implies that there exists $C \in (0, \infty)$ such that

$$\lim_{g \rightarrow \infty} \frac{V_{g,n} \sqrt{g}}{(2g - 3 + n)! (4\pi^2)^{2g-3+n}} = C. \quad (3.1)$$

This result will be generalized in §4.

We recall that for any $n \geq 0$ the results obtained in [M3] yield

$$\frac{V_{g,n}}{8\pi^2 g V_{g-1,n+1}} = 1 + O\left(\frac{1}{g}\right) \quad (3.2)$$

and

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + O\left(\frac{1}{g}\right) \quad (3.3)$$

as $g \rightarrow \infty$. The main ingredient of the proof is the following property of the intersection numbers:

$$[\tau_{d_1} \dots \tau_{d_n}]_{g,n} \leq [\tau_0^n]_{g,n} = V_{g,n} \quad (3.4)$$

for any $d = (d_1, \dots, d_n)$. Moreover,

$$\frac{[\tau_{d_1} \dots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1 + O\left(\frac{1}{g}\right), \quad (3.5)$$

as $g \rightarrow \infty$.

Remark 3.1. The same result holds if d_1, \dots, d_n grow slowly with g in such a way that

$$\frac{d_1 \dots d_n}{g} \rightarrow 0$$

as $g \rightarrow \infty$. In particular, (3.5) holds if $d_i = O(\log g)$ for each $i = 1, \dots, n$.

A stronger statement is formulated below:

Theorem 3.2. *Let $k, n \geq 1$, then*

(i)

$$\frac{[\tau_k \tau_0^{n-1}]_{g,n}}{V_{g,n}} = 1 + \frac{e_{n,k}^{(1)}}{g} + O\left(\frac{1}{g^2}\right),$$

as $g \rightarrow \infty$, where

$$e_{n,k}^{(1)} = -\frac{k^2 + (n-5/2)k - n/2 + 3/2}{\pi^2}.$$

(ii)

$$\begin{aligned} 8n[\tau_{k-1} \tau_0^{n-2}]_{g,n-1} &< [\tau_k \tau_0^{n-1}]_{g,n} - [\tau_{k+1} \tau_0^{n-1}]_{g,n} \leq \\ &\leq 16((n+k)V_{g,n-1} + kV_{g-1,n+1}). \end{aligned}$$

Remark 3.3. In general, one can show that for $k \geq 1$

$$\frac{[\tau_{k+1} \tau_{j_1} \dots \tau_{j_s} \tau_0^{n-1-s}]_{g,n}}{[\tau_k \tau_{j_1} \dots \tau_{j_s} \tau_0^{n-1-s}]_{g,n}} = 1 - \frac{2(k+j_1+\dots+j_s)+n-3/2}{\pi^2 g} + O\left(\frac{1}{g^2}\right).$$

Proof of Theorem 3.2. We will need the following simple fact. Let $\{r_i\}_{i=1}^{\infty}$ be a sequence of real numbers and $\{k_g\}_{g=1}^{\infty}$ be an increasing sequence of positive integers. Assume that for all g , and i we have $0 \leq c_{i,g} \leq c_i$, and $\lim_{g \rightarrow \infty} c_{i,g} = c_i$. If $\sum_{i=1}^{\infty} |c_i r_i| < \infty$, then

$$\lim_{g \rightarrow \infty} \sum_{i=1}^{k_g} r_i c_{i,g} = \sum_{i=1}^{\infty} r_i c_i. \quad (3.6)$$

To prove part (i) of the Theorem, it is sufficient to show that

$$\frac{[\tau_1 \tau_0^{n-1}]_{g,n}}{[\tau_0^n]_{g,n}} = 1 - \frac{n}{2\pi^2 g} + O\left(\frac{1}{g^2}\right),$$

and for $k \geq 1$

$$\frac{[\tau_{k+1} \tau_0^{n-1}]_{g,n}}{[\tau_k \tau_0^{n-1}]_{g,n}} = 1 - \frac{2k+n-3/2}{\pi^2 g} + O\left(\frac{1}{g^2}\right). \quad (3.7)$$

Here we use the recursive formula **(III)** to expand the difference $[\tau_k \tau_0^{n-1}]_{g,n} - [\tau_{k+1} \tau_0^{n-1}]_{g,n}$ in terms of the intersection numbers on $\overline{\mathcal{M}}_{g-1,n+1}$, $\overline{\mathcal{M}}_{g,n-1}$ and $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2}$. For the sake of brevity let us put

$$[\tau_k \tau_0^{n-1}]_{g,n} - [\tau_{k+1} \tau_0^{n-1}]_{g,n} = \tilde{A}_{k,g,n} + \tilde{B}_{k,g,n} + \tilde{C}_{k,g,n}, \quad (3.8)$$

where $\tilde{A}_{k,g,n}$, $\tilde{B}_{k,g,n}$, and $\tilde{C}_{k,g,n}$ are the terms corresponding to (2.2), (2.3) and (2.4) respectively. We will evaluate these terms separately.

1. Contribution from (2.2). By **(III)**, the numbers

$$[\tau_{k-1} \tau_0^{n-2}]_{g,n-1}, \dots, [\tau_{3g+n-4} \tau_0^{n-2}]_{g,n-1}$$

contribute to $[\tau_k \tau_0^{n-1}]_{g,n}$ and $[\tau_{k+1} \tau_0^{n-1}]_{g,n}$. In fact, it is easy to check that

$$\tilde{A}_{k,g,n} = 8 a_0 [\tau_{k-1} \tau_0^{n-2}]_{g,n-1} + 8 \sum_{i=1}^{3g-3+n-k} (a_{i+1} - a_i) [\tau_{k-1+i} \tau_0^{n-2}]_{g,n-1}.$$

The term $[\tau_{k-1} \tau_0^{n-2}]_{g,n-1}$ is non-zero only when $k \geq 1$. In order to calculate the asymptotic behavior of $\tilde{A}_{k,g,n}/V_{g,n-1}$ we simply apply (3.5) and (3.6). In view of Lemma 2.1, we get that

$$\lim_{g \rightarrow \infty} \frac{\tilde{A}_{k,g,n}}{V_{g,n-1}} = 8(n-1) \left(a_0 \delta + \sum_{i=0}^{\infty} (a_{i+1} - a_i) \right) = 8(n-1)(1/2 \delta + 1/2),$$

where $\delta = 0$ when $k = 0$, and otherwise $\delta = 1$. Thus

$$\frac{\tilde{A}_{k,g,n}}{V_{g,n-1}} = 8(n-1)(1/2 \delta + 1/2) + O\left(\frac{1}{g}\right) \quad (3.9)$$

as $g \rightarrow \infty$. On the other hand, Lemma 2.1 also implies that

$$\tilde{A}_{k,g,n} \leq 8(n-1)V_{g,n-1}. \quad (3.10)$$

2. Contribution from (2.3). Similarly, by **(III)**, the numbers $[\tau_i \tau_j \tau_0^{n-1}]_{g-1,n+1}$ contribute to $[\tau_k \tau_0^{n-1}]_{g,n}$ (resp. to $[\tau_{k+1} \tau_0^{n-1}]_{g,n}$) whenever $i + j \geq k - 2$ (resp. $i + j \geq k - 1$). To simplify the notation, let

$$T_{m,g,n} = \sum_{i+j=m} [\tau_i \tau_j \tau_0^{n-1}]_{g-1,n+1}$$

(we assume $T_{m,g,n} = 0$ for $m < 0$). Then

$$\tilde{B}_{k,g,n} = 16 (a_0 T_{k-2,g,n} + (a_1 - a_0) T_{k-1,g,n} + \dots + (a_{i+1} - a_i) T_{k-1+i,g,n} + \dots).$$

Note that by (3.5) as $g \rightarrow \infty$,

$$\lim_{g \rightarrow \infty} \frac{T_{m,g,n}}{V_{g-1,n+1}} = m + 1.$$

Now since $a_0 = 1/2$, Lemma 2.1 (iii), together with (3.5) and (3.6), implies for $k > 1$

$$\begin{aligned} \lim_{g \rightarrow \infty} \frac{\tilde{B}_{k,g,n}}{V_{g-1,n+1}} &= 16 \left(a_0(k-1) + \sum_{i=0}^{\infty} (i+k)(a_{i+1} - a_i) \right) = \\ &= 16 \left(\frac{k-1}{2} + \frac{k}{2} + 1/4 \right) = 16(k-1 + 1/4), \quad (3.11) \\ \lim_{g \rightarrow \infty} \frac{\tilde{B}_{1,g,n}}{V_{g-1,n+1}} &= 16 \sum_{i=0}^{\infty} (i+1)(a_{i+1} - a_i) = 4, \end{aligned}$$

and we also have

$$\tilde{B}_{k,g,n} \leq 16kV_{g-1,n+1}. \quad (3.12)$$

3. Contribution from (2.4). By the results obtained in [M3]

$$\sum_{\substack{I \amalg J = \{2, \dots, n\} \\ 0 \leq g' \leq g}} \frac{V_{g',|I|+1} \cdot V_{g-g',|J|+1}}{V_{g,n}} = O\left(\frac{1}{g^2}\right).$$

Put

$$S_{k_1, k_2, g, n} = \sum_{\substack{I \amalg J = \{2, \dots, n\} \\ 0 \leq g' \leq g}} \left[\tau_{k_1} \prod_{i \in I} \tau_i \right]_{g', |I|+1} \cdot \left[\tau_{k_2} \prod_{i \in J} \tau_i \right]_{g-g', |J|+1}.$$

Note that by (3.4) and recursion (Ia),

$$S_{k_1, k_2, g, n} \leq \sum_{\substack{I \amalg J = \{2, \dots, n\} \\ 0 \leq g' \leq g}} V_{g', |I|+1} \cdot V_{g-g', |J|+1} \leq V_{g, n-1}$$

Therefore, the contribution from the term (2.4) in (III) satisfies

$$\begin{aligned} 0 \leq \tilde{C}_{k,g,n} &\leq 16 \sum_{i=0}^{\infty} (a_i - a_{i+1}) \sum_{k_1+k_2=i+k} S_{k_1, k_2, g, n} \leq \\ &\leq 16 \sum_{i=0}^{\infty} (i+k)(a_i - a_{i+1}) V_{g, n-1} \leq 16(1/4 + k/2) V_{g, n-1}. \end{aligned}$$

Using (3.6), as in the cases **1** and **2** considered above, we see that the contribution from the term (2.4) in (III) becomes small as $g \rightarrow \infty$:

$$\frac{\tilde{C}_{k,g,n}}{V_{g,n}} = O\left(\frac{1}{g^2}\right), \quad (3.13)$$

and

$$\tilde{C}_{k,g,n} \leq 16(1+k)V_{g, n-1}. \quad (3.14)$$

Now, in view of (3.2), (3.3) and (3.5), equations (3.9), (3.11) and (3.13) imply that for $k \geq 1$

$$1 - \frac{[\tau_{k+1} \tau_0^{n-1}]_{g,n}}{[\tau_k \tau_0^{n-1}]_{g,n}} = \frac{2k + n - 3/2}{\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right)$$

and

$$1 - \frac{[\tau_1 \tau_0^{n-1}]_{g,n}}{[\tau_0^n]_{g,n}} = \frac{n}{2\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right).$$

Finally, the inequalities (3.10), (3.12) and (3.14) imply part (ii) of the Theorem. \square

Proof of Theorem 1.4. We start with proving (1.3). From (II),

$$\frac{2(2g - 2 + n)V_{g,n}}{V_{g,n+1}} = \sum_{l=1}^{3g-2+n} \frac{(-1)^{l-1} l \pi^{2l-2}}{(2l+1)!} \cdot \frac{[\tau_l, \tau_0^n]_{g,n+1}}{V_{g,n+1}}.$$

Differentiating $t^{-1} \sin t$ and putting $t = \pi$ we get

$$\sum_{l=1}^{\infty} \frac{(-1)^{l-1} l \pi^{2l-2}}{(2l+1)!} = \frac{1}{2\pi^2}.$$

Now we can use (3.6), and Theorem 3.2 in order to calculate the error term in $(2g - 2 + n)V_{g,n}/V_{g,n+1} - 1/4\pi^2$. Clearly,

$$\frac{(2g - 2 + n)V_{g,n}}{V_{g,n+1}} - \frac{1}{4\pi^2} = \sum_{l=1}^{3g-2+n} \frac{(-1)^{l-1} l \pi^{2l-2}}{(2l+1)!} \cdot \left(\frac{[\tau_l, \tau_0^n]_{g,n+1}}{V_{g,n+1}} - 1 \right). \quad (3.15)$$

Then Theorem 3.2, (i) implies that

$$\begin{aligned} \frac{(2g - 2 + n)V_{g,n}}{V_{g,n+1}} - \frac{1}{4\pi^2} &= \\ &= - \sum_{l=1}^{\infty} \frac{(-1)^{l-1} (l^2 + (n - 3/2)l - n + 1) l \pi^{2l-2}}{2(2l+1)! \pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right). \end{aligned}$$

On the other hand,

$$\sum_{l=1}^{\infty} \frac{(-1)^{l-1} l (l^2 + (n - 3/2)l - n/2 + 1) \pi^{2l}}{(2l+1)!} = -\frac{4n + \pi^2 - 8}{8\pi^2}.$$

Hence,

$$\frac{4\pi^2(2g - 2 + n)V_{g,n}}{V_{g,n+1}} = 1 + \frac{4n + \pi^2 - 8}{4\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right), \quad (3.16)$$

and

$$\frac{8\pi^2 g V_{g,n}}{V_{g,n+1}} = 1 + \left(\left(\frac{1}{\pi^2} - \frac{1}{2} \right) n + \frac{5}{4} - \frac{2}{\pi^2} \right) \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right).$$

We proceed with proving (1.4). First, we will check this estimate when $n \geq 2$, that is,

$$\frac{V_{g-1,n+4}}{V_{g,n+2}} = 1 - \frac{2n+1}{\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right).$$

From the recursion (1a) we get

$$\begin{aligned} \frac{V_{g-1,n+4}}{V_{g,n+2}} &= \frac{[\tau_1 \tau_0^{n+1}]_{g,n+2}}{V_{g,n+2}} \\ &\quad - \frac{6}{V_{g,n+2}} \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} V_{g_1,|I|+2} \cdot V_{g_2,|J|+2}. \end{aligned} \quad (3.17)$$

On the other hand, by Theorem 3.2 we have for $k = 1$

$$\frac{[\tau_1 \tau_0^{n+1}]_{g,n+2}}{[\tau_0^{n+2}]_{g,n+2}} = 1 - \frac{n+2}{2\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right),$$

and (3.3) implies

$$\begin{aligned} \frac{1}{V_{g,n+2}} \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} V_{g_1,|I|+2} \cdot V_{g_2,|J|+2} &= 2n \frac{V_{g,n+1}}{V_{g,n+2}} + O\left(\frac{1}{g^2}\right) \\ &= \frac{2n}{8\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right). \end{aligned}$$

Hence

$$\frac{V_{g-1,n+4}}{V_{g,n+2}} = 1 - \left(\frac{n+2}{2\pi^2} + \frac{12n}{8\pi^2}\right) \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right) = 1 - \frac{2n+1}{\pi^2 g} + O\left(\frac{1}{g^2}\right).$$

The remaining cases $n = 0, 1$ follow from (1.3) and (1.4) for $n \geq 2$. For instance, if $n = 1$

$$\frac{V_{g-1,3}}{V_{g,1}} = \frac{V_{g-1,4}}{V_{g,2}} \cdot \frac{V_{g,2}}{V_{g,1}} \cdot \frac{V_{g-1,3}}{V_{g-1,4}} = 1 + \frac{1}{\pi^2} \cdot \frac{1}{g} + O\left(\frac{1}{g^2}\right).$$

The case $n = 0$ can be treated similarly. \square

Theorem 1.4 immediately implies Corollary 1.5 about the asymptotic behavior of the ratio $V_{g+1,n}/V_{g,n}$ (see Introduction). An important consequence of Corollary 1.5, explained in Remark 1.6, is formula (3.1) announced at the beginning of this Section. As a byproduct of this statement we also get the following estimate that we will need later:

Lemma 3.4. *Fix $n_1, n_2, s \geq 0$. Then*

$$\sum_{\substack{g_1+g_2=g \\ 2g_i+n_i \geq s, i=1,2}} V_{g_1,n_1} \cdot V_{g_2,n_2} = O\left(\frac{V_{g,n_1+n_2}}{g^s}\right). \quad (3.18)$$

4 Error terms in the asymptotics expansions

In this section, we prove Theorem 1.2 using the following results:

Theorem 4.1. *We have the following asymptotic expansions as $g \rightarrow \infty$:*

- (i) *Given the integers $n, s \geq 1$, and $d = (d_1, \dots, d_n)$, there exist $e_{n,d}^{(1)}, \dots, e_{n,d}^{(s-1)}$ independent of g such that*

$$\frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} = 1 + \frac{e_{n,d}^{(1)}}{g} + \dots + \frac{e_{n,d}^{(s-1)}}{g^{s-1}} + O\left(\frac{1}{g^s}\right). \quad (4.1)$$

- (ii) *Given $n \geq 0, s \geq 1$, there exist $a_n^{(i)}, b_n^{(i)}$ $i = 1, \dots, s-1$, independent of g such that*

$$\frac{4\pi^2(2g-2+n)V_{g,n}}{V_{g,n+1}} = 1 + \frac{a_n^{(1)}}{g} + \dots + \frac{a_n^{(s-1)}}{g^{s-1}} + O\left(\frac{1}{g^s}\right), \quad (4.2)$$

$$\frac{V_{g,n}}{V_{g-1,n+2}} = 1 + \frac{b_n^{(1)}}{g} + \dots + \frac{b_n^{(s-1)}}{g^{s-1}} + O\left(\frac{1}{g^s}\right). \quad (4.3)$$

The coefficients of the above asymptotic expansions (4.1)–(4.3) can be characterized more precisely:

Theorem 4.2. *We have*

- (i) *For any fixed n and d the coefficient $e_{n,d}^{(i)}$ is a polynomial in $\mathbb{Q}[\pi^{-2}, \pi^2]$ of degree at most i .*
- (ii) *Each $a_n^{(i)}$ and $b_n^{(i)}$ is a polynomial in $\mathbb{Q}[n, \pi^{-2}, \pi^2]$ of degree i in n and of degree at most i in π^{-2} and π^2 .*

Remark 4.3. In the simplest case $[\tau_0 \tau_k]_{g,2}/V_{g,2}$ we have the following expansions:

$$\begin{aligned} \frac{[\tau_0 \tau_1]_{g,2}}{V_{g,2}} &= 1 - \frac{1}{\pi^2 g} + \left(\frac{1}{64} - \frac{5}{6\pi^2} + \frac{1}{\pi^4}\right) \cdot \frac{1}{g^2} + O\left(\frac{1}{g^3}\right), \\ \frac{[\tau_0 \tau_2]_{g,2}}{V_{g,2}} &= 1 - \frac{7}{2\pi^2 g} + \left(\frac{1}{64} - \frac{13}{6\pi^2} + \frac{1}{\pi^4}\right) \cdot \frac{1}{g^2} + O\left(\frac{1}{g^3}\right), \\ \frac{[\tau_0 \tau_k]_{g,2}}{V_{g,2}} &= 1 - \frac{2k^2 - k + 1}{2\pi^2 g} + \left(\frac{k^4}{2\pi^4} - \frac{13k^3}{6\pi^4} - \left(\frac{1}{2\pi^2} - \frac{27}{8\pi^4}\right) \cdot k^2 \right. \\ &\quad \left. + \left(\frac{1}{24\pi^2} - \frac{59}{24\pi^4}\right) \cdot k + \left(\frac{1}{64} - \frac{1}{4\pi^2} + \frac{19}{8\pi^4}\right)\right) \cdot \frac{1}{g^2} + O\left(\frac{1}{g^3}\right). \end{aligned}$$

We see that no (positive) powers of π^2 appear in these expansions. The term of order $1/g^2$ is a polynomial in k of degree 4 for $k \geq 3$ (computed numerically). However, for $k = 1, 2$ the general formula is off by $\frac{1}{8\pi^2} + \frac{5}{8\pi^4}$ and $\frac{5}{8\pi^4}$ for $k = 1$ and $k = 2$ respectively. This is a manifestation of the “boundary effect” in recursion (III). These results can be proved using Remark 3.3.

Remark 4.4. Note that a result similar to (4.2) and (4.3) holds for the inverse ratios $V_{g,n+1}/(8\pi^2 g V_{g,n})$ and $V_{g-1,n+2}/V_{g,n}$. This is because of the following simple fact. Let $\{w_g\}_{g=1}^\infty$ be a sequence of the form

$$w_g = 1 + \frac{u_1}{g} + \dots + \frac{u_{s-1}}{g^{s-1}} + O\left(\frac{1}{g^s}\right),$$

then

$$\frac{1}{w_g} = 1 + \frac{v_1}{g} + \dots + \frac{v_{s-1}}{g^{s-1}} + O\left(\frac{1}{g^s}\right),$$

where each v_i is a polynomial in u_1, \dots, u_i with integer coefficients. Moreover, if u_i is a polynomial of degree m_i in n , then v_k is a polynomial of degree at most $\max_{i+j=k}(m_i + m_j)$ in n .

Let us first outline some general ideas underlying the proofs of Theorems 4.1 and 4.2. All proofs are by induction in s and are similar to each other. We basically follow the same steps as in the course of proving Theorems 3.2 and 1.4.

Remark 4.5. Let $f : \mathbb{Z}_+ \rightarrow \mathbb{R}$ be a function such that $\lim_{g \rightarrow \infty} f(g)$ exists. We say that f has an expansion up to $O(1/g^s)$ if there exist $e_0, \dots, e_{s-1} \in \mathbb{R}$ so that

$$f(g) = e_0 + \frac{e_1}{g} + \dots + \frac{e_{s-1}}{g^{s-1}} + O\left(\frac{1}{g^s}\right).$$

Note that if f_1, \dots, f_k all have expansions of order s , the expansion of the product $f_1 \cdots f_k$ up to $1/g^s$ can easily be calculated in terms of the expansions of f_i .

In this section, we are interested in the expansions of the ratios $\frac{V_{g,n}}{V_{g-1,n+2}}$, $\frac{4\pi^2(2g-2+n)V_{g,n}}{V_{g,n+1}}$ and $\frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}}$. Some remarks are in order:

1. In general, given g', n' , in order to obtain the expansion of $\frac{V_{g-g',n-n'}}{V_{g,n}}$ up to $O(1/g^s)$, it is enough to know the expansions of $\frac{V_{g,k}}{V_{g-1,k+2}}$ and $\frac{4\pi^2(2g-2+k)V_{g,k}}{V_{g,k+1}}$ up to $O(1/g^{s-2g'-n'})$; this is simply because

$$\begin{aligned} \frac{V_{g-g',n-n'}}{V_{g,n}} &= \prod_{j=-n'+1}^{2g'} \frac{4\pi^2(2g-2g'+n-j+1)V_{g-g',n+j-1}}{V_{g-g',n+j}} \\ &\cdot \prod_{j=1}^{g'} \frac{V_{g-j,n+2j}}{V_{g-j+1,n+2j-2}} \cdot \prod_{j=-n'+1}^{2g'} \frac{1}{(4\pi^2(2g-2g'+n-j+1))}. \end{aligned} \quad (4.4)$$

2. Following (Ia) and (II), the expansion of $\frac{4\pi^2(2g-2+n)V_{g,n}}{V_{g,n+1}}$ up to $O(1/g^s)$ can be written explicitly in terms of the expansion of $\frac{[\tau_1 \tau_0^n]_{g,n+1}}{V_{g,n+1}}$ up to $O(1/g^s)$; see

(3.15). Similarly, by (3.8) and (4.4) the expansion of $\frac{V_{g,n}}{V_{g-1,n+2}}$ up to $O(1/g^s)$ can be written in terms of the expansions of $\frac{V_{g,n_1}}{V_{g-1,n_1+2}}$ and $\frac{4\pi^2(2g-2+n_1)V_{g,n_1}}{V_{g,n_1+1}}$ up to $O(1/g^{s-1})$, and the expansion of $\frac{[\tau_1 \tau_0^{n-1}]_{g,n}}{V_{g,n}}$ up to $O(1/g^s)$.

3. In view of **(III)**, the expansion of $\frac{[\tau_{d_1} \dots \tau_{d_n}]_{g,n}}{V_{g,n}}$ up to $O(1/g^s)$ can be written in terms of the expansions of $\frac{[\tau_{c_1} \dots \tau_{c_m}]_{g,m}}{V_{g,m}}$ up to $O(1/g^{s-1})$ and the expansions of $\frac{V_{g,n}}{V_{g-1,n+2}}$ and $\frac{4\pi^2(2g-2+n)V_{g,n}}{V_{g,n+1}}$ up to $O(1/g^{s-1})$. Actually, for our purposes it will be enough to obtain the expansion of $\frac{[\tau_{d_1+1} \dots \tau_{d_n}]_{g,n}}{[\tau_{d_1} \dots \tau_{d_n}]_{g,n}}$ up to $O(1/g^s)$. As in (3.8), we put

$$[\tau_{d_1} \dots \tau_{d_n}]_{g,n} - [\tau_{d_1+1} \dots \tau_{d_n}]_{g,n} = \tilde{A}_{d,g,n} + \tilde{B}_{d,g,n} + \tilde{C}_{d,g,n},$$

where $\tilde{A}_{d,g,n}$, $\tilde{B}_{d,g,n}$, and $\tilde{C}_{d,g,n}$ are the terms corresponding to (2.2), (2.3) and (2.4). Put

$$\frac{[\tau_{d_1} \dots \tau_{d_n}]_{g,n} - [\tau_{d_1+1} \dots \tau_{d_n}]_{g,n}}{V_{g,n}} = S_1 + S_2 + S_3, \quad (4.5)$$

where

$$\begin{aligned} S_1 &= \frac{1}{4\pi^2(2g-3+n)} \cdot \frac{4\pi^2(2g-3+n)V_{g,n-1}}{V_{g,n}} \cdot \frac{\tilde{A}_{d,g,n}}{V_{g,n-1}}, \\ S_2 &= \frac{1}{4\pi^2(2g-3+n)} \cdot \frac{4\pi^2(2g-3+n)V_{g-1,n+1}}{V_{g-1,n+2}} \cdot \frac{V_{g-1,n+2}}{V_{g,n}} \cdot \frac{\tilde{B}_{d,g,n}}{V_{g-1,n+1}}, \\ S_3 &= \frac{\tilde{C}_{d,g,n}}{V_{g,n}}. \end{aligned}$$

Similar to the case **1** in the proof of Theorem 3.2, we have

$$\frac{\tilde{A}_{d,g,n}}{V_{g,n-1}} = 8 \sum_{i=0}^{3g-3+n-d_1} (a_{i+1} - a_i) \frac{[\tau_{d_1+d_j+i-1} \tau_{d_2} \dots \widehat{\tau_{d_j}} \dots \tau_{d_n}]_{g,n-1}}{V_{g,n-1}} \quad (4.6)$$

(the hat means that the corresponding entry is omitted, and $a_{-1} = 0$). The case **2** of the same proof now reads

$$\frac{\tilde{B}_{d,g,n}}{V_{g-1,n+1}} = 16 (a_0 T_{k-2,g,n} + (a_1 - a_0) T_{k-1,g,n} + \dots + (a_{i+1} - a_i) T_{k-1+i,g,n} + \dots), \quad (4.7)$$

where

$$T_{m,g,n} = \frac{\sum_{i+j=m} [\tau_i \tau_j \tau_{d_2} \dots \tau_{d_n}]_{g-1,n+1}}{V_{g-1,n+1}}.$$

Similarly, according to (2.4), each term in $\tilde{C}_{d,g,n}$ has the form

$$\sum_{\substack{k_1+k_2= \\ =i+d_1-2}} (a_{i+1} - a_i) \left[\tau_{k_1} \prod_{i \in I} \tau_{d_i} \right]_{g', |I|+1} \cdot \left[\tau_{k_2} \prod_{i \in J} \tau_{d_i} \right]_{g-g', |J|+1}, \quad (4.8)$$

where $I \cup J = \{2, \dots, n\}$, and $0 \leq g' \leq g$. In order to obtain the expansions of S_1 and S_2 , we can use the expansions of ratios $\frac{[\tau_{c_1} \dots \tau_{c_{n-1}}]_{g,n-1}}{V_{g,n-1}}$ and $\frac{[\tau_{c_1} \dots \tau_{c_{n+1}}]_{g-1,n+1}}{V_{g-1,n+1}}$ up to $O(1/g^{s-1})$. What concerns the term S_3 , by Lemma 3.4 each product

$$\left[\tau_{k_1} \prod_{i \in I} \tau_{d_i} \right]_{g', |I|+1} \cdot \left[\tau_{k_2} \prod_{i \in J} \tau_{d_i} \right]_{g-g', |J|+1}$$

is of order $O(1/g^{s+1})$ unless either $2g' - 1 + |I| < s$ or $2g - 2g' + |J| - 1 < s$. In these cases we apply (4.4) to obtain the expansion of S_3 up to $O(1/g^s)$. Then we can use the expansions of $\frac{4\pi^2(2g-2+n)V_{g-1,n+2}}{V_{g-1,n+3}}$ (for S_1) and $\frac{V_{g,n}}{V_{g-1,n+2}}, \frac{4\pi^2(2g-3+n)V_{g-1,n+1}}{V_{g-1,n+2}}$ (for S_2), all up to $O(1/g^{s-1})$, to get the expansion of $\frac{[\tau_{d_1} \dots \tau_{d_n}]_{g,n} - [\tau_{d_1+1} \dots \tau_{d_n}]_{g,n}}{V_{g,n}}$ up to $O(1/g^s)$, which will complete the inductive step.

Remark 4.6. We will need the following basic facts to prove Theorems 4.1 and 4.2:

1. For any $k \geq 0$ the sum

$$\sum_{l=0}^{\infty} \frac{(-1)^l l^k \pi^{2l}}{(2l+1)!}$$

is a polynomial in π^2 of degree at most $2\lfloor k/2 \rfloor$ with rational coefficients (this can be easily seen by expanding $\frac{\sin x}{x}$ in the Taylor series, differentiating it and putting $x = \pi$).

2. For a polynomial $p(x) = \sum_{j=1}^m b_j x^j$ of degree m ,

$$\tilde{p}(x) = \sum_{i=1}^{\infty} (a_{i+1} - a_i) p(x+i)$$

is again a polynomial of degree m . The coefficient of $\tilde{p}(x)$ at x^j is equal to $\sum_{j+r \leq m} \binom{j+r}{j} b_{j+r} \cdot A(r)$ where $A(r) = \sum_{i=1}^{\infty} i^r (a_{i+1} - a_i)$.

3. Since ψ_i and $\kappa_1 = \frac{[\omega_{g,n}]}{2\pi^2}$ are rational classes (i.e., belong to $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, cf. [Wo], [AC]),

$$[\tau_{d_1} \dots \tau_{d_n}]_{g,n} \in \mathbb{Q} \cdot \pi^{6g-6+2n-2|d|},$$

where $|d| = \sum_{i=1}^n d_i$. In particular, $V_{g,n} = [\tau_0 \dots \tau_0]_{g,n}$ is a rational multiple of $\pi^{6g-6+2n}$ ([Wo], see also [M2] for a different point of view).

4. The function $S_m(n) = \sum_{i=1}^n i^m$ is a polynomial in n of degree $m+1$ with rational coefficients (Faulhaber's formula).

Proof of Theorem 4.1. First, we use **(Ia)**, **(II)** and **(III)** to prove the existence of $e_{n,d}^{(s)}$, $a_n^{(s)}$ and $b_n^{(s)}$. This is similar to what we did in the proofs of (1.3) and (1.4). In fact, instead of (i) we will prove a stronger statement: namely, there exist polynomials $Q_n^{(s)}(d_1, \dots, d_n)$ and $q_n^{(s)}(d_1, \dots, d_n)$ in variables d_1, \dots, d_n of degrees $s+1$ and s respectively such that for any $d = (d_1, \dots, d_n)$

$$\left| \frac{[\tau_{d_1} \cdots \tau_{d_n}]_{g,n}}{V_{g,n}} - 1 - \frac{e_{n,d}^{(1)}}{g} - \cdots - \frac{e_{n,d}^{(s)}}{g^s} \right| \leq \frac{Q_n^{(s)}(d_1, \dots, d_n)}{g^{s+1}}, \quad (4.9)$$

and

$$|e_{n,d}^{(s)}| \leq q_n^s(d_1, \dots, d_n). \quad (4.10)$$

These formulas follow from the two claims below:

Claim 1: *Formulas (4.9) and (4.10) for $s = r$ and formulas (4.2), (4.3) for $s < r$ imply (4.2) and (4.3) for $s = r$.*

In fact, from (3.15), (4.9) and (4.10) for $d = (l, 0, \dots, 0)$ we have

$$\frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} - \frac{1}{4\pi^2} = \frac{a_n^{(1)}}{g} + \cdots + \frac{a_n^{(s)}}{g^s} + O\left(\frac{1}{g^{s+1}}\right), \quad (4.11)$$

where

$$a_n^{(s)} = \sum_{l=1}^{\infty} \frac{(-1)^{l-1} l \pi^{2l-2}}{(2l+1)!} e_{n,l}^{(s)}, \quad (4.12)$$

with $d = (l, 0, \dots, 0)$. The existence of $a_n^{(s)}$ is guaranteed by the estimate

$$\sum_{l=N}^{\infty} \frac{(-1)^{l-1} l^{k+1} \pi^{2l-2}}{(2l+1)!} = O(e^{-N})$$

valid for any $k \geq 0$.

Similarly, we can use (3.17) to evaluate the error term in the expansion of $\frac{V_{g-1,n+4}}{V_{g,n+2}} - 1$. In this case we apply (4.9) with $d = (1, 0, \dots, 0)$. Note that by Lemma 3.4

$$\begin{aligned} & \frac{6}{V_{g,n+2}} \sum_{\substack{g_1+g_2=g \\ I \cup J = \{1, \dots, n\}}} V_{g_1, |I|+2} \cdot V_{g_2, |J|+2} = \\ & = \sum_{2j+i+2 \leq s} \binom{n}{i} \frac{V_{g-j,n+2-i}}{V_{g,n+2}} \times V_{j,i+2} + O\left(\frac{1}{g^{s+1}}\right). \end{aligned} \quad (4.13)$$

We can now use (4.4), and together with (4.2) and (4.3) for $s = r$ this yields the expansion of $\frac{V_{g-j,n+2-i}}{V_{g,n+2}}$ up to $O(1/g^{r+1})$.

Claim 2. *Formulas (4.2), (4.3), (4.9) and (4.10) for $s < r$ imply (4.9) and (4.10) for $s = r$.*

According to (4.5), we need to evaluate the contributions from the term S_1 , S_2 and S_3 up to $O(1/g^r)$. In view of (4.6), (4.7) and (4.8) we can use (4.9) for $s = r - 1$ to obtain the expansions of $\frac{\tilde{A}_{d,g,n}}{V_{g,n-1}}$ and $\frac{\tilde{B}_{d,g,n}}{V_{g-1,n+1}}$ up to $O(1/g^{r-1})$. Formula (4.6) now takes the form

$$\frac{\tilde{A}_{d,g,n}}{V_{g,n-1}} = 8 \sum_{i=0}^{\infty} (a_{i+1} - a_i) \left(1 + \frac{e_{n,d(i)}^{(1)}}{g} + \dots + \frac{e_{n,d'}^{(r-1)}}{g^{r-1}} + E_{d',r} \right), \quad (4.14)$$

where $d(i) = d_1 + d_j + i - 1, d_2, \dots, \hat{d}_j \dots, d_n$, and $E_{d',r} \leq \frac{Q_n^{r-1}(d')}{g^r}$. Note that by Lemma 2.1, (ii)

$$\sum_{i=N}^{\infty} (a_{i+1} - a_i) i^k = O(2^{-N}).$$

This allows us to calculate the contribution from S_1 up to $O(1/g^r)$. The other two terms S_2 and S_3 can be treated in a similar way. \square

In order to prove Theorem 4.2 we need two auxiliary lemmas:

Lemma 4.7. (i) Fix k ($0 < k \leq n$) and $d_1, \dots, d_k \in \mathbb{Z}_{\geq 0}$. Then for $d = (d_1, \dots, d_k, 0, \dots, 0)$ each term $e_{n,d}^{(s)}$ in the asymptotic expansion (4.1) is a polynomial in n of degree at most s .

(ii) Each $a_n^{(s)}$ and $b_n^{(s)}$ in (4.2) and (4.3) is a polynomial in n of degree s .

Proof. The proof is again by induction on s . We prove a slightly stronger version of (i):

(i') For given k and s , there exist polynomials $q_j(d_1, \dots, d_k)$, $j = 0, \dots, s$, such that the term $e_{n,d}^{(s)}$ has the form

$$e_{n,d}^{(s)} = \sum_{j=0}^s e_{d,j} n^j$$

with $|e_{d,j}| \leq q_j(d_1, \dots, d_k)$. In other words, the coefficients of $e_{n,d}^{(s)}$ considered as a polynomial in n grow at most polynomially in d_1, \dots, d_k . This would imply that

$$\sum_{i=1}^{\infty} (a_{i+1} - a_i) e_{d(i),j} < \infty,$$

where $d(i) = (d_1 + i, d_2, \dots, d_k, 0, \dots, 0)$.

Now, by (3.17) and (3.15) the statement (i') for $s = r$ implies part (ii) of the Lemma for $s = r$, that is clear in view of (4.12) and (4.13). Moreover, part (ii) for $s = r$ and the statement (i') for $s < r$ imply (i') for $s = r$. This follows from (4.5) by analyzing the contributions from the terms S_1 , S_2 and S_3 as in (4.6), (4.7) and (4.8). \square

Lemma 4.8. (i) Let n and k be fixed, and let $d = (d_1, \dots, d_k, d_{k+1}, \dots, d_n)$ with d_{k+1}, \dots, d_n fixed. Then there exists a polynomial $P_s \in \mathbb{R}[x_1, \dots, x_k]$ (depending on d_{k+1}, \dots, d_n) of degree $2s$ such that $e_{n,d}^{(s)} = P_s(d_1, \dots, d_k)$ provided $d_j \geq 2s$ for $j = 1, \dots, k$. The coefficient at each monomial $d_1^{\alpha_1} \dots d_k^{\alpha_k}$ in P_s is a linear rational combination of $\pi^{2s-2\lfloor \frac{|\alpha|+1}{2} \rfloor}, \dots, \pi^{-2s}$, where $|\alpha| = \alpha_1 + \dots + \alpha_k$. Moreover, for arbitrary d_j the difference $e_{n,d}^{(s)} - P_s(d_1, \dots, d_k)$ is a linear rational combinations of $\pi^{2s-2|d|}, \dots, \pi^{-2s}$.

(ii) Each $a_n^{(s)}$ and $b_n^{(s)}$ in (4.2) and (4.3) is a rational polynomial of degree at most s in $\mathbb{Q}[\pi^2, \pi^{-2}]$.

Proof. The proof is by induction in s and utilizes the same techniques that we have already used before, so we only sketch it here. The statement of the lemma follows from the following claims:

Claim 1: Part (ii) for $s < r$ implies that the coefficient at $1/g^r$ in the expansion of $V_{g_1, n_1+1} \cdot V_{g-g_1, n-n_1+1} / V_{g,n}$ is a polynomial of degree at most r in $\mathbb{Q}[\pi^2, \pi^{-2}]$. More precisely, when g_1, n_1 and n_2 are fixed

$$\frac{V_{g_1, n_1+1} \cdot V_{g-g_1, n-n_1+1}}{V_{g,n}} = \sum_{k=2g_1+n_1-1}^s \frac{c_{g_1, n_1}^{(k)}}{g^k} + O\left(\frac{1}{g^{s+1}}\right),$$

where $c_{g_1, n_1}^{(k)}$ is a polynomial of degree at most s in $\mathbb{Q}[\pi^2, \pi^{-2}]$.

This is a simple consequence of Remarks 4.6(2), 4.5(2) and formula (4.4).

Claim 2: Part (i) of the lemma for $s = r$ and part (ii) for $s < r$ imply part (ii) for $s = r$.

Note that by part (i), each element of the infinite sum (4.12) is a polynomial of degree at most r in $\mathbb{Q}[\pi^2, \pi^{-2}]$. On the other hand, when $l \geq r$ the coefficient $e_{n,l}^{(r)} = P_r(l)$ is a polynomial in l . Remark 4.6(1) and the properties of the coefficients of $P_r(l)$ imply that $a_n^{(r)}$ is a rational polynomial of degree at most r in $\mathbb{Q}[\pi^2, \pi^{-2}]$. The similar statement for $b_n^{(r)}$ follows from (3.17), Claim 1 and part (i) for $s = r$ (when $k = 1$ and $d = (1, 0, \dots, 0)$).

Claim 3: Part (ii) for $s < r$ and part (i) for $s < r$ imply part (i) for $s = r$.

First, we use (Ib) to find the polynomial P_r . We put $P_r(d_1, \dots, d_k) = e_{n+2,d}^{(r)}$ for $d = (0, 0, d_1, \dots, d_k, d_{k+1}, \dots, d_n)$ with $d_1, \dots, d_k \geq 2r$. Note that the number $[\tau_l, \tau_{d_1} \dots \tau_{d_k}]_{g, k+1} \neq 0$ only when $l \leq 3g - 2 + k - |d|$, $|d| = d_1 + \dots + d_k$. On the other hand, by Lemma 3.4, the term

$$\left[\tau_0^2 \tau_l \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+3} \cdot \left[\tau_0^2 \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+2}$$

in **(Ib)** is of order $O(1/g^{s+1})$ unless either $2g_1 + |I| + 1 < s$ or $2g_2 + |J| < s$. Similarly the term

$$\left[\tau_0 \tau_l \prod_{i \in I} \tau_{d_i} \right]_{g_1, |I|+2} \cdot \left[\tau_0^3 \prod_{i \in J} \tau_{d_i} \right]_{g_2, |J|+3}$$

in **(Ib)** is of order $O(1/g^{s+1})$ unless $2g_1 + |I| < s$ or $2g_2 + |J| + 1 < s$. For both terms we can explicitly calculate the expansions of the factors up to $O(1/g^{s+1})$ if we know their expansions up to $O(1/g^s)$.

Then Lemma 3.4, formula **(Ib)** and the induction hypothesis imply that

$$P_s(d_1 + 1, d_2, \dots, d_k) = P_s(d_1, \dots, d_k, 0, 0) + P'_s(d_1, \dots, d_k) + P''_s(d_1, \dots, d_k),$$

where P'_s is a polynomial when $d_1 \geq s-1, d_2 \geq s, \dots, d_k \geq s$, and $P''_s(d_1, \dots, d_k)$ is nontrivial only if $d_1, \dots, d_k \leq s$. The result follows from Lemma 4.7(i) and Claim 1. In the simplest case $k = 1, d_2 = \dots = d_n = 0$, and $d_1 = d \geq 2s$, the relation **(Ib)** implies that

$$[\tau_d \tau_0^{n-1}]_{g,n} = [\tau_s \tau_0^{n+2(d-s)-1}]_{g-(d-s), n+2(d-s)} + Q(d),$$

where Q is a polynomial in d .

Next, we use **(III)** to prove the statement about the coefficients of P_s . We explicitly calculate the expansion of $[\tau_{d_1} \dots \tau_{d_n}]_{g,n}$ using (4.5) and Remark 4.5. We evaluate the contributions from the terms S_1, S_2 and S_3 and show that there exist polynomials $Q_i^{(s)}, i = 1, 2, 3$, such that:

1. When d_1, \dots, d_k are large enough (i.e., $\geq 2s$), the coefficient $S_i^{(s)}$ at $1/g^s$ in S_i is equal to $Q_i^{(s)}(d_1, \dots, d_k)$;
2. The coefficient at the monomial $d_1^{\alpha_1} \dots d_k^{\alpha_k}$ in each $Q_i^{(s)}$ is a linear rational combination of $\pi^{2s-2[(|\alpha|+1)/2]}, \dots, \pi^{-2s}$, where $|\alpha| = \alpha_1 + \dots + \alpha_k$;
3. For all d_1, \dots, d_k , the difference $S_i^{(s)} - Q_i^{(s)}(d_1, \dots, d_k)$ is a linear rational combinations of $\{\pi^{2s-2|d|}, \dots, \pi^{-2s}\}$.

Contributions from S_1 . We use the induction hypothesis and Lemma 2.1 to expand the ratio $\tilde{A}_{d,g,n}/V_{g,n-1}$ up to the order $O(1/g^s)$ by expanding each $[\tau_{d_1+d_j+i-1} \tau_{d_2} \dots \widehat{\tau_{d_j}} \dots \tau_{d_n}]_{g,n-1}/V_{g,n-1}$ up to $O(1/g^{s-1})$. Put

$$q^{(m)}(d) = 8 \sum_{j=2}^n \sum_{i=0}^{\infty} (a_{i+1} - a_i) e_{n,d_j(i)}^{(m-1)},$$

where $d_j(i) = (d_1 + d_j + i - 1, d_2, \dots, \widehat{d_j}, \dots, d_n)$. Then following Remark 4.6(2), and (4.14)

$$S_1 = \sum_{j=1}^s \frac{Q_1^{(j)}(d)}{g^j} + O\left(\frac{1}{g^s}\right),$$

where

$$Q_1^{(j)}(d) = \sum_{j_1+j_2=j} q^{(j_1)}(d) \cdot a_{n-1}^{(j_2)}.$$

Now by the induction hypothesis and part (ii) of the lemma for $s < r$, both $q^{(m)}(d)$ and $Q_1^{(m)}(d)$ are polynomials in d_1, \dots, d_k of degree $2m$ whenever d_1, \dots, d_k are large enough. The coefficient at $d_1^{\alpha_1} \dots d_k^{\alpha_k}$ in $q^{(m)}(d)$ is a rational linear combination of the terms of the form $c_{\alpha_j(r)} \sum_{i=0}^{\infty} i^r (a_{i+1} - a_i)$, where $c_{\alpha_j(r)}$ is the coefficient at $x_1^{r+\alpha_1+\alpha_j} x_2^{\alpha_2} \dots \widehat{x_j^{\alpha_j}} \dots x_k^{\alpha_k}$ in $P^{(m-1)}(x_1, \dots, \widehat{x_j}, \dots, x_k)$. Now by Lemma 2.1, (iv) the sum $\sum_{i=0}^{\infty} i^r (a_{i+1} - a_i)$ is a rational linear combination of $\pi^2, \dots, \pi^{2\lceil r/2 \rceil}$, and by the induction hypothesis $c_{\alpha_j(r)}$ is a rational linear combination of $\pi^{2m-2-2r-|\alpha|}, \dots, \pi^{-2m-2}$. Therefore the coefficients of these polynomials are rational combinations of $\pi^{2m-|\alpha|}, \dots, \pi^{-2m}$.

Contributions from S_2 . In the same way, the induction hypothesis and Lemma 2.1 allow to expand $\widetilde{B}_{d,g,n}/V_{g-1,n+1}$ up to the order of $1/g^s$. Repeating the proof of case (2) of Theorem 3.2 we see that each term in the expansion is of the form (4.7). In the expansion of $T_{m,g,n}$ the term of order $1/g^m$ is a polynomial in d_1, \dots, d_n of degree $m+1$, whose coefficients satisfy the properties 1, 2, and 3 mentioned above.

Contributions from S_3 . What is different here compared to step 3 in the proof of Theorem 3.2, is that $\widetilde{C}_{d,g,n}$ contributes to the terms of order $1/g^2, \dots, 1/g^r$. However, by Lemma 3.4 these contributions can be evaluated using (4.4). More precisely, as in Remark 4.5, we can write S_3 as a sum of finitely many elements of the form

$$\begin{aligned} & \frac{[\tau_k \prod_{i \in I} \tau_{d_i}]_{g', |I|+1}}{V_{g', |I|+1}} \cdot \frac{V_{g-g', |J|+1} \cdot V_{g', |I|+1}}{V_{g,n}} \times \\ & \times \sum_{i=0}^{\infty} (a_{i+1} - a_i) \frac{[\tau_{d_1+i-2-k} \prod_{i \in J} \tau_{d_i}]_{g-g', |J|+1}}{V_{g-g', |J|+1}}, \end{aligned}$$

where $I \amalg J = \{2, \dots, n\}$, $0 \leq g' \leq g$, and $2g' - 1 + |I| \leq r$. The result now follows from the induction hypothesis on the behavior of the coefficient at $1/g^s$ in the expansion of $\frac{[\tau_{d_1+i-2-k} \prod_{i \in J} \tau_{d_i}]_{g-g', |J|+1}}{V_{g-g', |J|+1}}$ for $s < r$, together Claim 1 and Remark 4.6(2), (3). \square

Proof of Theorem 4.2. It is easy to see that part (i) of the theorem is a special case of Lemma 4.8 for $n = k$. Part (ii) is a consequence of Lemma 4.8(ii) and 4.7(ii). \square

Proof of Theorem 1.2. For a fixed $n \geq 0$, Theorem 4.1(ii) applied to the

obvious identity $\frac{V_{g+1,n}}{V_{g,n}} = \frac{V_{g+1,n}}{V_{g,n+2}} \cdot \frac{V_{g,n+2}}{V_{g,n+1}} \cdot \frac{V_{g,n+1}}{V_{g,n}}$ immediately yields

$$\begin{aligned} \frac{V_{g+1,n}}{V_{g,n}} &= (4\pi^2)^2 (2g+n-1)(2g+n-2) \times \\ &\times \left(1 - \frac{1}{2g}\right) \cdot \left(1 + \frac{r_n^{(2)}}{g^2} + \dots + \frac{r_n^{(s)}}{g^s} + O\left(\frac{1}{g^{s+1}}\right)\right) \end{aligned}$$

as $g \rightarrow \infty$. On the other hand, we have

$$V_{g,1} = \prod_{j=2}^{g-1} \frac{V_{j+1,1}}{V_{j,1}} \cdot V_{2,1},$$

and therefore the result of Theorem 1.2 for $n = 1$ is a consequence of the following:

Lemma 4.9. *Let $a_2, \dots, a_l \in \mathbb{R}$ and let $\{c_j\}_{j=1}^\infty$ be a positive sequence with the property*

$$c_j = 1 + \frac{a_2}{j^2} + \dots + \frac{a_s}{j^s} + O\left(\frac{1}{j^{s+1}}\right).$$

Then there exist b_1, \dots, b_{s-1} such that

$$\prod_{j=1}^g c_j = C_0 \left(1 + \frac{b_1}{g} + \dots + \frac{b_{s-1}}{g^{s-1}} + O\left(\frac{1}{g^s}\right)\right),$$

as $g \rightarrow \infty$, where $C_0 = \prod_{j=1}^\infty c_j$. Moreover, b_1, \dots, b_{s-1} are polynomials in a_2, \dots, a_s with rational coefficients.

Proof. First, we can write

$$c_j = R_j \left(1 + \frac{d_2}{j^2}\right) \dots \left(1 + \frac{d_l}{j^l}\right),$$

where d_2, \dots, d_l are polynomials in a_2, \dots, a_l and $R_j = 1 + O(1/j^{l+1})$ as $j \rightarrow \infty$. One can check that there exists a constant R such that

$$\prod_{j=1}^g R_j = R \left(1 + O\left(\frac{1}{g^l}\right)\right)$$

as $g \rightarrow \infty$. So it is enough to prove that

$$\prod_{j=1}^g \left(1 + \frac{d_k}{j^k}\right) = C_k \left(1 + \frac{p_k^{(1)}}{g} + \dots + \frac{p_k^{(l-1)}}{g^{l-1}} + O\left(\frac{1}{g^l}\right)\right)$$

as $g \rightarrow \infty$, where $p_k^{(1)}, \dots, p_k^{(l)}$ are polynomials in d_k with rational coefficients, and $C_k = \prod_{j=1}^\infty \left(1 + \frac{d_k}{j^k}\right)$, $k \geq 2$. It is enough to get bounds for the error term

$$E_{N,k} = \log \prod_{j=N}^\infty \left(1 + \frac{d_k}{j^k}\right) = \sum_{j=N}^\infty \log \left(1 + \frac{d_k}{j^k}\right). \quad (4.15)$$

Using the Taylor series $\log(1+x) = x - x^2/2 + x^3/3 - \dots$, we expand each term $\log\left(1 + \frac{d_k}{j^k}\right)$ up to the order j^{-l} .

Now we make use of the Euler-Maclaurin summation formula (cf. [Ed]):

$$\begin{aligned}\zeta(s) &= \sum_{i=1}^N \frac{1}{i^s} + \frac{1}{(s-1)N^{s-1}} + \frac{1}{2N^s} + \\ &+ \sum_{m=1}^{r-1} \frac{s(s+1)\dots(s+2m-2)B_{2m}}{(2m)!N^{s+2m-1}} + E_{2r}(s),\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta function, $B_{2m} = (-1)^{m+1}2(2m)! \frac{\zeta(2m)}{(2\pi)^{2m}}$ is the m th Bernoulli number, and the error term $E_{2r}(s)$ has an estimate

$$|E_{2r}(s)| < \left| \frac{s(s+1)\dots(s+2r-2)B_{2r}}{(\operatorname{Re}(s)+2r-1)(2r)!N^{s+2r-1}} \right|$$

(the formula holds for all s with $\operatorname{Re}(s) > -2r+1$). From here we get that for any integer $k > 1$

$$\begin{aligned}\sum_{i=N+1}^{\infty} \frac{1}{i^k} &= \frac{1}{(k-1)N^{k-1}} + \frac{1}{2N^k} + \\ &+ \sum_{m=1}^{[(l-k)/2]} \frac{B_{2m} \cdot k(k+1)\dots(k+2m-2)}{(2m)!N^{k+2m-1}} + O\left(\frac{1}{N^l}\right)\end{aligned}$$

as $N \rightarrow \infty$. Therefore, given l , there exist $q_k^{(k-1)}, \dots, q_k^{(l)}$ such that

$$E_{N,k} = \frac{q_k^{(k-1)}}{N^{k-1}} + \dots + \frac{q_k^{(l)}}{N^l} + O\left(\frac{1}{N^{l+1}}\right),$$

where each $q_k^{(i)}$ is a polynomial with rational coefficients in d_k . Now we can easily get hold on the error terms $e^{E_{N,k}}$:

$$e^{E_{N,k}} = \frac{p_k^{(k-1)}}{N^{k-1}} + \dots + \frac{p_k^{(l)}}{N^l} + O\left(\frac{1}{N^{l+1}}\right),$$

where $p_k^{(j)}$ is a polynomial in $q_k^{(i)}$, $i = k-1, \dots, l$, which implies the result. \square
As a result, we can write

$$V_{g,1} = C \frac{(2g-2)!(4\pi^2)^{2g-2}}{\sqrt{g}} \left(1 + \frac{c_1^{(1)}}{g} + \dots + \frac{c_1^{(k)}}{g^k} + O\left(\frac{1}{g^{k+1}}\right) \right),$$

as $g \rightarrow \infty$. On the other hand,

$$\frac{V_{g,n}}{C \cdot C_{g,n}} = \prod_{j=1}^{n-1} \frac{V_{g,j+1}}{4\pi^2(2g-2+j)V_{g,j}} \cdot \frac{V_{g,1}}{C \cdot C_{g,n}},$$

where $C_{g,n} = (2g - 3 + n)! (4\pi^2)^{2g-3+n} g^{-1/2}$, $C = \lim_{g \rightarrow \infty} \frac{V_{g,1}}{C_{g,1}}$. The following is elementary:

Fact. Let $\{f_i\}_{i=1}^\infty$ be a sequence of functions with the expansion

$$f_i(g) = 1 + \frac{p(1,i)}{g} + \dots + \frac{p(s,i)}{g^s} + O\left(\frac{1}{g^{s+1}}\right).$$

Assume that for a given j , $p(j,k)$ is a polynomial in k of degree j . Let

$$H(g,n) = \prod_{j=1}^n f_j(g).$$

Then

$$H(g,n) = 1 + \frac{h_1(i)}{g} + \dots + \frac{h_s(i)}{g^s} + O\left(\frac{1}{g^{s+1}}\right),$$

where for a given j , $h_j(k)$ is a polynomial in k of degree $2j$. Moreover the leading coefficient of $h_j(k)$ is equal to $\frac{l^j}{2^j j!}$, where l is the leading coefficient of the linear polynomial $p(1,j)$.

This fact is a consequence of an elementary observation. Given $m_1, \dots, m_k \in \mathbb{N}$, let

$$F_{m_1, \dots, m_k}(d) = \sum_{\substack{x_1, \dots, x_k \in \{1, \dots, n\}, \\ x_i \neq x_j}} x_1^{m_1} \dots x_k^{m_k}$$

is a polynomial in d of degree $(m_1 + 1) + \dots + (m_k + 1)$. Note that for $d < k$, $F_{m_1, \dots, m_k}(d) = 0$.

Now we use this observation for $f_j(g) = \frac{V_{g,j+1}}{4\pi^2(2g-2+j)V_{g,j}}$. In this case, (3.16) implies that $l = -1/\pi^2$. Hence the result follows from (4.2) and Theorem 4.2, (ii).

5 Asymptotics for variable n

In this section we discuss the asymptotics behavior of $V_{g,n(g)}$ in case when $n(g) \rightarrow \infty$ as $g \rightarrow \infty$ and prove Theorem 1.8 : if $n(g)^2/g \rightarrow 0$ as $g \rightarrow \infty$, then

$$\lim_{g \rightarrow \infty} \frac{V_{g,n(g)}}{C_{g,n(g)}} = C,$$

where $C_{g,n} = (2g - 3 + n)! (4\pi^2)^{2g-3+n} g^{-1/2}$ and $C = \lim_{g \rightarrow \infty} \frac{V_{g,0}}{C_{g,0}}$.

We need the following basic lemma:

Lemma 5.1. *There are universal constants $c_0, c_1, c_2, c_3, c_4 > 0$ such that for $g, n \geq 0$ the following inequalities hold:*

(i) for any $k \geq 1$,

$$c_0 \cdot \frac{n}{2g - 2 + n} \leq 1 - \frac{[\tau_k \tau_0^{n-1}]_{g,n}}{V_{g,n}} \leq c_1 \cdot \frac{nk^2}{2g - 2 + n},$$

(ii)

$$\left| \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} - \frac{1}{4\pi^2} \right| \leq c_2 \cdot \frac{n}{2g-2+n},$$

(iii)

$$\frac{V_{g-1,n+4}}{V_{g,n+2}} \leq 1 - c_4 \cdot \frac{n}{2g-2+n}.$$

Proof. The proof follows the same lines as the proofs of formulas (3.2), (3.3) and (3.5) (see [M3] for details). First, observe that

- Recursion **(III)** implies

$$[\tau_{k+1} \tau_0^{n-1}]_{g,n} \leq [\tau_k \tau_0^{n-1}]_{g,n} \leq V_{g,n}, \quad (5.1)$$

and we also have

$$b \leq \frac{[\tau_1 \tau_0^{n-1}]_{g,n}}{V_{g,n}}$$

where $b = \max\{a_i/a_{i+1}\}$, $i = 0, 1, \dots$;

- For $l \geq 0$

$$\frac{l \pi^{2l-2}}{(2l+1)!} \geq \frac{(l+1)\pi^{2l}}{(2l+3)!},$$

so that from **(II)** and (5.1) we have

$$b_0 \leq \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} \leq b_1, \quad (5.2)$$

where

$$b_0 = b \cdot \left(\frac{1}{6} - \frac{\pi^2}{60} \right), \quad b_1 = \sum_{l=1}^{\infty} \frac{l \pi^{2l-2}}{(2l+1)!}.$$

- By recursion **(Ia)**, for $n \geq 2$

$$\frac{V_{g-1,n+4}}{V_{g,n+2}} \leq 1. \quad (5.3)$$

Note that in view of (5.2) and (5.3), Theorem 3.2, (ii) implies the first inequality. In order to prove (ii) we will use (i) and (3.15). As a result, we get

$$\left| \frac{(2g-2+n)V_{g,n}}{V_{g,n+1}} - \frac{1}{4\pi^2} \right| \leq \sum_{l=1}^{\infty} \frac{l^3 \pi^{2l-2}}{(2l+1)!} \cdot \frac{n}{2g-2+n}.$$

The third inequality is a simple consequence of **(Ia)**. This bound can be obtained from the lower bound in (5.2) and (3.17). \square

Proof of Theorem 1.8. By Lemma 5.1,

$$\frac{(2g-2+n(g))V_{g,n(g)}}{V_{g,n(g)+1}} \rightarrow \frac{1}{4\pi^2}, \quad \frac{V_{g-1,n(g)+4}}{V_{g,n(g)+2}} \rightarrow 1,$$

when $n(g)/g \rightarrow 0$ as $g \rightarrow \infty$. From definition of $C_{g,n}$ immediately follows that

$$\frac{V_{g,n(g)}}{C_{g,n(g)}} = \frac{V_{g,n(g)}}{4\pi^2(2g-2+n(g))V_{g,n(g)-1}} \cdot \frac{V_{g,n(g)-1}}{4\pi^2(2g-3+n(g))V_{g,n(g)-2}} \cdots \frac{V_{g,0}}{C_{g,0}}.$$

Hence, by Theorem 1.2 and Lemma 5.1, (ii) we have

$$\begin{aligned} & (1 - c_2/g) \cdot (1 - 2 \cdot c_2/g) \cdots (1 - n(g) \cdot c_2/g) \cdot (C + O(1/g)) \leq \\ & \leq \frac{V_{g,n(g)}}{C_{g,n(g)}} \leq \\ & \leq (1 + c_2/g) \cdot (1 + 2 \cdot c_2/g) \cdots (1 + n(g) \cdot c_2/g)(C + O(1/g)), \end{aligned}$$

which implies the result. \square

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