PERVERSE COHERENT SHEAVES AND FOURIER-MUKAI TRANSFORMS ON SURFACES II

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Abstract. We study perverse coherent sheaves on the resolution of rational double points. As examples, we consider rational double points on 2-dimensional moduli spaces of stable sheaves on $K3$ and elliptic surfaces. Then we show that perverse coherent sheaves appears in the theory of Fourier-Mukai transforms. As an application, we generalize the Fourier-Mukai duality for $K3$ surfaces to our situation.

Contents

0. Introduction.

This is the second half part of our study of perverse coherent sheaves on surfaces. In the first part [Y7], we studied basic properties of the category of perverse coherent sheaves especially on the minimal resolution of a projective surface with rational double points. In this paper, we shall give several examples of perverse coherent sheaves on projective surfaces. In particular, we shall study Fourier-Mukai transforms associated to normal $K3$ surfaces and ellitpic surfaces.

In section 1, we collect some results in [Y7]. In section 2, we consider the Fourier-Mukai transforms on $K3$ surfaces. We first generalize known facts on the 2-dimensional moduli spaces of usual stable sheaves to those of stable perverse coherent sheaves. In particular, we shall show that the singularities of the moduli spaces $Y' := \overline{M}_H^v(v)$ are rational double points and the minimal resolutions $\pi' : X' \to Y'$ are constructed as $X' = \overline{M}_H^w(v)$, where w is a suitable parameter. We next define similar categories $\mathfrak{A}$ and $\mathfrak{A}^\mu$ to those in [Br4], and generalize results in [H]. In particular, we study the relation of Fourier-Mukai transforms and the

1991 *Mathematics Subject Classification.* 14D20.
The author is supported by the Grant-in-aid for Scientific Research (No. 18340010, No. 22340010), JSPS.

categories $\mathfrak{A}, \mathfrak{A}^\mu$ (Theorem 2.5.9). This result will be used to study Bridgeland's stable objects in [MYY]. We also prove the Fourier-Mukai duality (Theorem 2.6.1). Finally we give some conditions for the preservation of Gieseker stability conditions.

In section 3, we shall study Fourier-Mukai transforms on elliptic surfaces.

Fourier-Mukai transforms by equivariant coherent sheaves are treated in section 4. Let G be a finite group acting on a projective surface X. Assume that K_X is the pull-back of a line bundle on $Y := X/G$. We shall first construct the moduli space of G-sheaves (Theorem 4.2.4). In particular, we shall construct a minimal resolution X' of Y as a moduli space of stable G-sheaves. Then we can describe the exceptional divisors by a similar method as in the proof of [Y7, Thm.2.2.19]. We next show that the Fourier-Mukai transform $\mathbf{D}_G(X) \to \mathbf{D}(X')$ induces an equivalence $\operatorname{Coh}_G(X) \to {}^{-1}\operatorname{Per}(X'/Y)$ (McKay correspondence [VB]). Then by using this equivalence, we show that there are many moduli spaces of stable G-sheaves which induce Fourier-Mukai transforms, if X' is a $K3$ surface.

Notation.

(i) For a scheme X, $\operatorname{Coh}(X)$ denotes the category of coherent sheaves on X and $\mathbf{D}(X)$ the bounded derived category of $\operatorname{Coh}(X)$. We denote the Grothendieck group of X by $K(X)$.

(ii) Let $\mathcal{A}$ be a sheaf of $\mathcal{O}_X$-algebras on a scheme X which is coherent as an $\mathcal{O}_X$-module. Let $\operatorname{Coh}_{\mathcal{A}}(X)$ be the category of coherent $\mathcal{A}$-modules on X and $\mathbf{D}_{\mathcal{A}}(X)$ the bounded derived category of $\operatorname{Coh}_{\mathcal{A}}(X)$.

(iii) Assume that X is a smooth projective variety. Let E be an object of $\mathbf{D}(X)$. $E^\vee := \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ denotes the dual of E. We denote the rank of E by $\operatorname{rk} E$. For a fixed nef divisor H on X, $\deg(E)$ denotes the degree of E with respect to H. For $G \in K(X)$, $\operatorname{rk} G > 0$, we also define the twisted rank and degree by $\operatorname{rk}_G(E) := \operatorname{rk}(G^\vee \otimes E)$ and $\deg_G(E) := \deg(G^\vee \otimes E)$ respectively. We set $\mu_G(E) := \deg_G(E)/\operatorname{rk}_G(E)$, if $\operatorname{rk} E \neq 0$.

(iv) **Integral functor.** For two schemes X, Y and an object $\mathcal{E} \in \mathbf{D}(X \times Y)$, $\Phi^{\mathcal{E}}_{X \to Y} : \mathbf{D}(X) \to \mathbf{D}(Y)$ is the integral functor

$$\Phi^{\mathcal{E}}_{X \to Y}(E) := \mathbf{R}p_{Y*}(\mathcal{E} \overset{\mathbf{L}}{\otimes} p_X^*(E)),\ E \in \mathbf{D}(X), \tag{0.1}$$

where $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are projections. If $\Phi^{\mathcal{E}}_{X \to Y}$ is an equivalence, it is said to be the *Fourier-Mukai transform*.

(v) $\mathbf{D}(X)_{op}$ denotes the opposit category of $\mathbf{D}(X)$. We have a functor

$$\begin{array}{cccc} D_X : & \mathbf{D}(X) & \to & \mathbf{D}(X)_{op} \\ & E & \mapsto & E^\vee. \end{array}$$

(vi) Assume X is a smooth projective surface.

(a) We set $H^{ev}(X, \mathbb{Z}) := \bigoplus_{i=0}^2 H^{2i}(X, \mathbb{Z})$. In order to describe the element x of $H^{ev}(X, \mathbb{Z})$, we use two kinds of expressions: $x = (x_0, x_1, x_2) = x_0 + x_1 + x_2 \varrho_X$, where $x_0 \in \mathbb{Z}, x_1 \in H^2(X, \mathbb{Z}), x_2 \in \mathbb{Z}$, and $\int_X \varrho_X = 1$. For $x = (x_0, x_1, x_2)$, we set $\operatorname{rk} x := x_0$ and $c_1(x) = x_1$.

(b) We define a homomorphism

$$\begin{array}{cccc} \gamma : & K(X) & \to & \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z} \\ & E & \mapsto & (\operatorname{rk} E, c_1(E), \chi(E)) \end{array} \tag{0.2}$$

and set $K(X)_{\mathrm{top}} := K(X)/\ker \gamma$. We denote $E \mod \ker \gamma$ by $\tau(E)$. $K(X)_{\mathrm{top}}$ has a bilinear form $\chi(\ ,\ )$.

(c) **Mukai lattice.** We define a lattice structure $\langle\ ,\ \rangle$ on $H^{ev}(X, \mathbb{Z})$ by

$$\begin{aligned} \langle x, y \rangle :=& -\int_X x^\vee \cup y \\ =&(x_1, y_1) - (x_0 y_2 + x_2 y_0), \end{aligned} \tag{0.3}$$

where $x = (x_0, x_1, x_2)$ (resp. $y = (y_0, y_1, y_2)$) and $x^\vee = (x_0, -x_1, x_2)$. It is now called the *Mukai lattice*. Mukai lattice has a weight-2 Hodge structure such that the (p,q)-part is $\bigoplus_i H^{p+i,q+i}(X)$. We set

$$\begin{aligned} H^{ev}(X, \mathbb{Z})_{\mathrm{alg}} =& H^{1,1}(H^{ev}(X, \mathbb{C})) \cap H^{ev}(X, \mathbb{Z}) \\ \cong& \mathbb{Z} \oplus \operatorname{NS}(X) \oplus \mathbb{Z}. \end{aligned} \tag{0.4}$$

Let E be an object of $\mathbf{D}(X)$. If X is a $K3$ surface or $\operatorname{rk} E = 0$, we define the *Mukai vector* of E as

$$v(E) := \operatorname{rk}(E) + c_1(E) + (\chi(E) - \operatorname{rk}(E))\varrho_X \in H^{ev}(X, \mathbb{Z}). \tag{0.5}$$

Then for $E, F \in \mathbf{D}(X)$ such that the Mukai vectors are well-defined, we have

$$\chi(E, F) = -\langle v(E), v(F) \rangle. \tag{0.6}$$

(d) Since $\deg_G(E)$ is determined by the Chern character $\mathrm{ch}(E)$, we can also define $\deg_G(v)$, $v \in H^{ev}(X,\mathbb{Z})_{\mathrm{alg}}$ by using $E \in \mathbf{D}(X)$ with $v(E) = v$.

1. A summary of some results in [Y7].

1.1. **Perverse coherent sheaves.** For a convenience sake, we collect some results in [Y7] which will be used in this paper.

Let Y be a projective normal surface with at worst rational singularities and $\pi : X \to Y$ the minimal resolution. Let p_i, $i = 1, 2, ..., n$ be the singular points of Y and $Z_i := \pi^{-1}(p_i) = \sum_{j=1}^{t_i} a_{ij} C_{ij}$ their fundamental cycles. By the assumption, we have $R^1\pi_*(\mathcal{O}_X) = 0$ and C_{ij} are smooth rational curves on X.

We are interested in an abelian subcategory $\mathcal{C}$ of $\mathbf{D}(X)$ such that there is a locally free sheaf G on X satisfying

(1) $R^1\pi_*(G^\vee \otimes G) = 0$,

(2) $\mathbf{R}\pi_*(G^\vee \otimes \bullet)$ induces an equivalence $\mathcal{C} \cong \mathrm{Coh}_{\mathcal{A}}(Y)$, where $\mathcal{A} = \pi_*(G^\vee \otimes G)$ is a sheaf of $\mathcal{O}_Y$-algebras.

Thus

$$\mathcal{C} = \{E \in \mathbf{D}(X) | H^i(E) = 0, i \neq -1, 0,\ H^{-1}(E) \in S, H^0(E) \in T\}, \tag{1.1}$$

where

$$\begin{aligned} S :=& \{E \in \mathrm{Coh}(X) | \pi_*(G^\vee \otimes E) = 0\} \\ T :=& \{E \in \mathrm{Coh}(X) | R^1\pi_*(G^\vee \otimes E) = 0\} \end{aligned} \tag{1.2}$$

and $S \cap T = 0$.

Definition 1.1.1 (cf. [Y7, Prop. 2.1.1 (1)])**.** Let $\mathbf{b}_i := (b_{i1}, b_{i2}, \dots, b_{is_i})$, $i = 1, 2, ..., n$ be sequences of integers.

(1) We define a torsion pair (T, S) of $\mathrm{Coh}(X)$ such that

$$\begin{aligned} S :=& \{E \in \mathrm{Coh}(X) |\ E \text{ is generated by subsheaves of } \mathcal{O}_{C_{ij}}(b_{ij})\ \}, \\ T :=& \{E \in \mathrm{Coh}(X) | \operatorname{Hom}(E, \mathcal{O}_{C_{ij}}(b_{ij})) = 0\}. \end{aligned} \tag{1.3}$$

(2) $\mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ denotes the tilting of $\mathrm{Coh}(X)$ by (T, S).

$\mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ is an example of the category of perverse coherent sheaves ([Y7, Lem. 1.2.4]). If $\mathbf{b}_i = (-1, -1, ..., -1)$ for all i, then $\mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ is nothing but ${}^{-1}\mathrm{Per}(X/Y)$ in [Br3] and [VB].

We take a locally free sheaf G_0 on X such that $G_{0|C_{ij}} \cong \mathcal{O}_{C_{ij}}(b_{ij} + 1)^{\oplus \mathrm{rk}\, G_0}$. We set $\mathcal{A}_0 := \pi_*(G_0^\vee \otimes G_0)$.

Definition 1.1.2 ([Y7, Lem. 1.2.16], [Y7, Defn. 2.1.7])**.** (1)

$$A_0(\mathbf{b}_i) := \pi^{-1}(\pi_*(G_0^\vee \otimes \mathbb{C}_x)) \otimes_{\pi^{-1}(\mathcal{A}_0)} G_0 \tag{1.4}$$

is the unique line bundle on Z_i such that $A_0(\mathbf{b}_i)_{|C_{ij}} \cong \mathcal{O}_{C_{ij}}(b_{ij} + 1)$ for all j. $A_0(\mathbf{b}_i)$ is denoted by A_{p_i} in [Y7, Defn. 2.1.7].

(2) We also set $A_0(\mathbf{b}_i)^* := A_0(\mathbf{b}_i) \otimes \omega_{Z_i}$.

We collect easy facts on $A_0(\mathbf{b}_i)$ and $A_0(\mathbf{b}_i)^*$ which follow from [Y7, Lem. 1.2.22, Lem. 1.2.27].

Lemma 1.1.3. (1) (a) *For $E = A_0(\mathbf{b}_i)$, we have*

$$\operatorname{Hom}(E, \mathcal{O}_{C_{ij}}(b_{ij})) = \operatorname{Ext}^1(E, \mathcal{O}_{C_{ij}}(b_{ij})) = 0,\ 1 \le j \le t_i \tag{1.5}$$

and there is an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{1.6}$$

such that F is a successive extension of $\mathcal{O}_{C_{ij}}(b_{ij})$ and $x \in Z_i$.

(b) *Conversely if E satisfies these conditions, then $E \cong A_0(\mathbf{b}_i)$.*

(2) (a) *For $E = A_0(\mathbf{b}_i)^*$, we have*

$$\operatorname{Hom}(\mathcal{O}_{C_{ij}}(b_{ij}), E) = \operatorname{Ext}^1(\mathcal{O}_{C_{ij}}(b_{ij}), E) = 0,\ 1 \le j \le t_i \tag{1.7}$$

and there is an exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathbb{C}_x \longrightarrow 0 \tag{1.8}$$

such that F is a successive extension of $\mathcal{O}_{C_{ij}}(b_{ij})$ and $x \in Z_i$.

(b) *Conversely if E satisfies these conditions, then $E \cong A_0(\mathbf{b}_i)^*$.*

Proposition 1.1.4 (cf. [Y7, Cor. 1.2.24])**.** *The irreducible objects of* $\mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$ *are*

$$\begin{aligned} &\mathbb{C}_x\ (x \in X \setminus \cup_i Z_i) \text{ and} \\ &A_0(\mathbf{b}_i), \mathcal{O}_{C_{i1}}(b_{i1})[1], ..., \mathcal{O}_{C_{is_i}}(b_{is_i})[1]\ (1 \le i \le n) \end{aligned} \tag{1.9}$$

Proposition 1.1.5 ([Y7, Prop. 1.1.33]). *Let $\mathcal{C}$ be a category of perverse coherent sheaves on X. Let $\mathbf{I}_{yj}$ $(0 \le j \le s_y)$ be the irreducible objects of $\mathcal{C}$ such that $\pi(\operatorname{Supp}(\mathbf{I}_{yj})) = \{y\}$.*

(1) *Let G_1 be an object of $\mathbf{D}(X)$ such that $H^i(E) = 0$ for $i \ne -1, 0$ and satisfies*

$$(a)\ \operatorname{Hom}(G_1, \mathbf{I}_{yj}[p]) = 0, p \ne 0, \quad (b)\ \chi(G_1, \mathbf{I}_{yj}) > 0 \tag{1.10}$$

for all $y \in Y$ and $j = 0, 1, ..., s_y$. Then G_1 is a locally free sheaf on X such that $R^1\pi_*(G_1^\vee \otimes G_1) = 0$ and G_1 is a local projective generator of $\mathcal{C}$.

(2) *Assume that $\chi(G_1, \mathbf{I}_{yj}) > 0$ for all $y \in Y$ and $0 \le j \le s_y$. Then there is a local projective generator G' with $\tau(G') = 2\tau(G_1)$.*

Lemma 1.1.6 ([Y7, Lem. 1.1.14]). *Let $\mathcal{C}$ be a category of perverse coherent sheaves and G a locally free sheaf on X which gives a local projective generator of $\mathcal{C}$.*

(1) *We have a category of perverse coherent sheaves $\mathcal{C}^D$ such that $G^\vee$ is a local projective generator:*

$$\mathcal{C}^D = \{E \in \mathbf{D}(X) | \mathbf{R}\pi_*(G \otimes E) \in \operatorname{Coh}(Y)\}.$$

(2) *If E is a local projective object of $\mathcal{C}$, that is, $R^1\pi_*(E^\vee \otimes F) = 0$ for all $F \in \mathcal{C}$, then $E^\vee$ is a local projective object of $\mathcal{C}^D$.*

(3) *E is an irreducible object of $\mathcal{C}$ if and only if $E^\vee[2]$ is an irreducible object of $\mathcal{C}^D$.*

Definition 1.1.7 (cf. [Y7, Prop. 2.1.1 (2)]). $\operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)^*$ denotes the tilting of $\operatorname{Coh}(X)$ by the torsion pair (T^*, S^*):

$$\begin{aligned} S^* :=& \{E \in \operatorname{Coh}(X) |\ E \text{ is generated by subsheaves of } A_0(\mathbf{b}_i)^*\}, \\ T^* :=& \{E \in \operatorname{Coh}(X) | \operatorname{Hom}(E, A_0(\mathbf{b}_i)^*) = 0\}. \end{aligned} \tag{1.11}$$

Definition 1.1.8. For $\mathbf{b}_i = (b_{i1}, ..., b_{is_i})$ $(1 \le i \le n)$, we set $\mathbf{b}_i^D = (-b_{i1} - 2, ..., -b_{is_i} - 2)$.

Then $\operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)^D = \operatorname{Per}(X/Y, \mathbf{b}_1^D, ..., \mathbf{b}_n^D)^*$. In particular, $\operatorname{Per}(X/Y, \mathbf{b}_1^D, ..., \mathbf{b}_n^D)^*$ is the category of perverse coherent sheaves. For $\operatorname{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)^D$, the irreducible objects are

$$\begin{aligned} &\mathbb{C}_x (x \in X \setminus \cup_i Z_i) \text{ and} \\ &\mathcal{O}_{C_{i1}}(-b_{i1} - 2), ..., \mathcal{O}_{C_{is_i}}(-b_{is_i} - 2), A_0(\mathbf{b}_i^D)^*[1]\ (1 \le i \le n). \end{aligned} \tag{1.12}$$

1.2. **Stabilities.**

1.2.1. *Stability for perverse coherent sheaves.* We introduce the notion of semi-stability and constructed the moduli space as a projective scheme. We shall briefly recall parts of the notion. Let H be the pull-back of an ample divisor on Y. For a local projective generator G and a perverse coherent sheaf $E \in \mathcal{C}$, we have a G-twisted Hilbert polynomial $\chi(G, E(nH))$. If the degree is d, then E is of dimension d. A 2-dimensional object E is G-twisted semi-stable with respect to H if

$$\chi(G, F(nH)) \le \frac{\operatorname{rk} F}{\operatorname{rk} E} \chi(G, E(nH)),\ n \gg 0 \tag{1.13}$$

for all subsheaf F of E. We also define μ-semi-stability for a purely 2-dimensional object by comparing the coefficients of n in (1.13). If E is 1-dimensional, then the condition is

$$\chi(G, F) \le \frac{(H, c_1(F))}{(H, c_1(E))} \chi(G, E) \tag{1.14}$$

for all proper subobject F of E.

Definition 1.2.1. (1) For $\mathbf{e} \in K(X)_{\text{top}}$, $\overline{M}_H^G(\mathbf{e})$ is the moduli space of G-twisted semi-stable objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$ and $M_H^G(\mathbf{e})$ the open subscheme consisting of G-twisted stable objects.

(2) Let $\mathcal{M}_H(\mathbf{e})^{\mu\text{-ss}}$ (resp. $\mathcal{M}_H^G(\mathbf{e})^{ss}, \mathcal{M}_H^G(\mathbf{e})^s$) be the moduli stack of μ-semi-stable (resp. G-twisted semi-stable, G-twisted stable) objects E of $\mathcal{C}$ with $\tau(E) = \mathbf{e}$.

1.2.2. *Stability for 0-dimensional objects.* A 0-dimensional object E is (G, α)-twisted semi-stable, if

$$\frac{\chi(\alpha, F)}{\chi(G, F)} \le \frac{\chi(\alpha, E)}{\chi(G, E)} \tag{1.15}$$

for all subobject F of E. If $v(E) = \varrho_X$, then it is equivalent to the condition

$$\chi(\alpha, F) \le 0 \tag{1.16}$$

for all subobject F of E. In this case, the semi-stability is independent of the choice of G. We abbreviatedly say that E is α-semi-stable. (G, α)-twisted stability and α-stability is also defined in a usual way.

Definition 1.2.2. Let $\mathcal{M}_H^{G,\alpha}(v)$ be the moduli stack of (G, α)-semi-stable objects E with $v(E) = v$ and $\overline{M}_H^{G,\alpha}(v)$ the moduli space of (G, α)-semi-stable objects E. We also set $X^\alpha := \overline{M}_H^{G,\alpha}(\varrho_X)$.

Proposition 1.2.3. *There is an isomorphism $\psi : X^0 \to Y$ such that $\psi \circ \varphi : X \to Y$ coincides with π. In particular, X^0 is a normal projective surface.*

1.3. **Characterization of** $\mathrm{Per}(X/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$**.**

Proposition 1.3.1. *Let $\mathcal{C}$ be the category of perverse coherent sheaves. Then there exists X' and γ such that $X = (X')^\gamma$ and $\mathcal{C} = \Phi_{X' \to X}^{(\mathcal{E}^\gamma)^\vee[2]}(\mathrm{Per}(X'/Y, \mathbf{b}_1, ..., \mathbf{b}_n))$ if and only if there is a $\beta \in \varrho_X^\perp$ such that $\mathbb{C}_x$ are β-stable for all $x \in X$, where $\mathcal{E}^\gamma \in \mathbf{D}(X' \times (X')^\gamma)$ is the universal family of γ-stable objects of $\mathrm{Per}(X'/Y, \mathbf{b}_1, ..., \mathbf{b}_n)$.*

Since $X' \cong X$, $\Phi_{X' \to X}^{(\mathcal{E}^\gamma)^\vee[2]}$ is regarded as an auto-equivalence of $\mathbf{D}(X)$.

Proposition 1.3.2 ([Y7, Prop. 2.4.5])**.** *We set $v = (r, \xi, a) \in H^{ev}(X, \mathbb{Z})_{\mathrm{alg}}$, $r > 0$. Assume that $(\xi, D) \notin r\mathbb{Z}$ for all $D \in \oplus_{i,j}\mathbb{Z}[C_{ij}]$ with $(D^2) = -2$. Then there is a category of perverse coherent sheaves $\mathcal{C}(v)$ satisfying the following conditions:*

1. *There is a local projective generator G of $\mathcal{C}(v)$ such that G is a locally free sheaf on X with $v(G) = 2v$.*
2. *There is $\beta \in \varrho_X^\perp$ such that $\mathbb{C}_x \in \mathcal{C}(v)$ is β-stable for all $x \in X$.*

Corollary 1.3.3 ([Y7, Cor.2.5.5])**.** *Let X be a $K3$ surface with a birational morphism $\pi : X \to Y$, where Y is a normal surface. Let $v_0 = (r, \xi, a)$ be a primitive isotropic Mukai vector such that $r \not| (\xi, D)$ for all (-2)-curves D with $(D, H) = 0$. Let $\mathcal{C}(v_0)$ be the category in Proposition 1.3.2. Then $M_H^{v_0}(v_0) \neq \emptyset$.*

2. Fourier-Mukai transform on a $K3$ surface.

2.1. **Basic results on the moduli spaces of dimension 2.** Let Y be a normal $K3$ surface and $\pi : X \to Y$ the minimal resolution. Let $p_1, p_2, \dots, p_n$ be the singular points of Y and $Z_i := \pi^{-1}(p_i) = \sum_{j=0}^{s_i} a_{ij} C_{ij}$ the fundamental cycle, where C_{ij} are smooth rational curves on X and $a_{ij} \in \mathbb{Z}_{>0}$. We shall study moduli of stable objects in the category of perverse coherent sheaves $\mathcal{C}$ satisfying the following assumption.

Assumption 2.1.1. There is a $\beta \in \varrho_X^\perp \otimes \mathbb{Q}$ such that $\mathbb{C}_x$ is β-stable for all $x \in X$.

By Proposition 1.3.1, there are $\mathbf{b}_i := (b_{i1}, b_{i2}, \dots, b_{is_i}) \in \mathbb{Z}^{\oplus s_i}$ and an autoequivalence $\Phi_{X \to X}^{\mathcal{F}^\vee[2]} : \mathbf{D}(X) \to \mathbf{D}(X)$ such that $\Phi_{X \to X}^{\mathcal{F}^\vee[2]}(\mathrm{Per}(X/Y)) = \mathcal{C}$, where $\mathrm{Per}(X/Y) := \mathrm{Per}(X/Y, \mathbf{b}_1, \dots, \mathbf{b}_n)$ and $\mathcal{F}$ is the family of $\Phi_{X \to X}^{\mathcal{F}}(\beta)$-stable objects of $\mathrm{Per}(X/Y)$ in Proposition 1.3.1. We set

$$A_{ij} := \begin{cases} \Phi_{X \to X}^{\mathcal{F}^\vee[2]}(A_0(\mathbf{b}_i)), & j = 0, \\ \Phi_{X \to X}^{\mathcal{F}^\vee[2]}(\mathcal{O}_{C_{ij}}(b_{ij})[1]), & j > 0. \end{cases} \tag{2.1}$$

Throughout this section, we assume the following:

Assumption 2.1.2. $v_0 := r_0 + \xi_0 + a_0 \varrho_X$, $r_0 > 0, \xi_0 \in \mathrm{NS}(X)$ is a primitive isotropic Mukai vector such that $\langle v_0, v(A_{ij}) \rangle < 0$ for all i, j.

By Proposition 1.1.5 (2), we have the following.

Lemma 2.1.3. *There is a local projective generator G of $\mathcal{C}$ whose Mukai vector is $2v_0$. More generally, for a sufficiently small $\alpha \in (v_0^\perp \cap \varrho_X^\perp) \otimes \mathbb{Q}$, there is a local projective generator G of $\mathcal{C}$ such that $v(G) \in \mathbb{Q}_{>0}(v_0 + \alpha)$.*

Remark 2.1.4. By Proposition 1.3.2, Assumptions 2.1.1, 2.1.2 are weak.

Let H be the pull-back of an ample divisor on Y. For a sufficiently small $\alpha \in (v_0^\perp \cap \varrho_X^\perp) \otimes \mathbb{Q}$, we take a local projective generator G of $\mathcal{C}$ with $v(G) \in \mathbb{Q}_{>0}(v_0 + \alpha)$. We define $v_0 + \alpha$-twisted semi-stability in a usual way. Since it is equivalent to the G-twisted semi-stability, we have the moduli space $\overline{M}_H^{v_0+\alpha}(v_0)$. Let $M_H^{v_0+\alpha}(v_0)$ be the moduli space of $v_0 + \alpha$-stable objects. By Corollary 1.3.3, $M_H^{v_0}(v_0) \neq \emptyset$. Hence we see that $M_H^{v_0+\alpha}(v_0)$ is also non-empty. Then we have the following which is well-known for the moduli of stable sheaves on $K3$ surfaces.

Proposition 2.1.5. (1) *$M_H^{v_0+\alpha}(v_0)$ is a smooth surface. If α is general, then $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$ is projective.*
(2) *If $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$, then it is a $K3$ surface.*

For the structure of $\overline{M}_H^{v_0}(v_0)$, as in [OY], we have the following.

Theorem 2.1.6 (cf. [OY, Thm. 0.1])**.** (1) *$\overline{M}_H^{v_0}(v_0)$ is normal and the singular points $q_1, q_2, \dots, q_m$ of $\overline{M}_H^{v_0}(v_0)$ correspond to the S-equivalence classes of properly v_0-twisted semi-stable objects.*
(2) *For a suitable choice of α with $|\langle \alpha^2 \rangle| \ll 1$, there is a surjective morphism $\pi : \overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0) \to \overline{M}_H^{v_0}(v_0)$ which becomes a minimal resolution of the singularities.*

(3) *Let* $\bigoplus_{j\geq 0} E_{ij}^{\oplus a'_{ij}}$ *be the S-equivalence class corresponding to* q_i, *where* E_{ij} *are* v_0*-twisted stable objects.*
   (a) *Then the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\geq 0}$ *is of affine type* $\tilde{A}, \tilde{D}, \tilde{E}$.
   (b) *Assume that* $a'_{i0}=1$. *Then the singularity of* $\overline{M}_H^{v_0}(v_0)$ *at* q_i *is a rational double point of type* A, D, E *according as the type of the matrix* $(-\langle v(E_{ij}), v(E_{ik})\rangle)_{j,k\geq 1}$.

Remark 2.1.7. A (-2)-vector $u \in L := v_0^{\perp} \cap \widehat{H}^{\perp} \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$ is *numerically irreducible*, if there is no decomposition $u = \sum_i b_i u_i$ such that $u_i \in L$, $\langle u_i^2\rangle = -2$, $\operatorname{rk} u > \operatorname{rk} u_i > 0$, $b_i \in \mathbb{Z}_{>0}$. If u is numerically irreducible, as we shall see in Proposition 2.2.14, there is a v_0-twisted stable object E with $v(E) = u$. In particular, if there is a decomposition $v_0 = \sum_{i\geq 0} a_i u_i$ such that $u_i \in L$ are numerically irreducible, $\langle u_i^2\rangle = -2$, $\operatorname{rk} u_i > 0$ and $a_i \in \mathbb{Z}_{>0}$, then there are v_0-stable objects E_i such that $v(E_i) = u_i$, and hence $v_0 = v(\bigoplus_i E_i^{\oplus a_i})$. Thus the types of the singularities are determined by the sublattice L of $H^*(X,\mathbb{Z})$.

We shall give a proof of this theorem in subsection 2.2. We assume that $\alpha \in (v_0^{\perp} \cap \varrho_X^{\perp}) \otimes \mathbb{Q}$ is general and set $X' := M_H^{v_0+\alpha}(v_0)$. X' is a $K3$ surface. We have a morphism $\phi : X' \to \overline{M}_H^{v_0}(v_0)$. We shall explain some cohomological properties of the Fourier-Mukai transform associated to X'. Let $\mathcal{E}$ be a universal family as a twisted object on $X' \times X$. For simplicity, we assume that $\mathcal{E}$ is an untwisted object on $X' \times X$. But all results hold even if $\mathcal{E}$ is a twisted object. We set

$$\begin{aligned} G_1 :=& \mathcal{E}_{|\{x'\}\times X} \in K(X), \\ G_2 :=& \mathcal{E}^{\vee}_{|X'\times\{x\}} \in K(X'), \\ G_3 :=& \mathcal{E}_{|X'\times\{x\}} \in K(X') \end{aligned} \tag{2.2}$$

for some $x \in X$ and $x' \in X'$. We also set

$$w_0 := v(\mathcal{E}^{\vee}_{|X'\times\{x\}}) = r_0 + \widetilde{\xi}_0 + \widetilde{a}_0 \varrho_{X'}, \widetilde{\xi}_0 \in \mathrm{NS}(X'). \tag{2.3}$$

We set $\Phi^{\alpha} := \Phi^{\mathcal{E}^{\vee}}_{X\to X'}$ and $\widehat{\Phi}^{\alpha} := \Phi^{\mathcal{E}}_{X'\to X}$. Thus

$$\Phi^{\alpha}(x) := \mathbf{R}\operatorname{Hom}_{p_{X'}}(\mathcal{E}, p_X^*(x)), x \in \mathbf{D}(X), \tag{2.4}$$

and $\widehat{\Phi}^{\alpha} : \mathbf{D}(X') \to \mathbf{D}(X)$ by

$$\widehat{\Phi}^{\alpha}(y) := \mathbf{R}\operatorname{Hom}_{p_X}(\mathcal{E}^{\vee}, p_{X'}^*(y)), y \in \mathbf{D}(X'), \tag{2.5}$$

where $\operatorname{Hom}_{p_Z}(-,-) = p_{Z*}\mathcal{H}om_{\mathcal{O}_{X'\times X}}(-,-)$, $Z = X, X'$ are the sheaves of relative homomorphisms.

Theorem 2.1.8 ([Br2], [O])**.** Φ^{α} *is an equivalence of categories and the inverse is given by* $\widehat{\Phi}^{\alpha}[2]$.

Definition 2.1.9. (1) We set

$$\begin{array}{cccc} \delta : & \mathrm{NS}(X)\otimes\mathbb{Q} & \to & H^*(X,\mathbb{Q}) \\ & D & \mapsto & D + \frac{(D,\xi_0)}{r_0}\varrho_X. \end{array} \tag{2.6}$$

(2) For $D \in H^2(X,\mathbb{Q})$, we set

$$\begin{aligned} \widehat{D} :=& -\left[\Phi^{\alpha}\left(\delta(D)\right)\right]_1 \\ =& \left[p_{X'*}\left(\left(c_2(\mathcal{E}) - \frac{r_0-1}{2r_0}(c_1(\mathcal{E})^2)\right) \cup p_X^*(D)\right)\right]_1 \in H^2(X',\mathbb{Q}), \end{aligned} \tag{2.7}$$

where $[\ ]_1$ means the projection to $H^2(X',\mathbb{Q})$.

The following result is a consequence of [Y7, Lem. 1.4.6, Lem. 1.4.8].

Lemma 2.1.10 (cf. [Y5, Lem. 1.4])**.** $r_0\widehat{H}$ *is a nef and big divisor on* X' *which defines a contraction* $\pi' : X' \to Y'$ *of* X' *to a normal surface* Y'. *There is a morphism* $\psi : Y' \to \overline{M}_H^{v_0}(v_0)$ *such that* $\phi = \psi \circ \pi'$.

Proof. Let G be a local projective generator of $\mathcal{C}$ such that $\tau(G) = 2\tau(G_1)$ (Lemma 2.1.3). Applying [Y7, Lem. 1.4.6], we have an ample line bundle $\mathcal{L}(\zeta)$ on $\overline{M}_H^G(v_0) = \overline{M}_H^{v_0}(v_0)$. By the definition of $\widehat{H}$, $c_1(\phi^*(\mathcal{L}(\zeta))) = r_0\widehat{H}$ ([Y7, Lem. 1.4.8]). Hence our claim holds. □

We use H (resp. $\widehat{H}$) to define $\deg_{G_1}(E)$ (resp. $\deg_{G_i}(E')$ $(i=2,3)$) for $E \in \mathbf{D}(X)$ (resp. $E' \in \mathbf{D}(X')$).

Proposition 2.1.11 (cf. [Y5, Prop. 1.5])**.** (1) *Every element* $v \in H^*(X,\mathbb{Z})$ *can be uniquely written as*

$$v = lv_0 + a\varrho_X + d\left(H + \frac{1}{r_0}(H,\xi_0)\varrho_X\right) + \left(D + \frac{1}{r_0}(D,\xi_0)\varrho_X\right),$$

where

$$l = \frac{\operatorname{rk} v}{\operatorname{rk} v_0} = -\frac{\langle v, \varrho_X \rangle}{\operatorname{rk} v_0} \in \frac{1}{r_0}\mathbb{Z},$$

$$a = -\frac{\langle v, v_0 \rangle}{\operatorname{rk} v_0} \in \frac{1}{r_0}\mathbb{Z}, \tag{2.8}$$

$$d = \frac{\deg_{G_1}(v)}{\operatorname{rk} v_0 (H^2)} \in \frac{1}{r_0(H^2)}\mathbb{Z}$$

and $D \in H^2(X, \mathbb{Q}) \cap H^\perp$. *Moreover* $v \in v(\mathbf{D}(X))$ *if and only if* $D \in \operatorname{NS}(X) \otimes \mathbb{Q} \cap H^\perp$.

(2)

$$\begin{aligned} &\Phi^\alpha \left(l v_0 + a \varrho_X + \left(dH + D + \frac{1}{r_0}(dH + D, \xi_0) \varrho_X \right) \right) \\ =& l \varrho_{X'} + a w_0 - \left(d\widehat{H} + \widehat{D} + \frac{1}{r_0}(d\widehat{H} + \widehat{D}, \widetilde{\xi}_0) \varrho_{X'} \right) \end{aligned} \tag{2.9}$$

where $D \in H^2(X, \mathbb{Q}) \cap H^\perp$.

(3)

$$\deg_{G_1}(v) = -\deg_{G_2}(\Phi^\alpha(v)).$$

In particular, $\deg_{G_2}(w) \in \mathbb{Z}$ *for* $w \in H^*(X', \mathbb{Z})$ *and*

$$\min\{\deg_{G_1}(E) > 0 | E \in K(X)\} = \min\{\deg_{G_2}(F) > 0 | F \in K(X')\}.$$

2.2. **Proof of Theorem 2.1.6.** We shall choose a special α and study the structure of the moduli spaces. We first prove the following. The normalness of $\overline{M}_H^{v_0}(v_0)$ will be proved in Proposition 2.2.13.

Proposition 2.2.1. (1) $\psi : Y' \to \overline{M}_H^{v_0}(v_0)$ *is bijective.*

(2) *The singular points of* Y' *correspond to properly* v_0*-twisted semi-stable objects.*

(3) *Let* $\bigoplus_{j \geq 0} E_{ij}^{\oplus a'_{ij}}$ *be the* S*-equivalence class of a properly* v_0*-twisted semi-stable object, where* E_{ij} *are* v_0*-twisted stable. Then the matrix* $(-\langle v(E_{ij}), v(E_{ik}) \rangle)_{j,k \geq 0}$ *is of affine type* $\tilde{A}, \tilde{D}, \tilde{E}$. *We assume that* $a_{i0} = 1$. *Then* $\psi^{-1}(\bigoplus_{j \geq 0} E_{ij}^{\oplus a'_{ij}})$ *is a rational double point of type* A, D, E *according as the type of the matrix* $(-\langle v(E_{ij}), v(E_{ik}) \rangle)_{j,k \geq 1}$.

2.2.1. *Proof of Proposition 2.2.1.* We note that $M_H^{v_0}(v_0)$ is smooth, and $\phi : X' \to \overline{M}_H^{v_0}(v_0)$ and $\psi : Y' \to \overline{M}_H^{v_0}(v_0)$ are isomorphic over $M_H^{v_0}(v_0)$. Hence the singular points of Y' are in the inverse image of $\overline{M}_H^{v_0}(v_0) \setminus M_H^{v_0}(v_0)$. Thus we may concentrate on the locus of properly v_0-twisted semi-stable objects. The first claim of Proposition 2.2.1 (3) follows from the following.

Lemma 2.2.2. *Assume that* E *is* S*-equivalent to* $\bigoplus_{j \geq 0} E_{ij}^{\oplus a'_{ij}}$, *where* E_{ij} *are* v_0*-twisted stable objects. Then the matrix* $(-\langle v(E_{ij}), v(E_{ik}) \rangle)_{j,k \geq 0}$ *is of type* $\widetilde{A}, \widetilde{D}, \widetilde{E}$. *Moreover* $\langle v(E_{ij}), v(E_{kl}) \rangle = 0$, *if* $\bigoplus_{j \geq 0} E_{ij}^{\oplus a'_{ij}} \not\cong \bigoplus_{l \geq 0} E_{kl}^{\oplus a'_{kl}}$.

Proof. We note that $\operatorname{rk}(\bullet) : K(X) \to \mathbb{Z}$ satisfies $\operatorname{rk} E_{ij} > 0$ for all i, j. Since $\deg_{G_1}(E) = \chi(G_1, E) = 0$, $\deg_{G_1}(E_{ij}) = \chi(G_1, E_{ij}) = 0$, which implies that $v(E_{ij}) \in v_0^\perp \cap \delta(H)^\perp$. Since $(v_0^\perp \cap \delta(H)^\perp)/\mathbb{Z}v_0$ is negative definite, applying Lemma [Y7, Lem. 3.1.1 (1)], we see that the matrix is of type $\widetilde{A}, \widetilde{D}, \widetilde{E}$. We note that $\bigoplus_{j \geq 0} E_{ij}^{\oplus a'_{ij}} \not\cong \bigoplus_{l \geq 0} E_{kl}^{\oplus a'_{kl}}$ implies that $\{E_{i0}, E_{i1}, ..., E_{is'_i}\} \neq \{E_{k0}, E_{k1}, ..., E_{ks'_k}\}$. Since $\chi(E_{ij}, E_{kl}) > 0$ implies that $E_{ij} \cong E_{kl}$, we have $\{v(E_{i0}), v(E_{i1}), ..., v(E_{is'_i})\} \neq \{v(E_{k0}), v(E_{k1}), ..., v(E_{ks'_k})\}$. Then the second claim follows from [Y7, Lem. 3.1.1 (2)]. □

By this lemma, we may assume that $a'_{i0} = 1$ for all i. Then we can choose a sufficiently small $\alpha \in v_0^\perp$ such that $-\langle \alpha, v(E_{ij}) \rangle > 0$ for all $j > 0$. We have the following.

Lemma 2.2.3. *Let* E_{ij} *be* v_0*-stable objects in Theorem 2.1.6. Assume that* $-(\alpha, c_1(E_{ij})) > 0$ *for all* $j > 0$. *Let* F *be a* v_0*-semi-stable object such that* $v(F) = v(E_{i0} \oplus \bigoplus_{j > 0} E_{ij}^{\oplus b_j})$, $0 \leq b_j \leq a_{ij}$.

(1) *If* $v(F) \neq v_0$, *then* F *is* S*-equivalent to* $E_{i0} \oplus \bigoplus_{j > 0} E_{ij}^{\oplus b_j}$ *with respect to* v_0*-stability.*

(2) *Assume that* F *is* S*-equivalent to* $E_{i0} \oplus \bigoplus_{j > 0} E_{ij}^{\oplus b_j}$. *Then the following conditions are equivalent.*
   (a) F *is* $v_0 + \alpha$*-stable*
   (b) F *is* $v_0 + \alpha$*-semi-stable*
   (c) $\operatorname{Hom}(E_{ij}, F) = 0$ *for all* $j > 0$.

(3) *Assume that* F *is* $v_0 + \alpha$*-stable. For a non-zero homomorphism* $\phi : F \to E_{ij}$, $j > 0$, ϕ *is surjective and* $F' := \ker \phi$ *is a* $v_0 + \alpha$*-stable object.*

(4) *If there is a non-trivial extension*

$$0 \to F \to F'' \to E_{ij} \to 0 \tag{2.10}$$

and $b_k + \delta_{jk} \le a_{ik}$, *then* F'' *is a* $v_0 + \alpha$*-stable object, where* $\delta_{jk} = 0, 1$ *according as* $j \ne k$, $j = k$.

Proof. The proof is similar to that of [Y7, Lem. 2.2.17]. (1) Assume that F is S-equivalent to $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}}$, where F_{ij} are v_0-twisted stable objects. If $v(F) = v(\bigoplus_{j\ge 0} E_{ij}^{\oplus b_{ij}})$, $b_{i0} = 1$, then applying Lemma 2.2.2 to $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}} \oplus \bigoplus_{j\ge 0} E_{ij}^{\oplus (a_{ij}-b_{ij})}$ and $\bigoplus_{j\ge 0} E_{ij}^{\oplus a_{ij}}$, we get $\bigoplus_{j\ge 0} F_{ij}^{\oplus c_{ij}} \oplus \bigoplus_{j>0} E_{ij}^{\oplus (a_{ij}-b_{ij})} \cong \bigoplus_{j\ge 0} E_{ij}^{\oplus a_{ij}}$, which implies the claim. Then the proofs of (2), (3) and (4) are the same as of [Y7, Lem. 2.2.17]. □

Lemma 2.2.4. (1) *We set*

$$C'_{ij} := \{x' \in X' \mid \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E_{ij}) \ne 0\}, j > 0. \tag{2.11}$$

Then C'_{ij} *is a smooth rational curve.*

(2)

$$\phi^{-1}(\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}}) = \{x' \in X' \mid \operatorname{Hom}(E_{i0}, \mathcal{E}_{|\{x'\}\times X}) \ne 0\} = \cup_j C'_{ij}. \tag{2.12}$$

In particular, ϕ *and* ψ *are surjective.*

Proof. The proof is the same as in [Y7, Lem. 2.2.22]. □

We also have the following lemma whose proof is the same as of [Y7, Lem. 2.3.11].

Lemma 2.2.5. $\Phi^\alpha(E_{ij})[1]$ *is a line bundle on* C'_{ij}. *In particular,* $\langle v(E_{ij}), v(E_{kl})\rangle = (C'_{ij}, C'_{kl})$. *We define* b'_{ij} *by* $\Phi^\alpha(E_{ij}) = \mathcal{O}_{C'_{ij}}(b'_{ij})[-1]$.

This lemma shows that the configuration of $\{C'_{ij} | j > 0\}$ is of type A, D, E. Since $(\widehat{H}, C'_{ij}) = 0$, $\cup_j C'_{ij}$ is contracted to a rational double point of Y'. Hence Proposition 2.2.1 (2) and (3) hold. Since $\psi^{-1}(\bigoplus_{j\ge 0} E_{ij}^{\oplus a'_{ij}})$ is a point, ψ is injective. Thus Proposition 2.2.1 (1) also holds.

We shall prove the normality in Proposition 2.2.13.

2.2.2. *Perverse coherent sheaves on* X' *and the normality of* $\overline{M}_H^{v_0}(v_0)$. We set $Z'_i := \pi^{-1}(q_i) = \sum_{j=1}^{s'_i} a'_{ij} C'_{ij}$. Then E_{i0} is a subobject of $\mathcal{E}_{|\{x'\}\times X}$ for $x' \in Z'_i$ and we have an exact sequence

$$0 \to E_{i0} \to \mathcal{E}_{|\{x'\}\times X} \to F \to 0,\ x' \in Z'_i \tag{2.13}$$

where F is a v_0-twisted semi-stable object with $\operatorname{gr}(F) = \bigoplus_{j=1}^{s'_i} E_{ij}^{\oplus a'_{ij}}$. Then we get an exact sequence

$$0 \to \Phi^\alpha(F)[1] \to \Phi^\alpha(E_{i0})[2] \to \mathbb{C}_{x'} \to 0 \tag{2.14}$$

in $\operatorname{Coh}(X')$. Thus WIT_2 holds for E_{i0} with respect to Φ^α.

Definition 2.2.6. We set $A'_{i0} := \Phi^\alpha(E_{i0})[2]$ and $A'_{ij} := \Phi^\alpha(E_{ij})[2] = \mathcal{O}_{C'_{ij}}(b'_{ij})[1]$ for $j > 0$.

Lemma 2.2.7. (1) $\operatorname{Hom}(A'_{i0}, A'_{ij}[-1]) = \operatorname{Ext}^1(A'_{i0}, A'_{ij}[-1]) = 0$.

(2) *We set* $\mathbf{b}'_i := (b'_{i1}, b'_{i2}, \dots, b'_{is'_i})$. *Then* $A'_{i0} \cong A_0(\mathbf{b}'_i)$. *In particular,* $\operatorname{Hom}(A'_{i0}, \mathbb{C}_{x'}) = \mathbb{C}$ *for* $x' \in Z'_i$.

(3) *Irreducible objects of* $\operatorname{Per}(X'/Y', \mathbf{b}'_1, ..., \mathbf{b}'_m)$ *are*

$$A'_{ij}\ (1 \le i \le m, 0 \le j \le s'_i),\ \mathbb{C}_{x'}\ (x' \in X' \setminus \cup_i Z'_i). \tag{2.15}$$

Proof. (1) We have

$$\begin{aligned} \operatorname{Hom}(A'_{i0}, A'_{ij}[k]) &= \operatorname{Hom}(\Phi^\alpha(E_{i0})[2], \Phi^\alpha(E_{ij})[2+k]) \\ &= \operatorname{Hom}(E_{i0}, E_{ij}[k]) = 0 \end{aligned} \tag{2.16}$$

for $k = -1, 0$.

(2) By (2.14) and (1), we can apply Lemma 1.1.3 to prove $A'_{i0} = A_0(\mathbf{b}'_i) = A_{q_i}$. (3) is a consequence of (2) and Proposition 1.1.4 □

Definition 2.2.8. We set

$$\begin{aligned} \operatorname{Per}(X'/Y') &:= \operatorname{Per}(X'/Y', \mathbf{b}'_1, \dots, \mathbf{b}'_m), \\ \operatorname{Per}(X'/Y')^D &:= \operatorname{Per}(X'/Y', {\mathbf{b}'_1}^D, \dots, {\mathbf{b}'_m}^D)^*. \end{aligned} \tag{2.17}$$

Remark 2.2.9. Assume that $\alpha \in v_0^\perp$ satisfies $-\langle v(E_{ij}), \alpha\rangle < 0$, $j > 0$. Then $\Phi(E_{ij})[2] = \mathcal{O}_{C'_{ij}}(b''_{ij})$, $j > 0$ and $\Phi(E_{i0})[2] = A_0(\mathbf{b}''_i)[1]$ belong to $\operatorname{Per}(X'/Y', \mathbf{b}''_1, \dots, \mathbf{b}''_m)^*$, where $\mathbf{b}''_i = (b''_{i0}, ..., b''_{is'_i})$.

Lemma 2.2.10. *There is a local projecive generator G of* $\operatorname{Per}(X'/Y')$ *such that* $\tau(G) = 2\tau(G_2)$. *Moreover* $G^\vee$ *is a local projective generator of* $\operatorname{Per}(X'/Y')^D$.

Proof. Since $\chi(G_2, A_{ij}) = \chi(\mathbb{C}_x, E_{ij}) = \operatorname{rk} E_{ij} > 0$, we get our claim by Proposition 1.1.5 (2). The second claim follows from the definition of $\operatorname{Per}(X'/Y')^D$ and Lemma 1.1.6. □

Lemma 2.2.11. *Let E be an object of $\mathcal{C}$ such that E is G_1-twisted stable and* $\deg_{G_1}(E) = \chi(G_1, E) = 0$. *Then $E \cong E_{ij}$ or $E \cong \mathcal{E}_{|\{x'\}\times X}$, $x' \in X' \setminus \cup_i Z_i'$.*

Proof. Since $\chi(G_1, E) = 0$, there is a point $x' \in X'$ such that $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$ or $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) \neq 0$. Then E is a quotient object or a subobject of $\mathcal{E}_{|\{x'\}\times X}$, which implies the claim. □

Definition 2.2.12. (1) Let $\mathcal{C}_{v_0}$ be the full subcategory of $\mathcal{C}$ generated by E_{ij} and $\mathcal{E}_{|\{x'\}\times X}$, $x' \in X'$. That is $\mathcal{C}_{v_0}$ consists of v_0-twisted semi-stable objects E with $\deg_{G_1}(E) = \chi(G_1, E) = 0$.

(2) Let $\operatorname{Per}(X'/Y')_0$ be the full subcategory of $\operatorname{Per}(X'/Y')$ consisting of 0-dimensional objects.

Proposition 2.2.13. (1) $\Phi^\alpha[2]$ *induces an equivalence* $\mathcal{C}_{v_0} \to \operatorname{Per}(X'/Y')_0$.

(2) *Moreover $\Phi^\alpha[2]$ induces an isomorphism* $\mathcal{M}_H^{v_0+\beta}(v_0)^{ss} \cong \mathcal{M}_{\widehat{H}}^{G,\Phi^\alpha(\beta)}(\varrho_{X'})^{ss}$, *where* $\beta \in (v_0^\perp \cap \varrho_X^\perp) \otimes \mathbb{Q}$ *is sufficiently small and G an arbitrary projective generator of* $\operatorname{Per}(X'/Y')$.

(3) $\overline{M}_H^{v_0+\beta}(v_0) \cong \overline{M}_{\widehat{H}}^{G,\Phi^\alpha(\beta)}(\varrho_{X'})$. *In particular,* $\overline{M}_H^{v_0}(v_0)$ *is a normal surface.*

Proof. (1) We note that $\Phi^\alpha(E_{ij})[2] = A'_{ij}$ and $\Phi^\alpha(\mathcal{E}_{|\{x'\}\times X})[2] = \mathbb{C}_{x'}$, $x' \in X'$. Hence the claim holds. (2) We note that $E \in \mathcal{M}_H^{v_0}(v_0)^{ss}$ is $v_0 + \beta$-twisted semi-stable, if $\chi(\beta, F) = \chi(v_0 + \beta, F) \leq 0$ for all subsheaf F of E with $\deg_{G_1}(F) = \chi(G_1, F) = 0$. Since $\chi(\Phi^\alpha(\beta), \Phi^\alpha(F)) = \chi(\beta, F)$, $\Phi^\alpha(E)[2]$ is $(G_2, \Phi^\alpha(\beta))$-twisted semi-stable. Then (1.16) implies that $\Phi^\alpha(E)[2]$ is $(G, \Phi^\alpha(\beta))$-twisted semi-stable for any G. The first claim of (3) follows from (2). In the notation of Definition 1.2.2, $\overline{M}_{\widehat{H}}^{G,0}(\varrho_{X'}) \cong (X')^0$. Hence the second claim of (3) follows from Proposition 1.2.3. □

Proposition 2.2.14. *Let* $u \in H^{ev}(X, \mathbb{Z})_{\mathrm{alg}}$ *be a Mukai vector such that* $u \in v_0^\perp \cap \delta(H)^\perp$, $0 < \operatorname{rk} u < \operatorname{rk} v_0$ *and* $\langle u^2 \rangle = -2$. *Then* $u = \sum_j b_j v(E_{ij})$, $0 \leq b_j \leq a_{ij}$. *In particular,* $\overline{M}_H^{v_0}(u) \neq \emptyset$.

Proof. Since $u \in v_0^\perp \cap \delta(H)^\perp$, $\Phi^\alpha(u) = (0, D, b)$, $D \in \operatorname{NS}(X')$, $b \in \mathbb{Z}$ and $(D, \widehat{H}) = 0$. Since $(D^2) = -2$, D or $-D$ is an effective divisor supported on an exceptional locus Z_i'. Hence $\Phi^\alpha(u) \in \oplus_{j=0}^{s_i'} \mathbb{Z}\Phi^\alpha(E_{ij}) = \oplus_{j=1}^{s_i'} \mathbb{Z} C_{ij} \oplus \mathbb{Z}\varrho_X$. By the basic properties of the root systems of affine Lie algebra, $\Phi^\alpha(u) = c\Phi^\alpha(v_0) \pm \sum_{j>0} c_j \Phi^\alpha(E_{ij})$, $0 \leq c_j \leq a_{ij}$. Then $\operatorname{rk} u = cr \pm \sum_{j>0} c_j \operatorname{rk} E_{ij}$. Since $\sum_{j>0} c_j \operatorname{rk} E_{ij} \leq \sum_{j>0} a_{ij} \operatorname{rk} E_{ij} < r$, we get $u = \sum_{j>0} c_j v(E_{ij})$ or $u = v_0 - \sum_{j>0} c_j v(E_{ij})$. Therefore the claim holds. □

2.3. **Walls and chambers for the moduli spaces of dimension 2.**

We shall study the dependence of $\overline{M}_H^w(v_0)$ on w. We may assume that $w = v_0 + \alpha$, $\alpha \in \delta(H^\perp)$ (cf. [OY, sect. 1.1]). We set

$$\mathcal{U} := \left\{ u \in v(\mathbf{D}(X)) \,\middle|\, \begin{array}{l} \langle u^2 \rangle = -2, \langle v_0, u \rangle \leq 0, \langle \delta(H), u \rangle = 0, \\ 0 < \operatorname{rk} u < \operatorname{rk} v_0 \end{array} \right\}. \tag{2.18}$$

For a fixed v_0 and H, $\mathcal{U}$ is a finite set. For $u \in \mathcal{U}$, we define a wall $W_u \subset \delta(H^\perp) \otimes_{\mathbb{Q}} \mathbb{R}$ with respect to v by

$$W_u := \{\alpha \in \delta(H^\perp) \otimes \mathbb{R} \mid \langle v_0 + \alpha, u \rangle = 0\}. \tag{2.19}$$

A connected component of $\delta(H^\perp) \otimes_{\mathbb{Q}} \mathbb{R} \setminus \cup_{u \in \mathcal{U}} W_u$ is said to be a chamber.

Lemma 2.3.1. *If α does not lie on any wall W_u, $u \in \mathcal{U}$, then* $\overline{M}_H^{v_0+\alpha}(v_0) = M_H^{v_0+\alpha}(v_0)$. *In particular,* $\overline{M}_H^{v_0+\alpha}(v_0)$ *is a K3 surface.*

We are interested in the $v_0 + \alpha$-twisted stability with a sufficiently small $|\langle \alpha^2 \rangle|$. So we may assume that

$$u \in \mathcal{U}' := \{u \in \mathcal{U} \mid \langle v_0, u \rangle = 0\}. \tag{2.20}$$

For an $\alpha \in \delta(H^\perp)$ with $|\langle \alpha^2 \rangle| \ll 1$, let F be a $v_0 + \alpha$-twisted stable torsion free object such that

(i) $\langle v(F)^2 \rangle = -2$,
(ii) $\langle v(F), \delta(H) \rangle / \operatorname{rk} F = (c_1(F), H)/\operatorname{rk} F - (\xi_0, H)/r_0 = 0$ and
(iii) $\langle v_0, v(F) \rangle = \langle \alpha, v(F) \rangle = 0$.

By (i), F is a rigid torsion free object.

Proposition 2.3.2 ([OY, Prop. 1.12])**.** *We set* $\alpha^\pm := \pm\epsilon v(F) + \alpha$, *where* $0 < \epsilon \ll 1$. *Then T_F induces an isomorphism*

$$\begin{array}{ccc} \mathcal{M}_H^{v+\alpha^-}(v)^{ss} & \to & \mathcal{M}_H^{v+\alpha^+}(v)^{ss} \\ E & \mapsto & T_F(E) \end{array} \tag{2.21}$$

which preserves the S-equivalence classes. Hence we have an isomorphism

$$\overline{M}_H^{v+\alpha^-}(v) \to \overline{M}_H^{v+\alpha^+}(v). \tag{2.22}$$

Remark 2.3.3. In [OY], we considered the functor $T_F[-1]$.

Combining Proposition 2.3.2 with [Y7, Lem. 2.3.20], we get the following Corollary.

Corollary 2.3.4.

$$\Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^+}} \cong T_F \circ \Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^-}} \cong \Phi_{X'\to X}^{\mathcal{E}^{v_0+\alpha^-}} \circ T_A, \tag{2.23}$$

where $A := \Phi_{X\to X'}^{(\mathcal{E}^{v_0+\alpha^-})^\vee[2]}(F)$.

Assume that $\mathcal{E}^{v_0+\alpha}_{|\{x'\}\times X}$ is S-equivalent to $\bigoplus_i E_i'^{\oplus a_i'}$. Then $\alpha \in (\sum_i \mathbb{Q}v(E_i'))^\perp$.

Remark 2.3.5. If α belongs to exactly one wall W_u, $u \in \mathcal{U}'$, then there is a $v+\alpha$-twisted stable object F with $v(F) = u$. So we can apply Propositions 2.3.2. Moreover $A = \mathcal{O}_C(b)$, where C is a smooth rational curve defined by

$$C := \{x' \in X' \mid \operatorname{Ext}^2(\mathcal{E}^{v_0+\alpha^-}_{|\{x'\}\times X}, F) \neq 0\}. \tag{2.24}$$

Proposition 2.3.6. *Let G be an object of $\mathbf{D}(X)$ such that $\chi(G, E_{ij}) > 0$ for all i, j and*

$$\operatorname{Hom}(G, E_{ij}[k]) = \operatorname{Hom}(G, E[k]) = 0, k \neq 2 \tag{2.25}$$

for all $E \in M_H^{G_1}(v_0)$ and i, j. Assume that $\alpha \in \delta(H^\perp) \setminus \cup_{u\in\mathcal{U}'} W_u$ is sufficiently small.

(1) *$G^\alpha := \Phi^\alpha(G)$ is a locally free sheaf on X' and $\mathcal{A}' := \pi_*((G^\alpha)^\vee \otimes G^\alpha)$ is a reflexive sheaf on Y' which is independent of the choice of α.*
(2) *$\mathbf{R}\pi_*((G^\alpha)^\vee \otimes \_) \circ \Phi^\alpha : \mathbf{D}(X) \to \mathbf{D}_{\mathcal{A}'}(Y')$ is independent of the choice of α.*

Proof. We take a small $\alpha \in \delta(H^\perp)$ with $-\langle \alpha, v(E_{ij})\rangle > 0$, $j > 0$. By the base change theorem, G^α is a locally free sheaf on X'. Let A'_{ij} be objects of $\operatorname{Per}(X'/Y')$ in subsection 2.2. Then we have $\operatorname{Hom}(G^\alpha, A'_{ij}[k]) = 0$ for $k \neq 0$ and $\operatorname{Hom}(G^\alpha, A'_{ij}) \neq 0$. Assume that $\alpha' \in \delta(H^\perp)$ belongs to another chamber. We set $X'' := M_H^{v_0+\alpha'}(v_0)$. By Proposition 2.2.13 (2), $X'' \cong M_{\widehat{H}}^{G^\alpha, \Phi^\alpha(\alpha')}(\varrho_{X'})$ and $\mathcal{F} := \Phi_{X\to X'}^{(\mathcal{E}^\alpha)^\vee[2]}(\mathcal{E}^{\alpha'})$ is the universal family of $\Phi^\alpha(\alpha')$-twisted stable objects, where $\mathcal{E}^{\alpha'}$ is the universal family associated to α'. We have $\Phi^{\alpha'} = \Phi_{X'\to X''}^{\mathcal{F}^\vee[2]} \circ \Phi^\alpha$. In particular, $G^{\alpha'} = \Phi_{X'\to X''}^{\mathcal{F}^\vee[2]}(G^\alpha)$. Then the claim follows from [Y7, Prop. 2.3.4]. □

2.4. **A tilting appeared in** [Br4] **and its generalizations.** From now on, we assume that α satisfies $-\langle \alpha, v(E_{ij})\rangle > 0$ for all $j > 0$ and set

$$\Phi := \Phi^\alpha, \ \widehat{\Phi} := \widehat{\Phi}^\alpha. \tag{2.26}$$

By Proposition 2.3.6, the assumption is not essential.

Definition 2.4.1. We set

$$\mathfrak{C}_i := \begin{cases} \mathcal{C}, & i = 1, \\ \operatorname{Per}(X'/Y'), & i = 2, \\ \operatorname{Per}(X'/Y')^D, & i = 3. \end{cases} \tag{2.27}$$

For an object $E \in \mathfrak{C}_i$, we define the G_i-twisted Hilbert polynomial by

$$\chi(G_i, E(n)) := \sum_j (-1)^j \dim \operatorname{Hom}(G_i, E(n)[j]), \tag{2.28}$$

where $E(n) := E(nH)$, $i = 1$ and $E(n) := E(n\widehat{H})$, $i = 2, 3$.

Then Lemma 2.1.3 and Lemma 2.2.10 imply the following.

Lemma 2.4.2. *$\chi(G_i, E(n)) > 0$ for $E \neq 0$ and $n \gg 0$, that is, (i) $\operatorname{rk} E > 0$ or (ii) $\operatorname{rk} E = 0, \deg_{G_i}(E) > 0$ or (iii) $\operatorname{rk} E = \deg_{G_i}(E) = 0, \chi(G_i, E) > 0$.*

Definition 2.4.3. Let $E \neq 0$ be an object of $\mathfrak{C}_i$.

(1) There is a (unique) filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{2.29}$$

such that each $E_j := F_j/F_{j-1}$ is a torsion object or a torsion free G_i-twisted semi-stable object and

$$(\operatorname{rk} E_{j+1})\chi(G_i, E_j(n)) > (\operatorname{rk} E_j)\chi(G_i, E_{j+1}(n)), n \gg 0. \tag{2.30}$$

We call it the *Harder-Narasimhan filtration* of E.

(2) In the notation of (1), we set

$$
\begin{aligned}
\mu_{\max,G_i}(E) &:= \begin{cases} \mu_{G_i}(E_1), & \operatorname{rk} E_1 > 0 \\ \infty, & \operatorname{rk} E_1 = 0, \end{cases} \\
\mu_{\min,G_i}(E) &:= \begin{cases} \mu_{G_i}(E_s), & \operatorname{rk} E_s > 0 \\ \infty, & \operatorname{rk} E_s = 0. \end{cases}
\end{aligned} \tag{2.31}
$$

Remark 2.4.4. An object $E \neq 0$ has a torsion if and only if $\mu_{\max,G_i}(E) = \infty$ and E is a torsion object if and only if $\mu_{\min,G_i}(E) = \infty$.

We define several torsion pairs of $\mathfrak{C}_i$.

Definition 2.4.5. (1) Let $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{T}}_i^\mu$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{T}}_i^\mu$) if (i) E is a torsion object or (ii) $\mu_{\min,G_i}(E) > 0$ (resp. $\mu_{\min,G_i}(E) \geq 0$).

(2) Let $\mathfrak{F}_i^\mu$ (resp. $\overline{\mathfrak{F}}_i^\mu$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i^\mu$ (resp. $\overline{\mathfrak{F}}_i^\mu$) if $E = 0$ or E is a torsion free object with $\mu_{\max,G_i}(E) \leq 0$ (resp. $\mu_{\max,G_i}(E) < 0$).

Definition 2.4.6. (1) Let $\mathfrak{T}_i$ (resp. $\overline{\mathfrak{T}}_i$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{T}_i$ (resp. $\overline{\mathfrak{T}}_i$) if (i) E is a torsion object or (ii) for the Harder-Narasimhan filtration (2.29) of E, E_s satisfies $\mu_{G_i}(E_s) > 0$ or $\mu_{G_i}(E_s) = 0$ and $\chi(G_i, E_s) > 0$ (resp. $\mu_{G_i}(E_s) = 0$ and $\chi(G_i, E_s) \geq 0$).

(2) Let $\mathfrak{F}_i$ (resp. $\overline{\mathfrak{F}}_i$) be the full subcategory of $\mathfrak{C}_i$ such that $E \in \mathfrak{C}_i$ belongs to $\mathfrak{F}_i$ (resp. $\overline{\mathfrak{F}}_i$) if E is a torsion free object and for the Harder-Narasimhan filtration (2.29) of E, E_1 satisfies $\mu_{G_i}(E_1) < 0$ or $\mu_{G_i}(E_1) = 0$ and $\chi(G_i, E_1) \leq 0$ (resp. $\mu_{G_i}(E_1) = 0$ and $\chi(G_i, E_1) < 0$).

Definition 2.4.7. $(\mathfrak{T}_i^\mu, \mathfrak{F}_i^\mu)$, $(\overline{\mathfrak{T}}_i^\mu, \overline{\mathfrak{F}}_i^\mu)$, $(\mathfrak{T}_i, \mathfrak{F}_i)$ and $(\overline{\mathfrak{T}}_i, \overline{\mathfrak{F}}_i)$ are torsion pairs of $\mathfrak{C}_i$. We denote the tiltings of $\mathfrak{C}_i$ by $\mathfrak{A}_i^\mu$, $\overline{\mathfrak{A}}_i^\mu$, $\mathfrak{A}_i$ and $\overline{\mathfrak{A}}_i$ respectively.

We note that $\mathfrak{T}_1^\mu \subset \mathfrak{T}_1$. We shall study the condition $\mathfrak{T}_1^\mu = \mathfrak{T}_1$. We start with the following lemma.

Lemma 2.4.8. *Let E be a local projective generator of $\mathfrak{C}_i$. Then $\operatorname{Ext}^1(E, F) = 0$ for all 0-dimensional objects F of $\mathfrak{C}_i$. In particular, if E is a subobject of a torsion free object E' such that E'/E is 0-dimensional, then $E' = E$.*

Proof. We only treat the case where $i = 1$. Then $\mathbf{R}\pi_*(E^\vee \otimes F) = \pi_*(E^\vee \otimes F)$ is a 0-dimensional sheaf on Y. Hence we get $\operatorname{Ext}^1(E, F) = H^1(Y, \pi_*(E^\vee \otimes F)) = 0$. □

Lemma 2.4.9. *Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator of $\mathcal{C}$ for a general $x' \in X'$.*

(1) $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$.

(2) *Every μ-semi-stable object $E \in \mathcal{C}$ with $\deg_{G_1}(E) = \chi(G_1, E) = 0$ is G_1-twisted semi-stable. Moreover if E is G_1-twisted stable, then it is μ-stable.*

(3) *Let E be a μ-semi-stable object $E \in \mathcal{C}$ with $\operatorname{rk} E > 0$, $\deg_{G_1}(E) = \chi(G_1, E) = 0$. Then $\operatorname{Ext}^i(E, S) = 0$, $i \neq 0$ for any irreducible object $S \in \mathcal{C}$.*

(4) *$\mathcal{E}_{|\{x'\}\times X}$ is a local projective generator of $\mathcal{C}$ for any $x' \in X'$.*

Proof. (1) Let E be a μ-stable object of $\mathcal{C}$ with $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) > 0$. Since $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all $x' \in X'$, $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$ for all $x' \in X'$. Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator. By Lemma 2.4.8 and $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) \neq 0$, we get $E \cong \mathcal{E}_{|\{x'\}\times X}$. Therefore $\chi(G_1, E) \leq 0$ for all μ-stable object $E \in \mathcal{C}$ with $\deg_{G_1}(E) = 0$. Hence we get $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$.

(2) Let E' be a subobject of E with $\deg_{G_1}(E') = 0$. Then (1) implies that $\chi(G_1, E') \leq 0$. Hence E is G_1-twisted semi-stable. If E/E' is torsion free, then we also have $\chi(G_1, E/E') \leq 0$, which implies that $\chi(G_1, E') = \chi(G_1, E/E') = 0$. Thus E is properly G_1-twisted semi-stable. Therefore the second claim also holds.

(3) If $\operatorname{Ext}^1(S, E) = \operatorname{Ext}^1(E, S)^\vee \neq 0$, then a non-trivial extension

$$0 \to E \to E' \to S \to 0 \tag{2.32}$$

gives a μ-semi-stable object E' with $\chi(G_1, E') = \chi(G_1, S) > 0$. On the other hand, (1) implies that $\chi(G_1, E') \leq 0$. Therefore $\operatorname{Ext}^1(E, S) = 0$. Since S is a torsion object, $\operatorname{Ext}^2(E, S) \cong \operatorname{Hom}(S, E)^\vee = 0$.

(4) Since $\mathcal{E}_{|\{x'\}\times X}$ is a μ-semi-stable object with $\deg_{G_1}(\mathcal{E}_{|\{x'\}\times X}) = \chi(G_1, \mathcal{E}_{|\{x'\}\times X}) = 0$, $\mathcal{E}_{|\{x'\}\times X} \in \mathcal{C}$ and satisfies the assertion of (3). By Lemma 2.4.2, $\chi(\mathcal{E}_{|\{x'\}\times X}, S) = \chi(G_1, S) > 0$ for any irreducible object S. Then $\mathcal{E}_{|\{x'\}\times X}$ is locally free and is a local projective generator by Proposition 1.1.5. □

Remark 2.4.10. By the proof of Lemma 2.4.9, $\mathcal{E}_{|\{x'\}\times X}$, $x' \in X'$ is a local projective generator of $\mathcal{C}$ if $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$. Indeed if $\mathfrak{T}_1 = \mathfrak{T}_1^\mu$, then the same proofs of (2), (3) and (4) work.

2.5. **Equivalence between $\mathfrak{A}_1$ and $\mathfrak{A}_2^\mu$.**

Lemma 2.5.1. (1) *If $E \in \mathfrak{T}_1$, then $\operatorname{Hom}(E, E_{ij}) = \operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all i, j and $x' \in X'$.*

(2) *If $E \in \mathfrak{F}_1$, then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = 0$ for a general $x' \in X'$. In particular, $H^0(\Phi(E)) = 0$.*

Proof. (1) The first claim is obvious. (2) If there is a non-zero morphism $\phi : \mathcal{E}_{|\{x'\}\times X} \to E$, we see that ϕ is injective and $\operatorname{coker}\phi \in \mathfrak{F}_1$. By the induction on $\operatorname{rk} E$, we get the first claim. The second claim follows by the base change theorem. □

Lemma 2.5.2. *Let E be an object of $\mathcal{C}$.*

(1) *Assume that $\operatorname{Hom}(E_{ij}, E[q]) = \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E[q]) = 0$ for all i, j, $x' \in X'$ and $q > 0$. Then $\Phi(E) \in \operatorname{Per}(X'/Y')$.*

(2) *There is a complex*

$$0 \to W_0 \to W_1 \to W_2 \to 0 \tag{2.33}$$

such that W_i are local projective objects of $\operatorname{Per}(X'/Y')$ and $\Phi(E)$ is quasi-isomorphic to this complex.

(3) *$H^0({}^pH^2(\Phi(E))) = H^2(\Phi(E))$ and ${}^pH^0(\Phi(E)) \subset H^0(\Phi(E))$. In particular, ${}^pH^0(\Phi(E))$ is torsion free.*

(4) *If $\operatorname{Hom}(E, E_{ij}) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all $x' \in X'$, then ${}^pH^2(\Phi(E)) = 0$. In particular, if $E \in \mathfrak{T}_1$, then ${}^pH^2(\Phi(E)) = 0$.*

(5) *If $E \in \mathfrak{F}_1$, then ${}^pH^0(\Phi(E)) = 0$.*

Proof. (1) Since $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E[q]) = 0$ for all $x' \in X'$ and $q \neq 0$, the base change theorem implies that $H^q(\Phi(E)) = 0$ for $q \neq 0$ and $H^0(\Phi(E))$ is a locally free sheaf on X'. In particular, ${}^pH^q(\Phi(E)) = 0$ unless $q = 0, 1$. We note that $F \in \operatorname{Per}(X'/Y')$ is 0 if and only if $\operatorname{Hom}(F, A'_{ij}) = \operatorname{Hom}(F, A'_{i0}) = \operatorname{Hom}(F, \mathbb{C}_{x'}) = 0$ for all i, $j > 0$ and $x' \in X'$. Since

$$\begin{aligned} \operatorname{Hom}(\Phi(E)[q], \Phi(E_{ij})[2]) &\cong \operatorname{Hom}(E[q], E_{ij}[2]) \cong \operatorname{Hom}(E_{ij}, E[q])^\vee, \\ \operatorname{Hom}(\Phi(E)[q], \Phi(\mathcal{E}_{|\{x'\}\times X})[2]) &\cong \operatorname{Hom}(E[q], \mathcal{E}_{|\{x'\}\times X}[2]) \cong \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E[q])^\vee, \end{aligned} \tag{2.34}$$

we have ${}^pH^q(\Phi(E)) = 0$ for $q > 0$, which implies that $\Phi(E) \in \operatorname{Per}(X'/Y')$. Thus the claim (1) holds.

(2)

We take a resolution of E

$$0 \to V_{-2} \to V_{-1} \to V_0 \to E \to 0 \tag{2.35}$$

such that $V_{-k} = G(-n_k H)^{\oplus N_k}$, $n_k \gg 0$ for $k = 0, 1$, where G is a local projective generator of $\mathcal{C}$. By using the Serre duality, our choice of n_k implies that $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, V_{-k}[q]) = \operatorname{Hom}(E_{ij}, V_{-k}[q]) = 0$ for $q \neq 2$ and $k = 0, 1$. Then we also have $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, V_{-2}[q]) = \operatorname{Hom}(E_{ij}, V_{-2}[q]) = 0$ for $q \neq 2$. Hence $\Phi(V_{-k})[2]$, $k = 0, 1, 2$ are locally free sheaves on X'. Since $\operatorname{Hom}(\Phi(V_{-k})[2], A'_{ij}[q]) = \operatorname{Hom}(\Phi(V_{-k})[2], \Phi(E_{ij})[2+q]) = \operatorname{Hom}(V_{-k}, E_{ij}[q]) = 0$, $q > 0$, $W_{2-k} := \Phi(V_{-k})[2]$, $k = 0, 1, 2$ are local projective objects of $\operatorname{Per}(X'/Y')$ and the associated complex $W_\bullet$ defines the required complex.

(3) is obvious. (4) follows from the proof of (1) and Lemma 2.5.1 (1). (5) follows from (3) and Lemma 2.5.1 (2). □

Corollary 2.5.3. *For $F \in \operatorname{Per}(X'/Y')$, ${}^pH^q(\widehat{\Phi}(F)) = 0$ unless $q = 0, 1, 2$.*

Proof. For any $E \in \mathcal{C}$, Lemma 2.5.2 (2) implies that $\Phi(E)$ is generated by ${}^pH^q(\Phi(E))[-q]$ $(q = 0, 1, 2)$. Hence $\operatorname{Hom}(\widehat{\Phi}(F)[q], E) = \operatorname{Hom}(F, \Phi(E)[-q+2]) = 0$ for $q > 2$ and $\operatorname{Hom}(E, \widehat{\Phi}(F)[q]) = \operatorname{Hom}(\Phi(E), F[q-2]) = 0$ for $q < 0$, which implies our claim. □

Definition 2.5.4. (1) We set $\Phi^i(E) := {}^pH^i(\Phi(E)) \in \operatorname{Per}(X'/Y')$ and $\widehat{\Phi}^i(E) := {}^pH^i(\widehat{\Phi}(E)) \in \mathcal{C}$.

(2) We say that WIT_i holds for $E \in \mathcal{C}$ (resp. $F \in \operatorname{Per}(X'/Y')$) with respect to Φ (resp. $\widehat{\Phi}$), if $\Phi^j(E) = 0$ (resp. $\widehat{\Phi}^j(F)) = 0$) for $j \neq i$.

Lemma 2.5.5. *Let E be an object of $\mathcal{C}$.*

(1) *If WIT_0 holds for E with respect to Φ, then $E \in \mathfrak{T}_1$.*

(2) *If WIT_2 holds for E with respect to Φ, then $E \in \mathfrak{F}_1$. In particular, E is torsion free. Moreover if $\Phi^2(E)$ does not contain a 0-dimensional object, then $E \in \overline{\mathfrak{F}}_1^\mu$.*

Proof. For an object $E \in \mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.36}$$

such that $E_1 \in \mathfrak{T}_1$ and $E_2 \in \mathfrak{F}_1$. Applying Φ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi^0(E_1) & \longrightarrow & \Phi^0(E) & \longrightarrow & \Phi^0(E_2) & & \\ & \longrightarrow & \Phi^1(E_1) & \longrightarrow & \Phi^1(E) & \longrightarrow & \Phi^1(E_2) & & \\ & \longrightarrow & \Phi^2(E_1) & \longrightarrow & \Phi^2(E) & \longrightarrow & \Phi^2(E_2) & \longrightarrow & 0. \end{array} \tag{2.37}$$

By Lemma 2.5.2 (4),(5), $\Phi^0(E_2) = \Phi^2(E_1) = 0$. If WIT_0 holds for E, then we get $\Phi(E_2) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Phi(E_1) = 0$. Thus the first part of (2) holds. Assume that there is an exact sequence

$$0 \to E_2' \to E \to E_2'' \to 0 \tag{2.38}$$

such that E_2' is a μ-semi-stable object with $\deg_{G_1}(E_2') = 0$ and $E_2'' \in \overline{\mathfrak{F}}_1^\mu$. By the first part of (2), we get $\chi(G_1, E_2') \leq 0$. By Lemma 2.5.2 (5), $\Phi^0(E_2'') = 0$. Then we see that WIT_2 holds for E_2' and $\deg_{G_2}(\Phi^2(E_2')) = -\deg_{G_1}(E_2') = 0$. Since $\operatorname{rk}\Phi^2(E_2') = \chi(G_1, E_2') \leq 0$, $\Phi^2(E_2')$ is a 0-dimensional object. By our assumption, we get that $\Phi^1(E_2'') \to \Phi^2(E_2')$ is an isomorphism. By Lemma 5.1.1 in the appendix, we have $\widehat{\Phi}^0(\Phi^1(E_2'')) = 0$, which implies that $E_2' \cong \widehat{\Phi}^0(\Phi^2(E_2')) = 0$. □

Lemma 2.5.6. *For an object $E \in \mathcal{C}$, $\deg_{G_2}(\Phi^0(E)) \leq 0$ and $\deg_{G_2}(\Phi^2(E)) \geq 0$.*

Proof. We note that

$$\widehat{\Phi}(\Phi^0(E)) = \widehat{\Phi}^2(\Phi^0(E))[-2],\ \widehat{\Phi}(\Phi^2(E)) = \widehat{\Phi}^0(\Phi^2(E)) \tag{2.39}$$

and

$$\deg_{G_2}(\Phi^0(E)) = -\deg_{G_1}(\widehat{\Phi}^2(\Phi^0(E))),\ \deg_{G_2}(\Phi^2(E)) = -\deg_{G_1}(\widehat{\Phi}^0(\Phi^2(E))). \tag{2.40}$$

Since $\widehat{\Phi}^2(\Phi^0(E))$ satisfies WIT_0 with respect to Φ, $\widehat{\Phi}^2(\Phi^0(E)) \in \mathfrak{T}_1$, which implies that $\deg_{G_1}(\widehat{\Phi}^2(\Phi^0(E))) \geq 0$. Since $\widehat{\Phi}^0(\Phi^2(E))$ satisfies WIT_2 with respect to Φ, $\widehat{\Phi}^0(\Phi^2(E)) \in \mathfrak{F}_1$, which implies that $\deg_{G_1}(\widehat{\Phi}^0(\Phi^2(E))) \leq 0$. Therefore our claims hold. □

Lemma 2.5.7. (1) *If $F \in \mathfrak{T}_2^\mu$, then $\widehat{\Phi}^2(F) = 0$.*
(2) *If WIT_0 holds for $F \in \operatorname{Per}(X'/Y')$ with respect to $\widehat{\Phi}$, then $F \in \mathfrak{T}_2^\mu$.*
(3) *If $F \in \mathfrak{F}_2^\mu$, then $\widehat{\Phi}^0(F) = 0$.*
(4) *If WIT_2 holds for $F \in \operatorname{Per}(X'/Y')$ with respect to $\widehat{\Phi}$, then $F \in \mathfrak{F}_2^\mu$.*

Proof. (1) By Lemma 5.1.1 in the appendix, we have an exact sequence

$$F \to \Phi^0(\widehat{\Phi}^2(F)) \xrightarrow{\phi} \Phi^2(\widehat{\Phi}^1(F)) \to 0. \tag{2.41}$$

By Lemma 2.5.6, $\deg_{G_2}(\ker\phi) \leq 0$. Since $\Phi^0(\widehat{\Phi}^2(F))$ is torsion free, $\ker\phi$ is also torsion free. By our assumption of F, we have $\ker\phi = 0$. Then $\Phi^0(\widehat{\Phi}^2(F)) \cong \Phi^2(\widehat{\Phi}^1(F))$ satisfies WIT_0 and WIT_2, which implies that $\Phi^0(\widehat{\Phi}^2(F)) \cong \Phi^2(\widehat{\Phi}^1(F)) \cong 0$. Therefore $\widehat{\Phi}^2(F) = 0$.

(2) Assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0 \tag{2.42}$$

such that $F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. By (1), we have $\widehat{\Phi}^2(F_1) = 0$. By a similar exact sequence to (2.37), we see that WIT_0 holds for F_2 and $\deg_{G_1}(\widehat{\Phi}^0(F_2)) = -\deg_{G_2}(F_2) \geq 0$. On the other hand, since WIT_2 holds for $\widehat{\Phi}^0(F_2)$, Lemma 2.5.5 implies that $\widehat{\Phi}^0(F_2) \in \mathfrak{F}_1$. Hence $\deg_{G_1}(\widehat{\Phi}^0(F_2)) = 0$ and $\chi(G_1, \widehat{\Phi}^0(F_2)) \leq 0$. Since $\chi(G_1, \widehat{\Phi}^0(F_2)) = \operatorname{rk} F_2$, we have $\operatorname{rk} F_2 = 0$. Since $\mathfrak{F}_2^\mu$ contains no torsion object except 0, we conclude that $F_2 = 0$.

(3) By Lemma 5.1.1, we have an exact sequence

$$0 \to \Phi^0(\widehat{\Phi}^1(F)) \xrightarrow{\psi} \Phi^2(\widehat{\Phi}^0(F)) \to F. \tag{2.43}$$

By (2), $\Phi^2(\widehat{\Phi}^0(F)) \in \mathfrak{T}_2^\mu$, which implies that $\operatorname{coker}\psi = 0$. Then $\Phi^0(\widehat{\Phi}^1(F)) \cong \Phi^2(\widehat{\Phi}^0(F))$ satisfies WIT_0 and WIT_2, which implies that $\Phi^0(\widehat{\Phi}^1(F)) \cong \Phi^2(\widehat{\Phi}^0(F)) \cong 0$. Therefore $\widehat{\Phi}^0(F) = 0$.

(4) Assume that there is an exact sequence

$$0 \to F_1 \to F \to F_2 \to 0 \tag{2.44}$$

such that $0 \neq F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. By (3), $\widehat{\Phi}^0(F_2) = 0$. By a similar exact sequence to (2.37), we see that WIT_2 holds for F_1 and $\deg_{G_1}(\widehat{\Phi}^2(F_1)) = -\deg_{G_2}(F_1) \leq 0$. Moreover if $\operatorname{rk} F_1 > 0$, then $\deg_{G_1}(\widehat{\Phi}^2(F_1)) < 0$. On the other hand, since WIT_0 holds for $\widehat{\Phi}^2(F_1)$, Lemma 2.5.5 implies that $\widehat{\Phi}^2(F_1) \in \mathfrak{T}_1$. Hence $\operatorname{rk} F_1 = 0$ and $\deg_{G_1}(\widehat{\Phi}^2(F_1)) = 0$. Then $\widehat{\Phi}^2(F_1) \in \mathfrak{T}_1$ implies that $0 < \chi(G_1, \widehat{\Phi}^2(F_1)) = \operatorname{rk} F_1$, which is a contradiction. Therefore $F_1 = 0$. □

Lemma 2.5.8. (1) *Assume that $E \in \mathfrak{T}_1$. Then*
(a) $\Phi^0(E) \in \mathfrak{F}_2^\mu$.
(b) $\Phi^1(E) \in \mathfrak{T}_2^\mu$.
(c) $\Phi^2(E) = 0$.
(2) *Assume that $E \in \mathfrak{F}_1$. Then*
(a) $\Phi^0(E) = 0$.
(b) $\Phi^1(E) \in \mathfrak{F}_2^\mu$.
(c) $\Phi^2(E) \in \mathfrak{T}_2^\mu$.

Proof. We take a decomposition

$$0 \to F_1 \to \Phi^1(E) \to F_2 \to 0 \tag{2.45}$$

with $F_1 \in \mathfrak{T}_2^\mu$ and $F_2 \in \mathfrak{F}_2^\mu$. Applying $\widehat{\Phi}$, we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{\Phi}^0(F_1) &\longrightarrow \widehat{\Phi}^0(\Phi^1(E)) \longrightarrow \widehat{\Phi}^0(F_2) \\ \longrightarrow \widehat{\Phi}^1(F_1) &\longrightarrow \widehat{\Phi}^1(\Phi^1(E)) \longrightarrow \widehat{\Phi}^1(F_2) \\ \longrightarrow \widehat{\Phi}^2(F_1) &\longrightarrow \widehat{\Phi}^2(\Phi^1(E)) \longrightarrow \widehat{\Phi}^2(F_2) \longrightarrow 0. \end{aligned} \tag{2.46}$$

By Lemma 2.5.7, we have $\widehat{\Phi}^0(F_2) = \widehat{\Phi}^2(F_1) = 0$.

(1) Assume that $E \in \mathfrak{T}_1$. Then (a) follows from Lemma 2.5.7 (4), and (c) follows from Lemma 2.5.2 (4). We prove (b). We assume that $F_2 \neq 0$. By Lemma 5.1.1 and (c), we have $\widehat{\Phi}^2(\Phi^1(E)) = 0$. Then WIT_1 holds for F_2 and $\deg_{G_1}(\widehat{\Phi}^1(F_2)) = \deg_{G_2}(F_2) \le 0$. By Lemma 5.1.1, we have a surjective homomorphism

$$E \to \widehat{\Phi}^1(\Phi^1(E)). \tag{2.47}$$

Hence $\widehat{\Phi}^1(F_2)$ is a quotient object of E. Since $E \in \mathfrak{T}_1$, we see that $\deg_{G_1}(\widehat{\Phi}^1(F_2)) \ge 0$. Hence $\deg_{G_1}(\widehat{\Phi}^1(F_2)) = 0$. If $\operatorname{rk} \widehat{\Phi}^1(F_2) > 0$, then since $\chi(G_1, \widehat{\Phi}^1(F_2)) = -\operatorname{rk} F_2 < 0$, we get $E \notin \mathfrak{T}_1$. Hence $\operatorname{rk} \widehat{\Phi}^1(F_2) = 0$. Then $\chi(G_1, \widehat{\Phi}^1(F_2)) = -\operatorname{rk} F_2 < 0$ implies that the G_1-twisted Hilbert polynomial of $\widehat{\Phi}^1(F_2)$ is not positive. By Lemma 2.4.2, this is impossible. Therefore $F_2 = 0$.

(2) Assume that $E \in \mathfrak{F}_1$. By Lemma 2.5.2 and Lemma 2.5.7, (a) and (c) hold. We prove (b). Assume that $F_1 \neq 0$. By $\Phi^0(E) = 0$ and Lemma 5.1.1, we have $\widehat{\Phi}^0(\Phi^1(E)) = 0$. Then WIT_1 holds for F_1 and we have an injective morphism $\widehat{\Phi}^1(F_1) \to \widehat{\Phi}^1(\Phi^1(E)) \to E$. Assume that $\dim F_1 \ge 1$. Since $\deg_{G_1}(\widehat{\Phi}^1(F_1)) = \deg_{G_2}(F_1) > 0$, this is impossible. Assume that $\dim F_1 = 0$. Then $\chi(G_2, F_1) > 0$, which implies that $\operatorname{rk} \widehat{\Phi}^1(F_1) = -\chi(G_2, F_1) < 0$. This is a contradiction. Therefore $F_1 = 0$. ∎

The following is a generalization of a result in [H] (see Remark 2.5.10 below).

Theorem 2.5.9. *Φ induces an equivalence $\mathfrak{A}_1 \to \mathfrak{A}_2^\mu[-1]$. Moreover $\widehat{\Phi}^0(F) \in \overline{\mathfrak{F}}_1^\mu$ if $F \in \mathfrak{T}_2^\mu$ does not contain a 0-dimensional object.*

Proof. For $E \in \mathfrak{A}_1$, we have an exact sequence in $\mathfrak{A}_1$

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0. \tag{2.48}$$

Then we have an exact triangle

$$\Phi(H^{-1}(E))[2] \to \Phi(E[1]) \to \Phi(H^0(E))[1] \to \Phi(H^{-1}(E))[3]. \tag{2.49}$$

Hence $\Phi^i(E[1]) = 0$ for $i \neq -1, 0$ and we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \Phi^1(H^{-1}(E)) &\longrightarrow \Phi^{-1}(E[1]) \longrightarrow \Phi^0(H^0(E)) \\ \longrightarrow \Phi^2(H^{-1}(E)) &\longrightarrow \Phi^0(E[1]) \longrightarrow \Phi^1(H^0(E)) \longrightarrow 0. \end{aligned} \tag{2.50}$$

By Lemme 2.5.8, $\Phi^{-1}(E[1]) \in \mathfrak{F}_2^\mu$ and $\Phi^0(E[1]) \in \mathfrak{T}_2^\mu$. Therefore $\Phi(E[1]) \in \mathfrak{A}_2^\mu$.

Conversely for $F \in \mathfrak{A}_2^\mu$ and $E_1 \in \mathfrak{A}_1$, $\Phi(E_1)[1] \in \mathfrak{A}_2^\mu$ implies that

$$\begin{aligned} \operatorname{Hom}(\widehat{\Phi}(F)[1], E_1[p]) &= \operatorname{Hom}(F, (\Phi(E_1)[1])[p]) = 0,\ p < 0, \\ \operatorname{Hom}(E_1[p], \widehat{\Phi}(F)[1]) &= \operatorname{Hom}((\Phi(E_1)[1])[p], F) = 0,\ p > 0. \end{aligned} \tag{2.51}$$

Hence $\widehat{\Phi}(F)[1] \in \mathfrak{A}_1$. Therefore the first claim holds.

For the last claim, we note that there is an exact sequence

$$0 \to \Phi^0(\widehat{\Phi}^1(F)) \to \Phi^2(\widehat{\Phi}^0(F)) \to F \tag{2.52}$$

by Lemma 5.1.1. By Lemma 2.5.2 (3), $\Phi^0(\widehat{\Phi}^1(F))$ is torsion free. Hence $\Phi^2(\widehat{\Phi}^0(F))$ does not contain a 0-dimensional object. Then Lemma 2.5.5 (2) implies the claim. ∎

Remark 2.5.10. In [Y5], we gave a different proof of [H, Prop. 4.2]. Since we used different notations in [Y5], we explain the correspondence of the terminologies: Φ corresponds to $\mathcal{F}_{\mathcal{E}}$ in [Y5], $\mathfrak{A}_2^\mu$ corresponds to $\mathfrak{A}_1$ in [Y5, Thm. 2.1] and $\mathfrak{A}_1$ corresponds to $\mathfrak{A}_2$ or $\mathfrak{A}_2'$ in [Y5, Thm. 2.1, Prop. 2.7].

2.6. **Fourier-Mukai duality for a $K3$ surface.** In this subsection, we shall prove a kind of duality property between (X, H) and $(X', \widehat{H})$. In other words, we show that X is the moduli space of some objects on X' and H is the natural determinant line bundle on the moduli space.

Theorem 2.6.1. *Assume that $\mathbb{C}_x$ is β-stable for all $x \in X$ (Assumption 2.1.1).*

(1) *$\mathcal{E}_{|X' \times \{x\}} \in \operatorname{Per}(X'/Y')^D$ is $G_3 - \Phi(\beta)^\vee$-twisted stable for all $x \in X$ and we have an isomorphism $\phi : X \to M_{\widehat{H}}^{G_3 - \Phi(\beta)^\vee}(w_0^\vee)$ by sending $x \in X$ to $\mathcal{E}_{|X' \times \{x\}} \in M_{\widehat{H}}^{G_3 - \Phi(\beta)^\vee}(w_0^\vee)$. Moreover we have $H = \widehat{(\widehat{H})}$ under this isomorphism.*

(2) *Assume that $\mathcal{E}_{|\{x'\} \times X}$ is a μ-stable local projective generator of $\mathcal{C}$ for a general $x' \in X'$. Then $\mathcal{E}_{|X' \times \{x\}}$ is a μ-stable local projective generator of $\operatorname{Per}(X'/Y')^D$ for $x \in X \setminus \cup_i Z_i$.*

The proof is similar to that in [Y5, Thm. 2.2]. In particular, if $\mathcal{E}_{|\{x'\} \times X}$ is a μ-stable locally free sheaf for a general $x' \in X'$, then the same proof in [Y5] works. However if $\mathcal{E}_{|\{x'\} \times X}$ is not a μ-stable locally free sheaf for any $x' \in X'$, then we need to introduce a (contravariant) Fourier-Mukai transforms and study their properties. We set

$$\begin{aligned} \Psi(E) &:= \mathbf{R}\operatorname{Hom}_{p_{X'}}(p_X^*(E), \mathcal{E}) = \Phi(E)^\vee[-2],\ E \in \mathbf{D}(X), \\ \widehat{\Psi}(F) &:= \mathbf{R}\operatorname{Hom}_{p_X}(p_{X'}^*(F), \mathcal{E}),\ F \in \mathbf{D}(X'). \end{aligned} \tag{2.53}$$

We shall first study the properties of Ψ and $\widehat{\Psi}$ which are similar to those of Φ and $\widehat{\Phi}$.

We set

$$\begin{aligned} \Psi(E_{ij})[2] &= B_{ij}',\ j > 0 \\ \Psi(E_{i0})[2] &= B_{i0}'. \end{aligned} \tag{2.54}$$

Then the following claims follow from Definition 2.2.8 and Lemma 2.2.7.

Lemma 2.6.2. (1) *$B_{ij}' = \mathcal{O}_{C_{ij}'}(-b_{ij}' - 2) \in \operatorname{Per}(X'/Y')^D$ and $B_{i0}' = A_0(\mathbf{b'}^D)^*[1] \in \operatorname{Per}(X'/Y')^D$.*

(2) *Irreducible objects of $\operatorname{Per}(X'/Y')^D$ are*

$$B_{ij}'\ (1 \le i \le m, 0 \le j \le s_i'),\ \mathbb{C}_{x'}(x' \in X \setminus \cup_i Z_i'). \tag{2.55}$$

Lemma 2.6.3. (1) *Assume that $E \in \overline{\mathfrak{T}}_1$. Then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for a general $x' \in X'$.*

(2) *Assume that $E \in \overline{\mathfrak{F}}_1$. Then $\operatorname{Hom}(\mathcal{E}_{|\{x'\} \times X}, E) = 0$ for all $x' \in X'$.*

Proof. We only prove (1). Let E be a G_1-twisted stable object of $\mathcal{C}$. If $\deg_{G_1}(E) > 0$ or $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) > 0$, then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for all $x' \in X'$. Assume that $\deg_{G_1}(E) = 0$ and $\chi(G_1, E) = 0$. Then a non-zero homomorphism $E \to \mathcal{E}_{|\{x'\} \times X}$ is an isomorphism if $x' \notin \cup_i Z_i'$. Therefore $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}) = 0$ for a general $x' \in X'$. □

Lemma 2.6.4. *Let E be an object of $\mathcal{C}$.*

(1) *${}^pH^q(\Psi(E)) = 0$ for $q \ne 0, 1, 2$.*
(2) *$H^0({}^pH^2(\Psi(E))) = H^2(\Psi(E))$.*
(3) *${}^pH^0(\Psi(E)) \subset H^0(\Psi(E))$. In particular, ${}^pH^0(\Psi(E))$ is torsion free.*
(4) *If $\operatorname{Hom}(E, E_{ij}[2]) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}[2]) = 0$ for all $x' \in X'$, then ${}^pH^2(\Psi(E)) = 0$. In particular, if $E \in \overline{\mathfrak{F}}_1$, then ${}^pH^2(\Psi(E)) = 0$.*
(5) *If E satisfies $E \in \overline{\mathfrak{T}}_1$, then ${}^pH^0(\Psi(E)) = 0$.*

Proof. Let $W_\bullet$ be the complex in Lemma 2.5.2 (2). By Lemma 1.1.6, $W_i^\vee$ are local projective objects of $\operatorname{Per}(X'/Y')^D$. Since $\Psi(E)$ is represented by the complex $W_\bullet^\vee[-2]$, (1), (2) and (3) follow.

By Lemma 2.6.2, $F \in \operatorname{Per}(X'/Y')^D$ is 0 if and only if $\operatorname{Hom}(F, B_{ij}') = \operatorname{Hom}(F, \mathbb{C}_{x'}) = 0$ for all i, j and $x' \in X'$.

Since

$$\begin{aligned} \operatorname{Hom}(E, E_{ij}[2-p])^\vee &\cong \operatorname{Hom}(\Psi(E)[2-p], \Psi(E_{ij})[2]), \\ \operatorname{Hom}(E, \mathcal{E}_{|\{x'\} \times X}[2-p])^\vee &\cong \operatorname{Hom}(\Psi(E)[2-p], \Psi(\mathcal{E}_{|\{x'\} \times X})[2]), \end{aligned} \tag{2.56}$$

we have (4). (5) follows from (3) and Lemma 2.6.3 (1). □

As in the proof of Corollary 2.5.3, we have the following result by Lemma 2.6.4 (1).

Corollary 2.6.5. *${}^pH^q(\widehat{\Psi}(F)) = 0$ for $q \ne 0, 1, 2$ and $F \in \operatorname{Per}(X'/Y')^D$.*

Definition 2.6.6. We set $\Psi^i(E) := {}^pH^i(\Psi(E)) \in \operatorname{Per}(X'/Y')^D$ and $\widehat{\Psi}^i(E) := {}^pH^i(\widehat{\Psi}(E)) \in \mathcal{C}$.

Lemma 2.6.7. *Let E be an object of $\mathcal{C}$.*

(1) *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1$.*
(2) *If WIT_2 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{T}}_1$. If $\Psi^2(E)$ does not contain a 0-dimensional object, then $E \in \mathfrak{T}_1$.*

Proof. For an object E of $\mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.57}$$

such that $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \overline{\mathfrak{F}}_1$. Applying Ψ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Psi^0(E_2) & \longrightarrow & \Psi^0(E) & \longrightarrow & \Psi^0(E_1) \\ & \longrightarrow & \Psi^1(E_2) & \longrightarrow & \Psi^1(E) & \longrightarrow & \Psi^1(E_1) \\ & \longrightarrow & \Psi^2(E_2) & \longrightarrow & \Psi^2(E) & \longrightarrow & \Psi^2(E_1) \longrightarrow 0 \end{array} \tag{2.58}$$

By Lemma 2.6.4, we have $\Psi^0(E_1) = \Psi^2(E_2) = 0$. If WIT_0 holds for E, then we get $\Psi(E_1) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Psi(E_2) = 0$. Thus the first part of (2) holds. Assume that $\Psi^2(E)$ does not have a non-zero 0-dimensional subobject. We take a decomposition

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.59}$$

such that $E_1 \in \mathfrak{T}_1$ and E_2 is a G_1-twisted semi-stable object with $\deg_{G_1}(E_2) = \chi(G_1, E_2) = 0$. Then $\Psi^0(E_1) = \Psi^0(E_2) = \Psi^1(E_2) = 0$. In particular, WIT_2 holds for E_2 with respect to Ψ. Then $\Psi^2(E_2)$ is a torsion object with $\deg_{G_3}(\Psi^2(E_2)) = 0$, which implies that $\Psi^2(E_2)$ is 0-dimensional. Our assumption implies that $\Psi^1(E_1) \cong \Psi^2(E_2)$. By Lemma 5.1.2 and $\widehat{\Psi}^0(\Psi^0(E_1)) = 0$, we get $E_2 = \widehat{\Psi}^2(\Psi^2(E_2)) = \widehat{\Psi}^2(\Psi^1(E_1)) = 0$. □

Lemma 2.6.8. *Let E be a μ-semi-stable object with $\deg_{G_1}(E) = 0$. If WIT_0 holds for E, then $E = 0$.*

Proof. If WIT_0 holds for $E \neq 0$, then $\chi(G_1, E) = \operatorname{rk}\Psi(E) \geq 0$. On the other hand, Lemma 2.6.7 implies that $\chi(G_1, E) < 0$. Therefore $E = 0$. □

Lemma 2.6.9. *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1^{\mu}$.*

Proof. Assume that there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{2.60}$$

such that E_1 is a μ-semi-stable object with $\deg_{G_1}(E_1) = 0$ and $E_2 \in \overline{\mathfrak{F}}_1^{\mu}$. Then we have $\Psi^2(E_2) = 0$. By the exact sequence (2.58), WIT_0 holds for E_1. Then Lemma 2.6.8 implies that $E_1 = 0$. □

Lemma 2.6.10. *If $E \in \overline{\mathfrak{T}}_1^{\mu}$, then $\Psi^0(E) = 0$.*

Proof. We may assume that E is a μ-semi-stable object or a torsion object. If $\deg_{G_1}(E) > 0$ or a torsion object, then the claim holds by Lemma 2.6.4 (5). Assume that E is torsion free and $\deg_{G_1}(E) = 0$. By Lemma 5.1.2, we have an exact sequence

$$E \to \widehat{\Psi}^0(\Psi^0(E)) \to \widehat{\Psi}^2(\Psi^1(E)) \to 0. \tag{2.61}$$

By Lemma 2.6.9, $\widehat{\Psi}^0(\Psi^0(E)) \in \overline{\mathfrak{F}}_1^{\mu}$. Since E is a μ-semi-stable object with $\deg_{G_1}(E) = 0$, $E \to \widehat{\Psi}^0(\Psi^0(E))$ is a zero map. Then $\widehat{\Psi}^0(\Psi^0(E)) \cong \widehat{\Psi}^2(\Psi^1(E))$ satisfies WIT_0 and WIT_2, which implies that $\widehat{\Psi}^0(\Psi^0(E)) \cong \widehat{\Psi}^2(\Psi^1(E)) \cong 0$. Therefore $\Psi^0(E) = 0$. □

Lemma 2.6.11.

$$\deg_{G_3}(\Psi^0(E)) \leq 0,\ \deg_{G_3}(\Psi^2(E)) \geq 0. \tag{2.62}$$

Proof. We note that

$$\deg_{G_3}(\Psi^i(E)) = \deg_{G_1}(\widehat{\Psi}^i(\Psi^i(E))) \tag{2.63}$$

for $i = 0, 2$ by Lemma 5.1.2. Then the claim follows from Lemma 2.6.7. □

Proof of Theorem 2.6.1.

(1) We first prove the G_3-twisted semi-stability of $\mathcal{E}_{|X' \times \{x\}}$ for all $x \in X$. It is sufficient to prove the following lemma.

Lemma 2.6.12. *Let E be a 0-dimensional object of $\mathcal{C}$. Then WIT_2 holds for E with respect to Ψ and $\Psi^2(E)$ is a G_3-twisted semi-stable object such that $\deg_{G_3}(\Psi^2(E)) = \chi(G_3, \Psi^2(E)) = 0$. Moreover if E is irreducible, then $\Psi^2(E)$ is G_3-twisted stable.*

Proof. We first prove that E satisfies WIT_2 with respect to Ψ. We may assume that E is irreducible. Then we get $\mathrm{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for all x'. Hence $\Psi^0(E) = 0$. We shall prove that $\Psi^1(E) = 0$ by showing $\widehat{\Psi}^i(\Psi^1(E)) = 0$ for $i = 0, 1, 2$. By Lemma 5.1.2, $\widehat{\Psi}^2(\Psi^1(E)) = 0$ and we have an exact sequence

$$0 \to \widehat{\Psi}^0(\Psi^1(E)) \to \widehat{\Psi}^2(\Psi^2(E)) \to E \to \widehat{\Psi}^1(\Psi^1(E)) \to 0. \tag{2.64}$$

By Lemma 2.6.7 and Lemma 5.1.2, $\widehat{\Psi}^0(\Psi^1(E)) \in \overline{\mathfrak{F}}_1$ and $\widehat{\Psi}^2(\Psi^2(E)) \in \overline{\mathfrak{T}}_1$. Since E is 0-dimensional, $\widehat{\Psi}^0(\Psi^1(E))$ is μ-semi-stable and $\deg_{G_1}(\widehat{\Psi}^0(\Psi^1(E))) = \deg_{G_1}(\widehat{\Psi}^2(\Psi^2(E))) = 0$. By Lemma 2.6.8, $\widehat{\Psi}^0(\Psi^1(E)) = 0$. Since E is an irreducible object, $\widehat{\Psi}^2(\Psi^2(E)) = 0$ or $\widehat{\Psi}^1(\Psi^1(E)) = 0$. If $\widehat{\Psi}^2(\Psi^2(E)) = 0$, then $\Psi^2(E) = 0$. Since $\chi(G_1, E) > 0$, we get a contradiction. Hence we also have $\widehat{\Psi}^1(\Psi^1(E)) = 0$, which implies that $\Psi^1(E) = 0$. Therefore WIT_2 holds for E with respect to Ψ.

We next prove that $\Psi^2(E)$ is G_3-twisted semi-stable. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^2(E) \to F_2 \to 0 \tag{2.65}$$

such that $F_1 \in \mathrm{Per}(X'/Y')^D$, $\deg_{G_3}(F_1) \geq 0$ and $F_2 \in \mathrm{Per}(X'/Y')^D$. Applying $\widehat{\Psi}$ to this exact sequence, we get a long exact sequence

$$\begin{array}{cccccccc} 0 & \longrightarrow & \widehat{\Psi}^0(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^0(F_1) & \\ & \longrightarrow & \widehat{\Psi}^1(F_2) & \longrightarrow & 0 & \longrightarrow & \widehat{\Psi}^1(F_1) & \\ & \longrightarrow & \widehat{\Psi}^2(F_2) & \longrightarrow & E & \longrightarrow & \widehat{\Psi}^2(F_1) & \longrightarrow 0. \end{array} \tag{2.66}$$

By Lemma 5.1.2, WIT_2 holds for $\widehat{\Psi}^2(F_2)$. Hence $\widehat{\Psi}^2(F_2) \in \overline{\mathfrak{T}}_1$, in particular, we have $\deg_{G_1}(\widehat{\Psi}^2(F_2)) \geq 0$. By Lemma 5.1.2, WIT_0 holds for $\widehat{\Psi}^1(F_2) \cong \widehat{\Psi}^0(F_1)$. Hence $\widehat{\Psi}^1(F_2) \in \overline{\mathfrak{F}}_1$, which implies that $\deg_{G_1}(\widehat{\Psi}^1(F_2)) \leq 0$. Therefore $\deg_{G_1}(\widehat{\Psi}(F_2)) \geq 0$. On the other hand, $\deg_{G_1}(\widehat{\Psi}(F_2)) = \deg_{G_3}(F_2) \leq 0$. Hence $\widehat{\Psi}^1(F_2)$ is a μ-semi-stable object with $\deg_{G_1}(\widehat{\Psi}^1(F_2)) = 0$ and $\deg_{G_3}(F_2) = 0$. Then Lemma 2.6.8 implies that $\widehat{\Psi}^1(F_2) = 0$. If $\chi(G_3, F_2) \leq 0$, then $\mathrm{rk}\, \widehat{\Psi}^2(F_2) = \chi(G_3, F_2)$ implies that $\chi(G_3, F_2) = 0$ and $\widehat{\Psi}^2(F_2)$ is a torsion object. This in particular means that $\Psi^2(E)$ is G_2-twisted semi-stable. We further assume that E is irreducible. Since $\deg_{G_1}(\widehat{\Psi}^2(F_2)) = 0$, $\widehat{\Psi}^2(F_2)$ is a 0-dimensional object. Then WIT_2 holds for $\widehat{\Psi}^1(F_1)$, $\widehat{\Psi}^2(F_1)$ and $\widehat{\Psi}^2(F_2)$ with respect to Ψ. Since $\Psi^2(\widehat{\Psi}^1(F_1)) = 0$, $\widehat{\Psi}^1(F_1) = 0$. Then $\widehat{\Psi}^2(F_2) = 0$ or $\widehat{\Psi}^2(F_1) = 0$, which implies that $F_1 = 0$ or $F_2 = 0$. Therefore $\Psi^2(E)$ is G_3-twisted stable. □

We continue the proof of (1). Assume that there is an exact sequence in $\mathrm{Per}(X'/Y')^D$

$$0 \to F_1 \to \mathcal{E}_{|X'\times\{x\}} \to F_2 \to 0 \tag{2.67}$$

such that $\deg_{G_3}(F_1) = \chi(G_3, F_1) = 0$. By the proof of Lemma 2.6.12, WIT_2 holds for F_1 and F_2. Thus we get an exact sequence

$$0 \to \widehat{\Psi}^2(F_2) \to \mathbb{C}_x \to \widehat{\Psi}^2(F_1) \to 0 \tag{2.68}$$

Since $\mathbb{C}_x$ is β-stable, $\chi(\beta, \widehat{\Psi}^2(F_2)) < 0$, which implies that $\chi(-\Psi(\beta), F_2) > 0$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is $G_3 - \Psi(\beta)$-twisted stable. Then we have an injective morphism $\phi : X \to \overline{M}_{\widehat{H}}^{G_3+\alpha'}(w_0^\vee)$ by sending $x \in X$ to $\mathcal{E}_{|X'\times\{x\}}$, where $\alpha' = -\Psi(\beta)$. By a standard argument, we see that ϕ is an isomorphism. We note that $[\widehat{\Psi}(\widehat{H} + (\widehat{H}, \widetilde{\xi}_0)/r_0\varrho_{X'})]_1$ is the pull-back of the canonical polarization on $\overline{M}_{\widehat{H}}^{G_3}(w_0^\vee)$. Hence under the identification $M_{\widehat{H}}^{G_3+\alpha'}(w_0^\vee) \cong X$, $\widehat{(\widehat{H})} = H$.

(2) Assume that $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective generator for a general $x' \in X'$. By Lemma 2.6.14 (2) below, we only need to prove the μ-stability of $\mathcal{E}_{|X'\times\{x\}}$ for $x \in X \setminus \cup_i Z_i$. We shall study the exact sequence (2.65) in Lemma 2.6.12, where $E = \mathbb{C}_x$. We may assume that F_2 satisfies $\deg_{G_3}(F_2) = 0$ and $\chi(G_3, F_2) > 0$. Then WIT_2 holds for F_2 by the proof of Lemma 2.6.12. We shall first prove that $\widehat{\Psi}^1(F_1)$ does not contain a 0-dimensional object. Let T_1 be the 0-dimensional subobject of $\widehat{\Psi}^1(F_1)$. Then we have a surjective morphism $\Psi^2(\widehat{\Psi}^1(F_1)) \to \Psi^2(T_1)$. Since WIT_2 holds for T_1 with respect to Ψ and $\Psi^0(\widehat{\Psi}^0(F_1)) \to \Psi^2(\widehat{\Psi}^1(F_1))$ is surjective, we get $T_1 = 0$. By Lemma 2.6.7, $\widehat{\Psi}^2(F_2) \in \overline{\mathfrak{T}}_1$. Then Lemma 2.4.9 and $\deg_{G_1}(\widehat{\Psi}^2(F_2)) = 0$ imply that $\widehat{\Psi}^2(F_2)$ is an extension of a G_1-twisted semi-stable object E_1 with $\deg_{G_1}(E_1) = \chi(G_1, E_1) = 0$ by a 0-dimensional object T. Since $T \cap \widehat{\Psi}^1(F_1) = 0$, $T = \mathbb{C}_x$ or 0. By our assumption, $\Psi^2(E_1)$ is a torsion object. By the exact sequence

$$\Psi^2(E_1) \to F_2 \to \Psi^2(T) \to 0, \tag{2.69}$$

we have $\mathrm{rk}\, F_2 = (\mathrm{rk}\, \mathcal{E}_{|X'\times\{x\}}) \dim T$, which implies that $\mathrm{rk}\, F_2 = \mathrm{rk}\, \mathcal{E}_{|X'\times\{x\}}$ or $\mathrm{rk}\, F_2 = 0$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is μ-stable. □

Lemma 2.6.13. *If $\mathcal{E}_{|\{x'\}\times X}, x' \in X'$ and E_{ij} are locally free on an open subset X^0 of X, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\operatorname{Per}(X'/Y')^D$ for $x \in X^0$.*

Proof. We first note that $\mathcal{E}_{|X'\times\{x\}} \in \operatorname{Coh}(X')$ by Theorem 2.6.1. The claim follows from the following equalities:

$$\begin{aligned}\operatorname{Hom}(\mathcal{E}_{|X'\times\{x\}}, \mathbb{C}_{x'}[k]) &= \operatorname{Hom}(\Psi(\mathbb{C}_x), \Psi(\mathcal{E}_{|\{x'\}\times X})[k]) = \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, \mathbb{C}_x[k]) = 0,\\ \operatorname{Hom}(\mathcal{E}_{|X'\times\{x\}}, B'_{ij}[k]) &= \operatorname{Hom}(\Psi(\mathbb{C}_x), \Psi(E_{ij})[k]) = \operatorname{Hom}(E_{ij}, \mathbb{C}_x[k]) = 0\end{aligned} \tag{2.70}$$

for $x \in X^0$, $x' \in X'$ and $k \neq 0$. □

Lemma 2.6.14. (1) *If $X = Y$ and Y' is not smooth, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\operatorname{Per}(X'/Y')^D$ for all $x \in X$.*
(2) *If $\mathcal{E}_{|\{x'\}\times X}$ is a μ-stable local projective object of $\mathcal{C}$ for a general $x' \in X'$, then $\mathcal{E}_{|X'\times\{x\}}$ is a local projective generator of $\operatorname{Per}(X'/Y')^D$ for all $x \in X$.*

Proof. (1) We first note that $E_{ij} \in \operatorname{Coh}(X) = \mathcal{C}$ are locally free sheaves for all i, j. Assume that $E := \mathcal{E}_{|\{x'\}\times X}$ is not locally free for a point $x' \in X'$. Then we have a morphism from an open subscheme Q of $\operatorname{Quot}^n_{E^{\vee\vee}/X/\mathbb{C}}$ to X', where $n = \dim(E^{\vee\vee}/E)$. Since $\dim X' = 2$, this morphism is dominant. Hence $\mathcal{E}_{|\{x'\}\times X}$ is non-locally free for all $x' \in X'$. Since $\mathcal{E}_{|\{x'\}\times X}$ is locally free if x' belongs to the exceptional locus, $\mathcal{E}_{|\{x'\}\times X}$ is locally free for any $x' \in X'$. Then the claim follows from Lemma 2.6.13.

(2) The claim follows from Lemma 2.4.9 (3), (4) and the proof of Lemma 2.6.13. □

In the remaining of this subsection, we shall prove the following result.

Proposition 2.6.15. *$\Psi : \mathbf{D}(X) \to \mathbf{D}(X')_{op}$ induces an equivalence $\overline{\mathfrak{A}}_1^{\mu}[-2] \to (\overline{\mathfrak{A}}_3)_{op}$.*

We first note that the following two lemmas hold thanks to Theorem 2.6.1.

Lemma 2.6.16 (cf. Lemma 2.6.3, Lemma 2.6.4)**.** (1) *Assume that $F \in \overline{\mathfrak{T}}_3$. Then $\operatorname{Hom}(F, \mathcal{E}_{|X'\times\{x\}}) = 0$ for a general $x \in X$. In particular, $\widehat{\Psi}^0(F) = 0$.*
(2) *Assume that $F \in \overline{\mathfrak{F}}_3$. Then $\operatorname{Hom}(\mathcal{E}_{|X'\times\{x\}}, F) = 0$ for all $x \in X$. In particular, $\widehat{\Psi}^2(F) = 0$.*

Lemma 2.6.17 (cf. Lemma 2.6.7, Lemma 2.6.9, Lemma 2.6.10)**.** *Let F be an object of $\operatorname{Per}(X'/Y')^D$.*
(1) *If WIT_0 holds for F with respect to $\widehat{\Psi}$, then $F \in \overline{\mathfrak{F}}_3^{\mu} (\subset \overline{\mathfrak{F}}_3)$.*
(2) *If WIT_2 holds for F with respect to $\widehat{\Psi}$, then $F \in \overline{\mathfrak{T}}_3$. If $\widehat{\Psi}^2(F)$ does not contain a 0-dimensional subobject, then $F \in \mathfrak{T}_3$.*
(3) *If $F \in \overline{\mathfrak{T}}_3$, then $\widehat{\Psi}^0(F) = 0$.*

Lemma 2.6.18. (1) *Assume that $E \in \overline{\mathfrak{T}}_1^{\mu}$. Then*
  (a) $\Psi^0(E) = 0$.
  (b) $\Psi^1(E) \in \overline{\mathfrak{F}}_3$.
  (c) $\Psi^2(E) \in \overline{\mathfrak{T}}_3$. *Moreover if E does not contain a non-trivial 0-dimensional subobject, then $\Psi^2(E) \in \mathfrak{T}_3$.*
(2) *Assume that $E \in \overline{\mathfrak{F}}_1^{\mu}$. Then*
  (a) $\Psi^0(E) \in \overline{\mathfrak{F}}_3$.
  (b) $\Psi^1(E) \in \overline{\mathfrak{T}}_3$.
  (c) $\Psi^2(E) = 0$.

Proof. We take a decomposition

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0 \tag{2.71}$$

with $F_1 \in \overline{\mathfrak{T}}_3$ and $F_2 \in \overline{\mathfrak{F}}_3$. Applying $\widehat{\Psi}$, we have an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{\Psi}^0(F_2) & \longrightarrow & \widehat{\Psi}^0(\Psi^1(E)) & \longrightarrow & \widehat{\Psi}^0(F_1) \\ & \longrightarrow & \widehat{\Psi}^1(F_2) & \longrightarrow & \widehat{\Psi}^1(\Psi^1(E)) & \longrightarrow & \widehat{\Psi}^1(F_1) \\ & \longrightarrow & \widehat{\Psi}^2(F_2) & \longrightarrow & \widehat{\Psi}^2(\Psi^1(E)) & \longrightarrow & \widehat{\Psi}^2(F_1) \longrightarrow 0. \end{array} \tag{2.72}$$

By Lemma 2.6.16, we have $\widehat{\Psi}^0(F_1) = \widehat{\Psi}^2(F_2) = 0$.

(1) Assume that $\deg_{\min,G_1}(E) \geq 0$, that is, $E \in \overline{\mathfrak{T}}_1^{\mu}$. By Lemma 2.6.17 (2) and Lemma 2.6.10, (a) and the first claim of (c) hold. For the second claim of (c), by Lemma 2.6.17 (2), it is sufficient to prove that $\widehat{\Psi}^2(\Psi^2(E))$ does not contain a non-trivial 0-dimensional subobject. By the exact sequence

$$0 \to \widehat{\Psi}^0(\Psi^1(E)) \to \widehat{\Psi}^2(\Psi^2(E)) \to E \tag{2.73}$$

and the torsion-freeness of $\widehat{\Psi}^0(\Psi^1(E))$, we get our claim.

We prove (b). By Lemma 5.1.2 and (a), we have $\widehat{\Psi}^2(\Psi^1(E))=0$. Then WIT_1 holds for F_1. We have a surjective homomorphism

$$E \to \widehat{\Psi}^1(\Psi^1(E)). \tag{2.74}$$

Hence E has a quotient object $\widehat{\Psi}^1(F_1)$ with $\deg_{G_1}(\widehat{\Psi}^1(F_1))=-\deg_{G_3}(F_1)\le 0$. If $\deg_{G_1}(\widehat{\Psi}^1(F_1))<0$, then we see that $\operatorname{rk}\widehat{\Psi}^1(F_1)>0$ and $E\not\in\overline{\mathfrak{T}}_1^\mu$. Hence $\deg_{G_1}(\widehat{\Psi}^1(F_1))=-\deg_{G_3}(F_1)=0$. Then $F_1\in\overline{\mathfrak{T}}_3$ implies that $\operatorname{rk}\widehat{\Psi}^1(F_1)=-\chi(G_3,F_1)\le 0$. Since $\chi(G_1,\widehat{\Psi}^1(F_1))=-\operatorname{rk}F_1\le 0$, the G_1-twisted Hilbert polynomial of $\widehat{\Psi}^1(F_1)$ is 0. Therefore $F_1=0$.

(2) Assume that $\deg_{\max,G_1}(E)<0$, that is $E\in\overline{\mathfrak{F}}_1^\mu$. By Lemma 2.6.4 and Lemma 2.6.17, (a) and (c) hold. We prove (b). Since $\Psi^2(E)=0$, Lemma 5.1.2 implies that $\widehat{\Psi}^0\Psi^1(E)=0$. Hence WIT_1 holds for F_2 and we have an injective morphism $\widehat{\Psi}^1(F_2)\to\widehat{\Psi}^1(\Psi^1(E))\to E$. Since $\deg_{G_1}(\widehat{\Psi}^1(F_2))\ge 0$, we have $\widehat{\Psi}^1(F_2)=0$, which implies that $F_2=0$. □

Proof of Proposition 2.6.15.
For $E\in\overline{\mathfrak{A}}_1^\mu$, we have an exact sequence in $\overline{\mathfrak{A}}_1^\mu$

$$0\to H^{-1}(E)[1]\to E\to H^0(E)\to 0. \tag{2.75}$$

Then we have an exact triangle

$$\Psi(H^0(E))[2]\to\Psi(E[-2])\to\Psi(H^{-1}(E))[1]\to\Psi(H^0(E))[3]. \tag{2.76}$$

Hence $\Psi^i(E[-2])=0$ for $i\ne -1,0$ and we have an exact sequence

$$\begin{aligned} 0 &\longrightarrow \Psi^1(H^0(E)) \longrightarrow \Psi^{-1}(E[-2]) \longrightarrow \Psi^0(H^{-1}(E))\\ &\longrightarrow \Psi^2(H^0(E)) \longrightarrow \Psi^0(E[-2]) \longrightarrow \Psi^1(H^{-1}(E)) \longrightarrow 0. \end{aligned} \tag{2.77}$$

By Lemme 2.6.18, $\Psi^{-1}(E[-2])\in\overline{\mathfrak{F}}_3$ and $\Psi^0(E[-2])\in\overline{\mathfrak{T}}_3$. Therefore $\Psi(E[-2])\in(\overline{\mathfrak{A}}_3)_{op}$. □

Definition 2.6.19. (1) Let $\operatorname{Per}(X'/Y')^D_{w_0^\vee}$ be the full subcategory of $\operatorname{Per}(X'/Y')^D$ consisting of G_3-twisted semi-stable objects E with $\deg_{G_3}(E)=\chi(G_3,E)=0$.
(2) Let $\mathcal{C}_0$ (resp. $\operatorname{Per}(X'/Y')^D_0$) be the full subcategory of $\mathcal{C}$ (resp. $\operatorname{Per}(X'/Y')^D$) consisting of 0-dimensional objects.

Proposition 2.6.20. Ψ *induces the following correspondences:*

$$\begin{aligned} \mathcal{C}_0 &\cong (\operatorname{Per}(X'/Y')^D_{w_0^\vee})_{op},\\ \mathcal{C}_{v_0} &\cong (\operatorname{Per}(X'/Y')^D_0)_{op}. \end{aligned} \tag{2.78}$$

Proof. By Lemma 2.6.12, $\Psi^2(\mathcal{C}_0)$ is contained in $(\operatorname{Per}(X'/Y')^D_{w_0^\vee})_{op}$. By the proof of Lemma 2.2.11, we see that $\operatorname{Per}(X'/Y')^D_{w_0^\vee}$ is generated by $\Psi^2(A_{ij}), i,j\ge 0$ and $\Psi^2(\mathbb{C}_x)$, $x\in X\setminus\cup_i Z_i$. Thus the first claim holds.

We have an equivalence

$$\begin{array}{ccc} \operatorname{Per}(X'/Y')_0 & \to & (\operatorname{Per}(X'/Y')^D_0)_{op}\\ E & \mapsto & \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E,\mathcal{O}_X)[2]. \end{array} \tag{2.79}$$

Then the second claim is a consequence of Proposition 2.2.13 (1). □

2.7. **Preservation of Gieseker stability conditions.**

Proposition 2.7.1. *Let E be a G_1-twisted semi-stable object with $\deg_{G_1}(E)=0$ and $\chi(G_1,E)<0$. Then WIT_1 holds for E and $\Psi^1(E)$ is G_3-twisted semi-stable. In particular, we have an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss}\to\mathcal{M}_{\widehat{H}}^{G_3}(-\Psi(v))^{ss} \tag{2.80}$$

which preserves the S-equivalence classes, where $v=lv_0+a\varrho_X+(D+(D/r_0,\xi_0)\varrho_X)$, $l>0$, $a<0$.

Proof. We note that $E\in\overline{\mathfrak{F}}_1\cap\overline{\mathfrak{T}}_1^\mu$. By Lemma 2.6.4 and Lemma 2.6.18, WIT_1 holds for E and $\Psi^1(E)\in\overline{\mathfrak{F}}_3$. Assume that $\Psi^1(E)$ is not G_3-twisted stable. Then there is an exact sequence in $\operatorname{Per}(X'/Y')^D$

$$0\to F_1\to\Psi^1(E)\to F_2\to 0 \tag{2.81}$$

such that F_1 is a G_3-twisted stable object with $\deg_{G_3}(F_1)=0$ and

$$0>\frac{\chi(G_3,F_1)}{\operatorname{rk}F_1}\ge\frac{\chi(G_3,\Psi^1(E))}{\operatorname{rk}\Psi^1(E)}, \tag{2.82}$$

and $F_2\in\overline{\mathfrak{F}}_3$. By Lemma 2.6.16, $\widehat{\Psi}^2(F_1)=\widehat{\Psi}^2(F_2)=0$. Since $F_1,F_2\in\overline{\mathfrak{T}}_3^\mu$, Lemma 2.6.17 implies $\widehat{\Psi}^0(F_1)=\widehat{\Psi}^0(F_2)=0$. Then we have an exact sequence

$$0\to\widehat{\Psi}^1(F_2)\to E\to\widehat{\Psi}^1(F_1)\to 0. \tag{2.83}$$

Since

$$\begin{aligned}\frac{\chi(G_1,\widehat{\Psi}^1(F_1))}{\operatorname{rk}(\widehat{\Psi}^1(F_1))} &= \frac{\operatorname{rk} F_1}{\chi(G_3,F_1)}\\ &\leq \frac{\operatorname{rk}\Psi^1(E)}{\chi(G_3,\Psi^1(E))}=\frac{\chi(G_1,E)}{\operatorname{rk} E},\end{aligned} \tag{2.84}$$

we have

$$\frac{\chi(G_3,F_1)}{\operatorname{rk} F_1}=\frac{\chi(G_3,\Psi^1(E))}{\operatorname{rk}\Psi^1(E)}. \tag{2.85}$$

Hence $\Psi^1(E)$ is G_3-twisted semi-stable. Thus we have a morphism $\mathcal{M}_H^{G_1}(v)^{ss}\to\mathcal{M}_{\widehat{H}}^{G_3}(-\Psi(v))^{ss}$. It is easy to see that this morphism preserves the S-equivalence classes. By the symmetry of the conditions, we have the inverse morphism, which shows the second claim. □

The following is a generalization of [Y5, Thm. 1.7].

Proposition 2.7.2. *Let $w\in v(\mathbf{D}(X'))$ be a Mukai vector such that $\langle w^2\rangle\geq -2$ and*

$$w=lw_0+a\varrho_{X'}+\left(d\widehat{H}+\widehat{D}+\frac{1}{r_0}(d\widehat{H}+\widehat{D},\xi_0)\varrho_{X'}\right), \tag{2.86}$$

where $l\geq 0$, $a>0$ and $D\in \operatorname{NS}(X)\otimes\mathbb{Q}\cap H^{\perp}$. Assume that

$$\begin{aligned} d&>\max\{(4l^2r_0^3+1/(H^2)),2r_0^2l(\langle w^2\rangle-(D^2))\},\ \text{if } l>0,\\ a&>\max\{(2r_0+1),(\langle w^2\rangle-(D^2))/2+1\},\ \text{if } l=0.\end{aligned} \tag{2.87}$$

Then

1. $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}\cong\mathcal{M}_{\widehat{H}}^{G_2}(w)^{ss}$.
2. *$\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}$ consists of local projective generators.*
3. *If $(\widehat{H},G_2)$ is general with respect to w, then $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}\cong\mathcal{M}_{H+\epsilon}^{G_1}(\widehat{\Phi}(w))^{ss}$ for a sufficiently small relatively ample divisor ϵ.*

Proof. (1) We first note that $\mathcal{F}_{\mathcal{E}}$ in [Y5] corresponds to $\widehat{\Phi}$. Since [Y5, Thm. 2.1, Thm. 2.2] are replaced by Theorem 2.5.9, 2.6.1 and since [Y5, Prop. 2.8, Prop. 2.11] also hold for our case, the same proof of [Y5, Thm. 1.7] works for our case. More precisely, in order to show that $\Phi(F), F\in\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))$ does not contain a 0-dimensional subobject, we use the fact that WIT_0 holds for 0-dimensional object $E\in\operatorname{Per}(X'/Y')$ (see Proposition 2.2.13 (1)).

(2) The proof is the same as in the proof of [Y5, Rem. 2.3]. Let E be a μ-semi-stable object of $\mathcal{C}$ such that $v(E)=\widehat{\Phi}(w)$. We shall apply Proposition 1.1.5 to show the claim. If $\operatorname{Ext}^1(S,E)\neq 0$ for an irreducible object S of $\mathcal{C}$, then a non-trivial extension

$$0\to E\to E'\to S\to 0 \tag{2.88}$$

gives a μ-semi-stable object E' with $\chi(G_1,E')>\chi(G_1,E)$. By Proposition [Y5, Prop. 2.8, Prop. 2.11], we get a contradiction. Hence $\operatorname{Ext}^1(E,S)\cong\operatorname{Ext}^1(S,E)^{\vee}=0$ for any irreducible object S of $\mathcal{C}$. Since $\operatorname{Ext}^2(E,S)\cong\operatorname{Hom}(S,E)^{\vee}=0$, it is sufficient to prove that $\chi(S,E)>0$. We note that $\chi(S,E)=\chi(S,\widehat{\Phi}(w))=a\chi(S,G_1)+(c_1(S),D)$. Since $(H,c_1(S))=0$, we have $|(c_1(S),D)^2|\leq|(c_1(S)^2)(D^2)|=-2(D^2)$. Since $\chi(S,G_1)>0$, it is sufficient to prove that $a>\sqrt{-2(D^2)}$.

We first assume that $l>0$. Then $d(H^2)-1>4l^2r_0^3(H^2)$ and $d>2r_0^2l(\langle w^2\rangle-(D^2))=2r_0^2l(d^2(H^2)-2lar_0)$. Hence

$$a>\frac{d(d(H^2)-1/(2r_0^2l))}{2r_0l}>\frac{d}{2lr_0}4l^2r_0^3(H^2)=2dlr_0^2(H^2). \tag{2.89}$$

Hence $a>2(4l^2r_0^3)lr_0^2(H^2)=8r_0(lr_0)^3r_0(H^2)\geq 8$. If $-(D^2)\leq 4$, then $a>3>\sqrt{-2(D^2)}$. If $-(D^2)>4$, then $\langle w^2\rangle-(D^2)\geq -2-(D^2)>-(D^2)/2$. Hence

$$a>2dlr_0^2(H^2)>r_0(\langle w^2\rangle-(D^2))4(lr_0)^2r_0(H^2)>\sqrt{-2(D^2)}. \tag{2.90}$$

We next assume that $l=0$. Then $a>2r_0+1$ and $a>\langle w^2\rangle/2+1-(D^2)/2\geq -(D^2)/2$. If $-(D^2)\geq 8$, then $a>-(D^2)/2\geq\sqrt{-2(D^2)}$. If $-(D^2)<8$, then since $a\geq 2r_0+1+1/r_0$, $\sqrt{-2(D^2)}<4\leq a$.

Therefore $\chi(E,S)>0$ and E is a local projective generator of $\mathcal{C}$.

(3) By our assumption, $\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}=\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{\mu\text{-}ss}$ ([Y5, Cor. 2.14]) and H is a general polarization. Hence for $E\in\mathcal{M}_H^{G_1}(\widehat{\Phi}(w))^{ss}$ and a subobject E_1 of E, $\frac{(c_1(E),H)}{\operatorname{rk}E}=\frac{(c_1(E_1),H)}{\operatorname{rk}E_1}$ implies $\frac{c_1(E)}{\operatorname{rk}E}=\frac{c_1(E_1)}{\operatorname{rk}E_1}$. Let E be a μ-semi-stable sheaf of $v(E)=\widehat{\Phi}(w)$ with respect to H. We shall prove that $E\in\mathcal{C}$. We set

$$\Sigma:=\{A_{ij}[-1]\,|\,i,j\}\cap\operatorname{Coh}(X)$$

as in [Y7, Prop. 1.1.26]. We assume that $\operatorname{Hom}(E,F)\neq 0$ for $F\in\Sigma$. Then there is a μ-semi-stable sheaf $E'\in\mathcal{C}\cap\operatorname{Coh}(X)$ with respect to H fitting in an exact sequence

$$0\to E'\to E\to F'\to 0, \tag{2.91}$$

where $F'\in\mathcal{C}[-1]\cap\operatorname{Coh}(X)$. Then we see that $\chi(G_1,E')>\chi(G,E)$, which is a contradiction. Therefore $E\in\mathcal{C}$. Then we can easily see that E is μ-semi-stable in $\mathcal{C}$. □

Corollary 2.7.3. *If (G,H) is general with respect to v, then $M_H^G(v)$ is isomorphic to the moduli space of usual stable sheaves on a $K3$ surface.*

Proof. We first construct a primitive and isotropic Mukai vector u such that $\operatorname{rk}u>0$ and $(\operatorname{rk}G)c_1(u)-(\operatorname{rk}u)c_1(G^\vee)\in\mathbb{Z}H$: We first take a primitive isotropic Mukai vector t such that $t=lv(G^\vee)+a\varrho_X$. Then for a sufficiently small τ, $T:=M_H^{G^\vee+\tau}(t)$ is a $K3$ surface. Let $\mathcal{F}$ be the universal family on $T\times X$ as a twisted object. Then we have an equivalence $\Phi_{X\to T}^{\mathcal{F}^\vee}:\mathbf{D}(X)\to\mathbf{D}^\beta(T)$. We consider $\Pi:=\Phi_{T\to X}^{\mathcal{F}(nD)}\circ\Phi_{X\to T}^{\mathcal{F}^\vee}:\mathbf{D}(X)\to\mathbf{D}(X)$, $n\gg 0$, where we set $D:=\widehat{H}$. Then Π also induces a Hodge isometry $\Pi: H^*(X,\mathbb{Z})\to H^*(X,\mathbb{Z})$. By its construction, Π preserves the subspace $(\mathbb{Q}t+\mathbb{Q}H+\mathbb{Q}\varrho_X)\cap H^*(X,\mathbb{Z})$ and $\operatorname{rk}\Pi(\varrho_X)>0$ for $n\gg 0$. Hence $u:=\Pi(\varrho_X)$ satisfies the claim. Since $c_1(u)/\operatorname{rk}u-c_1(G^\vee)/\operatorname{rk}G^\vee\in\mathbb{Q}H$, $\chi(u,A_{ij}^\vee[2])/\operatorname{rk}u=\chi(G^\vee,A_{ij}^\vee[2])/\operatorname{rk}G$. By Proposition 1.1.5 (2), there is a local projective generator G_u of $\mathcal{C}^D$ with $v(G_u)=2u$. Since $\langle\Pi(\mathcal{O}_X),u\rangle=-1$, $X_1:=M_H^{u+\alpha}(u)$ is a fine moduli space of stable objects of $\mathcal{C}^D$. Since $\mathcal{C}$ satisfies Assumption 2.1.1, $\mathcal{C}^D$ also satisfies Assumption 2.1.1. Let $\mathcal{E}$ be the universal family on $X\times X_1$. By Theorem 2.6.1, we can regard $\mathcal{E}$ as a universal family of $v_0+\gamma$-twisted stable objects of $\operatorname{Per}(X_1/Y_1)^D$ with respect to H_1, where $Y_1:=\overline{M}_H^u(u)$, $H_1:=\widehat{H}$, $v_0=v(\mathcal{E}_{|\{x\}\times X_1})$ and γ is determined by α. Then $(M_{H_1}^{v_0+\gamma}(v_0),\widehat{H}_1)=(X,H)$. For $\widehat{\Phi}=\Phi_{X\to X_1}^{\mathcal{E}}$ and $\mathcal{M}_{\widehat{H}_1}^{u^\vee}(ve^{m\widehat{H}_1})^{ss}=\mathcal{M}_H^{u^\vee}(ve^{mH})^{ss}$, $m\gg 0$, we shall apply Proposition 2.7.2. Then $\mathcal{M}_H^{u^\vee}(v)^{ss}$ is isomorphic to a moduli stack of usual semi-stable sheaves on X_1. Since $\mathcal{M}_H^{u^\vee}(v)^{ss}=\mathcal{M}_H^G(v)^{ss}$, we get our claim. □

Since (2.87) is numerical, we can apply Proposition 2.7.2 to a family of $K3$ surfaces.

Example 2.7.4. Let $f:(\mathcal{X},\mathcal{H})\to S$ be a family of polarized $K3$ surfaces over S. Let $v_0:=(r,d\mathcal{H},a)$, $\gcd(r,a)=1$ be a family of isotropic Mukai vectors. We set $\mathcal{X}':=M_{\mathcal{X}/S}^{v_0}(v_0)$. Then we have a family of polarizations $\mathcal{H}'$ on $\mathcal{X}'$. Since $\gcd(r,a)=1$, there is a universal family $\mathcal{E}$ on $\mathcal{X}'\times_S\mathcal{X}$ and we have a family of Fourier-Mukai transforms $\Phi_{\mathcal{X}\to\mathcal{X}'}^{\mathcal{E}}:\mathbf{D}(\mathcal{X})\to\mathbf{D}(\mathcal{X}')$. Then we can apply Proposition 2.7.1 and Proposition 2.7.2 to families of moduli spaces over S.

We also give a generalization of [Y1, Thm. 7.6] based on Theorem 2.5.9 and Proposition 2.6.15. We set

$$d_{\min}:=\min\{\deg_{G_1}(F)>0|F\in\mathbf{D}(X)\}. \tag{2.92}$$

Proposition 2.7.5. *Assume that $\mathfrak{T}_1=\mathfrak{T}_1^\mu$. Let $v\in H^*(X,\mathbb{Z})$ be a Mukai vector of a complex such that $\deg_{G_1}(v)=d_{\min}$.*

(1) *If $\operatorname{rk}\Phi(v)\le 0$, then Φ induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss}\to\mathcal{M}_{\widehat{H}}^{G_2}(-\Phi(v))^{ss} \tag{2.93}$$

by sending E to $\Phi^1(E)$.

(2) *If $\operatorname{rk}\Psi(v)\ge 0$, then Ψ induces an isomorphism*

$$\mathcal{M}_H^{G_1}(v)^{ss}\to\mathcal{M}_{\widehat{H}}^{G_3}(\Psi(v))^{ss} \tag{2.94}$$

by sending E to $\Psi^2(E)$.

The proof is an easy exercise. We shall give a proof in [MYY], as an application of Bridgeland's stability condition.

Remark 2.7.6. In [Y6], we constructed actions of Lie algebras on the cohomology groups of some moduli spaces of stable sheaves. In particular, we constructed the action on the cohomology groups of some moduli spaces of stable objects of ${}^{-1}\operatorname{Per}(X/Y)$ in [Y6, Prop. 6.15]. Then a generalization of [Y6, Prop. 6.15] to the objects in $\operatorname{Per}(X'/Y')$ corresponds to the action in [Y6, Example 3.1.1] via Proposition 2.7.5.

2.7.1. We shall consider the category of perverse coherent sheaves which appears in a family of moduli spaces of stable sheaves.

Let T be a smooth manifold over $\mathbb{C}$ and we consider a flat family of polarized K3 surfaces $f:(\mathcal{X},\mathcal{H})\to T$ such that

(i) $(\mathcal{X}_{t_0},\mathcal{H}_{t_0})=(X,H)$, $t_0\in T$,
(ii) there are families of Mukai vectors $\mathbf{v}\in R^*\pi_*\mathbb{Z}$, $\mathbf{a}\in R^*\pi_*\mathbb{Q}$ with $\mathbf{v}_{t_0}=v$ and
(iii) $\operatorname{rk}\operatorname{Pic}(\mathcal{X}_t)=1$ for a point $t\in T$,

where $(\mathcal{X}_t, \mathcal{H}_t) := (\mathcal{X} \otimes k(t), \mathcal{H} \otimes k(t))$ and $k(t)$ is the residue field at $t \in T$. Replacing T by a suitable covering of T, we assume that there is a section of π and a locally free sheaf $\mathcal{G}$ on $\mathcal{X}$ with $v(\mathcal{G}_t) = \mathbf{v}_t$, $t \in T$. We consider the relative quot-scheme $g : \mathrm{Quot}^{\mathbf{v}}_{\mathcal{G}(-n\mathcal{H})^{\oplus N}/\mathcal{X}/T} \to T$ parametrizing all quotients $\mathcal{G}_t(-n\mathcal{H}_t)^{\oplus N} \to F$, $t \in T$ with $v(F) = \mathbf{v}_t$, where $N := \chi(\mathcal{G}_t, F(n\mathcal{H}_t))$. We set $Q := \mathrm{Quot}^{\mathbf{v}}_{\mathcal{G}(-n\mathcal{H})^{\oplus N}/\mathcal{X}/T}$. We denote the universal quotient sheaf by $\mathcal{F}$. We set

$$Q^{ss} := \{x \in Q |\ \mathcal{F}_x := \mathcal{F}_{|\mathcal{X}_t \times \{x\}} \text{ is } \mathbf{v}_t\text{-twisted semi-stable with respect to } \mathcal{H}_t,\ t = g(x)\ \}. \tag{2.95}$$

For $n \gg 0$, we have a relative coarse moduli space $\overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v}) := Q^{ss}/PGL(N) \to T$. Since T is defined over a field of characteristic 0, $\overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})_t = \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$ (cf. [MFK, Thm. 1.1]). Let $q : Q^{ss} \to \overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})$ be the quotient map. Assume that $\overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})_{t_0}$ is singular. Then $\mathcal{F}_{|\mathcal{X}_{t_0} \times Q^{ss}_{t_0}}$ is a locally free sheaf. Replacing T be an open neighborhood, we assume that $\mathcal{F}$ is locally free.

For a smooth curve C and a morphism $C \to T$, [OY, sect. 2.3] implies that $\overline{M}^{\mathbf{v}}_{(\mathcal{X}_C,\mathcal{H}_C)/C}(\mathbf{v}) \to C$ is flat over C. In particular the Hilbert polynomial of $\overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$ is independent of $t \in T$, which implies that $\overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v}) \to T$ is flat.

Definition 2.7.7. We set $G := PGL(N)$. For a G-linearized coherent sheaf E on $\mathcal{X} \times_T Q^{ss}$, $q_*(E)^G$ denotes the G-invariant part of $q_*(E)$.

By [MFK, Thm. A.1.1] or [Se, Thm. 2], $q_*(E)^G$ is a coherent $\mathcal{O}_{\mathcal{X} \times_T \overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})}$-module. Let V be a $GL(N)$-equivariant locally free sheaf on Q^{ss} such that $\mathbb{C}^\times (\subset GL(N))$ acts as a multiplication. For $\mathcal{B} := q_*(V^\vee \otimes V)$, we set $\mathcal{A} := \mathcal{B}^G$. $\mathcal{A}$ is a coherent $\mathcal{O}_{\overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})}$-module. Let $\mathrm{Spec}(A)$ be an affine neighborhood of T and I an ideal of A. By the exact sequence

$$0 \to I\mathcal{B} \to \mathcal{B} \to \mathcal{B}/I\mathcal{B} \to 0 \tag{2.96}$$

and the Reynolds operator R, we have an exact sequence

$$0 \to I(\mathcal{B}^G) \to \mathcal{B}^G \to (\mathcal{B}/I\mathcal{B})^G \to 0. \tag{2.97}$$

Thus $(\mathcal{B}/I\mathcal{B})^G \cong \mathcal{B}^G \otimes_A A/I$. Since $(\mathcal{B}/I\mathcal{B})^G$ is reflexive, it is torsion free as an A/I-module. In particular, $(\mathcal{B}/I\mathcal{B})^G$ is flat over A/I if $\mathrm{Spec}(A/I)$ is a smooth curve. Thus $\mathcal{A}_{|C}$ is flat over C for any smooth curve $C \subset T$. Then $\mathcal{A}$ is flat over T. We also see that $(1_{\mathcal{X}} \times q)_*(\mathcal{F} \otimes V^\vee)^G$ is a coherent $\mathcal{O}_{\mathcal{X} \times_T \overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})}$-module on $\mathcal{X} \times_T \overline{M}^{\mathbf{v}}_{(\mathcal{X},\mathcal{H})/T}(\mathbf{v})$ which is flat over T.

Let $Q_t(\mathbf{a}_t)^{ss} \subset Q^{ss}_t$ be the open subset such that $\mathcal{F}_x$ ($x \in Q_t(\mathbf{a}_t)^{ss}$) is $\mathbf{v}_t + \mathbf{a}_t$-twisted semi-stable and $q' : Q_t(\mathbf{a}_t)^{ss} \to \overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$ be the quotient map. We have a projective morphism $\pi : \overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t) \to \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$. Then we have a homomorphism

$$\iota : (1_{\mathcal{X}_t} \times q)_*(\mathcal{F}_t \otimes V_t^\vee)^G \to (1_{\mathcal{X}_t} \times \pi)_*((1_{\mathcal{X}_t} \times q')_*(\mathcal{F}_t \otimes V_t^\vee)^G). \tag{2.98}$$

Since $(1_{\mathcal{X}_t} \times q)_*(\mathcal{F}_t \otimes V_t^\vee)^G$ is a reflexive $\mathcal{O}_{\mathcal{X}_t \times \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)}$-module and $(1_{\mathcal{X}_t} \times \pi)_*((1_{\mathcal{X}_t} \times q')_*(\mathcal{F}_t \otimes V_t^\vee)^G)$ is a torsion free $\mathcal{O}_{\mathcal{X}_t \times \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)}$-module, ι is isomorphic. We also have an isomorphism

(2.99)

$$\iota^\vee : (1_{\mathcal{X}_t} \times q)_*((\mathcal{F}_t \otimes V_t^\vee)^\vee)^G \to (1_{\mathcal{X}_t} \times \pi)_*((1_{\mathcal{X}_t} \times q')_*((\mathcal{F}_t \otimes V_t^\vee)^\vee)^G) \cong (1_{\mathcal{X}_t} \times \pi)_*((1_{\mathcal{X}_t} \times q')_*((\mathcal{F}_t \otimes V_t^\vee)^G)^\vee).$$

Let $\mathcal{F}_t^\alpha$ and V_t^α be the twisted sheaves on $\mathcal{X}_t \times \overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$ and $\overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)$ defined by $\mathcal{F}_t$ and V_t. We have an equivalence $\Xi : \mathbf{D}(\mathcal{X}_t) \to \mathbf{D}^\alpha(\overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)) \cong \mathbf{D}_{\mathcal{E}nd(V_t^\alpha)}(\overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t))$ by

$$\Phi^{(\mathcal{F}_t^\alpha)^\vee}_{\mathcal{X}_t \to \overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)}(\bullet) \otimes V_t^\alpha = \Phi^{((1_{\mathcal{X}_t} \times q')_*(\mathcal{F}_t \otimes V_t^\vee)^G)^\vee}_{\mathcal{X}_t \to \overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)}(\bullet).$$

Hence we have an equivalence

$$\mathbf{D}(\mathcal{X}_t) \xrightarrow{\Xi} \mathbf{D}_{\mathcal{E}nd(V_t^\alpha)}(\overline{M}^{\mathbf{v}_t+\mathbf{a}_t}_{\mathcal{H}_t}(\mathbf{v}_t)) \xrightarrow{\mathbf{R}\pi_*} \mathbf{D}_{\mathcal{A}_t}(\overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)) \tag{2.100}$$

by $\Phi^{(1_{\mathcal{X}_t} \times q)_*((\mathcal{F}_t \otimes V_t^\vee)^\vee)^G}_{\mathcal{X}_t \to \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)}$.

Proposition 2.7.8. $(1_{\mathcal{X}} \times q)_*((\mathcal{F} \otimes V^\vee)^\vee)^G$ *defines a family of equivalences*

$$\Phi^{(1_{\mathcal{X}_t} \times q)_*((\mathcal{F}_t \otimes V_t^\vee)^\vee)^G}_{\mathcal{X}_t \to \overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)} : \mathbf{D}(\mathcal{X}_t) \to \mathbf{D}_{\mathcal{A}_t}(\overline{M}^{\mathbf{v}_t}_{\mathcal{H}_t}(\mathbf{v}_t)).$$

3.1. **Moduli of stable sheaves of dimension 2.** Let $Y \to C$ be a morphism from a normal projective surface to a smooth curve C such that a general fiber is an elliptic curve. Let $\pi : X \to Y$ be the minimal resolution. Then $\mathfrak{p} : X \to C$ is an elliptic surface over a curve C. We fix a divisor H on X which is the pull-back of a very ample divisor on Y. As in section 2, let $\mathcal{C}$ be the category of perverse coherent sheaves satisfying Assumption 2.1.1. We also use the notation A_{ij} in section 2. Let G_1 be a locally free sheaf on X which is a local projective generator of $\mathcal{C}$. Let $\mathbf{e} \in K(X)_{\text{top}}$ be the topological invariant of a locally free sheaf E of rank r and degree d on a fiber of $\mathfrak{p}$. Thus $\operatorname{ch}(\mathbf{e}) = (0, rf, d)$, where f is a fiber of $\mathfrak{p}$. Assume that $\mathbf{e}$ is primitive. Then $\overline{M}_H^{G_1}(\mathbf{e})$ consists of G_1-twisted stable objects, if $G_1 \in K(X)_{\text{top}} \otimes \mathbb{Q}$, $\operatorname{rk} G_1 > 0$ is general with respect to $\mathbf{e}$ and H. From now on, we assume that $\chi(G_1, \mathbf{e}) = 0$. By [OY, sect. 1.1], we do not lose generality.

Remark 3.1.1. We have $\overline{M}_H^{G_1}(\mathbf{e}) = \overline{M}_{H+nf}^{G_1}(\mathbf{e})$ for all n.

Lemma 3.1.2. *We set*

$$\mathbf{e}^\perp := \{E \in K(X)_{\text{top}} | \chi(E, \mathbf{e}) = 0\}. \tag{3.1}$$

(1) $-\chi(\ ,\ )$ *is symmetric on* $\mathbf{e}^\perp$.
(2) $M := (\mathbb{Z}\tau(G_1) + \mathbb{Z}\tau(\mathbb{C}_x) + \mathbb{Z}\mathbf{e})^\perp / \mathbb{Z}\mathbf{e}$ *is a negative definite even lattice of rank* $\rho(X) - 2$.

Proof. (1) For a divisor D, we set

$$\nu(D) := \tau(\mathcal{O}_X(D) - \mathcal{O}_X) - \frac{\chi(G_1, \mathcal{O}_X(D) - \mathcal{O}_X)}{\operatorname{rk} G_1} \tau(\mathbb{C}_x) \in K(X)_{\text{top}} \otimes \mathbb{Q}. \tag{3.2}$$

Then ν induces a homomorphism

$$\operatorname{NS}(X) \otimes \mathbb{Q} \to K(X)_{\text{top}} \otimes \mathbb{Q} \tag{3.3}$$

such that $\operatorname{rk}(\nu(D)) = 0$, $c_1(\nu(D)) = D$ and $\chi(G_1, \nu(D)) = 0$. For $E \in K(X) \otimes \mathbb{Q}$, we have an expression

$$\tau(E) = l\tau(G_1) + a\tau(\mathbb{C}_x) + \nu(D) \tag{3.4}$$

where $l, a \in \mathbb{Q}$ and $D \in \operatorname{NS}(X) \otimes \mathbb{Q}$. If $\chi(E, \mathbf{e}) = 0$, then D satisfies $(D, f) = 0$. Hence we have a decomposition

$$\mathbf{e}^\perp \otimes \mathbb{Q} = (\mathbb{Q}\tau(G_1) + \mathbb{Q}\tau(\mathbb{C}_x)) + \nu((\mathbb{Q}f)^\perp). \tag{3.5}$$

For $E, F \in K(X)$, we have

$$\chi(E, F) - \chi(F, E) = (\operatorname{rk} E c_1(F) - \operatorname{rk} F c_1(E), K_X). \tag{3.6}$$

Hence the claim (1) holds.

(2) By (3.5), the signature of $\mathbf{e}^\perp/\mathbb{Z}\mathbf{e}$ is $(1, \rho(X) - 1)$. We note that $\mathbb{Q}\tau(G_1) + \mathbb{Q}\tau(\mathbb{C}_x) \to (\mathbf{e}^\perp/\mathbb{Z}\mathbf{e}) \otimes \mathbb{Q}$ is injective and defines a subspace of signature $(1, 1)$. Hence M is negative definite. Since $(\mathbb{Z}\tau(\mathbb{C}_x) + \mathbb{Z}\mathbf{e})^\perp$ is an even lattice, we get our claim. $\square$

Lemma 3.1.3. $(H, c_1(\bullet)) : K(X)_{\text{top}} \to \mathbb{Z}$ *satisfies* $(H, c_1(E)) > 0$ *for 1-dimensional objects* E *of* $\mathcal{C}$.

Lemma 3.1.4. (1) *Assume that* G_1 *is general with respect to* $\mathbf{e}$ *and* H. *Then* $\overline{M}_H^{G_1}(\mathbf{e})$ *is a smooth elliptic surface over* C *and* $E \otimes K_X \cong E$ *for all* $E \in \overline{M}_H^{G_1}(\mathbf{e})$.
(2) *Let* E *be a* G_1*-twisted stable object such that* $\operatorname{Supp}(E) \subset \mathfrak{p}^{-1}(c)$, $c \in C$. *If* $\chi(G_1, E) = 0$ *and* $(c_1(E), H) < (c_1(\mathbf{e}), H)$, *then* $\chi(E, E) = 2$ *and* $E \otimes K_X \cong E$.

Proof. (1) In [Br1, Thm. 1.2], Bridgeland proved that $\overline{M}_H^{G_1}(\mathbf{e})$ is smooth and defines a Fourier-Mukai transform $\mathbf{D}(\overline{M}_H^{G_1}(\mathbf{e})) \to \mathbf{D}(X)$, if $G_1 = \mathcal{O}_X$ is general with respect to $\mathbf{e}$ and H. We can easily generalize the arguments in [Br1, sect. 4] to the moduli space $\overline{M}_H^{G_1}(\mathbf{e})$ of G_1-twisted semi-stable objects, if G_1 is general with respect to $\mathbf{e}$ and H. Then the claims follow.

(2) Since $\operatorname{Supp}(E) \subset \mathfrak{p}^{-1}(c)$ and $\chi(G_1, E) = 0$, we have $E \in (\mathbb{Z}\tau(\mathbb{C}_x) + \mathbb{Z}\tau(G_1) + \mathbb{Z}\mathbf{e})^\perp$. Since $0 < (c_1(E), H) < (c_1(\mathbf{e}), H)$, $\tau(E) \notin \mathbb{Z}\mathbf{e}$. Then Lemma 3.1.2 (2) implies

$$2 \le \chi(E, E) = \dim \operatorname{Hom}(E, E) + \dim \operatorname{Hom}(E, E \otimes K_X) - \dim \operatorname{Ext}^1(E, E). \tag{3.7}$$

Hence $\operatorname{Hom}(E, E \otimes K_X) \neq 0$. Since $K_X^{\otimes m} \in \mathfrak{p}^*(\operatorname{Pic}(C))$ for an integer m, we see that $E \otimes K_X$ is a G_1-twisted stable object with $\tau(E) = \tau(E \otimes K_X)$, which implies that $E \otimes K_X \cong E$ and $\chi(E, E) = 2$. $\square$

In the same way as in the proof of Theorem 2.1.6, we get the following results.

Corollary 3.1.5. (1) $\overline{M}_H^{G_1}(\mathbf{e})$ *is a normal surface and the singular points* $q_1, q_2, \dots, q_m$ *of* $\overline{M}_H^{G_1}(\mathbf{e})$ *correspond to the S-equivalence classes of properly* G_1*-twisted semi-stable objects.*

(2) *Let* $\bigoplus_{j=0}^{s_i'} E_{ij}^{\oplus a_{ij}'}$ *be the* S*-equivalence class corresponding to* q_i*. Then the matrix* $(\chi(E_{ij}, E_{ik}))_{j,k\geq 0}$ *is of affine type* $\tilde{A}, \tilde{D}, \tilde{E}$*. We assume that* $a_{i0} = 1$ *for all* i*. Then* $q_1, q_2, \ldots, q_m$ *are rational double points of type* A, D, E *according as the type of the matrices* $(\chi(E_{ij}, E_{ik}))_{j,k\geq 1}$*.*

(3) *We take a sufficiently small general* $\alpha \in K(X) \otimes \mathbb{Q}$ *such that* $\chi(\alpha, \mathbf{e}) = 0$*. Then* $\pi' : \overline{M}_H^{G_1+\alpha}(\mathbf{e}) \to \overline{M}_H^{G_1}(\mathbf{e})$ *is the minimal resolution.*

(4) *Assume that* $a_{i0}' = 1$ *for all* i *and* $\chi(\alpha, E_{ij}) < 0$ *for all* $j > 0$*. We set*

$$C_{ij}' := \{E \in M_H^{G_1+\alpha}(\mathbf{e}) | \operatorname{Hom}(E_{ij}, E) \neq 0\}. \tag{3.8}$$

Then C_{ij}' *is a smooth rational curve such that* $(C_{ij}', C_{i'j'}') = -\chi(E_{ij}, E_{i'j'})$ *and* ${\pi'}^{-1}(q_i) = \sum_{j\geq 1} a_{ij}' C_{ij}'$*.*

Remark 3.1.6. (1) In order to apply [Y7, Lem. 3.1.1], we need Lemma 3.1.3.

(2) In Theorem 2.1.6, we assume that $\chi(\alpha, E_{ij}) > 0$ for $j > 0$. So the definition of C_{ij}' is different from that in Lemma 2.2.4. For the smoothness of C_{ij}', we use the moduli of coherent systems (E, V), where $E \in M_H^{G_1+\alpha}(\mathbf{e})$ and V is a 1-dimensional subspace of $\operatorname{Hom}(E_{ij}, E)$.

From now on, we take an α in Corollary 3.1.5 (3) and set $X' := \overline{M}_H^{G_1+\alpha}(\mathbf{e})$, $Y' := \overline{M}_H^{G_1}(\mathbf{e})$.

Lemma 3.1.7. X' *is an elliptic surface over* C*.* $\mathfrak{q} : X' \to C$ *denotes the structure morphism of the elliptic fibration.*

Proof. For $E \in \overline{M}_H^{G_1}(\mathbf{e})$, $\operatorname{Div}(E) \in \operatorname{Hilb}_X^{rf}$ depends only on the S-equivalence class of E. Hence we have a morphism $g : Y' \to \operatorname{Hilb}_X^{rf}$. For a smooth fiber $\pi^{-1}(c)$ $(c \in C)$, $g^{-1}(r\pi^{-1}(c))$ is the moduli of stable vector bundles of rank r and degree d on $\pi^{-1}(c)$. Hence $g(Y')$ is a curve in $\operatorname{Hilb}_X^{rf}$. Let $\iota : C \to \operatorname{Hilb}_X^{df}$ be the map sending $c \in C$ to $r\pi^{-1}(c) \in \operatorname{Hilb}_X^{df}$. Then ι is injective and $g(Y') \cap \iota(C) \neq \emptyset$. Since $g(Y')$ and $\iota(C)$ are irreducible, $g(Y') = \iota(C)$ and we have a morphism $\widetilde{g} : Y' \to C$ such that $\iota \circ \widetilde{g} = g$. Therefore we have an elliptic fibration $\mathfrak{q} : X' \to Y' \to C$.

We next show that $K_{X'}$ is numerically trivial along the fibration $\mathfrak{q}$. The proof is similar to that in [Br1, Prop. 4.2]. For a reduced and irreducible curve D in a fiber of $\mathfrak{q}$, let E be a locally free sheaf on X'. Then $\chi(E_{|D}, E) = \chi(E, E_{|D} \otimes K_{X'}) = \chi(E, E_{|D}) + (\operatorname{rk} E)^2 (K_{X'}, D)$. Let $\mathcal{E}$ be a universal family of stable objects as a twisted object on $X' \times X$. Then $\operatorname{Supp}(H^i(\Phi_{X' \to X}^{\mathcal{E}}(E_{|D})) \subset \mathfrak{p}^{-1}(\mathfrak{q}(D))$ for all i implies that

$$\begin{aligned} &\chi(E_{|D}, E) - \chi(E, E_{|D}) \\ =&\chi(\Phi_{X' \to X}^{\mathcal{E}}(E_{|D}), \Phi_{X' \to X}^{\mathcal{E}}(E)) - \chi(\Phi_{X' \to X}^{\mathcal{E}}(E), \Phi_{X' \to X}^{\mathcal{E}}(E_{|D})) \\ =&(\operatorname{rk}(\Phi_{X' \to X}^{\mathcal{E}}(E_{|D})) c_1(\Phi_{X' \to X}^{\mathcal{E}}(E)) - \operatorname{rk}(\Phi_{X' \to X}^{\mathcal{E}}(E)) c_1(\Phi_{X' \to X}^{\mathcal{E}}(E_{|D})), f) = 0. \end{aligned} \tag{3.9}$$

Hence $(K_{X'}, D) = 0$, and the claim holds. □

Remark 3.1.8. It is easy to see that ι induces a injective homomorphism of Zariski tangent spaces. Hence ι is a closed immersion.

3.2. **Fourier-Mukai duality for an elliptic surface.** Let $\mathcal{E}$ be a universal family as a twisted sheaf on $X' \times X$. For simplicity, we assume that it is an untwisted sheaf. We set

$$\begin{aligned} \Psi(E) :=&\mathbf{R}\operatorname{Hom}_{p_{X'}}(p_X^*(E), \mathcal{E}) = \Phi_{X \to X'}^{\mathcal{E}^\vee}(E \otimes K_X)^\vee[-2],\ E \in \mathbf{D}(X), \\ \widehat{\Psi}(F) :=&\mathbf{R}\operatorname{Hom}_{p_X}(p_{X'}^*(F), \mathcal{E}),\ F \in \mathbf{D}(X'). \end{aligned} \tag{3.10}$$

Lemma 3.2.1. *Replacing* G_1 *by* $G_1 - n\mathbb{C}_x$, $n \gg 0$*, we can choose* $\det \Psi(G_1)^\vee \in \operatorname{Pic}(X')$ *as the pull-back of an ample line bundle on* W*. Let* $\widehat{H}$ *be a divisor with* $\mathcal{O}_{X'}(\widehat{H}) = \det \Psi(G_1)^\vee$*.*

Proof. We note that $c_1(\Psi(\mathbb{C}_x)) = rf$. Hence $\det \Psi(G_1 - n\mathbb{C}_x)^\vee = \det \Psi(G_1)^\vee(nrf)$. We set

$$\xi := mr \operatorname{rk} G_1 (H, f)(-G_1^\vee + (\operatorname{rk} G_1) n(n+m)(H^2)/2\varrho_X). \tag{3.11}$$

By [Y7, (1.112)], $\det p_{X'!}(\mathcal{E} \otimes p_X^*(\xi))$ is the pull-back of a polarization of Y' for $m \gg n \gg 0$. Since $\det \Psi(\xi^\vee) = \det p_{X'!}(\mathcal{E} \otimes p_X^*(\xi))$ and $-\operatorname{ch}(\xi^\vee) \equiv mr \operatorname{rk} G_1 (H, f) \operatorname{ch}(G_1) \mod \mathbb{Q}\varrho_X$, we get our claim. □

Lemma 3.2.2. *We set* $A_{ij}' := \Psi(E_{ij})[2]$*.*

(1) *There are* $\mathbf{b}_i' := (b_{i1}', b_{i2}', \ldots, b_{is_i'}')$, $i = 1, \ldots, m$ *such that*

$$\begin{aligned} A_{ij}' &= \mathcal{O}_{C_{ij}'}(b_{ij}')[1],\ j > 0 \\ A_{i0}' &= A_0(\mathbf{b}_i'). \end{aligned} \tag{3.12}$$

(2) *Irreducible objects of* $\operatorname{Per}(X'/Y', \mathbf{b}_1', \ldots, \mathbf{b}_m')$ *are*

$$A_{ij}' (1 \leq i \leq m, 0 \leq j \leq s_i'),\ \mathbb{C}_{x'} (x' \in X' \setminus \cup_i Z_i'). \tag{3.13}$$

Proof. It is sufficient to prove (1) by Proposition 1.1.4. By the choice of α, we have

$$\begin{aligned} \operatorname{Ext}^2(E_{ij}, \mathcal{E}_{|\{x'\}\times X}) =& 0,\ j>0, \\ \operatorname{Hom}(E_{i0}, \mathcal{E}_{|\{x'\}\times X}) =& 0 \end{aligned} \tag{3.14}$$

for all $x' \in X'$. Then the claim for $j>0$ follow from the proof of Corollary 3.1.5 (4). For $x' \in {\pi'}^{-1}(q_i)$, we have an exact sequence

$$0 \to F_i \to \mathcal{E}_{|\{x'\}\times X} \to E_{i0} \to 0, \tag{3.15}$$

where F_i is a G_1-twisted semi-stable object which is S-equivalent to $\bigoplus_{j>0} E_{ij}^{\oplus_j a'_{ij}}$. Applying Ψ, we have an exact sequence

$$0 \to \Psi(F_i)[1] \to A'_{i0} \to \mathbb{C}_{x'} \to 0. \tag{3.16}$$

It is easy to see that

$$\operatorname{Hom}(A'_{i0}, A'_{ij}[-1]) = \operatorname{Ext}^1(A'_{i0}, A'_{ij}[-1]) = 0. \tag{3.17}$$

By Lemma 1.1.3, we get $A'_{i0} = A_0(\mathbf{b}'_i)$. □

We define $\operatorname{Per}(X'/Y')$ and $\operatorname{Per}(X'/Y')^D$ as in subsection 2.2. Replacing G_1 by G'_1 with $\tau(G'_1) = \tau(G_1) - n\tau(\mathbb{C}_x)$, we may assume that $G_{1|\mathfrak{p}^{-1}(t)}$, $t \in C$ is a semi-stable vector bundle for a general $t \in C$. Indeed for a torsion free object G'_1 with $\operatorname{Ext}^2(G'_1, G'_1(-f))_0 = 0$, a deformation of G'_1 satisfies the claim (cf. [Y7, Proof of Prop. 2.1.1]). Then $L'_2 = \Psi(G_1)[1]$ is a torsion object of $\operatorname{Per}(X'/Y') \cap \operatorname{Coh}(X')$ such that $c_1(L_2) = \widehat{H}$. Indeed L'_2 is a coherent torsion sheaf on X'. Since $\operatorname{Hom}(L'_2, A'_{ij}[-1]) = \operatorname{Hom}(E_{ij}, G_1) = 0$, $L'_2 \in \operatorname{Per}(X'/Y')$.

Lemma 3.2.3. *Let L_1 be a line bundle on a smooth curve $C \in |H|$ and set $G_2 := \Psi(L_1)[1]$. Then we have*

$$\begin{aligned} \operatorname{Hom}(G_2, \mathbb{C}_{x'}[k]) &= 0, \quad k \neq 0, \\ \operatorname{Hom}(G_2, A'_{ij}[k]) &= 0, \quad k \neq 0, \\ \dim \operatorname{Hom}(G_2, A'_{ij}) &= (c_1(E_{ij}), H). \end{aligned} \tag{3.18}$$

In particular G_2 is a local projective generator of $\operatorname{Per}(X'/Y')$.

Proof. The claim follows from the following relations:

$$\begin{aligned} \operatorname{Hom}(G_2, \mathbb{C}_{x'}[k]) &= \operatorname{Hom}(\Psi(L_1)[1], \Psi(\mathcal{E}_{|\{x'\}\times X})[2+k]) \\ &= \operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, L_1[k+1]), \\ \operatorname{Hom}(G_2, A'_{ij}[k]) &= \operatorname{Hom}(\Psi(L_1)[1], \Psi(E_{ij})[2+k]) \\ &= \operatorname{Hom}(E_{ij}, L_1[k+1]). \end{aligned} \tag{3.19}$$

□

For a conveniense sake, we summalize the image of $\mathbb{C}_x[-2], \mathcal{E}_{|\{x'\}\times X}, G_1, L_1$ by Ψ:

$$\begin{aligned} \Psi(\mathbb{C}_x[-2]) &= \mathcal{E}_{|X'\times\{x\}}, \\ \Psi(\mathcal{E}_{|\{x'\}\times X}) &= \mathbb{C}_{x'}[-2], \\ \Psi(G_1) &= L_2[-1], \\ \Psi(L_1) &= G_2[-1]. \end{aligned} \tag{3.20}$$

Lemma 3.2.4. (1) *For $E \in \mathcal{C}$, there is a complex $W_\bullet : W_0 \to W_1 \to W_2$ of local projective objects W_i of* $\operatorname{Per}(X'/Y')$ *such that $\Psi(E) \cong W_\bullet$. In particular, ${}^pH^q(\Psi(E)) = 0$ for $q \neq 0,1,2$. We also have ${}^pH^q(\widehat{\Psi}(F)) = 0$ for $F \in \operatorname{Per}(X'/Y')$ and $q \neq 0,1,2$.*

(2) *For $E \in \mathcal{C}$, assume that* $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = \operatorname{Hom}(E, E_{ij}) = 0$ *for all $x' \in X'$ and i,j. Then there is a complex $W'_\bullet : W'_1 \to W'_2$ of local projective objects W'_i of* $\operatorname{Per}(X'/Y')$ *such that $\Psi(E) \cong W'_\bullet$. In particular, ${}^pH^0(\Psi(E)) = 0$ and ${}^pH^1(\Psi(E))$ is a torsion free object of* $\operatorname{Per}(X'/Y')$.

(3) *For $E \in \mathcal{C}$, assume that* $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[q]) = \operatorname{Hom}(E, E_{ij}[q]) = 0$ $(q = 0,1)$ *for all $x' \in X'$ and i,j. Then $\Psi(E)[2]$ is a local projective object of* $\operatorname{Per}(X'/Y')$.

Proof. (1) For $E \in \mathcal{C}$, we take a resolution $0 \to V_{-2} \to V_{-1} \to V_0 \to E \to 0$ in the proof of Lemma 2.5.2. Then $\operatorname{Hom}(V_k, \mathcal{E}_{|\{x'\}\times X}[q]) = \operatorname{Hom}(V_k, E_{ij}[q]) = 0$ $(q \neq 0)$ for all k, $x' \in X'$ and i,j. By Corollary 3.1.5, we have $K_{X'} \in {\pi'}^*(\operatorname{Pic}(Y'))$. Hence $A'_{ij} \otimes K_{X'} \cong A'_{ij}$. For $q \neq 0$, we have

$$\begin{aligned} 0 = \operatorname{Hom}(V_k, E_{ij}[q]) &= \operatorname{Hom}(\Psi(E_{ij})[q], \Psi(V_k)) \\ &= \operatorname{Hom}(\Psi(V_k), A'_{ij} \otimes K_{X'}[q])^\vee \\ &\cong \operatorname{Hom}(\Psi(V_k), A'_{ij}[q])^\vee. \end{aligned} \tag{3.21}$$

Thus $\Psi(V_k)$ are local projective objects. Hence $W_\bullet := \Psi(V_{-\bullet})$ is a desired complex.

The last claim follows by a similar argument to the proof of Corollary 2.5.3.

(2) For the complex $W_\bullet$ in (1), we shall prove that $W_0 \to W_1$ is injective and W_1/W_0 is a local projective object of $\operatorname{Per}(X'/Y')$. For this purpose, it is sufficient to show the surjectivity of $W_1^\vee \to W_0^\vee$ in $\operatorname{Per}(X'/Y')^D$ by Lemma 1.1.6. If it is not surjective, then $\operatorname{Hom}(\Psi(E)^\vee, (A'_{ij})^\vee[2]) \neq 0$ or $\operatorname{Hom}(\Psi(E)^\vee, \mathbb{C}_{x'}^\vee[2]) \neq 0$ by Lemma 1.1.6. On the other hand, we see that

$$\begin{aligned} \operatorname{Hom}(\Psi(E)^\vee, (A'_{ij})^\vee[2]) &= \operatorname{Hom}(A'_{ij}, \Psi(E)[2]) = \operatorname{Hom}(\Psi(E_{ij}), \Psi(E)) = \operatorname{Hom}(E, E_{ij}) = 0, \\ \operatorname{Hom}(\Psi(E)^\vee, \mathbb{C}_{x'}^\vee[2]) &= \operatorname{Hom}(\mathbb{C}_{x'}, \Psi(E)[2]) = \operatorname{Hom}(\Psi(\mathcal{E}_{|\{x'\}\times X}), \Psi(E)) = \operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0 \end{aligned} \tag{3.22}$$

by the assumption. Therefore our claim holds. (3) also follows from the proof of (2). □

Definition 3.2.5. We set $\Psi^i(E) := {}^pH^i(\Psi(E)) \in \operatorname{Per}(X'/Y')$ and $\widehat{\Psi}^i(E) := {}^pH^i(\widehat{\Psi}(E)) \in \mathcal{C}$.

Lemma 3.2.6. WIT_2 *with respect to* Ψ *holds for all 0-dimensional objects* E *of* $\mathcal{C}$ *and* $\Psi^2(E)$ *is* G_2*-twisted semi-stable. Moreover if* E *is an irreducible object, then* $\Psi(E)[2]$ *is a* G_2*-twisted stable object of* $\operatorname{Per}(X'/Y')$.

Proof. It is sufficient to prove the claim for all irreducible objects E of $\mathcal{C}$. Since $\mathcal{E}_{|\{x'\}\times X}$ and E_{ij} are purely 1-dimensional objects of $\mathcal{C}$, $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = \operatorname{Hom}(E, E_{ij}) = 0$ for all $x' \in X'$ and i,j. Hence $\Psi^0(E) = 0$ and $\Psi^1(E)$ is a torsion free object of $\operatorname{Per}(X'/Y')$ by Lemma 3.2.4. Since $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[1]) = 0$ if $\operatorname{Supp}(E) \cap \mathfrak{p}^{-1}(\mathfrak{p}(x')) = \emptyset$, $\Psi^1(E) = 0$. Therefore WIT_2 holds for all 0-dimensional objects of $\operatorname{Per}(X'/Y')$.

For the G_2-twisted stability of $\Psi(E)[2]$, we first note that $\operatorname{Supp}(\Psi(E)[2]) \subset \mathfrak{q}^{-1}(\mathfrak{p}(E))$ and $\chi(G_2, \Psi(E)[2]) = \chi(\Psi(L_1)[1], \Psi(E)[2]) = \chi(E, L_1[1]) = 0$. Since $(c_1(\Psi(E)[2]), \widehat{H}) = -\chi(\Psi(E), \Phi(G_1)[1]) = \chi(G_1, E) > 0$, $\Psi(E)$ is a 1-dimensional object of $\operatorname{Per}(X'/Y')$. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^2(E) \to F_2 \to 0 \tag{3.23}$$

such that $0 \neq F_1 \in \operatorname{Per}(X'/Y')$ and $F_2 \in \operatorname{Per}(X'/Y')$ with $\chi(G_2, F_2) \leq 0$. Applying $\widehat{\Psi}$ to this exact sequence, we get a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \widehat{\Psi}^0(F_2) \longrightarrow 0 \longrightarrow \widehat{\Psi}^0(F_1) \\ &\longrightarrow \widehat{\Psi}^1(F_2) \longrightarrow 0 \longrightarrow \widehat{\Psi}^1(F_1) \\ &\longrightarrow \widehat{\Psi}^2(F_2) \longrightarrow E \longrightarrow \widehat{\Psi}^2(F_1) \longrightarrow 0. \end{aligned} \tag{3.24}$$

Since $\operatorname{Supp}(F_1) \subset \operatorname{Supp}(\Psi(E)[2])$, $\Psi^i(\widehat{\Psi}^j(F_1))$ are torsion object of $\operatorname{Per}(X'/Y')$. By Lemma 2.5.2 (1), $\Psi^0(\widehat{\Psi}^0(F_1))$ is torsion free. Hence $\Psi^0(\widehat{\Psi}^0(F_1)) = 0$, which implies $\widehat{\Psi}^0(F_1) = 0$ by Lemma 5.1.2. Then (3.24) implies WIT_2 holds for F_2. Since $0 \geq \chi(G_2, F_2) = \chi(\widehat{\Psi}(F_2), \widehat{\Psi}(G_2)) = \chi(\widehat{\Psi}(F_2), L_1[-1]) = (H, c_1(\widehat{\Psi}^2(F_2))) \geq 0$, we get $\chi(G_2, F_2) = 0$ and $\widehat{\Psi}^2(F_2)$ is a 0-dimensional object. Then $\Psi(E)$ is purely 1-dimensional and $\widehat{\Psi}^1(F_1)$ is 0-dimensional. Since E is an irreducible object of $\mathcal{C}$, we have (i) $\widehat{\Psi}^2(F_1) = 0$ or (ii) $\widehat{\Psi}^2(F_1) \cong E$. Since WIT_2 holds for $\widehat{\Psi}^1(F_1)$ with respect to Ψ, the first case does not hold. If $\widehat{\Psi}^2(F_1) \cong E$, then $\widehat{\Psi}^1(F_1) \cong \widehat{\Psi}^2(F_2)$. Since $\widehat{\Psi}^0(F_1) = 0$, Lemma 5.1.2 implies that $\Psi^2(\widehat{\Psi}^1(F_1)) = 0$, which implies that $F_2 = \Psi^2(\widehat{\Psi}^2(F_2)) = 0$. Therefore $\Psi^2(E)$ is G_2-twisted stable. □

Theorem 3.2.7. *We set* $\mathbf{f} := \tau(\mathcal{E}_{|X'\times\{x\}})$*. Then* $\mathcal{E}_{|X'\times\{x\}}$ *is* $G_2 - \Psi(\beta)$*-twisted stable for all* $x \in X$ *and we have an isomorphism* $X \to M_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$ *by sending* $x \in X$ *to* $\mathcal{E}_{|X'\times\{x\}} \in M_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$.

Proof. By Lemma 3.2.6, $\mathcal{E}_{|X'\times\{x\}}$ is G_2-twisted semi-stable. If $\mathcal{E}_{|X'\times\{x\}}$ is not G_2-twisted stable, then $\mathcal{E}_{|X'\times\{x\}}$ is S-equivalent to $\bigoplus_j \Psi^2(A_{ij})^{\oplus a_{ij}}$. Let $F_1 \neq 0$ be a G_2-twisted stable subobject of $\mathcal{E}_{|X'\times\{x\}}$ such that $\chi(G_2, F_1) = 0$. Then F_1 is S-equivalent to $\bigoplus_j \Psi^2(A_{ij})^{\oplus b_{ij}}$ and $\widehat{\Psi}(F_1)[2]$ is a quotient object of $\mathbb{C}_x$. Since $\mathbb{C}_x$ is β-stable, $0 < \chi(\beta, \widehat{\Psi}(F_1)) = \chi(\Psi(\beta), F_1)$. Therefore $\mathcal{E}_{|X'\times\{x\}}$ is $G_2 - \Psi(\beta)$-twisted stable. Then we have an injective morphism $\phi : X \to \overline{M}_{\widehat{H}}^{G_2-\Psi(\beta)}(\mathbf{f})$ by sending $x \in X$ to $\mathcal{E}_{|X'\times\{x\}}$. By a standard argument, we see that ϕ is an isomorphism. □

3.3. **Tiltings of** $\mathcal{C}$**,** $\operatorname{Per}(X'/Y')$ **and their equivalence.** We set $\mathfrak{C}_1 := \mathcal{C}$ and $\mathfrak{C}_2 := \operatorname{Per}(X'/Y')$. In this subsection, we define tiltings $\overline{\mathfrak{A}}_1$, $\widehat{\mathfrak{A}}_2$ of $\mathfrak{C}_1$, $\mathfrak{C}_2$ and show that Ψ induces a (contravariant) equivalence between them. We first define the relative twisted degree of $E \in \mathfrak{C}_i$ by $\deg_{G_i}(E) := (c_1(G_i^\vee \otimes E), f)$, and define $\mu_{\max,G_i}(E)$, $\mu_{\min,G_i}(E)$ in a similar way.

Definition 3.3.1. (1) Let $\overline{\mathfrak{T}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that (i) E is a torsion object or (ii) E is torsion free and $\mu_{\min,G_i}(E) \geq 0$.

(2) Let $\overline{\mathfrak{F}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that (i) $E = 0$ or (ii) E is torsion free and $\mu_{\max,G_i}(E) < 0$.

Definition 3.3.2. (1) Let $\widehat{\mathfrak{T}}_i$ be the full subcategory of $\mathfrak{C}_i$ consisting of objects E such that $\operatorname{Supp}(E)$ is contained in fibers and there is no quotient object $E \to E'$ with $\chi(G_i, E') < 0$.

(2) We set

$$\begin{aligned}\widehat{\mathfrak{F}}_i :=&(\widehat{\mathfrak{T}}_i)^{\perp} \\ =&\{E \in \mathfrak{C}_i \mid \operatorname{Hom}(E', E) = 0, E' \in \widehat{\mathfrak{T}}_i\}.\end{aligned} \tag{3.25}$$

Remark 3.3.3. We have $\widehat{\mathfrak{F}}_i \supset \overline{\mathfrak{F}}_i$ and $\widehat{\mathfrak{T}}_i \subset \overline{\mathfrak{T}}_i$.

Definition 3.3.4. $(\overline{\mathfrak{T}}_i, \overline{\mathfrak{F}}_i)$ and $(\widehat{\mathfrak{T}}_i, \widehat{\mathfrak{F}}_i)$ are torsion pairs of $\mathfrak{C}_i$. We denote the tiltings by $\overline{\mathfrak{A}}_i$ and $\widehat{\mathfrak{A}}_i$ respectively.

Then we have the following equivalence:

Proposition 3.3.5. *Ψ induces an equivalence $\overline{\mathfrak{A}}_1[-2] \to (\widehat{\mathfrak{A}}_2)_{op}$.*

For the proof of this proposition, we need the following properties.

Lemma 3.3.6. (1) *Assume that $E \in \overline{\mathfrak{T}}_1$. Then $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for a general $x' \in X'$.*

(2) *Assume that $E \in \widehat{\mathfrak{F}}_1$. Then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = \operatorname{Hom}(E_{ij}, E) = 0$ for all $x' \in X'$. In particular if $E \in \overline{\mathfrak{F}}_1$, then $\operatorname{Hom}(\mathcal{E}_{|\{x'\}\times X}, E) = \operatorname{Hom}(E_{ij}, E) = 0$ for all $x' \in X'$.*

Proof. We only prove (1). If $\operatorname{rk} E = 0$, then obviously the claim holds. Let E be a torsion free object on X such that $E_{|f}$ is a semi-stable locally free sheaf with $\chi(G_1, E_{|f}) \geq 0$ for a general f. Then if there is a non-zero homomorphism $\varphi : E \to \mathcal{E}_{|\{x'\}\times X}$, then $\chi(G_1, E_{|f}) = 0$, φ is surjective and $E_{|f}$ is S-equivalent to $\mathcal{E}_{|\{x'\}\times X} \oplus \ker \varphi$, where $f = \mathfrak{p}^{-1}(\mathfrak{q}(x'))$. Therefore $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}) = 0$ for a general $x' \in \mathfrak{q}^{-1}(\mathfrak{p}(f)) \subset Y$. □

Lemma 3.3.7. *Let E be an object of $\mathcal{C} = \mathfrak{C}_1$.*

(1) *$H^0(\Psi^2(E)) = H^2(\Psi(E))$.*

(2) *$\Psi^0(E) \subset H^0(\Psi(E))$. In particular, $\Psi^0(E)$ is torsion free.*

(3) *If $\operatorname{Hom}(E, E_{ij}[2]) = 0$ for all i, j and $\operatorname{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[2]) = 0$ for all $x' \in X'$, then $\Psi^2(E) = 0$. In particular, if $E \in \widehat{\mathfrak{F}}_1$, then $\Psi^2(E) = 0$.*

(4) *If E satisfies $E \in \overline{\mathfrak{T}}_1$, then $\Psi^0(E) = 0$.*

Proof. It is a consequence of Lemma 3.2.4 and Lemma 3.3.6. □

Corollary 3.3.8. *If $E \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$, then ${}^pH^i(\Psi(E)) = 0$ for $i \neq 1$.*

Lemma 3.3.9. *Let E be an object of $\mathcal{C}$.*

(1) *If WIT_0 holds for E with respect to Ψ, then $E \in \overline{\mathfrak{F}}_1$.*

(2) *If WIT_2 holds for E with respect to Ψ, then $E \in \widehat{\mathfrak{T}}_1$.*

Proof. For an object E of $\mathcal{C}$, there is an exact sequence

$$0 \to E_1 \to E \to E_2 \to 0 \tag{3.26}$$

such that $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \overline{\mathfrak{F}}_1$. Applying Ψ to this exact sequence, we get a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Psi^0(E_2) & \longrightarrow & \Psi^0(E) & \longrightarrow & \Psi^0(E_1) \\ & \longrightarrow & \Psi^1(E_2) & \longrightarrow & \Psi^1(E) & \longrightarrow & \Psi^1(E_1) \\ & \longrightarrow & \Psi^2(E_2) & \longrightarrow & \Psi^2(E) & \longrightarrow & \Psi^2(E_1) & \longrightarrow & 0. \end{array} \tag{3.27}$$

By Lemma 3.3.7, we have $\Psi^0(E_1) = \Psi^2(E_2) = 0$. If WIT_0 holds for E, then we get $\Psi(E_1) = 0$. Hence (1) holds. If WIT_2 holds for E, then we get $\Psi(E_2) = 0$. Thus $E \in \overline{\mathfrak{T}}_1$. We take a decomposition

$$0 \to E_1' \to E \to E_2' \to 0 \tag{3.28}$$

such that $E_1' \in \widehat{\mathfrak{T}}_1$ and $E_2' \in \widehat{\mathfrak{F}}_1 \cap \overline{\mathfrak{T}}_1$. Then $\Psi^i(E_2') = 0$ for $i \neq 1$ by Corollary 3.3.8. Since $\Psi^0(E_1') = 0$, we also get $\Psi^1(E_2') = 0$. Therefore $E_2' = 0$. □

By Theorem 3.2.7, we have the following lemma.

Lemma 3.3.10. *Similar claims to Lemma 3.3.7, Corollary 3.3.8 and Lemma 3.3.9 hold for $\widehat{\Psi}$.*

Lemma 3.3.11. (1) *If $E \in \overline{\mathfrak{T}}_1$, then (1a) $\Psi^0(E) = 0$, (1b) $\Psi^1(E) \in \widehat{\mathfrak{F}}_2$ and (1c) $\Psi^2(E) \in \widehat{\mathfrak{T}}_2$.*

(2) *If $E \in \overline{\mathfrak{F}}_1$, then (2a) $\Psi^0(E) \in \widehat{\mathfrak{F}}_2$, (2b) $\Psi^1(E) \in \widehat{\mathfrak{T}}_2$ and (2c) $\Psi^2(E) = 0$.*

Proof. (1a) and (2c) follow from Lemma 3.3.7. (2a) is easy. (1c) By Lemma 5.1.2, WIT_2 holds for $\Psi^2(E)$ with respect to $\widehat{\Psi}$. By a similar claim of Lemma 3.3.9 (2), we get $\Psi^2(E) \in \widehat{\mathfrak{T}}_2$.

We next study $\Psi^1(E)$ for $E \in \mathcal{C}$. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0 \tag{3.29}$$

such that $F_1 \in \widehat{\mathfrak{T}}_2$ and $F_2 \in \widehat{\mathfrak{F}}_2$. Applying $\widehat{\Psi}$, we have a long exact sequence

$$\begin{aligned} 0 \longrightarrow \widehat{\Psi}^0(F_2) &\longrightarrow \widehat{\Psi}^0(\Psi^1(E)) \longrightarrow \widehat{\Psi}^0(F_1) \\ \longrightarrow \widehat{\Psi}^1(F_2) &\longrightarrow \widehat{\Psi}^1(\Psi^1(E)) \longrightarrow \widehat{\Psi}^1(F_1) \\ \longrightarrow \widehat{\Psi}^2(F_2) &\longrightarrow \widehat{\Psi}^2(\Psi^1(E)) \longrightarrow \widehat{\Psi}^2(F_1) \longrightarrow 0. \end{aligned} \tag{3.30}$$

By similar claims to Lemma 3.3.7, we have $\widehat{\Psi}^0(F_1) = \widehat{\Psi}^2(F_2) = 0$.

Assume that $E \in \overline{\mathfrak{T}}_1$. Since $\Psi^0(E) = 0$, Lemma 5.1.2 implies that $\widehat{\Psi}^2(\Psi^1(E)) = 0$. Hence WIT_1 holds for F_1. Since $0 \le \chi(G_2, F_1) = \chi(\widehat{\Psi}^1(F_1), L_1) = -(H, c_1(\widehat{\Psi}^1(F_1))) \le 0$, $\widehat{\Psi}^1(F_1)$ is a 0-dimensional object. If $F_1 \ne 0$, then since $\widehat{\Psi}^1(F_1) \ne 0$, we see that $0 < \chi(G_1, \widehat{\Psi}^1(F_1)) = \chi(F_1, L_2) = -(\widehat{H}, c_1(F_1)) \le 0$, which is a contradiction. Therefore $F_1 = 0$.

Assume that $E \in \overline{\mathfrak{F}}_1$. Since $\Psi^2(E) = 0$, Lemma 5.1.2 implies that $\widehat{\Psi}^0(\Psi^1(E)) = 0$. Hence WIT_1 holds for F_2. We have an injection $\widehat{\Psi}^1(\Psi^1(E)) \to E$. Since $\mu_{\max,G_1}(E) < 0$, $\mathrm{Hom}(E, \mathcal{E}_{|\{x'\}\times X}[1]) = 0$ for a general $x' \in X'$. Hence $\Psi^1(E)$ is zero on a generic fiber of $\mathfrak{p}$. Then $\widehat{\Psi}^1(\Psi^1(E))$ is a torsion object. Since E is torsion free, $\widehat{\Psi}^1(\Psi^1(E)) = 0$. Since $\widehat{\Psi}^0(F_1) = 0$, we get $\widehat{\Psi}^1(F_2) = 0$, which implies that $F_2 = 0$. □

Proof of Proposition 3.3.5.

It is sufficient to prove that $\Psi(\overline{\mathfrak{T}}_1[-2]), \Psi(\overline{\mathfrak{F}}_1[-1]) \subset (\widehat{\mathfrak{A}}_2)_{op}$. Then the claims follow from Lemma 3.3.11. □

3.4. **Preservation of Gieseker stability conditions.** We give a generalization of [Y1, Thm. 3.15]. We first recall the following well-known fact.

Lemma 3.4.1. (1) *Let E be a torsion free object of $\mathcal{C}$. Then E is G_1-twisted semi-stable with respect to $H + nf$, $n \gg 0$ if and only if for every proper subobject E' of E, one of the following conditions holds:*

(a)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} > \frac{(c_1(E'), f)}{\operatorname{rk} E'}, \tag{3.31}$$

(b)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} = \frac{(c_1(E'), f)}{\operatorname{rk} E'},\ \frac{(c_1(E), H)}{\operatorname{rk} E} > \frac{(c_1(E'), H)}{\operatorname{rk} E'}, \tag{3.32}$$

(c)

$$\frac{(c_1(E), f)}{\operatorname{rk} E} = \frac{(c_1(E'), f)}{\operatorname{rk} E'},\ \frac{(c_1(E), H)}{\operatorname{rk} E} = \frac{(c_1(E'), H)}{\operatorname{rk} E'},\ \frac{\chi(G_1, E)}{\operatorname{rk} E} \ge \frac{\chi(G_1, E')}{\operatorname{rk} E'}. \tag{3.33}$$

(2) *Let F be a 1-dimensional object of $\mathrm{Per}(X'/Y')$ with $(c_1(F), f) \ne 0$. Then F is G_2-twisted semi-stable with respect to $\widehat{H} + nf$, $n \gg 0$ if and only if for every proper subobject F' of F, one of the following conditions holds:*

(a)

$$(c_1(F'), f)\frac{\chi(G_2, F)}{(c_1(F), f)} > \chi(G_2, F') \tag{3.34}$$

(b)

$$(c_1(F'), f)\frac{\chi(G_2, F)}{(c_1(F), f)} = \chi(G_2, F'),\ (c_1(F'), \widehat{H})\frac{\chi(G_2, F)}{(c_1(F), \widehat{H})} > \chi(G_2, F'). \tag{3.35}$$

Lemma 3.4.2. *Let F be a purely 1-dimensional G_2-twisted semi-stable object such that $(c_1(F), f) > 0$ and $\chi(G_2, F) < 0$. Then WIT_1 holds for F with respect to $\widehat{\Psi}$ and $\widehat{\Psi}^1(F)$ is torsion free.*

Proof. By Lemma 3.4.1 (2), $F \in \widehat{\mathfrak{F}}_2$. Then WIT_1 holds for F by Lemma 3.3.10. Assume that there is an exact sequence

$$0 \to E_1 \to \widehat{\Psi}^1(F) \to E_2 \to 0 \tag{3.36}$$

such that E_1 is the torsion subobject of $\widehat{\Psi}^1(F)$. Since $\widehat{\Psi}^1(F)_{|f}$ is a semi-stable vector bundle of $\deg(G_1^\vee \otimes \widehat{\Psi}^1(F)_{|f}) = 0$ for a general fiber f of $\mathfrak{p}$, $\mathrm{Supp}(E_1)$ is contained in fibers. Since $E_1 \in \overline{\mathfrak{T}}_1$ and $E_2 \in \widehat{\mathfrak{F}}_1$, WIT_1

holds for E_1, E_2 and we have a quotient $F \to \Psi^1(E_1)$. By our assumption on F, we get $\chi(G_2, \Psi^1(E_1)) \geq 0$. On the other hand, $\chi(G_2, \Psi^1(E_1)) = \chi(E_1, L_1) = -(H, c_1(E_1)) \leq 0$. Hence E_1 is a 0-dimensional object. Then we get $0 < \chi(G_1, E_1) = \chi(\Psi^1(E_1), L_2) = -(\widehat{H}, c_1(\Psi^1(E_1))) \leq 0$, which is a contradiction. □

Lemma 3.4.3. *Let F be a 1-dimensional object of* $\mathrm{Per}(X'/Y')$. *Then*

$$\begin{aligned} r(c_1(F), f) &= \mathrm{rk}(\widehat{\Psi}(F)[1]), \\ (c_1(F), \widehat{H}) &= -\chi(F, L_2) = -\chi(G_1, \widehat{\Psi}(F)[1]), \\ \chi(G_2, F) &= \chi(\widehat{\Psi}(F)[1], L_1) = -(c_1(\widehat{\Psi}(F)[1]), H) + \mathrm{rk}(\widehat{\Psi}(F)[1])\chi(L_1). \end{aligned} \tag{3.37}$$

Proposition 3.4.4. *Let $w \in K(X')_{\mathrm{top}}$ be a topological invariant of a 1-dimensional object. Assume that $\chi(G_2, w) < 0$. Then for $n \gg 0$, we have an isomorphism*

$$\mathcal{M}^{G_1}_{H+nf}(\widehat{\Psi}(-w))^{ss} \to \mathcal{M}^{G_2}_{\widehat{H}+nf}(w)^{ss}, \tag{3.38}$$

which preserves the S-equivalence classes.

Proof. Let E be a G_1-twisted semi-stable object with $\tau(E) = \widehat{\Psi}(-w)$. Then since $E_{|f}$ is a semi-stable locally free sheaf with $d\,\mathrm{rk}\,E - r\deg(E_{|f}) = 0$ for a general fiber, we have $E \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$. By Corollary 3.3.8, WIT_1 holds for E with respect to Ψ. Assume that there is an exact sequence

$$0 \to F_1 \to \Psi^1(E) \to F_2 \to 0. \tag{3.39}$$

By Lemma 3.3.11, $\Psi^1(E) \in \widehat{\mathfrak{F}}_2$, which implies that $F_1 \in \widehat{\mathfrak{F}}_2$. Since $\mathrm{rk}\,\Psi^1(E) = 0$, $F_1, F_2 \in \overline{\mathfrak{T}}_2$. In particular, $F_1 \in \overline{\mathfrak{T}}_2 \cap \widehat{\mathfrak{F}}_2$. Then similar claim to Corollary 3.3.8 implies that WIT_1 holds for F_1. Hence we get an exact sequence

$$0 \to \widehat{\Psi}^1(F_2) \to E \xrightarrow{\varphi} \widehat{\Psi}^1(F_1) \to \widehat{\Psi}^2(F_2) \to 0. \tag{3.40}$$

By Lemma 3.3.11, $\widehat{\Psi}^2(F_2) \in \widehat{\mathfrak{T}}_1$. Hence $\mathrm{rk}\,\widehat{\Psi}^1(F_1) = \mathrm{rk}\,\mathrm{im}\,\varphi$. By (3.37), we have the following equivalences.

$$(c_1(F_1), f)\frac{\chi(G_2, \Psi^1(E))}{(c_1(F), f)} \leq \chi(G_2, F_1) \iff \mathrm{rk}\,\widehat{\Psi}^1(F_1)\frac{(c_1(E), H)}{\mathrm{rk}\,E} \geq (c_1(\widehat{\Psi}^1(F_1)), H), \tag{3.41}$$

$$(c_1(F_1), \widehat{H})\frac{\chi(G_2, \Psi^1(E))}{(c_1(\Psi^1(E)), \widehat{H})} \leq \chi(G_2, F_1) \iff -\chi(G_1, \widehat{\Psi}^1(F_1))\frac{\chi(G_2, \Psi^1(E))}{-\chi(G_1, E)} \leq \chi(G_2, F_1). \tag{3.42}$$

If the equality holds in (3.41), then $\chi(G_2, \Psi^1(E)) < 0$ implies that (3.42) is equivalent to

$$\frac{\chi(G_1, \widehat{\Psi}^1(F_1))}{\chi(G_1, E)} \geq \frac{\mathrm{rk}\,\widehat{\Psi}^1(F_1)}{\mathrm{rk}\,E} \tag{3.43}$$

which is equivalent to

$$\frac{\chi(G_1, \widehat{\Psi}^1(F_1))}{\mathrm{rk}\,\widehat{\Psi}^1(F_1)} \leq \frac{\chi(G_1, E)}{\mathrm{rk}\,E} \tag{3.44}$$

by $-\chi(G_1, E) > 0$. Since

$$\frac{\chi(G_1, \mathrm{im}\,\varphi(nH))}{\mathrm{rk}\,\mathrm{im}\,\varphi} \leq \frac{\chi(G_1, \widehat{\Psi}^1(F_1)(nH))}{\mathrm{rk}\,\widehat{\Psi}^1(F_1)},\ n \gg 0, \tag{3.45}$$

we see that φ is surjective and the equalities hold for (3.41), (3.42). Therefore $\Psi^1(E)$ is G_2-twisted semi-stable.

Conversely let F be a G_2-twisted semi-stable object with $\tau(F) = w$. By Lemma 3.4.2, WIT_1 holds for F with respect to $\widehat{\Psi}$ and $\widehat{\Psi}^1(F)$ is a torsion free object whose restriction to a general fiber is stable. If $\widehat{\Psi}^1(E)$ is not G_1-twisted semi-stable, then we have an exact sequence

$$0 \to E_1 \to \widehat{\Psi}^1(F) \to E_2 \to 0 \tag{3.46}$$

such that $E_i \in \overline{\mathfrak{T}}_1 \cap \widehat{\mathfrak{F}}_1$. By using Lemme 3.4.3, we get the following equivalences:

$$\frac{(c_1(\widehat{\Psi}^1(F)), H)}{\mathrm{rk}\,\widehat{\Psi}^1(F)} \leq \frac{(c_1(E_1), H)}{\mathrm{rk}\,E_1} \iff \frac{\chi(G_2, F)}{(c_1(F), f)} \geq \frac{\chi(G_2, \Psi^1(E_1))}{(c_1(\Psi^1(E_1)), f)}, \tag{3.47}$$

$$\frac{\chi(G_1, \widehat{\Psi}^1(F))}{\mathrm{rk}\,\widehat{\Psi}^1(F)} \leq \frac{\chi(G_1, E_1)}{\mathrm{rk}\,E_1} \iff \frac{(c_1(F), \widehat{H})}{(c_1(F), f)} \geq \frac{(c_1(\Psi^1(E_1)), \widehat{H})}{(c_1(\Psi^1(E_1)), f)}. \tag{3.48}$$

If the equality holds in (3.47), then (3.48) is equivalent to

$$\frac{\chi(G_2, F)}{(c_1(F), \widehat{H})} \geq \frac{\chi(G_2, \Psi^1(E_1))}{(c_1(\Psi^1(E_1)), \widehat{H})} \tag{3.49}$$

by $\chi(G_2, F) < 0$. Therefore $\widehat{\Psi}^1(F)$ is G_1-twisted semi-stable. □

4. A category of equivariant coherent sheaves.

4.1. **Morita equivalence for G-sheaves.** Let X be a smooth projective surface and G a finite group acting on X. Assume that $G \to \mathrm{Aut}(X)$ is injective and $\mathrm{Stab}(x)$, $x \in X$ acts trivally on $(K_X)_{|\{x\}}$, that is, K_X is the pull-back of a line bundle on $Y := X/G$. By our assumption, all elements of G have at most isolated fixed points sets. Let $R(G)$ be the representation ring of G and $(\quad,\quad)$ the natural inner product. Let $K_G(X)$ be the Grothendieck group of G-sheaves and $K_G(X)_{\mathrm{top}}$ its image to the Grothendieck group of topological G-vector bundles. Since we are mainly interested in surfaces with trivial canonical bundles, we denotes the topological invariant of $E \in \mathrm{Coh}_G(X)$ by $v(E) \in K_G(X)_{\mathrm{top}}$.

Definition 4.1.1. For G-sheaves E and F on X,

(1) $\mathrm{Ext}^i_G(E, F)$ is the G-invariant part of $\mathrm{Ext}^i(E, F)$.

(2) $\chi_G(E, F) := \sum_i (-1)^i \dim \mathrm{Ext}^i_G(E, F)$ is the Euler characteristic of the G-invariant cohomology groups of E, F. We also set $\chi_G(E) := \chi_G(\mathcal{O}_X, E)$.

Remark 4.1.2. (1) If $K_X \cong \mathcal{O}_X$ in $\mathrm{Coh}_G(X)$, then $\chi_G(\ ,\ )$ is symmetric.

(2) $\chi_G(E, F)$ is invariant for flat deformations of E, F:

Let $\mathcal{E}$ and $\mathcal{F}$ be a flat family of G-sheaves on X over S. By taking a suitable locally free resolution of $\mathcal{E}$, we see that $\mathbf{R}\,\mathrm{Hom}_{p_S}(\mathcal{E}, \mathcal{F})$ is represented by a complex $0 \to V_0 \to V_1 \to \cdots \to V_n \to 0$ of locally free sheaves V_i on S with G-actions, where $n = \dim X$. Since S is a scheme over $\mathbb{C}$, we have a decomposition $V_i = \oplus_j V_{ij} \otimes \rho_j$, where ρ_j are irreducible representations and V_{ij} are locally free sheaves on S with trivial G-actions. Hence $\chi_G(\mathcal{E}_{|\{s\}\times X}, \mathcal{F}_{|\{s\}\times X}) = \sum_i (-1)^i \operatorname{rk} V_{i0}$, where ρ_0 is the trivial representation.

Let $\varpi : X \to Y$ be the quotient map. We set

$$\varpi_*(\mathcal{O}_X)[G] := \left\{ \sum_{g\in G} f_g(x) g \,\middle|\, f_g(x) \in \varpi_*(\mathcal{O}_X) \right\}. \tag{4.1}$$

$\varpi_*(\mathcal{O}_X)[G]$ is an $\mathcal{O}_Y$-algebra whose multiplication is defined by

$$\left(\sum_{g\in G} f_g(x) g\right) \cdot \left(\sum_{g'\in G} f'_{g'}(x) g'\right) := \sum_{g,g'\in G} f_g(x) f'_{g'}(g^{-1}x) g g'. \tag{4.2}$$

We note that $\epsilon := \frac{1}{\#G}\sum_{g\in G} g$ satisfies $g\epsilon = \epsilon$ for all $g \in G$. By the injective homomorphism

$$\varpi_*(\mathcal{O}_X) \to \varpi_*(\mathcal{O}_X)\epsilon \ (\subset \varpi_*(\mathcal{O}_X)[G]), \tag{4.3}$$

we have an action of $\varpi_*(\mathcal{O}_X)[G]$ on $\varpi_*(\mathcal{O}_X)$:

$$\left(\sum_{g\in G} f_g(x) g\right) \cdot f(x) := \sum_{g\in G} f_g(x) f(g^{-1}x). \tag{4.4}$$

Thus we have a homomorphism

$$\varpi_*(\mathcal{O}_X)[G] \to \mathrm{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X)). \tag{4.5}$$

Lemma 4.1.3. $\varpi_*(\mathcal{O}_X)[G] \cong \mathrm{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$.

Proof. We first prove the claim over the smooth locus Y^{sm} of Y. We note that $\#\varpi^{-1}(y) = \#G$, $y \in Y^{\mathrm{sm}}$. We take a point $z \in \varpi^{-1}(y)$. Then $\varpi_*(\mathcal{O}_X)_{|y} = \mathcal{O}_{\varpi^{-1}(y)}$ is identified with $\oplus_{g\in G}\mathbb{C}_{gz}$ as $\mathbb{C}[G]$-modules. Let $\chi_u(x)$ be the characteristic function of a point $u \in X$. Then $\{\chi_{gz} | g \in G\}$ is the base of $\oplus_{g\in G}\mathbb{C}_{gz}$ and $f(x) \in \mathcal{O}_{\varpi^{-1}(y)}$ is decomposed into $f(x) = \sum_{g\in G} f(gz)\chi_{gz}(x)$. Since

$$(\chi_{g'z}(x)(g'g^{-1})) \cdot \left(\sum_{h\in G} f(hz)\chi_{hz}(x)\right) = f(gz)\chi_{g'z}(x), \tag{4.6}$$

we see that

$$(\varpi_*(\mathcal{O}_X)[G])_{|y} \to \mathrm{Hom}(\varpi_*(\mathcal{O}_X)_{|y}, \varpi_*(\mathcal{O}_X)_{|y}) \tag{4.7}$$

is an isomorphism. Since $\varpi_*(\mathcal{O}_X)[G]$ and $\mathrm{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$ are reflexive sheaves on Y, we get the claim. □

We set $\mathcal{A} := \varpi_*(\mathcal{O}_X)[G] \cong \operatorname{Hom}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X), \varpi_*(\mathcal{O}_X))$.

Lemma 4.1.4. *We have an equivalence*

$$\begin{array}{rccc} \varpi_* : & \operatorname{Coh}_G(X) & \cong & \operatorname{Coh}_{\mathcal{A}}(Y) \\ & E & \mapsto & \varpi_*(E) \end{array} \tag{4.8}$$

whose inverse is $\varpi^{-1} : \operatorname{Coh}_{\mathcal{A}}(Y) \to \operatorname{Coh}_G(X)$. *In particular, we have an isomorphism*

$$\operatorname{Hom}_G(E_1, E_2) = \operatorname{Hom}_{\mathcal{A}}(\varpi_*(E_1), \varpi_*(E_2)),\ E_1, E_2 \in \operatorname{Coh}_G(X). \tag{4.9}$$

Proof. Since the problem is local, we may assume that Y is affine. Then X is also affine. For $F \in \operatorname{Coh}_{\mathcal{A}}(Y)$, $H^0(Y, F)$ is a $H^0(Y, \varpi_*(\mathcal{O}_X))[G]$-module. Hence $H^0(X, \varpi^{-1}(F)) = H^0(Y, F)$ is a $H^0(X, \mathcal{O}_X)[G]$-module, which implies that $\varpi^{-1}(F) \in \operatorname{Coh}_G(X)$. Then it is easy to see that ϖ^{-1} is the inverse of ϖ_*. □

By Lemma 4.1.4, we have an equivalence $\varpi_* : \mathbf{D}_G(X) \to \mathbf{D}_{\mathcal{A}}(Y)$. In particular,

$$\chi_G(E_1, E_2) = \sum_i (-1)^i \dim \operatorname{Hom}_{\mathcal{A}}(\varpi_*(E_1), \varpi_*(E_2)[i]),\ E_1, E_2 \in \operatorname{Coh}_G(X). \tag{4.10}$$

For a representation $\rho : G \to GL(V_\rho)$ of G, we define a G-linearization on $\mathcal{O}_X \otimes V_\rho$ in a usual way. Thus we define the action of G on $\varpi_*(\mathcal{O}_X \otimes V_\rho)$ as

$$g \cdot (f(x) \otimes v) := f(g^{-1}x) \otimes gv,\ g \in G, f(x) \in \varpi_*(\mathcal{O}_X), v \in V_\rho. \tag{4.11}$$

Then $\mathcal{O}_X \otimes \mathbb{C}[G]$ is a G-sheaf such that $\varpi_*(\mathcal{O}_X \otimes \mathbb{C}[G]) = \mathcal{A}$ and we have a decomposition

$$\mathcal{O}_X \otimes \mathbb{C}[G] = \bigoplus_i (\mathcal{O}_X \otimes V_{\rho_i})^{\oplus \dim \rho_i}, \tag{4.12}$$

where ρ_i are irreducible representations of G.

Definition 4.1.5. For a G-sheaf E and a representation $\rho : G \to GL(V_\rho)$, $E \otimes \rho$ denotes the G-sheaf $E \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes V_\rho)$.

Since $\varpi_*(\mathcal{O}_X \otimes \rho_i)$ are direct summands of $\mathcal{A}$, we get the following lemma.

Lemma 4.1.6. (1) $\mathcal{A}_i := \varpi_*(\mathcal{O}_X \otimes \rho_i)$ *are local projective objects of* $\operatorname{Coh}_{\mathcal{A}}(Y)$.
(2) $\bigoplus_i \varpi_*(\mathcal{O}_X \otimes \rho_i)^{\oplus r_i}$ *is a local projective generator of* $\operatorname{Coh}_{\mathcal{A}}(Y)$ *if and only if* $r_i > 0$ *for all* i.

For a local projective generator $\mathcal{B}$ of $\operatorname{Coh}_{\mathcal{A}}(Y)$, we set $\mathcal{A}' := \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$. Then we have an equivalence

$$\begin{array}{ccc} \operatorname{Coh}_{\mathcal{A}}(Y) & \to & \operatorname{Coh}_{\mathcal{A}'}(Y) \\ E & \mapsto & \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, E). \end{array} \tag{4.13}$$

4.2. **Stability for G-sheaves.** Let α be an element of $R(G) \otimes \mathbb{Q}$.

Definition 4.2.1. Let $\mathcal{O}_X(1)$ be the pull-back of an ample line bundle on Y. A coherent G-sheaf E is α-stable, if E is purely d-dimensional and

$$\frac{\chi_G(F(n) \otimes \alpha^\vee)}{a_d(F)} < \frac{\chi_G(E(n) \otimes \alpha^\vee)}{a_d(E)},\ n \gg 0 \tag{4.14}$$

for all proper subsheaf $F \neq 0$, where $a_d(\bullet)$ is the coefficient of n^d of the Hilbert polynomial $\chi_G(\bullet(n) \otimes \alpha^\vee)$. We also define the α-semi-stability as usual.

Remark 4.2.2. Assume that $\alpha = \sum_i r_i \rho_i$, $r_i > 0$. We set $\mathcal{B} := \bigoplus_i \mathcal{A}_i^{\oplus r_i}$ and $\mathcal{A}' := \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$. Under the equivalence

$$\begin{array}{ccc} \operatorname{Coh}_G(X) & \to & \operatorname{Coh}_{\mathcal{A}'}(Y) \\ E & \mapsto & \mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E)), \end{array} \tag{4.15}$$

$$\chi_G(E(n) \otimes \alpha^\vee) = \chi(\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E))(n)) \tag{4.16}$$

implies that α-twisted stability of E corresponds to the stability of $\mathcal{A}'$-module $\mathcal{H}om_{\mathcal{A}}(\mathcal{B}, \varpi_*(E))$.

For a coherent G-sheaf E of dimension 0, we also have a refined notion of stability, which also comes from the stability of 0-dimensional objects in $\operatorname{Coh}_{\mathcal{A}}(Y)$.

Definition 4.2.3. Let ρ_{reg} be the regular representation of G. A coherent G-sheaf E of dimension 0 is $(\rho_{\text{reg}}, \alpha)$-stable, if

$$\frac{\chi_G(F \otimes \alpha^\vee)}{\chi_G(F \otimes \rho_{\text{reg}}^\vee)} < \frac{\chi_G(E \otimes \alpha^\vee)}{\chi_G(E \otimes \rho_{\text{reg}}^\vee)} \tag{4.17}$$

for a proper subsheaf $F \neq 0$.

By [S, Thm. 4.7] and [Y7, Prop. 1.6.1], we get the following theorem.

Theorem 4.2.4. *We take $v \in K_G(X)_{\mathrm{top}}$.*

(1) *Assume that $n\alpha$ contains every irreducible representation for a sufficiently large n. Then there is a coarse moduli space $\overline{M}_{\mathcal{O}_X(1)}^{\alpha}(v)$ of α-semi-stable G-sheaves E with $v(E) = v$. $\overline{M}_{\mathcal{O}_X(1)}^{\alpha}(v)$ is a projective scheme. We denote the open subscheme consisting of α-stable G-sheaves by $M_{\mathcal{O}_X(1)}^{\alpha}(v)$.*

(2) *Assume that v is a 0-dimensional vector. Then there is a coarse moduli space $\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v)$ of $(\rho_{\mathrm{reg}},\alpha)$-semi-stable G-sheaves E with $v(E) = v$. $\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v)$ is a projective scheme. We denote the open subscheme consisting of $(\rho_{\mathrm{reg}},\alpha)$-stable G-sheaves by $M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v)$.*

(3) *If $K_X \cong \mathcal{O}_X$ in $\mathrm{Coh}_G(X)$, then $M_{\mathcal{O}_X(1)}^{\alpha}(v)$ and $M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v)$ are smooth of dimension $-\chi_G(v,v)+2$ with holomorphic symplectic structures.*

Remark 4.2.5. There is another construction due to Inaba [In].

4.3. **Fourier-Mukai transforms for G-sheaves.** For a smooth point y of Y, $H^0(X, \mathcal{O}_{\varpi^{-1}(y)}) \cong \rho_{\mathrm{reg}}$ and $\mathcal{O}_{\varpi^{-1}(y)}$ is an irreducible object of $\mathrm{Coh}_G(X)$. Let v_0 be the topological invariant of $\mathcal{O}_{\varpi^{-1}(y)}$.

Lemma 4.3.1. *A 0-dimensional G-sheaf E is $(\rho_{\mathrm{reg}},0)$-twisted stable if and only if E is an irreducible object of $\mathrm{Coh}_G(X)$.*

Proof. Let E be a G-sheaf of dimension 0. Then $\chi_G(E \otimes \rho_{\mathrm{reg}}^{\vee}) / \chi_G(E \otimes \rho_{\mathrm{reg}}^{\vee}) = 1$. Hence the claim holds. □

Definition 4.3.2. Let G-Hilb_X^{ρ} be the G-Hilbert scheme parametrizing 0-dimensional subschemes Z of X such that $H^0(X, \mathcal{O}_Z) \cong V_{\rho}$.

Let $\rho_0, \rho_1, \dots, \rho_n$ be the irreducible representations of G. Assume that ρ_0 is trivial. We take an α such that $(\alpha, \rho_{\mathrm{reg}}) = 0$ and $(\alpha, \rho_i) < 0$ for $i > 0$.

Lemma 4.3.3. $M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0) = G\text{-}\mathrm{Hilb}_X^{\rho_{\mathrm{reg}}}$. *In particular,* $M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0) \neq \emptyset$.

Proof. Let E be a G-sheaf with $v(E) = v_0$. Since $\chi_G(\mathcal{O}_X \otimes \rho_0, E) = 1$, we have a homomorphism $\phi : \mathcal{O}_X \otimes \rho_0 \to E$. Then $H^0(\operatorname{im}\phi)$ contains a trivial representation, which implies that $\chi_G(\mathcal{O}_X \otimes \rho_0, \operatorname{im}\phi) \geq 1$. We note that E belongs to $M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)$ if and only if E does not contain a proper subsheaf F with $\chi_G(\mathcal{O}_X \otimes \rho_0, F) \geq 1$. Hence if $E \in M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)$, then $\operatorname{im}\phi = E$, which implies that $E \in G\text{-}\mathrm{Hilb}_X^{\rho_{\mathrm{reg}}}$. Conversely, if $E \in G\text{-}\mathrm{Hilb}_X^{\rho_{\mathrm{reg}}}$, then for a subsheaf F with $\chi_G(\mathcal{O}_X \otimes \rho_0, F) \geq 1$, $\mathrm{Hom}_G(\mathcal{O}_X \otimes \rho_0, F) \to \mathrm{Hom}_G(\mathcal{O}_X \otimes \rho_0, E)$ is isomorphic. Hence ϕ factors through F. Since E is generated by the image of ϕ, $F = E$. Thus E is stable. □

We set $X' := M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)$ and let $\mathcal{E} = \mathcal{O}_{\mathcal{Z}}$ be the universal family on $X' \times X$. Let $\phi : X' \to M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},0}(v_0)$ be the natural map.

Lemma 4.3.4. *Let E, F be G-sheaves of dimension 0.*

(1) *Assume that E is simple and is S-equivalent to $\oplus_i E_i$ with respect to $(\rho_{\mathrm{reg}},0)$-semi-stability. Then there is a point $y \in Y$ such that $\mathrm{Supp}(E_i) = \{y\}$ for all i.*

(2) $\chi_G(\mathcal{E}_{|\{x'\}\times X}, E) = 0$, $x' \in X'$.

(3) $E \otimes K_X \cong E$ *in* $\mathrm{Coh}_G(X)$. *In particular,* $\mathrm{Ext}_G^i(E,F) \cong \mathrm{Ext}_G^{2-i}(F,E)^{\vee}$.

(4) *If E, F are $(\rho_{\mathrm{reg}},0)$-twisted stable and $E \not\cong F$, then $\chi_G(E,F) \leq 0$. Moreover $\chi_G(E,F) = 0$ implies $\mathrm{Ext}_G^1(E,F) = 0$.*

Proof. (1) Assume that $\cup_i \mathrm{Supp}(\varpi_*(E_i)) = \{y_1, ..., y_t\}$. Then $\mathrm{Supp}(\varpi_*(E)) = \{y_1, ..., y_t\}$ and we have a decomposition $E \cong \oplus_{k=1}^t F_k$, where F_k are G-sheaves with $\mathrm{Supp}(\varpi_*(F_k)) = \{y_k\}$. If $t > 1$, then E is not simple. Therefore $t = 1$ and the claim holds.

(2) Since $\chi_G(\mathcal{E}_{|\{x'\}\times X}, E)$ is independent of the choice of x', we may assume that $\mathrm{Supp}(\varpi_*(E)) \cap \mathrm{Supp}(\varpi(\mathcal{E}_{|\{x'\}\times X})) = \emptyset$. Then we have $\mathrm{Hom}_G(\mathcal{E}_{|\{x'\}\times X}, E[k]) = 0$ for all k. Therefore the claim holds.

(3) Since K_X is the pull-back of a line bundle on Y and $\mathrm{Supp}(\varpi_*(E))$ is a finite set, we get $E \otimes K_X \cong E$ (cf. Lemma 4.3.10). By the Serre duality, we have $\mathrm{Ext}_G^i(E,F) \cong \mathrm{Ext}_G^{2-i}(F,E)^{\vee}$.

(4) By (3), $\mathrm{Ext}_G^2(E,F) \cong \mathrm{Hom}_G(F,E)^{\vee}$. If $\mathrm{Hom}_G(E,F) \neq 0$ or $\mathrm{Hom}_G(F,E) \neq 0$, then we see that $E \cong F$. Hence $\mathrm{Hom}_G(E,F) = \mathrm{Ext}_G^2(E,F) = 0$, which implies that $\chi_G(E,F) = -\dim \mathrm{Ext}_G^1(E,F) \leq 0$. □

Remark 4.3.5. For $\mathcal{E}_{|\{x'\}\times X}$, let $y \in Y$ be the support of $\varpi_*(\mathcal{E}_{|\{x'\}\times X})$. Then y depends only on $\phi(x')$.

Corollary 4.3.6. *Let E be a G-sheaf of dimension 0. Then the pairing*

$$\mathrm{Ext}_G^1(E,E) \times \mathrm{Ext}_G^1(E,E) \to \mathrm{Ext}_G^2(E,E) \cong \mathrm{Ext}_G^2(E, E \otimes K_X) \to H^2(X, K_X)$$

is non-degenerate. In particular, $\dim \mathrm{Ext}_G^1(E,E)$ *is even.*

Proof. By (3) and the Serre duality, we get the claim. □

We consider the Fourier-Mukai transform:

$$\begin{array}{rccc} \Phi: & \mathbf{D}_G(X) & \to & \mathbf{D}(X') \\ & E & \mapsto & \mathbf{R}p_{X'*}(\mathcal{E}\otimes p_X^*(E))^G. \end{array} \tag{4.18}$$

Then

$$\begin{array}{rccc} \widehat{\Phi}: & \mathbf{D}(X') & \to & \mathbf{D}_G(X) \\ & F & \mapsto & \mathbf{R}p_{X*}(\mathcal{E}^\vee[2]\otimes p_{X'}^*(F))\otimes K_{X'} \end{array} \tag{4.19}$$

is the quasi-inverse of Φ (cf. [Br2]). In particular, Φ induces an isomorphism $K_G(X)\to K(X')$ such that

$$\chi_G(E,F)=\chi(\Phi(E),\Phi(F)). \tag{4.20}$$

We note that $\Phi(v_0^\vee)=\varrho_{X'}$. Since $\chi_G(\ ,\ )$ is symmetric on $\varrho_{X'}^\perp$ and $\varrho_{X'}^\perp/\mathbb{Z}\varrho_{X'}$ is isometric to $(\mathrm{NS}(X'),-(\ ,\ ))$, $\chi_G(\ ,\ )$ is symmetric on $v_0^{\vee\perp}$ and the signature of $v_0^{\vee\perp}/\mathbb{Z}v_0^\vee$ is $(\dim K_G(X)-3,1)$, Let $C_1,C_2\in|\mathcal{O}_Y(n)|$, $n\gg 0$ be two smooth connected curves on the smooth locus of Y^{sm}. We set $L:=\varpi^*(\mathcal{O}_{C_1})\in v_0^\perp$. Then $\chi_G(L,L)=\chi(\mathcal{O}_{C_1},\mathcal{O}_{C_2})=-(C_1,C_2)<0$. Thus $L^\perp\cap v_0^{\vee\perp}/\mathbb{Q}v_0^\vee$ is negative definite. Therefore we get the following.

Lemma 4.3.7. (1) *$\chi_G(\ ,\ )$ is symmetric on $v_0^{\vee\perp}$ and $L^\perp\cap v_0^{\vee\perp}/\mathbb{Q}v_0^\vee$ is negative definite.*

(2) *Let E be a G-sheaf of dimension 0. Then $E\in L^\perp\cap v_0^{\vee\perp}$.*

Proof. (2) We find $C_1\in|\mathcal{O}_Y(n)|$ and $x'\in X'$ such that $\mathrm{Supp}(\mathcal{E}_{|\{x'\}\times X})\cap\mathrm{Supp}(E)=\emptyset$ and $\varpi(\mathrm{Supp}(E))\cap C_1=\emptyset$. Hence the claim holds. □

Let Y' be the normalization of the image of $\phi: M_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)\to\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},0}(v_0)$. Then we have a morphism $\pi: X'\to Y'$.

Proposition 4.3.8. (1) *$Y'\to\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},0}(v_0)$ is a bijective morphism.*

(2) *Let $\{p_1,p_2,\dots,p_l\}$ be the set of singular points of Y'. Then each p_i corresponds to S-equivalence classes of properly $(\rho_{\mathrm{reg}},0)$-twisted semi-stable G-sheaves. Let $\bigoplus_{j=0}^{s_i}E_{ij}^{\oplus a_{ij}}$ be the S-equivalence class corresponding to p_i. Then the matrix $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 0}$ is of affine type $\tilde{A},\tilde{D},\tilde{E}$.*

(3) *We can assume that $a_{i0}=1$ for all i. Then p_i is a rational double point of type A,D,E according as the type of the matrix $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 1}$.*

(4) *We assume that $a_{i0}=1$ for all i. For $j\neq 0$,*

$$C_{ij}:=\{x'\in X'\,|\,\mathrm{Hom}_G(E_{ij},\mathcal{E}_{|\{x'\}\times X})\neq 0\} \tag{4.21}$$

is a smooth rational curve and $\pi^{-1}(p_i)=\sum_{j>0}a_{ij}C_{ij}$.

Proof. We first note that $\chi_G(\mathcal{O}_X\otimes\rho,\bullet): K_G(X)\to\mathbb{Z}$ satisfies $\chi_G(\mathcal{O}_X\otimes\rho_{\mathrm{reg}},E)>0$ and $\chi_G(\mathcal{O}_X\otimes\rho_0,E)\geq 0$ for all 0-dimensional G-sheaves E.

Assume that $E\in\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)$ is S-equivalent to $\bigoplus_{j=0}^{s_i}E_{ij}^{\oplus a_{ij}}$ with respect to $(\rho_{\mathrm{reg}},0)$-twisted semi-stability. By Lemma 4.3.4 (4), $\chi_G(E_{ij},E_{ik})\leq 0$ if $j\neq k$. By Lemma 4.3.7, $\chi_G(E_{ij},E_{ij})>0$. Then the simpleness of E_{ij} and Corollary 4.3.6 imply $\chi_G(E_{ij},E_{ij})=2$. By [Y7, Lem. 3.1.1], $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 0}$ is of affine type $\tilde{A},\tilde{D},\tilde{E}$.

Since $H^0(X,\mathcal{O}_{\mathcal{Z}_{x'}})\cong\mathbb{C}[G]$, $x'\in X'$, we have

$$\sum_j a_{ij}\,\chi_G(\mathcal{O}_X\otimes\rho_0,E_{ij})=\chi_G(\mathcal{O}_X\otimes\rho_0,\bigoplus_j E_{ij}^{\oplus a_{ij}})=1. \tag{4.22}$$

Hence we may assume that $a_{i0}=1$, $H^0(X,E_{i0})\cong\rho_0$ and $H^0(X,E_{ij})$ does not contain a trivial representation, if $j\neq 0$. In particular, $\chi_G(E_{ij}\otimes\alpha^\vee)<0$ for $j>0$. Then the proof is similar to the proof of [Y7, Thm. 2.2.19] and [Y7, Lem. 2.2.22]. □

Remark 4.3.9. We can also show the claim (2) without using Φ. Assume that $E\in\overline{M}_{\mathcal{O}_X(1)}^{\rho_{\mathrm{reg}},\alpha}(v_0)$ is S-equivalent to $\bigoplus_{j=0}^{s_i}E_{ij}^{\oplus a_{ij}}$ with respect to $(\rho_{\mathrm{reg}},0)$-twisted semi-stability. By Lemma 4.3.4 (4), $\chi_G(E_{ij},E_{ik})\leq 0$ if $j\neq k$. For any j, we shall find $k\neq j$ such that $\chi_G(E_{ij},E_{ik})<0$. Assume that there is a decomposition $\{0,1,...,s_i\}=I_1\coprod I_2$ such that $\chi(E_{ij},E_{ik})=0$ for all $(j,k)\in I_1\times I_2$. By Lemma 4.3.4 (4), we have $\mathrm{Ext}^1(E_{ij},E_{ik})=0$ for all $(j,k)\in I_1\times I_2$. Then we see that $E\cong F_1\oplus F_2$, where F_1 is S-equivalent to $\bigoplus_{j\in I_1}E_{ij}^{\oplus a_{ij}}$ and F_2 is S-equivalent to $\bigoplus_{j\in I_2}E_{ij}^{\oplus a_{ij}}$. Since E is generated by $H^0(E)^G$ and $\dim H^0(E)^G=1$, we get a contradiction. Therefore there is $k\neq j$ with $\chi_G(E_{ij},E_{ik})<0$. By using Lemma 4.3.4 (2), we see that $\chi_G(E_{ij},E_{ij})>0$. Then the simpleness of E_{ij} and Corollary 4.3.6 imply $\chi_G(E_{ij},E_{ij})=2$. By [Y7, Rem. 3.1.2], $(\chi_G(E_{ij},E_{ij'}))_{j,j'\geq 0}$ is of affine type $\tilde{A},\tilde{D},\tilde{E}$.

Lemma 4.3.10. *For a point $x'\in X'$, $K_{X'}$ is trivial in a neighborhood of $\phi^{-1}(\phi(x'))$.*

Proof. We take a smooth section $C_1 \in |\pi_*(K_X(n))|$ with $y \notin C_1$. We also take a smooth section $C_2 \in |\mathcal{O}_Y(n)|$. Then $D_i := p_{X'}(\mathcal{Z} \cap (X' \times \varpi^{-1}(C_i)))$ are closed subset of X' such that $D_i \cap \phi^{-1}(\phi(x')) = \emptyset$ for $i = 1, 2$. We set $U := X' \setminus (D_1 \cup D_2)$. Then C_1 and C_2 define G-linearized homomorphisms $\mathcal{E} \otimes \mathcal{O}_X(-n) \to \mathcal{E} \otimes K_X$ and $\mathcal{E} \otimes \mathcal{O}_X(-n) \to \mathcal{E}$. By our choice of U, they are isomorphic on $U \times X$. We set $\mathcal{E}_U := \mathcal{E}_{|U \times X}$. Then we have

$$\operatorname{Ext}^2_{p_U}(\mathcal{E}_U, \mathcal{E}_U)^G \cong \operatorname{Ext}^2_{p_U}(\mathcal{E}_U, \mathcal{E}_U \otimes K_X)^G \cong (\operatorname{Hom}_{p_U}(\mathcal{E}_U, \mathcal{E}_U)^G)^\vee \cong \mathcal{O}_U. \tag{4.23}$$

Since $\operatorname{Ext}^2_{p_U}(\mathcal{E}_U, \mathcal{E}_U)^G \cong K_U^\vee$, the claim holds. □

We note that $p_{X'*}(\mathcal{O}_{\mathcal{Z}})$ is a locally free sheaf on X' with a G-action. We have a decomposition of $p_{X'*}(\mathcal{O}_{\mathcal{Z}})$ as G-sheaves:

$$p_{X'*}(\mathcal{O}_{\mathcal{Z}}) = \oplus_i \Phi(\mathcal{O}_X \otimes \rho_i) \otimes \rho_i^\vee. \tag{4.24}$$

For a G-sheaf E of dimension 0, $E^\vee = \mathcal{E}xt^2_{\mathcal{O}_X}(E, \mathcal{O}_X)[-2]$. Hence E is an irreducible object if and only if $E^\vee[2]$ is an irreducible object.

Lemma 4.3.11. *We set $F_{ij} := E_{ij}^\vee[2] \in \operatorname{Coh}_G(X)$.*

(1)

$$\Phi(F_{ij}) = \begin{cases} \mathcal{O}_{C_{ij}}(-1)[1], \ j > 0, \\ \mathcal{O}_{Z_i}, \ j = 0, \end{cases} \tag{4.25}$$

where $Z_i := \sum_j a_{ij} C_{ij}$ is the fundamental cycle of p_i.

(2) *$\Phi(\mathcal{O}_X \otimes \rho_i)$ is a locally free sheaf of rank $\dim \rho_i$ on X'. In particular, $\Phi(\mathcal{O}_X \otimes \rho_0) = \mathcal{O}_{X'}$.*

(3) *$\Phi(\mathcal{O}_X \otimes \rho_i)$ is a full sheaf ([E]).*

Proof. We consider the homomorphism $\psi : p_{X'*}(\mathcal{O}_{X' \times X}) \to p_{X'*}(\mathcal{O}_{\mathcal{Z}})$. For any point $x' \in X'$, $\psi_{x'} : H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{\mathcal{Z}_{x'}})$ is injective. Since $\operatorname{im} \psi \subset p_{X'*}(\mathcal{O}_{\mathcal{Z}})^G$, ψ is an isomorphism. Thus $\Phi(\mathcal{O}_X \otimes \rho_0) = \mathcal{O}_{X'}$. (2) is a consequence of (4.24). Then the proof of (1) is similar to the Fourier-Mukai transform on a $K3$ surface: We first show that $\Phi(F_{ij}) = \mathcal{O}_{C_{ij}}(b_{ij})[1]$ $(j > 0)$ for some $b_{ij} \in \mathbb{Z}$. Since $0 = \chi_G(\mathcal{O}_X \otimes \rho_0, F_{ij}) = \chi(\Phi(\mathcal{O}_X \otimes \rho_0), \Phi(F_{ij})) = -(b_{ij} + 1)$, we get $\Phi(F_{ij}) = \mathcal{O}_{C_{ij}}(-1)[1]$ for $j > 0$. Then we also get $\Phi(F_{i0}) = \mathcal{O}_{Z_i}$.

(3) We note that

$$\begin{aligned} \operatorname{Hom}(\Phi(\mathcal{O}_X \otimes \rho_i), \mathcal{O}_{C_{jk}}(-1)) &= \operatorname{Hom}(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{jk})[-1]) \\ &= \operatorname{Hom}_G(\mathcal{O}_X \otimes \rho_i, F_{jk}[-1]) = 0, \\ \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \mathcal{O}_{Z_j}) &= \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{j0})) \\ &= \operatorname{Ext}^1_G(\mathcal{O}_X \otimes \rho_i, F_{j0}) = 0. \end{aligned} \tag{4.26}$$

Hence $\Phi(\mathcal{O}_X \otimes \rho_i)$ is a full sheaf. □

We have

$$\Phi(\mathcal{O}_X \otimes \rho_i)_{|C_{jk}} \cong \mathcal{O}_{C_{jk}}^{\oplus(\dim \rho_i - k_{ijk})} \oplus \mathcal{O}_{C_{jk}}(1)^{\oplus k_{ijk}}, \tag{4.27}$$

where

$$\begin{aligned} k_{ijk} :=& (c_1(\Phi(\mathcal{O}_X \otimes \rho_i)), C_{jk}) \\ =& \dim \operatorname{Ext}^1(\Phi(\mathcal{O}_X \otimes \rho_i), \Phi(F_{jk})) \\ =& \dim \operatorname{Hom}_G(\mathcal{O}_X \otimes \rho_i, F_{jk}). \end{aligned} \tag{4.28}$$

Proposition 4.3.12. *Φ induces an equivalence*

$$\operatorname{Coh}_G(X) \to {}^{-1}\operatorname{Per}(X'/Y'). \tag{4.29}$$

Proof. It is sufficient to prove $\Phi(E) \in {}^{-1}\operatorname{Per}(X'/Y')$ for $E \in \operatorname{Coh}_G(X)$. We first prove that $H^i(\Phi(E)) = 0$ for $i \neq -1, 0$. Let E be a G-sheaf on X. Then there is an equivariant locally free resolution of E:

$$0 \to V_{-2} \to V_{-1} \to V_0 \to E \to 0. \tag{4.30}$$

Since $\Phi(V_i)$ are locally free sheaves on X' and

$$0 \to \Phi(V_{-2}) \to \Phi(V_{-1}) \to \Phi(V_0) \tag{4.31}$$

is exact on $X' \setminus \cup_i Z_i$, we get $H^i(\Phi(E)) = 0$ for $i \neq -1, 0$ and $\operatorname{Supp}(H^{-1}(\Phi(E))) \subset \cup_i Z_i$. Then we have

$$\begin{aligned} \operatorname{Hom}(H^0(\Phi(E)), \mathcal{O}_{C_{ij}}(-1)) &= \operatorname{Hom}(\Phi(E), \Phi(F_{ij})[-1]) \\ &= \operatorname{Hom}_G(E, F_{ij}[-1]) = 0, \ j > 0, \\ \operatorname{Hom}(\mathcal{O}_{Z_i}, H^{-1}(\Phi(E))) &= \operatorname{Hom}(\Phi(F_{i0}), \Phi(E)[-1]) \\ &= \operatorname{Hom}_G(F_{i0}, E[-1]) = 0. \end{aligned} \tag{4.32}$$

Hence $\Phi(E) \in {}^{-1}\operatorname{Per}(X'/Y')$. □

Remark 4.3.13. By the proof of Proposition 4.3.12, $H^{-1}(\Phi(E)) = 0$ if E does not contain a non-zero 0-dimensional sub G-sheaf.

Proposition 4.3.14. *For $\alpha = \sum_i r_i \rho_i$, $r_i > 0$, we set $P := \bigoplus_i \Phi(\mathcal{O}_X \otimes \rho_i)^{\oplus r_i}$.*
(1) *P is a local projective generator of ${}^{-1}\operatorname{Per}(X'/Y')$.*
(2) *A G-sheaf E is α-twisted stable if and only if $\Phi(E)$ is P-twisted stable.*

Proof. Since

$$\chi(P, \Phi(F_{jk})) = \sum_i r_i \chi_G(\mathcal{O}_X \otimes \rho_i, F_{jk}) = \sum_i r_i (\rho_i, H^0(X, F_{jk})) > 0 \tag{4.33}$$

for all j, k, (1) holds by Lemma 4.3.11 (3) and Proposition 1.1.5 (1). (2) is obvious. □

Example 4.3.15. Let X be an abelian surface. Then $G = \mathbb{Z}_2$ acts on X as the multiplication by (-1). Then the moduli of stable G-sheaves on X is isomorphic to the moduli space of stable objects of ${}^{-1}\operatorname{Per}(\operatorname{Km}(X)/Y)$, where $Y = X/G$ and $\operatorname{Km}(X) \to Y$ is the Kummer surface associated to X. By [Y7, sect. 2.5], it is a deformation of the moduli space of usual Gieseker semi-stable sheaves on a $K3$ surface.

Lemma 4.3.16. *$\overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0) \cong Y' \cong X/G$. In particular, $\overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0)$ is a normal surface with rational double points.*

Proof. We shall first show that $\overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0) \cong Y'$. By Proposition 4.3.14, $\overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0)$ is isomorphic to the moduli of 0-dimensional objects E of ${}^{-1}\operatorname{Per}(X'/Y')$ with $v(E) = v(\mathbb{C}_x)$. By [Y7, Lem. 2.2.12], we have the claim.

Let $\Delta \subset X \times X$ be the diagonal. Then $\mathcal{G} := \oplus_{g \in G} \mathcal{O}_{(1 \times g)^*(\Delta)}$ is a G-equivariant coherent sheaf on $X \times X$ which is flat over X. Since $v(\mathcal{G}_{|\{x\} \times X}) = v_0$, we have a morphism $\eta : X \to \overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0)$. We note that $\mathcal{G}_{|\{x\} \times X} \cong \mathcal{G}_{|\{g(x)\} \times X}$ for all $g \in G$ and $\mathcal{G}_{|\{x\} \times X} \cong \mathcal{G}_{|\{y\} \times X}$ if and only if $y \in Gx$. Hence η is G-invariant and we get an injective morphism $X/G \to \overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0)$. It is easy to see that $X/G \to \overline{M}_{\mathcal{O}_X(1)}^{v_0}(v_0)$ is an isomorphism. □

Corollary 4.3.17. *We set $P := \Phi(\mathcal{O}_X \otimes \mathbb{C}[G])$ and $\mathcal{A}' := \pi_*(P^\vee \otimes P)$. Under the isomorphism $Y' \cong Y$, we have an isomorphism $\pi_*(P) \cong \varpi_*(\mathcal{O}_X)$. Hence we have an isomorphism $\mathcal{A} \cong \mathcal{A}'$ as $\mathcal{O}_{Y'}$-algebras and we have the following commutative diagram.*

$$\begin{array}{ccc} \operatorname{Coh}_G(X) & \xrightarrow{\Phi} & {}^{-1}\operatorname{Per}(X'/Y') \\ {\scriptstyle \varpi_*}\downarrow & & \downarrow{\scriptstyle \mathbf{R}\pi_*(P^\vee \otimes (\ ))} \\ \operatorname{Coh}_{\mathcal{A}}(Y) & = & \operatorname{Coh}_{\mathcal{A}'}(Y) \end{array} \tag{4.34}$$

Proof. We set $R := \mathcal{O}_X \otimes \mathbb{C}[G]$. Since $\Phi(\mathcal{O}_X \otimes \mathbb{C}[G]) \cong \bigoplus_i \Phi(\mathcal{O}_X \otimes \rho_i)^{\oplus \dim \rho_i} \cong p_{X'*}(\mathcal{O}_{\mathcal{Z}})$, $\pi_*(P) \cong \pi_*(p_{X'*}(\mathcal{O}_{\mathcal{Z}}))$ is a reflexive sheaf. Since $\pi_*(p_{X'*}(\mathcal{O}_{\mathcal{Z}})) = \varpi_*(\mathcal{O}_X)$ on the smooth locus, we get an isomorphism $\pi_*(P) \cong \varpi_*(\mathcal{O}_X)$. Since $\mathcal{A}'$ is a reflexive sheaf on Y', we have $\mathcal{A}' \cong \operatorname{End}_{\mathcal{O}_{Y'}}(\pi_*(P))$. Therefore $\mathcal{A}' \cong \operatorname{End}_{\mathcal{O}_{Y'}}(\pi_*(P)) \cong \operatorname{End}_{\mathcal{O}_Y}(\varpi_*(\mathcal{O}_X)) \cong \mathcal{A}$.

Since $\varpi_*(R) = \mathcal{A}$ and every G-sheaf E has a locally free resolution

$$\cdots \to R(-n_{-2})^{\oplus N_{-2}} \to R(-n_{-1})^{\oplus N_{-1}} \to R(-n_0)^{\oplus N_0} \to E \to 0, \tag{4.35}$$

we get the commutative diagram. □

Since Φ induces an equivalence $\operatorname{Coh}_G(X) \to {}^{-1}\operatorname{Per}(X'/Y')$ (Proposition 4.3.14), for $F \in \operatorname{Coh}_G(X)$ such that $\Phi(F) \in {}^{-1}\operatorname{Per}(X'/Y')$ is a local projective generator, we can define F-twisted semi-stability, by replacing α by F in Definition 4.2.1. Obviously $F = \mathcal{O}_X \otimes \alpha$ coincides with the α-semi-stability in Definition 4.2.1. Then Theorem 4.2.4 is exytened for this semi-stability. For a topological invariant $v_0 \in K_G(X)$ such that v is primitive and $\chi_G(v_0, v_0) \le 2$, $M_{\mathcal{O}_X(1)}^F(v_0)$ denotes the moduli space of F-twisted stable G-sheaves E with the topological invariant v_0. Assume that X' is a $K3$ surface. Then for a general F, $M_{\mathcal{O}_X(1)}^F(v_0) \cong M_{\mathcal{O}_{X'}(1)}^{\Phi(F)}(\Phi(v_0))$ is smooth, projective and non-empty. In particular $M_{\mathcal{O}_X(1)}^F(v_0)$ is a $K3$ surface, if $\chi_G(v_0, v_0) = 0$. We set $X'' := M_{\mathcal{O}_{X'}(1)}^{\Phi(F)}(\Phi(v_0))$. If $\Phi(v_0) = (r, \xi, a)$ satisfies $0 < (\xi, C_{ij})$ and $(\xi, \sum_j a_{ij} C_{ij}) < r$ for all i, j and $\Phi(F) \in K(X') \otimes \mathbb{Q}$ is sufficiently close to v_0, then X' is a $K3$ surface. Assume that there is a universal family $\mathcal{F}$ on $X' \times X''$. Then $\mathcal{E}' := \widehat{\Phi}(\mathcal{F})$ is a flat family of stable G-sheaves and defines an equivalence $\Phi' : \mathbf{D}^G(X) \to \mathbf{D}(X'')$ such that $\Phi' = \Phi_{X' \to X''}^{\mathcal{E}'} \circ \Phi$. Thus there are many moduli spaces X'' of stable G-sheaves such that X'' are $K3$ surfaces and induce equivariant Fourier-Mukai transforms.

4.4. **Irreducible objects of** $\mathrm{Coh}_G(X)$**.** By Proposition 4.3.12, we will be able to study irreducible objects of $\mathrm{Coh}_G(X)$. In this subsection, we shall describe irreducible objects of $\mathrm{Coh}_G(X)$ by a more direct way. Let E be a G-sheaf of dimension 0. We may assume that $\mathrm{Supp}(E) = Gx$. Let H be the stabilizer of x and E_x the submodule of E whose support is x. Then E_x is a H-sheaf. We have a decomposition $H^0(X,E) = \oplus_{y\in Gx} H^0(X,E_y)$. Since $gH^0(X,E_x) = H^0(X,E_{gx})$, we have an isomorphism

$$H^0(X,E) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} H^0(X,E_x) \tag{4.36}$$

as G-modules. Then we have an equality of invariant subspaces:

$$H^0(X,E)^G = H^0(X,E_x)^H. \tag{4.37}$$

We shall prove

Lemma 4.4.1. *There is a bijection between*
(a) $\mathfrak{G} := \{E \in \mathrm{Coh}_G(X) | \mathrm{Supp}(E) = Gx,\ \mathrm{Stab}(x) = H\}$ *and*
(b) $\mathfrak{H} := \{F \in \mathrm{Coh}_H(X) | \mathrm{Supp}(F) = x\}$.

Proof. We define $r : \mathfrak{G} \to \mathfrak{H}$ by sending $E \in \mathfrak{G}$ to $E_x \in \mathfrak{H}$. For $F \in \mathfrak{H}$, we set $K := \ker(H^0(X,F)\otimes\mathcal{O}_X \to F)$. Then

$$s(F) := (\mathbb{C}[G] \otimes_{\mathbb{C}[H]} H^0(X,F)) \otimes \mathcal{O}_X / \sum_{g\in G} g(K) \tag{4.38}$$

is a G-sheaf such that $s(F)_x = F$. Hence we have a map $s : \mathfrak{H} \to \mathfrak{G}$ with $r \circ s = \mathrm{id}_{\mathfrak{H}}$. For $E \in \mathfrak{G}$, we also see that $s(E_x) \cong E$, and hence $s \circ r = \mathrm{id}_{\mathfrak{G}}$. Therefore our claim holds. □

If $H^0(X,F)$ is the regular representation of H, i.e., $H^0(X,F) \cong \mathbb{C}[H]$, then $H^0(X,E)$ is the regular representation of G.

Lemma 4.4.2. *Let E be a G-sheaf of dimension 0. Then E is irreducible if and only if* $\mathrm{Supp}(E) = Gx$ *and* $E_x \cong H^0(X,E_x) \otimes \mathbb{C}_x$.

Proof. For a G-sheaf of dimension 0, we take a point $x \in \mathrm{Supp}(E)$. We set $H := \mathrm{Stab}(x)$. Then $E \otimes (\oplus_{g\in G/H} \mathcal{O}_{gx})$ is a quotient G-sheaf. If E is irreducible, then $\mathrm{Supp}(E) = Gx$ and $E_x \cong H^0(X,E_x) \otimes \mathbb{C}_x$. Moreover $H^0(X,E_x)$ is an irreducible representation of H by Lemma 4.4.1. Conversely if $\mathrm{Supp}(E) = Gx$ and $E_x \cong H^0(X,E_x)\otimes\mathbb{C}_x$, then for any irreducible quotient F, we have $\mathrm{Supp}(F) = Gx$ and $F_x \cong H^0(X,F_x)\otimes\mathbb{C}_x$. Then E is irreducible if and only if $H^0(X,E_x)$ is an irreducible representation of H. Therefore our claim holds. □

Lemma 4.4.3. *Let E_1 and E_2 be irreducible G-sheaves such that* $\mathrm{Supp}(E_1) = \mathrm{Supp}(E_2) = Gx$ *and* $(E_i)_x = \rho_i \otimes \mathbb{C}_x$. *Then*

$$\begin{aligned} \chi_G(E_1,E_2) &= \chi_{\mathrm{Stab}(x)}(\rho_1 \otimes \mathbb{C}_x, \rho_2 \otimes \mathbb{C}_x) \\ &= (2\rho_1 - \rho_1 \otimes \rho_{\mathrm{nat}}, \rho_2), \end{aligned} \tag{4.39}$$

where $\rho_{\mathrm{nat}} : \mathrm{Stab}(x) \to SL_2(\mathbb{C})$ *is the natural representation of* $\mathrm{Stab}(x)$ *on the tangent space T_X at x.*

Proof. We note that $\chi_{\mathrm{Stab}(x)}((\oplus_{g\in G/\mathrm{Stab}(x)} \rho_i \otimes \mathbb{C}_{gx})/\rho_i \otimes \mathbb{C}_x, \rho_j \otimes \mathbb{C}_x) = 0$. By using an equivariant locally free resolution of E_1 and (4.37), we see that

$$\begin{aligned} \chi_G(E_1,E_2) =& \chi_{\mathrm{Stab}(x)}(E_1,(E_2)_x) \\ =& \chi_{\mathrm{Stab}(x)}((E_1)_x,(E_2)_x). \end{aligned} \tag{4.40}$$

Since $\sum_{i=0}^2 (-1)^i \mathcal{E}xt^i_{\mathcal{O}_X}(\mathbb{C}_x,\mathbb{C}_x) = \mathbb{C}_x - (T_X)_x + \det(T_X)_x$, we have $\sum_{i=0}^2 (-1)^i \dim \mathrm{Ext}^i(\mathbb{C}_x,\mathbb{C}_x) = 2\rho_{\mathrm{triv}} - \rho_{\mathrm{nat}}$, where ρ_{triv} is the trivial representation of $\mathrm{Stab}(x)$. Hence

$$\chi_{\mathrm{Stab}(x)}(\rho_1 \otimes \mathbb{C}_x, \rho_2 \otimes \mathbb{C}_x) = (2\rho_1 - \rho_1 \otimes \rho_{\mathrm{nat}}, \rho_2). \tag{4.41}$$

□

Lemma 4.4.4. *Let H be the stabilizer of $x \in X$. Let $\rho_0^H, \rho_1^H, ..., \rho_t^H$ be the irreducible representations of H. Then the matrix $(\chi_H(\rho_i^H \otimes \mathbb{C}_x, \rho_j^H \otimes \mathbb{C}_x))_{i,j}$ is of affine type $\tilde{A}, \tilde{D}, \tilde{E}$. In particular, $\chi_H(\rho\otimes\mathbb{C}_x, \rho\otimes\mathbb{C}_x)) \geq 0$ and $\chi_H(\rho \otimes \mathbb{C}_x, \rho \otimes \mathbb{C}_x) = 0$ implies $\rho \in \mathbb{Z}\rho_{\mathrm{reg}}^H$, where ρ is a representation of H.*

Proof. Since $H\text{-}\mathrm{Hilb}_X^{\rho_{\mathrm{reg}}^H}$ is projective, $\oplus_{h\in H} \mathbb{C}_{hz}$, $z \in X\setminus\{x\}$ deforms to $E \in H\text{-}\mathrm{Hilb}_X^{\rho_{\mathrm{reg}}^H}$ with $\mathrm{Supp}(E) = \{x\}$. Then E is S-equivalent to $\bigoplus_j (\rho_j^H)^{\oplus \dim \rho_j^H} \otimes \mathbb{C}_x$. Hence the claims hold by Remark 4.3.9. □

Proposition 4.4.5. (1) *Let E be a G-sheaf of dimension 0. Then $\chi_G(E,E) \geq 0$ and the equality implies $H^0(X,E) = \mathbb{C}[G]^{\oplus m}$.*

(2) *Let $E=\bigoplus_i E_i^{\oplus a_i}$ be a G-sheaf of dimension 0 such that $H^0(X,E)=\mathbb{C}[G]$, where E_i are irreducible G-sheaves with $E_i \ne E_j$ $(i\ne j)$. Then the matrix $(\chi_G(E_i,E_j))_{i,j}$ is of affine type $\tilde{A},\tilde{D},\tilde{E}$.*

Proof. (1) We may assume that E is a direct sum of irreducible G-sheaves. We have a decomposition $E\cong \oplus_i F_i$ such that $\mathrm{Supp}(F_i)=Gx_i$ and $Gx_i\ne Gx_j$ for $i\ne j$. Then $\chi_G(E,E)=\sum_i \chi_G(F_i,F_i)$. Hence we may assume that $\mathrm{Supp}(E)=Gx$, $x\in X$. Let H be the stabilizer of x. Then $E_x=H^0(X,E_x)\otimes \mathbb{C}_x$, $H^0(X,E)=\mathbb{C}[G]\otimes_{\mathbb{C}[H]} H^0(X,E_x)$ and $\chi_G(E,E)=\chi_H(E_x,E_x)$ by Lemma 4.4.3. Then the claim follows from Lemma 4.4.4.

(2) Assume that we have a decomposition $E\cong F_1\oplus F_2$ with $\chi_G(F_1,F_2)=0$. Since $\chi_G(F_1,F_1)+\chi_G(F_2,F_2)=\chi_G(E,E)=0$, (1) implies that $F_i\cong \mathbb{C}[G]^{\oplus m_i}$, $m_i>0$ for $i=1,2$. Thus $E\cong \mathbb{C}[G]^{\oplus(m_1+m_2)}$, which is a contradiction. Then Remark 4.3.9 implies the claim. □

5. Appendix.

5.1. **Spectral sequences.** Since $\widehat{\Phi}[2]$ and $\widehat{\Psi}$ are the inverses of Φ and Ψ respectively, we get the followng.

Lemma 5.1.1. *We have spectral sequences*

$$E_2^{p,q}=\Phi^p(\widehat{\Phi}^q(E))\Rightarrow E_\infty^{p+q}=\begin{cases}E, & p+q=2,\\ 0, & p+q\ne 2,\end{cases}\quad E\in \mathrm{Per}(X'/Y'), \tag{5.1}$$

$$E_2^{p,q}=\widehat{\Phi}^p(\Phi^q(F))\Rightarrow E_\infty^{p+q}=\begin{cases}F, & p+q=2,\\ 0, & p+q\ne 2,\end{cases}\quad F\in \mathcal{C}. \tag{5.2}$$

In particular,

(i) $\Phi^p(\widehat{\Phi}^0E))=0$, $p=0,1$.
(ii) $\Phi^p(\widehat{\Phi}^2(E))=0$, $p=1,2$.
(iii) *There is an injective homomorphism* $\Phi^0(\widehat{\Phi}^1(E))\to\Phi^2(\widehat{\Phi}^0(E))$.
(iv) *There is a surjective homomorphism* $\Phi^0(\widehat{\Phi}^2(E))\to\Phi^2(\widehat{\Phi}^1(E))$.

For the claims (i) to (iv), we also use Lemma 2.5.2 (2) and Corollary 2.5.3.

Lemma 5.1.2. *We have spectral sequences*

$$E_2^{p,q}=\Psi^p(\widehat{\Psi}^{-q}(E))\Rightarrow E_\infty^{p+q}=\begin{cases}E, & p-q=0,\\ 0, & p-q\ne 0,\end{cases}\quad E\in \mathrm{Per}(X'/Y')^D, \tag{5.3}$$

$$E_2^{p,q}=\widehat{\Psi}^p(\Psi^{-q}(F))\Rightarrow E_\infty^{p+q}=\begin{cases}F, & p-q=0,\\ 0, & p-q\ne 0,\end{cases}\quad F\in \mathcal{C}. \tag{5.4}$$

In particular,

(i) $\Psi^p(\widehat{\Psi}^2(E))=0$, $p=0,1$.
(ii) $\Psi^p(\widehat{\Psi}^0(E))=0$, $p=1,2$.
(iii) *There is an injective homomorphism* $\Psi^0(\widehat{\Psi}^1(E))\to\Psi^2(\widehat{\Psi}^2(E))$.
(iv) *There is a surjective homomorphism* $\Psi^0(\widehat{\Psi}^0(E))\to\Psi^2(\widehat{\Psi}^1(E))$.

For a convenience of the reader, we give a proof of Lemma 5.1.2.

Proof. By the exact triangles

$$\Psi^{\le 1}(E)[-1]\to\Psi(E)\to\Psi^2(E)[-2]\to\Psi^{\le 1}(E) \tag{5.5}$$

and

$$\Psi^0(E)\to\Psi^{\le 1}(E)[-1]\to\Psi^1(E)[-1]\to\Psi^0(E)[1], \tag{5.6}$$

we have exact triangles

$$\widehat{\Psi}(\Psi^{\le 1}(E))[1]\leftarrow\widehat{\Psi}(\Psi(E))\leftarrow\widehat{\Psi}(\Psi^2(E))[2]\leftarrow\widehat{\Psi}(\Psi^{\le 1}(E)) \tag{5.7}$$

and

$$\widehat{\Psi}(\Psi^0(E))\leftarrow\widehat{\Psi}(\Psi^{\le 1}(E))[1]\leftarrow\widehat{\Psi}(\Psi^1(E))[1]\leftarrow\widehat{\Psi}(\Psi^0(E))[-1]. \tag{5.8}$$

Since $\widehat{\Psi}(\Psi(E)) = E$, we have exact sequences

$$\begin{aligned}
&0 \leftarrow \widehat{\Psi}^1(\Psi^{\leq 1}(E)) \leftarrow E \leftarrow \widehat{\Psi}^2(\Psi^2(E)) \leftarrow \widehat{\Psi}^0(\Psi^{\leq 1}(E)) \leftarrow 0,\\
&\widehat{\Psi}^2(\Psi^{\leq 1}(E)) = \widehat{\Psi}^1(\Psi^2(E)) = \widehat{\Psi}^0(\Psi^2(E)) = 0,\\
&0 \leftarrow \widehat{\Psi}^2(\Psi^1(E)) \leftarrow \widehat{\Psi}^0(\Psi^0(E)) \leftarrow \widehat{\Psi}^1(\Psi^{\leq 1}(E)) \leftarrow \widehat{\Psi}^1(\Psi^1(E)) \leftarrow 0,\\
&\widehat{\Psi}^0(\Psi^{\leq 1}(E)) \cong \widehat{\Psi}^0(\Psi^1(E)),\\
&\widehat{\Psi}^1(\Psi^0(E)) = \widehat{\Psi}^2(\Psi^0(E)) = 0.
\end{aligned} \tag{5.9}$$

These give the data of the spectral sequence. □

References

[Br1] Bridgeland, T., *Fourier-Mukai transforms for elliptic surfaces,* J. reine angew. Math. **498** (1998), 115–133

[Br2] Bridgeland, T., *Equivalences of triangulated categories and Fourier-Mukai transforms,* Bull. London Math. Soc. **31** (1999), 25–34, math.AG/9809114

[Br3] Bridgeland, T., *Flops and derived categories,* Invent. Math. **147** (2002), 613-632.

[Br4] Bridgeland, T., *Stability conditions on K3 surfaces,* math.AG/0307164, Duke Math. J. **141** (2008), 241–291

[E] Esnault, H., *Reflexive modules on quotient surface singularities,* J. Reine Angew. Math. **362** (1985), 63–71

[Hr] Hartmann, H., *Cusps of the Kähler moduli space and stability conditions on K3 surfaces,* arXiv:1012.3121

[H] Huybrechts, D., *Derived and abelian equivalence of K3 surfaces,* math.AG/0604150, J. Algebraic Geom. **17** (2008), 375-400

[In] Inaba, M., *Moduli of stable objects in a triangulated category,* arXiv:math/0612078,J. Math. Soc. Japan **62** (2010), 395–429 Math. Ann. **317** (2000), 239–262

[MYY] Minamide, H., Yanagida, S., Yoshioka, K., *Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions,* arXiv:1106.5217

[Mu1] Mukai, S., *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J., **81** (1981), 153–175

[Mu2] Mukai, S., *On the moduli space of bundles on $K3$ surfaces I,* Vector bundles on Algebraic Varieties, Oxford, 1987, 341–413

[MFK] Mumford, D., Fogarty, J., Kirwan, F., *Geometric invariant theory,* Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) 34. Springer-Verlag, Berlin, 1994.

[O] Orlov, D., *Equivalences of derived categories and $K3$ surfaces,* alg-geom/9606006, Algebraic geometry, 7. J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381.

[OY] Onishi, N., Yoshioka, K., *Singularities on the 2-dimensional moduli spaces of stable sheaves on $K3$ surfaces*, math.AG/0208241, Internat. J. Math. **14** (2003), 837–864

[ST] Seidel, P., Thomas, R. P., *Braid group actions on derived categories of coherent sheaves,* Duke Math. Jour. **108** (2001), 37–108

[Se] Seshadri, C. S., *Geometric reductivity over arbitrary base,* Advances in Math. **26** (1977), 225–274.

[S] Simpson, C., *Moduli of representations of the fundamental group of a smooth projective variety I,* Publ. Math. I.H.E.S. **79** (1994), 47–129

[VB] Van den Bergh, M., *Three-dimensional flops and noncommutative rings*, Duke Math. J. 122 (2004), no. 3, 423–455.

[Y1] Yoshioka, K., *Moduli spaces of stable sheaves on abelian surfaces,* Math. Ann. **321** (2001), 817–884, math.AG/0009001

[Y2] Yoshioka, K., *Twisted stability and Fourier-Mukai transform I,* Compositio Math. **138** (2003), 261–288,

[Y3] Yoshioka, K., *Twisted stability and Fourier-Mukai transform II,* Manuscripta Math. **110** (2003), 433–465

[Y4] Yoshioka, K., *Moduli of twisted sheaves on a projective variety,* math.AG/0411538, Adv. Stud. Pure Math. **45** (2006), 1–30

[Y5] Yoshioka, K., *Stability and the Fourier-Mukai transform II,* Compositio Math. **145** (2009), 112–142

[Y6] Yoshioka, K., *An action of a Lie algebra on the homology groups of moduli spaces of stable sheaves,* arXiv:math/0605163, Adv. Stud. Pure Math. **58** (2010), 403–459

[Y7] Yoshioka, K., *Perverse coherent sheaves and Fourier-Mukai transforms on surfaces I.*

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