Infinite-dimensional p-adic groups, semigroups of double cosets, and inner functions on Bruhat–Tits buildings

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We construct p-adic analogs of operator colligations and their characteristic functions. Consider a p-adic group $\mathbf{G} = \mathrm{GL}(\alpha+k\infty, \mathbb{Q}_p)$, its subgroup $L = \mathrm{O}(k\infty, \mathbb{Z}_p)$, and the subgroup $\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p)$ embedded to L diagonally. We show that double cosets $\Gamma = \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ admit a structure of a semigroup, Γ acts naturally in $\mathbf{K}$-fixed vectors of any unitary representations of $\mathbf{G}$. For each double coset we assign a 'characteristic function', which sends a certain Bruhat–Tits building to another building (buildings are finite-dimensional); image of the distinguished boundary is contained in the distinguished boundary. The latter building admits a structure of (Nazarov) semigroup, the product in Γ corresponds to a point-wise product of characteristic functions.

1 Degeneration of Iwahori–Hecke type algebras in the infinite dimensional limit

1.1. Hypergroups of double cosets. Consider a group G and its compact subgroup K. Consider double cosets $K \setminus G/K$, i.e., the quotient of G with respect to the equivalence relation

$$g \sim k_1 g k_2, \qquad \text{where } k_1,\, k_2 \in K.$$

Each double coset $\mathfrak{g} = KgK$ is equipped with a unique probability measure $\mu_\mathfrak{g}$, which is invariant with respect to left and right translations by elements of K. Convolution of measures $\mu_\mathfrak{g}$, $\mu_\mathfrak{h}$ can be represented in the form

$$\mu_\mathfrak{g} * \mu_\mathfrak{h} = \int_{K\setminus G/K} \mu_\mathfrak{r}(\mathfrak{r}) d\sigma_{\mathfrak{g},\mathfrak{h}}(\mathfrak{r}),$$

where $\sigma_{\mathfrak{g},\mathfrak{h}}$ is a positive probability measure on $K \setminus G/K$. Thus we get a map

$$(\mathfrak{g}, \mathfrak{h}) \mapsto \sigma_{\mathfrak{g},\mathfrak{h}}$$

from $K\setminus G/K \times K \setminus G/K$ to the space of measures on $K\setminus G/K$. Such algebraic structures are called *hypergroups*[2]. Also the map $g \mapsto g^{-1}$ induces an involution $\mu \mapsto \mu^*$ on the hypergroup,

$$(\mu_\mathfrak{g} * \mu_\mathfrak{h})^* = \mu_\mathfrak{h}^* * \mu_\mathfrak{g}^*.$$

Remark. We reformulate this in two forms.

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[2]See, e.g., [1].

a) Denote by $\mathcal{M}(K\setminus G/K)$ the set of all (sign-indefinite) compactly supported measures on G, which are invariant with respect to left and right translations by elements of K. Then $\mathcal{M}(K\setminus G/K)$ is an algebra with respect to the convolution.

b) Let G be a locally compact group with two-side invariant Haar measure dg. Consider the set $C(K\setminus G/K)$ of compactly supported left-right K-invariant continuous functions on G. Then $C(K\setminus G/K)$ is an algebra with respect to the convolution. Sometimes it is called *(generalized) Iwahori–Hecke algebra.* ⊠

Let ρ be a unitary representation of G in a Hilbert space H. Denote by H^K the space of K-fixed vectors, by P^K the projection operator to H^K. Let $g\in\mathfrak{g}$. Define the operator $H^K\to H^K$ given by

$$\overline{\rho}(\mathfrak{g}):=P^K\rho(g)\Big|_{H^K}. \tag{1.1}$$

It is easy to see that $\overline{\rho}(\mathfrak{g})$ depends on the double coset and not on a representative g. The operators $\overline{\rho}(g)$ also can be expressed as

$$\overline{\rho}(g)=\int_{K\times K}\rho(k_1gk_1)dk_1\,dk_2\Big|_{H^K}=\int_K\rho(kg)\,dk\Big|_{H^K}.$$

Also, we have a representation of the hypergroup in H^K in the following sense:

$$\overline{\rho}(\mathfrak{g}*\mathfrak{h})=\int\overline{\rho}(\mathfrak{r})d\sigma_{\mathfrak{g},\mathfrak{h}}(\mathfrak{r}).$$

Several special cases of this construction are widely used in representation theory, in particular for the following pairs $G\supset K$:

— G is a real semisimple Lie group and K is the maximal compact subgroup; or G is a compact Lie group and K is a symmetric subgroup, [2], [3];

— G is a finite Chevalley group, K is a Borel subgroup, [4];

— G is a p-adic semisimple group and K is the Iwahori subgroup, [5].

Even for $(G,K)=(\mathrm{SL}(2,\mathbb{R}),\mathrm{SO}(2))$ the explicit expression for $\sigma_{\mathfrak{g},\mathfrak{h}}$ is nontrivial, see [6].

For smaller subgroups $K\subset G$ in semisimple groups, the hypergroups $K\setminus G/K$ became too complicated objects. For a noncompact subgroup K there is no *finite* $K\times K$-invariant measure on $K\setminus G/K$. On the other hand, a convolution of infinite measures is not defined (except few exotic cases).

In 1970s R.S.Ismagilov and G.I.Olshanski observed that the situation can drastically change for infinite-dimensional groups. Now we discuss a real archetype of our p-adic construction.

1.2. Colligations. Denote by $\mathrm{U}(\infty)$ the group of all *finitary*[3] infinite unitary matrices g . Denote by $\mathrm{O}(\infty)\subset\mathrm{U}(\infty)$ the group of real orthogonal

[3] An infinite matrix g is finitary, if $g-1$ has only finite number of nonzero matrix elements

matrices. We also use notation $\mathrm{U}(n+\infty)$ for the group of block finitary unitary matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $(n+\infty)\times(n+\infty)$. Consider double cosets

$$K \setminus G/K = \mathrm{O}(\infty)\setminus \mathrm{U}(n+\infty)/\mathrm{O}(\infty),$$

i.e., matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{U}(n+\infty)$ determined up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}, \qquad \text{where } u,\, v \in \mathrm{O}(\infty).$$

We call such equivalence classes by *colligations*[4].

There is no Haar measure on K, therefore there are no natural measures on double cosets KgK, therefore we can not repeat the construction of a hypergroup $K \setminus G/K$.

However, there is a natural multiplication

$$K\setminus G/K \,\times\, K\setminus G/K \;\to K\setminus G/K$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p & 0 & q \\ 0 & 1 & 0 \\ r & 0 & t \end{pmatrix} = \begin{pmatrix} ap & b & cq \\ cp & d & cq \\ r & 0 & cq \end{pmatrix}. \tag{1.2}$$

The matrix in the right-hand side has size $(n+\infty+\infty)$, we regard it as a matrix of size

$$\bigl(n+(\infty+\infty)\bigr)\times\bigl(n+(\infty+\infty)\bigr) = (n+\infty)\times(n+\infty).$$

Proposition 1.1 *The $\circ$-multiplication is a well-defined associative operation on $K\setminus G/K$.*

We also define an involution $\mathfrak{g}\mapsto\mathfrak{g}^*$ on $K\setminus G/K$ induced by the map $g\mapsto g^*$ (taking of adjoint operator). It is easy to verify the identity

$$(\mathfrak{g}\circ\mathfrak{h})^* = \mathfrak{h}^*\circ\mathfrak{g}^*.$$

Consider a unitary representation of $G = \mathrm{U}(n+\infty)$ in a Hilbert space H. As above consider the space H^K of K-fixed vectors in H and operators (1.1). The following *multiplicativity theorem* holds:

Theorem 1.2 (see [7], [8], Section IX.4) *For any $\mathfrak{g}$, $\mathfrak{h}\in K\setminus G/K$,*

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(\mathfrak{h}) = \overline{\rho}(\mathfrak{g}\circ\mathfrak{h}).$$

[4]This is a term from operator theory, a colligation (node) is the conjugacy class (1.10)

Also, for any $\mathfrak{g}$,

$$\overline{\rho}(\mathfrak{g}^*) = \overline{\rho}(\mathfrak{g})^*$$

These phenomena (semigroup structure on $K\backslash G/K$ and the multiplicativity) have no finite-dimensional analogs. However, for infinite-dimensional groups they are usual, see a discussion in Subsection 1.8.

1.3. Characteristic functions. We wish to describe the $\circ$-multiplication on more usual language. For a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we write the following equation

$$\begin{pmatrix} q_+ \\ \lambda y \\ q_- \\ y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} p_+ \\ x \\ p_- \\ \lambda x \end{pmatrix}, \tag{1.3}$$

where $\lambda \in \mathbb{C}$, x, $y \in \ell_2$, $p_\pm$, $q_\pm \in \mathbb{C}^n$.

Eliminate variables x, y from this system of equations, this is possible if

$$\det(\lambda^2 \overline{d} - d)$$

is not identical zero. We get a dependence

$$\begin{pmatrix} q_+ \\ q_- \end{pmatrix} = \chi_g(\lambda) \begin{pmatrix} p_+ \\ p_- \end{pmatrix},$$

where $\lambda \mapsto \chi_g(\lambda)$ is a matrix-valued rational function on $\mathbb{C}$. It is called a *characteristic function.*

A characteristic function $\chi_g(\lambda)$ depends only on a double coset $\mathfrak{g}$ containing g and not on g itself.

Theorem 1.3 *If $\chi_\mathfrak{g}(\lambda)$ and $\chi_\mathfrak{h}(\lambda)$ are well-defined, then*

$$\chi_{\mathfrak{g}\circ\mathfrak{h}}(\lambda) = \chi_\mathfrak{g}(\lambda)\chi_\mathfrak{h}(\lambda). \tag{1.4}$$

Also,

$$\chi_{\mathfrak{g}^*}(\lambda) = \chi_{\mathfrak{g}^*}(\lambda^{-1})^{-1}.$$

1.4. Reformulation. The language of Grassmannians. Fix λ. Consider the set $\mathcal{X}_\mathfrak{g}(\lambda)$ of all $(q_+, q_-; p_+, p_-) \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ such that there are x, y satisfying (1.3). Evidently, $\mathcal{X}_\mathfrak{g}(\lambda)$ is a linear subspace. Notice, that at a non-singular point of the function $\chi_\mathfrak{g}(\lambda)$, the subspace $\mathcal{X}_\mathfrak{g}(\lambda)$ is the graph of the operator $\chi_\mathfrak{g}(\lambda) : \mathbb{C}^n \to \mathbb{C}^n$.

Next, we extend the function $\mathcal{X}_\mathfrak{g}(\lambda)$ to the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$ in the following way. We write the equation

$$\begin{pmatrix} q_+ \\ y \\ q_- \\ 0 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} p_+ \\ 0 \\ p_- \\ x \end{pmatrix}, \tag{1.5}$$

and consider the set $\mathcal{X}_\mathfrak{g}(\infty)$ of all $(q_+, q_-; p_+, p_-) \in \mathbb{C}^{2n} \oplus \mathbb{C}^{2n}$ such that the equation (1.5) has a solution.

Theorem 1.4 a) $\dim \mathcal{X}_{\mathfrak{g}}(\lambda) = 2n$ *for all* $\lambda \in \overline{\mathbb{C}}$.

b) *For any* $\mathfrak{g}$ *the map* $\lambda \mapsto \mathcal{X}_{\mathfrak{g}}(\lambda)$ *is holomorphic on* $\overline{\mathbb{C}}$.

Emphasize that the characteristic function $\mathcal{X}_{\mathfrak{g}}(\lambda)$ is well-defined for all double cosets $\mathfrak{g}$.

Next, we explain how to interpret formula (1.4) on the language of Grassmannian.

Let V, W be linear spaces. We say that a *linear relation* $L : V \rightrightarrows W$ is a subspace $L \subset V \oplus W$.

EXAMPLE. Let $A : V \to W$ be a linear operator. Then its graph $\mathrm{graph}(A) \subset V \oplus W$ is a linear relation. The set of all linear subspaces in $V \oplus W$ consists of $\dim V + \dim W$ components. Graphs of operators constitute an open dense subspace in one of components. ⊠

Consider two linear relations $L : V \rightrightarrows W$, $M : W \rightrightarrows Y$. Define their *product* $LM : V \rightrightarrows Y$ as the set of $(r,p) \in V \oplus Y$ such that there exists $q \in W$ such that $(r,q) \in L$, $(q,p) \in M$.

Also, for a linear relation $L : V \rightrightarrows W$ we define the *kernel* $\ker L \subset V$ and the *indefinity* $\mathrm{indef}\, L \subset W$,

$$\ker L := L \cap (V \oplus 0), \qquad \mathrm{indef}\, L := L \cap (0 \oplus W).$$

Theorem 1.5 *For any* $\mathfrak{g}$, $\mathfrak{h}$ *and each* $\lambda \in \overline{\mathbb{C}}$,

$$\mathcal{X}_{\mathfrak{g}\circ\mathfrak{h}}(\lambda) = \mathcal{X}_{\mathfrak{g}}(\lambda)\, \mathcal{X}_{\mathfrak{h}}(\lambda).$$

1.5. Conditions for characteristic functions. We equip the space $\mathbb{C}^n \oplus \mathbb{C}^n$ with a standard skew-symmetric bilinear form determined by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We regard vectors (p_+, p_-) and (q_+, q_-) as elements of $\mathbb{C}^n \oplus \mathbb{C}^n$. Denote by $\mathrm{Sp}(2n, \mathbb{C})$ the group of operators preserving this form.

Equip the space $(\mathbb{C}^n \oplus \mathbb{C}^n) \oplus (\mathbb{C}^n \oplus \mathbb{C}^n)$ by the difference of skew-symmetric forms, i.e. by the form with matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We regard vectors (p_+, p_-, q_+, q_-) as elements of this space.

Proposition 1.6 (see [8], IX.4)

a) *Outside poles, values of* $\chi_{\mathfrak{g}}(\lambda)$ *are contained in the complex symplectic group* $\mathrm{Sp}(2n, \mathbb{C})$.

b) *The characteristic function* $\mathcal{X}_{\mathfrak{g}}(\lambda)$ *takes values in the Lagrangian Grassmannian*[5].

[5] Recall that a subspace L in a $2m$-dimensional linear space equipped with a nondegenerate skew-symmetric bilinear form is Lagrangian if the form vanishes on L and $\dim L = m$, see, e.g., [9], Section 3.1.

Second, consider the Hermitian form M on $\mathbb{C}^n \oplus \mathbb{C}^n$ determined by the matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Denote by $\mathrm{U}(n,n)$ the group of matrices preserving M.

We say that a linear operator A in $\mathbb{C}^n \oplus \mathbb{C}^n$ is an *M-contraction* (see, e.g., [9], Section 2.7), if for all vectors v we have

$$M(Av, Av) \leqslant M(v, v).$$

We say that A is an *M-dilatation* if $M(Av, Av) \geqslant M(v, v)$.

Also, equip the space $(\mathbb{C}^n \oplus \mathbb{C}^n) \oplus (\mathbb{C}^n \oplus \mathbb{C}^n)$ with the difference of Hermitian forms, i.e. with a form $\widetilde{M}$ given by

$$\begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

Proposition 1.7 (see [8], Section IX.4) *Let $\chi_{\mathfrak{g}}(\lambda)$ be well-defined. Then:*

a) *If $|\lambda| = 1$, then $\chi_{\mathfrak{g}}(\lambda) \in \mathrm{U}(n,n)$.*

b) *If $|\lambda| < 1$, then $\chi_{\mathfrak{g}}(\lambda)$ is an M-contraction.*

c) *If $|\lambda| > 1$, then $\chi_{\mathfrak{g}}(\lambda)$ is an M-dilatation.*

Proposition 1.8 (see [8], Section IX.4)

a) *If $|\lambda| = 1$, then the subspace $\mathcal{X}_{\mathfrak{g}}(\lambda)$ is $\widetilde{M}$-isotropic.*

b) *If $|\lambda| < 1$, then the form $\widetilde{M}$ is positive semi-definite on the subspace $\mathcal{X}_{\mathfrak{g}}(\lambda)$.*

c) *If $|\lambda| > 1$, then the form $\widetilde{M}$ is negative semi-definite on the subspace $\mathcal{X}_{\mathfrak{g}}(\lambda)$.*

d) *If $|\lambda| < 1$, then the from M is strictly positive definite on*[6] $\ker \mathcal{X}_{\mathfrak{g}}(\lambda)$. *Also M is negative definite on* $\operatorname{indef} \mathcal{X}_{\mathfrak{g}}(\lambda)$.

Characteristic functions also satisfy to the following condition of symmetry at 0

$$\chi_{\mathfrak{g}}(-\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \chi_{\mathfrak{g}}(\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1}. \tag{1.6}$$

On the language of Grassmannians this means

$$(p_+, p_-, q_+, q_-) \in \mathcal{X}(\lambda) \Leftrightarrow (p_+, -p_-, q_+, -q_-) \in \mathcal{X}(\lambda). \tag{1.7}$$

Theorem 1.9 *Any holomorphic map $\mathcal{X}$ from $\overline{\mathbb{C}}$ to the Lagrangian Grassmannian satisfying the conditions of Proposition* 1.8 *and condition* (1.7) *is a characteristic function of a double coset $\mathfrak{g}$.*

[6] This condition contains additional information only at points λ, where $\mathcal{X}(\lambda)$ is not a graph of an operator. By statement b) M is positive semi-definite on the kernel.

1.6. Central extension. A characteristic function is not sufficient for a reconstruction of a double coset, in fact matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \begin{matrix} \}n \\ \}\infty \\ \}\infty \end{matrix}$$

with fixed a, b, c, d and arbitrary e have the same characteristic function. Let us introduce an additional invariant. We write the equation

$$\begin{pmatrix} 0 \\ \lambda y \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} 0 \\ x \\ 0 \\ \lambda x \end{pmatrix},$$

as an equation for x, y. Denote by $n_{\mathfrak{g}}(\lambda)$ the dimension of the space of solutions of this equation. Then

— $n_{\mathfrak{g}}(\lambda) = 0$ for all but a finite number of values of λ;

— $n_{\mathfrak{g}}(\lambda) = 0$ if $|\lambda| \neq 1$;

— $n_{\mathfrak{g}}(\lambda) = n_{\mathfrak{g}}(-\lambda)$;

— $n_{\mathfrak{g}}(\pm 1) = \infty$.

Thus we get a finite set with multiplicities (we call it *divisor*).

Theorem 1.10 [7] *A double coset is uniquely determined by its characteristic function $\mathcal{X}$ and the divisor n.*

Theorem 1.11 [8], IX.4.5)

$$n_{\mathfrak{g}\circ\mathfrak{h}}(\lambda) = n_{\mathfrak{g}}(\lambda) + n_{\mathfrak{g}}(\mathfrak{h};\lambda) + \dim\Bigl(\operatorname{indef} \mathcal{X}_{\mathfrak{h}}(\lambda) \cap \ker \mathcal{X}_{\mathfrak{g}}(\lambda)\Bigr).$$

Double cosets corresponding matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e \end{pmatrix} \begin{matrix} \}n \\ \{\infty \\ \}\infty \end{matrix}$$

is the center of the semigroup $K \setminus G/K$. The quotient of $K \setminus G/K$ with respect to the center is isomorphic to the semigroup of rational matrix-values functions described above.

[7]This and previous statements are given in [8], .IX.4.8 without formal proof. In fact, a proof is contained in the same book, Addendum E. Precisely, in Subsection E.4 it is shown how to reduce our statements to the standard theorem (see [10]) 'pure unitary operator node is determined by its characteristic function'. In fact, we only need this theorem for finitary matrices and rational characteristic functions.

1.7. Degeneration of hypergroups of double cosets. Let $N > k$. Embed $\mathrm{U}(n+k)$ to $\mathrm{U}(n+k+N)$ by

$$\iota_N : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Embed $\mathrm{U}(k+N)$ to $\mathrm{U}(n+k+N)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix}.$$

Fix matrices $g = \begin{pmatrix} q & b \\ c & d \end{pmatrix}$, $h = \begin{pmatrix} p & q \\ r & t \end{pmatrix} \in \mathrm{U}(n+k)$. Then for $N > k$ a matrix $\iota_N(g) \circ \iota_N(h)$ is well-defined as an element of $\mathrm{U}(k+N) \setminus \mathrm{U}(n+k+N)/\mathrm{U}(k+N)$.

We equip the group $\mathrm{U}(n+k+N)$ with the metric induced by the operator norm in Euclidean $\mathbb{C}^{n+k+N}$.

Proposition 1.12 *Fix g, $h \in \mathrm{U}(n+k)$ as above. Consider the corresponding double cosets*

$$\mathfrak{g}_N, \mathfrak{h}_N \in \mathrm{U}(k+N) \setminus \mathrm{U}(n+k+N)/\mathrm{U}(k+N)$$

and the measure

$$\varkappa_N = \mu_{\mathfrak{g}_N} * \mu_{\mathfrak{h}_N}$$

Then for each $\varepsilon > 0$, $\delta > 0$ there exists N such that the measure $\varkappa_N$ of ε-neighborhood of $\iota_N(g) \circ \iota_N(h)$ is $> 1 - \delta$.

See [7], [11], [12].

1.8. Semigroups of double cosets. The first example of multiplication of double cosets was discovered by Ismagilov [13], he considered the group $G = \mathrm{SL}(2, k)$ over a non-Archimedian normed non locally compact field k. The subgroup K is the group $\mathrm{SL}(2, o)$ over integer elements of k. The double cosets are parametrized by non-negative integers $\mathbb{Z}_+$, and the operation $\circ$ is the usual addition. The multiplicativity theorem allows to classify spherical functions (see also [14]). Olshanski [15] showed that this semigroup is a limit of hypergroups $\mathrm{SL}(2, \mathbb{Z}_p) \setminus \mathrm{SL}(2, \mathbb{Q}_p)/\mathrm{SL}(2, \mathbb{Z}_p)$ as $p \to \infty$.

Next, consider a series of Riemannian symmetric spaces $G(n)/K(n)$ (an example is $\mathrm{U}(n)/\mathrm{O}(n)$). Olshanski [7], [11] showed that the same phenomena hold for any pair $G(k+\infty) \supset K(\infty)$. Also he described such semigroups for infinite symmetric groups. As far as we know description of such objects, they became a tool of the representation theory. On the other hand, it seems that such structure are interesting by themselves.

In [8], Section 8.5, the author observed that multiplications on $K \setminus G/K$ are quite usual for infinite-dimensional groups (see also [16], [17]). In fact this happened more-or-less always if K is one of the following groups:

1) K is a complete infinite unitary group, orthogonal group, or symplectic (quaterninic unitary) group (or a product of several copies of such groups);

2) K is the infinite symmetric group $S(\infty)$;

3) K is the group of automorphisms of a measure space;

These groups are infinite-dimensional imitation of compact groups (but they are neither compact, nor locally compact) apparently some other examples also exist (for instance, below we discuss $K = \mathrm{O}(\infty, \mathbb{Z}_p)$).

For precise general theorems, see [16], [17]. To explore them we need explicit descriptions of $K \setminus G/K$, such descriptions recently were obtained in [18], [20], [16], [17].

1.9. Inner functions. Recall a definition of inner functions.

1) A holomorphic function $f(z)$ in a unit disk $|z| < 1$ is called *inner*, if $|f(z)| < 1$ for $|z| < 1$ and

$$\lim_{r \to 1^-} |f(re^{i\theta})| = 1 \qquad \text{a.s. } \theta \in [0, 2\pi], \tag{1.8}$$

where $z = re^{i\theta}$ and r, θ are real[8]. On this topic, see, e.g., [21]. It can be shown that limit (1.8) can be replaced a.s. by the *nontangential limit*

$$\lim_{z \to e^{i\theta},\ \left|\arg \frac{e^{i\theta}-z}{e^{i\theta}}\right| \leqslant \pi/2-\varepsilon} f(z), \tag{1.9}$$

where $\varepsilon > 0$ is fixed (in fact we consider a limit over the angle whose vertex is $e^{i\theta}$, the bisector is $te^{i\theta}$, and the value of the angle is $\pi - 2\varepsilon$.

2) A homomorphic matrix-valued (operator-valued) function $f(z)$ in the unit disk is called *inner* if $\|f(z)\| \leqslant 1$ for $|z| < 1$ and boundary values of f on the circle are unitary (see Livshits [22], Potapov [23]). Consider an operator d closed to unitary (one of possible variants $\mathrm{rk}(dd^* - 1) = \mathrm{rk}(d^*d - 1) < \infty$) with $\|d\| = 1$. We are interested its properties up to conjugations $d \mapsto udu^{-1}$, where u is unitary. Build a larger *unitary* matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ including d as a block. We consider g up to the equivalence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}, \qquad \text{where } u \in \mathrm{U}(\infty). \tag{1.10}$$

Assign to g the expression (*characteristic function*) by

$$\chi(\lambda) = a + \lambda b(1 - \lambda d)^{-1} c.$$

Such functions (under some conditions on d) are inner functions $\theta(z)$ in the unit disk. Invariant subspaces of d are in one-to-one correspondence with divisors of θ in the class of inner functions. The product of inner functions corresponds to the product of conjugacy classes (1.10) by formula (1.2).

[8] We can not write a limit as $z \to e^{i\theta}$, an inner function can be discontinuous at all points of the circle.

3) More generally, consider a pseudo-Euclidean space. We say that a meromorphic matrix-valued function f in the disk is *inner* if it is indefinite contractive in the disk and pseudo-unitary on the unit circle. Such functions arise in the same context but the condition $\|d\| \leqslant 1$ is omitted.

The characteristic function of double cosets defined above are inner in this sense.

4) Denote by B_n the set of all $n \times n$ complex symmetric matrix with norm < 1; B_n also is an Hermitian symmetric space

$$B_n = \mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n),$$

its distinguished boundary (Shilov boundary) consist of unitary matrices.

In [20], [16] there were considered various semigroups of double cosets on infinite-dimensional classical groups. For instance, consider group $G = \mathrm{U}(\alpha + k\infty)$ consisting of block unitary matrices of size $\alpha + \infty + \cdots + \infty$. Consider its subgroup $L = \mathrm{U}(\infty)$ embedded to G in the block diagonal way. Consider the subgroup $K = \mathrm{O}(\infty) \subset L$ embedded to $\mathrm{U}(\infty)$ in the natural way. Then $K \setminus G/K$ is a semigroup. Characteristic functions [20] are inner functions in $B_k \times B_k$ taking values at the space of $2\alpha \times 2\alpha$-matrices. This means that values of a function are M-contractions inside $B_k \times B_k$ and are pseudounitary on the Shilov boundary $\mathrm{U}(n) \times \mathrm{U}(n)$. The product of double cosets corresponds to the product of characteristic functions.

It is possible to vary the definition and to regard a characteristic function as a map $B_k \times B_k \to B_{2\alpha}$.

1.10. Infinite-dimensional p-adic groups. Representation theory of infinite-dimensional classical groups (see, e.g., [24], [25], [7], [8], [26], [27], [16]) and infinite symmetric groups (see, e.g., [28], [29], [18]) exists and is well-developed. There were several recent works concerning infinite-dimensional classical groups over finite fields (see [30], [31], [32]).

Few is known about infinite-dimensional p-adic groups. There are the following works:

1) Work of Nazarov [33], [34] on the Weil representation of an infinite-dimensional group $\mathrm{Sp}(2\infty, \mathbb{Q}_p)$. Existence of such representation is more-or-less evident. However, the Weil representation of $\mathrm{Sp}(2n, \mathbb{R})$ and $\mathrm{Sp}(2\infty, \mathbb{R})$ admits a continuation to a certain complex domain Γ (if $n < \infty$, then Γ is a semigroup parametrized by complex symmetric $2n \times 2n$ matrices with norm < 1, see, e.g., [8], Section 4.2, [9], Section 5.1). Nazarov constructed an analog of Γ for p-adic case, see below Section 3 (for more details, see [9], Sections 10.7, 11.2)

2) A construction of Hua measures on p-adic Grassmannians and on the inverse limit of p-adic Grassmannians in [35]. This is an analog of inverse limits of compact symmetric spaces (see [36]) and of symmetric groups (see [29]). Recall that in latter two cases there exists a substantial harmonic analysis on such inverse limits, see [27], [29].

3) The group of diffeomorphisms of p-adic projective line is an object similar to the group of diffeomorphisms of the circle (many constructions of representations of the latter group survive in p-adic case, [37]).

1.11. A p-adic example. Here we briefly discuss a p-adic object, which is related to the topic of this paper but more simple. Let $\mathbb{Q}_p$ be a p-adic field, $\mathbb{Z}_p \subset \mathbb{Q}_p$ be the ring of p-adic integers. Denote by $\mathrm{GL}(\infty, \mathbb{Q}_p)$ the group of finitary invertible matrices over $\mathbb{Q}_p$. Consider conjugacy classes of $\mathrm{GL}(\alpha+\infty, \mathbb{Q}_p)$ with respect to the subgroup $\mathrm{GL}(\infty, \mathbb{Z}_p)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}, \qquad \text{where } u \in \mathrm{GL}(\infty, \mathbb{Z}_p). \tag{1.11}$$

Such conjugacy classes admit a natural $\circ$-multiplication by formula (1.2), this multiplication is a well-defined associative operation on the space of conjugacy classes. We wish to construct an analog of characteristic functions.

First, choose a sufficiently large m such that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is actually contained in $\mathrm{GL}(\alpha+m, \mathbb{Q}_p)$. Consider a lattice[9] $R \subset \mathbb{Q}_p^2$. For this lattice we consider the lattice

$$R \otimes \mathbb{Z}_p^m \subset \mathbb{Q}_p^2 \otimes \mathbb{Q}_p^m \simeq \mathbb{Q}_p^m \oplus \mathbb{Q}_p^m.$$

We write an equation

$$\begin{pmatrix} v \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ x \end{pmatrix}. \tag{1.12}$$

Next, consider the set $\chi(R)$ of all pairs $(v, u) \in \mathbb{Q}_p^\alpha \oplus \mathbb{Q}_p^\alpha$ for which there exists $y \oplus x \in R \otimes \mathbb{Z}_p^m$ such that the equality (1.12) is satisfied. Then $\chi(R)$ is a $\mathbb{Z}_p$-submodule in $\mathbb{Q}_p^\alpha \oplus \mathbb{Q}_p^\alpha$, which can be regarded as a relation $\mathbb{Q}_p^\alpha \rightrightarrows \mathbb{Q}_p^\alpha$. The $\circ$-product corresponds to point-wise product of functions $\chi(R)$ with values in relations.

We also point out that these functions are compatible with the structure of Bruhat–Tits buildings and are inner in a reasonable sense. Both phenomena are discussed below for more sophisticated objects.

1.12. Purpose of the paper. We wish to describe multiplication of double cosets on p-adic groups and to obtain analogs of characteristic functions. For a double coset we assign a simplicial map from a Bruhat–Tits building Ω to a Bruhat–Tits building Ξ such that the image of the distinguished boundary is contained in the distinguished boundary. We also have a structure of a semigroup on the set of vertices of the building Ξ (the Nazarov semigroup) and the product of double cosets corresponds to pointwise product of functions $\Omega \to \Xi$.

Our construction is not a final solution of the problem[10]

1.13. A non-properly understood link. In fact our main construction below is organized as an extension of rational maps of p-adic Grassmannians to simplicial maps of Bruhat–Tits buildings. Also, our construction admits an

[9] see a definition in Subsection 3.1

[10] First, we do not introduce an analog of the 'divisor'. Secondly, [19] suggests that complete data separating double cosets must contain a sequence of characteristic functions determined on the increasing sequence of buildings. A construction of complete data in a real case in [19] is based on classical invariant theory, which does not valid over the ring $\mathbb{Z}_p$.

automatic pass to algebraic extensions. Constructions of such type are investigated in theory of Berkovich analytic spaces, see, e.g., [39], [40]. However their extensions are rigid, and our extensions depend on additional data[11]. So I can not understand relations of our constructions and Berkovich theory.

1.14. Notation. Let

— A^t be the transposed matrix;

— 1_α, 1_V be the unit matrix of order α, the unit operator in a space V;

— $\mathbb{Q}_p$ be the p-adic field;

— $\mathbb{Z}_p$ be the ring of p-adic integers;

— $\mathbb{Q}_p^\times$, $\mathbb{C}^\times$ be multiplicative groups of $\mathbb{Q}_p$, $\mathbb{C}$.

We denote the standard character $\mathbb{Q}_p \to \mathbb{C}^\times$ by $\exp\{2\pi i a\}$. For $a = \sum_{\geqslant -N} a_j p^j$, where $a_j = 0, 1, \dots, p-1$, we set

$$\exp\{2\pi i a\} = \exp\Bigl\{2\pi i \sum_{j\geqslant -N} a_j p^j\Bigr\} := \exp\Bigl\{2\pi i \sum_{j: -1\geqslant j\geqslant -N} a_j p^j\Bigr\}$$

Below we define:

— the groups $\mathrm{GL}(n,\mathbb{Q}_p)$, $\mathrm{Sp}(2n,\mathbb{Q}_p)$, $\mathrm{Sp}(2n,\mathbb{Q}_p)$, $\mathrm{O}(n,\mathbb{Z}_p)$, $\mathrm{GL}(\infty,\mathbb{Z}_p)$, $\mathrm{Sp}(2\infty,\mathbb{Q}_p)$, etc., Subsection 2.1;

— $V_\pm$, formula (2.1);

— groups $\mathbf{G} = \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$, $\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p)$, Subsection 2.2;

— $\mathfrak{g} \star \mathfrak{h}$, the product of double cosets, Subsection 2.2;

— $\mathfrak{g}^*$, the involution on double cosets, Subsection 2.5;

— $R_\downarrow$, $R^\uparrow$, , Subsection 3.1;

— $R_j \nearrow R$, rigid convergence, 3.4;

— $\mathrm{LMod}(V)$, $\mathrm{LLat}(V)$, $\mathrm{LGr}(V)$, spaces of Lagrangian submodules, Subsection 3.5;

— $\Delta(V)$, $\mathrm{Bd}(V)$, buildings, Subsections 3.6, 3.8;

— $P : V \rightrightarrows W$, $\ker P$, $\operatorname{indef} P$, $\operatorname{dom} P$, $\operatorname{im} P$, $P^\square$, Subsection 3.9;

— Naz, $\overline{\mathrm{Naz}}$, **Naz**, the Nazarov category, Subsections 3.12; 3.14;

— We, the Weil representation, Subsection 3.16;

— $\chi_\mathfrak{g}(Q,T)$, a characteristic function, Subsection 4.1.

[11] Below rational maps of Grassmannians originate from double cosets

$$\mathrm{O}(\infty,\mathbb{Q}_p) \setminus \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)/\mathrm{O}(\infty,\mathbb{Q}_p)$$

(see Proposition 4.11) maps of Bruhat–Tits buildings from double cosets

$$\mathrm{O}(\infty,\mathbb{Z}_p) \setminus \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)/\mathrm{O}(\infty,\mathbb{Z}_p).$$

Therefore we get many maps of Bruhat–Tits buildings with the same restriction to a distinguished boundary, i.e., to the Grassmannian.

2 Multiplication of double cosets

2.1. Groups. By $V = \mathbb{Q}_p^n$ we denote linear spaces over $\mathbb{Q}_p$. Denote by $\mathrm{GL}(n, \mathbb{Q}_p) = \mathrm{GL}(V)$ the group of invertible linear operators in $\mathbb{Q}_p^n$; by $\mathrm{GL}(n, \mathbb{Z}_p)$ the group of all matrices g with integer elements, such that g^{-1} have integer elements.

Consider a space $V = \mathbb{Q}_p^{2n}$ equipped with a non-degenerate skew-symmetric bilinear form B_V, say $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The symplectic group $\mathrm{Sp}(2n, \mathbb{Q}_p)$ is the group of matrices preserving this form, $\mathrm{Sp}(2n, \mathbb{Z}_p)$ is the group of symplectic matrices with integer elements. We also denote

$$V_+ := \mathbb{Q}_p^n \oplus 0, \qquad V_- = 0 \oplus \mathbb{Q}_p^n. \tag{2.1}$$

Also, consider a space $\mathbb{Q}_p^n$ equipped with the standard symmetric bilinear form $(v, w) = \sum v_j w_j$. We denote by $\mathrm{O}(n, \mathbb{Q}_p)$ the group of all matrices preserving this form[12].

By $\mathrm{GL}(\infty, \mathbb{Q}_p)$ we denote the group of all infinite invertible matrices over $\mathbb{Q}_p$ such that $g - 1$ has only finite number of non-zero elements. We call such matrices *finitary*. We define $\mathrm{GL}(\infty, \mathbb{Z}_p)$, $\mathrm{Sp}(2\infty, \mathbb{Q}_p)$, $\mathrm{Sp}(2\infty, \mathbb{Z}_p)$, $\mathrm{O}(\infty, \mathbb{Z}_p)$ in the same way.

2.2. Multiplication of double cosets. Let

$$\mathbf{G} := \mathrm{GL}(\infty, \mathbb{Q}_p) := \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$$

be the group of finitary block $(\alpha + \infty + \dots + \infty) \times (\alpha + \infty + \dots + \infty)$- matrices (there are k copies of ∞). By $\mathbf{K}$ we denote the group

$$\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p)$$

embedded to $\mathbf{G}$ by the rule

$$\mathfrak{I} : u \mapsto \begin{pmatrix} 1_\alpha & 0 & \dots & o \\ 0 & u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u \end{pmatrix}, \tag{2.2}$$

where 1_α denotes the unit matrix of order α.

Remark. Certainly, $\mathbf{G} := \mathrm{GL}(\infty, \mathbb{Q}_p)$. But the notation of the type $\mathbf{G} := \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ allows us to indicate certain subgroups in $\mathbf{G}$. ⊠

We wish to define a structure of a semigroup on double cosets $\mathbf{K} \setminus \mathbf{G} / \mathbf{K}$.

Set

$$\Theta_N := \begin{pmatrix} 0 & 1_N & 0 \\ 1_N & 0 & 0 \\ 0 & 0 & 1_\infty \end{pmatrix} \in \mathbf{K}. \tag{2.3}$$

[12]There are several non-equivalent non-degenerate quadratic forms and several different orthogonal groups over $\mathbb{Q}_p$, however we consider only this group.

Let $\mathfrak{g}$, $\mathfrak{h} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$. Choose their representatives g, $h \in \mathbf{G}$. Consider the sequence

$$f_N := g\mathfrak{I}(\Theta_N)h$$

and double coset $\mathfrak{f}_N$ containing f_N.

Theorem 2.1 a) *The sequence $\mathfrak{f}_N$ of double cosets is eventually constant.*

b) *The limit $\mathfrak{f} := \lim_{N\to\infty} \mathfrak{f}_N$ does not depend on a choice of representatives g, h.*

c) *The product $\mathfrak{g} \star \mathfrak{h}$ in $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ obtained in this way is associative.*

These statements are simple, see proofs of parallel real statements in [16]. Also, it is easy to write an explicit formula for the product. For definiteness, set $k = 2$. Then

$$\begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} \star \begin{pmatrix} a' & b'_1 & b'_2 \\ c'_1 & d'_{11} & d'_{12} \\ c'_2 & d'_{21} & d'_{22} \end{pmatrix} =$$

$$= \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_\infty & 0 & 0 \\ 0 & 1_\infty & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_\infty \\ 0 & 0 & 0 & 1_\infty & 0 \end{pmatrix} \begin{pmatrix} a' & b'_1 & 0 & b'_2 & 0 \\ c'_1 & d'_{11} & 0 & d'_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c'_2 & d'_{21} & 0 & d'_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since a result is double coset, we can write the final matrix in different forms, say

$$\mathfrak{f} = \begin{pmatrix} aa' & | & b_1 & ab'_1 & & b_1 & ab'_1 \\ - & + & - & - & - & - & - \\ c_1a' & | & d_{11} & c_1b'_1 & & d_{12} & c_1b'_2 \\ c'_1 & | & 0 & d'_{11} & & 0 & d'_{12} \\ & | & & & & & \\ c_2a' & | & d_{21} & c_2b'_1 & & d_{22} & c_2b'_2 \\ c'_2 & | & 0 & d'_{21} & & 0 & d'_{22} \end{pmatrix} \tag{2.4}$$

or

$$\mathfrak{f} = \begin{pmatrix} aa' & | & ab'_1 & b_1 & & ab'_2 & b_2 \\ - & + & - & - & - & - & \\ c_1a' & | & c_1b'_1 & d_{11} & & c_1b'_2 & d_{12} \\ c'_1 & | & d'_{11} & 0 & & d'_{12} & 0 \\ & | & & & & & \\ c_2a' & | & c_2b'_1 & d_{21} & & c_2b'_2 & d_{22} \\ c'_2 & | & d'_{21} & 0 & & d'_{22} & 0 \end{pmatrix}.$$

2.3. Multiplicativity theorem. Let ρ be a unitary representation of $\mathbf{G}$, denote by $H^{\mathbf{K}}$ the subspace of all $\mathbf{K}$-fixed vectors. Denote by $P^{\mathbf{K}}$ the operator of orthogonal projection to $H^{\mathbf{K}}$. For $g \in \mathbf{G}$ consider the operator $\overline{\rho}(g) : H^{\mathbf{K}} \to H^{\mathbf{K}}$ given by

$$\overline{\rho}(g) := P^{\mathbf{K}} \rho(g)\Big|_{H^K}.$$

Obviously, $\overline{\rho}(g)$ is a function on double cosets $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$, therefore we can write $\overline{\rho}(\mathfrak{g})$.

Theorem 2.2 *For any unitary representation ρ, for all $\mathfrak{g}$, $\mathfrak{h} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ the following equality (the "multiplicativity theorem") holds,*

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(\mathfrak{h}) = \overline{\rho}(\mathfrak{g} \star \mathfrak{h}).$$

We give a proof in Section 6.

REMARK. Apparently the analog of Proposition 1.12 for p-adic case does not hold. □

2.4. Sphericity.

Proposition 2.3 *Let $\alpha = 0$. Then the pair $(\mathbf{G}, \mathbf{K})$ is spherical, i.e., for any irreducible unitary representation of $\mathbf{G}$ the dimension of the space of $\mathbf{K}$-fixed vectors is $\leqslant 1$.*

We omit a proof, it is the same as for infinite-dimensional real classical groups, see [16].

2.5. Involution. The map $g \mapsto g^{-1}$ induces an involution $\mathfrak{g} \mapsto \mathfrak{g}^*$ on $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$. Evidently,

$$(\mathfrak{g} \star \mathfrak{h})^* = \mathfrak{h}^* \star \mathfrak{g}^*.$$

Also, for any unitary representation ρ of $\mathbf{G}$ we have

$$\overline{\rho}(\mathfrak{g}^*) = \overline{\rho}(\mathfrak{g})^*.$$

2.6. Purpose of the work. Our aim is to describe this multiplication in more usual terms. More precisely, we wish to get p-adic analogs of multivariate characteristic functions constructed in [16], [20].

2.7. Structure of the paper. Section 3 contains preliminaries (lattices, Bruhat–Tits buildings, relations, the Weil representation of the Nazarov category). A main construction (characteristic functions of double cosets and their properties) is contained in Section 4. Proofs are given in Section 5.

In Section 6 we prove the multiplicativity theorem. Section 7 contains some constructions of representations. Theorem 7.5 shows a link between the characteristic function and operators $\overline{\rho}(\mathfrak{g})$.

3 Preliminaries. Submodules, relations, Bruhat–Tits buildings, Nazarov category, and Weil representation

A. Submodules and convergence

3.1. Modules. Below the term *submodule* means an $\mathbb{Z}_p$-submodule in a linear space $V = \mathbb{Q}_p^k$. For each submodule $R \subset \mathbb{Q}_p^k$ there is a (non-canonical) basis $e_i \in \mathbb{Q}_p^k$ such that

$$R = \mathbb{Q}_p e_1 \oplus \cdots \oplus \mathbb{Q}_p e_j \oplus \mathbb{Z}_p e_{j+1} \oplus \cdots \oplus \mathbb{Z}_p e_l. \tag{3.1}$$

If $j = k$ then R is a linear subspace. If $j = 0$, $l = k$, then we get a *lattice*. A formal definition is: a *lattice* R is a compact $\mathbb{Z}_p$-submodule such that $\mathbb{Q}_p R = \mathbb{Q}_p^k$. For details, see, e.g., [41].

Denote by $\mathrm{Mod}(V)$ the set of all submodules in V, by $\mathrm{Lat}(V)$ the space of all lattices. It is easy to see that

$$\mathrm{Lat}(V) \simeq \mathrm{GL}(V, \mathbb{Q}_p)/\mathrm{GL}(V, \mathbb{Z}_p).$$

For any submodule R denote by $R_\downarrow$ the maximal linear subspace in R. By $R^\uparrow$ we denote the minimal linear subspace containing R,

$$R_\downarrow \subset R \subset R^\uparrow$$

The image of R in the quotient space $R^\uparrow/R_\downarrow$ is a lattice.

Conversely, let $L \subset M$ be a pair of subspaces, $\pi : L \to L/M$ be the projection. Let $P \subset M/L$ be a lattice. Then $\pi^{-1}P$ is a submodule in $\mathbb{Q}_p^k$ and all submodules have such form.

3.2. Duality. For a p-adic linear space V we denote by V' the space of linear functionals on V. For a submodule $L \subset V$ define the dual module $L^\diamondsuit \subset V'$ as the set of all linear functionals $\ell \in V'$ such that

$$\ell(v) \in \mathbb{Z}_p \qquad \text{for all } v \in L$$

Notice that $L^{\diamondsuit\diamondsuit} = L$.

If L is a lattice, then $L^\diamondsuit$ is a lattice.

3.3. The Hausdorff convergence on $\mathrm{Mod}(V)$. Let $V = \mathbb{Q}_p^n$. We define a norm on V as

$$\|x\| = \max_j |x_j|.$$

Denote by $B(p^l)$ the ball with center at 0 of radius p^l.

Let K be a metric space, A, B be closed subsets. Define the *Hausdorff deviation* $\eta_B(A)$ as the supremum of distance between a ranging in A and B (a number $\eta_B(A)$ is a nonnegative real or ∞). The *Hausdorff* ∞[13] on the space of closed subset is defined by

$$h(A, B) = \max(\eta_A(B), \eta_B(A)).$$

Its restriction to the space of compact subsets is a metric. If K is compact then the space of its closed subsets is compact.

[13]We allow distance $= +\infty$

Now we introduce the topology on $\mathrm{Mod}(V)$. We say that R_j converges to R if for each l we have a convergence $B(p^l)\cap R_j \to B(p^l)\cap R$ in the sense of Hausdorff metric. Notice that this convergence is metrized, a (non-canonical) metric is given by

$$d(L,M)=\sum_{j=1}^{\infty}(2p)^{-j}h\bigl(L\cap B(p^l), M\cap B(p^l)\bigr).$$

Lemma 3.1 a) *The space* $\mathrm{Mod}(V)$ *is compact with respect to the Hausdorff topology.*

b) *The space* $\mathrm{Lat}(V)$ *is a discrete dense subset in* $\mathrm{Mod}(V)$.

Let us prove a). Choose a convergent subsequence from arbitrary sequence of submodules L_j. First, we choose a subsequence L_{j_k} such that $L_{j_k}\cap B(p^0)$ converges. From the latter sequence we choose a subsequence such that intersections with $B(p^1)$ converges. Etc.

3.4. Analog of the radial limit. We need an analog of the radial limit (1.8). Say that a sequence R_j of submodules *rigidly converges* to a submodule R (notation $R_j \nearrow R$) if

(A) for any compact subset $S\subset R$ we have $S\subset R_j$ starting some place.

(B) for each $\varepsilon>0$, for sufficiently large j the set R_j is contained in the ε-neighborhood of R.

Example. Let $V=\mathbb{Q}_p^2$. Let $R_j=p^{-k}\mathbb{Z}_p e_1\oplus p^k\mathbb{Z}_p e_2$. Then R_j rigidly converges to a line $\mathbb{Q}_p e_1$. Now let

$$S_j=\mathbb{Z}_p(p^{-k}e_1+e_2)\oplus p^k\mathbb{Z}_p e_2.$$

Then S_j converges to the line $\mathbb{Q}_p e_1$ in Hausdorff sense but not rigidly. ⊠

Evidently, we can reformulate the condition **(A)** as

$$\eta_R(R_j)\to 0.$$

Lemma 3.2 *The condition* **(B)** *is equivalent to*

$$\eta_{R^\lozenge}(R_j^\lozenge)\to 0.$$

Proof. Let us equip V' by the dual norm. Let S, $S_j\in V'$ and $\eta_S(S_j)\to 0$. For small $\varepsilon>0$ we have

$$S_j\subset S+B(\varepsilon). \tag{3.2}$$

Passing to the duals, we get

$$S_j^\lozenge \supset S^\lozenge\cap B(\varepsilon^{-1}). \tag{3.3}$$

But $S^\lozenge\cap B(\varepsilon^{-1})$ is an exhausting sequence of compact subsets in $S^\lozenge$. Also, (3.3) implies (3.2). □

Lemma 3.3 *If $R_j \nearrow R$, then $(R_j)_\downarrow \subset R_\downarrow$ and $(R_j)^\uparrow \supset R^\uparrow$ starting some j.*

PROOF. The first claim. For sufficiently large k we have $R \subset R_\downarrow + B(p^k)$, also $B(p^k) + B(\varepsilon) = B(p^k)$ for $\varepsilon \leqslant p^k$. Therefore for a large j we have

$$R_\downarrow + B(p^k) \supset R_j \supset (R_j)_\downarrow$$

But a subspace, which is contained in a tube neighborhood of a subspace $R_\downarrow$, is contained in $R_\downarrow$.

The second claim. We consider a compact subset $K \subset R$ generating $R^\uparrow$ as a $\mathbb{Q}_p$-subspace. Then $(R_j)^\uparrow$ contains K for sufficiently large j and therefore $(R_j)^\uparrow \supset R^\uparrow$. □

In particular, a *$\nearrow$-convergent sequence of linear subspaces is eventually constant.*

Lemma 3.4 a) *Let $L \subset V$ be a linear subspace. If $R_j \nearrow R$, then $(L \cap R_j) \nearrow (L \cap R)$.*

b) *Let $M \subset V$ be a linear subspace, denote by π the natural map $V \to V/M$. If $R_j \nearrow R$ then $\pi(R_j) \nearrow \pi(R)$.*

PROOF. a) Only condition **(B)** requires a proof, i.e., for each $\varepsilon > 0$ there exists N such that for $j \geqslant N$

$$R_j \cap L \subset (R \cap L) + B(\varepsilon).$$

It is easy to shown that there is a basis e_m in $\mathbb{Q}_p^n$ such that R has canonical form (3.1) and L is a linear span of several basis elements. Then for sufficiently big N we have

$$(R + p^N \oplus \mathbb{Z}_p e_j) \cap L \;\subset\; (R \cap L) + p^N \oplus \mathbb{Z}_p e_j.$$

Passing from the basis e_m to the standard basis in $\mathbb{Q}_p^n$ we get

$$(R + B(\delta)) \cap L \;\subset\; (R \cap L) + B(C\delta).,$$

where $C = C(R, L)$ is a constant. Now we take $\delta = \varepsilon/C$ and choose number k, starting which $R_j \subset R + B(\delta)$.

b) follows from a) by the duality. □

REMARK. $\nearrow$-convergence is not metrizable. ⊠

B. Bruhat–Tits buildings

3.5. Self-dual modules. For details, see [9], Sections 10.6–10.7. Consider a p-adic linear space $V \simeq \mathbb{Q}_p^{2n}$ equipped with a nondegenerate skew-symmetric bilinear form $B_V(\cdot,\cdot)$ (as above). We say that a subspace L is *isotropic* if $B_V(v, w) = 0$ for all v, $w \in V$. By $\mathrm{LGr}(V)$ we denote the set of all maximal isotropic (*Lagrangian*) subspaces in V (their dimensions $= n$).

By $L^\perp$ we denote the ortho-dual of a subspace L, i.e set of all vectors w such that $B_V(v,w)=0$ for all $v\in L$.

If P is a submodule, denote by $P^{\perp\!\!\perp}$ the *dual submodule*, i.e., the set of vectors w such that $B(v,w)\in\mathbb{Z}_p$ for all $v\in P$. If P is a subspace, then $P^{\perp\!\!\perp}=P^\perp$.

We say that a submodule $R\subset V$ is *isotropic* if $B_V(v,w)\in\mathbb{Z}_p$ for all v, $w\in R$.

Example. If R is a linear subspace, then R is isotropic in the usual sense. On the other hand, the lattice $\mathbb{Z}_p^{2n}$ is isotropic (and self-dual, see below). ⊠

We say that a submodule R is *self-dual* if it is a maximal isotropic submodule in V. Equivalently, $P^{\perp\!\!\perp}=P$. Denote by $\mathrm{LMod}(V)$ the set of all self-dual submodules in V, by $\mathrm{LLat}(V)$ the set of all self-dual lattices. It is easy to show that $\mathrm{Sp}(2n,\mathbb{Q}_p)$ acts on $\mathrm{LLat}(V)$ transitively and

$$\mathrm{LLat}(V)=\mathrm{Sp}(2n,\mathbb{Q}_p)/\mathrm{Sp}(2n,\mathbb{Z}_p).$$

Lemma 3.5 a) *For any self-dual submodule R the subspace $R_\downarrow$ is isotropic, and $R^\uparrow$ is the ortho-dual of $R_\downarrow$.*

b) *Let L ranges in the set of isotropic subspaces. Denote by $\pi: L^\perp\to L^\perp/L$ the natural projection map. Any self-dual submodule R has the form $\pi^{-1}S$, where S is a self-dual lattice in $L^\perp/L$.*

c) *The unique* $\mathrm{Sp}(V)$*-invariant of a self-dual module R is* $\dim R_\downarrow$.

These statement is obvious.

Sometimes it is convenient to reformulate a definition of an isotropic module. Define a bicharacter β on $V\times V$ by

$$\beta(x,y)=\exp\bigl\{2\pi i B(x,y)\bigr\}. \tag{3.4}$$

We say that a module P is *isotropic* if $\beta(x,y)=1$ on $P\times P$.

3.6. Almost self-dual modules. Let V and B be same as above. A submodule L in V is *almost self-dual* if it contains a self-dual module M and $B(v,w)\in p^{-1}\mathbb{Z}_p$ for all v, $w\in L$ (see, e.g., [8], Section 10.6). Notice that $L/M\simeq(\mathbb{Z}/p\mathbb{Z})^k$ with $k=0,1,\dots,n$. .

Lemma 3.6 a) *Any almost self-dual module can be reduced by a symplectic transformation to the form*

$$\begin{split}(p^{-1}\mathbb{Z}_p e_1\oplus\mathbb{Z}_p e_{n+1})\oplus\dots\oplus(p^{-1}\mathbb{Z}_p e_k\oplus\mathbb{Z}_p e_{n+k})\oplus\\ \oplus(\mathbb{Z}_p e_{k+1}\oplus\mathbb{Z}_p e_{n+k+1})\oplus\dots\oplus(\mathbb{Z}_p e_m\oplus\mathbb{Z}_p e_{n+m})\oplus\\ \oplus\mathbb{Q}_p e_{m+1}\oplus\dots\oplus\mathbb{Q}_p e_n.\end{split} \tag{3.5}$$

b) *The only* $\mathrm{Sp}(V)$*-invariants of an almost self-adjoint module R are* $\dim R_\downarrow$ *and the number k (rank of an Abelian group R/S, where S is a self-dual submodule in R. For almost self-dual lattices the only* $\mathrm{Sp}(V)$*-invariant is the volume of R, it is equal p^{-k}.*

3.7. Graph $\Delta(V)$. Consider a p-adic linear space V equipped with a nondegenerate skew-symmetric bilinear form B as above. We draw an oriented graph $\Delta(V)$. Vertices are almost self-dual modules in V. If $R \supset R'$, then we draw an arrow from R to R'.

If R, R' are connected by an arrow, then $R_{\downarrow} = (R')_{\downarrow}$ and $R^{\uparrow} = (R')^{\uparrow}$.

Any pair of lattices can be connected by a (non-oriented) way. Denote the subgraph whose vertices are all lattices by $\Delta_0(V)$.

More generally, fix an isotropic subspace L and consider the subgraph $\Delta_L(V)$ whose vertices are almost self-dual modules R such that $R_{\downarrow} = L$, $R^{\uparrow} = L^{\perp}$. We get a connected subgraph, moreover

$$\Delta_L(V) \simeq \Delta_0(L^{\perp}/L).$$

By definition,

$$\Delta(V) = \coprod_{L \text{ is isotropic subspace}} \Delta_L(V).$$

If $L \subset M$, then Δ_M is contained in the closure of Δ_L in the sense of $\nearrow$-convergence.

3.8. Bruhat–Tits buildings, for details, see [42], [8]. Now we consider all k-plets of vertices of $\Delta(V)$ that are pairwise connected by edges. For any such k-plet we draw a $(k-1)$-simplex with given vertices and edges. Faces of a simplex correspond to subsets of the k-plet. Thus we get a simplicial complex, denote it by $\mathrm{Bd}(V)$.

Consider the subgraph Δ_0. It can be shown that $k \leqslant n+1$ and each simplex is contained in an n-dimensional simplex. In this way we get a structure of an n-dimensional simplicial complex, it is called a *Bruhat–Tits building.* We denote it by $\mathrm{Bd}_0(V)$.

For a subgraph Δ_L we get a simplicial complex complex $\mathrm{Bd}_L(V)$ isomorphic $\mathrm{Bd}(L^{\perp}/L)$.

Below we use term '*distinguished boundary of a building*' for the Lagrangian Grassmannian, this is an counterpart of Shilov boundary.

C. Relations and Nazarov category

3.9. Relations. Let V, W be linear spaces. We say that a *relation* $P : V \rightrightarrows W$ is a submodule in $V \oplus W$.

Example. Let $A : V \to W$ be a linear operator. Then its graph is a relation. ⊠

Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be relations. We define their product $S = QP : V \rightrightarrows Y$ as the set of all $v \oplus y \in V \oplus Y$ for which there exists $w \in W$ such that $v \oplus w \in P$, $w \oplus y \in Q$.

For a relation $P : V \rightrightarrows W$ we define its *kernel* $\ker P \subset V$ as

$$\ker P = P \cap (V \oplus 0),$$

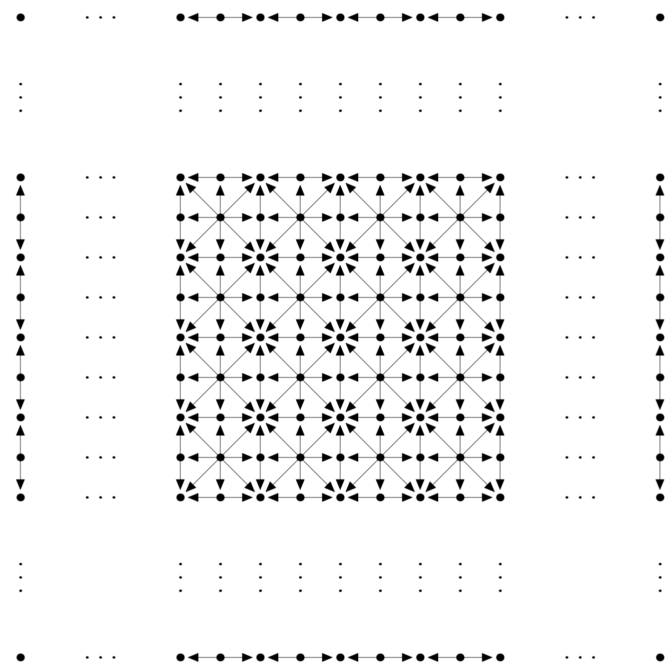

Figure 1: A reference to Subsections 3.4, 3.6. A subcomplex ('apartment') of the building $\mathrm{Bd}(\mathbb{Q}_p^4)$ corresponding to lattices of the form $R_1 \oplus \cdots \oplus R_4$, where R_j is a submodule in the line $\mathbb{Q}_p e_j$.
1) Vertices of the central piece of the subcomplex are almost self-dual lattices of the form $L = p^{k_1}\mathbb{Z}_p e_1 \oplus p^{k_2}\mathbb{Z}_p e_2 \oplus p^{l_1}\mathbb{Z}_p e_3 \oplus p^{l_2}\mathbb{Z}_p e_4$. They are almost self-dual iff $k_1 + l_1$, $k_2 + l_2$ are 0 or -1.
2) Four boundary pieces. Each piece corresponds to almost self-dual submodules containing a line $\mathbb{Q}_p e_j$, where $j = 1, 2, 3, 4$. For instance, for $j = 1$ such submodules have a form $M = \mathbb{Q}_p e_1 \oplus p^{m_2}\mathbb{Z}_p e_2 \oplus p^{l_2}\mathbb{Z}_p e_4$, where $m_2 + l_2 = 0$,1. A sequence of lattices $\nearrow$-converges to M only if $k_1 \to -\infty$ and $k_2 = m_2$ starting some place.
3) Four extreme points correspond to Lagrangian planes spanned by pairs of vectors (e_1, e_2), (e_1, e_4), (e_2, e_3), (e_3, e_4). A sequence of lattices $\nearrow$-converges to $\mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_4$ iff $k_1 \to +\infty$, $k_2 \to -\infty$.

the *indefiniteness* $\operatorname{indef} P \subset W$,

$$\operatorname{indef} P = P \cap (0 \oplus W),$$

the *domain of definiteness*

$$\operatorname{dom} P = \text{projection of } P \text{ to } V,$$

and the *image*

$$\operatorname{im} P = \text{projection of } P \text{ to } W.$$

We define the *pseudo-inverse* relation $P^{\square} : W \rightrightarrows V$ being the same submodule in $W \oplus V \simeq V \oplus W$. Evidently,

$$(PQ)^{\square} = Q^{\square} P^{\square}.$$

3.10. The definition of product. A reformulation. We keep the same notation. Consider the space $\mathcal{Z} := V \oplus W \oplus W \oplus Y$ and following submodules of $\mathcal{Z}$:

— the subspace $\mathcal{H}$ consisting of vectors $v \oplus w \oplus w \oplus y$;

— the subspace $\mathcal{A}$ consisting of vectors $0 \oplus w \oplus w \oplus 0$;

— the submodule $P \oplus Q \subset (V \oplus W) \oplus (W \oplus Y)$.

Then we do the following operations:

— take the intersection $R = \mathcal{H} \cap (P \oplus Q)$;

— take the map $\theta : \mathcal{H} \to \mathcal{H}/\mathcal{A} \simeq V \oplus Y$.

Then $QP = \theta(R)$.

3.11. Action on $\operatorname{Mod}(V)$**.** Let $P : V \rightrightarrows W$ be a relation, T be a submodule in V. We define the submodule $PT \subset W$ as the set of $w \in W$ such that there is $v \in T$ satisfying $v \oplus w \in P$.

Remark. We can consider a submodule $T \subset V$ as a relation $0 \rightrightarrows V$. Therefore we can regard $PT : 0 \rightrightarrows W$ as the product of relations $T : 0 \rightrightarrows V$ and $Q : V \rightrightarrows W$. ⊠

3.12. The Nazarov category. For a pair V, W of symplectic linear spaces we define a skew-symmetric bilinear form $B^{\ominus}$ on $V \oplus W$ by

$$B^{\ominus}(v \oplus w, v' \oplus w') = B_V(v, v') - B_W(w, w').$$

Denote by

— $\overline{\operatorname{Naz}}(V, W)$ the set of all self-dual submodules of $V \oplus W$;

— $\operatorname{Naz}(V, W)$ the set of $P \in \overline{\operatorname{Naz}}(V, W)$ such that $\ker P$ and $\operatorname{indef} P$ are compact.

Theorem 3.7 *Let $P \in \overline{\operatorname{Naz}}(V, W)$, let T be a self-dual submodule in V. Then the submodule $PT \subset W$ is self-dual.*

In [9], Theorem 10.7.2, the same statement is established under slightly stronger condition $P \in \mathrm{Naz}(V, W)$. In fact, a proof remains valid for $P \in \overline{\mathrm{Naz}}(V, W)$.

Theorem 3.8 a) *If $P \in \mathrm{Naz}(V, W)$, $Q \in \mathrm{Naz}(W, Y)$, then $QP \in \mathrm{Naz}(V, Y)$.*

b) *If $P \in \overline{\mathrm{Naz}}(V, W)$, $Q \in \overline{\mathrm{Naz}}(W, Y)$, then $QP \in \overline{\mathrm{Naz}}(V, Y)$.*

c) *If $P \in \mathrm{Naz}(V, W)$, $Q \in \mathrm{Naz}(W, Y)$ are lattices, then QP is a lattice.*

The statement a) was proved in Nazarov [33] (see also [9], Section 10.7), c) is obvious. The statement b) is a corollary of Theorem 3.7, see [9], Subsection 10.7.4.

Thus we get two similar categories[14], Naz and $\overline{\mathrm{Naz}}$. *The group of automorphisms of an object V is the symplectic group* $\mathrm{Sp}(V, \mathbb{Q}_p)$ (for both categories), an operator $V \to V$ is symplectic iff its graph is isotropic with respect to the form $B^\ominus$.

For $P \in \overline{\mathrm{Naz}}(V, W)$, we have

$$(\ker P)^{\perp\!\!\perp} = \operatorname{dom} P, \qquad (\operatorname{indef} P)^{\perp\!\!\perp} = \operatorname{im} P$$

$$\big((\ker P)_\downarrow\big)^{\perp} = (\operatorname{dom} P)^{\uparrow}, \qquad \big((\operatorname{indef} P)_\downarrow\big)^{\perp\!\!\perp} = (\operatorname{im} P)^{\uparrow},$$

3.13. Action of the Nazarov category on buildings.

Proposition 3.9 a) *Let $P \in \mathrm{Naz}(V, W)$, let T be an almost-self-dual lattice. Then $PT \subset W$ is an almost self-dual lattice.*

b) *Let $P \in \overline{\mathrm{Naz}}(V, W)$, let T be an almost-self-dual submodule. Then $PT \subset W$ is an almost self-dual submodule.*

The statement a) is [9], Proposition 10.7.5, a proof remains to be valid for the statement b) also.

Now, let Ξ, Σ be simplicial complexes, let $\mathrm{Vert}(\Xi)$, $\mathrm{Vert}(\Sigma)$ be their sets of vertices. We say, that a map[15] $\mathrm{Vert}(\Xi) \to \mathrm{Vert}(\Sigma)$ is *simplicial*, if for any simplex $\Delta \subset \Xi$ images of its vertices are are contained in one simplex of Σ. Notice, that we can extend a simplicial map to a map of complexes $\Xi \to \Sigma$ assuming that a map is affine on each face.

The following statement is a corollary of Proposition 3.9.

Theorem 3.10 a) *A morphism $P \in \mathrm{Naz}(V, W)$ induces simplicial map*

$$\mathrm{Bd}(V) \to \mathrm{Bd}(W)$$

sending

$$\mathrm{Bd}_0(V) \to \mathrm{Bd}_0(W).$$

[14] The Nazarov category is an analog of Krein–Shmulian type categories, see [8], [9]

[15] generally, non-injective.

b) *A morphism* $P \in \overline{\mathrm{Naz}}(V,W)$ *induces a simplicial map* $\mathrm{Bd}(V) \to \mathrm{Bd}(V)$*, sending* $\mathrm{Bd}(V)$ *to*

$$\mathrm{Bd}(V) \to \coprod_{\substack{M \text{ is isotropic subspace in } W \\ M \supset \mathrm{indef}(P)_{\downarrow}}} \mathrm{Bd}[M^{\perp}/M]. \tag{3.6}$$

REMARK. The map $T \to PT$ is contractive in an essentially stronger sense, see [43]. ⊠

Theorem 3.11 *Let* $P \in \overline{\mathrm{Naz}}(V,W)$*. The the induced map* $\mathrm{Bd}(V) \to \mathrm{Bd}(W)$ *is* $\nearrow$*-continuous, i.e., for a convergent sequence* $T_j \nearrow T$ *of almost self-dual modules, we have* $PT_j \nearrow PT$*.*

PROOF. We evaluate PT_j according procedure described in Subsection 3.10. By Lemma 3.4, both steps of the evaluation are continuous. □

D. Weil representation

The Weil representation is used below only in Section 7.

3.14. Extended Nazarov category. Now we add to the Nazarov category an infinite-dimensional object $V_{2\infty}$. This is the space of vectors

$$(x_1^+, x_2^+, \dots, x_1^-, x_2^-, \dots), \qquad \text{where } x_j^{\pm} \in \mathbb{Q}_p \text{ and } x_j^{\pm} \in \mathbb{Z}_p \text{ for almost all } j.$$

Notice that $V_{2\infty}$ is not a $\mathbb{Q}_p$-linear space but is a $\mathbb{Z}_p$-module.

We introduce a bicharacter $\beta(\cdot,\cdot)$ on $V_{2\infty} \oplus V_{2\infty}$ by

$$\beta(x,y) = \exp\Bigl[2\pi i \sum_{j=1}^{\infty} (x_j^+ y_j^- - x_j^- y_j^+)\Bigr] := \prod_{j=1}^{\infty} \exp\bigl\{2\pi i (x_j^+ y_j^- - x_j^- y_j^+)\bigr\}.$$

Notice that almost all factors of the product equal to 1. The sum in square brackets defining a symplectic form is not well defined, more precisely it is well defined modulo $\mathbb{Z}_p$.

Objects of the *extended Nazarov category* **Naz** are

— finite-dimensional spaces V equipped with skew-symmetric non-degenerate bilinear forms B_V and with the corresponding bicharacters β_V, see (3.4);

— the space $V_{2\infty}$.

Let V, W be two objects. We equip their direct sum with a bicharacter

$$\beta_{V\oplus W}(v \oplus w, v' \oplus w') = \frac{\beta_V(v,v')}{\beta_W(w,w')}.$$

A *morphism of the category* **Naz** is a self-dual submodule $P \subset V \oplus W$ such that $\ker P$ and $\mathrm{indef}\, P$ are compact.

Group $\mathbf{Sp}(2\infty, \mathbb{Q}_p)$ of automorphisms of $V_{2\infty}$ consists of $2\infty \times 2\infty$ matrices $r = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

— all but a finite number of matrix elements are integer;

— matrix elements a_{ij}, b_{ij}, c_{ij}, d_{ij} tend to 0 as $i \to \infty$ for fixed j; also they tend to 0 as $j \to \infty$ for fixed i;

— matrices r are symplectic in the usual sense,

$$r^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} r^t.$$

3.15. Heisenberg groups. For the sake of simplicity, set $p > 2$. Denote by $\mathbb{T}_p \subset \mathbb{C}^\times$ the group of complex roots of unity of degrees p, p^2, p^3,.... Let V be an object of the extended Nazarov category. We define the Heisenberg group $\mathrm{Heis}(V)$ as a central extension of the Abelian group V by $\mathbb{T}_p$ in the following way. As a set, $\mathrm{Heis}(V) \simeq V \times \mathbb{T}_p$. The multiplication is given by

$$(v, \lambda) \cdot (w, \mu) = \bigl(v + w, \lambda\mu \cdot \beta_V(v, w)\bigr).$$

Decompose $V = V_+ \oplus V_-$ as in (2.1). For a finite dimensional V we define a unitary representation Ψ of $\mathrm{Heis}(V)$ in $L^2(\mathbb{Q}_p^n)$ by the formula

$$\Psi(v^+ \oplus v^-, \lambda) f(x) = \lambda f(x + v^+) \exp\Bigl\{2\pi i\Bigl(\sum v_j^+ x_j + \frac{1}{2} \sum v_j^+ v_j^-\Bigr)\Bigr\}. \quad (3.7)$$

Next, consider the space $\mathcal{E}_\infty$ consisting of sequences $z = (z_1, z_2, \dots)$ such that $|z_j| \leqslant 1$ for all but a finite number of j. This space is an Abelian locally compact group, it admits a Haar measure. On the open subgroup $\mathbb{Z}_p^\infty \subset \mathcal{E}_\infty$, the Haar measure is a product of probability Haar measures on $\mathbb{Z}_p$. The whole space $\mathcal{E}_\infty$ is a countable disjoint union of sets $u + \mathbb{Z}_p^\infty$.

We define the representation of the group $\mathrm{Heis}(V_{2\infty})$ in $L^2(\mathcal{E}_\infty)$ by the same formula (3.7).

3.16. The Weil representation of the Nazarov category. Formal definition. See [33], [34], for finite-dimensional case, see [9], Chapter 11.

Theorem 3.12 *For a $2n$-dimensional object of the category* **Naz** *we assign the Hilbert space $\mathcal{H}(V) := L^2(\mathbb{Q}_p^n)$. For the object $V_{2\infty}$, we assign the Hilbert space $\mathcal{H}(V_{2\infty}) := L^2(\mathcal{E}_\infty)$.*

a) *Let V, W be objects of* **Naz**. *Let $P : V \rightrightarrows W$ be a morphism of category* **Naz**. *Then there is a unique up to a scalar factor bounded operator*

$$\mathrm{We}(P) : \mathcal{H}(V) \to \mathcal{H}(W)$$

such that

$$\Psi(w, 1)\mathrm{We}(P) = \mathrm{We}(P)\Psi(v, 1) \quad \textit{for all } v \oplus w \in P.$$

b) *Let V, W, Y be objects of* **Naz**. *Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be morphisms of* **Naz**. *Then*

$$\mathrm{We}(Q)\mathrm{We}(P) = s \cdot \mathrm{We}(QP),$$

where $s = s(P, Q) \in \mathbb{C}^\times$ is a nonzero scalar. In other words, we get a projective representation *of the category* **Naz**. *Also,*

$$\mathrm{We}(P^\square) = t \cdot \mathrm{We}(P)^*, \qquad t \in \mathbb{C}^\times.$$

For symplectic groups $\mathrm{Sp}(2n, \mathbb{Q}_p) = \mathrm{Aut}(\mathbb{Q}_p^{2n})$ the representation $\mathrm{We}(g)$ coincides with the Weil representation.

3.17. Explicit formulas for operators for some morphisms.

1) Let $V = W$ and P be a graph of a symplectic operator. There are simple formulas for some special symplectic matrices:

$$\mathrm{We}\begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix} f(z) = |\det A|^{1/2} f(zA); \tag{3.8}$$

$$\mathrm{We}\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(z) = \exp\{\pi i z B z^t\};$$

$$\mathrm{We}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(z) = \int_{\mathbb{Q}_p^n} f(x) \exp\{2\pi i x z^t\}\, dx.$$

Any element of $\mathrm{Sp}(2n, \mathbb{Q}_p)$ can be represented as a product of matrices of such forms, this allows to write an explicit formula for $\mathrm{We}(g)$ for any element $g \in \mathrm{Sp}(2n, \mathbb{Q}_p)$.

Denote by $I(x)$ the function on $\mathbb{Q}_p$ defined by

$$I(x) = \begin{cases} 1, & |x| \leqslant 1; \\ 0, & \text{otherwise}. \end{cases}$$

Next, we need some special non-invertible morphisms.

2) Let $V = \mathbb{Q}_p^{2n}$, $W = V \oplus Y$, where $Y = \mathbb{Q}_p^{2n}$ or $V_{2\infty}$. Denote by $Y(\mathbb{Z}_p)$ the lattice $\mathbb{Z}_p^{2n}$ or $\mathbb{Z}_p^{2\infty}$ respectively. Denote by

$$\lambda_W^V : V \rightrightarrows W$$

the direct sum of the graph $\mathrm{graph}(1_V)$ of the unit operator $1_V : V \to V$ and the lattice $Y(\mathbb{Z}_p) \subset Y$. Then

$$\mathrm{We}(\lambda_W^V)\, f(v_1, \dots, v_n, y_1, y_2, \dots) = f(v_1, \dots, v_n)\, I(y_1) I(y_2) \dots$$

3) Preserving the previous notation denote by

$$\theta_W^V : W \rightrightarrows W$$

the direct sum

$$\mathrm{graph}(1_V) \oplus (Y(\mathbb{Z}_p) \oplus Y(\mathbb{Z}_p)) \subset (V \oplus V) \oplus (Y \oplus Y).$$

Then

$$\theta_W^V = \lambda_W^V \left(\lambda_W^V\right)^*, \qquad \left(\theta_W^V\right)^2 = \theta_W^V, \qquad \left(\lambda_W^V\right)^* \lambda_W^V = 1_V. \tag{3.9}$$

The operator $\mathrm{We}(\theta_W^V)$ is the orthogonal projection to the space of functions of the form

$$f(v_1, \dots, v_n)\, I(y_1) I(y_2) \dots$$

3.18. General case. Any morphism of the category **Naz** can be represented as a product of morphisms of the types described above. Moreover, for finite dimensional V, W, any $P : V \rightrightarrows W$ can be represented as

$$P = (\lambda_Z^W)^* \cdot g \cdot \lambda_Z^V, \qquad g \in \mathrm{Sp}(Z),$$

where Z is sufficiently large ($\dim Z \geqslant 2 \max(\dim V, \dim W)$). In fact, the same decomposition holds for morphisms $Q : V_{2\infty} \to V_{2\infty}$, any Q can be represented as

$$Q = \theta^{V_{2\infty}}_{V_{2\infty} \oplus V_{2\infty}} \cdot g \cdot \theta^{V_{2\infty}}_{V_{2\infty} \oplus V_{2\infty}}, \qquad g \in \mathrm{Sp}(V_{2\infty} \oplus V_{2\infty}).$$

4 Characteristic function

Here we define characteristic functions of double cosets $\mathbf{K} \setminus \mathbf{G} / \mathbf{K}$ and formulate several theorems. Proofs are in the next section.

4.1. Construction. Consider the group

$$\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p) := \lim_{j \to \infty} \mathrm{GL}(\alpha + kj, \mathbb{Q}_p).$$

Let $g \in \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ actually be contained in $\mathrm{GL}(\alpha + km, \mathbb{Q}_p)$,

$$g = \begin{pmatrix} a & b_1 & \dots & b_k \\ c_1 & d_{11} & \dots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{k1} & \dots & d_{kk} \end{pmatrix} \in \mathrm{GL}(\alpha + km, \mathbb{Q}_p). \tag{4.1}$$

We write the following equation (this is an analog of (1.5), the analogy is important)

$$\begin{pmatrix} v^+ \\ y_1^+ \\ \vdots \\ y_k^+ \\ v^- \\ y_1^- \\ \vdots \\ y_k^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b_1 & \dots & b_k \\ c_1 & d_{11} & \dots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{k1} & \dots & d_{kk} \end{matrix} & \begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{matrix} & \begin{pmatrix} a & b_1 & \dots & b_k \\ c_1 & d_{11} & \dots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{k1} & \dots & d_{kk} \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ \vdots \\ x_k^+ \\ u^- \\ x_1^- \\ \vdots \\ x_k^- \end{pmatrix}. \tag{4.2}$$

Here $u^\pm$, $v^\pm \in \mathbb{Q}_p^\alpha$ and $x_j^\pm$, $y_j^\pm \in \mathbb{Q}_p^m$.

Before the exploring of this identity as (1.5), we need some preparations.

Define 3 spaces, $\mathcal{V}$, $\mathcal{H}$, ℓ_m:

1) Denote $\mathcal{V} := \mathbb{Q}_p^\alpha \oplus \mathbb{Q}_p^\alpha$. We regard $u = u^+ \oplus u^-$, $v = v^+ \oplus v^-$ as elements of $\mathcal{V}$. Equip $\mathcal{V}$ with the standard skew-symmetric bilinear form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2) Denote

$$\mathcal{H} := \mathcal{H}^+ \oplus \mathcal{H}^- = \mathbb{Q}_p^k \oplus \mathbb{Q}_p^k \tag{4.3}$$

and equip this space with the standard skew-symmetric bilinear form.

3) Denote by ℓ_m the space $\mathbb{Q}_p^m$ equipped with the standard symmetric bilinear form

$$(z, w) = \sum z_j w_j.$$

We regard $x_j^\pm$, $y_j^\pm$ as elements of this space.

Consider the tensor product $\mathcal{H} \otimes_{\mathbb{Q}_p} \ell_m$, vectors

$$\begin{pmatrix} x_1^+ & \dots & x_k^+ & x_1^- & \dots & x_k^- \end{pmatrix}, \quad \begin{pmatrix} y_1^+ & \dots & y_k^+ & y_1^- & \dots & y_k^- \end{pmatrix}$$

are regarded as elements of $\mathcal{H} \otimes \ell_m$. We equip $\mathcal{H} \otimes \ell_m$ with the tensor product of bilinear forms, this form is a skew-symmetric with matrix[16]

$$\begin{pmatrix} 0 & \dots & 0 & 1_m & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1_m \\ -1_m & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & -1_m & 0 & \dots & 0 \end{pmatrix}.$$

Thus the operator in (4.2) is an operator

$$\mathcal{V} \oplus (\mathcal{H} \otimes \ell_m) \quad \to \quad \mathcal{V} \oplus (\mathcal{H} \otimes \ell_m)$$

We equip the spaces $\mathcal{V} \oplus (\mathcal{H} \otimes \ell_m)$ with a skew-symmetric bilinear form that is a direct sum of forms in $\mathcal{V}$ and $\mathcal{H} \otimes \ell_m$. The matrix of this form is

$$\begin{pmatrix} 0 & 1_\alpha & 0 & 0 \\ -1_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{km} \\ 0 & 0 & -1_{km} & 0 \end{pmatrix}$$

Evidently, *operators* (4.2) *preserve this form*, i.e., they are contained in $\mathrm{Sp}\big(2(\alpha + km), \mathbb{Q}_p\big)$.

Now we start a description of characteristic functions.

[16] A tensor product of a symmetric and a skew-symmetric bilinear forms is a skew-symmetric bilinear form.

For any self-dual module $Q \subset \mathcal{H}$ we consider the self-dual module

$$Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m \subset \mathcal{H} \otimes \ell_m.$$

Notice, that $Q \otimes \mathbb{Z}_p^m$ is a direct sum of m copies of Q.

Definition 4.1 *Fix g. Fix self-dual submodules Q, $T \subset \mathcal{H}$. We define a relation*

$$\chi_g(Q,T) : \mathcal{V} \rightrightarrows \mathcal{V}$$

as the set of all $u \oplus v \in \mathcal{V} \oplus \mathcal{V}$ for which there exist $x \in Q \otimes \mathbb{Z}_p^m$, $y \in T \otimes \mathbb{Z}_p^m$ such that (4.2) *holds.*

4.2. An auxiliary definition.

Definition 4.2 *We say that some property of a double coset* holds in a general position *if for any sufficiently large m the set of points $g \in \mathrm{GL}(\alpha + km, \mathbb{Q}_p)$, where the property does not hold, is a proper algebraic subvariety in* $\mathrm{GL}(\alpha + km, \mathbb{Q}_p)$.

4.3. Basic properties of characteristic functions.

Lemma 4.3 *$\chi_g(Q,T)$ does not depend on a choice of m.*

Theorem 4.4 *If g_1, g_2 are contained in the same double coset $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$, then $\chi_{g_1}(Q,T) = \chi_{g_2}(Q,T)$.*

Thus, for any double coset $\mathfrak{g} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ we get a well-defined map

$$\chi_{\mathfrak{g}} : \ \mathrm{LMod}(\mathcal{H}) \times \mathrm{LMod}(\mathcal{H}) \quad \to \quad \Bigl\{\text{space of relations } \mathcal{V} \rightrightarrows \mathcal{V}\Bigr\}.$$

Therefore, we can write

$$\chi_{\mathfrak{g}}(Q,T), \qquad \text{where } \mathfrak{g} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}.$$

We say that $\chi_{\mathfrak{g}}(\cdot,\cdot)$ is the *characteristic function* of the double coset $\mathfrak{g}$.

Theorem 4.5 *$\chi_{\mathfrak{g}}(Q,T) \in \overline{\mathrm{Naz}}(\mathcal{V},\mathcal{V})$.*

Theorem 4.6 *The following identity holds*

$$\chi_{\mathfrak{g}\star\mathfrak{h}}(Q,T) = \chi_{\mathfrak{g}}(Q,T)\, \chi_{\mathfrak{h}}(Q,T),$$

in the right-hand side we have a product of relations

4.4. Refinement of Theorem 4.5. Fix a double coset $\mathfrak{g}$. Substituting $x^{\pm} = 0$, $y^{\pm} = 0$ to the equation (4.2), we get an equation for $u \oplus v \in \mathcal{V} \oplus \mathcal{V}$. The explicit form (see equation (5.3)) is

$$\begin{cases} v^+ = au^+ \\ 0 \ \ = c_j u^+, & \text{for all } j \\ u^- = a^t v^- \\ 0 \ \ = b_j^t v^-, & \text{for all } j \end{cases} \tag{4.4}$$

Denote by $\Lambda(\mathfrak{g}) \subset \mathcal{V} \oplus \mathcal{V}$ the linear subspace of solutions of this system.

Notice that

$$\ker \Lambda(\mathfrak{g}) = 0, \qquad \operatorname{indef} \Lambda(\mathfrak{g}) = 0$$

(since g is an invertible matrix).

For $\mathfrak{g}$ being in a general position $\Lambda(\mathfrak{g}) = 0$.

Proposition 4.7 a) *For any self-dual Q, $T \in \mathrm{LMod}(\mathcal{H})$,*

$$\chi_{\mathfrak{g}}(Q,T)_{\downarrow} \supset \Lambda(\mathfrak{g}), \qquad \chi_{\mathfrak{g}}(Q,T)^{\uparrow} \subset \Lambda(\mathfrak{g})^{\perp}.$$

b) *If Q, T are self-dual lattices, then*

$$\chi_{\mathfrak{g}}(Q,T)_{\downarrow} = \Lambda(\mathfrak{g}), \qquad \chi_{\mathfrak{g}}(Q,T)^{\uparrow} = \Lambda(\mathfrak{g})^{\perp}.$$

Corollary 4.8 *For $\mathfrak{g}$ being in a general position, we get a map*

$$\mathrm{LLat}(\mathcal{H}) \times \mathrm{LLat}(\mathcal{H}) \ \to \ \mathrm{LLat}(\mathcal{V} \oplus \mathcal{V}).$$

4.5. Values of characteristic functions on the distinguished boundary.

Theorem 4.9 *Let Q, T range in the Lagrangian Grassmannian $\mathrm{LGr}(\mathcal{H})$. Then*

a) *$\chi_{\mathfrak{g}}(Q,T)$ is a Lagrangian subspace in $\mathcal{V} \oplus \mathcal{V}$.*

b) *The map*

$$\chi_{\mathfrak{g}} : \mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H}) \to \mathrm{LGr}(\mathcal{V} \oplus \mathcal{V})$$

is rational.

c) *For $\mathfrak{g}$ being in a general position, $\chi_{\mathfrak{g}}(Q,T) \in \mathrm{Sp}(\mathcal{V}, \mathbb{Q}_p)$ a.s. on $\mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H})$.*

A precise description of the subset of $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$, where the last property holds, is given below in Subsection 5.9.

There is a more exotic statement in the same spirit.

Proposition 4.10 *For all $\mathfrak{g}$ for almost all $(Q,T) \in \mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H})$, the condition $(u^+ \oplus u^-) \oplus (v^+ \oplus v^-) \in \chi_{\mathfrak{g}}(Q,T)$ can be written as an equation*

$$\begin{pmatrix} v^+ \\ u^- \end{pmatrix} = Z(Q,T) \begin{pmatrix} v^- \\ u^+ \end{pmatrix}$$

there $Z(Q,T)$ is a symmetric matrix.

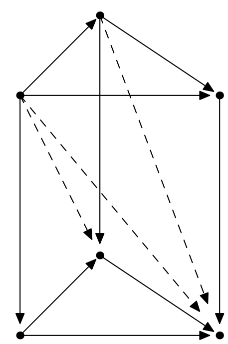

Figure 2: A reference to Subsection 4.6. A product of two simplices and additional arrows.

Point out that this can done *for all* $\mathfrak{g}$.

Proposition 4.11 *Let*

$$\mathfrak{g}_1,\ \mathfrak{g}_2 \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p) \setminus \mathbf{G}/\mathrm{O}(\infty, \mathbb{Z}_p)$$

be contained in the same double coset

$$\mathrm{O}(\infty, \mathbb{Q}_p) \setminus \mathbf{G}/\mathrm{O}(\infty, \mathbb{Q}_p),$$

then the restrictions of $\chi_{\mathfrak{g}_1}$ *and* $\chi_{\mathfrak{g}_2}$ *to* $\mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H})$ *coincide.*

4.6. Extension of characteristic function to buildings. Next, consider two almost self-dual submodules Q, T and apply to them the definition of characteristic function Q, T.

Proposition 4.12 *If* Q*,* T *are almost self-dual modules, then* $\chi_{\mathfrak{g}}(Q, T)$ *is almost self-dual.*

Now we construct an oriented graph $\Delta(\mathcal{H} \bowtie \mathcal{H})$. Vertices are ordered pairs (Q, T) of almost self-dual submodules in $\mathcal{H}$. We draw an arrow from (Q, T) to (Q', T') if $Q \supset Q'$, $T \supset T'$.

Consider the product of simplicial complexes $\mathrm{Bd}(\mathcal{H}) \times \mathrm{Bd}(\mathcal{H})$. It is polyhedral complex, whose cells are products of simplices. Two vertices (of this complex) (Q, T) and (Q', T') are connected by an arrow if $Q \supset Q'$ and $T = T'$ or $Q = Q'$ and $T \supset T'$. However, our rule from the previous paragraph produces more arrows, this provides a simplicial partition of each product of simplices (see, e.g., [44], Section 3.B). Finally, we get a $2k$-dimensional simplicial complex $\mathrm{Bd}(\mathcal{H} \bowtie \mathcal{H})$ (it also is a subcomplex of the complex $\mathrm{Bd}(\mathcal{H} \oplus \mathcal{H})$).

Let Φ, Ψ be two oriented graphs, assume that number of edges connecting any pair of vertices is $\leqslant 1$. We say that a map $\sigma : \mathrm{Vert}(\Phi) \to \mathrm{Vert}(\Psi)$ is a *morphism of graphs* if for any arrow $a \to b$ in Φ we have $\sigma(a) = \sigma(b)$ or there is an arrow $\sigma(a) \to \sigma(b)$.

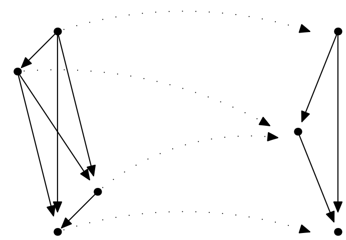

Figure 3: A reference to Subsection 4.6. A morphism of oriented graphs

Theorem 4.13 *A characteristic function $\chi_\mathfrak{g}$ is a morphism of oriented graphs*

$$\Delta(\mathcal{H} \bowtie \mathcal{H}) \to \Delta(\mathcal{V} \oplus \mathcal{V}). \tag{4.5}$$

4.7. Continuity.

Theorem 4.14 *Let Q_j, Q, T_j, T be almost self-dual modules. If $Q_j \nearrow Q$, $T_j \nearrow T$, then*

$$\chi_\mathfrak{g}(Q_j, T_j) \nearrow \chi_\mathfrak{g}(Q, T).$$

Notice that characteristic function can be discontinuous with respect to the Hausdorff convergence. Moreover, the restriction of $\chi_\mathfrak{g}$ to $\mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H})$ can be discontinuous in the topology of Grassmannian.

4.8. Involution.

Proposition 4.15 *If $u \oplus v \in \chi_\mathfrak{g}(Q,T)$, then $v \oplus u \in \chi_{\mathfrak{g}^*}(T,Q)$.*

4.9. Additional symmetry. For a nonzero $\lambda \in \mathbb{Q}_p^\times = \mathbb{Q}_p$, we define an operator $M(\lambda)$ in $\mathcal{H}$ given by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, by the same symbol we denote the operator $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ in the space $\mathcal{V}$.

Theorem 4.16

$$\chi_\mathfrak{g}\bigl(M(\lambda)Q, M(\lambda)T\bigr) = M(\lambda^{-1})\chi_\mathfrak{g}(Q,T)M(\lambda).$$

4.10. Remark. Another semigroup of double cosets. Consider the group $\widetilde{\mathbf{G}} = \mathrm{Sp}(2\alpha + 2k\infty, \mathbb{Q}_p)$ of symplectic matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $(\alpha + k\infty) + (\alpha + k\infty)$, $\widetilde{\mathbf{G}} \supset \mathbf{G}$. Consider its subgroup $\mathbf{G} = \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ consisting of matrices $\begin{pmatrix} g & 0 \\ 0 & g^{t-1} \end{pmatrix}$, consider the same $\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p) \subset \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$. Consider the semigroup of double cosets $\mathbf{K} \setminus \widetilde{\mathbf{G}}/\mathbf{K}$, the multiplication is determined as in Theorem 2.1.

We define characteristic function $\chi_{\widetilde{\mathfrak{g}}}(Q,T)$ in the same way, in formula (4.2) instead the matrix $\begin{pmatrix} g & 0 \\ 0 & g^{t-1}\end{pmatrix}$ we write a symplectic matrix $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{Sp}(2\alpha + 2k\infty, \mathbb{Q}_p)$.

Theorem 4.17 *All the statements of this section hold for $\chi_{\widetilde{\mathfrak{g}}}(Q,T)$ except Theorem* 4.16 *and Proposition 4.10*[17].

5 Proofs

5.1. Independence of representatives. To shorten expressions, set $k = 2$. Let $h \in \mathrm{O}(m, \mathbb{Z}_p)$, let $\mathfrak{I}(h)$ be given by (2.2). Then characteristic function of $g\mathfrak{I}(h)$ is determined by

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ y_1^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} a & b_1 h & b_2 h & & 0\quad 0\quad 0 & \\ c_1 & d_{11} h & d_{12} h & & 0\quad 0\quad 0 & \\ c_2 & d_{21} h & d_{22} h & & 0\quad 0\quad 0 & \\ 0 & 0 & 0 & \multicolumn{3}{c}{} \\ 0 & 0 & 0 & & \begin{pmatrix} a & b_1 h & b_2 h \\ c_1 & d_{11} h & d_{12} h \\ c_2 & d_{21} h & d_{22} h \end{pmatrix}^{t-1} & \\ 0 & 0 & 0 & & & \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ u^- \\ x_1^- \\ x_2^- \end{pmatrix}.$$

or

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ y_1^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} a & b_1 & b_2 & & 0\quad 0\quad 0 & \\ c_1 & d_{11} & d_{12} & & 0\quad 0\quad 0 & \\ c_2 & d_{21} & d_{22} & & 0\quad 0\quad 0 & \\ 0 & 0 & 0 & \multicolumn{3}{c}{} \\ 0 & 0 & 0 & & \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix}^{t-1} & \\ 0 & 0 & 0 & & & \end{pmatrix} \begin{pmatrix} u^+ \\ hx_1^+ \\ hx_2^+ \\ u^- \\ hx_1^- \\ hx_2^- \end{pmatrix}.$$

We introduce new variables $\widetilde{x}_1^{\pm} = hx_1^{\pm}$, $\widetilde{x}_2^{\pm} = hx_2^{\pm}$ and come to the equation for χ_g. Notice that modules $Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m$ are invariant with respect to $\mathrm{O}(m, \mathbb{Z}_p)$.

5.2. Proof of Proposition 4.11. Proof is the same, we only take $h \in \mathrm{O}(m, \mathbb{Q}_p)$. If $Q \subset \mathcal{H}$ is a subspace, then $Q \otimes \ell_m = Q \otimes \mathbb{Q}_p^m$ is a subspace, it is $\mathrm{O}(m, \mathbb{Q}_p)$-invariant.

5.3. Reformulation of definition. The equation (4.2) determines a linear subspace in

$$\Big(\mathcal{V} \oplus (\mathcal{H} \otimes \ell_m)\Big) \oplus \Big(\mathcal{V} \oplus (\mathcal{H} \otimes \ell_m)\Big).$$

We regard it as a linear relation

$$\xi : \Big((\mathcal{H} \otimes \ell_m) \oplus (\mathcal{H} \otimes \ell_m)\Big) \rightrightarrows \big(\mathcal{V} \oplus \mathcal{V}\big).$$

Then $\chi_{\mathfrak{g}}$ is the image of the submodule

$$\eta_{Q,T} = (Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m) \oplus (T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m)$$

[17] the system (4.4) also must be modified.

under ξ.

5.4. Immediate corollaries. The relation ξ is a morphism of the category $\overline{\text{Naz}}$. A module $\eta_{Q,T}$ is self-dual. By Theorem 3.7 the module $\xi\,\eta_{Q,T}$ is self-dual. Theorem 4.5 is proved.

The same argument implies Theorem 4.9.a and Proposition 4.12.

Also Lemma 4.3 became obvious.

5.5. Continuity (Theorem 4.14). We refer to Theorem 3.11.

5.6. Products. Proof of Theorem 4.6. To shorten notation, set $k = 2$. Let

$$g = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix} \in \mathrm{GL}(\alpha+2l, \mathbb{Q}_p), \qquad h = \begin{pmatrix} a' & b'_1 & b'_2 \\ c'_1 & d'_{11} & d'_{12} \\ c'_2 & d'_{21} & d'_{22} \end{pmatrix} \in \mathrm{GL}(\alpha+2m, \mathbb{Q}_p).$$

Let $v \oplus w \in \chi_{\mathfrak{g}}(Q,T)$, $u \oplus v \in \chi_{\mathfrak{h}}(Q,T)$. Then there are $x \in Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m$, $y \in T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m$ such that

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ y_1^- \\ y_2^- \end{pmatrix} = \begin{pmatrix} a' & b'_1 & b'_2 & 0 & 0 & 0 \\ c'_1 & d'_{11} & d'_{12} & 0 & 0 & 0 \\ c'_2 & d'_{21} & d'_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{3}{c}{\begin{pmatrix} a' & b'_1 & b'_2 \\ c'_1 & d'_{11} & d'_{12} \\ c'_2 & d'_{21} & d'_{22} \end{pmatrix}^{t-1}} \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ u^- \\ x_1^- \\ x_2^- \end{pmatrix}. \tag{5.1}$$

Also there are $X \in Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^l$, $Y \in T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^l$ such that

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ Y_2^+ \\ w^- \\ Y_1^- \\ Y_2^- \end{pmatrix} = \begin{pmatrix} a & b_1 & b_2 & 0 & 0 & 0 \\ c_1 & d_{11} & d_{12} & 0 & 0 & 0 \\ c_2 & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \multicolumn{3}{c}{\begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & d_{12} \\ c_2 & d_{21} & d_{22} \end{pmatrix}^{t-1}} \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \end{pmatrix} \begin{pmatrix} v^+ \\ X_1^+ \\ X_2^+ \\ v^- \\ X_1^- \\ X_2^- \end{pmatrix}. \tag{5.2}$$

We write (5.2) as

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ y_1^+ \\ Y_2^+ \\ y_2^+ \\ w^- \\ Y_1^- \\ y_1^- \\ Y_2^- \\ y_2^- \end{pmatrix} = \left(\begin{array}{ccccc|c} a & b_1 & 0 & b_2 & 0 & 0\;0\;0\;0\;0 \\ c_1 & d_{11} & 0 & d_{12} & 0 & 0\;0\;0\;0\;0 \\ 0 & 0 & 1 & 0 & 0 & 0\;0\;0\;0\;0 \\ c_2 & d_{21} & 0 & d_{22} & 0 & 0\;0\;0\;0\;0 \\ 0 & 0 & 0 & 0 & 1 & 0\;0\;0\;0\;0 \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{t-1} \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \end{array} \right) \begin{pmatrix} v^+ \\ X_1^+ \\ y_1^+ \\ X_2^+ \\ y_2^+ \\ v^- \\ X_1^- \\ y_1^+ \\ X_2^- \\ y_2^- \end{pmatrix}.$$

Applying (5.1) we come to

$$\begin{pmatrix} w^+ \\ Y_1^+ \\ y_1^+ \\ Y_2^+ \\ y_2^+ \\ w^- \\ Y_1^- \\ y_1^- \\ Y_2^- \\ y_2^- \end{pmatrix} = \left(\begin{array}{ccccc c} a & b_1 & 0 & b_2 & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ c_1 & d_{11} & 0 & d_{12} & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ 0 & 0 & 1 & 0 & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ c_2 & d_{21} & 0 & d_{22} & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ 0 & 0 & 0 & 0 & 1 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} a & b_1 & 0 & b_2 & 0 \\ c_1 & d_{11} & 0 & d_{12} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_2 & d_{21} & 0 & d_{22} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{t-1} \end{array} \right) \times$$

$$\times \left(\begin{array}{ccccc c} a' & 0 & b'_1 & 0 & b'_2 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ 0 & 1 & 0 & 0 & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ c'_1 & 0 & d'_{11} & 0 & d'_{12} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ 0 & 0 & 0 & 1 & 0 & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ c'_2 & 0 & d'_{21} & 0 & d'_{22} & \begin{matrix} 0 & 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} a' & 0 & b'_1 & 0 & b'_2 \\ 0 & 1 & 0 & 0 & 0 \\ c'_1 & 0 & d'_{11} & 0 & d'_{12} \\ 0 & 0 & 0 & 1 & 0 \\ c'_2 & 0 & d'_{21} & 0 & d'_{22} \end{pmatrix}^{t-1} \end{array} \right) \begin{pmatrix} u^+ \\ X_1^+ \\ x_1^+ \\ X_2^+ \\ x_2^+ \\ u^- \\ X_1^- \\ x_1^+ \\ X_2^- \\ x_2^- \end{pmatrix}$$

Now

$$X \oplus x \in Q \otimes (\mathbb{Z}_p^l \oplus \mathbb{Z}_p^m), \qquad Y \oplus y \in T \otimes (\mathbb{Z}_p^l \oplus \mathbb{Z}_p^m),$$

and we get $u \oplus w \in \chi_{\mathfrak{g} \star \mathfrak{h}}(Q, T)$. Thus,

$$\chi_{\mathfrak{g} \star \mathfrak{h}}(Q, T) \supset \chi_{\mathfrak{g}}(Q, T) \chi_{\mathfrak{h}}(Q, T).$$

But both sides are self-dual, therefore they coincide.

5.7. Morphisms of graphs (Theorem 4.13). Consider the map

$$\mathrm{LMod}(\mathcal{H}) \times \mathrm{LMod}(\mathcal{H}) \to \mathrm{LMod}(\mathcal{H} \otimes \ell_m) \times \mathrm{LMod}(\mathcal{H} \otimes \ell_m)$$

given by $(Q, T) \mapsto (Q \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m, T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p^m)$.

Lemma 5.1 *This map is a morphism of graphs*

$$\Delta(\mathcal{H} \bowtie \mathcal{H}) \to \Delta\bigl((\mathcal{H} \otimes \ell_m) \bowtie (\mathcal{H} \otimes \ell_m)\bigr).$$

This statement is obvious.

Next, we have an embedding of complexes

$$\mathrm{Bd}\bigl((\mathcal{H} \otimes \ell_m) \bowtie (\mathcal{H} \otimes \ell_m)\bigr) \to \mathrm{Bd}\bigl((\mathcal{H} \otimes \ell_m) \oplus (\mathcal{H} \otimes \ell_m)\bigr).$$

On the other hand, the linear relation ξ is a morphism of the category Naz. Therefore it induces a morphism of graphs $\Delta\bigl((\mathcal{H} \otimes \ell_m) \oplus (\mathcal{H} \otimes \ell_m)\bigr) \to \Delta(\mathcal{V} \oplus \mathcal{V})$, see [9], Proposition 10.7.6.

5.8. Proof of Proposition 4.7. We have

$$\operatorname{indef} \xi = \Lambda(\mathfrak{g}).$$

Therefore $\Lambda(\mathfrak{g}) \subset \xi\, \eta_{Q,T} \subset \Lambda(\mathfrak{g})^{\perp}$. This is the statement a) of Proposition 4.7.

Also, if R is a relation $\mathcal{V} \rightrightarrows W$, $Y \subset \mathcal{V}$ is a lattice, then $(RY)_{\downarrow} = (\operatorname{indef} R)_{\downarrow}$. This implies b).

5.9. Values on the distinguished boundary. Now let Q, T be Lagrangian subspaces in $\mathcal{H}$.

Proof of Proposition 4.10. Decompose $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \mathbb{Q}_p^\alpha \oplus \mathbb{Q}_p^\alpha$. A Lagrangian subspace $Q \subset \mathcal{H}$ of general position is a graph of an operator $\mathcal{H}^+ \to \mathcal{H}^-$, and matrix of this operator is symmetric (see, e.g., [9], Theorem 3.1.4). To shorten notation, set $k = 2$. The equation (4.2) can be written in the form

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ u^- \\ t_{11}x_1^+ + t_{12}x_2^+ \\ t_{12}x_1^+ + t_{22}x_2^+ \end{pmatrix} \begin{pmatrix} a & b_1 & b_2 & 0 & 0 & 0 \\ c_1 & d_{11} & d_{12} & 0 & 0 & 0 \\ c_2 & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & a^t & c_1^t & c_2^t \\ 0 & 0 & 0 & b_1^t & d_{11}^t & d_{21}^t \\ 0 & 0 & 0 & b_2^t & d_{12}^t & d_{22}^t \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ v^- \\ q_{11}y_1^+ + q_{12}y_2^+ \\ q_{12}y_1^+ + q_{22}y_2^+ \end{pmatrix}, \qquad (5.3)$$

We denote

$$\varkappa := \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix}, \qquad \tau := \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}$$

and write (5.3) as

$$v^+ = au^+ + bx^+ \qquad (5.4)$$
$$y^+ = cu^+ + dx^+ \qquad (5.5)$$
$$u^- = a^t v^- + c^t \varkappa y^+ \qquad (5.6)$$
$$\tau x^+ = b^t v^- + d^t \varkappa y^+. \qquad (5.7)$$

We regard lines (5.5),(5.7) as a system of equations for x^+, y^+. The matrix of the system is

$$\Omega(\varkappa, \tau) = \begin{pmatrix} -d & 1 \\ \tau & -d^t \varkappa \end{pmatrix}.$$

Evidently, the polynomial $\det \Omega(\varkappa, \tau)$ is not zero. Indeed, fix $\varkappa$ and take $\tau = p^{-N} \cdot 1$. If N is sufficiently large, then the determinant is $\neq 0$. Thus, outside the hypersurface

$$\det \Omega(\varkappa, \tau) = 0$$

we can express x^+ and y^+ as functions of u^+, v^-. After substitution of x^+, y^+ to (5.4),(5.6), we get a dependence of u^-, v^+ in u^+, v^-. □

This also proves Theorem 4.9.b (rationality of characteristic function).

Proof Theorem 4.9.c. Denote

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and write the equation (4.2) in the form

$$\begin{pmatrix} v^+ \\ y_1^+ \\ y_2^+ \\ v^- \\ q_{11}y_1^+ + q_{12}y_2^+ \\ q_{12}y_1^+ + q_{22}y_2^+ \end{pmatrix} \begin{pmatrix} a & b_1 & b_2 & 0 & 0 & 0 \\ c_1 & d_{11} & d_{12} & 0 & 0 & 0 \\ c_2 & d_{21} & d_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & A^t & C_1^t & C_2^t \\ 0 & 0 & 0 & B_1^t & D_{11}^t & D_{21}^t \\ 0 & 0 & 0 & B_2^t & D_{12}^t & D_{22}^t \end{pmatrix} \begin{pmatrix} u^+ \\ x_1^+ \\ x_2^+ \\ u^- \\ t_{11}x_1^+ + t_{12}x_2^+ \\ t_{12}x_1^+ + t_{22}x_2^+ \end{pmatrix},$$

or

$$v^+ = au^+ + bx^+ \tag{5.8}$$

$$y^+ = cu^+ + dx^+ \tag{5.9}$$

$$v^- = A^t u^- + C^t \tau x^+ \tag{5.10}$$

$$y_+ = B^t u^- + D^t \tau x^+. \tag{5.11}$$

We consider lines (5.9), (5.11) as equations for y^+, x^+. The matrix of the system is

$$\Xi(\varkappa, \tau) = \begin{pmatrix} 1 & -d \\ \varkappa & -D^t\tau \end{pmatrix}.$$

Its determinant equals

$$\det \Xi(\varkappa, \tau) = \det(-D^t\tau + \varkappa d).$$

If it is nonzero, we get a linear operator $u \mapsto v$. We come to the following statement:

Proposition 5.2 *If there exists a pair of symmetric matrices $\varkappa$, τ such that* $\det(-D^t\tau + \varkappa d) \neq 0$, *then* $\chi_{\mathfrak{g}}(Q, T) \in \mathrm{Sp}(\mathcal{V}, \mathbb{Q}_p)$ *a.s. on* $\mathrm{LGr}(\mathcal{H}) \times \mathrm{LGr}(\mathcal{H})$.

5.10. Involution. Proof of Proposition 4.15. We write the defining relation for $\chi_{g^{-1}}$,

$$\begin{pmatrix} v^+ \\ y^+ \\ v^- \\ y^- \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \end{pmatrix} \begin{pmatrix} u^+ \\ x^+ \\ u^- \\ x^- \end{pmatrix},$$

represent this in the form

$$\begin{pmatrix} u^+ \\ x^+ \\ u^- \\ x^- \end{pmatrix} = \begin{pmatrix} \begin{matrix} a & b \\ c & d \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{t-1} \end{pmatrix} \begin{pmatrix} v^+ \\ y^+ \\ v^- \\ y^- \end{pmatrix}$$

and come to desired statement.

5.11. Proof of Theorem 4.16. We write (4.2) as

$$\begin{pmatrix}u^+\\x^+\\u^-\\x^-\end{pmatrix}=\begin{pmatrix}\lambda^{-1}&&&\\&\lambda^{-1}&&\\&&\lambda&\\&&&\lambda\end{pmatrix}\begin{pmatrix}a&b&0&0\\c&d&0&0\\0&0&\multicolumn{2}{c}{\begin{pmatrix}a&b\\c&d\end{pmatrix}^{t-1}}\\0&0&&\end{pmatrix}\begin{pmatrix}\lambda&&&\\&\lambda&&\\&&\lambda^{-1}&\\&&&\lambda^{-1}\end{pmatrix}\begin{pmatrix}v^+\\y^+\\v^-\\y^-\end{pmatrix}$$

or

$$\begin{pmatrix}\lambda u^+\\\lambda x^+\\\lambda^{-1}u^-\\\lambda^{-1}x^-\end{pmatrix}=\begin{pmatrix}a&b&0&0\\c&d&0&0\\0&0&\multicolumn{2}{c}{\begin{pmatrix}a&b\\c&d\end{pmatrix}^{t-1}}\\0&0&&\end{pmatrix}\begin{pmatrix}\lambda v^+\\\lambda y^+\\\lambda^{-1}v^-\\\lambda^{-1}y^-\end{pmatrix}$$

5.12. Another reformulation of the definition of characteristic functions. Consider the space $W=\mathcal{V}\oplus(\mathcal{H}\otimes\ell_m)$. For any self-dual submodule $Q\subset\mathcal{H}$, consider the linear relation $\Lambda:\mathcal{V}\rightrightarrows W$ defined by

$$\Lambda_Q=1_{\mathcal{V}}\oplus(Q\otimes\mathbb{Z}_p^m)\;\subset\;(\mathcal{V}\oplus\mathcal{V})\oplus(Q\otimes\ell_m).$$

Then $\chi_{\mathfrak{g}}$ is a product of linear relations

$$\chi_{\mathfrak{g}}(Q,T)=(\Lambda_T)^{\square}\begin{pmatrix}g&0\\0&g^{t-1}\end{pmatrix}\Lambda_Q.$$

6 Multiplicativity theorem

Theorem 2.2 (multiplicativity theorem) formulated above is a representative of wide class of theorems, their proofs are standard, below we refer to proofs [8], Chapter VIII.

6.1. Corners of orthogonal matrices.

Lemma 6.1 *Let A be a $m\times m$ matrix with elements $\in\mathbb{Z}_p$. Then there exists N and a matrix* $\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{O}(m+N,\mathbb{Z}_p)$.

PROOF. Denote by $\mathbf{B}_m$ the set of all possible $m\times m$ left upper corners of matrices $g\in\mathrm{O}(\infty,\mathbb{Z}_p)$.

1) The set $\mathbf{B}_m$ is closed with respect to matrix products. Indeed, let

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{O}(m+N,\mathbb{Z}_p),\qquad\begin{pmatrix}A'&B'\\C'&D'\end{pmatrix}\in\mathrm{O}(m+N',\mathbb{Z}_p).$$

Then

$$\begin{pmatrix}A&B&0\\C&D&0\\0&0&1\end{pmatrix}\begin{pmatrix}A'&0&B'\\0&1&0\\C'&0&D'\end{pmatrix}=\begin{pmatrix}AA'&\dots&\dots\\\dots&\dots&\dots\\\dots&\dots&\dots\end{pmatrix}\in\mathrm{O}(m+N+N',\mathbb{Z}_p).$$

2) If $A\in\mathbf{B}_m$, $A'\in\mathbf{B}_n$, then $\begin{pmatrix}A&0\\0&A'\end{pmatrix}\in\mathbf{B}_{m+n}$.

3) It is more-or-less clear that for any $z \in \mathbb{Z}_p$ we have

$$\begin{pmatrix} z \end{pmatrix} \in \mathbf{B}_1, \qquad \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \in \mathbf{B}_2.$$

4) $\mathbf{B}_m$ contains matrices of permutations.

Now we can produce any matrix with integer elements. □

6.2. Admissible representations. Denote by $\mathbf{K}_m$ the subgroup in $\mathbf{K}$ consisting of matrices of the form $\begin{pmatrix} 1_m & 0 \\ 0 & * \end{pmatrix}$.

Let τ be a unitary representation of $\mathbf{K}$ in a Hilbert space H. Denote by $H(m)$ the subspace of $\mathbf{K}_m$-fixed vectors. Denote by $P(m)$ the operator of orthogonal projection to $H(m)$. We say, that τ is *admissible* if $\cup_m H(m)$ is dense in H.

We say, that a representation of $\mathbf{G}$ is $\mathbf{K}$-*admissible* if its restriction to $\mathbf{K}$ is admissible.

6.3. Continuation of representations. Denote by $\mathbf{B}_\infty$ the semigroup of all infinite matrices A such that:

a) $a_{ij} \in \mathbb{Z}_p$;

b) for each i the sequence a_{ij} tends to 0 as $j \to \infty$; for each j the sequence a_{ij} tends to 0 as $i \to \infty$.

We say that a sequence of matrices $A^{(j)} \in \mathbf{B}_\infty$ weakly converges to A if we have convergence of each matrix element, $a_{kl}^{(j)} \to a_{kl}$.

Denote by $\mathbf{O}(\infty, \mathbb{Z}_p)$ the group of all orthogonal matrices $\in \mathbf{B}_\infty$.

Lemma 6.2 *The group* $\mathrm{O}(\infty, \mathbb{Z}_p)$ *is dense in* $\mathbf{O}(\infty, \mathbb{Z}_p)$ *and in* $\mathbf{B}_\infty$*.*

PROOF. Let $S \in \mathbf{B}_\infty$. Consider its left upper corner of size $m \times m$. Consider $g_m \in \mathbb{O}(\infty, \mathbb{Z}_p)$ having the same left upper corner. Then g_m weakly converges to S, □

Theorem 6.3 a) *Let* τ *be a unitary representation of* $\mathbf{K} = \mathrm{O}(\infty, \mathbb{Z}_p)$*. The following conditions are equivalent:*

— τ is admissible;

— τ admits a weakly continuous extension to the group $\mathbf{O}(\infty, \mathbb{Z}_p)$*;*

— τ admits a weakly continuous extension to a representation $\widetilde{\tau}$ of the semigroup $\mathbf{B}_\infty$ such that $\widetilde{\tau}(A^t) = \widetilde{\tau}(A)^*$, $\|\widetilde{\tau}(A)\| \leqslant 1$ for all A.

b) *For an admissible representation τ,*

$$P(m) = \widetilde{\tau} \begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a statement in the spirit of [24]. We omit a proof, since it is a one-to-one repetition of proof of [8], Theorem VIII.1.4 about symmetric groups (admissibility implies semigroup continuation), the only new detail is Lemma 6.1). Admissibility follows from continuity by [8], Proposition VIII.1.3.

Corollary 6.4 *Denote*

$$\Theta_N^{(m)} = \begin{pmatrix} 1_m & 0 & 0 & 0 \\ 0 & 0 & 1_N & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & 0 & 1_\infty \end{pmatrix}.$$

The projector $P(m)$ is a weak limit of the sequence

$$P(m) = \lim_{N\to\infty} \tau(\Theta_N^{(m)}). \tag{6.1}$$

PROOF. The sequence $\Theta_N^{(m)} \in \mathrm{O}(\infty, \mathbb{Z}_p)$ weakly converges to the matrix $\begin{pmatrix} 1_m & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{B}_\infty$. We refer to the statement b) of the theorem. □

6.4. Proof of Theorem 2.2. We keep the notation of Subsection 2.3. Let $v \in H^{\mathbf{K}}$, $g \in G_j = \mathrm{GL}(\alpha + km, \mathbb{Q}_p)$, let $q \in \mathbf{K}_j$. Then

$$\rho(q)\rho(g)v = \rho(g)\rho(q)h = \rho(g)h,$$

i.e., $v \in H(j)$. Thus the subspace $\cup_j H(j)$ is $\mathbf{G}$-invariant. Its closure is an admissible representation of $\mathbf{G}$. In $\bigl(\cup_j H(j)\bigr)^\perp$ Theorem 2.2 holds by a trivial reason (the space of fixed vectors $\mathbf{K}$ is zero).

Thus, without loss of generality we can assume that ρ is admissible.

Now let g, $h \in \mathbf{G}$, let $\mathfrak{g}$, $\mathfrak{h} \in \mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ be the corresponding double cosets. Let $P = P(0)$ be the projector to $\mathbf{K}$-fixed vectors. Applying Corollary 6.4, we obtain

$$\overline{\rho}(\mathfrak{g})\overline{\rho}(h) = P\rho(g)P\rho(h) = \lim_{N\to\infty} P\rho(g)\rho(\mathfrak{I}(\Theta_N^{(0)}))\rho(h) = \lim_{N\to\infty} P\rho(g\mathfrak{I}(\Theta_N)h),$$

here $\mathfrak{I} : \mathbf{K} \to \mathbf{G}$ is the embedding (2.2). By the definition ($\Theta_N^{(0)}$ is Θ_N from Subsection 2.3), we get $\overline{\rho}(\mathfrak{g} \star \mathfrak{h})$.

6.5. Variation of construction. Train. We can define multiplication of double cosets

$$\mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q \times \mathbf{K}_q \setminus \mathbf{G}/\mathbf{K}_r \to \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_r.$$

In the definition of product of double cosets (Subsection 2.2), we simply change Θ_N by $\Theta_N^{(q)}$. An explicit formula of the product is the same (2.4). Thus we get a category (*train* $\mathcal{T}(\mathbf{G}, \mathbf{K})$ *of the pair* $(\mathbf{G}, \mathbf{K})$).

Next, for any unitary representation ρ of the group $\mathbf{G}$, a double coset $\mathfrak{g} \in \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q$ determines an operator $\overline{\rho}(\mathfrak{g}) : H(q) \to H(p)$ by the formula

$$\overline{\rho}(g) := P(q)\rho(g), \qquad g \in \mathfrak{g}.$$

For any

$$\mathfrak{g} \in \mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q \qquad \mathfrak{h} \in \mathbf{K}_q \setminus \mathbf{G}/\mathbf{K}_r,$$

the following identity holds

$$\rho(\mathfrak{g})\rho(\mathfrak{h}) = \rho(\mathfrak{g} \star \mathfrak{h}),$$

i.e., we get a representation of the category $\mathcal{T}(\mathbf{G}, \mathbf{K})$. Also,

$$\rho(\mathfrak{g}^*) = \rho(\mathfrak{g})^*, \qquad \|\rho(\mathfrak{g})\| \leqslant 1. \tag{6.2}$$

Also it can be shown that

Theorem 6.5 *This construction is a bijection between the set of* $\mathbf{K}$*-admissible unitary representations of* $\mathbf{G}$ *and the set of representations of the category* $\mathcal{T}(\mathbf{G}, \mathbf{K})$ *satisfying* (6.2).

We omit a proof, since it is the same as in [16]. □

Also the construction of characteristic functions and their properties survive for double cosets $\mathbf{K}_p \setminus \mathbf{G}/\mathbf{K}_q$.

7 Representations of the group G

7.1. Existence of representations. Let

$$\begin{pmatrix} a & b_1 & \dots & b_k \\ c_1 & d_{11} & \dots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ c_k & d_{k1} & \dots & d_{kk} \end{pmatrix} \in \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p).$$

Consider embedding $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p) \to \mathrm{Sp}(2(\alpha + k\infty), \mathbb{Q}_p)$ given by

$$\iota : g \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{t-1} \end{pmatrix}.$$

For any

$$r = \begin{pmatrix} r_{11} & \dots & r_{1\,2n} \\ \vdots & \ddots & \vdots \\ r_{2n\,1} & \dots & r_{2n\,2n} \end{pmatrix} \in \mathrm{Sp}(2k, \mathbb{Q}_p)$$

consider the matrix $\sigma(r) = 1_{2\alpha} \oplus (r \otimes 1_\infty)$,

$$\sigma(r) := \begin{pmatrix} 1_\alpha & 0 & \dots & 0 & 0 \\ 0 & r_{11} \cdot 1_\infty & \dots & 0 & r_{1k} \cdot 1_\infty \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1_\alpha & 0 \\ 0 & r_{11} \cdot 1_\infty & \dots & 0 & r_{1k} \cdot 1_\infty \end{pmatrix}$$

This matrix is not contained in $\mathrm{Sp}(2(\alpha + k\infty), \mathbb{Q}_p)$, because it is not finitary. However, the map

$$q \mapsto \sigma(r^{-1})\, q\, \sigma(r) \tag{7.1}$$

is an outer automorphism of $\mathrm{Sp}(2(\alpha+k\infty),\mathbb{Q}_p)$. Emphasize that this automorphism fixes the subgroup $\mathbf{K}=\mathrm{O}(\infty,\mathbb{Z}_p)$.

We consider the representation $\rho(r)$ of $\mathrm{GL}(\alpha+k\infty,\mathbb{Q}_p)$ given by the formula

$$\rho_r(g)=\mathrm{We}\bigl(\sigma(r^{-1})\iota(g)\sigma(r)\bigr),$$

where $\mathrm{We}(\cdot)$ is the Weil representation, see Subsection 3.16.

Recall that the Weil representation is projective.

Lemma 7.1 *The representation ρ_r is equivalent to a linear representation, i.e., there is a function* (a trivializer) $\gamma:\mathbf{G}\to\mathbb{C}^\times$ *such that $\gamma(g)\rho_r(g)$ is a linear representation.*

PROOF. First, the restriction of the Weil representation of $\mathrm{Sp}(2n,\mathbb{Q}_p)$ to $\mathrm{GL}(n,\mathbb{Q}_p)$ is linear, see (3.8). Therefore, restricting the Weil representation to each finite-dimensional group $G_j=\mathrm{GL}(\alpha+kj,\mathbb{Q}_p)$ we get a representation equivalent to a linear representation (for finite-dimensional groups the automorphism (7.1) is inner). Denote by $\gamma_j(g)$ the trivializer for G_j. Ratio $\gamma(g)_j/\gamma(g)_{j+1}$ of two trivializers is a character $G_j\to\mathbb{C}^\times$. All characters of $G_j\to\mathbb{C}^\times$ has the form $\varphi(\det h)$, where φ is a character $\mathbb{Q}^\times\to\mathbb{C}^\times$. Correcting $\gamma_{j+1}(g)\mapsto\gamma_{j+1}(g)\psi(\det g)$, we can assume that $\gamma_{j+1}(g)=\gamma_j(g)$ on G_j.

In this way we choose a trivializer γ on the whole group $\mathbf{G}$. Restriction of γ to $\mathrm{O}(\infty,\mathbb{Z}_p)$ must be a character on $\mathrm{O}(\infty,\mathbb{Z}_p)\to\mathbb{C}^\times$. The only non-trivial character is $\det(u)=\pm1$. We change the trivializer $\gamma(g)$ to $\det(g)\gamma(g)$. $\square$

Lemma 7.2 *In the model of Subsection* 3.16*, the subspace $L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}}$ of* $\mathbf{K}$*-fixed vectors of ρ_r coincides with the space of functions of the form*

$$f(z_1,\dots,z_\alpha)I(z_{\alpha+1})I(z_{\alpha+2})\dots$$

PROOF. Without loss of generality, we can set $\alpha=0$. We regard $\mathcal{E}_{k\infty}$ as the space of $\infty\times k$ matrices $Z=\{z_{ij}\}$ with elements in $\mathbb{Q}_p$ (all but a finite number of matrix elements are in $\mathbb{Z}_p$). The group $\mathbf{K}=\mathrm{O}(\infty,\mathbb{Z}_p)$ acts by left multiplications

$$\mathrm{We}(u)f(Z)=f(Zu).$$

We must show that $\prod_{ij}I(z_{ij})$ is a unique $\mathrm{O}(\infty,\mathbb{Z}_p)$-invariant function in $L^2(\mathcal{E}_{k\infty})$. Equivalently, $\mathbb{Z}_p^{k\infty}$ is a unique invariant subset of finite positive measure.

The group $\mathrm{O}(\infty,\mathbb{Z}_p)$ contains the group $S(\infty)$ of finitely supported permutations of the set $\mathbb{N}$. According zero-one law (see, e.g., [45], §4.1), the action of $S(\infty)$ on the set $\mathbb{Z}_p^{k\infty}\subset\mathcal{E}_{k\infty}$ is ergodic. Let $\Omega\subset\mathcal{E}_{k\infty}$ be an invariant set. Let $\xi\in\mathcal{E}_{k\infty}\setminus\mathbb{Z}_p^{k\infty}$. Assume that the measure of the set $\Omega\cap(\xi+\mathbb{Z}_p^{k\infty})$ is non-zero, say ν_0. Since Ω is $S(\infty)$-invariant, for any $\mathfrak{s}\in S(\infty)$, the set $\Omega\cap(\xi\mathfrak{s}+\mathbb{Z}_p^{k\infty})$ has the same measure ν_0. However there is a countable number of disjoint sets of the form $\xi\mathfrak{s}+\mathbb{Z}_p^{k\infty}$, therefore the measure of Ω is infinite. $\square$

Corollary 7.3 *Let $\alpha = 0$. Then the representation ρ_r contains a unique irreducible* **K***-spherical representation of* **G***.*

PROOF. We take the **G**-cyclic span of the unique **K**-fixed vector. $\square$

Next, consider the subgroup $\mathrm{GL}(1,\mathbb{Q}_p) \subset \mathrm{Sp}(2k,\mathbb{Q}_p)$ consisting of matrices $\begin{pmatrix} \lambda \cdot 1_k & 0 \\ 0 & \lambda^{-1} \cdot 1_k \end{pmatrix}$, where $\lambda \in \mathbb{Q}_p^\times$.

Lemma 7.4 *If r, $r' \in \mathrm{Sp}(2k,\mathbb{Q}_p)$ are contained in the same double coset*

$$\mathrm{GL}(1,\mathbb{Q}_p) \setminus \mathrm{Sp}(2k,\mathbb{Q}_p)/\mathrm{Sp}(2k,\mathbb{Z}_p),$$

then $\rho_r \simeq \rho_{r'}$.

PROOF. First, if $q \in \mathrm{GL}(1,\mathbb{Q}_p)$, then the automorphism (7.1) fixes the subgroup $\mathrm{GL}(\alpha + k\infty,\mathbb{Q}_p)$.

Second, if $t \in \mathrm{Sp}(2k,\mathbb{Z}_p)$, then $\sigma(t)$ is contained in the group **Sp** of automorphisms of the infinite object of the Nazarov category. Therefore the operator $\mathrm{We}(\sigma(t))$ is well-defined, it intertwines ρ_r and ρ_{rt}. $\square$

7.2. Relation of characteristic functions and representations. By Lemma 7.2, we can identify the space of **K**-fixed vectors of ρ_r and the space of the Weil representation of $\mathrm{Sp}(2\alpha,\mathbb{Q}_p)$.

Theorem 7.5 *The representation of the semigroup* $\mathbf{K} \setminus \mathbf{G}/\mathbf{K}$ *in the space of* **K***-fixed vectors of ρ_r is given by the formula*

$$\overline{\rho}_r(\mathfrak{g}) = s \cdot \mathrm{We}\big(\chi_{\mathfrak{g}}(r\mathbb{Z}_p^{2k}, r\mathbb{Z}_p^{2k})\big), \qquad s \in \mathbb{C}^\times.$$

PROOF. We use the notation and statements of Subsection 3.16. Let $\mathcal{V}$ and $\mathcal{H}$ be the same as in Section 4. Let $Y = \mathcal{V}_{2k\infty}$, $W = \mathcal{V} \oplus Y$. The operator of projection $\mathcal{H}(\mathcal{V} \oplus Y)$ to $\mathcal{H}(V \oplus Y)^{\mathbf{K}} \simeq \mathcal{H}(V)$ is $\mathrm{We}(\theta_W^V)$. Therefore

$$\overline{\rho}(\mathfrak{g}) = s' \cdot \mathrm{We}(\theta_W^V)\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\theta_W^V)$$

as an operator $L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}} \to L^2(\mathcal{E}_{\alpha+k\infty})^{\mathbf{K}}$. The operator

$$\mathrm{We}(\lambda_W^V) : L^2(\mathbb{Q}_p^\alpha) \to L^2(\mathcal{E}_{\alpha+k\infty})$$

is an operator of isometric embedding, the image is $\mathcal{H}(V \oplus V_{2k\infty})^{\mathbf{K}}$. Therefore we can write $\overline{\rho}(\mathfrak{g})$ as

$$\begin{aligned}\overline{\rho}(\mathfrak{g}) = s'' \cdot \mathrm{We}(\lambda_W^V)^*\mathrm{We}(\theta_W^V)\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\theta_W^V)\mathrm{We}(\lambda_W^V) = \\ = s''' \cdot \mathrm{We}(\lambda_W^V)^*\mathrm{We}(\sigma(r^{-1})\iota(g)\sigma(r))\mathrm{We}(\lambda_W^V) = \\ = s'''' \cdot \mathrm{We}\Big[(\lambda_W^V)^*\sigma(r^{-1})\iota(g)\sigma(r)\lambda_W^V\Big]. \quad (7.2)\end{aligned}$$

Next, $\sigma(r)\lambda_W^V : V \rightrightarrows V \oplus Y$ is a direct sum of $1_V \subset V \oplus V$ and the lattice in Y given by

$$\sigma(r)Y(\mathbb{O}) = \sigma(r)(H(\mathbb{O}) \otimes \mathbb{O}^\infty) = (rH(\mathbb{O})) \otimes \mathbb{O}^\infty).$$

We apply Subsection 5.12 for the expression in square brackets in (7.2).

7.3. A more general construction. Consider the embedding

$$\iota_l : \mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p) \to \mathrm{Sp}(2l\alpha + 2lk\infty, \mathbb{Q}_p)$$

given by

$$g \mapsto \begin{pmatrix} g & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & g & 0 & \dots & 0 \\ 0 & \dots & 0 & g^{t-1} & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & g^{t-1} \end{pmatrix}.$$

This is a $2l \times 2l$ block matrix, each block of this matrix has size $(\alpha + k\infty) \times (\alpha + k\infty)$.

Next, for a matrix $r \in \mathrm{Sp}(2kl, \mathbb{Q}_p)$ we take

$$\sigma(r) := 1_{2\alpha l} \oplus (r \otimes 1_\infty)$$

and consider the representation of $\mathrm{GL}(\alpha + k\infty, \mathbb{Q}_p)$ given by

$$\rho_r(g) = \mathrm{We}(\sigma(r)^{-1} \iota_l(g) \sigma(r)).$$

Set $\alpha = 0$. As above, each representation ρ_r of $\mathbf{G} = \mathrm{GL}(k\infty, \mathbb{Q}_p)$ contains a unique $\mathbf{K}$-spherical subrepresentation.

Conjecture 7.6 *Any* $\mathbf{K}$*-spherical representation of* $\mathrm{GL}(k\infty, \mathbb{Q}_p)$ *is a subrepresentation in* $\varphi(\det(g))\,\rho_r(g)$*, where* $\varphi = \varphi_r : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ *is a character. Representations* ρ_r *are parametrized by the set*

$$\bigcup_l \mathrm{GL}(l, \mathbb{Q}_p) \setminus \mathrm{Sp}(2kl, \mathbb{Q}_p)/\mathrm{Sp}(2kl, \mathbb{Z}_p).$$

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