CAYLEY GRAPHS GENERATED BY SMALL DEGREE POLYNOMIALS OVER FINITE FIELDS

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Abstract. We improve upper bounds of F. R. K. Chung and of M. Lu, D. Wan, L.-P. Wang, X.-D. Zhang on the diameter of some Cayley graphs constructed from polynomials over finite fields.

1. Introduction

Let $\mathcal{P}_d$ be the set of monic polynomials of degree d over a finite field $\mathbb{F}_q$ of q elements, that are powers of some irreducible polynomial, that is

$$\mathcal{P}_d = \{g \in \mathbb{F}_q[X] \ : \ \deg g = d,\ g = h^k, \\ h \in \mathbb{F}_q[X] \text{ monic and irreducible},\ k = 1, 2, \ldots,\ \}.$$

For a root α of an irreducible polynomial $f \in \mathbb{F}_q[X]$ of degree n, thus $\mathbb{F}_q(\alpha) = \mathbb{F}_{q^n}$, we define

$$\mathcal{E}(\alpha, d) = \{g(\alpha) \ : \ g \in \mathcal{P}_d\}.$$

It is easy to see that for $d < n$ we have

$$\#\mathcal{E}(\alpha, d) = \#\mathcal{P}_d = (1 + o(1))\frac{q^d}{d}$$

as $d \to \infty$, see also (3) below.

Following Lu, Wan, Wang and Zhang [6], we now define the directed Cayley graph $\mathfrak{G}(\alpha, d)$ on $q^n - 1$ vertices, labelled by the elements of $\mathbb{F}_{q^n}^*$, where for $u, v \in \mathbb{F}_{q^n}^*$ the edge $u \to v$ exists if and only if $u/v \in \mathcal{E}(\alpha, d)$. These graphs are similar to those introduced by Chung [1] however are a little spraser: they are $\#\mathcal{P}_d$-regular rather than q^d-regular as in [1].

It has been shown in [6] that the graphs $\mathfrak{G}(\alpha, d)$ have very attractive connectivity properties. In particular, we denote by $D(\alpha, d)$ the *diameter* of $\mathfrak{G}(\alpha, d)$. Using bounds of multiplicative character sum from [7, Theorem 2.1], Lu, Wan, Wang and Zhang [6] have shown that for $n < q^{d/2} + 1$ the graph $\mathfrak{G}(\alpha, d)$ is connected and its diameter satisfies the inequality

$$D(\alpha, d) \le \frac{2n}{d}\left(1 + \frac{2\log(n-1)}{d\log q - 2\log(n-1)}\right) + 1. \tag{1}$$

Here we augment the argument of [6] with some new combinatorial and analytic considerations and improve the bound (1).

First we assume that $d \ge 2$.

Theorem 1. *For $d \ge 2$ and a root α of an irreducible polynomial $f \in \mathbb{F}_q[X]$ of degree $\deg f = n$ with $2d+1 \le n < q^{d/2}+1$, we have*

$$D(\alpha, d) \le \frac{2n}{d}\left(1 + \frac{\log(n-1)-1}{d\log q - 2\log(n-1)}\right) + \frac{4\log(n-1)+7}{d\log q - 2\log(n-1)}.$$

For $d = 1$ the bound (1) is exactly the same as the bound of Wan [7, Theorem 3.3] which improves slightly the bound of Chung [1, Theorem 6]. For $d = 1$, we set $\Delta(\alpha) = D(\alpha, 1)$. For a sufficiently large q, Katz [4, Theorem 1] has improved the results of Chung [1] and showed that $\Delta(\alpha) \le n+2$, provided that $q \ge B(n)$ for some inexplicit function $B(n)$ of n. Furthermore, Cohen [2] shows that one can take $B(n) = (n(n+2)!)^2$ in the estimate of Katz [4].

We also use our idea in the case $d = 1$ and obtain an improvement of (1) and thus of the bounds of Chung [1, Theorem 6] and Wan [7, Theorem 3.3].

Theorem 2. *For a root α of an irreducible polynomial $f \in \mathbb{F}_q[X]$ of degree $\deg f = n$ with $3 \le n < q^{1/2}+1$, we have*

$$\Delta(\alpha) \le 2n\left(1 + \frac{\log(n-1)-1}{\log q - 2\log(n-1)}\right) + \frac{3\log(n-1)+3}{\log q - 2\log(n-1)}.$$

We use the same idea for the proofs of Theorems 1 and 2, however the technical details are slightly different.

We also note that the additive constants 7 and 3 in the bounds of Theorems 1 and 2, respectively, can be replaced by a slightly smaller (but fractional values).

To compre the bound (1) with Theorems 1 and 2, we assume that $n = q^{(\vartheta+o(1))d}$ for some fixed positive $\vartheta < 1/2$.

The Theorems 1 and 2, imply that for any $d \ge 1$,

$$D(\alpha, d) \le \left(\frac{2-2\vartheta}{1-2\vartheta} + o(1)\right)\frac{n}{d},$$

while (1) implies a weaker bound

$$D(\alpha, d) \le \left(\frac{2}{1-2\vartheta} + o(1)\right)\frac{n}{d}.$$

2. Preparation

We define the polynomial analogue of the von Mangoldt function as follows. For $g \in \mathbb{F}_q[X]$ we define

$$\Lambda(g) = \begin{cases} \deg h, & \text{if } g = h^k \text{ for some irreducible } h \in \mathbb{F}_q[X], \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{X}_n$ be the set of multiplicative characters of $\mathbb{F}_{q^n}$ and let $\mathcal{X}_n^* = \mathcal{X}_n \setminus \{\chi_0\}$ be the set of non-principal characters; we appeal to [3] for a background on the basic properties of multiplicative characters, such as orthogonality.

For any $\chi \in \mathcal{X}_n$ we also define the character sum

$$S_{\alpha,d}(\chi) = \sum_{g \in \mathcal{P}_d} \Lambda(g)\chi\left(g(\alpha)\right).$$

A simple combinatorial argument shows that for the principal character χ_0 we have

$$S_{\alpha,d}(\chi_0) = \sum_{g \in \mathcal{P}_d} \Lambda(g) = q^d, \tag{2}$$

see, for example, [5, Corollary 3.21].

As in [6], we recall that by [7, Theorem 2.1] we have:

Lemma 3. *For any $\chi \in \mathcal{X}_n^*$ we have*

$$|S_{\alpha,d}(\chi)| \le (n-1)q^{d/2}.$$

We also consider the set $\mathcal{I}_d$ of irreducible polynomials of degree d, that is,

$$\mathcal{I}_d = \{h \in \mathbb{F}_q[X] \ : \ \deg h = d, \ h \in \mathbb{F}_q[X] \text{ irreducible}\},$$

and the sums

$$T_{\alpha,d}(\chi) = \sum_{h \in \mathcal{I}_d} \chi\left(h(\alpha)\right).$$

Our new ingredient is the following bound "on average".

Lemma 4. *Let $m = \lceil n/d \rceil - 1$. Then*

$$\sum_{\chi \in \mathcal{X}_n} |T_{\alpha,d}(\chi)|^{2m} \le m!(q^n - 1)(\#\mathcal{I}_d)^m.$$

Proof. Using the orthogonality of characters, we see that

$$\sum_{\chi \in \mathcal{X}_n} |T_{\alpha,d}(\chi)|^{2m} = (q^n - 1)N,$$

where N is the number of solutions to the equation

$$h_1(\alpha)\dots h_m(\alpha) = h_{m+1}(\alpha)\dots h_{2m}(\alpha),$$

with some $h_1,\dots,h_{2m} \in \mathcal{I}_d$. Since $dm < n$ this implies the identity

$$h_1(X)\dots h_m(X) = h_{m+1}(X)\dots h_{2m}(X)$$

in the ring of polynomials over $\mathbb{F}_q$. Thus, using the uniqueness of polynomial factorisation, we obtain

$$W \le m!(\#\mathcal{I}_d)^m,$$

which concludes the proof. □

Finally, we recall the well-know formula (see, for example, [5, Theorem 3.25])

$$\#\mathcal{I}_d = \frac{1}{d}\sum_{s\mid d}\mu(s)q^{d/s}, \tag{3}$$

where $\mu(s)$ is the Möbius function, that is,

$$\mu(s) = \begin{cases} (-1)^\nu & \text{if } s \text{ is a product } \nu \text{ distinct primes,}\\ 0 & \text{otherwise.}\end{cases}$$

3. Proof of Theorem 1

Let as before $m = \lceil n/d \rceil - 1$. For an integer $k > 2m$ and $v \in \mathbb{F}_{q^n}^*$ we consider

$$M_k(\alpha, d; v) = \sum_{\substack{g_1,\dots,g_{k-2m}\in\mathcal{P}_d \quad h_1,\dots,h_{2m}\in\mathcal{I}_d\\ g_1(\alpha)\dots g_{k-2m}(\alpha)h_1(\alpha)\dots h_{2m}(\alpha)=v}} \Lambda(g_1)\dots\Lambda(g_{k-2m}).$$

Clearly, if for some k we have $M_k(\alpha, d; v) > 0$ for every $v \in \mathbb{F}_{q^n}^*$ then $D(\alpha, d) \le k$.

We now closely follow the same path as in the proof of [6, Theorem 15]. In particular, using the orthogonality of characters we write

$$M_k(\alpha, d; v) = \frac{1}{q^n-1}\sum_{g_1,\dots,g_{k-2m}\in\mathcal{P}_d}\ \sum_{h_1,\dots,h_{2m}\in\mathcal{I}_d}\Lambda(g_1)\dots\Lambda(g_{k-2m})$$
$$\sum_{\chi\in\mathcal{X}_n}\chi\left(g_1(\alpha)\dots g_{k-2m}(\alpha)h_1(\alpha)\dots h_{2m}(\alpha)v^{-1}\right).$$

Changing the order of summation, separating the term corresponding to χ_0, and recalling (2), we derive

$$M_k(\alpha, d; v) - \frac{q^{d(k-2m)}(\#\mathcal{I}_d)^{2m}}{q^n - 1} = \frac{1}{q^n-1}\sum_{\chi\in\mathcal{X}_n^*} \chi(v^{-1}) S_{\alpha,d}(\chi)^{k-2m} T_{\alpha,d}(\chi)^{2m}.$$

Therefore

$$\left|M_k(\alpha, d; v) - \frac{q^{d(k-2m)}(\#\mathcal{I}_d)^{2m}}{q^n - 1}\right| \le \frac{1}{q^n-1}\sum_{\chi\in\mathcal{X}_n^*} \left|S_{\alpha,d}(\chi)\right|^{k-2m} \left|T_{\alpha,d}(\chi)\right|^{2m}.$$

Using Lemma 3 and then (after extending the summation over all $\chi \in \mathcal{X}_n$) using Lemma 4, we derive

$$\left|M_k(\alpha, d; v) - \frac{q^{d(k-2m)}(\#\mathcal{I}_d)^{2m}}{q^n - 1}\right| \le m!(n-1)^{k-2m} q^{d(k/2-m)}(\#\mathcal{I}_d)^m. \tag{4}$$

Thus, if for some $v \in \mathbb{F}_{q^n}^*$ we have $M_k(\alpha, d; v) = 0$ then

$$\frac{q^{d(k-2m)}(\#\mathcal{I}_d)^{2m}}{q^n - 1} \le m!(n-1)^{k-2m} q^{d(k/2-m)}(\#\mathcal{I}_d)^m$$

or

$$\left(\frac{q^{d/2}}{n-1}\right)^k \le m!(n-1)^{-2m}(q^n-1)q^m(\#\mathcal{I}_d)^{-m}. \tag{5}$$

Now, as in the proof of [6, Theorem 9] we note that

$$\#\mathcal{I}_d \ge \frac{q^d}{d} - \frac{2q^{d/2}}{d}.$$

Hence (5) implies that

$$\left(\frac{q^{d/2}}{n-1}\right)^k \le m!(n-1)^{-2m} d^m (q^n-1)\left(1 - 2q^{-d/2}\right)^{-m}.$$

Note that since $n > 2d + 1$, we have $m \ge 2$. Hence, by the Stirling inequality,

$$m! \le \sqrt{2\pi} m^{m+1/2} e^{-m+1/12m} \le \sqrt{2\pi} m^{m+1/2} e^{-m+1/24}. \tag{6}$$

Thus, using that $m \le (n-1)/d$, we see that

$$m! d^m \le \sqrt{2\pi} m^{1/2} (n-1)^m e^{-m+1/24}. \tag{7}$$

Since $d \ge 2$ and $2d+1 \le n < q^{d/2}+1$ we have $q^{d/2} > 4$. Thus $q^{d/2} \ge 5$. Furthermore, since $m \le (n-1)/2 < q^{d/2}/2$, we also have

$$\left(1-2q^{-d/2}\right)^{-m} \le \left(1-2q^{-d/2}\right)^{-q^{d/2}/2} \le (1-2/5)^{-5/2} < 3.6. \tag{8}$$

Hence, recalling that $m \le (n-1)/d \le (n-1)/2$, we derive from (7) and (8) that

$$\begin{aligned}\left(\frac{q^{d/2}}{n-1}\right)^k &< 3.6\sqrt{2\pi}m^{1/2}(n-1)^{-m}q^n e^{-m+1/24}\\ &\le \sqrt{\pi}(n-1)^{-m+1/2}q^n e^{-m+1/24}\\ &\le \sqrt{\pi}\left(e(n-1)\right)^{-m+1/2} q^n e^{-11/24}.\end{aligned}$$

Since $m \ge (n-1)/d - 1$, we conclude that

$$m - \frac{1}{2} \ge \frac{n}{d} - 2.$$

Therefore,

$$\left(e(n-1)\right)^{-m+1/2} \le \left(e(n-1)\right)^{-n/d+2},$$

which finally implies

$$\begin{aligned}k &\le 2\frac{n\log q - (n/d-2)(1+\log(n-1)) + \log(3.6\sqrt{\pi}) - 11/24}{d\log q - 2\log(n-1)}\\ &\le 2\frac{n\log q - (n/d-2)(1+\log(n-1)) + 1.4}{d\log q - 2\log(n-1)}\\ &= \frac{2n}{d}\left(1 + \frac{\log(n-1)-1}{d\log q - 2\log(n-1)}\right) + \frac{4\log(n-1)+6.8}{d\log q - 2\log(n-1)},\end{aligned}$$

which concludes the proof.

4. Proof of Theorem 2

We now put $m = n-1$. Note that the set $\mathcal{P}_1$ is the set of q linear polynomials $X+u$, $u \in \mathbb{F}_q$. For an integer $k > 2m$ and $v \in \mathbb{F}_{q^n}^*$ we consider

$$N_k(\alpha; v) = \sum_{\substack{u_1,\ldots,u_k \in \mathbb{F}_q\\ (u_1+\alpha)\ldots(u_k+\alpha)=v}} 1.$$

Clearly, if for some k we have $N_k(\alpha; v) > 0$ for every $v \in \mathbb{F}_{q^n}^*$ then $\Delta(\alpha) \le k$.

Using the same argument as in the proof Theorem 1, we obtain the following analogue of (4)

$$\left|N_k(\alpha; v) - \frac{q^k}{q^n-1}\right| \le m!(n-1)^{k-2m}q^{k/2} = (n-1)!(n-1)^{k-2n+2}q^{k/2}.$$

Thus if for some $v \in \mathbb{F}_{q^n}^*$ we have $N_k(\alpha; v) = 0$ then

$$\left(\frac{q^{1/2}}{n-1}\right)^k \le (n-1)!(n-1)^{-2n+2}(q^n - 1). \tag{9}$$

The inequality (9) together with the Stirling inequality (6) imply that, for $n \ge 3$,

$$\left(\frac{q^{d/2}}{n-1}\right)^k \le \sqrt{2\pi}(n-1)^{-n+3/2} q^n e^{-n+1+1/12(n-1)}.$$

Using the inequality

$$\log\left(\sqrt{2\pi} e^{1+1/12(n-1)}\right) = \frac{25}{24} + \frac{1}{2}\log(2\pi) \le 2,$$

that holds for $n \ge 3$, we obtain

$$\begin{aligned} k &\le 2\frac{n\log q - (n-3/2)\log(n-1) - n + 2}{\log q - 2\log(n-1)} \\ &= 2n\left(1 + \frac{\log(n-1) - 1}{\log q - 2\log(n-1)}\right) + \frac{3\log(n-1) + 2}{\log q - 2\log(n-1)}, \end{aligned}$$

and the result now follows.

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