A MIXED MOCK MODULAR SOLUTION OF KANEKO – ZAGIER EQUATION

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Abstract. The notion of mixed mock modular forms was recently introduced by Don Zagier. We show that certain solutions of Kaneko - Zagier differential equation constitute simple yet non-trivial examples of this notion. That allows us to address a question posed by Kaneko and Koike on the (non)-modularity of these solutions.

1. Introduction

The differential equation

$$f''(\tau) - \frac{k+1}{6} E_2(\tau) f'(\tau) + \frac{k(k+1)}{12} E_2'(\tau) f(\tau) = 0 \tag{KZ_k}$$

initially appeared in the paper by Kaneko and Zagier [10] in the connection with supersingular j-invariants. Here and throughout $\tau = x + iy$ with $y > 0$ is a variable in the upper half-plane $\mathfrak{H}$, and as in [10, 6], the symbol $'$ denotes the differentiation with respect to $2\pi i \tau$, specifically $f' := (2\pi i)^{-1} df/d\tau$. The Eisenstein series E_k is defined by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \left(\sum_{d|n} d^{k-1} \right) q^n,$$

where $q := \exp(2\pi i \tau)$ and B_k is the k-th Bernoulli number. We denote by M_k the space of modular forms of weight k on $SL_2(\mathbb{Z})$. In particular, $E_k \in M_k$ for even integer $k > 2$, while $E_2 \notin M_k$ is a weight 2 quasi-modular form.

Equation (KZ_k) was further investigated in a series of papers by Kaneko and Koike [6, 7, 8, 9]. The general theory of differential equations predicts the existence of two linearly independent holomorphic solutions of (KZ_k) locally. In many cases two global linearly independent holomorphic solutions of this equation were found (see [10, Theorem 5] and [6, Theorem 1]), and these solutions turn out to be modular forms. In contrast, in the case when k is an even integer congruent to 0 or 4 modulo 6 (that is the case initially considered in [10]) only one solution, $f_k \in M_k$ was found. On the basis of numerical experiments, Kaneko and Koike conjecture in [6], in particular, that the second solution is not modular, and called for the investigation of its arithmetic nature. In fact, it is not obvious even that there exist a second global holomorphic solution on $\mathfrak{H}$ in this case. (The fact that all solutions already

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found are indeed global, i.e. are holomorphic functions on $\mathfrak{H}$, is clear since these solutions are written down explicitly in [10, 6].)

On the other hand, Zagier recently introduced a notion of *mixed mock modular forms*, which are certain holomorphic functions on $\mathfrak{H}$. The definition is given in a preprint by A.Dabholkar, S. Murthy, and D. Zagier [3, Section 7.3]. In his lecture at "Mock theta functions and applications in combinatorics, algebraic geometry, and mathematical physics" May 2009 conference at MPI Bonn, Zagier justified introduction of the new notion by important examples such as indefinite theta series and Gaiotto function related to super Liouville elliptic genus. These examples are quite complicated. The primary goal of this note is to present a rather simple yet non-trivial example: a second solution of equation KZ_k in the case when $k \equiv 4 \mod 6$. In contrast, in the case when $k \equiv 0 \mod 6$, although a second solution of (KZ_k) is also a holomorphic function on $\mathfrak{H}$, is bounded at $i\infty$, and has a quite similar behavior under the action of $SL_2(\mathbb{Z})$, this function is not a mixed mock modular form. We also shed some light on the question of Kaneko and Koike mentioned above in the case when $k \equiv 0$ or $4 \mod 6$ (see Corollary 1 below).

We introduce and discuss in some details Zagier's definition of mixed mock modular forms in Section 3, and we denote by $\mathbb{M}_{u,v}(\chi_1, \chi_2)$ the space of mixed mock modular forms of even integer weights (u, v) on $SL_2(\mathbb{Z})$ with Nebentypus (χ_1, χ_2). In order to state the principal result of this paper we introduce some notations, and provide a specific example of a mixed mock modular form now.

Let

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n\geq 1}(1 - q^n)$$

be Dedekind's η-function. Throughout χ is the homomorphism of $SL_2(\mathbb{Z})$ to the group of sixth roots of unity defined by

$$\chi(\sigma) = \frac{\eta^4(\sigma(\tau))}{(c\tau + d)^2\eta^4(\tau)} \quad \text{for} \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{1}$$

We denote by $M_k(\chi)$ the space of modular forms on $SL_2(\mathbb{Z})$ with Nebentypus χ, in particular, $\eta^4 \in M_2(\chi)$. Let h_4 be the meromorphic function on $\mathfrak{H}$ such that $\lim_{y\to\infty} h_4(\tau) = 0$, and

$$h_4' = \frac{\eta^{20}}{E_4^2}.$$

The existence of h_4 follows from Proposition 3 below. The function $f_4 = E_4 \in M_4$ is a solution of (KZ_k) for $k = 4$. We show, in particular, (see Proposition 2 and Proposition 4 for details) that $E_4h_4 \in \mathbb{M}_{4,0}(triv, \overline{\chi})$ is also a solution of (KZ_k) for $k = 4$. Here and throughout *triv* stays for the trivial character.

Define a holomorphic function $\mathcal{E}$ on $\overline{\mathfrak{H}}$ (an Eichler integral) by

$$\mathcal{E}(\tau) := 2\pi i \int_{i\infty}^{\tau} \eta^4(z)\, dz.$$

Theorem 1. *Let $k \geq 0$ be an even positive integer congruent to 0 or 4 modulo 6, and let $f_k = 1 + O(q)$ be the modular solution of (KZ_k).*

(i) *If $k \equiv 0 \mod 6$, then equation (KZ_k) admits a solution*

$$F_k = A_k\mathcal{E} + B_k$$

with $A_k \in M_k$ which is a non-zero multiple of f_k, and $B_k \in M_k(\chi)$.

(ii) *If $k \equiv 4 \mod 6$, then equation (KZ_k) admits a mixed mock modular form*

$$F_k \in \mathbb{M}_{k,0}(triv, \overline{\chi})$$

as a solution. Furthermore,

$$F_k = C_k h_4 + D_k$$

with $C_k \in M_k$ which is a non-zero multiple of f_k, and $D_k \in M_k(\overline{\chi})$.

Remark 1. In the case when $k \equiv 4 \mod 6$, one can represent the functions F_k as $C\widetilde{\mathcal{E}} + D$ with weakly holomorphic modular forms C and D, and an Eichler integral $\widetilde{\mathcal{E}}$ of a weakly holomorphic cusp form (cf. [2]). This representation, however, does not capture the behavior of F_k at cusps: it admits an exponential growth. In fact, being a mixed mock modular form, the functions F_k have no more than polynomial growth at cusps (see [3] and Section 3). This property is, of course, obvious in the case $k \equiv 0 \mod 6$ when we deal with a usual Eichler integral and holomorphic modular forms.

Theorem 1 allows us to derive a corollary concerning the modularity of the solutions of (KZ_k) since the transformation laws of both $\mathcal{E}(6\tau)$ and $h_4(6\tau)$ with respect to $\Gamma_0(36)$ are clear. The function $\mathcal{E}(6\tau)$ is an Eichler integral of a weight two primitive cusp form $\eta^4(6\tau)$. The function $h_4(6\tau)$ differs (cf. (9) below) by a (meromorphic) modular function on $\Gamma_0(36)$ from mock modular form M^+ whose shadow is a constant multiple of $\eta^4(6\tau)$. In both cases, we thus have group homomorphisms $\nu_i : \Gamma_0(36) \to \mathbb{C}$ with $i = 1, 2$ defined by

$$\nu_1(\sigma) = h_4(6\sigma(\tau)) - h_4(6\tau), \quad \nu_2(\sigma) = \mathcal{E}(6\sigma(\tau)) - \mathcal{E}(6\tau).$$

Since the images of both homomorphisms are non-degenerate lattices in $\mathbb{C}$, their kernels, $K_i = \ker(\nu_i)$ are normal subgroups of $\Gamma_0(36)$ of infinite indexes. (Equations (4) and (10) below imply that in fact $K_1 = K_2 = K$.) Clearly, the functions $\mathcal{E}(6\tau)$ and $h_4(6\tau)$ are modular on K, and are not modular on any bigger subgroup of $\Gamma_0(36)$.

It follows from Theorem 1 that, although an individual solution of (KZ_k) may be not modular on $\Gamma_0(36)$, the two-dimensional space of solutions is, in a certain sense, modular. In order to formulate that a bit nicer, we consider a slight modification of (KZ_k),

$$(\natural_k) \qquad f''(\tau) - (k+1)E_2(6\tau)f'(\tau) + \frac{k(k+1)}{2}E_2'(6\tau)f(\tau) = 0,$$

such that $f(\tau)$ is a solution of (KZ_k) if an only if $g(\tau) = f(6\tau)$ is a solution of ($\natural_k$).

Corollary 1. *Let $k \geq 0$ be an even positive integer congruent to 0 or 4 modulo 6.*

(i) *Let $G(\tau)$ be a solution of ($\natural_k$). Then for any $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(36)$ the function*

$$G_\sigma(\tau) := (c\tau + d)^{-k} G\left(\frac{a\tau + b}{c\tau + d}\right)$$

is also a solution of ($\natural_k$).

(ii) *There exists an infinite index normal subgroup $K \trianglelefteq \Gamma_0(36)$ such that every solution of ($\natural_k$) is modular of weight k (and trivial Nebentypus) on K, while no solution of ($\natural_k$) which is not a constant multiple of the modular solution f_k is modular on any normal subgroup $\Gamma \trianglelefteq \Gamma_0(36)$ of finite index.*

Remark 2. The modularity of the space of solutions claimed in Corollary 1(i) can be also proved in a straightforward way. For instance the argument in the proof of [6, Proposition 2] proves a stronger statement of the modularity of the space of solutions of (KZ_k) on $SL_2(\mathbb{Z})$. The non-modularity claim of Corollary 1(ii) answers a question posed in [6, Remark 2] on the non-modularity of these solutions.

We introduce necessary notations and discuss in some details Zagier's definition of mixed mock modular forms in Section 3, and prove Theorem 1 in Section 4. Both our proof of Theorem 1, and our way to derive Corollary 1 may be built on results from [4], where the existence of a certain mock modular form M^+ was shown by making use of general theory of weak harmonic Maass forms. In Section 2 of this paper we provide an alternative more explicit construction of M^+ which uses nothing but the classical theory of modular forms and elliptic curves along with some computer calculations.

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2. A weight zero weak harmonic Maass form

In this section, we construct of a certain weight zero weak harmonic Maass form M. A general definition of mock modular forms parallel to the definition of mixed mock modular forms is given in Section 3. For the purposes of this section we recall that a weight zero weak harmonic Maass form on a subgroup $\Gamma \subset SL_2(\mathbb{Z})$ (with trivial Nebentypus) is a Γ-invariant harmonic function M on $\mathfrak{H}$. Being a harmonic function, M decomposes into a sum $M = M^+ + M^-$ of a holomorphic function M^+ and anti-holomorphic function M^-. The holomorphic part M^+ of a weak harmonic Maass form M is called a mock modular form.

Let

$$g = \eta^4(6\tau) = q - 4q^7 + 2q^{13} + 8q^{19} - 5q^{25} - 4q^{31} - 10q^{37} + 8q^{43} + 9q^{49} + 14q^{61} - 16q^{67} - 10q^{73} + \dots$$

be the unique normalized cusp form in $S_2(\Gamma_0(36))$ of weight 2 on $\Gamma_0(36)$ with trivial character.

Proposition 1. *There exists a weight zero weak harmonic Maass form M such that*

$$\frac{\partial M}{\partial \tau} = \frac{dM^+}{d\tau} = 2\pi i \frac{E_4(6\tau)}{\eta^4(6\tau)}, \tag{2}$$

and

$$\frac{\partial M}{\partial \overline{\tau}} = \frac{dM^-}{d\overline{\tau}} = t\overline{g} \tag{3}$$

for a non-zero $t \in \mathbb{C}^*$.

Remark 3. The conditions (2) and (3) determine M up to addition of a constant while the condition (3) alone determines M up to addition of a weakly holomorphic (with poles only at cusps) modular function on Γ.

Remark 4. In [4], Proposition 1 was derived from general theory of weak harmonic Maass forms. We offer a different proof based on an explicit construction here.

Proof. The Eichler integral

$$\mathcal{E}_g(\tau) = -2\pi i \int_\tau^{i\infty} g(z)\, dz = \frac{\pi i}{3}\mathcal{E}(6\tau)$$

determines a homomorphism $\nu \;:\; \Gamma_0(36) \to \mathbb{C}$ by

$$\nu(\gamma) = \mathcal{E}_g(\gamma(\tau)) - \mathcal{E}_g(\tau).$$

(It easily follows from $g \in S_2(\Gamma_0(36))$ that $\nu(\gamma)$ does not depend on τ for $\gamma \in \Gamma_0(36)$.) The image of ν is a lattice $\Lambda \subset \mathbb{C}$, and the function $\mathcal{E}_g$ performs an isomorphism of smooth projective curves of genus one

$$\mathcal{E}_g \;:\; X_0(36) = \overline{\Gamma_0(36)\backslash\mathfrak{H}} \to \mathbb{C}/\Lambda.$$

(The CM elliptic curve $E = \mathbb{C}/\Lambda$ has Weierstrass equation $y^2 = x^3 + 1$.) We now consider Weierstrass ζ-function

$$\zeta(\Lambda, z) = \frac{1}{z} + \frac{1}{35}z^5 - \frac{1}{7007}z^{11} + \frac{1}{1440257}z^{17} - \frac{3}{886605265}z^{23} + \frac{4}{242582910115}z^{29} + \dots$$

associated with Λ. Although meromorphic function $\zeta(\Lambda, z)$ is not Λ-periodic, the function $\zeta^*(\Lambda, z) = \zeta(\Lambda, z) + \lambda\overline{z}$ is. Here (cf. [12]) $\lambda = -\pi/Vol(\Lambda)$, where $Vol(\Lambda)$ is the volume of a fundamental parallelogram of the lattice. It follows that the harmonic on $\mathfrak{H}$ function $\zeta^*(\Lambda, \mathcal{E}_g)$ is $\Gamma_0(36)$-invariant, and satisfies

$$\frac{\partial \zeta^*(\Lambda, \mathcal{E}_g)}{\partial \overline{\tau}} = 2\pi i \lambda \overline{g}$$

by construction.

The holomorphic on $\mathfrak{H}$ function

$$\zeta^*(\Lambda, \mathcal{E}_g)^+ := \zeta(\Lambda, \mathcal{E}_g) = q^{-1} + \frac{3}{5}q^5 + \frac{1}{11}q^{11} - \frac{5}{17}q^{17} - \frac{8}{23}q^{23} - \frac{1}{29}q^{29} + \frac{4}{5}q^{35} + \frac{11}{41}q^{41} - \frac{10}{47}q^{47} + \dots$$

is a mock modular form.

Let $K(\tau)$ be an holomorphic antiderivative of $E_4(6\tau)/\eta^4(6\tau)$,

$$K(\tau) = -\frac{1}{q} + \frac{244}{5}q^5 + \frac{3134}{11}q^{11} + \frac{18760}{17}q^{17} + \frac{84345}{23}q^{23} + \frac{306252}{29}q^{29} + \frac{140482}{5}q^{35} + \dots.$$

A quick computer calculation shows that

$$K + 12\zeta(\Lambda, \mathcal{E}_g) = 11q^{-1} + 56q^5 + 286q^{11} + 1100q^{17} + 3663q^{23} + 10560q^{29} + 28106q^{35} + \dots,$$

and absence of the denominators suggests that $K + 12\zeta(\Lambda, \mathcal{E}_g)$ is a weakly holomorphic (i. e. with its poles only in cusps) modular form. Indeed we have the identity

$$\frac{E_2(6\tau) - 2E_2(12\tau)}{g(\tau)} + \frac{3}{2}\frac{E_2(6\tau) - 3E_2(18\tau)}{g(\tau)} - 3\frac{E_2(6\tau) - 6E_2(36\tau)}{g(\tau)} = K + 12\zeta(\Lambda, \mathcal{E}_g) \tag{4}$$

which requires only a finite number of q-expansion coefficients to check since the derivatives of all terms are weakly holomorphic weight two modular forms on $\Gamma_0(36)$. We thus conclude that the function

$$M := -12\zeta^*(\Lambda, \mathcal{E}_g) + \frac{E_2(6\tau) - 2E_2(12\tau)}{g(\tau)} + \frac{3}{2}\frac{E_2(6\tau) - 3E_2(18\tau)}{g(\tau)} - 3\frac{E_2(6\tau) - 6E_2(36\tau)}{g(\tau)}$$

satisfies all requirements of Proposition 1. $\square$

3. Mixed Mock Modular Forms

In this section, we introduce both mock modular forms and mixed mock modular forms in a way which allows us to reveal the similarities and differences between these objects. We try to follow the exposition given by Don Zagier in a lecture at MPI (see also [3, Section 7.3]). For the sake of brevity and clarity we limit ourselves with even integral weights and cuspidal shadows here. The case of half-integral weights is quite similar, and possible presence of Eisenstein series as shadows requires only a little more work. (For an alternative definition of mock modular forms and a detailed discussion of their properties and important applications see [11].)

Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a congruence subgroup. For a finite order group homomorphism $\psi : \Gamma \to \mathbb{C}^*$, and a non-negative even integer k, we use standard notations $M_k(\Gamma, \psi)$ (resp. $S_k(\Gamma, \psi)$) for the spaces of holomorphic modular (resp. cusp) forms of weight k on Γ with Nebentypus ψ. For even integers u and v, denote by $\mathfrak{M}^!_{u+v}(\Gamma, \psi)$ the linear space of real-analytic functions $\mathcal{F}$ on the upper half-plane $\mathfrak{H}$ which satisfy the transformation law

$$\mathcal{F}\left(\frac{a\tau + b}{c\tau + d}\right) = \psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(c\tau + d)^{u+v}\mathcal{F}(\tau) \quad \text{for every} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tag{5}$$

and have at most linear exponential growth at cusps. The subspace $\mathfrak{M}_{u+v}(\Gamma, \psi) \subset \mathfrak{M}^!_{u+v}(\Gamma, \psi)$ consists of those functions which have at most polynomial growth at cusps. In the case $\Gamma = SL_2(\mathbb{Z})$, which is of primary interest for the purposes of this note, the latter condition simplifies to the existence of the limit

$$\lim_{\Im(\tau) \to \infty} \mathcal{F}(\tau). \tag{6}$$

It is easy to check that the linear operator

$$\partial_v := \Im(\tau)^v \frac{\partial}{\partial \overline{\tau}}$$

takes functions satisfying the transformation law (5) to functions Φ satisfying the transformation law

$$\Phi\left(\frac{a\tau + b}{c\tau + d}\right) = \psi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(c\tau + d)^u (c\overline{\tau} + d)^{2-v}\Phi(\tau) \quad \text{for every} \quad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \tag{7}$$

The linear space of all real-analytic functions Φ on $\mathfrak{H}$ which satisfy (7) contains the subspace

$$W(\chi_1, \chi_2) := M_u(\Gamma, \chi_1) \otimes \overline{S_{2-v}(\Gamma, \overline{\chi_2})} \quad (\text{with } \chi_1\chi_2 = \psi),$$

In order to have a non-empty space $W(\chi_1, \chi_2)$ we shall assume that $u \geq 0$ and $v \leq 0$.

Let us firstly consider the special case $u = 0$ in order to construct *mock modular forms.* In this case $M_u(\Gamma, \chi_1)$ is empty unless χ_1 is trivial, and contains nothing but constants in the latter case. We thus end up with

$$W(triv, \psi) = \overline{S_{2-v}(\Gamma, \overline{\psi})}.$$

Consider the space $\mathcal{H}^!_v(\Gamma, \psi)$,

$$\mathcal{H}^!_v(\Gamma, \psi) := \partial_v^{-1}\left(W(triv, \psi)\right) \subset \mathfrak{M}^!_v(\Gamma, \psi),$$

defined as the preimage of $W(triv, \psi)$ under ∂_v in $\mathfrak{M}^!_v(\Gamma, \psi)$. This is the space of weak harmonic Maass forms on Γ with Nebentypus ψ of weight v. Note that [1, Proposition 3.5] implies that

$$\mathcal{H}^!_v(\Gamma, \psi) \bigcap \mathfrak{M}_v(\Gamma, \psi) = \emptyset. \tag{8}$$

For $g \in S_{2-v}(\Gamma, \overline{\psi})$ define

$$g^*(\tau) = (2i)^{-v} \int_{-\overline{\tau}}^{i\infty} \overline{g(-\overline{z})}(\tau + z)^{-v}\, dz.$$

Then $\partial_v(g^*) = \overline{g} \in \overline{S_{2-v}(\Gamma, \overline{\psi})}$.

If now $F \in \mathcal{H}^!_v(\Gamma, \psi)$ we let

$$\overline{g_F} = \partial_v(F) \in \overline{S_{2-v}(\Gamma, \overline{\psi})},$$

and the real-analytic on $\mathfrak{H}$ function $F - g_F^*$ is holomorphic on $\mathfrak{H}$ since

$$\frac{\partial}{\partial \overline{\tau}}\left(F - g_F^*\right) = 0$$

by construction. The function $F - g_F^*$ being the canonically defined holomorphic part of F is referred to as a *mock modular form* of weight v with a shadow of g, and we denote the linear space of these mock modular forms by $\mathbb{M}_v(\Gamma, \psi)$.

We now consider the case $u > 0$ in order to present a parallel constuction of *mixed mock modular forms.* Similarly to the previous case, we define $\mathcal{H}^!_{u,v}(\Gamma, \chi_1, \chi_2)$ as the preimage of $W(\chi_1, \chi_2)$ under ∂_v in $\mathfrak{M}^!_{u+v}(\Gamma, \psi)$:

$$\mathcal{H}^!_{u,v}(\Gamma, \chi_1, \chi_2) := \partial_v^{-1}\left(W(\chi_1, \chi_2)\right) \subset \mathfrak{M}^!_{u+v}(\Gamma, \psi).$$

If we proceed as previously, we end up with the linear space of canonically defined holomorphic parts of functions $F \in \mathcal{H}^!_{u,v}(\Gamma, \chi_1, \chi_2)$ which (see [1]) is nothing but $M^!_u(\Gamma, \chi_1) \otimes \mathbb{M}_{2-v}(\Gamma, \chi_2)$, and this space does *not* deserve any special name.

However, there is a substantial difference from the case when $u = 0$: an analog of (8) does not need to hold, we define

$$\mathcal{H}_{u,v}(\Gamma, \chi_1, \chi_2) := \mathcal{H}^!_{u,v}(\Gamma, \chi_1, \chi_2) \bigcap \mathfrak{M}_{u+v}(\Gamma, \psi),$$

and apply a construction similar to the that in the case $u = 0$ to the space $\mathcal{H}_{u,v}(\Gamma, \chi_1, \chi_2)$. Specifically, for $F \in \mathcal{H}_{u,v}(\Gamma, \chi_1, \chi_2)$ we have that $\partial_v(F) \in W(\chi_1, \chi_2)$, therefore

$$\partial_v(F) = \sum_j f_j \overline{g}_j,$$

where the sum is finite, $f_j \in M_u(\Gamma, \chi_1)$ and $\overline{g_j} \in \overline{S_{2-v}(\Gamma, \overline{\chi_2})}$. Again the real-analytic on $\mathfrak{H}$ function

$$\mathcal{F}_F := F - \sum_j f_j g_j^*$$

satisfies

$$\frac{\partial \mathcal{F}_F}{\partial \overline{\tau}} = \frac{\partial}{\partial \overline{\tau}} \left(F - g_F^* \right) = 0,$$

and is therefore holomorphic on $\mathfrak{H}$. Moreover, since F has at most polynomial growth at cusps, so does $\mathcal{F}_F$. The function $\mathcal{F}_F = F - \sum_j f_j g_j^*$ being the canonically defined holomorphic part of F is referred to as a *mixed mock modular form* of weight (u, v) and Nebentypus (χ_1, χ_2). We denote by $\mathbb{M}_{u,v}(\chi_1, \chi_2)$ the linear space of mixed mock modular forms. We emphasize that, by construction, all mixed mock modular forms are holomorphic on the upper half-plane $\mathfrak{H}$, and, in contrast to the case of mock modular forms, have at most polynomial growth at cusps. For this reason, no mock modular form is a mixed mock modular form. However, a product of a mock modular form with a cusp form may be a mixed mock modular form if the cusp form has zeros of sufficiently large order in all cusps.

There is another space of functions which are also holomorphic on the upper half-plane and transform quite similarly to mixed mock modular forms. For a modular form $g \in S_{2-v}(\Gamma, \chi_2)$, denote by $\mathcal{E}_g$ the Eichler integral

$$\mathcal{E}_g(\tau) = \int_{i\infty}^{\tau} g(z)\, dz.$$

The linear space of functions

$$\{ A\mathcal{E} + B \mid A \in M_u(\Gamma, \chi_1), \;\; B \in M_{u+v}(\Gamma, \chi_1\chi_2) \}$$

shows up naturally if one allows weakly holomorphic cusp forms in the definition of $W(\chi_1, \chi_2)$ (i.e. considers $\widetilde{W}(\chi_1, \chi_2) := M_u(\Gamma, \chi_1) \otimes \overline{S_{2-v}^!(\Gamma, \overline{\chi_2})} \supset W(\chi_1, \chi_2)$ instead). Despite of this similarity, we do not call these functions mixed mock modular forms.

The following proposition provides us with an example which illustrates non-triviality of the notion of mixed mock modular form. We will later make use of this example.

Proposition 2. *There exists a meromorphic function h_4 on $\mathfrak{H}$ such that*

$$h_4'(\tau) = \frac{\eta^{20}(\tau)}{E_4^2(\tau)}.$$

We have that

$$F_4(\tau) := E_4(\tau) h_4(\tau) \in \mathbb{M}_{4,0}(triv, \overline{\chi})$$

Remark 5. Our description of mixed mock modular forms obviously implies that $\mathbb{M}_{k,0}(triv, \psi) \supseteq M_k(\psi)$. Furthermore, it implies that $\mathbb{M}_{*,0}(triv, \psi) = \oplus_l \mathbb{M}_{l,0}(triv, \psi)$ is a graded vector space over $M_* = \oplus_k M_k$. In particular, Proposition 2 implies that in order to prove $F_k \in \mathbb{M}_{k,0}(triv, \psi)$ claimed in Theorem 1(ii), it suffices to prove that $C_k \in M_k$ and $D_k \in M_k(\overline{\chi})$.

Proof. Let M be the weight zero weak harmonic Maass form constructed in Proposition 1. Remark 3 allows us to assume that the Fourier expansion $M^+(\tau) = -q^{-1} + O(q)$ has no constant term.

The product $E_4(\tau)M(\tau/6)$ thus satisfies (5) with $u+v=4$, and the character χ of $\Gamma = SL_2(\mathbb{Z})$ with values in 6-th roots of unity defined by (1),

$$E_4(\tau)M(\tau/6) \in \mathfrak{M}_4^!(SL_2(\mathbb{Z}), \overline{\chi}).$$

Note that $E_4(\tau)M(\tau/6) \notin \mathfrak{M}_4(SL_2(\mathbb{Z}), \overline{\chi})$, and thus extracting a holomorphic part we will not produce a mixed mock modular form out of this function. However, we claim that

$$\mathcal{F} := E_4(\tau)M(\tau/6) + \frac{E_6(\tau)}{\eta(\tau)^4} \in \mathfrak{M}_4(SL_2(\mathbb{Z}), \overline{\chi}).$$

It suffices to check that the limit (6) exists. Indeed,

$$\lim_{\Im(\tau)\to\infty} \mathcal{F}(\tau) = \lim_{\Im(\tau)\to\infty} \left(E_4(\tau)M^+(\tau/6) + \frac{E_6(\tau)}{\eta(\tau)^4} + M^-(\tau/6) \right)$$

exists because $\lim_{\Im(\tau)\to\infty} M^-(\tau/6) = 0$, and the function M^+ has a Fourier expansion $M^+ = -q^{-1/6} + O(q^{5/6})$.

We put $v=0$ (and $u=4$), and note that

$$\partial_0\left(E_4(\tau)M(\tau/6)\right) = \frac{t}{6} E_4(\tau)\overline{\eta^4(\tau)} \in M_4 \otimes \overline{S_2(SL_2(\mathbb{Z}), \chi)} = W(triv, \overline{\chi}).$$

We thus conclude that the holomorphic part of $\mathcal{F}$ defined by

$$-576F_4(\tau) := \mathcal{F}(\tau) - E_4(\tau)M^-(\tau/6) = E_4(\tau)M^+(\tau/6) + \frac{E_6(\tau)}{\eta(\tau)^4} \in \mathbb{M}_{4,0}(triv, \overline{\chi})$$

is a mixed mock modular form. We now define a meromorphic function h_4 on $\mathfrak{H}$ by

$$-576h_4(\tau) := \frac{F_4(\tau)}{E_4(\tau)} = M^+(\tau/6) + \frac{E_6(\tau)}{\eta^4(\tau)E_4(\tau)}, \tag{9}$$

and find that

$$h_4'(\tau) = -\frac{1}{576}\left(\frac{F_4(\tau)}{E_4(\tau)}\right)' = -\frac{1}{3456}\frac{E_4(\tau)}{\eta^4(\tau)} - \frac{1}{576}\left(\frac{E_6(\tau)}{\eta^4(\tau)E_4(\tau)}\right)' = \frac{\eta^{20}(\tau)}{E_4^2(\tau)}, \tag{10}$$

the latter identity was found in [5].

□

4. Proof of Theorem 1

Let k be an even positive integer such that $k \equiv 0 \mod 6$ or $k \equiv 4 \mod 6$. Recall that by [10, Theorem 5] and [6, Theorem 1], equation (KZ_k) has a one-dimensional space of modular solutions generated by

$$f_k := 1 + O(q) \in M_k.$$

The following statement will help us to construct a second solution as a function on $\mathfrak{H}$

Proposition 3. *There exists a meromorphic function h_k on $\mathfrak{H}$ such that*

$$h_k' = \frac{\eta^{4(k+1)}}{f_k^2}.$$

Proof. All poles of $\eta^{4(k+1)}/f_k^2$ are located in the interior of the upper half-plane at the points z such that $f(z)=0$. The order of these poles is always 2 since, being a non-zero solution of (KZ_k), the function f may have only simple zeroes. We thus have Laurent expansion around such a point z

$$\frac{\eta^{4(k+1)}}{f_k^2}=\frac{c_{-2}}{(\tau-z)^2}+\frac{c_{-1}}{\tau-z}+c_0+c_1(\tau-z)+\dots.$$

It suffices to prove that $c_{-1}=0$. We multiply the above equation by $(\tau-z)^2$ and take the derivative (recall that $'=(2\pi i)^{-1}d/d\tau$) to obtain

$$2(\tau-z)h_k'+2\pi i(\tau-z)^2h_k''=c_{-1}+2c_0(\tau-z)+\dots,$$

therefore

$$c_{-1}=\left(2(\tau-z)h_k'+2\pi i(\tau-z)^2h_k''\right)|_{\tau=z}.$$

Since

$$\eta'=\frac{1}{24}E_2\eta,$$

we use the definition of h_k' to obtain that

$$\frac{h_k''}{h_k'}=-2\frac{f_k'}{f_k}+\frac{k+1}{6}E_2, \tag{11}$$

and we conclude that

$$c_{-1}=2\pi i\left((\tau-z)h_k'\left(\frac{1}{\pi i}+(\tau-z)\frac{k+1}{6}E_2-2(\tau-z)\frac{f_k'}{f_k}\right)\right)\Bigg|_{\tau=z}. \tag{12}$$

We now write $f_k=(\tau-z)\phi$, therefore

$$\begin{aligned}f_k'&=\frac{1}{2\pi i}\phi+(\tau-z)\phi'\\ f_k''&=\frac{1}{\pi i}\phi'+(\tau-z)\phi''\\ (\tau-z)\frac{f_k'}{f_k}&=\frac{1}{2\pi i}+(\tau-z)\frac{\phi'}{\phi},\end{aligned}$$

and produce a differential equation for ϕ

$$(\tau-z)\phi''+\left(\frac{1}{\pi i}-(\tau-z)\frac{k+1}{6}E_2\right)\phi'+\left((\tau-z)\frac{k(k+1)}{12}E_2'-\frac{1}{2\pi i}\frac{k+1}{6}E_2\right)\phi=0$$

out of (KZ_k). We specialize at $\tau=z$ to obtain that

$$2\phi'(z)=\frac{k+1}{6}E_2\phi(z).$$

We now plug all this data into (12) to obtain that

$$c_{-1}=2\pi i\left((\tau-z)^2h_k'\left(-2\frac{\phi'}{\phi}+\frac{k+1}{6}E_2\right)\right)\Bigg|_{\tau=z}=0$$

as required since the function $(\tau-z)^2h_k'$ is holomorphic at $\tau=z$. $\square$

Proposition 3 along with the fact that h_k' vanishes exponentially at $i\infty$ allows us to conclude that

$$h_k(\tau) = 2\pi i \int_{i\infty}^{\tau} h_k'(z)\ dz$$

makes sense, does not depend on the path of integration as soon as the path misses the zeros of f_k, and defines a meromorphic function on $\mathfrak{H}$.

In the next proposition we make use of the function h_k in order to construct the second holomorphic solution of (KZ_k). This proposition was shown to the author by M. Kaneko.

Proposition 4. *The function*

$$F_k(\tau) := h_k(\tau) f_k(\tau)$$

is holomorphic on the upper half-plane $\mathfrak{H}$*, and satisfies* (KZ_k)*.*

Proof. By construction, h_k is meromorphic with poles of order 1 which are located at the zeroes of f_k, therefore F_k is holomorphic.

In order to check that $F_k = h_k f_k$ satisfies (KZ_k) we simply plug the function into the equation:

$$\begin{aligned}(h_k f_k)'' - \frac{k+1}{6} E_2 (h_k f_k)' + \frac{k(k+1)}{12} E_2'(h_k f_k) \\ = h_k (f_k'' - \frac{k+1}{6} E_2 f_k' + \frac{k(k+1)}{12} E_2' f_k \\ = f_k h_k'' + (2 f_k' - \frac{k+1}{6} E_2 f_k) h_k' = 0,\end{aligned}$$

where the latter equality follows from (11).

□

We will make use of the inductive structure of the solutions of (KZ_k) investigated in [6, Section 3]. Since the function h_k in Proposition 3 is non-constant, and equation (KZ_k) locally has a two-dimensional space of solutions, Proposition 4 allows us to conclude that the whole space of solutions is a two-dimensional space spanned by F_k and f_k. We denote this space by L_k, and define a linear map μ_k from L_k to holomprohic functions on $\mathfrak{H}$ by

$$\mu_k(F) := \frac{[F, E_4]}{\eta^{24}} = \frac{kFE_4' - 4F'E_4}{\eta^{24}}.$$

Proposition 5. *For even integer $k \geq 6$ congruent to 0 or 4 modulo 6, the map μ_k performs an isomorphism $\mu_k\ :\ L_k \to L_{k-6}$.*

Proof. For every $F \in L_k$ we have that $\mu_k(F) \in L_{k-6}$ by a result of Kaneko and Koike [6, Proposition 1(i)]. It thus suffices to check that μ_k has trivial kernel. In other words, it suffices to check that the function $E_4^{k/4}$ is not a solution of (KZ_k). That follows by a straightforward calculation making use of the standard formulas

$$E_2' = \frac{1}{12}\left(E_2^2 - E_4\right), \quad E_4' = \frac{1}{3}\left(E_2 E_4 - E_6\right), \quad E_6' = \frac{1}{2}\left(E_2 E_6 - E_4^2\right).$$

□

We now turn to the proof of Theorem 1.

Proof of Theorem 1. To this end we have two-dimensional spaces L_k of holomorphic on $\mathfrak{H}$ solutions of (KZ_k) along with isomorphisms $\mu_k : L_k \to L_{k-6}$. Since $f_k \in M_k$, we have that $\mu_k(f_k) \in M_{k-6}$ by standard properties of Rankin - Cohen bracket. Moreover, a result of Kaneko and Koike [6, Proposition 1(ii)] implies that

$$\mu_k(f_k) = \alpha_k f_{k-6}$$

with $\alpha_k \neq 0$. Since $\mu_k(F_k) \in L_{k-6}$, we have that

$$\mu_k(F_k) = \gamma_k F_{k-6} + \delta_k f_{k-6}$$

with $\gamma_k \neq 0$ because μ_k is an isomorphism. We also have that

$$\mu_k(F_k) = \mu_k(f_k h_k) = \frac{[f_k, E_4]h_k - 4f_k h_k' E_4}{\eta^{24}} = \mu_k(f_k)\frac{F_k}{f_k} - \frac{4E_4\eta^{4(k-5)}}{f_k}.$$

Combining the three equations above, we the inductive relation

$$\alpha_k F_k = \gamma_k F_{k-6}\frac{f_k}{f_{k-6}} + \delta_k f_k + \frac{4E_4\eta^{4(k-5)}}{f_{k-6}} \tag{13}$$

with $\alpha_k\gamma_k \neq 0$. We take into the account the following description of the solutions of (KZ_k) for $k = 0$ and $k = 4$

$$f_0 = 1, \quad F_0 = \mathcal{E}, \quad f_4 = E_4, \quad F_4 = E_4 h_4,$$

and make use of (13) for an inductive argument to conclude that

$$F_k = \begin{cases} A_k\mathcal{E} + B_k & \text{if } k \equiv 0 \mod 6 \\ C_k h_4 + D_k & \text{if } k \equiv 4 \mod 6, \end{cases}$$

where A_k and C_k are non-zero multiples of the corresponding functions f_k. We also have by an inductive argument that the function B_k (resp. D_k) transform like a modular form from $M_k(\chi)$ (resp. $M_k(\overline{\chi})$). We still need to show that these functions do not have poles in the interior of $\mathfrak{H}$ (the fact that they are bounded at $i\infty$ easily follows from (13) by induction). In the case when $k \equiv 0 \mod 6$, that follows from the holomorphicity in the interior of $\mathfrak{H}$ of both F_k (by Proposition 4) and $A_k\mathcal{E}$. In the case when $k \equiv 4 \mod 6$, the function F_k is again holomorphic in the interior of $\mathfrak{H}$ by Proposition 4, and we now claim that so is $C_k h_4$. Indeed, the modular form f_k (and, therefore, C_k) is divisible by E_4 (see e.g. [10, Equation (2)]):

$$\frac{C_k}{E_4} \in M_{k-4}(SL_2(\mathbb{Z})).$$

Therefore

$$C_k h_4 = \frac{C_k}{E_4} E_4 h_4$$

is holomorphic in $\mathfrak{H}$.

The last assertion to prove is $F_k \in \mathbb{M}_{k,0}(triv, \overline{\chi})$ when $k \equiv 4 \mod 6$. This now follows from Proposition 2 (cf. Remark 5). $\square$

Remark 6. It was communicated to the author by Prof. Kaneko that the results from [6] imply the following explicit expressions for the quantities and functions involved into our proof of Theorem 1.

$$\alpha_k = \frac{288k(k-4)}{k-5}, \quad \delta_k = 0, \quad \gamma_k = -\frac{2}{3}(k-5).$$

In the case $k \equiv 0 \mod 6$, put $k = 6n$. Then we have that

$$A_k = \frac{\binom{-\frac{1}{6}}{n}^2}{432^n \binom{-\frac{1}{3}}{n}} f_k, \quad B_k = \eta^4 \widetilde{B}_{k-2}$$

with a modular form $\widetilde{B}_{k-2} \in M_{k-2}$ of weight $k-2$ on $SL_2(\mathbb{Z})$.

In the case when $k \equiv 4 \mod 6$, put $k = 6n+4$. Then we have that

$$C_k = \frac{\binom{-\frac{5}{6}}{n}^2}{432^n \binom{-\frac{5}{3}}{n}} f_k, \quad D_k = \eta^{20} \widetilde{D}_{k-10}$$

with a modular form $\widetilde{D}_{k-10} \in M_{k-10}$ of weight $k-10$ on $SL_2(\mathbb{Z})$.

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