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by

Thomas Ashton
Andrey Mudrov

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Thomas Ashton
Andrey Mudrov

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of Leicester
University Road
LE1 7RH Leicester
UK

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Thomas Ashton and Andrey Mudrov

Department of Mathematics,

University of Leicester,

University Road, LE1 7RH Leicester, UK

e-mail: am405@le.ac.uk

Abstract

Let $\mathfrak{g}$ be a complex orthogonal or symplectic Lie algebra and $\mathfrak{g}' \subset \mathfrak{g}$ the Lie subalgebra of rank $\mathrm{rk}\, \mathfrak{g}' = \mathrm{rk}\, \mathfrak{g} - 1$ of the same type. We give an explicit construction of generators of the Mickelsson algebra $Z_q(\mathfrak{g}, \mathfrak{g}')$ in terms of Chevalley generators via the R-matrix of $U_q(\mathfrak{g})$.

Mathematics Subject Classifications: 81R50, 81R60, 17B37.

Key words: Mickelsson algebras, quantum groups, R-matrix, lowering/raising operators.

1 Introduction

In the mathematics literature, lowering and rasing operators operators are known as generators of step algebras, which were originally introduced by Mickelsson [1] for reductive pairs of Lie algebras, $\mathfrak{g}' \subset \mathfrak{g}$. These algebras naturally act on $\mathfrak{g}'$-singular vectors in $U(\mathfrak{g})$-modules and are important in representation theory, [2, 3].

The general theory of step algebras for classical universal enveloping algebras was developed in [2, 4] and was extended to the special linear and orthogonal quantum groups in [5]. They admit a natural description in terms of extremal projectors, [4], introduced for classical groups in [6, 7] and extended to the quantum group case in [8]. It is known that the step algebra $Z(\mathfrak{g}, \mathfrak{g}')$ is generated by the image of the orthogonal complement $\mathfrak{g} \ominus \mathfrak{g}'$ under the extremal projector of the $\mathfrak{g}'$. Another description of lowering/raising operators for classical groups was obtained in [9, 10, 11, 12] in an explicit form of polynomials in $\mathfrak{g}$.

A generalization of the results of [9, 10] to quantum $\mathfrak{gl}(n)$ can be found in [13]. In this special case, the lowering operators can be also conveniently expressed through "modified commutators" in the Chevalley generators of $U(\mathfrak{g})$ with coefficients in the field of fractions of $U(\mathfrak{h})$. Extending [11, 12] to a general quantum group is not straightforward, since there are no immediate candidates for the nilpotent triangular Lie subalgebras $\mathfrak{g}_\pm$ in $U_q(\mathfrak{g})$. We suggest such a generalization, where the lack of $\mathfrak{g}_\pm$ is compensated by the entries of the universal R-matrix with one leg projected to the natural representation. Those entries are nicely expressed through modified commutators in the Chevalley generators turning into elements of $\mathfrak{g}_\pm$ in the quasi-classical limit. Their commutation relation with the Chevalley generators modify the classical commutation relations with $\mathfrak{g}_\pm$ in a tractable way. This enabled us to generalize the results of [9, 10, 11, 12] and construct generators of Mickelsson algebras for the non-exceptional quantum groups.

1.1 Quantized universal enveloping algebra

In this paper, $\mathfrak{g}$ is a complex simple Lie algebra of type B, C or D. The case of $\mathfrak{gl}(n)$ can be easily derived from here due to the natural inclusion $U_q\big(\mathfrak{gl}(n)\big) \subset U_q(\mathfrak{g})$, so we do not pay special attention to it. We choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the canonical inner product $(.,.)$ on $\mathfrak{h}^*$. By R we denote the root system of $\mathfrak{g}$ with a fixed subsystem of positive roots $\mathrm{R}^+ \subset \mathrm{R}$ and the basis of simple roots $\Pi^+ \subset \mathrm{R}^+$. For every $\lambda \in \mathfrak{h}^*$ we denote by h_λ its image under the isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}$, that is $(\lambda, \beta) = \beta(h_\lambda)$ for all $\beta \in \mathfrak{h}^*$. We put $\rho = \frac{1}{2}\sum_{\alpha \in \mathrm{R}^+} \alpha$ for the Weyl vector.

Suppose that $q \in \mathbb{C}$ is not a root of unity. Denote by $U_q(\mathfrak{g}_\pm)$ the $\mathbb{C}$-algebra generated by $e_{\pm\alpha}$, $\alpha \in \Pi^+$, subject to the q-Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_{q_{\alpha_i}} e_{\pm\alpha_i}^{1-a_{ij}-k} e_{\pm\alpha_j} e_{\pm\alpha_i}^k = 0,$$

where $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, $i, j = 1, \ldots, n = \mathrm{rk}\, \mathfrak{g}$, is the Cartan matrix, $q_\alpha = q^{\frac{(\alpha,\alpha)}{2}}$, and

$$\left[\begin{array}{c} m \\ k \end{array}\right]_q = \frac{[m]_q!}{[k]_q![m-k]_q!}, \quad [m]_q! = [1]_q \cdot [2]_q \ldots [m]_q.$$

Here and further on, $[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$ whenever $q^{\pm z}$ make sense.

Denote by $U_q(\mathfrak{h})$ the commutative $\mathbb{C}$-algebra generated by $q^{\pm h_\alpha}$, $\alpha \in \Pi^+$. The quantum group $U_q(\mathfrak{g})$ is a $\mathbb{C}$-algebra generated by $U_q(\mathfrak{g}_\pm)$ and $U_q(\mathfrak{h})$ subject to the relations

$$q^{h_\alpha} e_{\pm\beta} q^{-h_\alpha} = q^{\pm(\alpha,\beta)} e_{\pm\beta}, \quad [e_\alpha, e_{-\beta}] = \delta_{\alpha,\beta} \frac{q^{h_\alpha} - q^{-h_\alpha}}{q_\alpha - q_\alpha^{-1}}.$$

Remark that $\mathfrak{h}$ is not contained in $U_q(\mathfrak{g})$, still it is convenient for us to keep reference to $\mathfrak{h}$.

Fix the comultiplication in $U_q(\mathfrak{g})$ as in [14]:

$$\Delta(e_\alpha) = e_\alpha \otimes q^{h_\alpha} + 1 \otimes e_\alpha, \quad \Delta(e_{-\alpha}) = e_{-\alpha} \otimes 1 + q^{-h_\alpha} \otimes e_{-\alpha},$$
$$\Delta(q^{\pm h_\alpha}) = q^{\pm h_\alpha} \otimes q^{\pm h_\alpha},$$

for all $\alpha \in \Pi^+$.

The subalgebras $U_q(\mathfrak{b}_\pm) \subset U_q(\mathfrak{g})$ generated by $U_q(\mathfrak{g}_\pm)$ over $U_q(\mathfrak{h})$ are quantized universal enveloping algebras of the Borel subalgebras $\mathfrak{b}_\pm = \mathfrak{h} + \mathfrak{g}_\pm \subset \mathfrak{g}$.

The Chevalley generators e_α can be extended to a set of higher root vectors e_β for all $\beta \in \mathrm{R}$. A normally ordered set of root vectors generate a Poincaré-Birkhoff-Witt (PBW) basis of $U_q(\mathfrak{g})$ over $U_q(\mathfrak{h})$, [14]. We will use $\mathfrak{g}_\pm$ to denote the vector space spanned by $\{e_{\pm\beta}\}_{\beta \in \mathrm{R}^+}$.

The universal R-matrix is an element of a certain extension of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. We heavily use the intertwining relation

$$\mathcal{R}\Delta(x) = \Delta^{op}(x)\mathcal{R}, \tag{1.1}$$

between the coproduct and its opposite for all $x \in U_q(\mathfrak{g})$. Let $\{\varepsilon_i\}_{i=1}^n \subset \mathfrak{h}^*$ be the standard orthonormal basis and $\{h_{\varepsilon_i}\}_{i=1}^n$ the corresponding dual basis in $\mathfrak{h}$. The exact expression for $\mathcal{R}$ can be extracted from [14], Theorem 8.3.9, as the ordered product

$$\mathcal{R} = q^{\sum_{i=1}^n h_{\varepsilon_i} \otimes h_{\varepsilon_i}} \prod_\beta \exp_{q_\beta}\{(1 - q_\beta^{-2})(e_\beta \otimes e_{-\beta})\} \in U_q(\mathfrak{b}_+) \hat{\otimes} U_q(\mathfrak{b}_-), \tag{1.2}$$

where $\exp_q(x)=\sum_{k=0}^{\infty}q^{\frac{1}{2}k(k+1)}\frac{x^k}{[k]_q!}$.

We use the notation $e_i=e_{\alpha_i}$ and $f_i=e_{-\alpha_i}$ for $\alpha_i\in\Pi^+$, in all cases apart from $i=n$, $\mathfrak{g}=\mathfrak{so}(2n+1)$, where we set $f_n=[\frac{1}{2}]_q e_{-\alpha_n}$. The reason for this is two-fold. Firstly, the natural representation can be defined through the classical assignment on the generators, as given below. Secondly, we get rid of $q_{\alpha_n}=q^{\frac{1}{2}}$ and can work over $\mathbb{C}[q]$, as the relations involved turn into

$$[e_n,f_n]=\frac{q^{h_{\alpha_n}}-q^{-h_{\alpha_n}}}{q-q^{-1}},$$

$$f_n^3f_{n-1}-(q+1+q^{-1})f_n^2f_{n-1}f_n+(q+1+q^{-1})f_nf_{n-1}f_n^2-f_{n-1}f_n^3=0.$$

It is easy to see that the square root of q disappears from the corresponding factor in the presentation (1.2).

In what follows, we regard $\mathfrak{gl}(n)\subset\mathfrak{g}$ to be the Lie subalgebra with the simple roots $\{\alpha_i\}_{i=1}^{n-1}$ and $U_q\bigl(\mathfrak{gl}(n)\bigr)$ the corresponding quantum subgroup in $U_q(\mathfrak{g})$.

Consider the natural representation of $\mathfrak{g}$ in the vector space $\mathbb{C}^N$. We use the notation $i'=N+1-i$ for all integers $i=1,\ldots,N$. The assignment

$$\pi(e_i)=e_{i,i+1}\pm e_{i'-1,i'},\quad \pi(f_i)=e_{i+1,i}\pm e_{i',i'-1},\quad \pi(h_{\alpha_i})=e_{ii}-e_{i+1,i+1}+e_{i'-1,i'-1}-e_{i'i'},$$

for $i=1,\ldots,n-1$, defines a direct sum of two representations of $\mathfrak{gl}(n)$ for each sign. It extends to the natural representation of the whole $\mathfrak{g}$ by

$$\pi(e_n)=e_{n,n+1}\pm e_{n'-1,n'},\quad \pi(f_n)=e_{n+1,n}\pm e_{n',n'-1},\quad \pi(h_{\alpha_n})=e_{nn}-e_{n'n'},$$

$$\pi(e_n)=e_{nn'},\quad \pi(f_n)=e_{n'n},\quad \pi(h_{\alpha_n})=2e_{nn}-2e_{n'n'},$$

$$\pi(e_n)=e_{n-1,n'}\pm e_{n,n'+1},\quad \pi(f_n)=e_{n',n-1}\pm e_{n'+1,n},\quad \pi(h_{\alpha_n})=e_{n-1,n-1}+e_{nn}-e_{n'n'}-e_{n'+1,n'+1},$$

respectively, for $\mathfrak{g}=\mathfrak{so}(2n+1)$, $\mathfrak{g}=\mathfrak{sp}(2n)$, and $\mathfrak{g}=\mathfrak{so}(2n)$.

Two values of the sign give equivalent representations. The choice of minus corresponds to the standard representation that preserves the bilinear form with entries $C_{ij}=\delta_{i'j}$, for $\mathfrak{g}=\mathfrak{so}(N)$, and $C_{ij}=\mathrm{sign}(i'-i)\delta_{i'j}$, for $\mathfrak{g}=\mathfrak{sp}(N)$. However, we fix the sign to + in order to simplify calculations. The above assignment also defines representations of $U_q(\mathfrak{g})$.

2 R-matrix of non-exceptional quantum groups

Define $\check{\mathcal{R}}=q^{-\sum_{i=1}^{n}h_{\varepsilon_i}\otimes h_{\varepsilon_i}}\mathcal{R}$. Denote by $\check{R}^-=(\pi\otimes\mathrm{id})(\check{\mathcal{R}})\in\mathrm{End}(\mathbb{C}^N)\otimes U_q(\mathfrak{g}_-)$ and by $\check{R}^+=(\pi\otimes\mathrm{id})(\check{\mathcal{R}}_{21})\in\mathrm{End}(\mathbb{C}^N)\otimes U_q(\mathfrak{g}_+)$. In this section, we deal only with $\check{R}^-$ and suppress the label "$-$" for simplicity, $\check{R}=\check{R}^-$.

Denote by N_+ the ring of all upper triangular matrices in $\mathrm{End}(\mathbb{C}^N)$ and by N'_+ its ideal spanned by e_{ij}, $i < j+1$.

Lemma 2.1. *One has*

$$\check{R} = 1 \otimes 1 + (q^{1+\delta_{1n}} - q^{-1-\delta_{1n}}) \sum_{i=1}^{n} \pi(e_i) \otimes f_i \mod N'_+ \otimes U_q(\mathfrak{g}_-),$$

where δ_{1n} is present only for $\mathfrak{g} = \mathfrak{sp}(2n)$.

Proof. For all positive roots α, β the matrix $\pi(e_\alpha e_\beta)$ belongs to N'_+. Also, $\pi(e_\beta) \in N'_+$ for all $\beta \in \mathrm{R}^+ \backslash \Pi^+$. Therefore, the only terms that contribute to $\mathrm{Span}_{\varepsilon_i - \varepsilon_j \in \Pi^+}\{e_{ij} \otimes U_q(\mathfrak{g}_-)\}$ are those of degree 1 from the series $\exp_{q_\alpha}(1 - q_\alpha^{-2})(e_\alpha \otimes e_{-\alpha})$ with $\alpha \in \Pi^+$. □

Write $\check{R} = \sum_{i,j=1}^{N} e_{ij} \otimes \check{R}_{ij}$, where $\check{R}_{ij} = 0$ for $i > j$. Due to the $\mathfrak{h}$-invariance of $\check{R}$, the entry $\check{R}_{ij} \in U_q(\mathfrak{g}_-)$ carries weight $\varepsilon_j - \varepsilon_i$.

For all $\mathfrak{g}$, we have $f_{k,k+1} = f_k = f_{k'-1,k'}$ once $k < n$ and $f_{n,n+1} = f_n = f_{n+1,n'}$ for $\mathfrak{g} = \mathfrak{so}(2n+1)$, $f_{n-1,n'} = f_n = f_{n,n'+1}$ for $\mathfrak{g} = \mathfrak{so}(2n)$, and $f_{nn'} = [2]_q f_n$ for $\mathfrak{g} = \mathfrak{sp}(2n)$. We present explicit expressions for the entries f_{ij} in terms of modified commutators in Chevalley generators, $[x,y]_a = xy - ayx$, where a is a scalar; we also put $\bar{q} = q^{-1}$.

Proposition 2.2. *Suppose that $\varepsilon_i - \varepsilon_j \in \mathrm{R}^+ \backslash \Pi^+$. Then the elements f_{ij} are given by the following formulas:*
For all $\mathfrak{g}$ and $i + 1 < j \leqslant \frac{N+1}{2}$:

$$f_{ij} = [f_{j-1}, \ldots [f_{i+1}, f_i]_{\bar{q}} \ldots]_{\bar{q}}, \quad f_{j'i'} = [\ldots [f_i, f_{i+1}]_{\bar{q}}, \ldots f_{j-1}]_{\bar{q}}. \tag{2.3}$$

Furthermore,

- *For $\mathfrak{g} = \mathfrak{so}(2n+1)$: $f_{nn'} = (q-1) f_n^2$ and*

$$f_{i,n+1} = [f_n, f_{i,n}]_{\bar{q}}, \quad f_{n+1,i'} = [f_{n',i'}, f_n]_{\bar{q}}, \quad i < n,$$

$$f_{ij'} = q^{\delta_{ij}} [f_{n+1,j'}, f_{i,n+1}]_{\bar{q}^{\delta_{ij}}}, \quad i, j < n.$$

- *For $\mathfrak{g} = \mathfrak{sp}(2n)$: $f_{nn'} = [2]_q f_n$ and*

$$f_{in'} = [f_n, f_{in}]_{\bar{q}^2}, \quad f_{ni'} = [f_{n'i'}, f_n]_{\bar{q}^2}, \quad i < n,$$

$$f_{ij'} = q^{\delta_{ij}} [f_{nj'}, f_{in}]_{\bar{q}^{1+\delta_{ij}}}, \quad i, j < n.$$

- *For $\mathfrak{g}=\mathfrak{so}(2n)$: $f_{nn'}=0$ and*

$$f_{in'}=[f_n,f_{i,n-1}]_{\bar q},\quad f_{ni'}=[f_{n'+1,i'},f_n]_{\bar q},\quad i<n-2,$$

$$f_{ji'}=q^{\delta_{ij}}[f_{ni'},f_{j,n}]_{\bar q^{1+\delta_{ij}}},\quad i,j\leqslant n-1.$$

Proof. The proof is a direct calculation with the use of the identity

$$(f_\alpha\otimes 1)\check{\mathcal{R}}-\check{\mathcal{R}}(f_\alpha\otimes 1)=\check{\mathcal{R}}(q^{-h_\alpha}\otimes f_\alpha)-(q^{h_\alpha}\otimes f_\alpha)\check{\mathcal{R}},$$

which follows from the intertwining axiom (1.1) for $x=f_\alpha$. This allows us to construct the elements f_{ij} by induction starting from f_α, $\alpha\in\Pi^+$. □

For each $\alpha\in\Pi^+$, denote by $P(\alpha)$ the set of ordered pairs $l,r=1,\ldots,N$, with $\varepsilon_l-\varepsilon_r=\alpha$. We call such pairs simple.

Proposition 2.3. *The matrix entries $f_{i,j}\in U_q(\mathfrak{g}_-)$ such that $\varepsilon_i-\varepsilon_j\notin\Pi^+$ satisfy the identity*

$$[e_\alpha,f_{ij}]=\sum_{(l,r)\in P(\alpha)}\bigl(f_{il}\delta_{jr}q^{h_\alpha}-q^{-h_\alpha}\delta_{il}f_{rj}\bigr),$$

for all simple positive roots α.

Proof. The proof is a straightforward calculation based on the intertwining relation (1.1), which is equivalent to

$$(1\otimes e_\alpha)\check{\mathcal{R}}-\check{\mathcal{R}}(1\otimes e_\alpha)=\check{\mathcal{R}}(e_\alpha\otimes q^{h_\alpha})-(e_\alpha\otimes q^{-h_\alpha})\check{\mathcal{R}},$$

for $x=e_\alpha$, $\alpha\in\Pi^+$. Alternatively, one can use the expressions for f_{ij} from Proposition 2.2. □

3 Mickelsson algebras

Consider the Lie subalgebra $\mathfrak{g}'\subset\mathfrak{g}$ corresponding to the root subsystem $\mathrm{R}_{\mathfrak{g}'}\subset\mathrm{R}_{\mathfrak{g}}$ generated by α_i, $i>1$, and let $\mathfrak{h}'\subset\mathfrak{g}'$ denote its Cartan subalgebra. Let the triangular decomposition $\mathfrak{g}'_-\oplus\mathfrak{h}'\oplus\mathfrak{g}'_+$ be compatible with the triangular decomposition of $\mathfrak{g}$. Recall the definition of step algebra $Z_q(\mathfrak{g},\mathfrak{g}')$ of the pair $(\mathfrak{g},\mathfrak{g}')$. Consider the left ideal $J=U_q(\mathfrak{g})\mathfrak{g}'_+$ and its normalizer $\mathcal{N}=\{x\in U_q(\mathfrak{g}):e_\alpha x\subset J,\forall\alpha\in\Pi^+_{\mathfrak{g}'}\}$. By construction, J is a two-sided ideal in the algebra $\mathcal{N}$. Then $Z_q(\mathfrak{g},\mathfrak{g}')$ is the quotient $\mathcal{N}/J$.

For all $\beta_i \in \mathrm{R}^+_{\mathfrak{g}} \backslash \mathrm{R}^+_{\mathfrak{g}'}$ let e_{β_i} be the corresponding PBW generators and let Z be the vector space spanned by $e^{k_l}_{-\beta_l} \dots e^{k_1}_{-\beta_1} e_0^{k_0} e^{m_1}_{\beta_1} \dots e^{m_l}_{\beta_l}$, were $e_0 = q^{h_{\alpha_1}}$, $k_i \in \mathbb{Z}_+$, and $k_0 \in \mathbb{Z}$. The PBW factorization $U_q(\mathfrak{g}) = U_q(\mathfrak{g}'_-) Z U_q(\mathfrak{h}') U_q(\mathfrak{g}'_+)$ gives rise to the decomposition

$$U_q(\mathfrak{g}) = Z U_q(\mathfrak{h}') \oplus (\mathfrak{g}'_- U_q(\mathfrak{g}) + U_q(\mathfrak{g}) \mathfrak{g}'_+).$$

Proposition 3.1 ([5], Theorem 1)**.** *The projection $U_q(\mathfrak{g}) \to Z U_q(\mathfrak{h}')$ implements an embedding of $Z_q(\mathfrak{g}, \mathfrak{g}')$ in $Z U_q(\mathfrak{h}')$.*

Proof. The statement is proved in [5] for the orthogonal and special linear quantum groups but the arguments apply to symplectic groups too. □

It is proved within the theory of extremal projectors that generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ are labeled by the roots $\beta \in \mathrm{R}_{\mathfrak{g}} \backslash \mathrm{R}_{\mathfrak{g}'}$ plus $z_0 = q^{h_{\alpha_1}}$. We calculate them in the subsequent sections, cf. Propositions 3.5 and 3.9.

3.1 Lowering operators

In what follows, we extend $U_q(\mathfrak{g})$ along with its subalgebras containing $U_q(\mathfrak{h})$ over the field of fractions of $U_q(\mathfrak{h})$ and denote such an extension by hat, e.g. $\hat{U}_q(\mathfrak{g})$. In this section we calculate representatives of the negative generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ in $\hat{U}_q(\mathfrak{b}_-)$.

Set $h_i = h_{\varepsilon_i} \in \mathfrak{h}$ for all $i = 1, \dots, N$ and introduce $\eta_{ij} \in \mathfrak{h} + \mathbb{C}$ for $i, j = 1, \dots, N$, by

$$\eta_{ij} = h_i - h_j + (\varepsilon_i - \varepsilon_j, \rho) - \frac{1}{2} \|\varepsilon_i - \varepsilon_j\|^2. \tag{3.4}$$

Here $\|\mu\|$ is the Euclidean norm on $\mathfrak{h}^*$.

Lemma 3.2. *Suppose that $(l, r) \in P(\alpha)$ for some $\alpha \in \Pi^+$. Then*

i) if $l < r < j$, then $\eta_{lj} - \eta_{rj} = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r)$,

ii) if $i < l < r$, then $\eta_{li} - \eta_{ri} = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r)$,

iii) $\eta_{lr} = h_\alpha$.

Proof. We have $(\alpha, \rho) = \frac{1}{2}\|\alpha\|^2$ for all $\alpha \in \Pi^+$. This proves iii). Further, for $\varepsilon_l - \varepsilon_r = \alpha$:

$$\begin{aligned} \eta_{lj} - \eta_{rj} &= h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_j - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_j - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_j - \varepsilon_r), \quad r < j, \\ \eta_{li} - \eta_{ri} &= h_\alpha + \frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\varepsilon_i - \varepsilon_r\|^2 - \frac{1}{2}\|\varepsilon_i - \varepsilon_r - \alpha\|^2 = h_\alpha + (\alpha, \varepsilon_i - \varepsilon_r), \quad i < l, \end{aligned}$$

which proves i) and ii). □

We call a strictly ascending sequence $\vec{m} = (m_1, \dots, m_s)$ of integers a route from m_1 to m_s. We write $m < \vec{m}$ and $\vec{m} < m$ for $m \in \mathbb{Z}$ if, respectively, $m < \min \vec{m}$ and $\max \vec{m} < m$. More generally, we write $\vec{m} < \vec{k}$ if $\max \vec{m} < \min \vec{k}$. In this case, a sequence $(\vec{m}, \vec{k})$ is a route from $\min \vec{m}$ to $\max \vec{k}$.

Given a route $\vec{m} = (m_1, \dots, m_s)$, define the product $f_{\vec{m}} = f_{m_1,m_2} \cdots f_{m_{s-1},m_s} \in U_q(\mathfrak{g}_-)$. Consider a free right $\hat{U}_q(\mathfrak{h})$-module Φ_{1m} generated by $f_{\vec{m}}$ with $1 \leqslant \vec{m} \leqslant j$ and define an operation $\partial_{lr} \colon \Phi_{1j} \to \hat{U}_q(\mathfrak{b}_-)$ for $(l,r) \in P(\alpha)$ as follows. Assuming $1 \leqslant \vec{\ell} < l < r < \vec{\rho} < j$, set

$$
\begin{aligned}
\partial_{lr} f_{(\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho})} &= f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} [\eta_{lj} - \eta_{rj}]_q, \\
\partial_{lr} f_{(\vec{\ell},l)} f_{(l,\vec{\rho})} &= -f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} q^{-\eta_{lj}+\eta_{rj}}, \\
\partial_{lr} f_{(\vec{\ell},r)} f_{(r,\vec{\rho})} &= f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} q^{\eta_{lj}-\eta_{rj}}, \\
\partial_{lr} f_{\vec{m}} &= 0, \qquad l \notin \vec{m}, \quad r \notin \vec{m}.
\end{aligned}
$$

Extend ∂_{lr} to entire Φ_{1j} by $\hat{U}_q(\mathfrak{h})$-linearity. Let $p \colon \Phi_{1j} \to \hat{U}(\mathfrak{g})$ denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules.

Lemma 3.3. *For all $\alpha \in \Pi^+$ and all $x \in \Phi_{1j}$, $e_\alpha \circ p(x) = \sum_{(l,r)\in P(\alpha)} \partial_{lr} x \mod \hat{U}_q(\mathfrak{g}) e_\alpha$.*

Proof. A straightforward analysis based on Proposition 2.3 and Lemma 3.2. □

To simplify the presentation, we suppress the symbol of projection p in what follows.

Introduce elements $A^j_r \in \hat{U}_q(\mathfrak{h})$ by

$$
A^j_r = \frac{q - q^{-1}}{q^{-2\eta_{rj}} - 1}, \tag{3.5}
$$

for all $r, j \in [1, N]$ subject to $r < j$. For each simple pair (l,r) we define (l,r)-chains as

$$
f_{(\vec{\ell},l)} f_{(l,\vec{\rho})} A^j_l + f_{(\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho})} A^j_l A^j_r + f_{(\vec{\ell},r)} f_{(r,\vec{\rho})} A^j_r, \quad f_{(\vec{\ell},l)} f_{l,j} A^j_l + f_{(\vec{\ell},j)}, \tag{3.6}
$$

where $1 \leqslant \vec{\ell} < l$ and $r < \vec{\rho} \leqslant j$. Remark that $f_{(l,r)} = \left[\frac{(\alpha,\alpha)}{2}\right]_q e_{-\alpha}$, where $\alpha = \varepsilon_l - \varepsilon_r$.

Lemma 3.4. *The operator ∂_{lr} annihilates (l,r)-chains.*

Proof. Applying ∂_{lr} to the 3-chain in (3.6), we get

$$
f_{(\vec{\ell},l)} f_{(r,\vec{\rho})} (-q^{-\eta_{lj}+\eta_{rj}} A^j_l + [\eta_{lj} - \eta_{rj}]_q A^j_l A^j_r + q^{\eta_{lj}-\eta_{rj}} A^j_r).
$$

The factor in the brackets turns zero on substitution of 3.5.

Now apply ∂_{lj} to the right expression in (3.6) and get

$$
f_{(\vec{\ell},l)} ([h_\alpha]_q A^j_l + q^{h_\alpha}) = f_{(\vec{\ell},l)} (\frac{q^{h_\alpha} - q^{-h_\alpha}}{q^{-2\eta_{lj}} - 1} + q^{h_\alpha}) = f_{(\vec{\ell},l)} \frac{[h_\alpha - \eta_{lj}]_q}{[-\eta_{lj}]_q} = 0,
$$

so long as $\eta_{lj} = h_\alpha$ by Lemma 3.2. □

Given a route $\vec{m} = (m_1, \dots, m_s)$, put $A^j_{\vec{m}} = A^j_{m_1} \cdots A^j_{m_s} \in \hat{U}_q(\mathfrak{h})$ (and $A^j_{\vec{m}} = 1$ for the empty route) and define

$$z_{-j+1} = \sum_{1<\vec{m}<j} f_{(1,\vec{m},j)} A^j_{\vec{m}} \in \hat{U}_q(\mathfrak{b}_-), \quad j = 2, \dots, N, \tag{3.7}$$

where the summation is taken over all possible $\vec{m}$ subject to the specified inequalities plus the empty route.

Proposition 3.5. *$e_\alpha z_{-j} = 0 \mod \hat{U}_q(\mathfrak{g}) e_\alpha$ for all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and $j = 1, \dots, N-1$.*

Proof. Thanks to Lemma 3.3, we can reduce consideration to the action of operators ∂_{lr}, with $(l,r) \in P(\alpha)$. According to the definition of ∂_{lr} the summands in (3.7) that survive the action of ∂_{lr} can be organized into a linear combination of (l,r)-chains with coefficients in $\hat{U}_q(\mathfrak{h})$. By Lemma 3.4 they are killed by ∂_{lr}. □

The elements z_{-i}, $i = 1, \dots, N-1$, belong to the normalizer $\mathcal{N}$ and form the set of negative generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the negative part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by z_{-i}, $i = 1, \dots, N-2$.

3.2 Raising operators

In this section we construct positive generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$, which are called raising operators. Consider an algebra automorphism $\omega\colon U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined on the generators by $f_\alpha \leftrightarrow e_\alpha$, $q^{\pm h_\alpha} \mapsto q^{\mp h_\alpha}$. For $i < j$, let g_{ji} be the image of f_{ij} under this isomorphism. The natural representation restricted to $U_q(\mathfrak{g}_\pm)$ intertwines ω and matrix transposition. Since $(\omega \otimes \omega)(\check{\mathcal{R}}) = \check{\mathcal{R}}_{21}$, the matrix $\check{R}^+ = (\pi \otimes \mathrm{id})(\check{\mathcal{R}}_{21})$ is equal to $1 \otimes 1 + (q - q^{-1}) \sum_{i<j} e_{ji} \otimes g_{ji}$.

Lemma 3.6. *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $i > 1$, $e_\alpha g_{i1} = \sum_{(l,r)\in P(\alpha)} \delta_{il} g_{r1} \mod \hat{U}_q(\mathfrak{g}) e_\alpha$.*

Proof. Follows from the intertwining property of the R-matrix. □

Consider the right $\hat{U}_q(\mathfrak{h})$-module Ψ_{i1} freely generated by $f_{(\vec{m},k)} g_{k1}$ with $i \leqslant \vec{m} < k$. We define operators $\partial_{lr}\colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$ similarly as we did it for Φ_{1j}. For a simple pair $(l,r) \in P(\alpha)$, put

$$\partial_{l,r} f_{(\vec{m},k)} g_{k1} = \begin{cases} f_{(\vec{m},l)} g_{r1}, & l = k, \\ \big(\partial_{l,r} f_{(\vec{m},k)}\big) g_{k1}, & l \neq k, \end{cases} \qquad i \leqslant \vec{m} < r.$$

The Cartan factors appearing in $\partial_{lr} f_{(\vec{m},k)}$ depend on h_α. When pushed to the right-most position, h_α is shifted by $(\alpha, \varepsilon_1 - \varepsilon_r)$. We extend ∂_{lr} to an action on Ψ_{i1} by the requirement that ∂_{lr} commutes with the right action of $\hat{U}_q(\mathfrak{h})$. Let p denote the natural homomorphism of $\hat{U}_q(\mathfrak{h})$-modules, $p\colon \Psi_{i1} \to \hat{U}_q(\mathfrak{g})$. One can prove the following analog of Lemma 3.3.

Lemma 3.7. *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $x \in \Psi_{i1}$, $e_\alpha \circ p(x) = \sum_{(l,r)\in P(\alpha)} \partial_{lr} x \mod \hat{U}_q(\mathfrak{g}) e_\alpha$.*

Proof. Straightforward. □

We suppress the symbol of projection p to simplify the formulas.

Define σ_i for all $i = 1, \ldots, N$ as follows. For $i < j$ let $\|i - j\|$ be the number of simple positive roots entering $\varepsilon_i - \varepsilon_j$. For all $i, k = 2, \ldots, N$, $i < k$, put

$$A^i_k = \frac{q^{\eta_{k1} - \eta_{i1}}}{[\eta_{i1} - \eta_{k1}]_q}, \quad B^i_k = \frac{(-1)^{\|i-k\|}}{[\eta_{i1} - \eta_{k1}]_q},$$

For each $(l, r) \in \mathrm{P}(\alpha)$, where $\alpha \in \Pi^+_{\mathfrak{g}'}$, define 3-chains as

$$f_{(i,\vec{m},l)} g_{l1} B^i_l + f_{(i,\vec{m},l)} f_{(l,r)} g_{r1} A^i_l B^i_r + f_{(i,\vec{m},r)} g_{r1} B^i_r, \tag{3.8}$$

with $i < \vec{m} < l < r \leqslant N$ and

$$f_{(i,\vec{\ell},l)} f_{(l,\vec{\rho},k)} g_{k1} A^i_l + f_{(i,\vec{\ell},l)} f_{(l,r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_l A^i_r + f_{(i,\vec{\ell},r)} f_{(r,\vec{\rho},k)} g_{k1} A^i_r \tag{3.9}$$

with $i < \vec{\ell} < l < r < \vec{\rho} < k \leqslant N$. The 2-chains are defined as

$$g_{i1} + f_{(i,r)} g_{r1} B^i_r, \quad f_{(i,\vec{m},k)} g_{k1} + f_{(i,r)} f_{(r,\vec{m},k)} g_{k1} A^i_r \tag{3.10}$$

where r is such that $\varepsilon_i - \varepsilon_r \in \Pi^+_{\mathfrak{g}'}$ and $i < r < \vec{m} < k \leqslant N$. In all cases the empty routes $\vec{m}$ are admissible.

Lemma 3.8. *For all $\alpha \in \Pi^+_{\mathfrak{g}'}$ and all $(l, r) \in P(\alpha)$ the (l, r)-chains are annihilated by ∂_{lr}.*

Proof. Suppose that $i = l$ and apply ∂_{ir} to the left 2-chain in (3.10). The result is

$$g_{r1} + [h_\alpha]_q g_{r1} B^i_r = g_{r1}(1 + [h_\alpha + (\alpha, \varepsilon_1 - \varepsilon_r)]_q B^i_r) = g_{r1}(1 + [\eta_{i1} - \eta_{r1}]_q B^i_r) = 0,$$

by Lemma 3.2. Applying ∂_{ir} to the right 2-chain in (3.10) we get

$$f_{(r,\vec{m},k)} g_{k1}(-q^{-\eta_{i1}+\eta_{r1}} + [\eta_{i1} - \eta_{r1}]_q A^i_r) = 0.$$

Now consider 3-chains. The action of ∂_{lr} on the (3.9) produces

$$-f_{(i,\vec{\ell},l)} q^{-h_\alpha} f_{(r,\vec{\rho},k)} g_{k,1} A^i_l + f_{(i,\vec{\ell},l)} [h_\alpha]_q f_{(r,\vec{\rho},k)} g_{k,1} A^i_l A^i_r + f_{(i,\vec{\ell},l)} q^{h_\alpha} f_{(r,\vec{\rho},k)} g_{k,1} A^i_r,$$

which turns zero since $-q^{\eta_{r1} - \eta_{l1}} A^i_l + [\eta_{l1} - \eta_{r1}]_q A^i_l A^i_r + q^{\eta_{l1} - \eta_{r1}} A^i_r = 0$. The action of ∂_{lr} on (3.8) yields

$$f_{(i,\vec{m},l)} g_{r1} B^i_l + f_{(i,\vec{m},l)} [h_\alpha] g_{r1} A^i_l B^i_r + f_{(i,\vec{m},l)} q^{h_\alpha} g_{r1} B^i_r.$$

This is vanishing since $B^i_l + [\eta_{l1} - \eta_{r1}] A^i_l B^i_r + q^{\eta_{l1} - \eta_{r1}} B^i_r = B^i_l + \frac{[\eta_{i1} - \eta_{r1}]_q}{[\eta_{i1} - \eta_{l1}]_q} B^i_r = 0.$ □

Given a route $\vec{m} = (m_1, \ldots, m_k)$ such that $i < \vec{m}$ let $A^i_{\vec{m}}$ denote the product $A^i_{m_1} \ldots A^i_{m_k}$. Introduce elements $z_i \in \hat{U}_q(\mathfrak{g}_-)\mathfrak{g}_+$ of weight $\varepsilon_1 - \varepsilon_i$ by

$$z_{i-1} = g_{i1} + \sum_{i<\vec{m}<k\leqslant N} f_{(i,\vec{m},k)} g_{k1} A^i_{\vec{m}} B^i_k, \quad i = 2, \ldots, N.$$

Again, the summation includes empty $\vec{m}$.

Proposition 3.9. $e_\alpha z_i = 0 \mod \hat{U}_q(\mathfrak{g}) e_\alpha$, *for all* $\alpha \in \Pi^+_{\mathfrak{g}'}$ *and* $i = 1, \ldots, N-1$.

Proof. By Lemma 3.6, the vectors $g_{2'1}$ and $z_{N-1} = g_{1'1}$ are normalizing the left ideal $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, so is $z_{N-2} = g_{2'1} + f_1 g_{1'1} B^{1'}_{2'}$. Once the cases $i = 2', 1'$ are proved, we further assume $i < 2'$. In view of Lemma 3.7, it is sufficient to show that z_{i-1} is killed, modulo $\hat{U}_q(\mathfrak{g})\mathfrak{g}'_+$, by all ∂_{lr} such that $\varepsilon_l - \varepsilon_r \in \Pi^+_{\mathfrak{g}'}$. Observe that z_{i-1} can be arranged into a linear combination of chains, which are killed by ∂_{lr}, as in Lemma 3.8. □

The elements z_i, $i = 1, \ldots, N-1$, belong to the normalizer $\mathcal{N}$. They form the set of positive generators of $Z_q(\mathfrak{g}, \mathfrak{g}')$ for symplectic $\mathfrak{g}$. In the orthogonal case, the positive part of $Z_q(\mathfrak{g}, \mathfrak{g}')$ is generated by z_i, $i = 1, \ldots, N-2$.

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