ON AN ANALOGUE OF THE JAMES CONJECTURE

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Abstract. We give a counterexample to the most optimistic analogue (due to Kleshchev and Ram) of the James conjecture for Khovanov-Lauda-Rouquier algebras associated to simply-laced Dynkin diagrams. The first counterexample occurs in type A_5 for $p = 2$ and involves the same singularity used by Kashiwara and Saito to show the reducibility of the characteristic variety of an intersection cohomology D-module on a quiver variety. Using recent results of Polo one can give counterexamples in type A in all characteristics.

Dedicated to Jimbo

1. Introduction

A basic question in representation theory asks for the dimensions of the simple modules for the symmetric group over an arbitrary field. Despite over a century's effort even a conjectural formula remains out of reach.

In 1990 James conjectured that, under explicit lower bounds on the characteristic, the decomposition matrices for Hecke algebras at a root of unity should be trival [J]. A second major breakthrough occured with the work of Lascoux, Leclerc and Thibon [LLT] who formulated conjectures subsequently proved by Ariki [A] and Grojnowski [G]. They show that the projective covers of simple modules for Hecke algebras at a root of unity are encoded by the canonical basis of Fock space for the quantum affine algebra associated to the affine special linear group. This solves the Hecke algebra problem, so the James conjecture is the only barrier to a major advance in our understanding of simple modules for the symmetric group.

Over the last decade these ideas have been gathered under the umbrella of categorification and the important role played by Khovanov-Lauda-Rouquier (KLR) algebras has become clear. This is mainly thanks to the Brundan-Kleshchev isomorphism [BK1]: the group algebras of symmetric groups, as well as Hecke algebras at roots of unity are isomorphic to cyclotomic KLR algebras. It follows that the representation theory of symmetric groups and Hecke algebras aquires a non-trivial grading. (The reader is referred to [K] for a survey of these and related ideas.)

The Brundan-Kleshchev isomorphism identifies the symmetric group algebra in characteristic p and Hecke algebras at a p^{th} root of unity with KLR algebras associated to a cyclic Dynkin quiver with p nodes (considered over a field of characteristic p and 0 respectively). Using this isomorphism the James conjecture may be translated into the natural statement that the decomposition numbers for these KLR algebras are trivial, under certain explicit lower bounds on p.

The upshot is that KLR algebras associated to cyclic quivers could help us solve one of the oldest questions in representation theory. This begs the quesion: what do KLR algebras associated to Dynkin quivers mean? One might hope that their modular representation theory is simpler, and that we can garner a clue as to how to approach the cyclic case. The following conjecture was proposed by Kleshchev and Ram [KR, Conjecture 7.3] as a finite analogue of the James conjecture:

Conjecture 1.1. *The decomposition numbers for KLR algebras associated to Dynkin quivers are trivial.*

The purpose of this note is to explain what this conjecture means geometrically, and give a counterexample. Over a field of characteristic zero it is a theorem of Varagnolo-Vasserot [VV] and Rouquier [R] that KLR algebras are the Ext algebras of Lusztig sheaves on the moduli space of quiver representations. Hence understanding projective modules over KLR algebras is the same as understanding the decomposition of Lusztig sheaves with coefficients of characteristic zero. The Decomposition Theorem [BBD] allows one to translate this into a combinatorial problem. Maksimau [M] has recently extended this result: KLR algebras are the Ext algebras of Lusztig sheaves over $\mathbb{Z}$. Here the decomposition theorem is missing, but Kleshchev and Ram's conjecture is equivalent to the statement that the decomposition of Lusztig sheaves should remain the same with coefficients in any field.

This in turn is equivalent to asking that the stalks and costalks of intersection cohomology complexes on moduli spaces of quiver representations be free of torsion. (Or equivalently, that parity sheaves are isomorphic to intersection cohomology complexes.) A number of people had hopes that similar statements would be true on flag varieties in type A. In fact, the two questions turn out be equivalent, by an observation of Kashiwara and Saito.

In 2004 Braden gave counterexamples to these hopes on the flag variety of GL_8 [B1]. In this note we explain the relation between these questions and give the first counterexample to Conjecture 1.1. It occurs for the Dynkin quiver of type A_5 with coefficients of characteristic 2. We also explain how one can use recent results of Polo to give counterexamples for type A Dynkin quivers in all characteristics. Unfortunately, these examples are large: to get a counterexample in characteristic p using the approach that we discuss here one needs to consider a KLR algebra associate to a Dykin quiver of type A_{8p-1} and dimension vector $(1, 2, \dots, 4p, \dots, 2, 1)$. Hence the prospects of simple-minded attempts to use KLR algebras to actually understanding what is going on seem a little dim!

Finally, let us note that the first counterexample $p = 2$ and the A_5 quiver is related to a certain subvariety in the moduli space of quiver representations which was used by Kashiwara and Saito to give a counterexample to a conjecture of Kazhdan and Lusztig on the irreducibility of the characteristic cycle [KS]. In fact, the examples that we discuss in this paper all have reducible characteristic cycles, by a result of Vilonen and the author [VW]. Hence Kleshchev and Ram's conjecture would have been implied by Kazhdan and and Lusztig's, had it been correct.

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2. Brauer reciprocity for graded rings

In the modular representation theory of finite groups an important role is played by Brauer reciprocity. Brauer reciprocity is an equivalence between the two problems of determining the composition factors of the reduction modulo p of a simple module in characteristic 0, and decomposing the lift of a projective module in characteristic p to characteristic 0. It will be important for us because it rephrases the question of decomposition numbers for simple modules over KLR algebras into questions about projective modules. As we will see, the question of decomposing lifts of projective modules to characteristic 0 has a straightforward geometric interpretation.

2.1. **Graded algebras and modules.** Throughout we fix a prime number p and let $\mathbb{F} = \mathbb{F}_p$, $\mathbb{O} = \mathbb{Z}_p$ and $\mathbb{K} = \mathbb{Q}_p$ (the finite field with p elements, the p-adic integers, and the p-adic numbers respectively). All of the results of this paper remain valid for any p-modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$.

Now let $H_{\mathbb{O}} = \bigoplus_{i\in\mathbb{Z}} H_i$ denote a finitely generated graded $\mathbb{O}$-algebra, such that H^i is a finitely generated free $\mathbb{O}$-module for all $i \in \mathbb{Z}$. We set $H_{\mathbb{F}} := \mathbb{F} \otimes_{\mathbb{O}} H$ and $H_{\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{O}} H$. These are also finitely generated graded algebras, finite dimensional in each degree. We assume in addition that there exists a graded polynomial subring $A_{\mathbb{O}} = \mathbb{O}[X_1, \dots, X_m] \subset H_{\mathbb{O}}$ with generators of positive degree such that:

(1) $A_{\mathbb{O}}$ is contained in the centre of $H_{\mathbb{O}}$;
(2) $A_{\mathbb{O}}$ is a summand of $H_{\mathbb{O}}$ as an $\mathbb{O}$-module;
(3) $H_{\mathbb{O}}$ is a finitely generated module over $A_{\mathbb{O}}$.

For $k \in \{\mathbb{F}, \mathbb{K}\}$ we set $A_k := A \otimes_{\mathbb{O}} k$ which is also a graded polynomial ring contained in the centre of H_k. Write A_k^+ for the ideal of polynomials of positive degree. Any graded simple H_k-module is annihilated by A_k^+ (by the graded Nakayama lemma) and hence is a module over $H_k/(A_k^+)H_k$, a finite dimensional graded algebra by our assumption (3). We conclude that H_k has finitely many graded simple modules up to shifts, all of which are finite dimensional.

For $k \in \{\mathbb{K}, \mathbb{O}, \mathbb{R}\}$ let $\operatorname{Rep} H_k$ denote the abelian category of all finitely generated graded H_k-modules. That is, objects of $\operatorname{Rep} H_k$ are graded left H_k-modules, and morphisms are homogenous of degree zero. Let $\operatorname{Rep}^f H_k$ denote the full subcategory of modules which are finite dimensional over k, and let $\operatorname{Proj} H_k$ denote the full subcategory of projective modules which are finitely generated over H_k. Given $M = \bigoplus M^i \in \operatorname{Rep} H_k$ let $M(j)$ denote its shift, given by $M(j)^i := M^{j+i}$. Given $M, N \in \operatorname{Rep} H_k$ set $\operatorname{Hom}^\bullet(M, N) := \bigoplus \operatorname{Hom}(M, N(i))$. Recall that a *graded category* is a category equipped with a self-equivalence. We view $\operatorname{Rep} H_k$ as a graded category with respect to the self-equivalence $M \mapsto M(1)$.

2.2. **Grothendieck groups.** The Grothendieck groups of $\operatorname{Rep}^f H_k$ and $\operatorname{Proj} H_k$ will be denoted by $[\operatorname{Rep}^f H_k]$ and $[\operatorname{Proj} H_k]$ respectively. They are naturally $\mathbb{Z}[v^{\pm1}]$-modules by declaring $v[M] := [M(1)]$. Given $P \in \operatorname{Proj} H_k$ and $M \in \operatorname{Rep}^f H_k$ we obtain a pairing $\langle -, - \rangle : [\operatorname{Proj} H_k] \times [\operatorname{Rep}^f H_k] \to \mathbb{Z}[v^{\pm1}]$ by setting

$$\langle [P], [M] \rangle := \underline{\dim} \operatorname{Hom}^\bullet(P, M),$$

here $\underline{\dim}\, V := \dim V^i v^{-i} \in \mathbb{N}[v^{\pm1}]$ for a finite dimensional graded vector space V.

It is known [NO, Corollary 9.6.7, Theorem 9.6.8] that every simple module over $H_k/(A_k^+)H_k$ admits a grading,[1] and this grading is unique up to isomorphism and shifts. By the Krull-Schmidt property every graded simple modules L admits a projective cover P_L. If we fix a choice $L_1, \dots, L_m$ of graded simple modules and let $P_1, \dots, P_m$ be their projective covers then the classes $[L_1], \dots, [L_m]$ (resp. $[P_1], \dots, [P_m]$ give a free $\mathbb{Z}[v^{\pm 1}]$-basis for $[\mathrm{Rep}\, H_k]$ (resp. $[\mathrm{Proj}\, H_k]$). Moreover, these bases are dual under $\langle -, - \rangle$.

2.3. **Decomposition maps.** Given any $M \in \mathrm{Rep}^f H_{\mathbb{K}}$ one can always find an $H_{\mathbb{O}}$-stable lattice $M_{\mathbb{O}} \subset M$ by applying $H_{\mathbb{O}}$ to a set of generators for M. One can argue as in [Se, Part III] that the class $[\mathbb{F} \otimes_{\mathbb{O}} M_{\mathbb{O}}]$ in the Grothendieck group of $\mathrm{Rep}^f H_{\mathbb{F}}$ does not depend on the choice of lattice. In this way one obtains the decomposition map between the Grothendieck groups:

$$d : [\mathrm{Rep}^f(H_{\mathbb{K}})] \to [\mathrm{Rep}^f(H_{\mathbb{F}})].$$

One the other hand, standard arguments (e.g. [F, Theorem 12.3]) show that idempotents in $H_{\mathbb{F}}$ lift to $H_{\mathbb{O}}$. It follows that given any projective module $P_{\mathbb{F}}$ over $H_{\mathbb{F}}$ there exists a projective module $P_{\mathbb{O}}$ over $H_{\mathbb{O}}$ such that $\mathbb{F} \otimes_{\mathbb{O}} P_{\mathbb{O}} \cong P_{\mathbb{F}}$. Moreover, $P_{\mathbb{O}}$ is unique up to (non-unique) isomorphism by Nakayama's lemma. This process gives us the extension map:

$$\begin{aligned} e : [\mathrm{Proj}(H_{\mathbb{F}})] &\to [\mathrm{Proj}(H_{\mathbb{K}})] \\ [P_{\mathbb{F}}] &\mapsto [\mathbb{K} \otimes_{\mathbb{O}} P_{\mathbb{O}}] \end{aligned}$$

2.4. **Brauer reciprocity.** One way of phrasing Brauer reciprocity is that d and e are adjoint with respect to the canonical pairing (the proof is identical to [Se, Part III]):

Lemma 2.1. *We have*

$$\langle e([P]), [L] \rangle = \langle [P], d([L]) \rangle.$$

for all $P \in \mathrm{Proj}\, H_{\mathbb{F}}$ *and* $L \in \mathrm{Rep}^f H_{\mathbb{K}}$.

Equivalently the matrices (with entries in $\mathbb{Z}[v^{\pm 1}]$) for d and e in the basis $\{[L_i]\}_{i=1}^m$ and $\{[P_i]\}_{i=1}^m$ are transposes of one another.

3. Quiver varieties and parity sheaves

In this section we recall briefly the definition of moduli spaces of quiver representations and explain why it makes sense to study parity sheaves on these spaces. Using a result of Maksimau [M] we then explain why describing the characters of parity sheaves on these spaces is equivalent to describing the indecomposable projective modules over Khovanov-Lauda-Rouquier algebras.

With the (possible) exception of § 3.6 and § 3.7 the material in this section is standard. The material concerning constructible sheaves on moduli of quiver representations is due to Lusztig [L1, L2] (see also [Sch] for an excellent survey). For the relation to KLR algebras see [VV, R, M].

[1]Here is an intuitive explanation of this fact, which was explained to me by Ivan Loseu. Let C be a finite-dimensional graded algebra. The grading on C is equivalent to an action of the multiplicative group on C. A module is gradable if and only if its twist by any element of the multiplicative group yields an isomorphic module. Now twisting preserves simple modules and their moduli are discrete (because C is finite-dimensional). Hence they are fixed, and hence gradable.

3.1. **Moduli of quiver representations.** Let Γ denote a quiver with vertex set I. Recall that a representation of Γ is an I-graded vector space $V = \oplus V_i$ together with linear maps $V_i \to V_j$ for each arrow $i \to j$ of Q. Let $\operatorname{Rep} Q$ denote the abelian category of complex representations of Γ. A *dimension vector* is an element of the monoid $\mathbb{N}[I]$ of formal $\mathbb{N}$-linear combination of I. Given an I-graded vector space V its dimension vector is $\underline{\dim} V = \sum(\dim V_i)i \in \mathbb{N}[I]$.

Fix a dimension vector $d = \sum d_i i \in \mathbb{N}[I]$ and a complex I-graded vector space V with $\dim V = d$. Consider the space

$$E_V := \prod_{i \to j} \operatorname{Hom}(V_i, V_j)$$

where the product is over all arrows $i \to j$ in Q. Then E_V is the space of all representations of the quiver Q of dimension vector d together with fixed isomorphism with V_i at each vertex $i \in I$. If we let

$$G_V := \prod_{i \in I} GL(V_i)$$

be the group of grading preserving automorphisms of V then then G_V acts on E_V by $(g_i) \cdot (\alpha_{i \to j}) := (g_j \alpha_{i \to j} g_i^{-1})$. The points of the quotient space $G_d \setminus E_V$ correspond to isomorphism classes of representations of Q of dimension vector d.

3.2. **Flags and proper maps.** Let us fix a dimension vector $d \in \mathbb{N}[I]$ and a sequence $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ such that $\sum i_j = d$. Given a representation W with $\underline{\dim} W = d$, a *flag on W of type* $\mathbf{i}$ is a flag

$$W^\bullet = (0 \subset W^1 \subset \dots W^m = W)$$

of subrepresentations of W such that W^j / W^{j-1} is isomorphic to the simple representation concentrated at the vertex i_j for all $1 \le j \le m$. Consider the space

$$E_V(\mathbf{i}) := \{(W^\bullet, W) \mid W \in E_V \text{ and } W^\bullet \text{ is a flag on } W \text{ of type } \mathbf{i}\}.$$

There is a natural embedding of $E_V(\mathbf{i})$ into a product of E_V and a product of partial flag varieties, and so $E_V(\mathbf{i})$ has the structure of a complex algebraic variety. One may define a G_V-action on $E_V(\mathbf{i})$ via $g \cdot (W^\bullet, W) := (gW^\bullet, g \cdot W)$. Consider the natural projection

$$\pi_{\mathbf{i}} : E_V(\mathbf{i}) \to E_V.$$

Then π is clearly proper and G_V-equivariant. It is also not too difficult to see that $E_V(\mathbf{i}, \mathbf{a})$ is smooth.

Example 3.1. If Γ consists of a single vertex i and a single loop and $\mathbf{i} = (i, i, \dots, i)$ then

$$\pi_{\mathbf{i}} : E_V(\mathbf{i}) \to E_V$$

is the Springer resolution of the nilpotent cone in $\operatorname{End}(V)$.

3.3. **Constructible sheaves and KLR algebras.** Fix a dimension vector d and an I-graded vector space V with $\underline{\dim} V = d$. In what follows we wish to consider constructible sheaves on $G_V \setminus E_V$. However, as this space is usually poorly behaved, we instead consider the G_V-equivariant geometry of the space E_V. Alternatively we could (and probably should) consider the quotient stack $[G_V \setminus E_V]$ as in the elegant treatment of Rouquier [R]. The difference is essentially aesthetic.

Let us fix a commutative ring of coefficients $\Bbbk$ and consider $D_d := D^b_{G_V}(E_V; \Bbbk)$ the G_V-equivariant bounded constructible derived category of E_V with coefficients in $\Bbbk$ [BL].

Set $\mathrm{Seq}(d) := \{(i_1, \dots, i_{\Bbbk}) \in I^k \mid \sum i_j = d\}$. Given any $\mathbf{i} \in \mathrm{Seq}(d)$ we can consider the proper map

$$\pi_{\mathbf{i}} : E_V(\mathbf{i}) \to E_V.$$

defined in the previous subsection. The direct image $\mathcal{L}_{\mathbf{i}} := \pi_{\mathbf{i}!}\underline{\Bbbk}$ of the equivariant constant sheaf on $E_V(\mathbf{i})$ in $D^b_{G_V}(E_V(\mathbf{i}), \Bbbk)$ is called a *Lusztig sheaf*. We set

$$\mathcal{L}_V := \bigoplus_{\mathbf{i} \in \mathrm{Seq}(d)} \mathcal{L}_{\mathbf{i}} \in D_d.$$

To d we may also associate a Khovanov-Lauda-Rouquier algebra $R(d)$ (see [KL, R]). It is a graded algebra with idempotents $e(\mathbf{i})$ corresponding to each $\mathbf{i} \in \mathrm{Seq}(d)$. It is free over $\mathbb{Z}$ and we denote by $R(d)_{\Bbbk}$ the algebra obtained by extension of scalars to our ring $\Bbbk$.

The following result explains the relation between Khovanov-Lauda-Rouquier algebras and the geometry of the moduli space of quiver representations:

Theorem 3.2. *One has an isomorphism of graded rings*

$$R(d)_{\Bbbk} \cong \mathrm{Ext}^{\bullet}_{D_d}(\mathcal{L}_V).$$

Under this isomomorphism $e(\mathbf{i})$ is mapped to the projection to $\mathcal{L}_{\mathbf{i}}$.

Proof. If $\Bbbk$ is a field of characteristic zero this is proved in [VV] and [R, § 5]. For an arbitrary ring $\Bbbk$ this is [M, Theorem 2.5] (see also [M, Theorem 2.11]). □

3.4. **Lusztig sheaves and projective modules.** It is well-known and easily proved that given an object X in an Karoubian additive category then $\mathrm{Hom}(-, X)$ gives an equivalence

$$\langle X \rangle^{op}_{\oplus} \xrightarrow{\sim} \mathrm{proj}\, E$$

where $E := \mathrm{End}(X)$, $\langle X \rangle^{op}_{\oplus}$ denotes the opposite category of the full additive Karoubian subcategory generated by X, and proj denotes the category of finitely generated projective modules over E (see e.g. [Kr, § 1.5]).

One can extend this observation to graded categories (i.e. categories equipped with a self-equivalence $M \mapsto TM$). If X is an object in a graded Karoubian additive category then $\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(-, T^m X)$ gives an equivalence of graded categories

$$\langle X \rangle^{op}_{T,\oplus} \xrightarrow{\sim} \mathrm{Proj}\, E$$

where $\langle X \rangle^{op}_{T,\oplus}$ denotes the opposite category of the full additive Karoubian subcategory generated by $T^m X$ for all $m \in \mathbb{Z}$, and E denotes $\bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(X, T^m X)$, naturally a graded algebra[2] and $\mathrm{Proj}\, E$ denotes the graded category of finitely generated projective modules over E (viewed as a graded category with self-equivalence $M \mapsto M(1)$).

Let us forget the triangulated structure on D_d and view it simply as a graded additive category, with self-equivalence given by $\mathcal{F} \mapsto \mathcal{F}[1]$. If we apply the above observations together with Theorem 3.2 (with V and d as in the previous section) we see that $\mathrm{Hom}^{\bullet}(-, \mathcal{L}_V)$ yields an equivalence

$$\langle \mathcal{L}_V \rangle^{op}_{[1],\oplus} \xrightarrow{\sim} \mathrm{Proj}\, R(d)_{\Bbbk}.$$

[2]given $f : X \to T^m X$ and $g : X \to T^n X$ their product is given by $T^n f \circ g : X \to T^{m+n} X$.

Moreover, because Theorem 3.2 is true for any ring of coefficients, the above equivalence is compatible with extension of scalars.

3.5. **Moduli of representations of Dynkin quivers.** We say that Q is a *Dynkin quiver* if the graph underlying Q is a simply-laced Dynkin diagram. If Q is a Dynkin quiver we identify I with the simple roots of the corresponding simply laced root system and write $R^+ \subset \mathbb{N}[I]$ for the positive roots. Recall Gabriel's theorem: a quiver has finitely many indecomposable representations if and only if the underlying graph of Q is a Dynkin diagram, in which case we have a bijection:

$$\underline{\dim} : \left\{ \begin{array}{c} \text{indececomposable} \\ \text{representations of } Q \end{array} \right\}_{/\cong} \xrightarrow{\sim} R^+.$$

Given a positive root $\alpha \in R^+$ we denote by I_α the corresponding indecomposable representation, which is well-defined up to isomorphism.

Recall that orbits of G_V on E_V correspond to isomorphism classes of representations of Q. Hence G_V has finitely many orbits on E_V if and only if there are finitely many isomorphism classes of representations of Q with dimension vector d. This is the case for all dimension vectors if and only if Γ is a Dynkin quiver.[3]

From now on let us assume that Q is a Dynkin quiver and fix a dimension vector d. By Gabriel's theorem and the Krull-Schmidt theorem we see that G_V-orbits on E_V are classified by tuples:

$$\Lambda_d := \{\lambda = (\lambda_\alpha)_{\alpha \in R^+} \mid \sum \lambda_\alpha \alpha = d\}$$

(λ_α gives the multiplicity of the indecomposable representation I_α as a direct summand of the isomorphism class of representation). Rephrasing this we have a stratification of E_V by G_V-orbits

$$E_V = \bigsqcup_{\lambda \in \Lambda_d} X_\lambda$$

where $X_\lambda := \{W \in E_V \mid W \cong \bigoplus_{\alpha \in R^+} I_\alpha^{\oplus \lambda_\alpha} \text{ in } \operatorname{Rep} Q\}$.

3.6. **Parity sheaves on quiver moduli.** We now recall the notion of a parity complex and sheaf [JMW]. As above we assume that Q is a Dynkin quiver and fix the stratification

$$E_V = \bigsqcup_{\lambda \in \Lambda_d} X_\lambda$$

of E_V by G_V-orbits. Given any $\lambda \in \Lambda_d$ we denote by $i_\lambda : X_\lambda \hookrightarrow E_V$ the inclusion of X_λ. Let $? \in \{!, *\}$. We say that a complex $\mathcal{F} \in D_d$ is *?-even* if the cohomology sheaves of $i_\lambda^? \mathcal{F}$ are local systems of free $\Bbbk$-modules which vanish in odd degree. We say that a comples $\mathcal{F} \in D_d$ is *even* if it is both $*$- and !-even. We say that $\mathcal{F}$ is *parity* if we can write it admits a decomposition $\mathcal{F} \cong \mathcal{F}_0 \oplus \mathcal{F}_1$ with both $\mathcal{F}_0$ and $\mathcal{F}[1]$ even. A complex $\mathcal{F}$ is a *parity sheaf* if it is parity, self-dual and indecomposable.

Now suppose that $\Bbbk$ is a complete local ring, so that D_d is a Krull-Schmidt category (see [JMW, § 2.1]). We have the following classification result:

Theorem 3.3. *For all $\lambda \in \Lambda_d$ there exists up to isomomorphism at most one parity sheaf $\mathcal{E}(\lambda, \Bbbk)$ with $\operatorname{supp} \mathcal{E}(\lambda, \Bbbk) = \overline{X_\lambda}$.*

[3]For this statement one needs slightly more than Gabriel's thoerm. It is known (see e.g. [C-B]) that a quiver has finitely many isomorphism classes of representations of any fixed dimension vector if and only if it is Dynkin.

Proof. The theorem is an immediate from the theory of parity sheaves, once we have established that our stratified G_V-variety E_V satisfies [JMW, (2.1) and (2.2)] and that all G_V-equivariant local systems on X_λ are constant. Both these statements hold by [M, Lemma 3.7]. □

For a general Dynkin quiver one does not know if Lusztig sheaves are parity (see [M, Conjecture 1.3]). The problem is that one does not know of the fibres of the maps $\pi_{\mathbf{i}}$ have vanishing odd cohomology. If Q is of type A then this has been established by Maksimau (see [M, Corollary 3.36]):

Theorem 3.4. *Suppose that Q is of type A. Then for all $\lambda \in \Lambda_d$ there exists a parity sheaf $\mathcal{E}(\lambda, \Bbbk)$ with $\operatorname{supp} \mathcal{E}(\lambda, \Bbbk) = \overline{X_\lambda}$. Moreover, $\mathcal{L}_V$ is parity and for all $\lambda \in \Lambda_d$, a shift of $\mathcal{E}(\lambda, \Bbbk)$ occurs as a direct summand of $\mathcal{L}_V$.*

Remark 3.5. Actually, Maksimau establishes the existence of parity sheaves for any Dynkin quiver. Surprisingly, it seems difficult to show that they occur as summands of Lusztig sheaves.

3.7. **Parity sheaves and the Kleshchev-Ram conjecture.** In this section we assume that Q is a Dynkin quiver of type A. We fix a dimension vector d. Recall our p-modular system $(\mathbb{K}, \mathbb{O}, \mathbb{F})$ from §2.1.

Recall that the KLR algebra $R(d)$ is a graded ring, with each graded component free and finite rank over $\mathbb{Z}$. Moreover $R(d)$ contains a polynomial subring and the symmetric polynomials yield a subring $A \subset R(d)$ in the centre of $R(d)$. In fact, A is equal to the centre of $R(d)$ [KL, Theorem 2.9] and hence is a summand of $R(d)$ as a $\mathbb{Z}$-module. Finally, $R(d)$ is free of finite rank over $R(d)$ by [KL, Corollary 2.11]. Hence if we set $H_{\mathbb{O}} := R(d) \supset A_{\mathbb{O}}$ then $H_{\mathbb{O}}$ satisfies the conditions of §2.1.

On the other hand, if we apply the observations in §3.4 for $\Bbbk \in \{\mathbb{K}, \mathbb{O}, \mathbb{F}\}$ we obtain an equivalence of graded categories

$$\langle \mathcal{L}_V \rangle^{op}_{[1],\oplus} \xrightarrow{\sim} \operatorname{Proj} R(d)_{\Bbbk}. \tag{3.1}$$

By the Theorem 3.4 of Maksimau the indecomposable summands of $\mathcal{L}_V$ coincide (up to shifts) with the parity sheaves. We conclude that the indecomposable graded projective modules (and hence also the graded simple modules) are parametrised up to shift by Λ_d.

Remark 3.6. This fact has been established algebraically for any Dynkin quiver by Kleshchev and Ram [KR].

The following is the main result of this section:

Theorem 3.7. *The following are equivalent:*

1. *The Kleshchev-Ram conjecture holds for $R(d)$: the decomposition map*
$$[\operatorname{Rep}^f R(d)_{\mathbb{K}}] \to [\operatorname{Rep}^f R(d)_{\mathbb{F}}]$$
is trivial.
2. *For all $\lambda \in \Lambda_d$ the parity complex $\mathcal{E}(\lambda, \mathbb{O}) \otimes^L_{\mathbb{Z}_p} \mathbb{K}$ is indecomposable.*
3. *For all $\lambda \in \Lambda_d$ the stalks and costalks of the intersection cohomology complex $\mathbf{IC}(\overline{X_\lambda}, \mathbb{O})$ are free of p-torsion.*

Proof. By Brauer reciprocity (Lemma 2.1) statement (1) is equivalent to the extension map

$$e : [\operatorname{Proj} R(d)_{\mathbb{F}}] \to [\operatorname{Proj} R(d)_{\mathbb{K}}]$$

being the identity. Now the equivalence of (1) and (2) follows from the fact that (3.1) is compatible with extension of scalars.

It remains to show the equivalence of (2) and (3).

(2) ⇒ (3): If $\mathbb{k} = \mathbb{K}$ then we can apply the decomposition theorem [BBD] to conclude the $\mathcal{L}_V$ is isomorphic to a direct sum of shifts of intersection cohomology complexes. We conclude that $\mathcal{E}(\lambda, \mathbb{K}) \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{K})$. By the uniqueness of parity sheaves, $\mathcal{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K}$ is indecomposable if and only if $\mathcal{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K} \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{K})$.

Let $\mathcal{E}$ be a complex of sheaves of $\mathbb{O}$-modules with torsion free stalks and costalks. Then $\mathcal{E} \otimes_{\mathbb{O}} \mathbb{K} \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{K})$ if and only if $\mathcal{E} \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{O})$, as follows from the characterisation of $\mathbf{IC}(\overline{X_\lambda}, \mathbb{O})$ in terms of stalks and costalks [BBD, Proposition 2.1.9 and § 3.3].

Putting these two observations together, we conclude that $\mathcal{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K}$ is indecomposable if and only if $\mathcal{E}(\lambda, \mathbb{O}) \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{O})$. Hence (2) implies (3), because the stalks and costalks of $\mathcal{E}(\lambda, \mathbb{O})$ are free of p-torsion by definition.

(3) ⇒ (2): If $\mathbf{IC}(\overline{X_\lambda}, \mathbb{O})$ has torsion free stalks and costalks then it is parity (because $\mathbf{IC}(\overline{X_\lambda}, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K} \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{K})$ is). Hence $\mathbf{IC}(\overline{X_\lambda}, \mathbb{O}) \cong \mathcal{E}(\lambda, \mathbb{O})$ and $\mathcal{E}(\lambda, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K} \cong \mathbf{IC}(\overline{X_\lambda}, \mathbb{O}) \otimes_{\mathbb{O}} \mathbb{K}$ is indecomposable. Hence (3) implies (2). □

Remark 3.8. The advantange of condition (3) in the above theorem is that it is a purely topological condition, and hence we can use it to move the Kleshchev-Ram conjecture to a question about equivalent singularities in the flag variety, where more is known and calculations are easier.

Remark 3.9. See [W, Proposition 3.11] for results along similar lines to the above theorem.

4. Counterexamples

In this section we assemble some known results due to Braden and Polo and use Theorem 3.7 to give counterexamples to the Kleshchev-Ram conjecture for quivers of type A.

4.1. **Quiver varieties and flag varieties.** We briefly recall a construction (which I learnt from [KS, §8.1]) which relates an open subvariety inside the moduli of representations of a type A quiver to singularities of Schubert varieties in the flag variety of type A.

Fix $n \geq 0$ and consider the quiver Q of type A_{2n-1} with the following orientation:

$$1 \to 2 \to \dots \to n \leftarrow n+1 \leftarrow \dots \leftarrow 2n-1.$$

Consider the dimension vector $d = (1, 2, \dots, n, \dots, 2, 1)$. Let I denote the vertices of Q and let V denote an I-graded vector space with dimension vector d. Inside E_V consider the open subvariety U consising of representations all of whose arrows are injective. The product of the automorphisms of V_i for $i \neq n$ act freely on U and the quotient is isomorphic to the space of pairs on flags on $V_n \cong \mathbb{C}^n$, with the natural action of $G = GL(V_n)$.

Hence, after fixing a Borel subgroup $B \subset G$, the singularities of the closures of G_V-orbits on U are equivalent to the singularities of closures of G orbits on $G/B \times G/B$, or equivalently to the singularities of Schubert varieties in G/B. Combining these observations with Theorem 3.7 we deduce that if the Kleshchev-Ram conjecture holds then stalks and costalks of all intersection cohomology complexes of Schubert varieties in G/B are torsion free. The first counterexamples to this statement were given by Braden in 2004 [B1]. He gave examples of 2-torsion in the costalks of intersection cohomology complexes on the flag variety of GL_8.

For several years since Braden's announcement of his results the existence of $p \neq 2$ torsion was not known. Recently Polo [P] has proved that for all prime numbers p there exists a Schubert variety in the flag variety of $GL_{4p}(\mathbb{C})$ whose intersection cohomology complex has p-torsion in its costalk. It then follows from Theorem 3.7 that the Kleshchev-Ram conjecture is false for all primes.

The modules involved in the above counterexamples are enormous (the first counterexample involves a quiver of type A_{15}!). It seems unlikely that one could verify these counterexamples algebraically, even with the help of a powerful computer. The rest of this paper is devoted to the description of the Kashiwara-Saito singularity, which occurs in a quiver of type A_5, and is small enough that one (i.e. Jon Brundan) can verify algebraically that it provides a counterexample.

4.2. **The Kashiwara-Saito singularity.** Let Q denote the A_5 quiver:

$$Q = 1 \to 2 \to 3 \to 4 \to 5.$$

We identify the vertices I of Q with the simple roots $\alpha_1, \dots, \alpha_5$ of a root system R of type A_5. For $1 \leq i \leq j \leq 5$ let $\alpha_{ij} := \alpha_i + \dots + \alpha_j$. Then the positive roots of R are $\{\alpha_{ij} \mid 1 \leq i \leq j \leq 5\}$. Consider the dimension vector $d = 2\alpha_1 + 4\alpha_2 + 4\alpha_3 + 4\alpha_4 + 2\alpha_5$ and let V be an I-graded vector space with $\underline{\dim}\, V = d$.

As described in § 3.5, G_V orbits on E_V are parametrised by the set

$$\Lambda_d := \left\{ \sum_{1 \leq i \leq j \leq 5} \lambda_{ij}\alpha_{ij} \mid \sum \lambda_{ij}\alpha_{ij} = d \right\}.$$

Consider

$$\sigma := \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{45} + \alpha_{14} + \alpha_{35}$$
$$\pi := 2(\alpha_{33} + \alpha_{12} + \alpha_{34} + \alpha_{24}).$$

then $\pi, \sigma \in \Lambda_d$ and we let X_π and X_σ denote the corresponding G_V-orbits on E_V.

Proposition 4.1. **IC**$(\overline{X_\sigma}, \mathbb{Z})$ *has 2-torsion in its costalk at any point of* X_π.

Remark 4.2. It is a result due to Kashiwara and Saito [KS, Theorem 7.2.1] that the characteristic cycle of the intersection cohomology D-module on $\overline{X_\pi}$ is reducible. In fact, the characteristic cycle is equal to the sum of the fundamental classes of the closures of the conormal bundles to X_π and X_σ [B2].

Proof. Let $M_n(\mathbb{C})$ denote the space of $n \times n$-complex matrices over $\mathbb{C}$. Consider the space S of matrices $M_i \in M_2(\mathbb{C})$ for $i \in \mathbb{Z}/4\mathbb{Z}$ satisfying the two conditions

$$\operatorname{rank} M_i \leq 1 \text{ for } i \in \mathbb{Z}/4\mathbb{Z}, \tag{4.1}$$

$$M_i M_{i+1} = 0 \text{ for } i \in \mathbb{Z}/4\mathbb{Z}. \tag{4.2}$$

Clearly S is an affine variety. One can show that its dimension is 8. We call S (or more precisely the singularity of S at $0 \in S$) the *Kashiwara-Saito* singularity. It is known [KS, Lemma 2.2.2] that the singularity of $\overline{X_\sigma}$ at X_π is smoothly equivalent to the singularity of S at 0.

Let $G = GL_8(\mathbb{C})$ and B denote the subgroup of upper triangular matrices, and identify the Weyl group of G with the permutation matrices. Consider the Schubert variety $X_x = \overline{BxB/B}$ where x is the permutation $x = 62845173$. Then it is not difficult to check that the singularity of X_x along the Schubert variety ByB/B where $y = 21654387$ is smoothly equivalent the singularity of S at 0 (see [KS, Example 8.3.1]).

Because the stalks and costalks of the intersection cohomology complexes are invariant (up to a shift) under smooth equivalence, the proposition follows once one knows that $\mathbf{IC}(X_x, \mathbb{Z})$ has 2-torsion in its costalk at y. This has been verified by Braden (using similar techniques to those used in [B1]), by the author (using generators and relations for Soergel bimodules [EW]) and by Polo [P] (his examples of torsion for all primes p gives this example for $p = 2$). □

By Theorem 3.7 we conclude that the projective module $P(\sigma, \mathbb{Z}_2)$ for $R(d)_{\mathbb{Z}_2}$ corresponding to $\mathcal{E}(\sigma, \mathbb{Z}_2)$ is decomposable when we tensor with $\mathbb{Q}_2$. Hence the Kleshchev-Ram conjecture fails for $R(d)$. One may show that

$$\mathcal{E}(\sigma, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathbb{Q}_2 \cong \mathbf{IC}(\overline{X_\sigma}, \mathbb{Q}_2) \oplus \mathbf{IC}(\overline{X_\pi}, \mathbb{Q}_2).$$

By Brauer reciprocity it follows that the simple module for $R(d)$ indexed by π becomes reducible when one reduces modulo 2. In fact, in $[\operatorname{Rep} R(d)_{\mathbb{F}_2}]$ one has

$$[L(\pi, \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathbb{F}_2] = [L(\pi, \mathbb{F}_2)] + [L(\sigma, \mathbb{F}_2)].$$

where $L(\pi, \mathbb{Z}_2)$ denotes an integral form of the $R(d)_{\mathbb{Q}_2}$-module labelled by π, and $L(\pi, \mathbb{F}_2)$ and $L(\sigma, \mathbb{F}_2))$ denote the simple $[\operatorname{Rep} R(d)_{\mathbb{F}_2}]$-modules labelled by σ and π respectively. This has been verified by direct algebraic computations by Jon Brundan (aided by his computer) and Alexander Kleshchev (with his bare hands) and has recently become available [BK2, Example 2.20].

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