AFFINE CONES OVER FANO THREEFOLDS AND ADDITIVE GROUP ACTIONS

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ABSTRACT. We address the following question:
When an affine cone over a smooth Fano threefold admits an effective action of the additive group?

In this paper we deal with Fano threefolds of index 1 and Picard number 1. Our approach is based on a geometric criterion from [KPZ], which relates the existence of an additive group action on the cone over a smooth projective variety X with the existence of an open polar cylinder $U \simeq Z \times \mathbb{A}^1$ in X. Non-trivial families of Fano threefolds carrying a cylinder were found in [KPZ]. Here we provide new such examples.

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INTRODUCTION

All varieties in this paper are defined over $\mathbb{C}$. It is known [KPZ] that the affine cone over any smooth del Pezzo surface of degree $d \geq 4$ anticanonically embedded in $\mathbb{P}^d$ admits an effective $\mathbb{G}_a$-action. The existence of a $\mathbb{G}_a$-action on the affine cone over a projective variety X depends upon the polarization chosen. However, if $\mathrm{Pic}(X) \simeq \mathbb{Z}$, then all polarizations are proportional and so all the affine cones over X simultaneously admit or do not admit a $\mathbb{G}_a$-action.

On the other hand, under the assumption $\mathrm{Pic}(X) \simeq \mathbb{Z}$ it is natural to restrict to Fano varieties X only, since otherwise X is not uniruled and so the affine cones over X do not admit a $\mathbb{G}_a$-action, see [KPZ]. Consider, for instance, a Fano variety X

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with Picard number one which contains the affine space $\mathbb{A}^n$ as a Zariski open subset. Clearly, every affine cone over X admits a $\mathbb{G}_a$-action. This applies e.g. to $\mathbb{P}^n$, the smooth quadric Q in $\mathbb{P}^{n+1}$, or the Fano threefold X_5 of index 2 and degree 5. In [KPZ, 5.1-5.2] we found two more families of rational Fano threefolds X with Picard number one such that every affine cone over X admits a $\mathbb{G}_a$-action. Namely, these are the smooth intersections of two quadrics in $\mathbb{P}^5$ and the Fano threefolds X_{22} of genus 12. In the next theorem we provide two more such families. Given a Fano threefold X, we let $\tau(X)$ denote the Fano scheme of X that is, the component of the Hilbert scheme parameterizing the lines on X.

Theorem 0.1. *Let X be a Fano threefold of genus $g = 9$ or 10 with*

$$\mathrm{Pic}(X) = \mathbb{Z} \cdot (-K_X)\,.$$

If the scheme $\tau(X)$ is not smooth, then the affine cone over X under any projective embedding $X \hookrightarrow \mathbb{P}^N$ admits an effective $\mathbb{G}_a$-action. The Fano threefolds with a non-smooth scheme $\tau(X)$ form a codimension one subvariety in the corresponding moduli space.

Let us make the following observation. It is known [Pr3] that the automorphism group of a Fano threefold X as in Theorem 0.1 is finite. It follows that for any affine cone over X, the group of its linear automorphisms is one-dimensional, while the whole automorphism group is infinite-dimensional, see [KPZ, §§2-3].

A geometric construction used in the proof of Theorem 0.1 involves a line L on X, which corresponds to a non-smooth point of $\tau(X)$. Besides, in Theorems 3.3 and 3.6 we provide families of examples, which evoke instead a smooth point $[L] \in \tau(X)$. It seems plausible that the latter families are not contained in the former ones. A natural question arises whether the conclusion of Theorem 0.1 remains true for any Fano threefold of genus $g = 9$ or 10 with Picard number 1. We expect, however, that the answer is negative.

The proof of Theorem 0.1 is based on the following geometric criterion. Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety. We say that X possesses a *polar* $\mathbb{A}^1$*-cylinder* U if there exists an effective $\mathbb{Q}$-divisor D on X such that $D \sim_{\mathbb{Q}} H$, where H stands for the hyperplane section, and

$$U = Y \setminus \mathrm{supp}\ D \cong Z \times \mathbb{A}^1$$

for some quasiprojective variety Z. We let $\mathrm{AffCone}(X)$ denote the affine cone over X.

Theorem 0.2. ([KPZ, Theorem 3.9]) *If X as above possesses a polar $\mathbb{A}^1$-cylinder $U \to Z$ with $\mathrm{Pic}(Z) = 0$, then $\mathrm{AffCone}(X)$ admits an effective $\mathbb{G}_a$-action.*

Vice versa, if $\mathrm{AffCone}(X)$ admits an effective $\mathbb{G}_a$-action, then there exists in X an open set $U = Y \setminus \mathrm{supp}\ D$, where D is as before, isomorphic to the total space of a line bundle.

Specifying Theorem 0.2 we deduce the following corollary.

Corollary 0.3. *Let X be a smooth subvariety in $\mathbb{P}^n$ with $\mathrm{Pic}(X) \simeq \mathbb{Z}$. Then $\mathrm{AffCone}(X)$ admits an effective $\mathbb{G}_a$-action if and only if there exists in X an open cylinder $U \simeq Z \times \mathbb{A}^1$.*

Proof. Indeed, since $\mathrm{Pic}(X) \simeq \mathbb{Z}$, every cylinder in X is polar. Since a line bundle over Z is locally trivial, shrinking Z if necessary we may assume that it is trivial. □

We apply this criterion to smooth Fano threefolds of index one and with Picard number one. Thus Theorem 0.1 follows from Theorem 3.1 which says that every Fano threefold X satisfying the assumptions of Theorem 0.1 has a cylinder.

Section 1 contains a brief overview on Fano threefolds, with a special accent on the rationality problem. Besides, we collect here some useful facts on the variety of lines in a Fano threefold. In Section 2 we describe two standard constructions, which give all Fano threefolds of genus 9 and 10. Sometimes the proofs are hardly accessible in the literature, so we provide them here. The main Theorems 0.1[1], 3.3, and 3.6 are proven in Section 3.

1. Generalities on Fano threefolds

We recall that a Fano variety is a smooth projective variety X with an ample anticanonical class $-K_X$. The Fano index $r = i(X)$ is defined via $-K_X = rH$, where $H \in \mathrm{Pic}(X)$ is a primitive ample divisor class. It is well known that $r \le \dim X + 1$. We write $X = X_d$ for a Fano threefold of degree d, where $d = H^3$. The genus g of X is defined via $2g - 2 = -K_X^3 \, (= dr^3)$.

1.1. **Classification of Fano threefolds: rationality.** Any Fano threefold X has index $r \le 4$. Furthermore,

- if $r = 4$ then $X \simeq \mathbb{P}^3$;
- if $r = 3$ then $X \simeq Q$, where Q is a smooth quadric in $\mathbb{P}^4$.

We assume in the sequel that $\mathrm{Pic}(X) \simeq \mathbb{Z}$.

- If $r = 2$ then the degree of X varies in the range $d = 1, \ldots, 5$. More precisely,
  (1) if $d = 1$ then X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$. Such a threefold X is non-rational [Tyu], [Gr];
  (2) if $d = 2$ then X is a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$. Such a threefold X is non-rational [Vo];
  (3) if $d = 3$ then X is a cubic hypersurface in $\mathbb{P}^4$, which is known to be non-rational [CG];
  (4) if $d = 4$ then $X = X_{2\cdot 2}$ is an intersection of two quadrics in $\mathbb{P}^5$. Such a threefold is rational [IPr];
  (5) if $d = 5$ then $X = X_5$ is a linear section (by $\mathbb{P}^6$) of the Grassmanian $G(2, 5)$ under its Plücker embedding in $\mathbb{P}^9$. Such a threefold is rational and unique up to isomorphism [IPr].
- If $r = 1$ then the genus of X varies in the range $g = 2, \ldots, 10$ and 12. More precisely,
  (a) If $g = 2, 3, 5$, or 8, then the threefold X is non-rational (see [Is_3], [IPu] for $g = 2$, [IM], [Is_3] for $g = 3$, [Be] for $g = 5$, [Is_3] and [CG] for $g = 8$);
  (b) if $g = 4$ or 6 then a general threefold X is non-rational [Be], [IPu], [Tyu];
  (c) if $g = 7, 9, 10$, or 12 then X is rational [IPr].

We are interested in Fano threefolds which possess a cylinder. By the Castelnuovo rationality criterion for surfaces, such a threefold must be rational. Of course, if X contains the affine space $\mathbb{A}^3$ as an open subset then it has a cylinder. Besides the projective space $\mathbb{P}^3$, a smooth quadric Q in $\mathbb{P}^4$, and the Fano threefold X_5, also certain threefolds X_{22} contain $\mathbb{A}^3$ [Fur]. The latter threefolds form a subvariety of codimension

[1]See also Theorem 3.1.

two in the moduli space of all the X_{22}, which has dimension 6. In contrast, a cylinder exists in every Fano threefold X_{22} or $X_{2\cdot 2}$ [KPZ, §5]. In Theorem 3.1 below we describe families of Fano threefolds with a cylinder among the X_{16} ($g=9$) and the X_{18} ($g=10$).

The question arises whether every rational Fano threefold carries a cylinder; in particular, whether this is true for all the threefolds X_{12} ($g=7$), X_{16} and X_{18}.

1.2. **Families of lines on Fano threefolds.** In the sequel we need the following facts.

Theorem 1.1 ([Sh_1], [Re_1], [Is_2, Ch. 3, §2], [IPr, §4.2])**.** *Let $X = X_{2g-2}$ be a Fano threefold of genus $g \ge 3$ with $\operatorname{Pic}(X) = \mathbb{Z}\cdot(-K_X)$, anticanonically embedded in $\mathbb{P}^{g+1}$. Then the following hold.*

(1) *There is a line L on X.*
(2) *For the normal bundle $\mathscr{N}_{L/X}$ there are the following possibilities:*

$$(\alpha)\quad \mathscr{N}_{L/X} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1), \quad \textit{or}$$

$$(\beta)\quad \mathscr{N}_{L/X} \simeq \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2)\,.$$

(3) *The scheme $\tau(X)$ is of pure dimension 1.*
(4) *The scheme $\tau(X)$ is smooth and reduced at a point $[L] \in \tau(X)$ if and only if the corresponding line L is of type (α).*
(5) *For $g \ge 7$ any line L on X meets at most a finite number of other lines L_i on X.*

Remark 1.2. Let $g = 9$ or 10 and $\operatorname{Pic}(X) = \mathbb{Z}\cdot(-K_X)$. According to [$\mathrm{Pr}_2$] and [GLN] every irreducible component of the scheme $\tau(X)$ is generically reduced. Thus for a Fano threefold X as in Theorem 0.1, the set of non-smooth points of the scheme $\tau(X)$ is at most finite. On the other hand, for a general Fano threefold X of this type, the scheme $\tau(X)$ is an irreducible smooth curve [Pr_1, §3.2], [Il, Cor. 5.1.b].

2. Fano threefolds of genera 9 and 10

We need the following lemma.

Lemma 2.1. *(a) Any smooth curve Γ of degree 7 and genus 3 in $\mathbb{P}^3$ lies on a unique (irreducible) cubic surface $F = F(\Gamma)$ in $\mathbb{P}^3$.*
(b) For any smooth, linearly non-degenerate curve Γ of degree 7 and genus 2 in $\mathbb{P}^4$, the quadrics containing Γ form a linear pencil, say, $\mathcal{Q}$. The base locus of this pencil is an irreducible quartic surface $F = F(\Gamma)$ in $\mathbb{P}^4$.

Proof. We provide a proof in the case $g = 10$, the case $g = 9$ being similar. Let $\mathscr{I}_\Gamma$ be the ideal sheaf of $\Gamma \subseteq \mathbb{P}^4$. Using the exact sequence

$$0 \longrightarrow \mathscr{I}_\Gamma(2) \longrightarrow \mathscr{O}_{\mathbb{P}^4}(2) \longrightarrow \mathscr{O}_\Gamma(2) \longrightarrow 0$$

by Riemann-Roch we obtain that $\dim H^0(\mathscr{I}_\Gamma(2)) \ge 2$. Hence there is a pencil of quadrics $\mathcal{Q}$ through Γ.

Assume to the contrary that there exist three linearly independent quadrics Q_1, Q_2, and $Q_3 \subseteq \mathbb{P}^4$ passing through Γ. Then $Q_1 \cap Q_2 \cap Q_3 = \Gamma + L$ (as a scheme), where L is a line. Consider the exact sequence

$$0 \longrightarrow \mathscr{O}_{\Gamma\cup L} \longrightarrow \mathscr{O}_\Gamma \oplus \mathscr{O}_L \longrightarrow \mathscr{F} \longrightarrow 0\,, \tag{2.1.1}$$

where the quotient sheaf $\mathscr{F}$ is supported on $\Gamma \cap L$. Since

$$\chi(\mathscr{O}_{\Gamma\cup L}) = -4 \quad \text{and} \quad \chi(\mathscr{O}_\Gamma \oplus \mathscr{O}_L) = \chi(\mathscr{O}_\Gamma) + \chi(\mathscr{O}_L) = 0\,,$$

we obtain by (2.1.1)

$$\#(\Gamma \cap L) = \dim H^0(\mathscr{F}) = \chi(\mathscr{O}_\Gamma \oplus \mathscr{O}_L) - \chi(\mathscr{O}_{\Gamma\cup L}) = 4\,.$$

Thus L must be a 4-secant line of Γ. Hence the projection with center L would map Γ to a plane cubic, a contradiction.

Let us show finally that F is irreducible. Indeed, otherwise Γ would be contained in an irreducible surface F' of degree ≤ 3 in $\mathbb{P}^4$. Since Γ is assumed to be linearly non-degenerate, F' must be a cubic surface. By [GH, Ch. 4, §3], either F' is a cone or $F' \simeq \mathbb{F}_1$. Proceeding as at the beginning of the proof, it is easily seen that in both cases $h^0(\mathscr{I}_\Gamma(2)) \ge h^0(\mathscr{I}_{F'}(2)) \ge 3$. Hence there is a two-dimensional family of quadrics passing through Γ, which leads to a contradiction as before. □

In 2.3–2.6 below we deal with the following setting.

Setup 2.2. We consider the following two cases:

(i) For $g = 9$, we let $W = \mathbb{P}^3$ and $\Gamma \subseteq \mathbb{P}^3$ be a smooth non-hyperelliptic curve of degree 7 and genus 3.

(ii) For $g = 10$, we let $W = Q \subseteq \mathbb{P}^4$ be a smooth quadric and Γ be a smooth curve of degree 7 and genus 2 on Q.

In both cases, we let $F = F(\Gamma)$ denote the corresponding surface from Lemma 2.1.

In the next proposition we list the possibilities for such a surface F.

Proposition 2.3. *In the notation and assumptions as in* 2.1*–2.2 we let* $g = 9$ *in case (a) of Lemma 2.1 and* $g = 10$ *in case (b). Then the surface* $F = F(\Gamma) \subseteq \mathbb{P}^{g-6}$ *belongs to one of the following classes.*

(1) $F \subseteq \mathbb{P}^{g-6}$ *is a normal del Pezzo surface with at worst Du Val singularities; or*

(2) $F \subseteq \mathbb{P}^{g-6}$ *is a non-normal scroll, whose singular locus* $\Lambda = \mathrm{Sing}(F)$ *is a double line. Furthermore, the normalization* F' *of* F *is a smooth scroll* F' *of the minimal degree* $g-6$ *in* $\mathbb{P}^{g-5}$*, and the normalization map* $\nu : F' \to F$ *is induced by the projection from a point* $P \in \mathbb{P}^{g-5} \setminus F'$*. The restriction* $\nu|_{\nu^{-1}(\Lambda)} : \nu^{-1}(\Lambda) \to \Lambda$ *is a ramified double cover. There are the following possibilities.*

   (a) *If* $g = 9$ *then* $F' \simeq \mathbb{F}_1$*, the embedding* $F' \subseteq \mathbb{P}^4$ *is defined by the linear system* $|\Sigma + 2\ell|$ *on* $\mathbb{F}_1$*, where* $\Sigma \subseteq \mathbb{F}_1$ *is the exceptional section and* ℓ *is a ruling, and* $\nu^{-1}(\Lambda) \sim \Sigma + \ell$ *is a reduced conic on* $F' \subseteq \mathbb{P}^4$*, which is either smooth or degenerate.*
   If $g = 10$ *then one of the following hold.*

   (b) $F' \simeq \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$*, the embedding* $F' \subseteq \mathbb{P}^5$ *is defined by the linear system* $|\Sigma + 2\ell|$*, and* $\nu^{-1}(\Lambda) \sim \Sigma$ *is a smooth conic on* $F' \subseteq \mathbb{P}^5$*; or*

   (b′) $F' \simeq \mathbb{F}_2$*, the embedding* $F' \subseteq \mathbb{P}^5$ *is defined by the linear system* $|\Sigma + 3\ell|$*, and* $\nu^{-1}(\Lambda) \sim \Sigma + \ell$ *is a reduced degenerate conic on* $F' \subseteq \mathbb{P}^5$*.*

Proof. Since F is a complete intersection, it is Gorenstein. By the adjunction formula $\omega_F \simeq \mathscr{O}_F(-1)$, i.e. F is (possibly non-normal) del Pezzo surface.

If F is normal, then by [HW] F is either a surface described in (1), or a cone over an elliptic curve $C \subseteq \mathbb{P}^{g-7}$ of degree $g-6$. Assume to the contrary that F is a cone.

Let $\xi : \tilde{F} \to F$ be the blowup of the vertex. Then $\tilde{F}$ is a smooth ruled surface over C. Let as before Σ and ℓ be the exceptional section and a ruling, respectively, with $\Sigma^2 = -k$. Letting $M = \xi^* \mathscr{O}_F(1)$ and $\tilde{\Gamma}$ be the proper transform of Γ on $\tilde{F}$, we can write $M \equiv \Sigma + k\ell$ and $\tilde{\Gamma} \equiv a\Sigma + b\ell$. Then

$$0 = M \cdot \Sigma, \qquad g - 6 = M^2 = k, \qquad \Sigma^2 = -k = 6 - g\,,$$

$$7 = \tilde{\Gamma} \cdot M = b, \quad \text{and} \quad \tilde{\Gamma} \cdot \Sigma = a(6-g) + 7 \geq 0\,.$$

Since $\tilde{\Gamma} \simeq \Gamma$ is not an elliptic curve, $a \geq 2$. This is only possible for $g = 9$, $a = 2$, and so $k = 3$. On the other hand, by adjunction

$$2g(\tilde{\Gamma}) - 2 = (\tilde{\Gamma} + K_{\tilde{F}}) \cdot \tilde{\Gamma} = 8,$$

a contradiction, since $g(\tilde{\Gamma}) = g(\Gamma) \leq 3$.

If F is non-normal then by [Na, Theorem 8], [Re_3], [Dol, 9.2.1], F is a projection of a normal surface F' of the minimal degree $g-6$ in $\mathbb{P}^{g-5}$. It is well known (see e.g., [GH, Ch. 4, §3, p. 525]) that $F' \subseteq \mathbb{P}^{g-5}$ is either a Veronese surface $F'_4 \subseteq \mathbb{P}^5$, or the image of a Hirzebruch surface $\mathbb{F}_n$ under the map given by the linear system $|\Sigma + k\ell|$, where $2k - n = g - 6$ and $k \geq n$. The case of the Veronese surface is impossible because the degree of every curve on $F'_4 \subseteq \mathbb{P}^5$ is even. Thus $F' \simeq \mathbb{F}_n$. Let $\Gamma' \subseteq \mathbb{F}_n$ be the proper transform of Γ on F'. We can write $\Gamma' \sim a\Sigma + b\ell$, where $a \geq 2$ and $b \geq na$. Note that in the case $g = 9$ we have $a \geq 3$, since Γ is assumed being non-hyperelliptic, see 2.2. It is easily seen that the remaining possibilities are as in (2). □

The following corollary is immediate.

Corollary 2.4. *In the notation of Proposition* 2.3(2)*, the class of* Γ' *in the Picard group of the normalization* $F' \simeq \mathbb{F}_n$ *is as follows:*

(a) $g = 9$, $F' \simeq \mathbb{F}_1$, $\Gamma' \sim 3\Sigma + 4\ell$;
(b) $g = 10$, $F' \simeq \mathbb{F}_0$, $\Gamma' \sim 2\Sigma + 3\ell$;
(b′) $g = 10$, $F' \simeq \mathbb{F}_2$, $\bar{\Gamma}' \sim 2\Sigma + 5\ell$.

In all cases Λ *is a* $(13-g)$*-secant line of* Γ *i.e., a* 3*-secant if* $g = 10$ *and* 4*-secant if* $g = 9$.

Now we can strengthen part (b) of Lemma 2.1.

Lemma 2.5. *In case (b) of Lemma* 2.1 *the pencil* $\mathcal{Q}$ *contains a smooth quadric.*

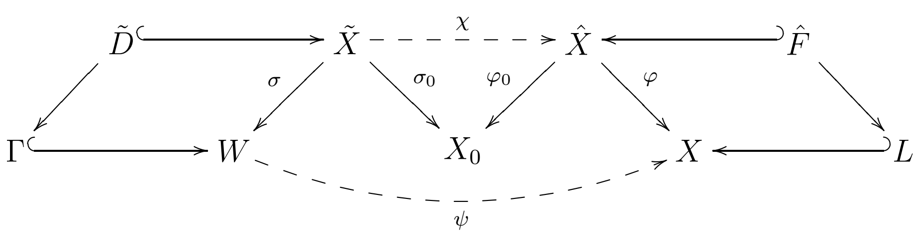
Proof. Assume to the contrary that every quadric $Q \in \mathcal{Q}$ is singular. By Bertini Theorem a general member $Q \in \mathcal{Q}$ is smooth outside F. Since F is a complete intersection, every member $Q \in \mathcal{Q}$ is smooth at the points of $F \setminus \mathrm{Sing}(F)$. If F has at worst isolated singularities, then so does every quadric $Q \in \mathcal{Q}$. Moreover, in this case they all must have a common singularity. Hence F should be a cone, which contradicts Proposition 2.3.

Thus under our assumption F must have non-isolated singularities. Moreover, by Proposition 2.3(2) F must be singular along a line Λ. If some quadric $Q \in \mathcal{Q}$ is singular along Λ, then F is again a cone, which is impossible. Thus we may assume that every quadric $Q \in \mathcal{Q}$ has an isolated singular point $P \in \Lambda$. Fixing such a quadric Q, we can choose an affine chart in $\mathbb{P}^4$ with coordinates $x_1, \ldots, x_4$ centered at P so that Λ is given by $x_1 = x_2 = 0$ and Q is given by $x_1x_3 + x_2x_4 = 0$. There is a quadric $Q' \in \mathcal{Q}$ given by $x_1u(x_1, x_2, x_4) + x_2v(x_1, x_2, x_3, x_4) = 0$, where u and v are linear forms. Since F is singular along Λ, at every point of Λ the Jacobian matrix of these two quadratic forms has rank ≤ 1. Therefore $x_3 v(0,0,x_3,x_4) = x_4 u(0,0,x_4)$ for all x_3, x_4. This implies that $v(0,0,x_3,x_4) = u(0,0,x_4) = 0$. So Q' is given by $x_1(ax_1+bx_2)+x_2(cx_1+dx_2) = 0$ for some $a,b,c,d \in \mathbb{C}$. Hence Q' is a cone with vertex Λ. Therefore $F = Q \cap Q'$ is a cone with vertex $P = (0,0,0,0) \in \Lambda$, which again gives a contradiction and concludes the proof. $\square$

In the case of a curve Γ lying on a smooth surface F, the following result can be found in [Is_1]. In the present more general form, the result was announced without proof in [IPr, Theorems 4.3.3 and 4.3.7]. Besides, we can quote an explanation in [IPr, 4.3.9(ii)] as to why the assumption in 2.2(i) that the curve Γ is non-hyperelliptic is important. The details of the proof can be found in an unpublished thesis [Pr_1] (in Russian). For the reader's convenience, we reproduce them below; see also the (unpublished) notes [BL].

Theorem 2.6. *In the notation as in Setup 2.2 there exists a Sarkisov link*

(2.6.2)



where σ is the blowup of Γ, σ_0 and φ_0 are the anticanonical maps onto $X_0 \subseteq \mathbb{P}^{g-1}$, χ is a flop, $X = X_{2g-2}$ is a smooth Fano threefold of genus g with $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$ anticanonically embedded in $\mathbb{P}^{g+1}$, and φ is the blowup of a line L on X. The exceptional divisor $\hat{F}$ of φ is a proper transform of the surface $F = F(\Gamma) \subseteq W$. The exceptional divisor $\tilde{D}$ of σ is a proper transform of a divisor $D \in |-(12-g)K_X - (25-2g)L|$. The map ψ^{-1} is the double projection with center L that is, a map given by the linear system $|A - 2L|$ on X, where $A \sim -K_X$ is a hyperplane section.

Proof. Let $\sigma : \tilde{X} \to W$ be the blowup of Γ. Let $\tilde{D}$ be the exceptional divisor and let $H^* = \sigma^* H$, where H is the positive generator of $\mathrm{Pic}(W) \simeq \mathbb{Z}$. We have (see e.g. [IPr, Lemma 2.2.14])

(2.6.3) $$(H^*)^3 = g-8, \qquad (H^*)^2\cdot\tilde{D} = 0, \qquad H^*\cdot\tilde{D}^2 = -H\cdot\Gamma = -7\,,$$

and

$$\tilde{D}^3 = -\deg\mathscr{N}_{\Gamma/W} = \begin{cases} -23 & \text{if } g = 10, \\ -32 & \text{if } g = 9. \end{cases}$$

Letting $\tilde{F} \subseteq \tilde{X}$ be the proper transform of F we get $\tilde{F} \sim (12-g)H^* - \tilde{D}$. The divisor classes $-K_{\tilde{X}} \sim (13-g)H^* - \tilde{D}$ and $\tilde{F}$ form a basis of $\mathrm{Pic}(\tilde{X}) \simeq \mathbb{Z}\oplus\mathbb{Z}$. We have

(2.6.4) $$-K_{\tilde{X}}^3 = 2g-6 > 0, \quad (-K_{\tilde{X}})^2\cdot\tilde{F} = 3, \quad -K_{\tilde{X}}\cdot\tilde{F}^2 = -2, \quad \text{and} \quad \tilde{F}^3 = g-13\,.$$

We need the following fact.

Claim 2.7. *The divisor $-K_{\tilde{X}}$ is nef and big.*

Proof. Since $-K_{\tilde{X}}^3 = 2g-6>0$, the divisor $-K_{\tilde{X}}$ is big. To establish that it is also nef, we consider the case $g=10$; the proof in the case $g=9$ is similar. From the exact sequence

$$0 \longrightarrow \mathscr{I}_\Gamma(3) \longrightarrow \mathscr{O}_W(3) \longrightarrow \mathscr{O}_\Gamma(3) \longrightarrow 0$$

we obtain by Riemann-Roch

$$\dim H^0(\mathscr{I}_\Gamma(3)) \ge \dim H^0(\mathscr{O}_W(3)) - \dim H^0(\mathscr{O}_\Gamma(3)) = 10\,.$$

The members of the linear system $|-K_{\tilde{X}}|$ are proper transforms of the members of the linear system $|-K_W| = |\mathscr{O}_W(3)|$ passing through Γ. Hence

$$\dim|-K_{\tilde{X}}| \ge 9\,. \tag{2.7.5}$$

Applying Lemma 2.1 it is easily seen that the only reducible members $\tilde{G}\in|-K_{\tilde{X}}|$ are those of the form $\tilde{G}=\tilde{F}+H^*$. Hence such divisors form a linear subsystem in $|-K_{\tilde{X}}|$ of codimension ≥ 5.

Assume to the contrary that there exists an irreducible curve $\tilde{C}$ on $\tilde{X}$ with $\tilde{C}\cdot(-K_{\tilde{X}})<0$, and let $C=\sigma(\tilde{C})\subseteq W$. Since $g(\Gamma)=2$, the curve Γ does not admit any 4-secant line. Indeed, otherwise the projection from this line would send Γ isomorphically to a plane cubic, which is impossible. Since

$$\#(C\cap\Gamma) = \tilde{C}\cdot\tilde{D} > 3H^*\cdot\tilde{C} = 3\deg C \ge 3\,,$$

the curve C cannot be a line. If C is contained in a plane $\Pi\subseteq\mathbb{P}^4$ then by the same argument

$$\#(\Pi\cap\Gamma) \ge \#(C\cap\Gamma) > 3\deg C \ge 6\,.$$

Since $\deg\Gamma=7$ and Γ is linearly non-degenerate, we get a contradiction. Thus C is not contained in a plane and so $\deg C\ge 3$. Assume that C is contained in some hyperplane $\Theta\subseteq\mathbb{P}^4$. Then as above

$$\#(\Theta\cap\Gamma) \ge \#(C\cap\Gamma) > 3\deg C \ge 9\,,$$

which again leads to a contradiction because $\deg\Gamma=7$. Therefore C is linearly non-degenerate and $\deg C\ge 4$.

On the other hand, F contains a line, say, Υ. Let $\tilde{\Upsilon}\subseteq\tilde{X}$ be its proper transform. We have $\tilde{\Upsilon}\cdot(-K_{\tilde{X}})\le 3=\Upsilon\cdot(-K_W)$. Therefore, fixing four general points on $\tilde{\Upsilon}$, a member $\tilde{M}\in|-K_{\tilde{X}}|$ passing through these points is forced to contain $\tilde{\Upsilon}$. The family of all such members has codimension at most 4, while degenerate ones vary in a family of codimension at least five, as we observed before. Hence there exists an irreducible divisor $\tilde{M}\in|-K_{\tilde{X}}|$ containing $\tilde{\Upsilon}$. By our assumption $\tilde{M}\cdot\tilde{C}<0$, and then also $\tilde{F}\cdot\tilde{C}=\tilde{M}\cdot\tilde{C}-H^*\cdot\tilde{C}<0$. Thus the intersection $\tilde{M}\cap\tilde{F}$ contains $\tilde{C}\cup\tilde{\Upsilon}$ and so by (2.6.3)

$$\deg(C+\Upsilon) = (\tilde{C}+\tilde{\Upsilon})\cdot H^* \le \tilde{M}\cdot\tilde{F}\cdot H^* = -K_{\tilde{X}}\cdot\tilde{F}\cdot H^* = (3H^*-\tilde{D})\cdot(2H^*-\tilde{D})\cdot H^* = 5\,.$$

It follows that $\deg C=4$, so $C\subseteq\mathbb{P}^4$ is a rational normal quartic curve. Every quadric in the linear system $H^0(\mathscr{I}_{C\cup\Gamma}(2))$ contains $C\cup\Gamma$. Picking two distinct points on Γ let us consider the family of quadrics from $H^0(\mathscr{I}_C(2))$ passing through these points. It has dimension four. Such a quadric cuts Γ in $13+2=15$ points, hence contains it.

An easy computation gives $\dim H^0(\mathscr{I}_C(2))=6$. It follows that

$$\dim H^0(\mathscr{I}_{C\cup\Gamma}(2)) \ge 6-2 = 4\,.$$

However, the latter contradicts Lemma 2.1(b). This shows that in the case $g = 10$, the divisor $-K_{\tilde{X}}$ is nef. The case $g = 9$ can be treated similarly. □

By the Base Point Freeness Theorem we deduce the following.

Corollary 2.8. *For some $n > 0$ the linear system $|-nK_{\tilde{X}}|$ defines a birational morphism $\sigma_0 : \tilde{X} \to X_0 \subseteq \mathbb{P}^N$ whose image is a Fano threefold with at worst Gorenstein canonical singularities. Moreover $-K_{\tilde{X}} = \sigma_0^*(-K_{X_0})$.*

Our next claim is as follows.

Claim 2.9. *The morphism σ_0 is small, i.e. it does not contract any divisor.*

Proof. Assume that σ_0 contracts a prime divisor $\Xi \sim \alpha(-K_{\tilde{X}}) - \beta\tilde{F}$. Then by (2.6.4)

$$0 = \Xi \cdot (-K_{\tilde{X}})^2 = (2g-6)\alpha - 3\beta\,.$$

This yields $\beta = (2g/3 - 2)\alpha$. Since $\Xi \neq \tilde{F}$ and $-K_{\tilde{X}}$ is nef by 2.7, we have

$$0 \leq \Xi \cdot \tilde{F} \cdot (-K_{\tilde{X}}) = 3\alpha + 2\beta = \alpha(4g/3 - 1)\,.$$

Hence $\alpha > 0$. Furthermore,

$$\Xi \sim \alpha(2g^2/3 - 11g + 37)H^* + \alpha(2g/3 - 3)\tilde{D}\,.$$

Since $\sigma_*\Xi$ is effective we must have $2g^2/3 - 11g + 37 \geq 0$, a contradiction. □

The following corollary is standard.

Corollary 2.10. *In the notation as above, X_0 has at worst isolated compound Du Val singularities.*

Following the techniques outlined in [IPr, §4.1] we can now finish the proof of Theorem 2.6.

End of the proof of 2.6. If $-K_{\tilde{X}}$ is ample then the map σ_0 is an isomorphism. In this case we let $\hat{X} = \tilde{X} = X_0$ and χ to be the identity map. Otherwise by [Kol] the contraction $\sigma_0 : \tilde{X} \to X_0$ can be completed to a flop triangle as in diagram (2.6.2). Here φ_0 is another small resolution of X_0. Let $\hat{C} \subseteq \hat{X}$ and $\tilde{C} \subseteq \tilde{X}$ be the flopped and the flopping curves, respectively. Then χ induces an isomorphism $\tilde{X} \setminus \tilde{C} \simeq \hat{X} \setminus \hat{C}$.

In both cases the divisor $-K_{\hat{X}} = \varphi^*(-K_{X_0})$ is nef and big. Further, we have

$$-K_{\hat{X}}^3 = -K_{\tilde{X}}^3 = 2g-6, \quad (-K_{\hat{X}})^2 \cdot \hat{F} = (-K_{\tilde{X}})^2 \cdot \tilde{F} = 3, \quad -K_{\hat{X}} \cdot \hat{F}^2 = -K_{\tilde{X}} \cdot \tilde{F}^2 = -2.$$

Since $\operatorname{Pic}(\hat{X}) \simeq \operatorname{Pic}(\tilde{X})$ is of rank 2 the Mori cone $\operatorname{NE}(\hat{X})$ is generated by two extremal rays. One of them has the form $\mathbb{R}_+[T]$, where T is a curve in the fiber of σ (resp., φ_0) if χ is an isomorphism (resp., not an isomorphism). Let $R \subseteq \operatorname{NE}(\hat{X})$ be the second extremal ray. Since $-K_{\hat{X}}$ is nef and big, R is K-negative. By [Mo] there exists a contraction $\varphi : \hat{X} \to X$ of R.

Since $-K_{\tilde{X}} - \tilde{F} = \sigma^*\mathscr{O}(1)$ is nef we have $(-K_{\tilde{X}} - \tilde{F}) \cdot \tilde{C} > 0$. Therefore $\tilde{F} \cdot \tilde{C} < 0$ and $\hat{F} \cdot \hat{C} > 0$. Since $-K_{\hat{X}} \cdot \hat{F}^2 = -2 < 0$, the divisor $\hat{F}$ is not nef. Hence $\hat{F} \cdot R < 0$ that is, the ray R is not nef. By the classification of extremal rays [Mo], φ is a birational divisorial contraction. Moreover, the φ-exceptional divisor coincides with $\hat{F}$. If $\varphi\colon \hat{X} \to X$ contracts $\hat{F}$ to a point, then by [Mo]

$$(-K_{\hat{X}})^2 \cdot \hat{F} = 4, \quad 2 \quad \text{or} \quad 1\,.$$

On the other hand, $(-K_{\hat{X}})^2 \cdot \hat{F} = 3$, a contradiction. Hence $\varphi\colon \hat{X} \to X$ contracts $\hat{F}$ to a curve Z. In this case both X and Z are smooth and φ is the blowup of Z [Mo]. Moreover, X is a Fano threefold of Fano index $r = 1$, 2, 3 or 4. The group $\operatorname{Pic}\hat{X}$ is generated by $\hat{F}$ and

$$-\frac{1}{r}\varphi^* K_X = \frac{1}{r}(-K_{\hat{X}} + \hat{F})\,.$$

Therefore, the subgroup generated by $\tilde{F}$ and $-K_{\tilde{X}}$ has index r in $\operatorname{Pic}\tilde{X} \simeq \operatorname{Pic}\hat{X}$. This implies that $r = 1$. We have

$$\begin{aligned}(-K_X)^3 = (-K_{\hat{X}}) \cdot (-K_{\hat{X}} + \hat{F})^2 &= \\ = (-K_{\hat{X}})^3 + 2\hat{F} \cdot (-K_{\hat{X}})^2 + (-K_{\hat{X}}) \cdot F'^2 &= \\ = 2g - 6 + 6 - 2 = 2g - 2\,,\end{aligned}$$

i.e. X is a Fano threefold of genus g. Furthermore,

$$\deg Z = -K_X \cdot Z = (-K_{\hat{X}} + \hat{F}) \cdot \hat{F} \cdot (-K_{\hat{X}}) = 3 - 2 = 1\,,$$

i.e. $Z \subseteq X$ is a line. Now an easy computation shows that $\hat{F}^3 \neq \tilde{F}^3$, so χ is not an isomorphism.

By [Is_2, Prop. 3] the linear system $|-K_{\hat{X}}|$ defines a birational map and X_0 is a Fano threefold with at worst isolated Gorenstein terminal singularities. In particular, $|-K_{X_0}|$ is very ample. Hence the linear system $|-K_{\tilde{X}}| = \sigma_0^*|-K_{X_0}|$ is base point free and defines the map σ_0.

Finally, Γ is (as a scheme) the base locus of the linear subsystem $\sigma_*|-K_{\tilde{X}}| \subseteq |\mathscr{O}_W(13-g)|$. It remains to show that in the case $g = 9$ the curve Γ is not hyperelliptic. Assume the converse. It was shown already that Γ does not admit a 5-secant line. On the other hand, by [GH, Ch. 2, §5] Γ admits a 4-secant line, say, N. The projection from N defines a linear system of degree 3 and dimension ≥ 1 on Γ. Hence the curve Γ is hyperelliptic and trigonal. However, this is impossible, since otherwise the linear systems g^1_2 and g^1_3 on Γ define a birational morphism $\Gamma \to \mathbb{P}^1 \times \mathbb{P}^1$ whose image is a divisor of bidegree $(2, 3)$. This contradicts the assumption that $g(\Gamma) = 3$. Now the proof of Theorem 2.6 is completed. □

Corollary 2.11. *In the notation as above we have $X \setminus D \simeq W \setminus F$.*

In the next proposition we describe the flopped and the flopping curves in (2.6.2).

Proposition 2.12. *In the notation as above we let $\tilde{C} \subseteq \tilde{X}$ and $\hat{C} \subseteq \hat{X}$ be the flopping and the flopped curve, respectively. Then the following hold.*

(1) *Any irreducible component $\hat{C}_i \subseteq \hat{X}$ either is a proper transform of a line $L_i \neq L$ on X meeting L, or (in the case where L is of type (β)) is the negative section Σ of the ruled surface $\hat{F} \simeq \mathbb{F}_3$.*

(2) *The curve $\hat{C}$ is a disjoint union of the $\hat{C}_i$'s.*

(3) *For any $\hat{C}_i$ we have*

$$\mathscr{N}_{\hat{C}_i/\hat{X}} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1) \quad \text{or} \quad \mathscr{N}_{\hat{C}_i/\hat{X}} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2)\,.$$

It follows that χ coincides with the Reid's pagoda [Re_2] near each $\hat{C}_i$.

(4) *The curve $\tilde{C}$ in $\tilde{X}$ is a disjoint union of the $\tilde{C}_i$'s, where each $\tilde{C}_i$ is the proper transform of a $(13-g)$-secant line of Γ.*

Proof. Recall that $\hat{C}$ and $\tilde{C}$ are exceptional loci of φ_0 and σ_0, respectively [Kol]. The assertion (1) is proven in [Is_1, Proposition 3, (iv)] and [IPr, Proposition 4.3.1], while (2) and (3) in [Cu, Proposition 4] and [Cu, Corollary 12, Theorem 13], respectively. By virtue of (3) $\tilde{C}$ is a disjoint union of its irreducible components. Finally $(-K_{\hat{X}}-\hat{F})\cdot\hat{C}_i = -1$. Therefore $1 = (-K_{\tilde{X}}-\tilde{F})\cdot\tilde{C}_i = \sigma^*\mathscr{O}_W(1)\cdot\tilde{C}_i$. So $\sigma(\tilde{C}_i)$ is a line. Since $-K_{\tilde{X}}\cdot\tilde{C}_i = 0$, this line must be $(13-g)$-secant. □

Finally in the next theorem we provide a criterion as to when the surface F in Theorem 2.6 is normal.

Theorem 2.13. *In the notation of Theorems 1.1 and* 2.6*, the surface F is normal if and only if L is a line of type (α) on X.*

Proof. We use the notation of Proposition 2.12. Assume that L is of type (β), and let $\tilde{C}_0$ denote the proper transform on $\tilde{X}$ of the negative section Σ of the ruled surface $\hat{F} \simeq \mathbb{F}_3$. By Remark 5.13 in [$\mathrm{Re}_2$], $\tilde{F}$ is not normal along $\tilde{C}_0$. Since $\tilde{C}_0$ is a smooth rational curve, σ is an isomorphism at a general point of $\tilde{C}_0$. So F is also non-normal along $\sigma(\tilde{C}_0)$.

Assume to the contrary that L is of type (α), while F is non-normal. Then F is singular along a line Λ. Clearly $\Lambda \neq \Gamma$, so $\tilde{F}$ is also non-normal and singular along $\sigma^{-1}(\Lambda)$. The map χ is an isomorphism near a general ruling $\hat{f} \subseteq \hat{F} \simeq \mathbb{F}_1$. Letting $\tilde{f} = \chi^{-1}(\hat{f})$, the surface $\tilde{F}$ is smooth along $\tilde{f}$ and $\sigma_0(\tilde{f}) = \varphi_0(\hat{f})$ is a line on $\sigma_0(\tilde{F}) = \varphi_0(\hat{F}) \simeq \mathbb{F}_1$. Let $l \subseteq F$ be a general line on a non-normal scroll F and $\tilde{l}$ be its proper transform on $\tilde{F}$. An easy computation shows that $\sigma_0(\tilde{l})$ is again a line on $\sigma_0(\tilde{F}) = \varphi_0(\hat{F}) \simeq \mathbb{F}_1$. Thus we may suppose that $\tilde{l} = \tilde{f}$. On the other hand, $\tilde{l} \cap \mathrm{Sing}(\tilde{F}) \neq \emptyset$, a contradiction. □

3. Constructions of cylinders

In this section we prove Theorem 0.1. Recall that under its assumptions $X = X_{2g-2}$ is a Fano threefold in $\mathbb{P}^{g+1}$ of genus $g = 9$ or 10 with $\mathrm{Pic}(X) = \mathbb{Z}\cdot(-K_X)$, having a non-smooth Fano scheme $\tau(X)$. By virtue of Corollary 0.3 the first assertion of Theorem 0.1 is equivalent to the following one.

Theorem 3.1. *Under the assumptions of Theorem 0.1 the variety X contains a cylinder.*

Proof. Assuming that the scheme $\tau(X)$ is not smooth at a point $[L] \in \tau(X)$, it suffices to construct a cylinder in $W \setminus F$ (see Corollary 2.11).

By Theorem 1.1(4) L is a line of type (β) on X. According to Theorem 2.13 the surface F is non-normal, and so by Proposition 2.3 $\Lambda = \mathrm{Sing}(F)$ is a double line on F. Consider the following diagram:

$$\begin{array}{ccc} & \bar{W} & \\ {}^{p}\swarrow & & \searrow^{q} \\ W & \overset{\xi}{\dashrightarrow} & \mathbb{P}^{g-8} \end{array}$$

where ξ is the projection from Λ, p is the blowup of Λ, and $q = \xi \circ p$. We show below that q is a $\mathbb{P}^{11-g}$-bundle over $\mathbb{P}^{g-8}$. Let $\bar{E} \subseteq \bar{W}$ be the exceptional divisor and $\bar{F} \subseteq \bar{W}$ be the proper transform of F.

In the case $g = 10$ the fibers of ξ are intersections of our smooth quadric $W \subseteq \mathbb{P}^4$ (see 2.2) with planes in $\mathbb{P}^4$ containing Λ. Therefore q is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, whose fibers are the proper transforms of lines in $W \subseteq \mathbb{P}^4$ meeting Λ. The morphism $q : \bar{W} \to \mathbb{P}^2$ is given by the linear system $|p^*\mathscr{O}_W(1) - \bar{E}|$. Since $\bar{F} \sim 2p^*\mathscr{O}_W(1) - 2\bar{E}$, the image $q(\bar{F}) = \xi(F)$ is a conic on $\mathbb{P}^2$. Since $\mathscr{N}_{\Lambda/W} \simeq \mathscr{O}_\Lambda \oplus \mathscr{O}_\Lambda(1)$, the $\mathbb{P}^1$-bundle $\bar{E} \to \Lambda$ is that of the Hirzebruch surface $\mathbb{F}_1 \to \mathbb{P}^1$. Moreover, its negative section $\bar{\Sigma}$ is a fiber of q. It follows that the open set $W \setminus F \simeq \bar{W} \setminus (\bar{F} \cup \bar{E})$ is an $\mathbb{A}^1$-bundle over $\mathbb{P}^2 \setminus q(\bar{F} \cup \bar{\Sigma})$. By [KM, Theorem 2] and [KW, Theorem], this bundle is trivial over a Zariski open subset $Z \subseteq \mathbb{P}^2 \setminus q(\bar{F} \cup \bar{\Sigma})$. This gives a cylinder contained in $W \setminus F$ and also a cylinder on X.

In the case $g = 9$ the fibers of ξ are planes in $W = \mathbb{P}^3$. The intersection of such a plane with the cubic surface F consists of the double line Λ and a residual line l. Therefore q is a $\mathbb{P}^2$-bundle over $\mathbb{P}^1$, and $\bar{F} \cup \bar{E}$ intersects each fiber along a pair of lines.

More precisely, we have $\bar{E} \cong \mathbb{F}_0$ and $\bar{F} \cong \mathbb{F}_1$ (see Proposition 2.3(2a)). Furthermore, $q|_{\bar{E}}$ and $q|_{\bar{F}}$, respectively, yield $\mathbb{P}^1$-bundles with rulings being lines in the fibers of q. By a simple computation we obtain that $\bar{F}|_{\bar{E}} \sim 2\bar{\Sigma} + \bar{l}$, where $\bar{\Sigma}$ (resp. $\bar{l}$) is a section (a ruling, respectively) of the trivial $\mathbb{P}^1$-bundle $\bar{E} \to \Lambda$. Notice that $\bar{\Sigma}$ is a line in a fiber of q and $\bar{l}$ is a section of q. The finite map $p|_{\bar{F}} : \bar{F} \to F$ yields a normalization of F. For the curve $\bar{F}|_{\bar{E}}$ there are the following two possibilities :

(i) $\bar{F}|_{\bar{E}} = \Delta_1$, where $\Delta_1 \in |2\bar{\Sigma} + \bar{l}|$ is irreducible, or
(ii) $\bar{F}|_{\bar{E}} = \bar{\Sigma} + \Delta_0$, where $\Delta_0 \in |\bar{\Sigma} + \bar{l}|$ is a diagonal.

We claim that $W \setminus F \simeq \bar{W} \setminus (\bar{F} \cup \bar{E})$ contains a cylinder. In what follows we deal with case (ii) only; (i) can be treated in a similar fashion. There exists exactly one fiber of q, say $\bar{\Pi}_\infty$, such that $\bar{E}, \bar{F}$ and $\bar{\Pi}_\infty$ meet along a common line. Blowing up $\bar{W}^\circ := \bar{W} \setminus \bar{\Pi}_\infty$ along the irreducible curve $\bar{E} \cap \bar{F} \cap \bar{W}^\circ$, we obtain an $\mathbb{F}_1$-bundle $\hat{\pi} : \hat{W}^\circ \to \mathbb{A}^1$ together with the proper transforms $\hat{F}^\circ$ and $\hat{E}^\circ$ on $\hat{W}^\circ$ of $\bar{F}$ and $\bar{E}$, respectively. The exceptional divisor $\hat{E}_1^\circ$ is ruled over $\mathbb{A}^1$ with rulings being the (-1)-curves in the fibers isomorphic to $\mathbb{F}_1$. There is a natural $\mathbb{P}^1$-bundle structure $\rho : \hat{W}^\circ \to \hat{E}_1^\circ$ which defines in each fiber of ρ the ruling $\mathbb{F}_1 \to \mathbb{P}^1$. The map ρ sends $\hat{E}^\circ$ and $\hat{F}^\circ$ to the intersections $\hat{E}^\circ \cap \hat{E}_1^\circ$ and $\hat{F}^\circ \cap \hat{E}_1^\circ$, respectively. The complement $\hat{W}^\circ \setminus (\hat{E}_1^\circ \cup \hat{E}^\circ \cup \hat{F}^\circ) \simeq \bar{W} \setminus (\bar{E} \cup \bar{F} \cup \bar{\Pi}_\infty) \simeq W \setminus (F \cup \Pi_\infty)$ is again a $\mathbb{P}^1$-bundle over $\hat{E}_1^\circ \setminus (\hat{E}^\circ \cup \hat{F}^\circ)$, where $\Pi_\infty := p_*(\bar{\Pi}_\infty)$. This bundle is trivial over a Zariski open subset $Z \subseteq \hat{E}_1^\circ$, and admits a tautological section defined by $\hat{E}_1^\circ \hookrightarrow \hat{W}^\circ$. After trivialization the map $\rho : \rho^{-1}(Z) \to Z$ becomes the first projection $Z \times \mathbb{P}^1 \to Z$. The second projection of the tautological section defines a morphism $f : Z \to \mathbb{P}^1$. The automorphism $t \longmapsto (t - f(z))^{-1}$ of $Z \times \mathbb{P}^1$ sends this section to the constant section 'at infinity'. The $\mathbb{A}^1$-bundle $\rho : \hat{W}^\circ \setminus \hat{E}_1^\circ \to \hat{E}_1^\circ$ being trivial over Z it defines a cylinder $\rho^{-1}(Z) \setminus \hat{E}_1^\circ \simeq Z \times \mathbb{A}^1$, as required. □

Proof of Theorem 0.1. The first assertion of Theorem 0.1 is a consequence of Theorems 1.1(4), 2.13, and 3.1. Let us show the second one. Recall that the automorphism group of a Fano threefold of genus $g = 9$ or 10 with $\mathrm{Pic}(X) = (-K_X) \cdot \mathbb{Z}$ is finite [Pr_3].

Fix a moduli space $\mathscr{M}_g$ of the Fano threefolds of genus $g = 9$ or 10 with $\mathrm{Pic}(X) = (-K_X) \cdot \mathbb{Z}$. It can be defined using GIT, and is unique up to a birational equivalence. Let $\mathscr{ML}_g$ be the moduli space of pairs (X, L), where X is a Fano threefold as above and L is a line on X. Consider a natural projection $\pi : \mathscr{ML}_g \to \mathscr{M}_g$ whose fiber over a point $[X] \in \mathscr{M}_g$ (which corresponds to X) is isomorphic to $\tau(X)$. By Theorem 1.1(3) we have $\dim \mathscr{M}_g = \dim \mathscr{ML}_g - 1$. By Theorem 2.6 $\mathscr{ML}_g$ is isomorphic to the moduli space of embedded curves $\Gamma \subseteq W$ of degree 7 and genus $g(\Gamma) = 12 - g$.

Let further $\mathscr{M}'_g \subseteq \mathscr{M}_g$ be the closed subvariety formed by all Fano threefolds X whose Fano scheme $\tau(X)$ is non-smooth, and let $\mathscr{ML}'_g \subseteq \mathscr{ML}_g$ be the subvariety formed by all pairs (X, L) such that L is of type (β). Then $\mathscr{M}'_g = \pi(\mathscr{ML}'_g)$. Since such a Fano threefold X contains at most a finite number of (β)-lines (see Remark 1.2) we have $\dim \mathscr{M}'_g = \dim \mathscr{ML}'_g$. Now the second assertion of Theorem 0.1 is immediate in view of the following claim. □

Claim 3.2. *Let $\mathscr{H}_g$ be the Hilbert scheme parameterizing the curves Γ on W of degree 7 and arithmetic genus $p_a(\Gamma) = 12 - g$. Then $\dim \mathscr{H}_g = 91 - 7g$. If the surface $F = F(\Gamma)$ is smooth along Γ, then $\mathscr{H}_g$ is smooth at the corresponding point. Furthermore, the subscheme of $\mathscr{H}_g$ parameterizing the curves Γ with $F(\Gamma)$ non-normal, has codimension 2.*

Proof. Assuming that $F(\Gamma)$ is smooth along Γ, we consider an exact sequence of normal bundles with base Γ

$$0 \longrightarrow \mathscr{N}_{\Gamma/F} \longrightarrow \mathscr{N}_{\Gamma/W} \longrightarrow \mathscr{N}_{F/W}|_{\Gamma} \longrightarrow 0\,. \tag{3.2.6}$$

Taking into account the relations

$$\deg \mathscr{N}_{\Gamma/F} = 2g(\Gamma) - 2 + \deg \Gamma \quad \text{and} \quad \deg \mathscr{N}_{F/W}|_{\Gamma} = \Gamma \cdot F\,,$$

we obtain by (3.2.6) that $H^1(\mathscr{N}_{\Gamma/W}) = 0$ and $\dim H^0(\mathscr{N}_{\Gamma/W}) = 91 - 7g$. Now the first two assertions follow by the standard facts of the deformation theory.

The proof of the last assertion is just a parameter count. By Corollary 2.4 the dimension of the family of curves Γ with a non-normal surface $F(\Gamma)$ equals 13 and 11 in cases (a) and (b)-(b′), respectively, while the family of all non-normal surfaces F is of codimension $15 - g$. □

Second construction. In this and the next subsections we describe some families of Fano threefolds of genera 9 and 10 carrying a cylinder, which plausibly are not covered by Theorem 0.1. In this subsection we prove the following theorem.

Theorem 3.3. *In the notation as in Setup 2.2 and Theorem 2.6, in the case $g = 10$ the threefold X contains a cylinder whenever the surface F has a singularity worse than the Du Val singularity of type A_1.*

Proof. Assume that the surface $F \subseteq W \subseteq \mathbb{P}^4$ is singular, where W is as before a smooth quadric in $\mathbb{P}^4$ and F is a complete intersection quartic surface in W. Let $P \in F$ be a singular point. There is a commutative diagram

$$\begin{array}{ccc} & \bar W & \\ {}^{p}\swarrow & & \searrow^{q} \\ W & \overset{\xi}{\dashrightarrow} & \mathbb{P}^3 \end{array} \tag{3.3.7}$$

where ξ is the projection from P and p is the blowup of P. Let $\bar E \subseteq \bar W$ be the exceptional divisor and $\bar F \subseteq \bar W$ the proper transform of F. Then $\Pi = q(\bar E)$ is a plane in $\mathbb{P}^3$, while the birational morphism q is the blowup of a conic $C \subseteq \Pi$. Furthermore, let $H_P = W \cap T_{P,W}$ be the tangent hyperplane section and $\bar H_P \subseteq \bar W$ be its proper transform. Then $\bar H_P$ is the q-exceptional divisor. Now let $\bar F \subseteq \bar W$ be the proper transform of F and let $F^\circ = q(\bar{F})$. It is easily seen that $F^\circ \subseteq \mathbb{P}^3$ is a quadric. Obviously, $W \setminus (F \cup H_P) \simeq \mathbb{P}^3 \setminus (F^\circ \cup \Pi)$. Note that $\bar{F} \cap \bar{E}$ is the exceptional divisor of $p_{\bar{F}} : \bar{F} \to F$ and $\bar{F} \cap \bar{E} \simeq F^\circ \cap \Pi$.

Now assume that the singularity $P \in F$ is worse than a Du Val singularity of type A_1. Then $\bar{F} \cap \bar{E} \simeq F^\circ \cap \Pi$ cannot be a smooth conic. So it is either a pair of crossing lines or a double line. In any case $\mathbb{P}^3 \setminus (F^\circ \cup \Pi)$ admits a cylinder by the arguments in the proof of Theorem 3.1 for $g = 9$. Indeed, $F^\circ \cup \Pi$ can be regarded as a cubic surface singular along a line. □

Consider, for instance, the following construction.

Example 3.4. Let $\Gamma_0 \subseteq \mathbb{P}^2$ be a plane quartic curve with a node P_1. Pick a pair of distinct general points P_2, $P_3 \in \Gamma_0$. Let $F_1 \to \mathbb{P}^2$ be the blowup of P_1, P_2, P_3 and let E_i be the corresponding exceptional divisors. Let $\Gamma_1 \subseteq F_1$ be the proper transform of Γ_0, and let $P_4 = \Gamma_1 \cap E_2$ (this is a single point). Let $F_2 \to F_1$ be the blowup of P_4, E_4 be the exceptional divisor, and $\Gamma_2 \subseteq F_2$ be the proper transform of Γ_1. Take a general point $P_5 \in E_4$. Letting $F_3 \to F_2$ be the blowup of P_5 and $\Gamma_3 \subseteq F_3$ be the proper transform of Γ_2, we see that F_3 is a weak del Pezzo surface of degree 4 [Dol, ch. 8] containing two (-2)-curves C_2 and C_4 that meet at a point. These are the proper transforms of E_2 and E_4, respectively. The anticanonical image of F_3 is a del Pezzo surface $F \subseteq \mathbb{P}^4$ with a Du Val singularity of type A_2, which is the image of $C_2 \cup C_4$. Since $\Gamma_3 \cdot (C_2 + C_4) = 1$, the image Γ of Γ_3 is a smooth curve of genus 2 and degree 7. Thus (F, Γ) satisfies the conditions of Theorem 3.3. More precisely, the complement $W \backslash F$ contains a cylinder, and the center $\Gamma \subseteq F$ of the blow-up $\sigma : \tilde{X} \to W$ is such that one can reach a pair (X, D) consisting of a Fano threefold $X = X_{18}$ $(g = 10)$ and an irreducible divisor D on X, which is the proper transform of $\sigma^{-1}(\Gamma)$ on X, with $X \backslash D \simeq W \backslash F$.

Remark 3.5. The construction (3.3.7) works as well in the case of a non-normal F. We believe that in this case there are several cylinder structures on X, and hence the Makar-Limanov invariant of any affine cone over X vanishes.

Third construction. In this subsection we construct a cylinder in the complement of an irreducible cubic surface $F \subseteq \mathbb{P}^3$ under certain restrictions on the singularities of F. In [Oh] some families of cubic surfaces F in $\mathbb{P}^3$ were found such that the complement $\mathbb{P}^3 \setminus F$ contains an $\mathbb{A}^2$-cylinder. However, sometimes an $\mathbb{A}^1$-cylinder exists while an $\mathbb{A}^2$-cylinder does not.

Theorem 3.6. *In the notation as in 2.2–2.6, in the case $g = 9$ the threefold X contains a cylinder whenever the cubic surface $F \subseteq \mathbb{P}^3$ has a singular point of type A_3 or worse. There exists a family of Fano threefolds X satisfying these assumptions.*

This theorem follows from the next proposition and Example 3.16 below.

Proposition 3.7. *Let F be an irreducible cubic surface in $\mathbb{P}^3$. Then the complement $\mathbb{P}^3 \setminus F$ contains a cylinder whenever the surface F has a singularity worse than the Du Val A_2 singularity.*

Before dwelling in the proof, let us mention an application of this result.

Remark 3.8. We observe that, whenever the complement of a cubic surface in $\mathbb{P}^3$ contains a cylinder, this complement admits an effective $\mathbb{G}_a$-action. This applies e.g. to the singular cubic surfaces as in Proposition 3.7 or in Lemma 3.10 below.

More generally, let X be a normal affine variety such that the class group $\mathrm{Cl}(X)$ is a torsion group, and let $U \simeq \mathbb{A}^1 \times Z$ be an $\mathbb{A}^1$-cylinder in X. We claim that X admits an effective $\mathbb{G}_a$-action along the corresponding $\mathbb{A}^1$-fibration. Indeed, consider the $\mathbb{G}_a$-action on U by shifts on the second factor, and let $\partial \in \mathrm{Der}(\mathcal{O}(U))$ be the corresponding locally nilpotent derivation. By our assumption, a multiple of the effective reduced divisor $D = X \setminus U$ is principal i.e., $mD = \mathrm{div}(f)$ for some $f \in \mathcal{O}(X)$ and $m \in \mathbb{N}$. Clearly, $f \in \ker(\partial)$ since f does not vanish on the $\mathbb{A}^1$-rulings of U. Hence $f^N\delta$ is well defined and locally nilpotent on $\mathcal{O}(X)$ for N sufficiently large (cf. [KPZ, Proposition 3.5]). □

We start the proof of Proposition 3.7 with several remarks and lemmas.

Remarks 3.9. (1) Any non-normal, irreducible cubic surface F in $\mathbb{P}^3$ different from a cone is a scroll in lines with a double line [Na], [Re$_3$] (cf. Proposition 2.3). The proof of Proposition 3.7 goes for such a surface F in the same way as that of Theorem 3.1.

(2) If F has a singular point P of multiplicity ≥ 3, then F is a cone over a plane cubic curve. So the projection $\mathbb{P}^3 \setminus \{P\} \to \mathbb{P}^2$ with center P determines a (linear) cylinder structure over an appropriate open set $Z \subseteq \mathbb{P}^2$.

(3) In case (1) F does not admit any isolated singularity. In fact, if F has a Du Val singularity then all its singular points are at most isolated Du Val singularities. The classification of all possible sets of Du Val singularities on cubic surfaces in $\mathbb{P}^3$ is as follows (see e.g., [BW] or [Dol])[2]:

$$(nA_1),\ n=1,\ldots,4,\ (nA_2),\ n=1,2,3,\ (A_3),\ (A_4),\ (A_5),$$

$$(nA_1, A_2),\ (nA_1, A_3),\ n=1,2,\ (A_1, 2A_2),\ (A_1, A_4),\ (A_1, A_5),$$

$$(D_4),\ (D_5),\ (E_6)\,.$$

In the proof of Proposition 3.7 we use the following simple observation.

Lemma 3.10. *Let F be a cubic surface in $\mathbb{P}^3$, L a line on F, and Π_λ ($\lambda \in \mathbb{P}^1$) the pencil of planes through L. Suppose that for a general $\lambda \in \mathbb{P}^1$*

$$\Pi_\lambda \cap F = L + C_\lambda, \quad \textit{where} \quad C_\lambda \cap L = 2P, \tag{3.10.8}$$

i.e. C_λ is a plane conic tangent to L at a point P. Then $\mathbb{P}^3 \setminus F$ contains a cylinder.

Proof. Blowing up $\mathbb{P}^3$ with center L yields a diagram

$$\begin{array}{ccc} & \tilde{\mathbb{P}}^3 & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{P}^3 & \overset{\xi}{\dashrightarrow} & \mathbb{P}^1 \end{array}$$

where p is the blowup of L, ξ is the projection with center L, and q is a $\mathbb{P}^2$-bundle. Let $\tilde{F}$ be the proper transform of F on $\tilde{\mathbb{P}}^3$ and $\tilde{E} \subseteq \tilde{\mathbb{P}}^3$ be the exceptional divisor of p. We

[2] The coefficients in the list mean the number of singular points of a given type.

fix a member, say, Π_∞ of our pencil, and we let $\tilde{\Pi}_\infty$ denote its proper transform on $\tilde{\mathbb{P}}^3$. In $\tilde{\mathbb{P}}^3$ we consider the open set

$$\tilde{U} = \tilde{\mathbb{P}}^3 \setminus (\tilde{\Pi}_\infty \cup \tilde{E}) \simeq \mathbb{P}^3 \setminus \Pi_\infty \simeq \mathbb{A}^3 .$$

Let h be a regular function on $\tilde{U}$ which defines the affine surface $\tilde{F} \cap \tilde{U}$. Consider further a rational map

$$\delta = (q, h) : \tilde{\mathbb{P}}^3 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 .$$

Its restriction to the open set $\tilde{U} \setminus \tilde{F}$ is regular, while the restriction to a general fiber $\tilde{\Pi}_\lambda \setminus (\tilde{E} \cup \tilde{F})$ of $q|\tilde{U}$ defines an $\mathbb{A}^1$-fibration. Hence δ defines as well an $\mathbb{A}^1$-fibration over a Zariski open subset of $\mathbb{P}^1 \times \mathbb{P}^1$. By[KM, Theorem 2] and [KW, Theorem] there exists a cylinder in $\mathbb{P}^3 \setminus F$ compatible with this $\mathbb{A}^1$-fibration.

□

Remarks 3.11. (1) The construction in the proof yields a cylinder in conics with a unique base point P. Such a cylinder can exist only if $P \in F$ is a singular point.

(2) If F as in Lemma 3.10 is singular at P, then there is a line L on F through P. Indeed, in an affine chart centered at P the equation of F can be written as $f_2 + f_3 = 0$, where f_2 and f_3 are homogeneous forms of degree 2 and 3, respectively. The system of equations $f_2 = f_3 = 0$ defines 6 lines on F through P, counting with multiplicities.

(3) Suppose that for a triple (F, L, P) as before, the pencil Π_λ does not satisfy the assumptions of Lemma 3.10. Then in an appropriate affine chart with coordinates (x, y, z) centered at a singular point P of F, the surface F can be given by equation

$$xy + zg(x, y, z) = 0 .$$

Since the quadratic part is of rank ≥ 2, in this case (F, P) is an A_n-singularity. These observations lead to the following corollary.

Corollary 3.12. *If (F, P) as before is a Du Val singularity not of type A_n, then $\mathbb{P}^3 \setminus F$ contains an $\mathbb{A}^1$-cylinder in conics with a unique base point P.*

It remains to determine for which A_n-singularities (F, P) of cubic surfaces the complement $\mathbb{P}^3 \setminus F$ contains a cylinder.

Lemma 3.13. *Let F be a cubic surface in $\mathbb{P}^3$ with an A_n-singularity $P \in F$. If $n \geq 3$ then the complement $\mathbb{P}^3 \setminus F$ contains a cylinder.*

Proof. Suppose that $n \geq 3$, and let $f = f_2 + f_3 = 0$ be an equation of F in a local affine chart (x, y, z) centered at P. If $\operatorname{rk} f_2 = 1$ then (F, P) is of type D_n or E_6. If $\operatorname{rk} f_2 = 2$ then (F, P) is non-normal or of type A_n $(n \geq 2)$, and if $\operatorname{rk} f_2 = 3$ then (F, P) is of type A_1. In the former case, the result follows from Corollary 3.12. The case $n \leq 2$ is eliminated by our assumption. In the second case, we can reduce the equation to the form

$$f = xy + g_3(x, y) + g_2(x, y)z + g_1(x, y)z^2 + cz^3 = 0 ,$$

where g_i is a homogeneous form of degree i. We claim that $c = 0$. Indeed, let the blowup of $\mathbb{P}^3$ at P be given in an affine chart as $(x, y, z) \longmapsto (xz, yz, z)$, with the exceptional divisor $z = 0$. Then the equation of the proper transform F' of F in this chart is

$$xy + g_3(x, y)z + g_2(x, y)z + g_1(x, y)z + cz = 0 .$$

Since $n > 2$ and the surface F' should acquire a singular point of type A_{n-2} at the origin, we conclude that $c = 0$.

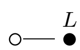
Furthermore, we may assume that $L = \{x = y = 0\}$. Consider the pencil of planes $\Pi_\lambda = \{y = \lambda x\}$ through L. We have $\Pi_\lambda \cap F = L + C_\lambda$, where $L \cap C_\lambda = \{x = 0,\ z^2 = 0\}$ has a double point. Now the conclusion follows by Lemma 3.10. □

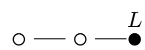
Remark 3.14. In the case where $P \in F$ is an A_1 or A_2 singularity and L is a line on F through P, there is no plane Π_λ through L such that the residual conic on the section $\Pi_\lambda \cap F$ is tangent to L at P.

Remark 3.15. For a cubic surface $F \subseteq \mathbb{P}^3$ the following are equivalent:

(1) F has a singularity worse than Du Val A_2 singularity,
(2) there exists a line L on F such that the pair (F, L) is not purely log terminal (PLT).

Indeed, assuming that all singularities of F are of type A_1 or A_2, consider a line L on F. Since L is smooth, for any singular point $P \in F$ the dual graph of the minimal resolution has the form

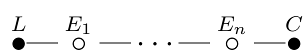


or


Thus (F, L) is PLT by the classification of the PLT singularities of surfaces. Hence (2) implies (1).

To show the converse, assume that (F, L) is PLT. Again by the classification of the PLT singularities, and because there is a line passing through any singular point of F, the surface F is normal and has only A_n-singularities. Take L as in Lemma 3.10, and let $P \in L$ be a singular point of F. For a general plane Π passing through L we have $\Pi \cap F = L + C$, where C is a smooth conic tangent to L at P. Then the pair $(F, C + L)$ is not log canonical (LC) at P.

On the other hand, we claim that the dual graph of the minimal resolution of $(F, C + L)$ has the form



Consequently, the pair $(F, C + L)$ is LC at P, a contradiction.

To show the claim we consider the minimal resolution $\mu : \tilde{F} \to F$ of the A_n-singularity (F, P) and its fundamental cycle $Z = \sum_{i=1}^n E_i$. Since L and C both are smooth and pass through P we have $L \cdot Z = 1 = C \cdot Z$. Hence C and L are both attached at the end vertices of the dual resolution chain $\sum_{i=1}^n E_i$. It remains to show that they are attached at the opposite end vertices. Write $\mu^*(C + L) = C' + L' + \sum_{i=1}^n \alpha_i E_i$, where $\alpha_i > 0$, $i = 1, \ldots, n$, are the vanishing orders on the E_i of the pullback to $\tilde{F}$ of the local equation of the Cartier divisor $C + L$ on F. Taking intersections with E_i, $i = 1, \ldots, n$, yields a system

$$-2\alpha_1 + \alpha_2 = -\delta_1,\ \alpha_1 - 2\alpha_2 + \alpha_3 = -\delta_2,\ \alpha_2 - 2\alpha_3 + \alpha_4 = -\delta_3,\ \ldots,\ \alpha_{n-1} - 2\alpha_n = -\delta_n\,,$$

where $\delta_i = (C' + L') \cdot E_i \in \{0, 1, 2\}$. We have $\sum_{i=1}^n \delta_i = 2$. Assuming that $\delta_1 > 0$ and summing up the equations we obtain $-(\alpha_1 + \alpha_n) = -2$, hence $\alpha_1 = \alpha_n = 1$. Plugging in this in our system we find $\alpha_2 + \delta_1 = 2$, where $\alpha_2 > 0$ and $\delta_1 > 0$, hence $\alpha_2 = 1 = \delta_1$. From the second equation we deduce

$$\alpha_3 = 2\alpha_2 - \alpha_1 - \delta_2 = 1 - \delta_2 > 0\,,$$

hence $\delta_2 = 0$ and $\alpha_3 = 1$, and so on. By recursion, finally we get

$$\delta_1 = 1,\ \delta_2 = \ldots = \delta_{n-1} = 0, \quad \text{and} \quad \delta_n = 1\,.$$

Now the claim follows.

The next example fixes the second part of Theorem 3.6.

Example 3.16. Let us construct a pair (F, Γ), where Γ is a smooth curve of degree 7 and genus 3 in the smooth locus of a cubic surface F in $\mathbb{P}^3$ with a unique singular point $\operatorname{Sing}(F) = \{P\}$, such that (F, P) is an A_3-singularity.

Consider a smooth quartic curve $\bar{\Gamma}$ in $\mathbb{P}^2$. Blowing up a point P_0 on $\bar{\Gamma}$ and three infinitesimally near points P_1, P_2, P_3 on the subsequent proper transforms of $\bar{\Gamma}$, and then also a point $P_4 \neq P_0$ on $\bar{\Gamma}$ and an extra point $P_5 \in \mathbb{P}^2 \setminus \bar{\Gamma}$, we obtain a smooth surface $\tilde{F}$, a chain of rational curves $\mathcal{L} = E_0 + E_1 + E_2$ on F with $E_i^2 = -2$, $i = 0, 1, 2$, which consists of the first three components appeared in the exceptional locus, and a smooth curve $\tilde{\Gamma}$ on $\tilde{F}$ of genus 3 and anticanonical degree 7, disjoint with $\mathcal{L}$. Blowing down $\mathcal{L}$ leads to a singular cubic surface F with a unique singular point of type A_3 anticanonically embedded in $\mathbb{P}^3$. The image Γ of $\tilde{\Gamma}$ on F is a desired curve.

The following observation shows however that not any cubic surface with a deep singularity is available for our purposes.

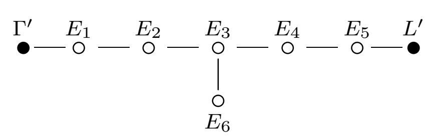
Remark 3.17. By construction, the criterion of Theorem 3.6 on the existence of a cylinder in $X = X_{16} \subseteq \mathbb{P}^{10}$ $(g = 9)$ is valid as long as the cubic surface F in $\mathbb{P}^3$ as in 2.2–2.6 contains a smooth curve Γ of genus 3 and degree 7. However, there is no such curve Γ on a cubic surface F with an isolated conic singularity or a Du Val E_6 singularity (although by Proposition 3.7 in this case $\mathbb{P}^3 \setminus F$ contains an $\mathbb{A}^1$-cylinder). In other words, a normal cubic surface with a conic or an E_6 singularity cannot appear via the Sarkisov link as in 2.6. In the conic case, this follows from Proposition 2.3.

Suppose further that $F \subseteq \mathbb{P}^3$ is a cubic surface with a Du Val E_6 singularity P. Let L be a line on F passing through P. Then $\operatorname{Cl}(F) \simeq \mathbb{Z}[L]$, where $L^2 = 1/3$. Since $\deg(\Gamma) = 7$ we have $\Gamma \sim 7L$ and so $\Gamma^2 = 49/3$. It follows that $P \in \Gamma$.

Consider the minimal resolution $\sigma : \tilde{F} \to F$, and let

$$Z = E_1 + 2E_2 + 3E_3 + 2E_4 + E_5 + 2E_6$$

be the fundamental cycle supported on the exceptional divisor $E = \sum_{i=1}^{6} E_i$ of σ with the dual graph



Since L and Γ are both smooth and pass through P, we have $Z \cdot \Gamma' = 1 = Z \cdot L'$, where Γ' and L' are the proper transforms of Γ and L on $\tilde{F}$, respectively. In the minimal resolution graph, Γ' and L' must be both attached at the end vertices E_1 or E_5. We claim that they are not attached to the same vertex.

Suppose to the contrary that $\Gamma' \cdot E_1 = 1 = L' \cdot E_1$. We use the notion of a *different* (see e.g. [Sh$_2$], [Pr$_4$])

$$\operatorname{Diff}_C(0) = (K_F + C)|_C - K_C\,,$$

where C is a curve on F smooth at P. By adjunction,

$$(K_F + L) \cdot L = -2 + \deg \mathrm{Diff}_L(0) \quad \text{and} \quad (K_F + \Gamma) \cdot \Gamma = 4 + \deg \mathrm{Diff}_\Gamma(0)\,,$$

where $\mathrm{Diff}_L(0) = \mathrm{Diff}_\Gamma(0)$ because of the local analytic invariance of the different [Pr_4]. We have $(K_F + L) \cdot L = -2/3$ and so $\deg \mathrm{Diff}_L(0) = 4/3$. On the other hand,

$$(K_F + \Gamma) \cdot \Gamma = (-3L + 7L) \cdot 7L = 28L^2 = 28/3\,.$$

We deduce that $\deg \mathrm{Diff}_\Gamma(0) = 16/3 \neq \deg \mathrm{Diff}_L(0)$, a contradiction. Thus we may assume that $\Gamma' \cdot E_1 = 1 = L' \cdot E_5$. Then by the symmetry of the resolution graph, it follows that $\deg \mathrm{Diff}_L(0) = \deg \mathrm{Diff}_\Gamma(0)$, which is again absurd by the same computation as above.

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