INTEGERS WITHOUT LARGE PRIME FACTORS: FROM RAMANUJAN TO DE BRUIJN[1]

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Abstract

A small survey of work done on estimating the number of integers without large prime factors up to around the year 1950 is provided. Around that time N.G. de Bruijn published results that dramatically advanced the subject and started a new era in this topic.

– In memoriam: Nicolaas Govert ('Dick') de Bruijn (1918-2012)

1. Introduction

Let $P(n)$ denote the largest prime divisor of n. We set $P(1) = 1$. A number n is said to be *y-friable*[2] if $P(n) \leq y$. We let $S(x, y)$ denote the set of integers $1 \leq n \leq x$ such that $P(n) \leq y$. The cardinality of $S(x, y)$ is denoted by $\Psi(x, y)$. We write $y = x^{1/u}$; that is, $u = \log x/\log y$.

Fix $u > 0$. In 1930, Dickman [14] proved that

$$\lim_{x\to\infty} \frac{\Psi(x, x^{1/u})}{x} = \rho(u), \tag{1}$$

with

$$\rho(u) = \rho(N) - \int_N^u \frac{\rho(v-1)}{v} dv, \quad (N < u \leq N+1,\ N = 1, 2, 3, \ldots),$$

and $\rho(u) = 1$ for $0 < u \leq 1$ (see Figure 1). It is left to the reader to show that we have

$$\rho(u) = \begin{cases} 1 & \text{for } 0 \leq u \leq 1; \\ \frac{1}{u}\int_0^1 \rho(u-t)dt & \text{for } u > 1. \end{cases} \tag{2}$$

[1]This paper is a slightly extended version of [22] (in a non-research publication) and is reproduced here with permission from the publishers.

[2]In the older literature one usually finds y-smooth. Friable is an adjective meaning easily crumbled or broken.

The function $\rho(u)$ in the literature is either called the *Dickman function* or the *Dickman-de Bruijn function*.

This survey concerns the work done on friable integers up to the papers of de Bruijn [7, 8] that appeared around 1950 and dramatically advanced the subject. A lot of the early work was carried out by number theorists from India (with the earliest contributor being Ramanujan).

De Bruijn [7] improved on (1) by establishing a result that, together with the best currently known estimate for the prime counting function (due to I.M. Vinogradov and Korobov in 1958), yields the following result.

Theorem 1. *The estimate*

$$\Psi(x,y) = x\rho(u)\Big\{1 + O^3\Big(\frac{\log(u+1)}{\log y}\Big)\Big\}, \tag{3}$$

holds for $1 \le u \le \log^{3/5-\epsilon} y$, *that is*, $y > \exp(\log^{5/8+\epsilon} x)$.

De Bruijn's most important tool in his proof of this result is the *Buchstab equation* [10],

$$\Psi(x,y) = \Psi(x,z) - \sum_{y<p\le z} \Psi(\frac{x}{p}, p), \tag{4}$$

where $1 \le y < z \le x$. The Buchstab equation is easily proved on noting that the number of integers $n \le x$ with $P(n) = p$ equals $\Psi(x/p, p)$. Given a good estimate for $\Psi(x,y)$ for $u \le h$, it allows one to obtain a good estimate for $u \le h+1$.

De Bruijn [8] complemented Theorem 1 by an asymptotic estimate for $\rho(u)$. That result has as a corollary that, for $u \ge 3$,

$$\rho(u) = \exp\Big\{-u\Big\{\log u + \log_2 u - 1 + \frac{\log_2 u - 1}{\log u} + O\Big(\big(\frac{\log_2 u}{\log u}\big)^2\Big)\Big\}\Big\}, \tag{5}$$

which will suffice for our purposes. Note that (5) implies that, as $u \to \infty$,

$$\rho(u) = \frac{1}{u^{u+o(u)}}, \quad \rho(u) = \Big(\frac{e+o(1)}{u\log u}\Big)^u,$$

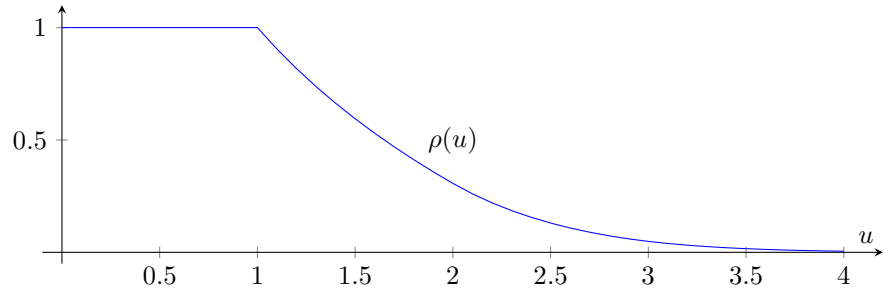
formulas that suffice for most purposes and are easier to remember. For a more detailed description of this and other work of de Bruijn in analytic number theory, we refer to Moree [23].

2. Results on $\rho(u)$

Note that $\rho(u) > 0$, for if not, then because of the continuity of $\rho(u)$ there is a smallest zero $u_0 > 1$ and then substituting u_0 in (2) we easily arrive at a contradiction.

[3]The reader not familiar with the Landau-Bachmann O-notation we refer to wikipedia or any introductory text on analytic number theory, e.g., Tenenbaum [38]. Instead of $\log\log x$ we sometimes write $\log_2 x$, instead of $(\log x)^A$, $\log^A x$.

Figure 1: The Dickman-de Bruijn function $\rho(u)$



Note that for $u > 1$ we have

$$\rho'(u) = -\frac{\rho(u-1)}{u}. \tag{6}$$

It follows that $\rho(u) = 1 - \log u$ for $1 \le u \le 2$. For $2 \le u \le 3$, $\rho(u)$ can be expressed in terms of the dilogarithm. However, with increasing u one has to resort to estimating $\rho(u)$ or finding a numerical approximation.

Since $\rho(u) > 0$ we see from (6) that $\rho(u)$ is strictly decreasing for $u > 1$. From this and (2) we then find that $u\rho(u) \le \rho(u-1)$, which upon using induction leads to $\rho(u) \le 1/[u]!$ for $u \ge 0$. It follows that $\rho(u)$ tends to zero fast as u tends to infinity.

Ramaswami [32] proved that

$$\rho(u) > \frac{C}{u4^u\Gamma(u)^2},\ u \ge 1,$$

for a suitable constant C, with Γ the Gamma function. By Stirling's formula we have $\log\Gamma(u) \sim u\log u$ and hence the latter inequality is, for u large enough, improved on by the following inequality due to Buchstab [10]:

$$\rho(u) > \exp\Big\{-u\Big\{\log u + \log_2 u + 6\frac{\log_2 u}{\log u}\Big\}\Big\},\quad (u \ge 6). \tag{7}$$

But just as (7) yields a great improvement over the Ramaswami inequality, in its turn, De Bruijn's inequality (5) also achieves a large improvement over (7).

3. S. Ramanujan (1887-1920) and the Friables

In his first letter (January 16th, 1913) to Hardy (see, e.g., [3]), one of the most famous letters in all of mathematics, Ramanujan claims that

$$\Psi(n,3) = \frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2\log 3}. \tag{8}$$

The formula is of course intended as an approximation, and there is no evidence to show how accurate Ramanujan supposed it to be. Hardy [18, pp. 69-81] in his lectures on Ramanujan's work gave an account of an interesting analysis that can be made to hang upon the above assertion. We return to this result in Section 6 (on the $\Psi(x,y)$ work of Pillai).

In the so-called Lost Notebook [30] we find at the bottom half of page 337:

$\phi(x)$ is the no. of nos of the form

$$2^{a_2}\cdot 3^{a_3}\cdot 5^{a_5}\cdots p^{a_p} \qquad p\le x^{\epsilon}$$

not exceeding x.

$$\tfrac{1}{2}\le\epsilon\le 1, \qquad \phi(x)\sim x\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}\Big\}$$

$$\tfrac{1}{3}\le\epsilon\le\tfrac{1}{2}, \qquad \phi(x)\sim x\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac12}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}$$

$$\begin{aligned}\tfrac{1}{4}\le\epsilon\le\tfrac{1}{3}, \qquad \phi(x)\sim x\Big\{1 \quad &-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac12}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &-\int_{\epsilon}^{\frac13}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}\end{aligned}$$

$$\begin{aligned}\tfrac{1}{5}\le\epsilon\le\tfrac{1}{4}, \qquad \phi(x)\sim x \quad &\Big\{1-\int_{\epsilon}^{1}\frac{d\lambda_0}{\lambda_0}+\int_{\epsilon}^{\frac12}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &-\int_{\epsilon}^{\frac13}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\\ &+\int_{\epsilon}^{\frac14}\frac{d\lambda_3}{\lambda_3}\int_{\lambda_3}^{\frac{1-\lambda_3}{3}}\frac{d\lambda_2}{\lambda_2}\int_{\lambda_2}^{\frac{1-\lambda_2}{2}}\frac{d\lambda_1}{\lambda_1}\int_{\lambda_1}^{1-\lambda_1}\frac{d\lambda_0}{\lambda_0}\Big\}\end{aligned}$$

and so on.

In the book by Andrews and Berndt [1, §8.2] it is shown that Ramanujan's assertion is equivalent to (1) with

$$\rho(u)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}I_k(u),$$

where

$$I_k(u)=\int_{\substack{t_1,\dots t_k\ge 1\\ t_1+\dots+t_k\le u}}\frac{dt_1}{t_1}\cdots\frac{dt_k}{t_k}.$$

This is one of the many examples where one could say that Ramanujan "reached with his hand from the grave to snatch a theorem, in this case from Dickman whose work came at least ten years after that of Ramanujan (see Berndt [2]). Chowla and Vijayaragahavan [13] seemed to have been the first to rigorously prove (1) with $\rho(u)$ expressed as a sum of iterated integrals (see Section 8). The asymptotic behaviour of the integrals $I_k(u)$ has been studied by Soundararajan [35].

Ramanujan's claim is reminiscent of the following result of Chamayou [11]: If $x_1, x_2, x_3, \cdots$ are independent random variables uniformly distributed in $(0, 1)$, and $u_n = x_1 + x_1x_2 + \ldots + x_1x_2 \cdots x_n$, then u_n converges in probability to a limit u_∞ and u_∞ has a probability distribution with density function $\rho(t)e^{-\gamma}$, where γ denotes Euler's constant.

4. I.M. Vinogradov (1891-1983) and the Friables

The first person to have an application for $\Psi(x, y)$ estimates seems to have been Ivan Matveyevich Vinogradov [39] in 1927. Let $k \geq 2$ be a prescribed integer and $p \equiv 1 (\text{mod } k)$ a prime. The k-th powers in $(\mathbb{Z}/p\mathbb{Z})^*$ form a subgroup of order $(p-1)/k$ and so the existence of $g_1(p, k)$ follows, the least k-th power non-residue modulo a prime p. Suppose that $y < g_1(p, k)$; then $S(x, y)$ consists of k-th power residues only. It follows that

$$\Psi(x, y) \leq \#\{n \leq x \ : \ n \equiv a^k (\text{mod } p) \text{ for some } a\}.$$

The idea, now, is to use good estimates for the quantities on both sides of the inequality sign in order to deduce an upper bound for $g_1(p, k)$.

Vinogradov [39] showed that $\Psi(x, x^{1/u}) \geq \delta(u)x$ for $x \geq 1$, $u > 0$, where $\delta(u)$ depends only on u and is positive. He applied this to show that if $m \geq 8$, $k > m^m$, and $p \equiv 1 (\text{mod } k)$ is sufficiently large, then

$$g_1(p, k) < p^{1/m}. \tag{9}$$

See Norton [24] for an historical account of the problem of determining $g_1(p, k)$ and original results.

5. K. Dickman (1861-1947) and the Friables

Karl Dickman was active in the Swedish insurance business during the end of the nineteenth century and the beginning of the twentieth century [20].

As already mentioned Dickman proved (1) and in the same paper[4] gave an heuris-

[4]Several sources falsely claim that Dickman wrote only one mathematical paper. He also wrote [15].

tic argument to the effect that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{2\le n\le x}\frac{\log P(n)}{\log n}=\int_0^\infty\frac{\rho(u)}{(1+u)^2}du. \tag{10}$$

Denote the integral above by λ. Dickman argued that $\lambda\approx 0.62433$. Mitchell [21] in 1968 computed that $\lambda = 0.62432998854\ldots$. The interpretation of Dickman's heuristic is that for an average integer with m digits, its greatest prime factor has about λm digits. The constant λ is now known as the *Golomb-Dickman constant*, as it arose independently in research of Golomb and others involving the largest cycle in a random permutation.

De Bruijn [7] in 1951 was the first to prove (10). He did this using his $\Lambda(x,y)$-function, an approximation of $\Psi(x,y)$, that he introduced in the same paper.

6. S.S. Pillai (1901-1950) and the Friables

Subbayya Sivasankaranarayana Pillai (1901-1950) was a number theorist who worked on problems in classical number theory (Diophantine equations, Waring's problem, etc.). Indeed, he clearly was very much inspired by the work of Ramanujan. He tragically died in a plane crash near Cairo while on his way to the International Congress of Mathematicians (ICM) 1950, which was held at Harvard University.

Pillai wrote two manuscripts on friable integers, [26, 27], of which [26] was accepted for publication in the Journal of the London Mathematical Society, but did not appear in print. Also [27] was never published in a journal.

In [26] (see also [29, pp. 481-483]), Pillai investigates $\Psi(x,y)$ for y fixed. Let $p_1,p_2,\ldots,p_k$ denote all the different primes not exceeding y. Notice that $\Psi(x,y)$ equals the cardinality of the set

$$\{(e_1,\ldots,e_k)\in\mathbb{Z}^k : e_i\ge 0,\ \sum_{i=1}^k e_i\log p_i\le x\}.$$

Thus $\Psi(x,y)$ equals the number of lattice points in a k-dimensional tetrahedron with sides of length $\log x/\log 2,\ldots,\log x/\log p_k$. This tetrahedron has volume

$$\frac{1}{k!}\prod_{p\le y}\Big(\frac{\log x}{\log p}\Big).$$

Pillai shows that

$$\Psi(x,y)=\frac{1}{k!}\prod_{p\le y}\Big(\frac{\log x}{\log p}\Big)\Big(1+(1+o(1))\frac{k\log(p_1p_2\ldots p_k)}{2\log x}\Big).$$

If $\rho_1, \ldots, \rho_k$ are positive real numbers and ρ_1/ρ_2 is irrational, then the same estimate with $\log p_i$ replaced by ρ_i holds for

$$\{(e_1, \ldots, e_k) \in \mathbb{Z}^k : e_i \geq 0, \ \sum_{i=1}^{k} e_i \rho_i \leq x\}.$$

This was proved by Specht [36] (after whom the Specht modules are named); see also Beukers [4]. A much sharper result than those of Pillai and Specht was obtained in 1969 by Ennola [16] (see also Norton [24, pp. 24-26]). In this result Bernoulli numbers make their appearance.

Note that Pillai's result implies that

$$\Psi(x, 3) = \frac{1}{2} \frac{\log(2x) \log(3x)}{\log 2 \log 3} + o(\log x), \tag{11}$$

and that the estimate

$$\Psi(x, 3) = \frac{\log^2 x}{2 \log 2 \log 3} + o(\log x)$$

is false. Thus Ramanujan's estimate (8) is more precise than the trivial estimate $\log^2 x/(2 \log 2 \log 3)$. Hardy [18, §5.13] showed that the error term $o(\log x)$ in (11) can be replaced by $o(\log x/ \log_2 x)$. In the proof of this he uses a result of Pillai [25] (see also [28, pp. 53-61]), saying that given $0 < \delta < 1$, one has $|2^x - 3^y| > 2^{(1-\delta)x}$ for all integers x and y with $x > x_0(\delta)$ sufficiently large.

In [27] (see also [29, pp. 515-517]), Pillai claims that, for $u \geq 6$, $B/u < \rho(u) < A/u$, with $0 < B < A$ constants. He proves this result by induction assuming a certain estimate for $\rho(6)$ holds. However, this estimate for $\rho(6)$ does not hold. Indeed, the claim contradicts (5) and is false.

Since Pillai reported on his work on the friables at conferences in India and stated open problems there, his influence on the early development of the topic was considerable. For example, one of the questions he raised was whether $\Psi(x, x^{1/u}) = O(x^{1/u})$ uniformly for $u \leq (\log x)/\log 2$. This question was answered in the affirmative by Ramaswami [32].

7. R.A. Rankin (1915-2001) and the Friables

In his work on the size of gaps between consecutive primes, Robert Alexander Rankin [34] in 1938 introduced a simple idea to estimate $\Psi(x, y)$ which turns out to be remarkably effective and can be used in similar situations. This idea is now called "Rankin's method" or "Rankin's trick." The starting point is the observation that for any $\sigma > 0$,

$$\Psi(x, y) \leq \sum_{n \in S(x,y)} (\frac{x}{n})^\sigma \leq x^\sigma \sum_{P(n) \leq y} \frac{1}{n^\sigma} = x^\sigma \zeta(\sigma, y), \tag{12}$$

where

$$\zeta(s,y)=\prod_{p\le y}(1-p^{-s})^{-1},$$

is the partial Euler product up to y for the Riemann zeta function $\zeta(s)$. Recall that, for $\Re s>1$,

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p}\frac{1}{1-p^{-s}}.$$

By making an appropriate choice for σ and estimating $\zeta(\sigma,y)$ using analytic prime number theory, a good upper bound for $\Psi(x,y)$ can be found. For example, the choice $\sigma=1-1/(2\log y)$ leads to

$$\zeta(\sigma,y)\ll\exp\Big\{\sum_{p\le y}\frac{1}{p^{\sigma}}\Big\}\le\exp\Big\{\sum_{p\le y}\frac{1}{p}+O\Big((1-\sigma)\sum_{p\le y}\frac{\log p}{p}\Big)\Big\}\ll\log y,$$

which gives rise to

$$\Psi(x,y)\ll x\mathrm{e}^{-u/2}\log y. \tag{13}$$

As a further example let us try to estimate $\Psi(x,\log^A x)$ for $A>1$. Letting $\sigma=1-1/A$, we get

$$\log\zeta(\sigma,y)\ll\sum_{p\le y}p^{-\sigma}=\sum_{p\le y}\frac{p^{1/A}}{p}\ll\frac{y^{1/A}}{\log y}\ll\frac{\log x}{\log\log x}.$$

This estimate together with (12) yields

$$\Psi(x,\log^A x)\le x^{1-1/A+O(1/\log\log x)}. \tag{14}$$

From a result of de Bruijn from 1966 [9] it follows that, actually, equality holds in (14).

8. A.A. Bukhshtab (1905-1990) and the Friables

Aleksandr Adol'fovich Bukhshtab[5]'s most important contribution is the equation (4) now named after him. A generalization of it plays an important role in sieve theory. Buchstab [10] in 1949 proved (1) and gave both Dickman's differential-difference equation as well as the result

$$\rho(u)=1+\sum_{n=1}^{N}(-1)^n\int_n^u\int_{n-1}^{t_1-1}\int_{n-2}^{t_2-1}\cdots\int_1^{t_{n-1}-1}\frac{dt_n dt_{n-1}\cdots dt_1}{t_1t_2\cdots t_n}, \tag{15}$$

for $N\le u\le N+1$ and $N\ge 1$ an integer, simplifying Chowla and Vijayaragahavan's expression (they erroneously omitted the term $n=N$). Further, Buchstab established inequality (7) and applied his results to show that the exponent in Vinogradov's result (9) can be roughly divided by two.

[5]Buchstab in the German spelling.

9. V. Ramaswami and the Friables

V. Ramaswami[6] [31] showed that

$$\Psi(x, x^{1/u}) = \rho(u)x + O_U(\frac{x}{\log x})$$

for $x > 1$, $1 < u \leq U$, and remarked that the error term is best possible. He sharpened this result in [32] and showed there that, for $u > 2$,

$$\Psi(x, x^{1/u}) = \rho(u)x + \sigma(u)\frac{x}{\log x} + O(\frac{x}{\log^{3/2} x}), \tag{16}$$

with $\sigma(u)$ defined similarly to $\rho(u)$. Indeed, it turns out that

$$\sigma(u) = (1 - \gamma)\rho(u - 1),$$

but this was not noticed by Ramaswami. In [33] Ramaswami generalized his results to $B_l(m, x, y)$ which counts the number of integers $n \leq x$ with $P(n) \leq y$ and $n \equiv l(\text{mod } m)$ [7]. Norton [24, pp. 12-13] points out some deficiencies in this paper and gives a reproof [24, §4] of Ramaswami's result on $B_l(m, x, x^{1/u})$ generalizing (16).

From de Bruijn's paper [7, Eqs. (5.3), (4.6)] one easily derives the following generalization of Ramaswami's results[8]:

Theorem 2. *Let $m \geq 0$, $x > 1$, and suppose $m + 1 < u < \sqrt{\log x}$. Then*

$$\Psi(x, y) = x \sum_{r=0}^{m} a_r \frac{\rho^{(r)}(u)}{\log^r y} + O_m\big(\frac{x}{\log^{m+1} y}\big),$$

where $\rho^{(r)}(u)$ is the r-th derivative of $\rho(u)$ and $a_0, a_1, \ldots$ are the coefficients in the power series expansion

$$\frac{z}{1+z}\zeta(1+z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

with $|z| < 1$.

It is well-known (see, e.g., Briggs and Chowla [5]) that around $s = 1$ the Riemann zeta function has the Laurent series expansion

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\gamma_k(s-1)^k,$$

[6] Ramaswami worked at Andhra University until his death in 1961. The author will be grateful for further biographical information.

[7] Buchstab [10] was the first to investigate $B_l(m, x, y)$.

[8] The notation O_m indicates that the implied constant might depend on m.

with γ_k the k-th Stieltjes constant and $\gamma_0 = \gamma$ Euler's constant. Using this we find that $a_0 = 1$ and $a_1 = \gamma - 1$. Thus Theorem 2 yields (16) with $\sigma(u) = (1-\gamma)\rho(u-1)$ for the range $2 < u < \sqrt{\log x}$. For $u > \sqrt{\log x}$ the estimate (16) in view of (5) reduces to

$$\Psi(x, x^{1/u}) \ll x \log^{-3/2} x,$$

which easily follows from (13).

10. S. Chowla (1907-1995) and the Friables

The two most prominent number theorists in the period following Ramanujan were S.S. Pillai and Sarvadaman Chowla. They kept in contact through an intense correspondence [37]. Chowla in his long career published hunderds of research papers.

Chowla and Vijayaragahavan [13] expressed $\rho(u)$ as an iterated integral and gave a formula akin to (15). De Bruijn [6] established some results implying that $\Psi(x, \log^h x) = O(x^{1-1/h+\epsilon})$ for $h > 2$. An easier reproof of the latter result was given by Chowla and Briggs [12].

11. Summary

It seems that the first person to look at friable integers was Ramanujan, starting with his first letter to Hardy (1913), Ramanujan also seems to have been the first person to arrive at the Dickman-de Bruijn function $\rho(u)$. Pillai generalized some of Ramanujan's work and spoke about it at conferences in India, which likely induced a small group of Indian number theorists to work on friable integers. Elsewhere in the same period (1930-1950) only incidental work was done on the topic. Around 1950 N.G. de Bruijn published his ground-breaking papers [7, 8]. Soon afterwards the Indian number theorists stopped publishing on friable integers.

It should also be said that the work on friable integers up to 1950 seems to contain more mistakes than more recent work. Norton [24] points out and corrects many of these mistakes.

Further reading. As a first introduction to friable numbers we highly recommend Granville's 2008 survey [17]. It has a special emphasis on friable numbers and their role in algorithms in computational number theory. Mathematically more demanding is the 1993 survey by Hildebrand and Tenenbaum [19]. Chapter III.5 in Tenenbaum's book [38] deals with $\rho(u)$ and approximations to $\Psi(x, y)$ by the saddle point method.

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