# SPHERICAL ACTIONS ON FLAG VARIETIES

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Abstract. For every finite-dimensional vector space $V$ and every $V$-flag variety $X$ we list all connected reductive subgroups in $\mathrm{GL}(V)$ acting spherically on $X$.

## 1. Introduction

Throughout this paper we fix an algebraically closed field $\mathbb{F}$ of characteristic 0, which is the ground field for all objects under consideration. By $\mathbb{F}^\times$ we denote the multiplicative group of $\mathbb{F}$.

Let $K$ be a connected reductive algebraic group and let $X$ be a $K$-variety (that is, an algebraic variety together with a regular action of $K$). The action of $K$ on $X$, as well as the variety $X$ itself, is said to be *spherical* (or *$K$-spherical* when one needs to emphasize the acting group) if $X$ is irreducible and a Borel subgroup of $K$ has an open orbit in $X$. Every finite-dimensional $K$-module that is spherical as a $K$-variety is said to be a *spherical $K$-module*. Spherical varieties possess various remarkable properties; a review of them can be found, for instance, in the monograph by Timashev [Tim].

Describing all spherical varieties for a fixed group $K$ is an important and interesting problem, and by now considerable results have been achieved in solving it. A review of these results can be also found in the monograph [Tim]. Along with the problem mentioned above one can also consider an opposite one, namely, for a given algebraic variety $X$ find all connected reductive subgroups in the automorphism group of $X$ that act spherically on $X$. In this paper we consider this problem in the case where $X$ is a generalized flag variety.

A *generalized flag variety* is a homogeneous space of the form $G/P$, where $G$ is a connected reductive group and $P$ is a parabolic subgroup of $G$. We call the variety $G/P$ *trivial* if $P = G$ and *nontrivial* otherwise. The following facts are well known:

(1) all generalized flag varieties of a fixed group $G$ are exactly all complete (and also all projective) homogeneous spaces of $G$;

(2) the center of $G$ acts trivially on $G/P$;

(3) the natural action of $G$ on $G/P$ is spherical.

Let $\mathscr{F}(G)$ denote the set of all (up to a $G$-equivariant isomorphism) nontrivial generalized flag varieties of a fixed group $G$. For every generalized flag variety $X$ we denote by $\operatorname{Aut} X$ its automorphism group.

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2010 *Mathematics Subject Classification.* 14M15, 14M27.

*Key words and phrases.* Algebraic group, flag variety, spherical variety, nilpotent orbit.

The first author was supported by the RFBR grant no. 12-01-00704, the SFB 701 grant of the University of Bielefeld (in 2013), the "Oberwolfach Leibniz Fellows" programme of the Mathematisches Forschungsinstitut Oberwolfach (in 2013), and Dmitry Zimin's "Dynasty" Foundation (in 2014). The second author was supported by the Guest Programme of the Max-Planck Institute for Mathematics in Bonn (in 2012–2013).

In [Oni, Theorem 7.1] (see also [Dem]) the automorphism groups of all generalized flag varieties were described. This description implies that, for every $X \in \mathscr{F}(G)$, $\operatorname{Aut} X$ is an affine algebraic group and its connected component of the identity $(\operatorname{Aut} X)^0$ is semisimple and has trivial center. Moreover, it turns out that in most of the cases (all exceptions are listed in [Oni, Table 8]; see also [Dem, § 2]), including the case $G = \mathrm{GL}_n$, the natural homomorphism $G \to (\operatorname{Aut} X)^0$ is surjective and its kernel coincides with the center of $G$. In any case one has $X \in \mathscr{F}((\operatorname{Aut} X)^0)$, therefore the problem of describing all spherical actions on generalized flag varieties reduces to the following one:

**Problem 1.1.** *For every connected reductive algebraic group $G$ and every variety $X \in \mathscr{F}(G)$, list all connected reductive subgroups $K \subset G$ acting spherically on $X$.*

Since the center of $G$ acts trivially on any variety $X \in \mathscr{F}(G)$, in solving Problem 1.1 the group $G$ without loss of generality may be assumed semisimple when it is convenient.

The main goal of this paper is to solve Problem 1.1 in the case $G = \mathrm{GL}_n$ (the results are stated below in Theorems 1.6 and 1.7).

Let us list all cases known to the authors where Problem 1.1 is solved.

1) $G$ and $X$ are arbitrary and $K$ is a Levi subgroup of a parabolic subgroup $Q \subset G$. In this case the condition that the variety $G/P \in \mathscr{F}(G)$ be $K$-spherical is equivalent to the condition that the variety $G/P \times G/Q$ be $G$-spherical, where $G$ acts diagonally (see Lemma 5.4). All $G$-spherical varieties of the form $G/P \times G/Q$ have been classified. In the case where $P, Q$ are maximal parabolic subgroups this was done by Littelmann in [Lit]. In the cases $G = \mathrm{GL}_n$ and $G = \mathrm{Sp}_{2n}$ the classification follows from results of the Magyar, Weyman, and Zelevinsky papers [MWZ1] and [MWZ2], respectively. (In fact, in [MWZ1] and [MWZ2] for $G = \mathrm{GL}_n$ and $G = \mathrm{Sp}_n$, respectively, the following more general problem was solved: describe all sets $X_1, \ldots, X_k \in \mathscr{F}(G)$ such that $G$ has finitely many orbits under the diagonal action on $X_1 \times \ldots \times X_k$.) Finally, for arbitrary groups $G$ the classification was completed by Stembridge in [Stem]. The results of this classification for $G = \mathrm{GL}_n$ will be essentially used in this paper and are presented in § 5.2.

2) $G$ and $X$ are arbitrary and $K$ is a symmetric subgroup of $G$ (that is, $K$ is the subgroup of fixed points of a nontrivial involutive automorphism of $G$). In this case the classification was obtained in the paper [HNOO].

3) $G$ is an exceptional simple group, $X = G/P$ for a maximal parabolic subgroup $P \subset G$, and $K$ is a maximal reductive subgroup of $G$. This case was investigated in the preprint [Nie].

Let $G$ be an arbitrary connected reductive group, $\mathfrak{g} = \operatorname{Lie} G$, and $K \subset G$ an arbitrary connected reductive subgroup.

We now discuss the key idea utilized in this paper. Let $P \subset G$ be a parabolic subgroup and let $N \subset P$ be its unipotent radical. Put $\mathfrak{n} = \operatorname{Lie} N$. Consider the adjoint action of $G$ on $\mathfrak{g}$. In view of a well-known result of Richardson (see [Rich, Proposition 6(c)]), for the induced action $P : \mathfrak{n}$ there is an open orbit $O_P$. We put

$$\mathcal{N}(G/P) = GO_P \subset \mathfrak{g}. \tag{1.1}$$

It is easy to see that $\mathcal{N}(G/P)$ is a nilpotent (that is, containing 0 in its closure) $G$-orbit in $\mathfrak{g}$.

**Definition 1.2.** We say that two varieties $X_1, X_2 \in \mathscr{F}(G)$ are *nil-equivalent* (notation: $X_1 \sim X_2$) if $\mathcal{N}(X_1) = \mathcal{N}(X_2)$.

It is well-known that every $G$-orbit in $\mathfrak{g}$ is endowed with the canonical structure of a symplectic variety. It turns out (see Theorem 2.6) that a variety $X \in \mathscr{F}(G)$ is $K$-spherical if and only if the action $K : \mathcal{N}(X)$ is coisotropic (see Definition 2.2) with respect to the symplectic structure on $\mathcal{N}(X)$. This immediately implies the following result.

**Theorem 1.3.** *Suppose that $X_1, X_2 \in \mathscr{F}(G)$ and $X_1 \sim X_2$. Then the following conditions are equivalent:*
(a) *the action $K : X_1$ is spherical;*
(b) *the action $K : X_2$ is spherical.*

Let $[\![X]\!]$ denote the nil-equivalence class of a variety $X \in \mathscr{F}(G)$. The inclusion relation between closures of nilpotent orbits in $\mathfrak{g}$ defines a partial order $\preccurlyeq$ on the set $\mathscr{F}(G)/\sim$ of all nil-equivalence classes in the following way: for $X_1, X_2 \in \mathscr{F}(G)$ the relation $[\![X_1]\!] \preccurlyeq [\![X_2]\!]$ (or $[\![X_2]\!] \succcurlyeq [\![X_1]\!]$) holds if and only if the orbit $\mathcal{N}(X_1)$ is contained in the closure of the orbit $\mathcal{N}(X_2)$. We shall also write $[\![X_1]\!] \prec [\![X_2]\!]$ (or $[\![X_2]\!] \succ [\![X_1]\!]$) when $[\![X_1]\!] \preccurlyeq [\![X_2]\!]$ but $[\![X_1]\!] \ne [\![X_2]\!]$.

Using a result from the resent paper [Los] by Losev, in § 2.4 we shall prove the following theorem.

**Theorem 1.4.** *Suppose that $X_1, X_2 \in \mathscr{F}(G)$ and $[\![X_1]\!] \prec [\![X_2]\!]$. If the action $K : X_2$ is spherical, then so is the action $K : X_1$.*

Theorems 1.3 and 1.4 yield the following method for solving Problem 1.1: at the first step, for each class $[\![X]\!] \in \mathscr{F}(G)/\sim$ that is a minimal element with respect to the order $\preccurlyeq$, find all connected reductive subgroups $K \subset G$ acting spherically on $X$; at the second step, using the lists of subgroups $K$ obtained at the first step, carry out the same procedure for all nil-equivalence classes that are on the "next level" with respect to the order $\preccurlyeq$; and so on.

We recall that there is the natural partial order $\leqslant$ on the set $\mathscr{F}(G)$. This order can be defined as follows. Fix a Borel subgroup $B \subset G$. Let $X_1, X_2 \in \mathscr{F}(G)$. Then $X_1 = G/P_1$ and $X_2 = G/P_2$ where $P_1, P_2$ are uniquely determined parabolic subgroups of $G$ containing $B$. By definition, the relation $X_1 \leqslant X_2$ (resp. $X_1 < X_2$) holds if and only if $P_1 \supset P_2$ (resp. $P_1 \supsetneq P_2$). It is easy to show that the following analogue of Theorem 1.4 is valid for the partial order $\leqslant$: if $X_1 < X_2$ and the action $K : X_2$ is spherical, then so is the action $K : X_1$. Now let $N_i$ be the unipotent radical of $P_i$, $i = 1, 2$. Then the condition $P_1 \supsetneq P_2$ implies that $N_1 \subsetneq N_2$, from which one easily deduces that the orbit $\mathcal{N}(X_1)$ is contained in the closure of the orbit $\mathcal{N}(X_2)$. As $\dim \mathcal{N}(X_i) = 2 \dim N_i$ for $i = 1, 2$ (see, for instance, [CM, Theorem 7.1.1]), one has $\mathcal{N}(X_1) \ne \mathcal{N}(X_2)$. Thus, if $X_1 < X_2$ then $[\![X_1]\!] \prec [\![X_2]\!]$. In particular, if $[\![X]\!]$ is a minimal element of the set $\mathscr{F}(G)/\sim$ with respect to the partial order $\preccurlyeq$, then $X$ is a minimal element of the set $\mathscr{F}(G)$ with respect to the partial order $\leqslant$.

We note that for a semisimple group $G$ of rank $n$ the set $\mathscr{F}(G)$ contains exactly $n$ minimal elements with respect to the partial order $\leqslant$. On the other hand, using known results on nilpotent orbits in the classical Lie algebras (see [CM, §§5–7]), one can show that for a simple group $G$ of type $\mathsf{A}_n$, $\mathsf{B}_n$, or $\mathsf{C}_n$ the set $\mathscr{F}(G)/\sim$ contains only one minimal element with respect to the partial order $\preccurlyeq$ and for a simple group $G$ of type $\mathsf{D}_n$ ($n \geqslant 4$) the set $\mathscr{F}(G)/\sim$ contains two (for $n = 2k+1$) or three (for $n = 2k$) minimal elements with respect to the partial order $\preccurlyeq$. This shows that the partial order $\preccurlyeq$ turns out to be much more effective in solving Problem 1.1 than the partial order $\leqslant$.

We now turn to $V$-flag varieties, which are the main objects in our paper. In order to fix the definition of these varieties that is convenient for us, we need the following notion. A *composition* of a positive integer $d$ is a tuple of positive integers $(a_1, \ldots, a_s)$ satisfying the condition

$$a_1 + \ldots + a_s = d.$$

We say that a composition $\mathbf{a} = (a_1, \ldots, a_s)$ is *trivial* if $s = 1$ and *nontrivial* if $s \geqslant 2$.

Let $V$ be a finite-dimensional vector space of dimension $d$ and let $\mathbf{a} = (a_1, \ldots, a_s)$ be a composition of $d$. The *$V$-flag variety* (or simply the *flag variety*) corresponding to $\mathbf{a}$ is the set of tuples $(V_1, \ldots, V_s)$, where $V_1, \ldots, V_s$ are subspaces of $V$ satisfying the following conditions:

(a) $V_1 \subset \ldots \subset V_s = V$;

(b) $\dim V_i/V_{i-1} = a_i$ for $i = 1, \ldots, s$ (here we suppose that $V_0 = \{0\}$).

We note that $\dim V_i = a_1 + \ldots + a_i$ for all $i = 1, \ldots, s$.

For every composition $\mathbf{a} = (a_1, \ldots, a_s)$, we denote by $\mathrm{Fl}_{\mathbf{a}}(V)$ the $V$-flag variety corresponding to $\mathbf{a}$. If $\mathbf{a}$ is nontrivial, then along with $\mathrm{Fl}_{\mathbf{a}}(V)$ we shall also use the notation $\mathrm{Fl}(a_1, \ldots, a_{s-1}; V)$.

The following facts are well known:

(1) every $V$-flag variety $X$ is equipped with a canonical structure of a projective algebraic variety and the natural action of $\mathrm{GL}(V)$ on $X$ is regular and transitive;

(2) up to a $\mathrm{GL}(V)$-equivariant isomorphism, all $V$-flag varieties are exactly all generalized flag varieties of the group $\mathrm{GL}(V)$.

In view of fact (2), in what follows all notions and notation introduced for generalized flag varieties will be also used for $V$-flag varieties.

It is easy to see that a $V$-flag variety $\mathrm{Fl}_{\mathbf{a}}(V)$ is trivial (resp. nontrivial) if and only if the composition $\mathbf{a}$ is trivial (resp. nontrivial).

For $1 \leqslant k \leqslant d$ we consider the composition $\mathbf{c}_k$ of $d$, where $\mathbf{c}_k = (k, d-k)$ for $1 \leqslant k \leqslant d-1$ and $\mathbf{c}_d = (d)$. The variety $\mathrm{Fl}_{\mathbf{c}_k}(V)$ is said to be a *Grassmannian*, we shall use the special notation $\mathrm{Gr}_k(V)$ for it. Points of this variety are in one-to-one correspondence with $k$-dimensional subspaces of $V$. The point corresponding to a subspace $W \subset V$ will be denoted by $[W]$. It is easy to see that $\mathrm{Gr}_d(V)$ consists of the single point $[V]$ and $\mathrm{Gr}_1(V)$ is nothing else than the projective space $\mathbb{P}(V)$.

In the present paper we implement the above-discussed general method of solving Problem 1.1 for $G = \mathrm{GL}(V)$. In this case there is a well-known description of the map from $\mathscr{F}(\mathrm{GL}(V))$ to the set of nilpotent orbits in $\mathfrak{gl}(V)$, as well as the partial order on the latter set (see details in § 3). In particular, the following proposition holds (see Corollary 3.5):

**Proposition 1.5.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two compositions of* $d$*. The following conditions are equivalent:*

(a) *the varieties* $\mathrm{Fl}_{\mathbf{a}}(V)$ *and* $\mathrm{Fl}_{\mathbf{b}}(V)$ *are nil-equivalent;*

(b) $\mathbf{a}$ *and* $\mathbf{b}$ *can be obtained from each other by a permutation* (*in particular,* $\mathbf{a}$ *and* $\mathbf{b}$ *contain the same number of elements*).

In view of Theorem 1.3, Proposition 1.5 implies the following theorem.

**Theorem 1.6.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two compositions of* $d$ *obtained from each other by a permutation and let* $K \subset \mathrm{GL}(V)$ *be an arbitrary connected reductive subgroup. The following conditions are equivalent:*

(a) *the action* $K : \mathrm{Fl}_{\mathbf{a}}(V)$ *is spherical;*

(b) *the action* $K : \mathrm{Fl}_{\mathbf{b}}(V)$ *is spherical.*

Theorem 1.6 reduces the problem of describing all spherical actions on $V$-flag varieties to the case of varieties $\mathrm{Fl}_{\mathbf{a}}(V)$ such that the composition $\mathbf{a} = (a_1, \ldots, a_s)$ satisfies the inequalities $a_1 \leqslant \ldots \leqslant a_s$.

It is easy to see that the $K$-sphericity of $\mathbb{P}(V)$ is equivalent to the sphericity of the $(K \times \mathbb{F}^\times)$-module $V$, where $\mathbb{F}^\times$ acts on $V$ by scalar transformations. Therefore a description of all spherical actions on $\mathbb{P}(V)$ is a trivial consequence of the known classification of spherical modules that was obtained in the papers [Kac], [BR], and [Lea]. (This classification plays a key role in our paper and is presented in § 5.1.) As $\mathbb{P}(V) = \mathrm{Gr}_1(V)$, to complete the description of all spherical actions on $V$-flag varieties it suffices to restrict ourselves to the case of varieties $\mathrm{Fl}_{\mathbf{a}}(V)$ such that the composition $\mathbf{a}$ is nontrivial and distinct from $(1, d-1)$.

Before we state the main result of this paper, we need to introduce one more notion and some additional notation.

Let $K_1, K_2$ be connected reductive groups, $U_1$ a $K_1$-module, and $U_2$ a $K_2$-module. Consider the corresponding representations

$$\rho_1 \colon K_1 \to \mathrm{GL}(U_1) \quad \text{and} \quad \rho_2 \colon K_2 \to \mathrm{GL}(U_2).$$

Following Knop (see [Kn2, § 5]), we say that the pairs $(K_1, U_1)$ are $(K_2, U_2)$ *geometrically equivalent* if there exists an isomorphism $U_1 \xrightarrow{\sim} U_2$ identifying the groups $\rho_1(K_1) \subset \mathrm{GL}(U_1)$ and $\rho_2(K_2) \subset \mathrm{GL}(U_2)$. In other words, the pairs $(K_1, U_1)$ and $(K_2, U_2)$ are geometrically equivalent if and only if they define the same linear group. For example, every pair $(K, U)$ is geometrically equivalent to the pair $(K, U^*)$ (where $U^*$ is the $K$-module dual to $U$), the pair $(\mathrm{SL}_2, \mathrm{S}^2\mathbb{F}^2)$ is geometrically equivalent to the pair $(\mathrm{SO}_3, \mathbb{F}^3)$, and the pair $(\mathrm{SL}_2 \times \mathrm{SL}_2, \mathbb{F}^2 \otimes \mathbb{F}^2)$ is geometrically equivalent to the pair $(\mathrm{SO}_4, \mathbb{F}^4)$.

Let $K$ be a connected reductive subgroup of $\mathrm{GL}(V)$ and let $K'$ be the derived subgroup of $K$. We denote by $C$ the connected component of the identity of the center of $K$. Let $\mathfrak{X}(C)$ denote the character group of $C$ (in additive notation). We regard $V$ as a $K$-module and fix a decomposition $V = V_1 \oplus \ldots \oplus V_r$ into a direct sum of simple $K$-submodules. For every $i = 1, \ldots, r$ we denote by $\chi_i$ the character of $C$ via which $C$ acts on $V_i$.

The following theorem is the main result of this paper.

**Theorem 1.7.** *Suppose that* $\mathbf{a} = (a_1, \ldots, a_s)$ *is a nontrivial composition of* $d$ *such that* $a_1 \leqslant \ldots \leqslant a_s$ *and* $\mathbf{a} \neq (1, d-1)$. *Then the variety* $\mathrm{Fl}_{\mathbf{a}}(V)$ *is* $K$*-spherical if and only if the following conditions hold:*

(1) *the pair* $(K', V)$*, considered up to a geometrical equivalence, and the tuple* $(a_1, \ldots, a_{s-1})$ *are contained in Table* 1*;*

(2) *the group* $C$ *satisfies the conditions listed in the fourth column of Table* 1.

Theorem 1.7 is a union of Theorems 6.1, 6.5, 6.6, and 6.7.

Let us explain the notation and conventions used in Table 1. In each case we assume that the $i$-th factor of $K'$ acts on the $i$-th direct summand of $V$. Further, we assume that the group of type $\mathrm{SL}_q$ (resp. $\mathrm{Sp}_{2q}$, $\mathrm{SO}_q$) naturally acts on $\mathbb{F}^q$ (resp. $\mathbb{F}^{2q}$, $\mathbb{F}^q$). In row 4

Table 1

| No. | $(K', V)$ | $(a_1, \ldots, a_{s-1})$ | Conditions on $C$ | Note |
|---|---|---|---|---|
| | | $s = 2$ (Grassmannians) | | |
| 1 | $(\mathrm{SL}_n, \mathbb{F}^n)$ | $(k)$ | | $n \geqslant 4$ |
| 2 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$ | $(k)$ | | $n \geqslant 2$ |
| 3 | $(\mathrm{SO}_n, \mathbb{F}^n)$ | $(k)$ | | $n \geqslant 4$ |
| 4 | $(\mathrm{Spin}_7, \mathbb{F}^8)$ | $(2)$ | | |
| 5 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n} \oplus \mathbb{F}^1)$ | $(k)$ | | $n \geqslant 2$ |
| 6 | $(\mathrm{SL}_n \times \mathrm{SL}_m, \mathbb{F}^n \oplus \mathbb{F}^m)$ | $(k)$ | $\chi_1 \neq \chi_2$ for $n = m = k$ | $n \geqslant m \geqslant 1$,<br>$n + m \geqslant 4$ |
| 7 | $(\mathrm{Sp}_{2n} \times \mathrm{SL}_m, \mathbb{F}^{2n} \oplus \mathbb{F}^m)$ | $(2)$ | $\chi_1 \neq \chi_2$ for $m = 2$ | $n, m \geqslant 2$ |
| 8 | $(\mathrm{Sp}_{2n} \times \mathrm{SL}_m, \mathbb{F}^{2n} \oplus \mathbb{F}^m)$ | $(3)$ | $\chi_1 \neq \chi_2$ for $m \leqslant 3$ | $n, m \geqslant 2$ |
| 9 | $(\mathrm{Sp}_4 \times \mathrm{SL}_m, \mathbb{F}^4 \oplus \mathbb{F}^m)$ | $(k)$ | $\chi_1 \neq \chi_2$ for $k = m = 4$ | $m, k \geqslant 4$ |
| 10 | $(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^{2m})$ | $(2)$ | $\chi_1 \neq \chi_2$ | $n \geqslant m \geqslant 2$ |
| 11 | $(\mathrm{SL}_n \times \mathrm{SL}_m$,<br>$\mathbb{F}^n \oplus \mathbb{F}^m \oplus \mathbb{F}^1)$ | $(k)$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$<br>lin. ind. for $k = n$<br>$\chi_2 \neq \chi_3$ for $m \leqslant k < n$ | $n \geqslant m \geqslant 1$,<br>$n \geqslant 2$ |
| 12 | $(\mathrm{SL}_n \times \mathrm{SL}_m \times \mathrm{SL}_l$,<br>$\mathbb{F}^n \oplus \mathbb{F}^m \oplus \mathbb{F}^l)$ | $(2)$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$<br>lin. ind. for $n = 2$<br>$\chi_2 \neq \chi_3$ for<br>$n \geqslant 3$, $m \leqslant 2$ | $n \geqslant m \geqslant l \geqslant 1$,<br>$n \geqslant 2$ |
| 13 | $(\mathrm{Sp}_{2n} \times \mathrm{SL}_m \times \mathrm{SL}_l$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^m \oplus \mathbb{F}^l)$ | $(2)$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$<br>lin. ind. for $m \leqslant 2$<br>$\chi_1 \neq \chi_3$ for<br>$m \geqslant 3, l \leqslant 2$ | $n \geqslant 2$,<br>$m \geqslant l \geqslant 1$ |
| 14 | $(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{SL}_l$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^{2m} \oplus \mathbb{F}^l)$ | $(2)$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$<br>lin. ind. for $l \leqslant 2$<br>$\chi_1 \neq \chi_2$ for $l \geqslant 3$ | $n \geqslant m \geqslant 2$,<br>$l \geqslant 1$ |
| 15 | $(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2l}$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^{2m} \oplus \mathbb{F}^{2l})$ | $(2)$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$<br>lin. ind. | $n \geqslant m \geqslant l \geqslant 2$ |
| | | $s \geqslant 3$ | | |
| 16 | $(\mathrm{SL}_n, \mathbb{F}^n)$ | $(a_1, \ldots, a_{s-1})$ | | $n \geqslant 3$ |
| 17 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$ | $(1, a_2)$ | | $n \geqslant 2$ |
| 18 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$ | $(1, 1, 1)$ | | $n \geqslant 2$ |
| 19 | $(\mathrm{SL}_n, \mathbb{F}^n \oplus \mathbb{F}^1)$ | $(a_1, \ldots, a_{s-1})$ | $\chi_1 \neq \chi_2$ for $s = n + 1$ | $n \geqslant 2$ |
| 20 | $(\mathrm{SL}_n \times \mathrm{SL}_m, \mathbb{F}^n \oplus \mathbb{F}^m)$ | $(1, a_2)$ | $\chi_1 \neq \chi_2$ for $n = 1 + a_2$ | $n \geqslant m \geqslant 2$ |
| 21 | $(\mathrm{SL}_n \times \mathrm{SL}_2, \mathbb{F}^n \oplus \mathbb{F}^2)$ | $(a_1, a_2)$ | $\chi_1 \neq \chi_2$ for<br>$n = 4$, $a_1 = a_2 = 2$ | $n \geqslant 4$,<br>$a_1 \geqslant 2$ |
| 22 | $(\mathrm{Sp}_{2n} \times \mathrm{SL}_m$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^m)$ | $(1, 1)$ | $\chi_1 \neq \chi_2$ for $m \leqslant 2$ | $n \geqslant 2$,<br>$m \geqslant 1$ |
| 23 | $(\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$,<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^{2m})$ | $(1, 1)$ | $\chi_1 \neq \chi_2$ | $n \geqslant m \geqslant 2$ |

the group $\mathrm{Spin}_7$ acts on $\mathbb{F}^8$ via the spin representation. If the description of the tuple $(a_1, \ldots, a_{s-1})$ given in the third column contains parameters, then these parameters may take any admissible values (that is, any values such that $a_1 \leqslant \ldots \leqslant a_s$ and $\mathbf{a} \neq (1, d-1)$). In particular, in rows 16 and 19 any composition $\mathbf{a}$ satisfying the above restrictions is possible. The empty cells in the fourth column mean that there are no conditions on $C$, that is, the characters $\chi_1, \ldots, \chi_r$ may be arbitrary. The abbreviation "lin. ind." stands for "are linearly independent in $\mathfrak{X}(C)$".

Our proof of Theorem 1.7 is based on an analysis of the partial order $\preccurlyeq$ on the set $\mathscr{F}(\mathrm{GL}(V))/\sim$. Here the starting points are the following facts:

(1) if $X \in \mathscr{F}(\mathrm{GL}(V))$ then $[\![X]\!] \succcurlyeq [\![\mathbb{P}(V)]\!]$;

(2) if $X \in \mathscr{F}(\mathrm{GL}(V))$, $[\![X]\!] \neq [\![\mathbb{P}(V)]\!]$, and $d \geqslant 4$, then $[\![X]\!] \succcurlyeq [\![\mathrm{Gr}_2(V)]\!]$.

In view of Theorem 1.4, facts (1) and (2) imply the following results, which hold for any nontrivial $V$-flag variety $X$ and any connected reductive subgroup $K \subset \mathrm{GL}(V)$:

(1′) if $X$ is a $K$-spherical variety then so is $\mathbb{P}(V)$ (see Proposition 3.7);

(2′) if $X$ is a $K$-spherical variety, $[\![X]\!] \neq [\![\mathbb{P}(V)]\!]$, and $d \geqslant 4$, then $\mathrm{Gr}_2(V)$ is a $K$-spherical variety (see Proposition 3.9).

Assertion (1′) means that a necessary condition for $K$-sphericity of a nontrivial $V$-flag variety is that $V$ be a spherical $(K \times \mathbb{F}^\times)$-module, where $\mathbb{F}^\times$ acts on $V$ by scalar transformations (see Corollary 3.8). Assertion (2′) implies that the first step in the proof of Theorem 1.7 is a description of all spherical actions on $\mathrm{Gr}_2(V)$.

We note that assertion (1′) was in fact proved in [Pet, Theorem 5.8] using the ideas presented in this paper.

The list of subgroups of $\mathrm{GL}(V)$ acting spherically on $\mathrm{Gr}_2(V)$ (see Theorems 6.1 and 6.5) turns out to be substantially shorter than that of subgroups acting spherically on $\mathbb{P}(V)$. This makes the subsequent considerations easier and enables us to complete the description of all spherical actions on nontrivial $V$-flag varieties that are nil-equivalent to neither $\mathbb{P}(V)$ nor $\mathrm{Gr}_2(V)$ (see Theorems 6.6 and 6.7).

The present paper is organized as follows. In § 2 we recall some facts on Poisson and symplectic varieties and then, using them, we prove the $K$-sphericity criterion of a generalized flag variety $X$ in terms of the action $K : \mathcal{N}(X)$. (This criterion implies Theorem 1.3.) At the end of § 2 we prove Theorem 1.4. In § 3 we study the nil-equivalence relation and the partial order on nil-equivalence classes in the case $G = \mathrm{GL}(V)$. We also discuss a transparent interpretation of this partial order in terms of Young diagrams. In § 4 we collect all auxiliary results that will be needed in our proof of Theorem 1.7. In § 5 we present two known classifications that will be used in the proof of Theorem 1.7: the first one is the classification of spherical modules and the second one is the classification of Levi subgroups in $\mathrm{GL}(V)$ acting spherically on $V$-flag varieties. We prove Theorem 1.7 in § 6. At last, Appendix A contains the most complicated technical proofs of some statements from § 4.

The authors express their gratitude to E. B. Vinberg and I. B. Penkov for useful discussions, as well as to the referee for a careful reading of the previous version of this paper and valuable comments.

**Basic notation and conventions.** In this paper all varieties, groups, and subgroups are assumed to be algebraic. All topological terms relate to the Zariski topology. The Lie algebras of groups denoted by capital Latin letters are denoted by the corresponding small Gothic letters. All algebras (except for Lie algebras) are assumed to be associative, commutative, and with identity. All vector spaces are assumed to be finite-dimensional.

Let $V$ be a vector space. Any nondegenerate skew-symmetric bilinear form on $V$ will be called a *symplectic form*. If $\Omega$ is a fixed symplectic form on $V$, then for every subspace $W \subset V$ we shall denote by $W^{\perp}$ the skew-orthogonal complement to $W$ with respect to $\Omega$.

All the notation and conventions used in Table 1 will be also used in all other tables appearing in this paper.

Notation:

$|X|$ is the cardinality of a finite set $X$;

$V^*$ is the space of linear functions on a vector space $V$

$\mathrm{S}^2V$ is the symmetric square of a vector space $V$;

$\wedge^2 V$ is the exterior square of a vector space $V$;

$\langle v_1, \ldots, v_k \rangle$ is the linear span of vectors $v_1, \ldots, v_k$ of a vector space $V$;

$G : X$ denotes an action of a group $G$ on a variety $X$;

$G_x$ is the stabilizer of a point $x \in X$ under an action $G : X$;

$G'$ is the derived subgroup of a group $G$;

$G^0$ is the connected component of the identity of a group $G$;

$\mathfrak{X}(G)$ is the character group of a group $G$ (in additive notation);

$\mathrm{S}(\mathrm{L}_n \times \mathrm{L}_m)$ is the subgroup in $\mathrm{GL}_{n+m}$ equal to $(\mathrm{GL}_n \times \mathrm{GL}_m) \cap \mathrm{SL}_{n+m}$;

$\mathrm{S}(\mathrm{O}_n \times \mathrm{O}_m)$ is the subgroup in $\mathrm{O}_{n+m}$ equal to $(\mathrm{O}_n \times \mathrm{O}_m) \cap \mathrm{SO}_{n+m}$;

$\overline{Y}$ is the closure of a subset $Y$ of a variety $X$;

$\operatorname{rk} G$ is the rank of a reductive group $G$, that is, the dimension of a maximal torus of $G$;

$X^{\mathrm{reg}}$ is the set of regular points of a variety $X$;

$\mathbb{F}[X]$ is the algebra of regular functions on a variety $X$;

$\mathbb{F}(X)$ is the field of rational functions on a variety $X$;

$A^G$ is the algebra of invariants of an action of a group $G$ on an algebra $A$;

$\operatorname{Quot} A$ is the field of fractions of an algebra $A$ without zero divisors;

$\operatorname{Spec} A$ is the spectrum of a finitely generated algebra $A$ without nilpotents, that is, the affine algebraic variety whose algebra of regular functions is isomorphic to $A$;

$T_xX$ is the tangent space of a variety $X$ at a point $x$;

$T_x^*X = (T_xX)^*$ is the cotangent space of a variety $X$ at a point $x$;

$T^*X$ is the cotangent bundle of a smooth variety $X$;

$P^{\top}$ is the transpose matrix of a matrix $P$.

## 2. Generalized flag varieties and nilpotent orbits

Throughout this section we fix an arbitrary connected reductive group $G$ and an arbitrary connected reductive subgroup $K \subset G$.

2.1. **Poisson and symplectic varieties.** In this subsection we gather all the required information on Poisson and symplectic varieties. The information here is taken from [Vin, §§ II.1–II.3].

An algebra $A$ is said to be a *Poisson algebra* if $A$ is equipped with a bilinear anticommutative operation $\{\,\cdot\,,\cdot\,\}\colon A \times A \to A$ (called a *Poisson bracket*) satisfying the identities

$$\{f, gh\} = \{f, g\}h + \{f, h\}g, \qquad \text{(Leibniz identity)}$$
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \qquad \text{(Jacobi identity)}$$

for any elements $f, g, h \in A$.

An irreducible variety $M$ is said to be a *Poisson variety* if the structure sheaf of $M$ is endowed with a structure of a sheaf of Poisson algebras. We note that the Poisson structure on $M$ is uniquely determined by the Poisson bracket induced in the field $\mathbb{F}(M)$.

A smooth irreducible variety $M$ together with a nondegenerate closed 2-form $\omega$ is said to be a *symplectic variety*. In this situation, the 2-form $\omega$ is said to be the *structure 2-form*.

Let $M$ be a Poisson variety. There is a unique bivector field $\mathcal{B}$ on $M^{\mathrm{reg}}$ with the following property:

$$\{f, g\} = \mathcal{B}(df, dg) \tag{2.1}$$

for any $f, g \in \mathbb{F}(M)$. The bivector $\mathcal{B}$ is said to be the *Poisson bivector*. If $\mathcal{B}$ is nondegenerate at each point of a nonempty open subset $Z \subset M^{\mathrm{reg}}$, then the 2-form $\omega = (\mathcal{B}^{\top})^{-1}$ (the formula relates the matrices of $\omega$ and $\mathscr{B}$ in dual bases of the spaces $T_p^*M$ and $T_pM$ for each point $p \in M$) defines a symplectic structure on $Z$.

Conversely, if $M$ is a symplectic variety with structure 2-form $\omega$ then the bivector $\mathcal{B} = (\omega^{\top})^{-1}$ defines a Poisson structure on $M$ by the formula (2.1), so that every symplectic variety is Poisson.

A morphism $\varphi \colon M \to M'$ of Poisson varieties is said to be *Poisson* if for every open subset $U \subset M'$ the corresponding homomorphism $\varphi^* \colon \mathbb{F}[U] \to \mathbb{F}[\varphi^{-1}(U)]$ is a homomorphism of Poisson algebras. If $\varphi$ is surjective, then $\varphi$ is Poisson if and only if the pushforward of the Poisson bivector on $M$ coincides with the Poisson bivector on $M'$. In particular, if $\varphi \colon M \to M'$ is a Poisson morphism of symplectic varieties that is a covering, then the pullback of the structure 2-form on $M'$ coincides with the structure 2-form on $M$.

The space $\mathfrak{g}^*$ is endowed with a natural Poisson structure. In view of the Leibnitz identity, this structure is uniquely determined by specifying the values of the Poisson bracket on linear functions, that is, on the space $(\mathfrak{g}^*)^* \simeq \mathfrak{g}$. For $\xi, \eta \in \mathfrak{g}$ one defines $\{\xi, \eta\} = [\xi, \eta]$.

It is well known that for an arbitrary $G$-orbit $O$ in $\mathfrak{g}^*$ the restriction of the Poisson bivector to $O$ is well defined and nondegenerate, which induces a symplectic structure on $O$. This structure is determined by the 2-form $\omega_O$ given at a point $\alpha \in \mathfrak{g}^*$ by the formula

$$\omega_O(\mathrm{ad}^*(\xi)\alpha, \mathrm{ad}^*(\eta)\alpha) = \alpha([\xi, \eta]),$$

where $\xi, \eta \in \mathfrak{g}$ and $\mathrm{ad}^*$ is the coadjoint representation of the algebra $\mathfrak{g}$. The form $\omega_O$ is said to be the *Kostant–Kirillov form*.

Let $X$ be a smooth irreducible variety. Then its cotangent bundle

$$T^*X = \{(x, p) \mid x \in X, p \in T_x^*X\}$$

is also a smooth irreducible variety. There is a canonical symplectic structure on $T^*X$. The structure 2-form $\omega_X$ can be expressed in the form $\omega_X = -d\theta$, where $\theta$ is the 1-form defined as follows. Let $\pi \colon T^*X \to X$ be the canonical projection and let $d\pi$ be its differential. Let $\xi$ be a tangent vector to $T^*X$ at a point $(x, p)$. Then $\theta(\xi) = p(d\pi(\xi))$.

2.2. **Some properties of symplectic $G$-varieties.** Let $X$ be a smooth irreducible $G$-variety. The action $G : X$ naturally induces an action $G : T^*X$ preserving the structure 2-form $\omega_X$.

Let $\mathscr{V}(X)$ be the Lie algebra of vector fields on $X$. The action of $G$ on $X$ defines the Lie algebra homomorphism $\tau_X\colon \mathfrak{g} \to \mathscr{V}(X)$ taking each element $\xi \in \mathfrak{g}$ to the corresponding velocity field on $X$. For all $\xi \in \mathfrak{g}$ and $x \in X$ let $\xi x$ denote the value of the field $\tau_X(\xi)$ at $x$.

The map $\Phi\colon T^*X \to \mathfrak{g}^*$ given by the formula

$$(x,p) \mapsto [\xi \mapsto p(\xi x)], \qquad \text{where } x \in X, p \in T^*_x X, \xi \in \mathfrak{g},$$

is said to be the *moment map.*

**Proposition 2.1** (see [Vin, § II.2.3, Proposition 2])**.** *The map $\Phi$ is a $G$-equivariant Poisson morphism.*

Let $V$ be a vector space with a given symplectic form $\Omega$. A subspace $W \subset V$ is said to be *isotropic* if the restriction of $\Omega$ to $W$ is identically zero and *coisotropic* if the skew-orthogonal complement of $W$ is isotropic.

Let $M$ be a symplectic variety with structure 2-form $\omega$. A smooth irreducible locally closed subvariety $Z \subset M$ is said to be *coisotropic* if the subspace $T_z Z$ is coisotropic in $T_z M$ for each point $z \in Z$.

**Definition 2.2.** An action $G : X$ preserving the structure 2-form $\omega$ is said to be *coisotropic* if orbits of general position for this action are coisotropic.

The following theorem is implied by [Kn1, Theorem 7.1], see also [Vin, § II.3.4, Theorem 2, Corollary 1].

**Theorem 2.3.** *Let $X$ be a smooth irreducible $G$-variety. The following conditions are equivalent:*
(a) *the action $G : X$ is spherical;*
(b) *the action $G : T^*X$ is coisotropic.*

A subset $A'$ of a Poisson algebra $A$ is said to be *Poisson-commutative* if $\{f, g\} = 0$ for all $f, g \in A'$.

**Proposition 2.4** (see [Vin, § II.3.2, Proposition 5])**.** *Let $M$ be a symplectic $G$-variety such that the structure 2-form is $G$-invariant. The following conditions are equivalent:*
(a) *the action $G : M$ is coisotropic;*
(b) *the field $\mathbb{F}(M)^G$ is Poisson-commutative.*

2.3. **The $K$-sphericity criterion for a generalized flag variety.** The main result of this subsection is Theorem 2.6.

We shall identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a bilinear form given by the trace of the product of two linear operators in a fixed faithful linear representation of $\mathfrak{g}$.

Let $P \subset G$ be a parabolic subgroup and let $N$ be the unipotent radical of $P$. Recall that in Introduction we defined the nilpotent orbit $\mathcal{N}(G/P) \subset \mathfrak{g}$, see (1.1). Put $o = eP \in G/P$. Let $\Phi_P\colon T^*(G/P) \to \mathfrak{g}$ be the moment map corresponding to the natural action $G : G/P$.

**Proposition 2.5.** *The image of $\Phi_P$ coincides with the closure of the orbit $\mathcal{N}(G/P) \subset \mathfrak{g}$. Moreover, $\Phi_P$ is finite over $\mathcal{N}(G/P)$.*

*Proof.* In view of the identifications $T^*_o(G/P) \simeq (\mathfrak{g}/\mathfrak{p})^* \simeq \mathfrak{n}$, it follows from [Vin, § II.2.3, Example 5] that $\Phi_P(T^*_o(G/P)) = \mathfrak{n}$. This together with the $G$-equivariance of the map $\Phi_P$

(see Proposition 2.1) implies that the image of $\Phi_P$ coincides with the subset $G\mathfrak{n} \subset \mathfrak{g}$ and, in particular, is irreducible. By [Rich, Proposition 6(b)] the set $G\mathfrak{n}$ is closed in $\mathfrak{g}$ and has dimension $2\dim\mathfrak{n}$. By its definition, the orbit $\mathcal{N}(G/P)$ is dense in $G\mathfrak{n}$, therefore $\dim\mathcal{N}(G/P) = \dim T^*(G/P)$. The latter means that the map $\Phi_P$ is finite over $\mathcal{N}(G/P)$. □

**Theorem 2.6.** *The following conditions are equivalent:*
(a) *the action $K : G/P$ is spherical;*
(b) *the action $K : \mathcal{N}(G/P)$ is coisotropic.*

*Proof.* By Proposition 2.5 the subset $U = \Phi_P^{-1}(\mathscr{N}(G/P))$ is open in $T^*(G/P)$ and the map $\Phi_P|_U : U \to \mathcal{N}(G/P)$ is a covering. It follows from Proposition 2.1 that $\Phi_P|_U$ is a $G$-equivariant Poisson morphism of symplectic varieties, therefore the image of the structure 2-form on $\mathcal{N}(G/P)$ coincides with the structure 2-form on $U$. Consequently, the action $K : \mathcal{N}(G/P)$ is coisotropic if and only if the action $K : U$ is so or, equivalently, the action $K : T^*(G/P)$ is so. In view of Theorem 2.3 the latter holds if and only if the action $K : G/P$ is spherical. □

2.4. **Proof of Theorem 1.4.** By Theorem 2.6, the proof of Theorem 1.4 reduces to that of the following proposition.

**Proposition 2.7.** *Let $O_1, O_2$ be two nilpotent $G$-orbits in $\mathfrak{g}$ such that $O_1 \subset \overline{O}_2$. If the action $K : O_2$ is coisotropic then so is the action $K : O_1$.*

In order to prove this proposition, we shall need Lemma 2.8 and Proposition 2.9 given below.

**Lemma 2.8** (see [BK, Proposition 1.2])**.** *For every $G$-orbit $O \subset \mathfrak{g}$ the algebra $\mathbb{F}[O]$ is the integral closure of the algebra $\mathbb{F}[\overline{O}]$ in the field $\mathbb{F}(O)$. In particular, $\mathbb{F}[O]$ is integrally closed.*

**Proposition 2.9.** *For every $G$-orbit $O \subset \mathfrak{g}$, one has $\mathbb{F}(O)^K = \mathrm{Quot}(\mathbb{F}[O]^K)$.*

*Proof.* Lemma 2.8 implies that the variety $\operatorname{Spec}\mathbb{F}[O]$ is normal. Now the required result follows from [Los, Corollary 3.4.1] and [Los, Theorem 1.2.4, part 1]. □

*Proof of Proposition* 2.7. Suppose that the action $K : O_2$ is coisotropic. Then by Proposition 2.4 the field $\mathbb{F}(O_2)^K$ is Poisson-commutative. Therefore the algebra $\mathbb{F}[\overline{O}_2]^K$ is Poisson-commutative as well. Consider the restriction map $\mathbb{F}[\overline{O}_2] \to \mathbb{F}[\overline{O}_1]$. It is surjective and a $K$-module homomorphism, hence the image of $\mathbb{F}[\overline{O}_2]^K$ coincides with $\mathbb{F}[\overline{O}_1]^K$. It follows that the latter algebra is Poisson-commutative. Let us show that the algebra $\mathbb{F}[O_1]^K$ is also Poisson-commutative. Let $f \in \mathbb{F}[O_1]^K$ be an arbitrary element. It follows from Lemma 2.8 that $f$ satisfies an equation of the form

$$f^n + c_{n-1}f^{n-1} + \ldots + c_1 f + c_0 = 0, \tag{2.2}$$

where $n \geqslant 1$ and $c_0, \ldots, c_{n-1} \in \mathbb{F}[\overline{O}_1]$. Applying the operator of "averaging over $K$" (which is also known as the Reynolds operator; see [PV, § 3.4]) we may assume that $c_0, \ldots, c_{n-1} \in \mathbb{F}[\overline{O}_1]^K$. Moreover, we shall assume that the number $n$ is minimal among all equations of the form (2.2). For an arbitrary element $g \in \mathbb{F}[\overline{O}_1]^K$ we have $\{c_i, g\} = 0$

for all $i = 0, \ldots, n-1$. Therefore, applying the Poisson bracket with $g$ to both sides of (2.2), we obtain

$$(nf^{n-1} + (n-1)c_{n-1}f^{n-2} + \ldots + c_1)\{f, g\} = 0.$$

Since $n$ is minimal, the expression $nf^{n-1} + (n-1)c_{n-1}f^{n-2} + \ldots + c_1$ is different from zero, hence $\{f, g\} = 0$. Consequently, $\{\mathbb{F}[O_1]^K, \mathbb{F}[\overline{O}_1]^K\} = 0$. Applying the same argument to an arbitrary element $g \in \mathbb{F}[O_1]^K$, again we get $\{f, g\} = 0$, hence the algebra $\mathbb{F}[O_1]^K$ is Poisson-commutative. Then Proposition 2.9 implies that the field $\mathbb{F}(O_1)^K$ is also Poisson-commutative. Applying Proposition 2.4 we find that the action $K : O_1$ is coisotropic. $\square$

*Remark* 2.10. In the case $G = \mathrm{GL}_n$ (or $\mathrm{SL}_n$) the proof of Proposition 2.7 simplifies. Namely, as was proved in [KP], in this case the closure of every $G$-orbit in $\mathfrak{g}$ is normal. Hence by Lemma 2.8 we have $\mathbb{F}[O_1] = \mathbb{F}[\overline{O}_1]$, and so $\mathbb{F}[O_1]^K = \mathbb{F}[\overline{O}_1]^K$.

## 3. The partial order on the set of nil-equivalence classes of $V$-flag varieties

Throughout this section we fix a vector space $V$ of dimension $d$.

3.1. **Nilpotent orbits in $\mathfrak{gl}(V)$.** A composition $(a_1, \ldots, a_s)$ of $d$ is said to be a *partition* if $a_1 \geqslant \ldots \geqslant a_s$.

The following fact is well known.

**Theorem 3.1.** *There is a bijection between the nilpotent orbits in $\mathfrak{gl}(V)$ and the partitions of $d$. Under this bijection, the orbit corresponding to a partition $(a_1, \ldots, a_s)$ consists of all matrices whose Jordan normal form has zeros on the diagonal and the block sizes are $a_1, \ldots, a_s$ up to a permutation.*

For every partition $\mathbf{a} = (a_1, \ldots, a_s)$ of $d$ we denote by $O_{\mathbf{a}}$ the corresponding nilpotent orbit in $\mathfrak{gl}(V)$.

We now introduce a partial order on the set of partitions of $d$ in the following way. For two partitions $\mathbf{a} = (a_1, \ldots, a_s)$ and $\mathbf{b} = (b_1, \ldots, b_t)$ we write $\mathbf{a} \preccurlyeq \mathbf{b}$ (or $\mathbf{b} \succcurlyeq \mathbf{a}$) if

$$a_1 + \ldots + a_i \leqslant b_1 + \ldots + b_i \text{ for all } i = 1, \ldots, d.$$

(In this formula we put $a_j = 0$ for $j > s$ and $b_j = 0$ for $j > t$.)

**Theorem 3.2** (see [CM, Theorem 6.2.5]). *Let $\mathbf{a}$ and $\mathbf{b}$ be two partitions of $d$. The following conditions are equivalent:*

(a) $O_{\mathbf{a}} \subset \overline{O}_{\mathbf{b}}$;

(b) $\mathbf{a} \preccurlyeq \mathbf{b}$.

3.2. **The correspondence between $V$-flag varieties and nilpotent orbits in $\mathfrak{gl}(V)$.** For each composition $\mathbf{a} = (a_1, \ldots, a_s)$ of $d$ one defines the *dual* partition $\mathbf{a}^\top = (\widehat{a}_1, \ldots, \widehat{a}_t)$ of $d$ by the following rule:

$$\widehat{a}_i = |\{j \mid a_j \geqslant i\}|, \quad i = 1, \ldots, t.$$

Obviously, for every composition $\mathbf{b}$ of $d$ obtained from $\mathbf{a}$ by a permutation one has $\mathbf{a}^\top = \mathbf{b}^\top$. Besides, it is not hard to see that the operation $\mathbf{a} \mapsto \widehat{\mathbf{a}}$ is an involution on the set of partitions.

**Proposition 3.3** (see [CM, Lemma 6.3.1]). *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two partitions of* $d$. *The following conditions are equivalent:*

(a) $\mathbf{a} \preccurlyeq \mathbf{b}$;

(b) $\mathbf{a}^\top \succcurlyeq \mathbf{b}^\top$.

Let $\mathbf{a} = (a_1, \dots, a_s)$ be an arbitrary composition of $d$. We denote by $\mathbf{a}^\natural$ the partition of $d$ obtained from $\mathbf{a}$ by arranging its elements in the non-increasing order.

We fix a basis $e_1, \dots, e_d$ of $V$. Let $P_\mathbf{a}$ be the parabolic subgroup of $\mathrm{GL}(V)$ that is the stabilizer of the point $(V_1, \dots, V_s) \in \mathrm{Fl}_\mathbf{a}(V)$, where $V_i = \langle e_1, \dots, e_{a_1+\ldots+a_i} \rangle$ for $i = 1, \dots, s$. It is easy to see that, relatively to the basis $e_1, \dots, e_d$, $P_\mathbf{a}$ consists of all nondegenerate block upper-triangular matrices whose diagonal blocks have sizes $a_1, \dots, a_s$.

**Proposition 3.4.** *One has* $\mathcal{N}(\mathrm{Fl}_\mathbf{a}(V)) = O_{\mathbf{a}^\top}$.

*Proof.* Put $\mathbf{c} = \mathbf{a}^\natural$. It is easy to see that Levi subgroups of $P_\mathbf{a}$ and $P_\mathbf{c}$ are conjugate in $\mathrm{GL}(V)$. Then from [JR, Theorem 2.7] (see also [CM, Theorem 7.1.3]) it follows that $\mathcal{N}(\mathrm{Fl}_\mathbf{a}(V)) = \mathcal{N}(\mathrm{Fl}_\mathbf{c}(V))$. Further, by [Kra, § 2.2, Theorem] (see also [CM, Theorem 7.2.3]) one has $\mathcal{N}(\mathrm{Fl}_\mathbf{c}(V)) = O_{\mathbf{c}^\top}$. Since $\mathbf{a}^\top = \mathbf{c}^\top$, we obtain $\mathcal{N}(\mathrm{Fl}_\mathbf{a}(V)) = O_{\mathbf{a}^\top}$. □

**Corollary 3.5.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two compositions of* $d$. *The following conditions are equivalent:*

(a) $\mathrm{Fl}_\mathbf{a}(V) \sim \mathrm{Fl}_\mathbf{b}(V)$;

(b) $\mathbf{a}^\natural = \mathbf{b}^\natural$.

*Proof* follows from Proposition 3.4 and the fact that $\mathbf{a}^\top = \mathbf{b}^\top$ if and only if $\mathbf{a}^\natural = \mathbf{b}^\natural$. □

**Corollary 3.6.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two compositions of* $d$. *The following conditions are equivalent:*

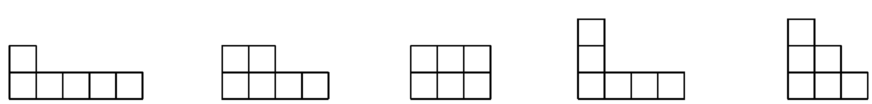
(a) $\llbracket \mathrm{Fl}_\mathbf{a}(V) \rrbracket \preccurlyeq \llbracket \mathrm{Fl}_\mathbf{b}(V) \rrbracket$;

(b) $\mathbf{a}^\top \preccurlyeq \mathbf{b}^\top$;

(c) $\mathbf{a}^\natural \succcurlyeq \mathbf{b}^\natural$.

*Proof.* Equivalence of (a) and (b) follows from Theorem 3.2 and Proposition 3.4. Equivalence of (b) and (c) follows from Proposition 3.3. □

3.3. **Young diagrams of $V$-flag varieties.** With every partition $\mathbf{a} = (a_1, \dots, a_s)$ we associate the left-aligned Young diagram $\mathrm{YD}(\mathbf{a})$ whose $i$-th row from the bottom contains $a_i$ boxes. With every variety $\mathrm{Fl}_\mathbf{a}(V)$ we associate the Young diagram $\mathrm{YD}(\mathbf{a}^\natural)$. As an example, we show the Young diagrams of some $\mathbb{F}^6$-flag varieties in Figure 1.



$\mathbb{P}(\mathbb{F}^6)$ $\quad$ $\mathrm{Gr}_2(\mathbb{F}^6)$ $\quad$ $\mathrm{Gr}_3(\mathbb{F}^6)$ $\quad$ $\mathrm{Fl}(1,1;\mathbb{F}^6)$ $\quad$ $\mathrm{Fl}(1,2;\mathbb{F}^6)$

FIGURE 1

The partial order on the set of nil-equivalence classes of $V$-flag varieties admits a transparent interpretation in terms of Young diagrams. Namely, let $\mathbf{a}$ and $\mathbf{b}$ be two compositions of $d$. Then, by Corollary 3.6, the condition $\llbracket \mathrm{Fl}_\mathbf{a}(V) \rrbracket \preccurlyeq \llbracket \mathrm{Fl}_\mathbf{b}(V) \rrbracket$ is equivalent to $\mathbf{a}^\natural \succcurlyeq \mathbf{b}^\natural$. According to the description of the partial order on the set of partitions given in

§ 3.1, the latter can be interpreted as follows: the diagram $\mathrm{YD}(\mathbf{a}^\natural)$ can be obtained from the diagram $\mathrm{YD}(\mathbf{b}^\natural)$ by *crumbling*, that is, by moving boxes from upper rows to lower ones. For example, using this interpretation it is easy to construct the Hasse diagram for the partial order on the set of nil-equivalence classes of $\mathbb{F}^6$-flag varieties appearing in Figure 1. This diagram is shown in Figure 2.

$$\begin{tikzcd}[column sep=tiny]
& \llbracket \operatorname{Fl}(1,2;\mathbb{F}^6) \rrbracket \arrow[dl, dash] \arrow[dr, dash] & \\
\llbracket \operatorname{Fl}(1,1;\mathbb{F}^6) \rrbracket \arrow[dr, dash] & & \llbracket \operatorname{Gr}_3(\mathbb{F}^6) \rrbracket \arrow[dl, dash] \\
& \llbracket \operatorname{Gr}_2(\mathbb{F}^6) \rrbracket \arrow[d, dash] & \\
& \llbracket \mathbb{P}(\mathbb{F}^6) \rrbracket &
\end{tikzcd}$$

FIGURE 2

3.4. **Necessary sphericity conditions for actions on $V$-flag varieties.** Let $K \subset \mathrm{GL}(V)$ be a connected reductive subgroup. In this subsection, making use of the description of the partial order on the set $\mathscr{F}(\mathrm{GL}(V))/\!\sim$ given in §§ 3.2, 3.3, we obtain two necessary conditions for $V$-flag varieties to be $K$-spherical (see Propositions 3.7 and 3.9). These conditions will be starting points in the proof of Theorem 1.7.

Let $\mathbf{a}$ be a nontrivial composition of $d$.

**Proposition 3.7** (see also [Pet, Theorem 5.8])**.** *If the variety $\operatorname{Fl}_{\mathbf{a}}(V)$ is $K$-spherical, then so is the variety $\mathbb{P}(V)$.*

*Proof.* One has $\mathbb{P}(V) = \operatorname{Fl}_{\mathbf{b}}(V)$ where $\mathbf{b} = (1, d-1)$. As $\mathbf{a} \ne (d)$, one has $\mathbf{b}^\natural \succcurlyeq \mathbf{a}^\natural$. In view of Corollary 3.6 the latter implies that $\llbracket \operatorname{Fl}_{\mathbf{a}}(V) \rrbracket \succcurlyeq \llbracket \mathbb{P}(V) \rrbracket$. It remains to apply Theorem 1.4. □

**Corollary 3.8.** *If $\operatorname{Fl}_{\mathbf{a}}(V)$ is a $K$-spherical variety, then $V$ is a spherical $(K \times \mathbb{F}^\times)$-module (where $\mathbb{F}^\times$ acts on $V$ by scalar transformations).*

**Proposition 3.9.** *If $\operatorname{Fl}_{\mathbf{a}}(V)$ is a $K$-spherical variety, $\llbracket \operatorname{Fl}_{\mathbf{a}}(V) \rrbracket \ne \llbracket \mathbb{P}(V) \rrbracket$, and $d \geqslant 4$, then $\operatorname{Gr}_2(V)$ is a $K$-spherical variety.*

*Proof.* We have $\operatorname{Gr}_2(V) = \operatorname{Fl}_{\mathbf{b}}(V)$, where $\mathbf{b} = (2, d-2)$. As $d \geqslant 4$, we have $\mathbf{b}^\natural = (d-2, 2)$. Next, the hypothesis implies that the partition $\mathbf{a}^\natural$ is different from $(d)$ and $(d-1, 1)$. The latter means that $\mathbf{b}^\natural \succcurlyeq \mathbf{a}^\natural$, whence by Corollary 3.6 we obtain $\llbracket \operatorname{Fl}_{\mathbf{a}}(V) \rrbracket \succcurlyeq \llbracket \operatorname{Gr}_2(V) \rrbracket$. The proof is completed by applying Theorem 1.4. □

*Remark* 3.10. For $d = 3$ one has $\operatorname{Gr}_2(V) \sim \mathbb{P}(V)$.

## 4. TOOLS

In this section we collect all auxiliary results that will be needed in our proof of Theorem 1.7.

4.1. **Spherical varieties and spherical subgroups.** Let $K$ be a connected reductive group, $B$ a Borel subgroup of $K$, and $X$ a spherical $K$-variety. The following proposition is well known; we supply it with a proof for the reader's convenience.

**Proposition 4.1.** *One has*

$$\dim K + \operatorname{rk} K \geqslant 2\dim X. \tag{4.1}$$

*Proof.* Since there is an open $B$-orbit in $X$, one has $\dim B \geqslant \dim X$. To complete the proof it remains to notice that $2\dim B = \dim K + \operatorname{rk} K$. □

In what follows we shall need the following notion. A subgroup $H \subset K$ is said to be *spherical* if the homogeneous space $K/H$ is a spherical $K$-variety. It is easy to see that $H$ is spherical if and only if $H^0$ is so.

4.2. **Homogeneous bundles.** Let $G$ be a group, $H$ a subgroup of $G$, and $X$ a $G$-variety. Suppose that there is a surjective $G$-equivariant morphism $\varphi\colon X \to G/H$. Let $Y$ denote the fiber of $\varphi$ over the point $o = eH$. Evidently, $Y$ is an $H$-variety. In this situation we say that $X$ is a *homogeneous bundle over $G/H$ with fiber $Y$* (or simply a *homogeneous bundle over $G/H$*).

Since $\varphi$ is $G$-equivariant, we have the following facts:

(1) every $G$-orbit in $X$ meets $Y$;

(2) for $g \in G$ and $y \in Y$ the condition $gy \in Y$ holds if and only if $g \in H$.

Facts (1) and (2) imply that for every $G$-orbit $O \subset X$ the intersection $O \cap Y$ is a nonempty $H$-orbit.

**Proposition 4.2.** *The map $\iota\colon O \mapsto O \cap Y$ is a bijection between $G$-orbits in $X$ and $H$-orbits in $Y$. Moreover, for every $G$-orbit $O \subset X$ one has*

$$\dim O - \dim(O \cap Y) = \dim X - \dim Y = \dim G/H.$$

*Proof.* It is easy to see that the map inverse to $\iota$ takes an arbitrary $H$-orbit $Y_0 \subset Y$ to the $G$-orbit $GY_0 \subset X$. Now consider an arbitrary $G$-orbit $O \subset X$ and an arbitrary point $y \in O \cap Y$. Making use of fact (2), we obtain $G_y \subset H$, whence $O \simeq G/G_y$ and $O \cap Y \simeq H/G_y$. Consequently, $\dim O - \dim(O \cap Y) = \dim G - \dim H$. Since all fibers of $\varphi$ are isomorphic to $Y$ (all of them are $G$-shifts of $Y$), we have $\dim X - \dim Y = \dim G/H$, hence the required equalities. □

**Corollary 4.3.** *There is an open $G$-orbit in $X$ if and only if there is an open $H$-orbit in $Y$.*

4.3. **Supplementary information on $V$-flag varieties.** Let $V$ be a vector space of dimension $d$ and let $\mathbf{a} = (a_1, \ldots, a_s)$ be a nontrivial composition of $d$. We put $m = a_1 + \ldots + a_{s-1} = d - a_s$ and consider the vector space

$$U = \underbrace{V \oplus \ldots \oplus V}_{m} \simeq V \otimes \mathbb{F}^m.$$

The natural $(\mathrm{GL}(V) \times \mathrm{GL}_m)$-module structure on $V \otimes \mathbb{F}^m$ is transferred to $U$ so that $\mathrm{GL}(V)$ acts diagonally on $U$ and the action of $\mathrm{GL}_m$ on $U$ is given by the formula

$$(g, (v_1, \ldots, v_m)) \mapsto (v_1, \ldots, v_m)g^\top,$$

where $g \in \mathrm{GL}_m$ and $(v_1, \ldots, v_m) \in U$.

Consider the open subset $U_0 \subset U$ formed by all tuples $(v_1, \ldots, v_m)$ of linearly independent vectors. Evidently, $U_0$ is a $(\mathrm{GL}(V) \times \mathrm{GL}_m)$-stable subset. There is the natural $\mathrm{GL}(V)$-equivariant surjective map

$$\rho \colon U_0 \to \mathrm{Fl}_{\mathbf{a}}(V)$$

taking a tuple $u = (v_1, \ldots, v_m) \in U_0$ to the tuple of subspaces $\rho(u) = (V_1, \ldots, V_s)$, where $V_i = \langle v_1, \ldots, v_{a_1+\ldots+a_i} \rangle$ for $i = 1, \ldots, s-1$ and $V_s = V$.

Let $e_1, \ldots, e_m$ be the standard basis in $\mathbb{F}^m$. We denote by $Q_{\mathbf{a}}$ the subgroup of $\mathrm{GL}_m$ preserving each of the subspaces

$$\langle e_1, \ldots, e_{a_1+\ldots+a_i} \rangle, \qquad i = 1, \ldots, s-1.$$

It is easy to see that the fibers of $\rho$ are exactly the orbits of the group $Q_{\mathbf{a}}$.

**Proposition 4.4.** *Let $G \subset \mathrm{GL}(V)$ be an arbitrary subgroup and let $\mathbf{a}$ be a nontrivial composition of $d$. The following conditions are equivalent:*

(a) *there is an open $G$-orbit in $\mathrm{Fl}_{\mathbf{a}}(V)$;*

(b) *there is an open $(G \times Q_{\mathbf{a}})$-orbit in $V \otimes \mathbb{F}^m$.*

*Proof.* It suffices to prove that the existence of an open $G$-orbit in $\mathrm{Fl}_{\mathbf{a}}$ is equivalent to the existence of an open $(G \times Q_{\mathbf{a}})$-orbit in $U_0$. The latter is implied by the fact that the fibers of $\rho$ are exactly the $Q_{\mathbf{a}}$-orbits. ☐

**Corollary 4.5.** *Suppose that $K \subset \mathrm{GL}(V)$ is a connected reductive subgroup and*

$$\mathbf{a}(m) = (\underbrace{1, \ldots, 1}_{m}, d-m),$$

*where $0 < m < d$. Then the following conditions are equivalent:*

(a) *$\mathrm{Fl}_{\mathbf{a}(m)}(V)$ is a $K$-spherical variety;*

(b) *$V \otimes \mathbb{F}^m$ is a spherical $(K \times \mathrm{GL}_m)$-module.*

*Proof.* Observe that $Q_{\mathbf{a}(m)}$ is nothing else than a Borel subgroup of $\mathrm{GL}_m$. It remains to apply Proposition 4.4 with $G$ being a Borel subgroup of $K$. ☐

The following result is also a particular case of [Ela, Lemma 1].

**Corollary 4.6.** *Let $0 < m < d$ and let $K \subset \mathrm{GL}(V)$ be an arbitrary subgroup. Suppose that for the natural action of $K \times \mathrm{GL}_m$ on $V \otimes \mathbb{F}^m$ there is a point $z \in V \otimes \mathbb{F}^m$ with stabilizer $H$ such that the orbit of $z$ is open. Then for the action of $K$ on $\mathrm{Gr}_m(V)$ the orbit of $\rho(z)$ is open and the stabilizer $K_{\rho(z)}$ equals $p(H)$, where $p \colon K \times \mathrm{GL}_m \to K$ is the projection to the first factor. Moreover, $p(H) \simeq H$.*

*Remark* 4.7. It follows from what we have said in this subsection that for an arbitrary nontrivial composition $\mathbf{a}$ of $d$ the variety $\mathrm{Fl}_{\mathbf{a}}(V)$ is the geometric quotient of $U_0$ by the action of $Q_{\mathbf{a}}$; see [PV, § 4.2 and Theorem 4.2].

4.4. **Sphericity of some actions on Grassmannians.** The following particular cases of spherical actions on Grassmannians are well known.

**Proposition 4.8.** *For $n \geqslant 2$ and $1 \leqslant m \leqslant 2n-1$, the action of $\mathrm{Sp}_{2n}$ on $\mathrm{Gr}_m(\mathbb{F}^{2n})$ is spherical.*

**Proposition 4.9.** *For $n \geqslant 3$ and $1 \leqslant m \leqslant n-1$, the action of* $\mathrm{SO}_n$ *on* $\mathrm{Gr}_m(\mathbb{F}^n)$ *is spherical.*

Proofs of Propositions 4.8 and 4.9 can be found, for instance, in [HNOO, § 5.2] and [HNOO, § 5.1], respectively.

4.5. **A method for verifying sphericity of some actions.** Let $K$ be a connected reductive group, $B$ a Borel subgroup of $K$, and $X$ a spherical $K$-variety.

**Definition 4.10.** We say that a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ *have property* (P) if the orbit $Kx$ is open in $X$ and for every pair $(Z, \varphi)$, where $Z$ is an irreducible $K$-variety and $\varphi\colon Z \to X$ is a surjective $K$-equivariant morphism with irreducible fiber over $x$, the following conditions are equivalent:
(1) the variety $Z$ is $K$-spherical;
(2) the variety $\varphi^{-1}(x) \subset Z$ is $L$-spherical.

**Proposition 4.11.** *Suppose that a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ are such that the orbit $Bx$ is open in $X$ and $B_x^0$ is a Borel subgroup of $L$. Then $x$ and $L$ have property* (P).

*Proof.* Let $Z$ be an arbitrary irreducible $K$-variety and let $\varphi\colon Z \to X$ be a surjective $K$-equivariant morphism with irreducible fiber over $x$. Since the orbit $Bx$ is open in $X$, the set $Z_0 = \varphi^{-1}(Bx)$ is open in $Z$ and is a homogeneous bundle over $Bx$. By Corollary 4.3, the existence in $Z_0$ of an open $B$-orbit is equivalent to the existence in $\varphi^{-1}(x)$ of an open $B_x$-orbit. As $B_x^0$ is a Borel subgroup of $L$, the latter condition is equivalent to sphericity of the action $L : \varphi^{-1}(x)$. □

The following theorem is implied by results of Panyushev's paper [Pan].

**Theorem 4.12.** *For every spherical $K$-variety $X$ there exist a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ having property* (P).

*Proof.* It follows from [Pan, Theorem 1] that there exist a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ such that the orbit $Bx$ is open in $X$ and $B_x^0$ is a Borel subgroup of $L$. Then by Proposition 4.11 the point $x$ and the subgroup $L$ have property (P). □

In the remaining part of this subsection we find explicitly a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ having property (P) for the following two cases:
(1) $K = \mathrm{SL}_n$, $X = \mathrm{Gr}_m(\mathbb{F}^n)$, where $n \geqslant 2$ and $1 \leqslant m \leqslant n-1$;
(2) $K = \mathrm{Sp}_{2n}$, $X = \mathrm{Gr}_m(\mathbb{F}^{2n})$, where $n \geqslant 2$ and $1 \leqslant m \leqslant 2n-1$.
The explicit form of $x$ and $L$ in these cases will be many times used in § 6. Proposition 4.13 corresponds to case (1), Proposition 4.14 and Corollary 4.15 correspond to case (2) with $m = 2k$, Proposition 4.16 and Corollary 4.17 correspond to case (2) with $m = 2k+1$. Propositions 4.13, 4.14, and 4.16 are the most complicated statements of this paper from the technical viewpoint, therefore we postpone their proofs until Appendix A.

**Proposition 4.13.** *Suppose that $n \geqslant 2$, $1 \leqslant m \leqslant n-1$, $V = \mathbb{F}^n$, $K = \mathrm{SL}_n$, and $X = \mathrm{Gr}_m(V)$. Then there are a point $[W] \in X$ and a connected reductive subgroup $L \subset K_{[W]}$ satisfying the following conditions:*
(1) *the point $[W]$ and the subgroup $L \subset K$ have property* (P)*;*
(2) $L \simeq \mathrm{S}(\mathrm{L}_m \times \mathrm{L}_{n-m})$*;*

(3) *the pair* $(L,W)$ *is geometrically equivalent to the pair* $(\mathrm{GL}_m,\mathbb{F}^m)$*;*
(4) *the pair* $(L,V)$ *is geometrically equivalent to the pair* $(\mathrm{S}(\mathrm{L}_m\times\mathrm{L}_{n-m}),\mathbb{F}^n)$*.*

*Proof.* See Appendix A. □

**Proposition 4.14.** *Suppose that* $n\geqslant 2$*,* $1\leqslant k\leqslant n/2$*,* $V=\mathbb{F}^{2n}$*,* $K=\mathrm{Sp}_{2n}$*, and* $X=\mathrm{Gr}_{2k}(V)$*. Then there are a point* $[W]\in X$ *and a connected reductive subgroup* $L\subset K_{[W]}$ *satisfying the following conditions:*

(1) *the point* $[W]$ *and the subgroup* $L\subset K$ *have property* (P)*;*
(2) $L=L_1\times\ldots\times L_k\times L_{k+1}$*, where* $L_i\simeq\mathrm{SL}_2$ *for* $i=1,\ldots,k$ *and* $L_{k+1}\simeq\mathrm{Sp}_{2n-4k}$*;*
(3) *there is a decomposition* $W=W_1\oplus\ldots\oplus W_k$ *into a direct sum of* $L$*-modules so that:*
  (3.1) $\dim W_i=2$ *for* $i=1,\ldots,k$*;*
  (3.2) *for every* $i=1,\ldots,k$ *the group* $L_i$ *acts trivially on all summands* $W_j$ *with* $j\neq i$*;*
  (3.3) *for every* $i=1,\ldots,k$ *the pair* $(L_i,W_i)$ *is geometrically equivalent to the pair* $(\mathrm{SL}_2,\mathbb{F}^2)$*;*
(4) *there is a decomposition* $V=V_1\oplus\ldots\oplus V_k\oplus V_{k+1}$ *into a direct sum of* $L$*-modules so that:*
  (4.1) $\dim V_i=4$ *for* $i=1,\ldots,k$ *and* $\dim V_{k+1}=2n-4k$*;*
  (4.2) *for every* $i=1,\ldots,k+1$ *the group* $L_i$ *acts trivially on all summands* $V_j$ *with* $j\neq i$*;*
  (4.3) *for every* $i=1,\ldots,k$ *the pair* $(L_i,V_i)$ *is geometrically equivalent to the pair* $(\mathrm{SL}_2,\mathbb{F}^2\oplus\mathbb{F}^2)$*, where* $\mathrm{SL}_2$ *acts diagonally, and the pair* $(L_{k+1},V_{k+1})$ *is geometrically equivalent to the pair* $(\mathrm{Sp}_{2n-4k},\mathbb{F}^{2n-4k})$*.*

(*For* $n=2k$ *the group* $L_{k+1}$ *and the space* $V_{k+1}$ *should be regarded as trivial.*)

*Proof.* See Appendix A. □

**Corollary 4.15.** *In the hypotheses and notation of Proposition* 4.14 *suppose that a point* $[W]\in\mathrm{Gr}_{2k}(V)$ *and a group* $L\subset K_{[W]}$ *satisfy conditions* (1)–(4)*. Then, for the variety* $\mathrm{Gr}_{2n-2k}(V)$*, the point* $[W^\perp]$ *and the group* $L$ *have property* (P)*, and the pair* $(L,W^\perp)$ *is geometrically equivalent to the pair* $(L,W\oplus V_{k+1})$*, where* $L$ *acts diagonally.*

*Proof.* Let $\Omega$ be a $K$-invariant symplectic form on $V$. There is the natural $K$-equivariant isomorphism $\mathrm{Gr}_{2k}(V)\simeq\mathrm{Gr}_{2n-2k}(V)$ taking each $2k$-dimensional subspace of $V$ to its skew-orthogonal complement with respect to $\Omega$. In view of condition (1) this implies that the point $[W^\perp]$ and the group $L$ have property (P). Further, since the $K$-orbit of $[W]$ is open in $\mathrm{Gr}_{2k}(V)$, the restriction of $\Omega$ to the subspace $W$ is nondegenerate. Hence $V=W\oplus W^\perp$, which by conditions (3) and (4) uniquely determines the $L$-module structure on $W^\perp$. □

**Proposition 4.16.** *Suppose that* $n\geqslant 2$*,* $0\leqslant k\leqslant(n-1)/2$*,* $V=\mathbb{F}^{2n}$*,* $K=\mathrm{Sp}_{2n}$*, and* $X=\mathrm{Gr}_{2k+1}(V)$*. Then there are a point* $[W]\in X$ *and a connected reductive subgroup* $L\subset K_{[W]}$ *satisfying the following conditions:*

(1) *the point* $[W]$ *and the subgroup* $L\subset K$ *have property* (P)*;*
(2) $L=L_0\times L_1\times\ldots\times L_k\times L_{k+1}$*, where* $L_0\simeq\mathbb{F}^\times$*,* $L_i\simeq\mathrm{SL}_2$ *for* $i=1,\ldots,k$ *and* $L_{k+1}\simeq\mathrm{Sp}_{2n-4k-2}$*;*
(3) *there is a decomposition* $W=W_0\oplus W_1\oplus\ldots\oplus W_k$ *into a direct sum of* $L$*-modules so that:*

(3.1) $\dim W_0 = 1$ *and* $\dim W_i = 2$ *for* $i = 1, \ldots, k$;

(3.2) *for every* $i = 0, 1, \ldots, k$ *the group* $L_i$ *acts trivially on all summands* $W_j$ *with* $j \neq i$;

(3.3) *the pair* $(L_0, W_0)$ *is geometrically equivalent to the pair* $(\mathbb{F}^\times, \mathbb{F}^1)$ *and for every* $i = 1, \ldots, k$ *the pair* $(L_i, W_i)$ *is geometrically equivalent to the pair* $(\mathrm{SL}_2, \mathbb{F}^2)$;

(4) *there is a decomposition* $V = V_0 \oplus V_1 \oplus \ldots \oplus V_k \oplus V_{k+1}$ *into a direct sum of* $L$*-modules so that:*

(4.1) $\dim V_0 = 2$, $\dim V_i = 4$ *for* $i = 1, \ldots, k$, *and* $\dim V_{k+1} = 2n - 4k - 2$;

(4.2) *for every* $i = 0, 1, \ldots, k+1$ *the group* $L_i$ *acts trivially on all summands* $V_j$ *with* $j \neq i$;

(4.3) *the pair* $(L_0, V_0)$ *is geometrically equivalent to the pair* $(\mathbb{F}^\times, \mathbb{F}^1 \oplus \mathbb{F}^1)$ *with the action* $(t, (x_1, x_2)) \mapsto (tx_1, t^{-1}x_2)$, *for every* $i = 1, \ldots, k$ *the pair* $(L_i, V_i)$ *is geometrically equivalent to the pair* $(\mathrm{SL}_2, \mathbb{F}^2 \oplus \mathbb{F}^2)$ *with* $\mathrm{SL}_2$ *acting diagonally, and the pair* $(L_{k+1}, V_{k+1})$ *is geometrically equivalent to the pair* $(\mathrm{Sp}_{2n-4k-2}, \mathbb{F}^{2n-4k-2})$.

*(For* $n = 2k+1$ *the group* $L_{k+1}$ *and the space* $V_{k+1}$ *should be regarded as trivial.)*

*Proof.* See Appendix A. □

**Corollary 4.17.** *Under the hypotheses and notation of Proposition* 4.16 *suppose that a point* $[W] \in \mathrm{Gr}_{2k+1}(V)$ *and a group* $L \subset K_{[W]}$ *satisfy conditions* (1)–(4)*. Then, for the variety* $\mathrm{Gr}_{2n-2k-1}(V)$, *the point* $[W^\perp]$ *and the group* $L$ *have property* (P) *and the pair* $(L, W^\perp)$ *is geometrically equivalent to the pair* $(L, W \oplus V_{k+1})$, *where* $L$ *acts diagonally.*

*Proof.* Let $\Omega$ be a $K$-invariant symplectic form on $V$. There is a natural $K$-equivariant isomorphism $\mathrm{Gr}_{2k+1}(V) \simeq \mathrm{Gr}_{2n-2k-1}(V)$ taking each $(2k+1)$-dimensional subspace of $V$ to its skew-orthogonal complement with respect to $\Omega$. In view of condition (1) this implies that the point $[W^\perp]$ and the group $L$ have property (P). Further, since the $K$-orbit of $[W]$ is open in $\mathrm{Gr}_{2k+1}(V)$, the restriction of $\Omega$ to the subspace $W$ has rank $2k$. Hence $\dim(W \cap W^\perp) = 1$, which by condition (4) implies $W \cap W^\perp = W_0$. Now the $L$-module structure on $W^\perp$ is uniquely determined by conditions (3) and (4). □

4.6. **Some sphericity conditions for actions on $V$-flag varieties.** Let $K$ be a connected reductive group, $V$ a $K$-module, and $V = V_1 \oplus V_2$ a decomposition of $V$ into a direct sum of two (not necessarily simple) nontrivial $K$-submodules.

**Proposition 4.18.** *Let* $1 \leqslant k \leqslant \dim V_1$, $Z = \mathrm{Gr}_k(V)$, *and* $X = \mathrm{Gr}_k(V_1)$.

(a) *Suppose that* $Z$ *is a* $K$*-spherical variety. Then* $X$ *is also a* $K$*-spherical variety.*

(b) *Suppose that* $X$ *is a* $K$*-spherical variety. Suppose that a point* $[W_0] \in X$ *and a connected reductive subgroup* $L \subset K_{[W_0]}$ *have property* (P)*. Then the following conditions are equivalent:*

(1) $Z$ *is a* $K$*-spherical variety;*

(2) $W_0^* \otimes V_2$ *is a spherical* $L$*-module.*

*Proof.* Let $p$ denote the projection of $V$ to $V_1$ along $V_2$. Let $Z_0 \subset Z$ be the open $K$-stable subset consisting of all points $[U]$ with $\dim p(U) = k$. The projection $p$ induces a surjective $K$-equivariant morphism $\varphi \colon Z_0 \to X$. For each point $[W] \in X$ the fiber $\varphi^{-1}([W])$ consists of all points $[U]$ with $p(U) = W$, whence

$$\varphi^{-1}([W]) \simeq \mathrm{Hom}(W, V_2) \simeq W^* \otimes V_2.$$

It is easy to see that the $K$-sphericity of $Z_0$ implies that of $X$, which proves part (a). To complete the proof of part (b), it remains to make use of property (P) for the point $[W_0]$ and the group $L$. ☐

*Remark* 4.19. It is easy to see that $X$ is realized as a $K$-stable subvariety of $Z$. Then Proposition 4.18(a) is also implied by the following well-known fact: every irreducible $K$-stable subvariety of a spherical $K$-variety is spherical (see, for instance, [Tim, Proposition 15.14]).

**Corollary 4.20.** *Suppose that $k = \dim V_1$ and $Z = \mathrm{Gr}_k(V)$. Then the following conditions are equivalent:*
(1) *$Z$ is a $K$-spherical variety;*
(2) *$V_1^* \otimes V_2$ is a spherical $K$-module.*

*Proof.* In this situation $X$ consists of the single point $[V_1]$. Evidently, this point and the group $K$ have property (P). ☐

**Proposition 4.21.** *Suppose that $\dim V \geqslant 4$ and $Z = \mathrm{Gr}_2(V)$ is a $K$-spherical variety. Then $V_2 \otimes \mathbb{F}^2$ is a spherical $(K \times \mathrm{GL}_2)$-module.*

*Proof.* We first consider the case $\dim V_1 \geqslant 2$. Put $X = \mathrm{Gr}_2(V_1)$. Proposition 4.18(a) yields that $X$ is a spherical $K$-variety. By Theorem 4.12 there are a point $x \in X$ and a connected reductive subgroup $L \subset K_x$ having property (P). Then Proposition 4.18(b) implies that $V_2 \otimes \mathbb{F}^2$ is a spherical $L$-module, hence a spherical $(K \times \mathrm{GL}_2)$-module.

We now consider the case $\dim V_1 = 1$. Then $\dim V_2 \geqslant 3$. For each two-dimensional subspace $W \subset V$ put $W_1 = W \cap V_2$ and let $W_2$ denote the projection of $W$ to $V_2$ along $V_1$.

Let $Z_0 \subset Z$ be the open $K$-stable subset consisting of all points $[W]$ with $\dim W_1 = 1$ and $\dim W_2 = 2$. It is easy to see that the morphism

$$Z_0 \to \mathrm{Fl}(1,1;V_2), \quad [W] \mapsto (W_1, W_2, V_2),$$

is surjective and $K$-equivariant, hence the $K$-sphericity of $\mathrm{Gr}_2(V)$ implies that of $\mathrm{Fl}(1,1;V_2)$. Then Corollary 4.5 implies that $V_2 \otimes \mathbb{F}^2$ is a spherical $(K \times \mathrm{GL}_2)$-module. ☐

**Proposition 4.22.** *Let $k_1 \geqslant 1$, $k_2 \geqslant 1$, $k_1 + k_2 \leqslant \dim V_1$. Put $Z = \mathrm{Fl}(k_1,k_2;V)$, $X = \mathrm{Gr}_{k_1+k_2}(V_1)$ and suppose that $X$ is a $K$-spherical variety. Suppose that a point $[W_0] \in X$ and a connected reductive subgroup $L \subset K_{[W_0]}$ have property* (P). *Then the following conditions are equivalent:*
(1) *$Z$ is a $K$-spherical variety;*
(2) *$(W_0^* \otimes V_2) \times \mathrm{Gr}_{k_1}(W_0)$ is an $L$-spherical variety ($L$ acts diagonally).*

*Proof.* Let $p$ denote the projection of $V$ to $V_1$ along $V_2$. Let $Z_0 \subset Z$ be the open $K$-stable subset consisting of all points $(U_1, U_2, V) \in Z$ with $\dim p(U_2) = k_1 + k_2$. Then $p$ induces the surjective $K$-equivariant morphism $\varphi\colon Z_0 \to X$ taking a point $(U_1, U_2, V)$ to $p(U_2)$. For every point $[W] \in X$, the fiber $\varphi^{-1}([W])$ is isomorphic to

$$\mathrm{Hom}(W, V_2) \times \mathrm{Gr}_{k_1}(W) \simeq (W^* \otimes V_2) \times \mathrm{Gr}_{k_1}(W).$$

To complete the proof it remains to make use of property (P) for $[W_0]$ and $L$. ☐

## 5. Known classifications used in the paper

5.1. **Classification of spherical modules.** In this subsection we present the classification of spherical modules obtained in the papers [Kac], [BR], and [Lea].

Let $K$ be a connected reductive group and let $C$ be the connected component of the identity of the center of $K$. For every simple $K$-module $V$ we consider the character $\chi \in \mathfrak{X}(C)$ via which $C$ acts on $V$.

**Theorem 5.1.** [Kac, Theorem 3] *A simple $K$-module $V$ is spherical if and only if the following conditions hold:*
(1) *up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 2;*
(2) *the group $C$ satisfies the conditions listed in the fourth column of Table 2.*

Table 2

| No. | $K'$ | $V$ | Conditions on $C$ | Note |
|---|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | $\chi \neq 0$ for $n = 1$ | $n \geqslant 1$ |
| 2 | $\mathrm{SO}_n$ | $\mathbb{F}^n$ | $\chi \neq 0$ | $n \geqslant 3$ |
| 3 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | | $n \geqslant 2$ |
| 4 | $\mathrm{SL}_n$ | $\mathrm{S}^2\mathbb{F}^n$ | $\chi \neq 0$ | $n \geqslant 3$ |
| 5 | $\mathrm{SL}_n$ | $\wedge^2\mathbb{F}^n$ | $\chi \neq 0$ for $n = 2k$ | $n \geqslant 5$ |
| 6 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \otimes \mathbb{F}^m$ | $\chi \neq 0$ for $n = m$ | $n, m \geqslant 2$<br>$n + m \geqslant 5$ |
| 7 | $\mathrm{SL}_2 \times \mathrm{Sp}_{2n}$ | $\mathbb{F}^2 \otimes \mathbb{F}^{2n}$ | $\chi \neq 0$ | $n \geqslant 2$ |
| 8 | $\mathrm{SL}_3 \times \mathrm{Sp}_{2n}$ | $\mathbb{F}^3 \otimes \mathbb{F}^{2n}$ | $\chi \neq 0$ | $n \geqslant 2$ |
| 9 | $\mathrm{SL}_n \times \mathrm{Sp}_4$ | $\mathbb{F}^n \otimes \mathbb{F}^4$ | $\chi \neq 0$ for $n = 4$ | $n \geqslant 4$ |
| 10 | $\mathrm{Spin}_7$ | $\mathbb{F}^8$ | $\chi \neq 0$ | |
| 11 | $\mathrm{Spin}_9$ | $\mathbb{F}^{16}$ | $\chi \neq 0$ | |
| 12 | $\mathrm{Spin}_{10}$ | $\mathbb{F}^{16}$ | | |
| 13 | $\mathsf{G}_2$ | $\mathbb{F}^7$ | $\chi \neq 0$ | |
| 14 | $\mathsf{E}_6$ | $\mathbb{F}^{27}$ | $\chi \neq 0$ | |

Let us give some comments and explanations for Table 2. In rows 3–8 the restrictions in the column "Note" are imposed in order to avoid coincidences (up to a geometric equivalence) of the respective $K'$-modules with $K'$-modules corresponding to other rows. In rows 10 and 11, the group $K'$ acts on $V$ via the spin representation. In row 12, the group $K'$ acts on $V$ via a (any of the two) half-spin representation. At last, in rows 13 and 14 the group $K'$ acts on $V$ via a faithful representation of minimal dimension.

Let $V$ be a $K$-module. We say that $V$ is *decomposable* if there exist connected reductive subgroups $K_1, K_2$, a $K_1$-module $V_1$, and a $K_2$-module $V_2$ such that the pair $(K, V)$ is geometrically equivalent to the pair $(K_1 \times K_2, V_1 \oplus V_2)$. We note that in this situation $V_1 \oplus V_2$ is a spherical $(K_1 \times K_2)$-module if and only if $V_1$ is a spherical $K_1$-module and $V_2$ is a spherical $K_2$-module. We say that $V$ is *indecomposable* if $V$ is not decomposable. At last, we say that $V$ is *strictly indecomposable* if $V$ is an indecomposable $K'$-module. Evidently, every simple $K$-module is strictly indecomposable.

Let $V$ be a $K$-module and let $V = V_1 \oplus \ldots \oplus V_r$ be a decomposition of $V$ into a direct sum of simple $K$-submodules. For every $i = 1, \ldots, r$ we denote by $\chi_i$ the character of $C$ via which $C$ acts on $V_i$.

**Theorem 5.2** ([BR], [Lea])**.** *In the above notation suppose that $V$ is a nonsimple $K$-module. Then $V$ is a strictly indecomposable spherical $K$-module if and only if $r = 2$ and the following conditions hold:*

(1) *up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 3;*

(2) *the group $C$ satisfies the conditions listed in the third column of Table 3.*

Let us explain some notation in Table 3. In the second column the pair $(K', V)$ is arranged in two levels, with $K'$ in the upper level and $V$ in the lower one. Further, each factor of $K'$ acts diagonally on all components of $V$ with which it is connected by an edge. In row 12 the symbols $\mathbb{F}^8_\pm$ stand for the spaces of the two half-spin representations of $\mathrm{Spin}_8$.

Let $V$ be a $K$-module such that, up to a geometrical equivalence, the pair $(K', V)$ is contained in one of Tables 2 or 3. Using the information in the column "Conditions on $C$", to $V$ we assign a multiset (that is, a set whose members are considered together with their multiplicities) $I(V)$ consisting of several characters of $C$ in the following way:

(1) $I(V) = \varnothing$ if there are no conditions on $C$;

(2) $I(V) = \{\psi_1 - \psi_2\}$ if the condition on $C$ is of the form "$\psi_1 \ne \psi_2$" for some $\psi_1, \psi_2 \in \mathfrak{X}(C)$;

(3) $I(V) = \{\chi_1, \chi_2\}$ if the condition on $C$ is of the form "$\chi_1, \chi_2$ lin. ind."

In the above notation, $V$ is a spherical $K$-module if and only if all characters in $I(V)$ are linearly independent in $\mathfrak{X}(C)$.

Let $V$ be an arbitrary $K$-module. It is easy to see that there are a decomposition $V = W_1 \oplus \ldots \oplus W_p$ into a direct sum of $K$-submodules (not necessarily simple) and connected semisimple normal subgroups $K_1, \ldots, K_p \subset K'$ (some of them are allowed to be trivial) with the following properties:

(1) $W_i$ is a strictly indecomposable $K$-module for all $i = 1, \ldots, p$;

(2) the pair $(K', V)$ is geometrically equivalent to the pair

$$(K_1 \times \ldots \times K_p, W_1 \oplus \ldots \oplus W_p).$$

The theorem below provides a sphericity criterion for the $K$-module $V$. This theorem is a reformulation of [BR, Theorem 7]; see also [Lea, Theorem 2.6].

**Theorem 5.3.** *In the above notation, $V$ is a spherical $K$-module if and only if the following conditions hold:*

(1) *$W_i$ is a spherical $K$-module for all $i = 1, \ldots, p$;*

(2) *all the $|I(W_1)| + \ldots + |I(W_p)|$ characters in the multiset $I(W_1) \cup \ldots \cup I(W_p)$ are linearly independent in $\mathfrak{X}(C)$.*

5.2. **Classification of Levi subgroups in $\mathrm{GL}(V)$ acting spherically on $V$-flag varieties.** Let $G$ be an arbitrary connected reductive group. Let $P, Q \subset G$ be parabolic subgroups and let $K$ be a Levi subgroup of $P$.

The following lemma is known to specialists; for the reader's convenience we provide it together with a proof.

**Lemma 5.4.** *The following conditions are equivalent:*

(a) *$G/Q$ is a $K$-spherical variety;*

(b) *$G/P \times G/Q$ is a spherical variety with respect to the diagonal action of $G$.*

TABLE 3

| No. | $(K', V)$ | Conditions on $C$ | Note |
|---|---|---|---|
| 1 | $\mathrm{SL}_n$<br>$\mathbb{F}^n \oplus \mathbb{F}^n$ | $\chi_1, \chi_2$ lin. ind. for $n = 2$;<br>$\chi_1 \neq \chi_2$ for $n \geqslant 3$ | $n \geqslant 2$ |
| 2 | $\mathrm{SL}_n$<br>$(\mathbb{F}^n)^* \oplus \mathbb{F}^n$ | $\chi_1 \neq -\chi_2$ | $n \geqslant 3$ |
| 3 | $\mathrm{SL}_n$<br>$\mathbb{F}^n \oplus \wedge^2 \mathbb{F}^n$ | $\chi_2 \neq 0$ for $n = 2k$<br>$\chi_1 \neq -\frac{n-1}{2}\chi_2$ for $n = 2k+1$ | $n \geqslant 4$ |
| 4 | $\mathrm{SL}_n$<br>$(\mathbb{F}^n)^* \oplus \wedge^2 \mathbb{F}^n$ | $\chi_2 \neq 0$ for $n = 2k$<br>$\chi_1 \neq \frac{n-1}{2}\chi_2$ for $n = 2k+1$ | $n \geqslant 4$ |
| 5 | $\mathrm{SL}_n \times \mathrm{SL}_m$<br>$\mathbb{F}^n \oplus (\mathbb{F}^n \otimes \mathbb{F}^m)$ | $\chi_1 \neq 0$ for $n \leqslant m-1$<br>$\chi_1, \chi_2$ lin. ind. for $n = m, m+1$<br>$\chi_1 \neq \chi_2$ for $n \geqslant m+2$ | $n, m \geqslant 2$ |
| 6 | $\mathrm{SL}_n \times \mathrm{SL}_m$<br>$(\mathbb{F}^n)^* \oplus (\mathbb{F}^n \otimes \mathbb{F}^m)$ | $\chi_1 \neq 0$ for $n \leqslant m-1$<br>$\chi_1, \chi_2$ lin. ind. for $n = m, m+1$<br>$\chi_1 \neq -\chi_2$ for $n \geqslant m+2$ | $n \geqslant 3, m \geqslant 2$ |
| 7 | $\mathrm{SL}_2 \times \mathrm{Sp}_{2n}$<br>$\mathbb{F}^2 \oplus (\mathbb{F}^2 \otimes \mathbb{F}^{2n})$ | $\chi_1, \chi_2$ lin. ind. | $n \geqslant 2$ |
| 8 | $\mathrm{SL}_n \times \mathrm{SL}_2 \times \mathrm{SL}_m$<br>$(\mathbb{F}^n \otimes \mathbb{F}^2) \oplus (\mathbb{F}^2 \otimes \mathbb{F}^m)$ | $\chi_1, \chi_2$ lin. ind. for $n = m = 2$<br>$\chi_2 \neq 0$ for $n \geqslant 3$ and $m = 2$ | $n \geqslant m \geqslant 2$ |
| 9 | $\mathrm{SL}_n \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m}$<br>$(\mathbb{F}^n \otimes \mathbb{F}^2) \oplus (\mathbb{F}^2 \otimes \mathbb{F}^{2m})$ | $\chi_1, \chi_2$ lin. ind. for $n = 2$<br>$\chi_2 \neq 0$ for $n \geqslant 3$ | $n, m \geqslant 2$ |
| 10 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_2 \times \mathrm{Sp}_{2m}$<br>$(\mathbb{F}^{2n} \otimes \mathbb{F}^2) \oplus (\mathbb{F}^2 \otimes \mathbb{F}^{2m})$ | $\chi_1, \chi_2$ lin. ind. | $n \geqslant m \geqslant 2$ |
| 11 | $\mathrm{Sp}_{2n}$<br>$\mathbb{F}^{2n} \oplus \mathbb{F}^{2n}$ | $\chi_1, \chi_2$ lin. ind. | $n \geqslant 2$ |
| 12 | $\mathrm{Spin}_8$<br>$\mathbb{F}^8_+ \oplus \mathbb{F}^8_-$ | $\chi_1, \chi_2$ lin. ind. | |

*Proof.* It is well known that there is a Borel subgroup $B \subset G$ with the following properties:

(1) the set $BP$ is open in $G$;

(2) the group $B_K = B \cap P$ is a Borel subgroup of $K$.

For the action $B : G/P$, the group $B_K$ is exactly the stabilizer of the point $o = eP$ and the orbit $O = Bo \simeq B/B_K$ is open. Consider the open subset $O \times G/Q$ in $G/P \times G/Q$.

This subset is a homogeneous bundle over $B/B_K$ with fiber $G/Q$, see § 4.2. Applying Corollary 4.3 we find that the existence of an open $B$-orbit in $O \times G/Q$ is equivalent to the existence of an open $B_K$-orbit in $G/Q$, which implies the required result. □

As was already mentioned in Introduction, there is a complete classification of all $G$-spherical varieties of the form $G/P \times G/Q$. Below, using Lemma 5.4, we reformulate the results of this classification in the case $G = \mathrm{GL}(V)$ and thereby list all cases where a Levi subgroup $K \subset \mathrm{GL}(V)$ acts spherically on a $V$-flag variety $\mathrm{Fl}_{\mathbf{a}}(V)$ (see Theorem 5.5).

Let $V$ be a vector space of dimension $d$ and let $\mathbf{d} = (d_1, \ldots, d_r)$ be a nontrivial composition of $d$ such that $d_1 \leqslant \ldots \leqslant d_r$. Fix a decomposition

$$V = V_1 \oplus \ldots \oplus V_r$$

into a direct sum of subspaces, where $\dim V_i = d_i$ for all $i = 1, \ldots, r$. Put

$$K_{\mathbf{d}} = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_r) \subset \mathrm{GL}(V).$$

Let $Q$ denote the stabilizer in $\mathrm{GL}(V)$ of the point

$$(V_1, V_1 \oplus V_2, \ldots, V_1 \oplus \ldots \oplus V_r) \in \mathrm{Fl}_{\mathbf{d}}(V).$$

Clearly, $Q$ is a parabolic subgroup of $\mathrm{GL}(V)$ and $K_{\mathbf{d}}$ is a Levi subgroup of $Q$. It is well known that every Levi subgroup of $\mathrm{GL}(V)$ is conjugate to a subgroup of the form $K_{\mathbf{d}}$.

The following theorem follows from results of the paper [MWZ1], see also [Stem, Corollary 1.3.A].

**Theorem 5.5.** *Suppose that $\mathbf{a} = (a_1, \ldots, a_s)$ is a nontrivial composition of $d$ such that $a_1 \leqslant \ldots \leqslant a_s$. Then the variety $\mathrm{Fl}_{\mathbf{a}}(V)$ is $K_{\mathbf{d}}$-spherical if and only if the pair of compositions $(\mathbf{d}, \mathbf{a})$ is contained in Table 4.*

Table 4

| No. | $\mathbf{d}$ | $\mathbf{a}$ |
|---|---|---|
| 1 | $(d_1, d_2)$ | $(a_1, a_2)$ |
| 2 | $(2, d_2)$ | $(a_1, a_2, a_3)$ |
| 3 | $(d_1, d_2, d_3)$ | $(2, a_2)$ |
| 4 | $(d_1, d_2)$ | $(1, a_2, a_3)$ |
| 5 | $(1, d_2, d_3)$ | $(a_1, a_2)$ |
| 6 | $(1, d_2)$ | $(a_1, \ldots, a_s)$ |
| 7 | $(d_1, \ldots, d_r)$ | $(1, a_2)$ |

## 6. Proof of theorem 1.7

We divide the proof of Theorem 1.7 into several steps. As follows from Proposition 3.9, the first step of the proof is a description of all spherical actions on the variety $\mathrm{Gr}_2(V)$. This is done in § 6.1 (in the case where $V$ is a simple $K$-module) and § 6.2 (in the case where $V$ is a nonsimple $K$-module). At the next step we classify all spherical actions on arbitrary Grassmannians, see § 6.3. Finally, in § 6.4 we list all spherical actions on $V$-flag varieties that are not Grassmannians.

We recall that the statement of Theorem 1.7 includes the following objects:

$V$ is a vector space of dimension $d$;

$K$ is a connected reductive subgroup of $\mathrm{GL}(V)$;
$C$ is the connected component of the identity of the center of $K$.
Next, there is a decomposition

$$V = V_1 \oplus \ldots \oplus V_r \tag{6.1}$$

into a direct sum of simple $K$-submodules and for every $i = 1, \ldots, r$ the group $C$ acts on $V_i$ via a character denoted by $\chi_i$.

We fix all the above-mentioned objects and notation until the end of this section.

6.1. **Spherical actions on $\mathrm{Gr}_2(V)$ in the case where $V$ is a simple $K$-module.** The goal of this subsection is to prove the following theorem.

**Theorem 6.1.** *Suppose that $d \geqslant 4$ and $V$ is a simple $K$-module. Then the variety $\mathrm{Gr}_2(V)$ is $K$-spherical if and only if, up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 5.*

Table 5

| No. | $K'$ | $V$ | Note |
|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | $n \geqslant 4$ |
| 2 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | $n \geqslant 2$ |
| 3 | $\mathrm{SO}_n$ | $\mathbb{F}^n$ | $n \geqslant 4$ |
| 4 | $\mathrm{Spin}_7$ | $\mathbb{F}^8$ | |

*Proof.* First of all we note that, since $V$ is a simple $K$-module, the center of $K$ acts trivially on $\mathrm{Gr}_2(V)$. Therefore the variety $\mathrm{Gr}_2(V)$ is $K$-spherical if and only if it is $K'$-spherical.

If $\mathrm{Gr}_2(V)$ is a spherical $K'$-variety then $V$ is a spherical $(K' \times \mathbb{F}^\times)$-module by Corollary 3.8. Theorem 5.1 implies that, up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 2.

As was already mentioned in Introduction, every flag variety of the group $\mathrm{SL}_n$ is spherical. Thus for every $n \geqslant 4$ the variety $\mathrm{Gr}_2(\mathbb{F}^n)$ is $\mathrm{SL}_n$-spherical.

The variety $\mathrm{Gr}_2(\mathbb{F}^{2n})$ is $\mathrm{Sp}_{2n}$-spherical for $n \geqslant 2$ by Proposition 4.8.

The variety $\mathrm{Gr}_2(\mathbb{F}^n)$ is $\mathrm{SO}_n$-spherical for $n \geqslant 4$ by Proposition 4.9.

Let us show that the variety $\mathrm{Gr}_2(\mathbb{F}^8)$ is $\mathrm{Spin}_7$-spherical. It is known (see [Ela, Table 6, row 4] or [SK, § 5, Proposition 26]) that under the natural action of the group $\mathrm{Spin}_7 \times \mathrm{GL}_2$ on $\mathbb{F}^8 \otimes \mathbb{F}^2$ there is an open orbit and the Lie algebra of the stabilizer of any point of this orbit is isomorphic to $\mathfrak{gl}_3$. Applying Corollary 4.6, we find that under the action of the group $\mathrm{Spin}_7$ on $\mathrm{Gr}_2(\mathbb{F}^8)$ there is an open orbit $O$ and the Lie algebra of the stabilizer of any point $x \in O$ is still isomorphic to $\mathfrak{gl}_3$. It follows that the connected component of the identity of the stabilizer of any point $x \in O$ is isomorphic to $\mathrm{GL}_3$. It is well known (see, for instance, [Krä, Table 1]) that $\mathrm{GL}_3$ is a spherical subgroup of $\mathrm{Spin}_7$.

We now prove that the variety $\mathrm{Gr}_2(V)$ is not $K'$-spherical for the pairs $(K', V)$ in rows 4–9, 11–14 of Table 2.

Applying Proposition 4.1, we obtain the following necessary condition for $\mathrm{Gr}_2(V)$ to be $K'$-spherical:

$$\dim K' + \operatorname{rk} K' \geqslant 4d - 8. \tag{6.2}$$

A direct check shows that inequality (6.2) does not hold for the pairs $(K', V)$ in rows 4, 5, 11–14 of Table 2.

**Lemma 6.2.** *Suppose that $L$ is a connected reductive subgroup of $\mathrm{SL}_n \times \mathrm{SL}_m$, where $n \geqslant m \geqslant 2$. Then the variety $\mathrm{Gr}_2(\mathbb{F}^n \otimes \mathbb{F}^m)$ is $L$-spherical if and only if $n = m = 2$ and $L = \mathrm{SL}_2 \times \mathrm{SL}_2$.*

*Proof.* If $n = m = 2$ and $L = \mathrm{SL}_2 \times \mathrm{SL}_2$, then the pair $(L, \mathbb{F}^2 \otimes \mathbb{F}^2)$ is geometrically equivalent to the pair $(\mathrm{SO}_4, \mathbb{F}^4)$. By Proposition 4.9 the action of $\mathrm{SO}_4$ on $\mathrm{Gr}_2(\mathbb{F}^4)$ is spherical.

We now prove that the $L$-sphericity of the variety $\mathrm{Gr}_2(\mathbb{F}^n \otimes \mathbb{F}^m)$ implies $n = m = 2$. Obviously, if $\mathrm{Gr}_2(\mathbb{F}^n \otimes \mathbb{F}^m)$ is $L$-spherical then it is also $(\mathrm{SL}_n \times \mathrm{SL}_m)$-spherical. Therefore it suffices to prove the required assertion in the case $L = \mathrm{SL}_n \times \mathrm{SL}_m$.

We divide our subsequent consideration into two cases.

*Case* 1. $n > 2m$. We show that in this case the variety $\mathrm{Gr}_2(\mathbb{F}^n \otimes \mathbb{F}^m)$ is not $(\mathrm{SL}_n \times \mathrm{SL}_m)$-spherical. For $k = n, m$ let $T_k$ denote the group of all upper-triangular matrices in $\mathrm{GL}_k$ and put $B_k = T_k \cap \mathrm{SL}_k$, so that $B_k$ is a Borel subgroup of $\mathrm{SL}_k$. In view of Proposition 4.4 it suffices to prove that the space $W = \mathbb{F}^n \otimes \mathbb{F}^m \otimes \mathbb{F}^2$ contains no open orbit for the action of $B_n \times B_m \times \mathrm{GL}_2$ or, equivalently, for the action of $T_n \times T_m \times \mathrm{SL}_2$. In turn, the latter will hold if we prove that the codimension of an orbit of general position for the action of $T_n \times T_m$ on $W$ is at least 4.

We represent $W$ as the set of pairs of $(n \times m)$-matrices. Then the action $T_n \times T_m : W$ is described by the formula $((T, U), (P, Q)) \mapsto (TPU^\top, TQU^\top)$, where $(T, U) \in T_n \times T_m$, $(P, Q) \in W$. It is not hard to check that, acting by the group $T_n \times T_m$, one can reduce a pair $(P, Q)$ from a suitable open subset of $W$ to a uniquely determined canonical form $(P', Q')$, where

$$P' = \left(\begin{array}{c} 0 \\ \hline \\ 0 \\ \\ \hline \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right), \qquad Q' = \left(\begin{array}{c} 0 \\ \hline \begin{matrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & * & 1 \end{matrix} \\ \hline \\ * \\ \\ \end{array}\right).$$

(Each of the matrices $P', Q'$ is divided into three blocks: the upper one consists of the upper $n - 2m$ rows, the middle one consists of the following $m$ rows, and the lower one consists of the last $m$ rows.) It follows that the codimension of a $(T_n \times T_m)$-orbit of general position in $W$ equals $m^2 + (m-1)(m-2)/2 \geqslant m^2 \geqslant 4$ as required.

*Case* 2. $m \leqslant n \leqslant 2m$. Suppose that $n = 2m - l$, where $0 \leqslant l \leqslant m$. If the variety $\mathrm{Gr}_2(\mathbb{F}^n \otimes \mathbb{F}^m)$ is $(\mathrm{SL}_n \times \mathrm{SL}_m)$-spherical, then applying (6.2) we get the inequality

$$(2m - l)^2 - 1 + m^2 - 1 + (2m - l) - 1 + m - 1 \geqslant 4(2m - l)m - 8,$$

which takes the form

$$3m^2 - 3m - 4 \leqslant l^2 - l \tag{6.3}$$

after transformations. Since $0 \leqslant l \leqslant m$, inequality (6.3) implies that $3m^2 - 3m - 4 \leqslant m^2$. Hence $2m^2 - 3m - 4 \leqslant 0$ and so $m = 2$. Then $l = 0, 1$ or $2$, but the first two cases do not occur by (6.3). Thus $l = 2$, that is, $n = 2$.

It remains to show that the action $L : \mathrm{Gr}_2(\mathbb{F}^2 \otimes \mathbb{F}^2)$ is not spherical for every proper reductive subgroup $L \subset \mathrm{SL}_2 \times \mathrm{SL}_2$. Indeed, in this case we have $\operatorname{rk} L \leqslant 2$ and $\dim L \leqslant 5$, hence inequality (6.2) does not hold.

The proof of the lemma is completed. □

Lemma 6.2 immediately implies that the variety $\mathrm{Gr}_2(V)$ is not $K'$-spherical for the pairs $(K', V)$ in rows 6–9 of Table 2, which completes the proof of Theorem 6.1. □

6.2. **Spherical actions on $\mathrm{Gr}_2(V)$ in the case where $V$ is a nonsimple $K$-module.** In this subsection we suppose that $r \geqslant 2$. Here the main result is Theorem 6.5.

**Proposition 6.3.** *Suppose that $\mathrm{Gr}_2(V)$ is a spherical $K$-variety. Then for every $i = 1, \ldots, r$ the pair $(K', V_i)$ is geometrically equivalent to either of the pairs* $(\mathrm{SL}_n, \mathbb{F}^n)$ $(n \geqslant 1)$ *or* $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$ $(n \geqslant 2)$.

*Proof.* It follows from Proposition 4.21 that $V_i \otimes \mathbb{F}^2$ is a spherical $(K \times \mathrm{GL}_2)$-module for every $i = 1, \ldots, r$. The proof is completed by applying Theorem 5.1. □

**Proposition 6.4.** *Suppose that $\mathrm{Gr}_2(V)$ is a spherical $K$-variety. Then every simple normal subgroup in $K'$ acts nontrivially on at most one summand of decomposition* (6.1).

*Proof.* Assume that there is a simple normal subgroup $K_0 \subset K'$ acting nontrivially on two different summands of decomposition (6.1). Without loss of generality we shall assume that $K_0$ acts nontrivially on $V_1$ and $V_2$. (We note that $\dim V_1 \geqslant 2$ and $\dim V_2 \geqslant 2$ in this case.) Proposition 4.18(a) implies that the variety $\mathrm{Gr}_2(V_1 \oplus V_2)$ is $K$-spherical. Making use of Proposition 6.3, we find that, up to a geometrical equivalence, the pair $(K', V_1 \oplus V_2)$ is contained in Table 6, where $K'$ is assumed to act diagonally on $V_1$ and $V_2$ in all cases.

Table 6

| No. | $K'$ | $V_1$ | $V_2$ | Note |
|---|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | $\mathbb{F}^n$ | $n \geqslant 2$ |
| 2 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | $(\mathbb{F}^n)^*$ | $n \geqslant 3$ |
| 3 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | $\mathbb{F}^{2n}$ | $n \geqslant 2$ |

For each case in Table 6 we put $Z = \mathrm{Gr}_2(V_1 \oplus V_2)$ and $X = \mathrm{Gr}_2(V_1)$. Let us show that the variety $Z$ is not $K$-spherical. We consider all the three cases separately.

*Case* 1. If $n = 2$ then, by Corollary 4.20, $Z$ is a $K$-spherical variety if and only if $V_1^* \otimes V_2$ is a spherical $(C \times K')$-module, where $K$ acts diagonally and $C$ acts via the character $\chi_2 - \chi_1$. It follows that $\mathbb{F}^2 \otimes \mathbb{F}^2$ is a spherical $(\mathrm{SL}_2 \times \mathbb{F}^\times)$-module, where $\mathrm{SL}_2$ acts diagonally and $\mathbb{F}^\times$ acts by scalar transformations. The latter is false since the indicated module does not satisfy inequality (4.1).

In what follows we suppose that $n \geqslant 3$. Applying Proposition 4.13 to $X$ and then Proposition 4.18(b) to $Z$ and $X$, we find a point $[W] \in X$ and a group $L \subset (K')_{[W]}$ with the following properties:

(1) $L \simeq \mathrm{S}(\mathrm{L}_2 \times \mathrm{L}_{n-2})$;

(2) the pair $(L, W)$ is geometrically equivalent to the pair $(\mathrm{GL}_2, \mathbb{F}^2)$;

(3) the pair $(L, V_2)$ is geometrically equivalent to the pair $(\mathrm{S}(\mathrm{L}_2 \times \mathrm{L}_{n-2}), \mathbb{F}^2 \oplus \mathbb{F}^{n-2})$;

(4) the $K$-sphericity of $Z$ is equivalent to the sphericity of the $(C \times L)$-module $W^* \otimes V_2$, where $L$ acts diagonally and $C$ acts via the character $\chi_2 - \chi_1$.

In view of the $\mathrm{SL}_2$-module isomorphisms $(\mathbb{F}^2)^* \simeq \mathbb{F}^2$ and $\mathbb{F}^2 \otimes \mathbb{F}^2 \simeq \mathrm{S}^2\mathbb{F}^2 \oplus \mathbb{F}^1$, the sphericity of the $(C \times L)$-module $W^* \otimes V_2$ implies that the $(\mathrm{SL}_2 \times \mathrm{SL}_{n-2} \times (\mathbb{F}^\times)^2)$-module

$$\mathrm{S}^2\mathbb{F}^2 \oplus (\mathbb{F}^2 \otimes \mathbb{F}^{n-2})$$

is spherical, where $\mathrm{SL}_2$ acts diagonally on $\mathrm{S}^2\mathbb{F}^2$ and $\mathbb{F}^2$, $\mathrm{SL}_{n-2}$ acts on $\mathbb{F}^{n-2}$, and $(\mathbb{F}^\times)^2$ acts on each of the direct summands by scalar transformations. By Theorem 5.2 the indicated module is not spherical.

*Case* 2. Using an argument similar to that in Case 1 for $n \geqslant 3$ we deduce from the condition of $Z$ being $K$-spherical that the $(\mathrm{SL}_2 \times \mathrm{SL}_{n-2} \times (\mathbb{F}^\times)^2)$-module

$$\mathrm{S}^2\mathbb{F}^2 \oplus (\mathbb{F}^2 \otimes (\mathbb{F}^{n-2})^*)$$

is spherical, where $\mathrm{SL}_2$ acts diagonally on $\mathrm{S}^2\mathbb{F}^2$ and $\mathbb{F}^2$, $\mathrm{SL}_{n-2}$ acts on $(\mathbb{F}^{n-2})^*$, and $(\mathbb{F}^\times)^2$ acts on each of the direct summands by scalar transformations. By Theorem 5.2 the indicated module is not spherical.

*Case* 3. If the variety $Z$ is $K$-spherical then $Z$ is also $(C \times \mathrm{SL}_{2n})$-spherical (where $\mathrm{SL}_{2n}$ acts diagonally on $V_1 \oplus V_2$). As was shown in Case 1, the latter is false. □

**Theorem 6.5.** *Suppose that $d \geqslant 4$ and $r \geqslant 2$. Then the variety $\mathrm{Gr}_2(V)$ is $K$-spherical if and only if the following conditions hold:*

(1) *up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 7;*

(2) *the group $C$ satisfies the conditions listed in the fourth column of Table 7.*

TABLE 7

| No. | $K'$ | $V$ | Conditions on $C$ | Note |
|---|---|---|---|---|
| 1 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m$ | $\chi_1 \neq \chi_2$ for $n = m = 2$ | $n \geqslant m \geqslant 1$, $n + m \geqslant 4$ |
| 2 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m$ | $\chi_1 \neq \chi_2$ for $m = 2$ | $n \geqslant 2$, $m \geqslant 1$ |
| 3 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m}$ | $\chi_1 \neq \chi_2$ | $n \geqslant m \geqslant 2$ |
| 4 | $\mathrm{SL}_n \times \mathrm{SL}_m \times \mathrm{SL}_l$ | $\mathbb{F}^n \oplus \mathbb{F}^m \oplus \mathbb{F}^l$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$ lin. ind. for $n = 2$; <br> $\chi_2 \neq \chi_3$ for $n \geqslant 3$, $m \leqslant 2$ | $n \geqslant m \geqslant l \geqslant 1$, $n \geqslant 2$ |
| 5 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m \times \mathrm{SL}_l$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m \oplus \mathbb{F}^l$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$ lin. ind. for $m \leqslant 2$ <br> $\chi_1 \neq \chi_3$ for $m \geqslant 3, l \leqslant 2$ | $n \geqslant 2$, $m \geqslant l \geqslant 1$ |
| 6 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{SL}_l$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m} \oplus \mathbb{F}^l$ | $\chi_2 - \chi_1, \chi_3 - \chi_1$ lin. ind. for $l \leqslant 2$ <br> $\chi_1 \neq \chi_2$ for $l \geqslant 3$ | $n \geqslant m \geqslant 2$, $l \geqslant 1$ |
| 7 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m} \times \mathrm{Sp}_{2l}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m} \oplus \mathbb{F}^{2l}$ | $\chi_1 - \chi_2, \chi_1 - \chi_3$ lin. ind. | $n \geqslant m \geqslant l \geqslant 2$ |

*Proof.* Put $U = V_2 \oplus \ldots \oplus V_r$. Let $K_1$ (resp. $K_2$) be the image of $K'$ in $\mathrm{GL}(V_1)$ (resp. $\mathrm{GL}(U)$).

If the variety $\mathrm{Gr}_2(V)$ is spherical with respect to the action of $K$, then it is also spherical with respect to the action of $\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_r)$. Then it follows from Theorem 5.5 that

TABLE 8

| Case | References | $(M, W)$ |
|---|---|---|
| 1, $n{=}m{=}2$ | 4.20 | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 1, $n \geqslant 3$ | 4.13, 4.18(b) | $(\mathrm{GL}_2, \mathbb{F}^2)$ |
| 2 | 4.14, 4.18(b) | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 3 | 4.14, 4.18(b) | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 4, $n = 2$ | 4.20 | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 4, $n \geqslant 3$ | 4.13, 4.18(b) | $(\mathrm{GL}_2, \mathbb{F}^2)$ |
| 5 | 4.14, 4.18(b) | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 6 | 4.14, 4.18(b) | $(\mathrm{SL}_2, \mathbb{F}^2)$ |
| 7 | 4.14, 4.18(b) | $(\mathrm{SL}_2, \mathbb{F}^2)$ |

$r \leqslant 3$. Applying Propositions 6.3 and 6.4 we find that, up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 7. The subsequent reasoning is similar for each of the cases in Table 7; the key points of the arguments are gathered in Table 8. First, applying an appropriate combination of statements 4.13, 4.14, 4.18(b), and 4.20 (see the column "References") to the varieties $Z = \mathrm{Gr}_2(V)$ and $X = \mathrm{Gr}_2(V_1)$, we find a connected reductive subgroup $L \subset K$ and an $L$-module $R$ with the following property: $Z$ is a spherical $K$-variety if and only if $R$ is a spherical $L$-module. After that the sphericity of the $L$-module $R$ is verified using Theorems 5.1 and 5.2. Since the group $C$ acts trivially on $X$, we have $C \subset L$. Therefore to describe the action $L : R$ it suffices to describe the actions $(L \cap K') : R$ and $C : R$. In all the cases we have $L \cap K' = M \times K_2$ for some subgroup $M \subset K_1$. Moreover, $R = W^* \otimes U$, where $M$ acts on $W^*$ and $K_2$ acts on $U$. For each of the cases, up to a geometrical equivalence, the pair $(M, W)$ is indicated in the third column of Table 8. The action of $C$ on $W$ is the same as on $V_1$ and the action of $C$ on $U$ coincides with the initial one. □

6.3. **Spherical actions on Grassmannians.** In this subsection we complete the description of spherical actions on Grassmannians initiated in §§ 6.1, 6.2. The main result of this subsection is the following theorem.

**Theorem 6.6.** *Suppose that $d \geqslant 6$ and $3 \leqslant k \leqslant d/2$. Then the variety $X = \mathrm{Gr}_k(V)$ is $K$-spherical if and only if the following conditions hold:*

1. *up to a geometrical equivalence, the pair $(K', V)$ is contained in Table 9;*
2. *the number $k$ satisfies the conditions listed in the fourth column of Table 9;*
3. *the group $C$ satisfies the conditions listed in the fifth column of Table 9.*

In the fourth column of Table 9 the empty cells mean that $k$ may be any number such that $3 \leqslant k \leqslant d/2$.

*Proof of Theorem* 6.6. We first consider the case $r = 1$, that is, the case where $V$ is a simple $K$-module. In this situation the center of $K$ acts trivially on $\mathrm{Gr}_k(V)$, hence $\mathrm{Gr}_k(V)$ is $K$-spherical if and only if it is $K'$-spherical.

If $\mathrm{Gr}_k(V)$ is $K'$-spherical, then by Proposition 3.9 and Theorem 6.1 the pair $(K', V)$ is geometrically equivalent to one of the pairs in Table 5.

Since every flag variety of the group $\mathrm{SL}_n$ is $\mathrm{SL}_n$-spherical, then so is $\mathrm{Gr}_k(\mathbb{F}^n)$.

For $n \geqslant 3$ and $3 \leqslant k \leqslant n$ the variety $\mathrm{Gr}_k(\mathbb{F}^{2n})$ is $\mathrm{Sp}_{2n}$-spherical by Proposition 4.8.

Table 9

| No. | $K'$ | $V$ | Conditions on $k$ | Conditions on $C$ | Note |
|---|---|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | | | $n \geqslant 6$ |
| 2 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | | | $n \geqslant 3$ |
| 3 | $\mathrm{SO}_n$ | $\mathbb{F}^n$ | | | $n \geqslant 6$ |
| 4 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m$ | | $\chi_1 \neq \chi_2$ for $n = m = k$ | $n \geqslant m$, $n + m \geqslant 6$ |
| 5 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^1$ | | | $n \geqslant 3$ |
| 6 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m$ | $k = 3$ | $\chi_1 \neq \chi_2$ for $m \leqslant 3$ | $n, m \geqslant 2$ |
| 7 | $\mathrm{Sp}_4 \times \mathrm{SL}_m$ | $\mathbb{F}^4 \oplus \mathbb{F}^m$ | $k \geqslant 4$ | $\chi_1 \neq \chi_2$ for $k = m = 4$ | $m \geqslant 4$ |
| 8 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m \oplus \mathbb{F}^1$ | | $\chi_2 - \chi_1, \chi_3 - \chi_1$ lin. ind. for $k = n$; $\chi_2 \neq \chi_3$ for $m \leqslant k < n$ | $n \geqslant m \geqslant 1$, $n + m \geqslant 5$ |

For $n \geqslant 6$ and $3 \leqslant k \leqslant n/2$ the variety $\mathrm{Gr}_k(\mathbb{F}^n)$ is $\mathrm{SO}_n$-spherical by Proposition 4.9.

For $3 \leqslant k \leqslant 4$ the variety $\mathrm{Gr}_k(\mathbb{F}^8)$ is not $\mathrm{Spin}_7$-spherical since inequality (4.1) does not hold in this case.

We now consider the case $r \geqslant 2$. By Proposition 3.9 the $K$-sphericity of $\mathrm{Gr}_k(V)$ implies the $K$-sphericity of $\mathrm{Gr}_2(V)$. Applying Theorems 6.5 and 5.5 we find that the pair $(K', V)$ is geometrically equivalent to one of the pairs in Table 10.

Table 10

| No. | $K'$ | $V$ | Note |
|---|---|---|---|
| 1 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m$ | $n \geqslant m \geqslant 1$, $n + m \geqslant 6$ |
| 2 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m$ | $n \geqslant 2$, $m \geqslant 1$, $2n + m \geqslant 6$ |
| 3 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m}$ | $n \geqslant m \geqslant 2$ |
| 4 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m \oplus \mathbb{F}^1$ | $n \geqslant m \geqslant 1$, $n + m \geqslant 5$ |
| 5 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m \oplus \mathbb{F}^1$ | $n \geqslant 2$, $m \geqslant 1$ |
| 6 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m} \oplus \mathbb{F}^1$ | $n \geqslant m \geqslant 2$ |

For each case in Table 10 we denote by $K_1$ (resp. $K_2$) the first (resp. second) factor of $K'$. We also put

$$U = V_2 \oplus \ldots \oplus V_r.$$

Up to changing the order of factors of $K$ (along with simultaneously interchanging the first and second summands of $V$), the pair $(K_1, V_1)$ fits in at least one of the cases listed in the second column of Table 11. In all these cases the subsequent reasoning is similar. At first, applying an appropriate combination of statements 4.13–4.18, 4.20 (see the third column of Table 11) to $Z = \mathrm{Gr}_k(V)$ and $X = \mathrm{Gr}_k(V_1)$, we find a connected reductive subgroup $L \subset K$ and an $L$-module $R$ with the following property: $Z$ is a spherical $K$-variety if and only if $R$ is a spherical $L$-module. After that the sphericity of the $L$-module $R$ is verified using Theorems 5.1, 5.2, and 5.3. Since $C$ acts trivially on $X$, we have $C \subset L$. Therefore to describe the action $L : R$ it suffices to describe the actions $(L \cap K') : R$ and $C : R$. In all the cases we have $L \cap K' = M \times K_2$ for some subgroup $M \subset K_1$. Moreover, $R = W^* \otimes U$ where $M$ acts on $W^*$ and $K_2$ acts on $U$. For each of the cases, up to a geometrical equivalence, the pair $(M, W)$ is indicated in the fourth column of Table 11.

TABLE 11

| No. | $(K_1, V_1)$ | Case | $(M, W)$ |
|---|---|---|---|
| 1 | $(\mathrm{SL}_n, \mathbb{F}^n)$, $k = n$ | 4.20 | $(\mathrm{SL}_n, \mathbb{F}^n)$ |
| 2 | $(\mathrm{SL}_n, \mathbb{F}^n)$, $k < n$ | 4.13, 4.18(b) | $(\mathrm{GL}_k, \mathbb{F}^k)$ |
| 3 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$, $k \leqslant n, k = 2l$ | 4.14, 4.18(b) | $(\underbrace{\mathrm{SL}_2 \times \ldots \times \mathrm{SL}_2}_{l}, \underbrace{\mathbb{F}^2 \oplus \ldots \oplus \mathbb{F}^2}_{l})$ |
| 4 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$, $k \leqslant n$, $k = 2l+1$ | 4.16, 4.18(b) | $(\mathbb{F}^\times \times \underbrace{\mathrm{SL}_2 \times \ldots \times \mathrm{SL}_2}_{l}, \mathbb{F}^1 \oplus \underbrace{\mathbb{F}^2 \oplus \ldots \oplus \mathbb{F}^2}_{l})$ |
| 5 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$, $n < k < 2n$, $2n - k = 2l$ | 4.15, 4.18(b) | $(\underbrace{\mathrm{SL}_2 \times \ldots \times \mathrm{SL}_2}_{l} \times \mathrm{Sp}_{2n-4l}, \underbrace{\mathbb{F}^2 \oplus \ldots \oplus \mathbb{F}^2}_{l} \oplus \mathbb{F}^{2n-4l})$ |
| 6 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$, $n < k < 2n$, $2n - k = 2l+1$ | 4.17, 4.18(b) | $(\mathbb{F}^\times \times \underbrace{\mathrm{SL}_2 \times \ldots \times \mathrm{SL}_2}_{l} \times \mathrm{Sp}_{2n-4l-2}, \mathbb{F}^1 \oplus \underbrace{\mathbb{F}^2 \oplus \ldots \oplus \mathbb{F}^2}_{l} \oplus \mathbb{F}^{2n-4l-2})$ |
| 7 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$, $k = 2n$ | 4.20 | $(\mathrm{Sp}_{2n}, \mathbb{F}^{2n})$ |

The action of $C$ on $W$ is the same as on $V_1$ and the action of $C$ on $U$ coincides with the initial one. $\square$

6.4. **Completion of the classification.** In this subsection we classify all spherical actions on $V$-flag varieties that are not Grassmannians (see Theorem 6.7). Thereby we complete the proof of Theorem 1.7.

Let $\mathbf{a} = (a_1, \ldots, a_s)$ be a composition of $d$ such that $a_1 \leqslant \ldots \leqslant a_s$ and $s \geqslant 3$. (The latter exactly means that $\mathrm{Fl}_{\mathbf{a}}(V)$ is not a Grassmannian.)

**Theorem 6.7.** *The variety $\mathrm{Fl}_{\mathbf{a}}(V)$ is $K$-spherical if and only if the following conditions hold:*

(1) *the pair $(K', V)$, considered up to a geometrical equivalence, and the tuple $(a_1, \ldots, a_{s-1})$ are contained in Table 12;*

(2) *the group $C$ satisfies the conditions listed in the fifth column of Table 12.*

In the proof of Theorem 6.7 we shall need several auxiliary results.

**Proposition 6.8.** *Suppose that $n \geqslant 3$, $V = \mathbb{F}^{2n}$, $K = \mathrm{Sp}_{2n}$, and $2 \leqslant k \leqslant n-1$. Then the variety $\mathrm{Fl}(1, k; V)$ is $K$-spherical.*

*Proof.* Put $Z = \mathrm{Fl}(1, k; V)$, $X = \mathbb{P}(V)$ and consider the natural $K$-equivariant morphism $\varphi \colon Z \to X$. Using Proposition 4.16 and then Definition 4.10, we find that there are a point $[W] \in X$ and a connected reductive subgroup $L \subset K_{[W]}$ with the following properties:

(1) the pair $(L, V/W)$ is geometrically equivalent to the pair $(\mathbb{F}^\times \times \mathrm{Sp}_{2n-2}, \mathbb{F}^1 \oplus \mathbb{F}^{2n-2})$;

(2) the $K$-sphericity of $Z$ is equivalent to the $L$-sphericity of $\varphi^{-1}([W]) \simeq \mathrm{Gr}_k(V/W)$.

Table 12

| No. | $K'$ | $V$ | $(a_1,\dots,a_{s-1})$ | Conditions on $C$ | Note |
|---|---|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | $(a_1,\dots,a_{s-1})$ | | $n\geqslant 3$ |
| 2 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | $(1,a_2)$ | | $n\geqslant 2$ |
| 3 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | $(1,1,1)$ | | $n\geqslant 2$ |
| 4 | $\mathrm{SL}_n$ | $\mathbb{F}^n\oplus\mathbb{F}^1$ | $(a_1,\dots,a_{s-1})$ | $\chi_1\ne\chi_2$ for $s=n+1$ | $n\geqslant 2$ |
| 5 | $\mathrm{SL}_n\times\mathrm{SL}_m$ | $\mathbb{F}^n\oplus\mathbb{F}^m$ | $(1,a_2)$ | $\chi_1\ne\chi_2$ for $n=1+a_2$ | $n\geqslant m\geqslant 2$ |
| 6 | $\mathrm{SL}_n\times\mathrm{SL}_2$ | $\mathbb{F}^n\oplus\mathbb{F}^2$ | $(a_1,a_2)$ | $\chi_1\ne\chi_2$ for $n=4$ and $a_1=a_2=2$ | $n\geqslant 4$, $a_1\geqslant 2$ |
| 7 | $\mathrm{Sp}_{2n}\times\mathrm{SL}_m$ | $\mathbb{F}^{2n}\oplus\mathbb{F}^m$ | $(1,1)$ | $\chi_1\ne\chi_2$ for $m\leqslant 2$ | $n\geqslant 2$, $m\geqslant 1$ |
| 8 | $\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2m}$ | $\mathbb{F}^{2n}\oplus\mathbb{F}^{2m}$ | $(1,1)$ | $\chi_1\ne\chi_2$ | $n\geqslant m\geqslant 2$ |

By Theorems 6.5 and 6.6 the variety $\mathrm{Gr}_k(V/W)$ is $L$-spherical, which completes the proof. $\square$

**Proposition 6.9.** *Suppose that $n\geqslant 3$, $V=\mathbb{F}^{2n}$, and $K=\mathrm{Sp}_{2n}$. Then the variety $\mathrm{Fl}(2,2;V)$ is not $K$-spherical.*

*Proof.* Put $Z=\mathrm{Fl}(2,2;V)$, $X=\mathrm{Gr}_2(V)$ and consider the natural $K$-equivariant morphism $\varphi\colon Z\to X$. Applying Proposition 4.14 and taking into account Definition 4.10, we find that there are a point $[W]\in X$ and a connected reductive subgroup $L\subset K_{[W]}$ with the following properties:

(1) the pair $(L,V/W)$ is geometrically equivalent to the pair

$$(\mathrm{SL}_2\times\mathrm{Sp}_{2n-4},\mathbb{F}^2\oplus\mathbb{F}^{2n-4});$$

(2) the $K$-sphericity of $Z$ is equivalent to the $L$-sphericity of $\varphi^{-1}([W])\simeq\mathrm{Gr}_2(V/W)$.

By Theorem 6.5 the variety $\mathrm{Gr}_2(V/W)$ is not $L$-spherical, which completes the proof. $\square$

**Proposition 6.10.** *Suppose that $n\geqslant 2$, $m\geqslant 1$, $V_1=\mathbb{F}^{2n}$, $V_2=\mathbb{F}^m$, $V=V_1\oplus V_2$, $K_1=\mathrm{Sp}_{2n}$, $K_2=\mathrm{GL}_m$, and $K=K_1\times K_2$. Then the variety $\mathrm{Fl}(1,2;V)$ is not $K$-spherical.*

*Proof.* Put $Z=\mathrm{Fl}(1,2;V)$ and $X=\mathrm{Gr}_3(V_1)$. Applying Proposition 4.16 (for $n\geqslant 3$) or Corollary 4.17 (for $n=2$) to $X$ and then Proposition 4.22 to $Z$ and $X$, we find that there are a point $[W]\in X$ and a connected reductive subgroup $L\subset K_{[W]}$ with the following properties:

(1) $L=L_1\times K_2$, where $L_1\subset K_1$;

(2) $L_1\simeq\mathbb{F}^\times\times\mathrm{SL}_2$ for $2\leqslant n\leqslant 3$ and $L_1\simeq\mathbb{F}^\times\times\mathrm{SL}_2\times\mathrm{Sp}_{2n-6}$ for $n\geqslant 4$;

(3) the pair $(L_1,W)$ is geometrically equivalent to the pair $(\mathbb{F}^\times\times\mathrm{SL}_2,\mathbb{F}^1\oplus\mathbb{F}^2)$;

(4) the $K$-sphericity of $Z$ is equivalent to the $L$-sphericity of $(W^*\otimes V_2)\times\mathbb{P}(W)$, where $L_1$ acts diagonally on $W^*$ and $\mathbb{P}(W)$, and $K_2$ acts on $V_2$.

It is easy to see that the $L$-sphericity of $(W^*\otimes V_2)\times\mathbb{P}(W)$ is equivalent to the sphericity of the $(L_1\times K_2\times\mathbb{F}^\times)$-module $(W^*\otimes V_2)\oplus W$, where $L_1$ acts diagonally on $W^*$ and $W$, $K_2$ acts on $V_2$, and $\mathbb{F}^\times$ acts on the summand $W$ by scalar transformations. Applying Theorem 5.2 we find that the indicated module is not spherical, which completes the proof. $\square$

*Proof of Theorem* 6.7. Throughout this proof, the description of the partial order $\preccurlyeq$ on the set $\mathscr{F}(\mathrm{GL}(V))/\sim$ (see Corollary 3.6 and § 3.3) as well as Theorem 1.4 will be use without extra explanation.

Since $\mathrm{Fl}_{\mathbf{a}}(V)$ is not a Grassmannian, we have

$$[\![\mathrm{Fl}_{\mathbf{a}}(V)]\!] \succcurlyeq [\![\mathrm{Fl}(1,1;V)]\!].$$

Therefore the $K$-sphericity of $\mathrm{Fl}_{\mathbf{a}}(V)$ implies the $K$-sphericity of $\mathrm{Fl}(1,1;V)$. The following proposition provides a complete classification of pairs $(K, V)$ for which $K$ acts spherically on $\mathrm{Fl}(1,1;V)$.

**Proposition 6.11.** *For $d \geqslant 3$, the variety $\mathrm{Fl}(1,1;V)$ is $K$-spherical if and only if the following conditions hold:*

(1) *up to a geometrical equivalence, the pair $(K', V)$ is contained in Table* 13*;*

(2) *the group $C$ satisfies the conditions listed in the fourth column of Table* 13.

TABLE 13

| No. | $K'$ | $V$ | Conditions on $C$ | Note |
|---|---|---|---|---|
| 1 | $\mathrm{SL}_n$ | $\mathbb{F}^n$ | | $n \geqslant 3$ |
| 2 | $\mathrm{Sp}_{2n}$ | $\mathbb{F}^{2n}$ | | $n \geqslant 2$ |
| 3 | $\mathrm{SL}_n$ | $\mathbb{F}^n \oplus \mathbb{F}^1$ | $\chi_1 \neq \chi_2$ for $n = 2$ | $n \geqslant 2$ |
| 4 | $\mathrm{SL}_n \times \mathrm{SL}_m$ | $\mathbb{F}^n \oplus \mathbb{F}^m$ | $\chi_1 \neq \chi_2$ for $n = m = 2$ | $n \geqslant m \geqslant 2$ |
| 5 | $\mathrm{Sp}_{2n} \times \mathrm{SL}_m$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^m$ | $\chi_1 \neq \chi_2$ for $m \leqslant 2$ | $n \geqslant 2$, $m \geqslant 1$ |
| 6 | $\mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$ | $\mathbb{F}^{2n} \oplus \mathbb{F}^{2m}$ | $\chi_1 \neq \chi_2$ | $n, m \geqslant 2$ |

*Proof.* By Corollary 4.5 the action $K : \mathrm{Fl}(1,1;V)$ is spherical if and only if $V \otimes \mathbb{F}^2$ is a spherical $(K \times \mathrm{GL}_2)$-module. Now the required result follows from Theorems 5.1 and 5.2. $\square$

In view of Proposition 6.11, to complete the proof of Theorem 6.7 it remains to find all $K$-spherical $V$-flag varieties $X$ with $[\![X]\!] \succ [\![\mathrm{Fl}(1,1;V)]\!]$ for all the cases in Table 13. In what follows we consider each of these cases separately.

*Case* 1. $K' = \mathrm{SL}_n, V = \mathbb{F}^n$. We have $[\![X]\!] \preccurlyeq [\![\mathrm{Fl}_{\mathbf{b}}(V)]\!]$ where $\mathbf{b} = (1, \ldots, 1)$. By Corollary 4.5 and Theorem 5.1 the variety $\mathrm{Fl}_{\mathbf{b}}(V)$ is $K$-spherical, hence so is $X$.

*Case* 2. $K' = \mathrm{Sp}_{2n}, V = \mathbb{F}^{2n}$. If $s = 3$ and $a_1 = 1$, then $X$ is $K$-spherical by Proposition 6.8. If $s = 3$ and $a_1 \geqslant 2$, then $[\![X]\!] \succcurlyeq [\![\mathrm{Fl}(2,2;V)]\!]$, and so $X$ is not $K$-spherical by Proposition 6.9. If $s = 4$ and $a_1 = a_2 = a_3 = 1$, then $X$ is $K$-spherical by Corollary 4.5 and Theorem 5.1. If $s = 4$ and $a_3 \geqslant 2$, then $[\![X]\!] \succcurlyeq [\![\mathrm{Fl}(2,2;V)]\!]$, and so $X$ is not $K$-spherical by Proposition 6.9. In what follows we assume that $s \geqslant 5$. Then $[\![X]\!] \succcurlyeq [\![\mathrm{Fl}(1,1,1,1;V)]\!]$. The variety $\mathrm{Fl}(1,1,1,1;V)$ is not $K$-spherical by Corollary 4.5 and Theorem 5.1, hence $X$ is not $K$-spherical either.

*Case* 3. $K' = \mathrm{SL}_n, V = \mathbb{F}^n \oplus \mathbb{F}^1$. If $s = n+1$, that is, $\mathbf{a} = (1, \ldots, 1)$, then by Corollary 4.5 and Theorem 5.2 the variety $X$ is $K$-spherical if and only if $\chi_1 \neq \chi_2$. In what follows we assume that $s \leqslant n$. Then $[\![X]\!] \preccurlyeq [\![\mathrm{Fl}_{\mathbf{b}}(V)]\!]$, where $\mathbf{b} = (1, \ldots, 1, 2)$. By Corollary 4.5 and Theorem 5.2 the variety $\mathrm{Fl}_{\mathbf{b}}(V)$ is $K'$-spherical, hence $X$ is $K$-spherical.

*Case* 4. $K' = \mathrm{SL}_n \times \mathrm{SL}_m$, $V = \mathbb{F}^n \oplus \mathbb{F}^m$, $n \geqslant m \geqslant 2$. Put $K_1 = \mathrm{SL}_n$, $K_2 = \mathrm{SL}_m$, $V_1 = \mathbb{F}^n$, $V_2 = \mathbb{F}^m$. If $s \geqslant 4$ then $[\![X]\!] \succcurlyeq [\![\mathrm{Fl}(1,1,1;V)]\!]$. By Corollary 4.5 and Theorem 5.2

the variety $\mathrm{Fl}(1,1,1;V)$ is not $K$-spherical, hence $X$ is not $K$-spherical either. In what follows we assume that $s=3$. Put $k=a_1+a_2$. We consider the following two subcases: $a_1=1$ and $a_1 \geqslant 2$.

*Subcase* 4.1. $a_1=1$. Then $k \leqslant n$. In view of Propositions 4.13 and 4.22 there are a point $[W] \in \mathrm{Gr}_k(V_1)$ and a subgroup $L \subset (K_1)_{[W]}$ with the following properties:

(1) the pair $(L,W)$ is geometrically equivalent to the pair $(\mathrm{GL}_k, \mathbb{F}^k)$ for $k<n$ and the pair $(\mathrm{SL}_k, \mathbb{F}^k)$ for $k=n$;

(2) the $K$-sphericity of $X$ is equivalent to the $(C \times L \times K_2)$-sphericity of the variety $(W^* \otimes V_2) \times \mathbb{P}(W)$, where $L$ acts diagonally on $W^*$ and $W$, $K_2$ acts on $V_2$, and $C$ acts on $W^*$, $V_2$, and $W$ via the characters $-\chi_1$, $\chi_2$, and $\chi_1$, respectively.

It follows from (2) that the $K$-sphericity of $X$ is equivalent to the sphericity of the $(C \times L \times K_2 \times \mathbb{F}^\times)$-module $(W^* \otimes V_2) \oplus W$, where $C$, $L$, and $K_2$ act as described in (2) and $\mathbb{F}^\times$ acts on the summand $W$ by scalar transformations. Applying Theorem 5.2 we find that the indicated module is not spherical when $k=n$, $\chi_1=\chi_2$ and is spherical in all other cases.

*Subcase* 4.2. $a_1 \geqslant 2$. By Theorem 5.5 the $K$-sphericity of $X$ implies that $m=2$. Then $k \leqslant n$ and the equality is attained if and only if $a_1=a_2=2$ and $n=4$. In view of Propositions 4.13 and 4.22 there are a point $[W] \in \mathrm{Gr}_k(V_1)$ and a subgroup $L \subset (K_1)_{[W]}$ with the following properties:

(1) the pair $(L,W)$ is geometrically equivalent to the pair $(\mathrm{GL}_k, \mathbb{F}^k)$ for $k<n$ and the pair $(\mathrm{SL}_k, \mathbb{F}^k)$ for $k=n$;

(2) the $K$-sphericity of $X$ is equivalent to the $(C \times L \times K_2)$-sphericity of the variety $Y=(W^* \otimes V_2) \times \mathrm{Gr}_{a_1}(W)$ be, where $L$ acts diagonally on $W^*$ and $\mathrm{Gr}_{a_1}(W)$, $K_2$ acts on $V_2$, and $C$ acts on $W^*$ and $V_2$ via the characters $-\chi_1$ and $\chi_2$, respectively.

Applying Proposition 4.13 to $\mathrm{Gr}_{a_1}(W)$ and then considering the natural projection $Y \to \mathrm{Gr}_{a_1}(W)$, we find that there is a subgroup $L_1 \subset L$ with the following properties:

(1) the pair $(L_1,W)$ is geometrically equivalent to the pair $(\mathrm{GL}_{a_1} \times \mathrm{GL}_{a_2}, \mathbb{F}^{a_1} \oplus \mathbb{F}^{a_2})$ for $k<n$ and the pair $(\mathrm{S}(\mathrm{L}_{a_1} \times \mathrm{L}_{a_2}), \mathbb{F}^{a_1} \oplus \mathbb{F}^{a_2})$ for $k=n$;

(2) the $(C \times L \times K_2)$-sphericity of $Y$ is equivalent to the sphericity of the $(C \times L_1 \times K_2)$-module $W^* \otimes V_2$ on which $C$ and $K_2$ act as described above and $L_1$ acts on $W^*$.

By Theorem 5.2 the indicated module is not spherical when $k=n$, $\chi_1=\chi_2$ and is spherical in all other cases.

*Case* 5. $K' = \mathrm{Sp}_{2n} \times \mathrm{SL}_m$, $V = \mathbb{F}^{2n} \oplus \mathbb{F}^m$, $n \geqslant 2$, $m \geqslant 1$. It follows from the condition $[\![X]\!] \succ [\![\mathrm{Fl}(1,1;V)]\!]$ that $[\![X]\!] \succcurlyeq [\![\mathrm{Fl}(1,2;V)]\!]$. The variety $\mathrm{Fl}(1,2;V)$ is not $K$-spherical by Proposition 6.10, hence $X$ is not $K$-spherical either.

*Case* 6. $K' = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2m}$, $V = \mathbb{F}^{2n} \oplus \mathbb{F}^{2m}$, $n \geqslant m \geqslant 2$. If $X$ is $K$-spherical then $X$ is also $(\mathrm{Sp}_{2n} \times \mathrm{SL}_{2m})$-spherical. As was shown in Case 5, the latter is false.

The proof of Theorem 6.7 and hence that of Theorem 1.7 is completed. □

## Appendix A. Proofs of Propositions 4.13, 4.14, and 4.16

*Proof of Proposition* 4.13. Let $e_1, \ldots, e_n$ be a basis of $V$. Put $W = \langle e_1, \ldots, e_m \rangle$, and $W' = \langle e_{m+1}, \ldots, e_n \rangle$. Denote by $L$ the subgroup of $K$ preserving each of the subspaces $W$ and $W'$. It is easy to see that the point $[W] \in X$ and the group $L \subset K_{[W]}$ satisfy conditions (2)–(4). It remains to show that condition (1) also holds.

For each $i = 1, \ldots, n$ we introduce the subspace $V_i = \langle e_n, \ldots, e_{n-i+1} \rangle \subset V$. The stabilizer in $K$ of the flag $(V_1, \ldots, V_n)$ is a Borel subgroup of $K$, we denote it by $B$. It is not hard to check that the group $B_{[W]}$ is a Borel subgroup of $L$.

Computations show that $\dim B_{[W]} = n(n+1)/2 - m(n-m) - 1$, whence

$$\dim B[W] = \dim B - \dim B_{[W]} = m(n-m) = \dim X,$$

and so the orbit $B[W]$ is open in $X$.

Applying Proposition 4.11 to the groups $K$, $B$, $L$ and the point $[W]$ we find that condition (1) holds. □

In the proofs of Propositions 4.14 and 4.16 we shall need the following notion.

Let $U$ be a finite-dimensional vector space with a given symplectic form $\Omega$ on it and let $\dim U = 2m$. A basis $e_1, \ldots, e_{2m}$ of $U$ will be called *standard* if the matrix of $\Omega$ has the form

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

in this basis.

*Proof of Proposition* 4.14. Let $\Omega$ be a symplectic form on $V$ preserved by $K$. We fix a decomposition into a skew-orthogonal direct sum

$$V = W \oplus W' \oplus R,$$

where $\dim W = \dim W' = 2k$, $\dim R = 2n - 4k$, and the restriction of $\Omega$ to each of the subspaces $W, W', R$ is nondegenerate. Let $e_1, \ldots, e_{2k}$ be a standard basis in $W$ and let $e'_1, \ldots, e'_{2k}$ be a standard basis in $W'$. If $n > 2k$ (that is, $R$ is nontrivial), then we fix a linearly independent set of vectors $r_1, \ldots, r_{n-2k}$ in $R$ that generates a maximal isotropic subspace in $R$. For each $i = 1, \ldots, k$ we introduce the two-dimensional subspaces

$$W_i = \langle e_{2i-1}, e_{2i} \rangle \subset W \quad \text{and} \quad W'_i = \langle e'_{2i-1}, e'_{2i} \rangle \subset W'.$$

For each $i = 1, \ldots, 2k$ we put $f_i = e_i + (-1)^i e'_i$. If $n > 2k$ then for each $j = 1, \ldots, n-2k$ we put $f_{2k+j} = r_j$. For each $i = 1, \ldots, n$ we introduce the subspace $F_i = \langle f_1, \ldots, f_i \rangle \subset V$. A direct check shows that $\Omega(f_i, f_j) = 0$ for all $i, j = 1, \ldots, n$, hence $F_n$ is a maximal isotropic subspace of $V$. Consequently, the stabilizer in $K$ of the isotropic flag $(F_1, \ldots, F_n)$ is a Borel subgroup of $K$, we denote it by $B$.

Put $H = K_{[W]}$. Clearly, $H$ preserves the subspace $W^\perp = W' \oplus R$.

Put $V_i = W_i \oplus W'_i$ for $i = 1, \ldots, k$ and $V_{k+1} = R$. Then

$$V = V_1 \oplus \ldots \oplus V_k \oplus V_{k+1}. \tag{A.1}$$

We define the group $L$ to be the stabilizer in $H$ of the flag $(F_2, F_4, \ldots, F_{2k})$. For each $i = 1, \ldots, k$ the $L$-invariance of the subspace $F_i$ implies the $L$-invariance of its projections to $W$ and $W^\perp$, hence both subspaces

$$W_1 \oplus \ldots \oplus W_i \quad \text{and} \quad W'_1 \oplus \ldots \oplus W'_i$$

are invariant with respect to $L$. It follows that $L$ preserves each of the subspaces $W_1, \ldots, W_k$, $W'_1, \ldots, W'_k$, hence it preserves each of the subspaces $V_1, \ldots, V_k, V_{k+1}$. At last, for each $i = 1, \ldots, k$ the $L$-invariance of the subspaces $V_i$ and $F_i$ implies the $L$-invariance of the subspace $V_i \cap F_i = \langle f_{2i-1}, f_{2i} \rangle$.

For each $i = 1, \ldots, k, k+1$ let $L_i$ denote the subgroup in $L$ consisting of all transformations acting trivially on all summands of decomposition (A.1) except for $V_i$. Then $L = L_1 \times \ldots \times L_k \times L_{k+1}$.

For a fixed $i \in \{1, \ldots, k\}$, the $L$-invariance of the subspace $\langle f_{2i-1}, f_{2i} \rangle$ implies that $L_i$ diagonally acts on the direct sum $W_i \oplus W'_i$ transforming the bases $(e_{2i-1}, e_{2i})$ and $(-e'_{2i-1}, e'_{2i})$ in the same way. Consequently, $L_i \simeq \mathrm{SL}_2$ and the pair $(L_i, V_i)$ is geometrically equivalent to the pair $(\mathrm{SL}_2, \mathbb{F}^2 \oplus \mathbb{F}^2)$ with the diagonal action of $\mathrm{SL}_2$.

The above arguments show that the point $[W] \in X$ and the group $L \subset K_{[W]}$ satisfy conditions (2)–(4). We now prove that condition (1) also holds.

Let us show that the group $B_{[W]} = H \cap B = L \cap B$ is a Borel subgroup of $L$. It is easy to see that $B_{[W]} = B_1 \times \ldots \times B_k \times B_{k+1}$, where $B_i = B \cap L_i$ for all $i = 1, \ldots, k, k+1$. For every $i = 1, \ldots, k$ the group $B_i$ is the stabilizer in $L_i$ of the line $\langle f_{2i-1} \rangle$ and the group $B_{k+1}$ is the stabilizer in $L_{k+1}$ of the maximal isotropic flag

$$(\langle r_1 \rangle, \langle r_1, r_2 \rangle, \ldots, \langle r_1, r_2, \ldots, r_{n-2k} \rangle)$$

in $V_{k+1}$. From this one deduces that $B_i$ is a Borel subgroup of $L_i$ for all $i = 1, \ldots, k, k+1$, hence $B_{[W]}$ is a Borel subgroup of $L$.

Computations show that $\dim B_{[W]} = (n-2k)^2 + n$, whence

$$\dim B[W] = \dim B - \dim B_{[W]} = n^2 - (n-2k)^2 = 2k(2n-2k) = \dim X,$$

and so the orbit $B[W]$ is open in $X$.

Applying Proposition 4.11 to the groups $K$, $B$, $L$ and the point $[W]$ we find that condition (1) holds. $\square$

*Proof of Proposition* 4.16. Let $\Omega$ be a symplectic form on $V$ preserved by $K$. We fix a decomposition into a skew-orthogonal direct sum

$$V = U_0 \oplus U \oplus U' \oplus R,$$

where $\dim U_0 = 2$, $\dim U = \dim U' = 2k$, $\dim R = 2n - 4k - 2$ and the restriction of $\Omega$ to each of the subspaces $U_0, U, U', R$ is nondegenerate. We choose a standard basis $e_0, e'_0$ in $U_0$, a standard basis $e_1, \ldots, e_{2k}$ in $U$, and a standard basis $e'_1, \ldots, e'_{2k}$ in $U'$. If $n > 2k+1$ (that is, $R$ is nontrivial), then we fix a linearly independent set of vectors $r_1, \ldots, r_{n-2k-1}$ in $R$ that generates a maximal isotropic subspace in $R$. We put $W_0 = \langle e_0 \rangle$ and for each $i = 1, \ldots, k$ we introduce the two-dimensional subspaces

$$W_i = \langle e_{2i-1}, e_{2i} \rangle \subset U \quad \text{and} \quad W'_i = \langle e'_{2i-1}, e'_{2i} \rangle \subset U'.$$

We put $f_0 = e'_0$. Next, for each $i = 1, \ldots, 2k$ we put $f_i = e_i + (-1)^i e'_i$. At last, if $n > 2k+1$ then for each $j = 1, \ldots, n-2k-1$ we put $f_{2k+j} = r_j$. For each $i = 0, 1, \ldots, n-1$ we introduce the subspace $F_i = \langle f_0, \ldots, f_i \rangle \subset V$. A direct check shows that $\Omega(f_i, f_j) = 0$ for all $i, j = 0, \ldots, n-1$, hence $F_n$ is a maximal isotropic subspace in $V$. Consequently, the stabilizer in $K$ of the isotropic flag $(F_0, \ldots, F_{n-1})$ is a Borel subgroup of $K$, we denote it by $B$.

Put $W = W_0 \oplus U$ and $H = K_{[W]}$.

Put $V_0 = U_0$, $V_i = W_i \oplus W'_i$ for $i = 1, \ldots, k$ and $V_{k+1} = R$. Then

$$V = V_0 \oplus V_1 \oplus \ldots \oplus V_k \oplus V_{k+1}. \tag{A.2}$$

We define the group $L$ to be the stabilizer in $H$ of the flag $(F_0, F_2, F_4, \ldots, F_{2k})$.

Since the subspaces $W$ and $\langle e_0' \rangle = F_0$ are $L$-invariant, it follows that the subspace $\langle e_0' \rangle \oplus W = U_0 \oplus U$ is also $L$-invariant, hence so is its skew-orthogonal complement $(U_0 \oplus U)^\perp = U' \oplus R$. Thus $V$ admits the decomposition $V = \langle e_0' \rangle \oplus W \oplus (U' \oplus R)$ into a direct sum of three $L$-invariant subspaces.

For each $i = 1, \ldots, k$ the $L$-invariance of the subspace $F_i$ implies the $L$-invariance of its projections to $W$ (along $\langle e_0' \rangle \oplus U' \oplus R$) and $U' \oplus R$ (along $\langle e_0' \rangle \oplus W$), hence both subspaces

$$W_1 \oplus \ldots \oplus W_i \quad \text{and} \quad W_1' \oplus \ldots \oplus W_i'$$

are invariant with respect to $L$. This implies that $L$ preserves each of the subspaces $W_1, \ldots, W_k$, $W_1', \ldots, W_k'$, hence it preserves $W_0$ and each of the subspaces $V_0, V_1, \ldots, V_k, V_{k+1}$. At last, for each $i = 1, \ldots, k$ the $L$-invariance of the subspaces $V_i$ and $F_i$ implies the $L$-invariance of the subspace $V_i \cap F_i = \langle f_{2i-1}, f_{2i} \rangle$.

For each $i = 0, 1, \ldots, k, k+1$ let $L_i$ denote the subgroup in $L$ consisting of all transformations acting trivially on all summands of decomposition (A.2) except for $V_i$. Then $L = L_0 \times L_1 \times \ldots \times L_k \times L_{k+1}$.

Since $L$ preserves each of the two one-dimensional subspaces $W_0 = \langle e_0 \rangle$ and $\langle e_0' \rangle$, it follows that $L_0$ acts on the direct sum $\langle e_0 \rangle \oplus \langle e_0' \rangle$ diagonally, multiplying $e_0$ and $e_0'$ by mutually inverse numbers. Hence $L_0 \simeq \mathbb{F}^\times$ and the pair $(L_0, V_0)$ is geometrically equivalent to the pair $(\mathbb{F}^\times, \mathbb{F}^1 \oplus \mathbb{F}^1)$ with the action $(t, (x_1, x_2)) \mapsto (tx_1, t^{-1}x_2)$. Next, for a fixed $i \in \{1, \ldots, k\}$, the $L$-invariance of the subspace $\langle f_{2i-1}, f_{2i} \rangle$ implies that $L_i$ diagonally acts on $W_i \oplus W_i'$ transforming the bases $(e_{2i-1}, e_{2i})$ and $(-e_{2i-1}', e_{2i}')$ in the same way. Consequently, $L_i \simeq \mathrm{SL}_2$ and the pair $(L_i, V_i)$ is geometrically equivalent to the pair $(\mathrm{SL}_2, \mathbb{F}^2 \oplus \mathbb{F}^2)$ with the diagonal action of $\mathrm{SL}_2$.

The above arguments show that the point $[W] \in X$ and the group $L \subset K_{[W]}$ satisfy conditions (2)–(4). We now prove that condition (1) also holds.

Let us show that the group $B_{[W]} = H \cap B = L \cap B$ is a Borel subgroup of $L$. It is easy to see that $B_{[W]} = B_0 \times B_1 \times \ldots \times B_k \times B_{k+1}$, where $B_i = B \cap L_i$ for all $i = 1, \ldots, k, k+1$. Evidently, $B_0 = L_0$. Next, for every $i = 1, \ldots, k$ the group $B_i$ is the stabilizer in $L_i$ of the line $\langle f_{2i-1} \rangle$ and the group $B_{k+1}$ is the stabilizer in $L_{k+1}$ of the maximal isotropic flag

$$(\langle r_1 \rangle, \langle r_1, r_2 \rangle, \ldots, \langle r_1, r_2, \ldots, r_{n-2k-1} \rangle)$$

in $V_{k+1}$. From this one deduces that $B_i$ is a Borel subgroup of $L_i$ for all $i = 0, 1, \ldots, k, k+1$, hence $B_{[W]}$ is a Borel subgroup of $L$.

Computations show that $\dim B_{[W]} = (n - 2k - 1)^2 + n$, whence

$$\dim B[W] = \dim B - \dim B_{[W]} = \\ n^2 - (n - 2k - 1)^2 = (2k + 1)(2n - 2k - 1) = \dim X,$$

and so the orbit $B[W]$ is open in $X$.

Applying Proposition 4.11 to the groups $K$, $B$, $L$ and the point $[W]$, we find that condition (1) holds. $\square$

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