

THE NUMBER OF FRAMINGS OF A KNOT IN A 3-MANIFOLD

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ABSTRACT. In view of the self-linking invariant, the number $|K|$ of framed knots in S^3 with given underlying knot K is infinite. In fact, the second author previously defined affine self-linking invariants and used them to show that $|K|$ is infinite for every knot in an orientable manifold unless the manifold contains a connected sum factor of $S^1 \times S^2$; the knot K need not be zero-homologous and the manifold is not required to be compact.

We show that when M is orientable, the number $|K|$ is infinite unless K intersects a non-separating sphere at exactly one point, in which case $|K| = 2$; the existence of a non-separating sphere implies that M contains a connected sum factor of $S^1 \times S^2$.

For knots in nonorientable manifolds we show that if $|K|$ is finite, then K is disorienting, or there is an isotopy from the knot to itself which changes the orientation of its normal bundle, or it intersects some embedded S^2 or $\mathbb{R}P^2$ at exactly one point, or it intersects some embedded S^2 at exactly two points in such a way that a closed curve consisting of an arc in K between the intersection points and an arc in S^2 is disorienting.

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1. INTRODUCTION

We work in the smooth category where the word “smooth” means C^∞ . Throughout the paper M denotes a manifold of dimension 3. It is not necessarily compact, orientable or without boundary. A *knot* in M is an embedding of a circle S^1 . A *framing* of a knot is a normal vector field along the knot in M . An *isotopy* of ordinary (respectively, framed) knots is a path in the space of ordinary (respectively, framed) embedded connected curves.

For an unframed knot K in M we are interested in the number $|K|$ of isotopy classes of framed knots that correspond to the isotopy class of K when one forgets the framing. Given a framed knot K_f and $i \in \mathbb{Z}$, let K_f^i denote the framed knot obtained by adding i extra positive full-twists to the framing of K_f , if $i \geq 0$; and by adding $|i|$ extra negative twists to the framing of K_f if $i < 0$. Clearly every isotopy class of framed knots corresponding to K contains K_f^i for some $i \in \mathbb{Z}$. However what could happen, and actually does happen for knots in some manifolds, is that K_f^i and K_f^j are isotopic for $i \neq j$. In this case of course $|K|$ is finite.

When K is a classical knot in \mathbb{R}^3 or, more generally, when K is a zero-homologous knot in an orientable manifold, the number of framings $|K|$ is infinite as the framed knots K_f^i can be distinguished by the self-linking numbers. On the other hand, self-linking numbers do not exist when the knot is nonzero-homologous. Still intuitively one expects that $|K| = \infty$ for many $K \subset M$. The second author [1] used the technique of Vassiliev-Goussarov invariants to define the affine self-linking number that generalizes the self-linking number to the case of nonzero-homologous framed

knots in an orientable manifold. In particular, the number of framings $|K|$ is infinite for all knots in orientable manifolds that are not realizable as a connected sum $(S^1 \times S^2) \# M'$; equivalently, the number of framings of a knot in an orientable manifold M is infinite unless M contains a nonseparating 2-sphere. For compact orientable manifolds this result was stated earlier without proof by Hoste and Przytycki in their paper [3, Remark on page 488]. They referred to the paper of McCullough [4] on the mapping class group of a 3-manifold for the idea of the proof.

If K intersects a nonseparating sphere in an orientable manifold at exactly one point then $|K| = 2$, see [3, page 487] and later works [1, Theorem 2.6], [2, Theorem 4.1.1]. Note that for a framed knot K_f in an orientable manifold, the framed knots K_f and K_f^1 are not isotopic and $|K|$ is even. Indeed, in order to prove the claim it suffices to observe that the spin structure takes different values on the loops in the principle SO_3 bundle corresponding to these two framed knots. The orientable 3-manifold M is parallelizable and the orthonormal 3-framing at a point of a framed knot consists of the framing vector, the normalized velocity vector of the curve, and the unique unit-length vector orthogonal to them such that the resulting 3-frame gives the chosen orientation of M . This argument admits a generalization for knots in nonorientable manifolds.

Lemma 1.1. *For a knot K in a (possibly nonorientable) manifold M of dimension 3, the number $|K|$ is either infinite or even ≥ 2 .*

Proof of Lemma 1.1. Let ξ be the fiber bundle over M with fiber over $p \in M$ consisting of pairs of orthonormal vectors in $T_p M$. It is a principle SO_3 -bundle. Given a framed knot K in M , there is a canonical lift K' of K to the total space P of ξ ; it is constructed by associating with $p \in K$ the pair of the velocity vector of K at p and the framing vector of K at p . Since the fiber of ξ is $\mathbb{R}P^3$, there is a canonical one-dimensional vector bundle E over P that restricts over each fiber to a canonical vector bundle over $\mathbb{R}P^3$. The vector bundle E can be constructed by means of the Borel construction. Indeed, let V be the total space of the canonical line bundle over $\mathbb{R}P^3$. Then E is the total space of the Borel construction

$$P \times_{SO_3} V \longrightarrow M;$$

in other words E is the quotient of $P \times V$ by the relation generated by equivalences $(xg, y) \sim (x, gy)$ where $g \in SO_3$. Let w_1 denote the first Stiefel-Whitney class of E . It is a non-trivial cohomology class in P . The evaluation of w_1 on the lift K' is an invariant of a regular homotopy of K . Passing from K_f to K_f with an extra twist of the framing changes the value of w_1 by one. This implies the claim. \square

We show that the 2006 work of McCullough [5] immediately implies the following.

Theorem 1.2. *Let K be a knot in an orientable 3-manifold M . If $|K| \neq \infty$, then K intersects some nonseparating 2-sphere at exactly one point.*

In the case of a non-orientable manifold M we say that a knot K in M is *disorienting* if it is orientation reversing or equivalently if its normal bundle is non-trivial.

Theorem 1.3. *If K is disorienting or if there is an isotopy from K to itself which changes the orientation of its normal bundle, then $|K| = 2$.*

Proof. If K is disorienting, it suffices to observe that the Dehn twist of a Klein bottle (which is the boundary of the regular neighborhood of K) along a meridian

in one direction is isotopic to the Dehn twist along the same meridian in the opposite direction.

If there is an isotopy from K to itself which changes the orientation of the normal bundle, then under this isotopy a knot K with an extra twist of its framing will become a knot with an extra twist of its framing in the opposite direction. \square

In the rest of the section we will consider the case where K is not disorienting and it does not admit an isotopy to itself that reverses the orientation of its normal bundle.

To formulate a counterpart of Theorem 1.1 in this case, we need to introduce an invariant associated with a pair of intersection points between an embedded oriented sphere $S^2 \subset M$ and a framed knot $K_f \subset M$. Without loss of generality, by slightly perturbing K_f if necessary, we may assume that K intersects S^2 transversally. Let p and q be two intersection points between K and S^2 . Choose a closed path γ in M that follows from p to q along K and then from q to p along S^2 . We observe that either for any choice of γ it is disorienting or for any choice of γ it is not.

Theorem 1.4. *Let K be a knot in a non-orientable manifold M . Suppose that K is not disorienting and there is no isotopy of K to itself which changes the orientation of the normal bundle of K and that $|K| < \infty$. Then either*

- a:** *the knot crosses some embedded two-sided $\mathbb{R}P^2$ at one point; or*
- b:** *the knot crosses some embedded 2-sphere at one point; or*
- c:** *the knot K crosses some embedded oriented 2-sphere S^2 at two points and a closed curve consisting of an arc in K between the two intersection points and an arc in S^2 is disorienting.*

Finally, we show that in the cases [a] and [b] the number of framings of K is 2.

Theorem 1.5. *If K intersects transversally at a unique point an embedded sphere or two-sided projective plane, then $|K| = 2$.*

2. PROOFS

Proof of Theorem 1.2. To begin with let us assume that M is a compact manifold possibly with boundary. If the number $|K|$ of framings is finite, then there is an isotopy from K_f to K_f^i for some i . Let U denote a thin solid torus neighborhood of K bounded by $T^2 = \partial U$. By the isotopy extension theorem there is a diffeomorphism of M which is identity in a neighborhood of the boundary components of M and which is an i -th power of a Dehn twist along the meridian curve $C = C_1$ on U . Put D_1 to be a meridian disk in U whose boundary is the meridian curve C_1 . Put $M' = M \setminus \mathring{U}$. Then $\partial M' = \partial M \sqcup T^2$.

McCullough [5, Theorem 1] says the following: Let M' be a compact orientable 3-manifold which admits a homeomorphism which is Dehn twists on the boundary about a collection C_1, \dots, C_n of simple closed curves in $\partial M'$. Then for each i , either C_i bounds a disk in M' , or for some $j \neq i$, C_j and C_i cobound an incompressible annulus in M' .

In our case the collection C_1, \dots, C_n of closed simple curves consists of only one curve C_1 , and therefore C_1 bounds a disk D_2 in M' . The disks D_1, D_2 have a common boundary and together they form a nonseparating 2-sphere that intersects K transversally at exactly one point. This completes the proof for compact M .

When M is noncompact it suffices to notice that the isotopy from K_f to K_f^i is contained within a compact submanifold of M , and if this submanifold contains a nonseparating 2-sphere intersecting K at exactly one point, then so does M . \square

McCullough states in [5, page 1332] without proof that using a more elaborate machinery his [5, Theorem 1] mentioned above can be generalized to nonorientable manifolds adding a possibility of Dehn twists about (two-sided) Möbius bands. Also when M is nonorientable, an annulus could meet the boundary in such a way that a Dehn twist along it is isotopic on the boundary to an even power of a Dehn twist along one of the boundary curves. (In this case one of the two components of the annulus boundary does not have to be an element of the collection of the curves C_i [6].)

First proof of Theorem 1.4. The argument is similar to one in the proof of Theorem 1.2 except that now we apply the generalization of the McCullough theorem to the case of nonorientable manifolds.

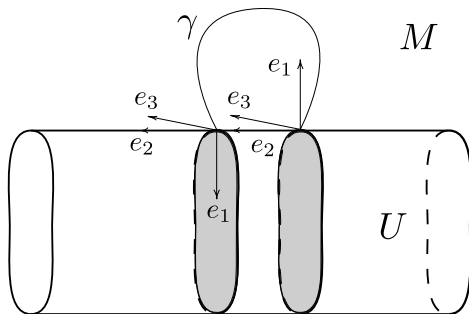


FIGURE 1. Disorienting curve γ .

The case of a Möbius band bounding C_i in the generalization of the McCullough theorem corresponds to the case [a] in Theorem 1.4. We may assume that the projective plane $\mathbb{R}P^2$ in [a] is two-sided in M since otherwise it can be replaced with the boundary $S^2 \subset M$ of its tubular neighborhood; the bounding sphere S^2 satisfies the requirements of [c]. The case of a disc bounding C_i in the generalization of the McCullough theorem corresponds to [b]. Finally, let U be a tubular neighborhood of K , and suppose that there exists an annulus in $M \setminus \mathring{U}$ that meets the boundary in such a way that a Dehn twist along it is isotopic to an even power of a Dehn twist of ∂U . Then, capping off the embedded annulus with two discs in U results in a sphere satisfying the requirements of [c], see Figure 1. \square

Theorem 1.4 can also be proven using Theorem 1.2, without assuming the generalization of the McCullough theorem to nonorientable manifolds. Since the proof of the generalization of the McCullough Theorem to nonorientable manifolds is not in the literature and appears to be quite involved, we provide an alternative proof below.

Remark 2.1. In our proof below we use a lift of K to the orientation double cover of M . If K is disorienting, then the lift disintegrates.

If there is an isotopy of K which changes the orientation of its normal bundle, then the automorphism of $M \setminus \mathring{U}$ obtained as a result of isotopy extension is not a Dehn twist on the boundary torus of $M \setminus \mathring{U}$ and we cannot apply Theorem 1.2.

This is why we considered these two cases separately in Theorem 1.3.

Second proof of Theorem 1.4. Choose a framed knot K_f corresponding to K . Since $|K| < \infty$, there exists an ambient isotopy h of M that takes K_f to itself with a different framing.

Let \tilde{M} be the orientation double cover of M and $\pi : \tilde{M} \rightarrow M$ the projection. Since K is not disorienting, K_f has two framed lifts K_1 and K_2 in \tilde{M} . Furthermore, the isotopy h of M taken twice lifts to an isotopy of \tilde{M} that takes K_1 to itself but with a different framing. Since \tilde{M} is orientable, by the McCullough theorem [5, Theorem 1] and the argument in the proof of Theorem 1.2, either K_1 intersects an embedded sphere in $\tilde{M} \setminus K_2$, or each of the knots K_1 and K_2 intersects the same embedded sphere at a unique point.

Suppose that K_1 intersects a sphere S^2 in $\tilde{M} \setminus K_2$. Then $\pi|S^2$ intersects K transversally at a unique point. Therefore, if U is a thin solid torus neighborhood of K , then the circle $S^1 = \pi S^2 \cap \partial U$ is contractible in $M \setminus \mathring{U}$. By the Loop Theorem, there is an embedded disc D in $M \setminus \mathring{U}$ bounding S^1 . Together with the disc $\pi S^2 \cap U$ it forms an embedded sphere in M intersecting K transversally at a unique point.

Suppose now that the knots K_1 and K_2 each intersect the same embedded sphere $S^2 \subset \tilde{M}$ at one point. Then K crosses the projected sphere πS^2 at two points. By a general position argument, we may assume that the projected sphere is an immersed sphere with at most double points; there are no triple points since $\tilde{M} \rightarrow M$ is a double covering. In what follows we will denote the immersion $\pi|S^2$ by f . By the definition of the orientation double cover, a curve consisting of an arc in K and an arc in the immersed sphere πS^2 is disorienting. Our aim is to modify f by surgery to get an embedding of either a sphere or projective plane satisfying one of the three conclusions of Theorem 1.4.

Let C be a path component of double points of the immersion f in M .

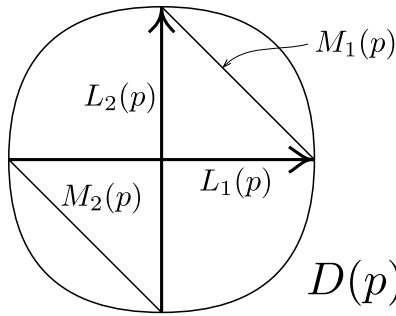


FIGURE 2. Segments $M_1(p)$ and $M_2(p)$.

If $f^{-1}C$ consists of one circle, then the tubular neighborhood of C in M is non-orientable. In particular, the component C is non-contractible in M , and therefore $f^{-1}C$ is non-contractible in the cylinder $S^2 \setminus f^{-1}\mathring{U}$, where U is a thin solid torus neighborhood of K . There is an obvious surgery eliminating the self-intersection circle of $f(S^2)$. Under this surgery we remove the cylindrical neighborhood V of

$f^{-1}(C)$ in S^2 and attach two Möbius bands M_1 and M_2 along the new boundary components of S^2 . The resulting surface F consists of two copies of $\mathbb{R}P^2$. The new map $f' : F \rightarrow M$ outside M_1 and M_2 remains the same as the map f outside V . Let us now describe $f'|_{M_1}$ and $f'|_{M_2}$. Let $D(p)$ be a normal disc in the tubular neighborhood U of K passing through $p \in K$. The disc $D(p)$ intersects $f(S^2)$ along two segments $L_1(p)$ and $L_2(p)$ that we can orient so that the monodromy along K exchanges oriented segments $L_1(p)$ and $L_2(p)$. Let $M_1(p)$ and $M_2(p)$ respectively denote the two segments in $D(p)$ joining the terminal and initial points of the oriented segments $L_1(p)$ and $L_2(p)$. We define f' on M_1 and M_2 so that its image is the union of all segments $M_1(p)$ and $M_2(p)$ where p ranges over all points $p \in K$.

After the surgery along C we obtain two immersed copies of $\mathbb{R}P^2$. We discard one of them. The remaining immersed copy of $\mathbb{R}P^2$ intersects K transversally and the number of components of its double points is less than before the surgery. We observe that the restriction of f' to the Möbius band M_i in the remaining copy of $\mathbb{R}P^2$ is an embedding. Therefore, if f' has a component C' of double points whose inverse image $f'^{-1}C'$ consists of one circle, then $f'^{-1}C'$ is a non-trivial circle in the cylinder $\mathbb{R}P^2 \setminus (M_i \cup f'^{-1}\mathring{U})$ and therefore the argument can be iterated.

Eventually we obtain an immersion of a surface to M —which we continue to denote by f —such that for each component C of double points of f the inverse image $f^{-1}(C)$ consists of two circles C_1 and C_2 . The source manifold of f is either $\mathbb{R}P^2$ or S^2 depending on whether we performed at least one surgery attaching a Möbius band or not. *To simplify notation, we will assume that the source manifold of f is $\mathbb{R}P^2$; the case of S^2 is similar.*

As above we deduce that C_1 and C_2 belong to the cylinder Q obtained from $\mathbb{R}P^2$ by removing $f^{-1}\mathring{U}$ and the attached Möbius band. There are several possible cases that we need to consider.

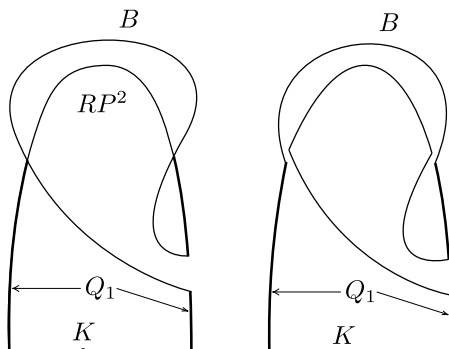


FIGURE 3. The first case.

First, if C_1 is non-trivial (noncontractible) in Q and C_2 is trivial, then C_1 breaks the cylinder Q into two closed cylinders Q_1 and Q_2 . Let Q_1 be the cylinder whose boundary intersects the boundary of $f^{-1}(U)$. Now C_2 can lie in either Q_1 or Q_2 .

First suppose C_2 is in Q_1 . Then we do surgery which exchanges the Möbius strip bounded by C_1 and the disc bounded by C_2 . The result is $\mathbb{R}P^2$ intersected at one point by K but with fewer number of double point components.

Now suppose that C_2 is in Q_2 . Then we do surgery that glues the disk bounded by C_2 to C_1 . The result is S^2 intersected at one point by K with a fewer number of double point components.

Second, if both C_1 and C_2 are non-trivial in Q , then each of C_1 and C_2 cut Q into two cylinders. There is a surgery resulting in an immersion of a copy of $\mathbb{R}P^2$ intersecting K at a unique point and a closed surface in $M \setminus K$. We discard the closed surface and obtain a new map f with fewer number of components of double points.

Finally if both C_1 and C_2 are trivial in Q , then C_1 cuts Q into a disc D_1 and its complement $Q \setminus D_1$, while C_2 cuts Q into a disc D_2 and its complement $Q \setminus D_2$. There is a surgery resulting in a map of a copy of the cylinder obtained from Q by exchanging the two discs D_1 and D_2 . The result is $\mathbb{R}P^2$ intersected at one point by K but with fewer number of double point components.

We induct on the number of double point components and after finitely many steps we obtain a desired embedded surface. \square

Remark 2.2. The above proof of Theorem 1.4 produces an embedded sphere or projective plane S in M which is the projection of a sphere or projective plane $\tilde{S} \subset \tilde{M}$, where \tilde{S} intersects each lift K_i of K_f at most once. If S is a sphere then \tilde{S} is a sphere as well. If S is a sphere that intersects K_f in two points, it must satisfy the conditions of [c]. Suppose S does not satisfy [c]. Then every loop γ of the form $\gamma_1\gamma_2$, where γ_1 is a path from p to q along K and γ_2 is a path from q to p along S , would lift to a loop in \tilde{M} , and hence S would lift to a sphere intersected twice by K_1 or twice by K_2 . Since \tilde{S} intersects each K_i at most once, this is impossible.

Proof of Theorem 1.5. Let us show that if case [a] occurs then $|K| = 2$. Consider a thin neighborhood of $\mathbb{R}P^2$ that is crossed by K_f at one point. We assume that this neighborhood is $[0, 1] \times \mathbb{R}P^2$. We draw a curve $\alpha : [0, 1] \rightarrow \mathbb{R}P^2$ that is a small figure 8 equipped with the normal vector field. We assume that K_f intersects $[0, 1] \times \mathbb{R}P^2$ as follows: the level $t \times \mathbb{R}P^2$ is intersected at the point $\alpha(t)$ and the direction of the framing vector of K_f at this point projects to $\mathbb{R}P^2$ to the direction of the normal vector to α at $\alpha(t)$.

Now we construct an isotopy of the knot K_f to the knot with two extra twists of the framing (of the same sign) by describing a homotopy $\alpha_\tau, \tau \in [0, 1]$ of the curve $\alpha = \alpha_0$. This isotopy will fix the framed knot outside of the set $[\epsilon, 1 - \epsilon] \times \mathbb{R}P^2$.

Consider the covering $p : S^2 \rightarrow \mathbb{R}P^2$ and one of the two lifts β of α to S^2 . This β is a small figure 8 having a locally well-defined winding number 0. On a sphere the curve with the winding number 0 is regular homotopic to the curve with the winding number 2. Let $\beta_\tau, \tau \in [0, 1]$ be this homotopy. We assume that it fixes β on the neighborhood $\beta([0, \epsilon]) \sqcup \beta((1 - \epsilon, 1])$. Put $\alpha_\tau = p \circ \beta_\tau$. One checks that the framed knot whose intersection with $[0, 1] \times \mathbb{R}P^2$ is described by the curve α_1 has two more twists of framings of the same sign than the knot K_f . (We can not say whether this knot is K_f^2 or K_f^{-2} because M is nonorientable and $K_f^{\pm 2}$ are not well defined.) So $|K| \leq 2$. Since $|K| \geq 2$ by Lemma 1.1, we have $|K| = 2$.

If case [b] occurs the proof is similar and simpler than the one above. Alternatively one can use the generalization of [3, page 487] and later works [1, Theorem 2.6], [2, Theorem 4.1.1]. \square

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