BANACH ANALYTIC SETS AND A NON-LINEAR VERSION OF THE LEVI EXTENSION THEOREM

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Abstract. We prove a certain non-linear version of the Levi extension theorem for meromorphic functions. This means that the meromorphic function in question is supposed to be extendable along a sequence of complex curves, which are arbitrary, not necessarily straight lines. Moreover, these curves are not supposed to belong to any finite dimensional analytic family. The conclusion of our theorem is that nevertheless the function in question meromorphically extends along an (infinite dimensional) analytic family of complex curves and its domain of existence is a pinched domain filled in by this analytic family.

1. Introduction

1.1. Statement of the main result. By (λ, z) we denote the standard coordinates in $\mathbb{C}^2$. For $\varepsilon > 0$ consider the following ring domain

$$R_{1+\varepsilon} = \{(\lambda, z) \in \mathbb{C}^2 : 1-\varepsilon < |\lambda| < 1+\varepsilon, |z| < 1\} = A_{1-\varepsilon,1+\varepsilon} \times \Delta, \tag{1.1}$$

i.e., $R_{1+\varepsilon}$ is the product of the annulus $A_{1-\varepsilon,1+\varepsilon} := \{z \in \mathbb{C} : 1-\varepsilon < |\lambda| < 1+\varepsilon\}$ with the unit disk Δ. Let a sequence of holomorphic functions $\{\varphi_k : \Delta_{1+\varepsilon} \to \Delta\}_{k=1}^{\infty}$ be given such that φ_k converge uniformly on $\Delta_{1+\varepsilon}$ to some $\varphi_0 : \Delta_{1+\varepsilon} \to \Delta$. We say that such sequence is a ***test sequence*** if $(\varphi_k - \varphi_0)|_{\partial\Delta}$ doesn't vanish for $k >> 0$ and

$$\mathsf{VarArg}_{\partial\Delta}(\varphi_k - \varphi_0) \qquad \text{stays bounded when} \qquad k \to +\infty. \tag{1.2}$$

Denote by C_k the graph of φ_k in $\Delta_{1+\varepsilon} \times \Delta$, by C_0 the graph of φ_0.

Theorem 1. *Let f be a meromorphic function on $R_{1+\varepsilon}$ and $\{\varphi_k\}_{k=1}^{\infty}$ a test sequence such that for every k the restriction $f|_{C_k \cap R_{1+\varepsilon}}$ is well defined and extends to a meromorphic function on the curve C_k and that the number of poles counting with multiplicities of these extensions is uniformly bounded. Then there exists an analytic family of holomorphic graphs $\{C_\alpha\}_{\alpha \in \mathcal{A}}$ parameterized by a Banach ball $\mathcal{A}$ of infinite dimension such that:*

i) $f|_{C_\alpha \cap R_{1+\varepsilon}}$ extends to a meromorphic function on C_α for every $\alpha \in \mathcal{A}$ and the number of poles counting with multiplicities of these extensions is uniformly bounded.

ii) Moreover f meromorphically extends as a function of two variables (λ, z) to the pinched domain $\mathcal{P} := \mathsf{Int}\left(\bigcup_{\alpha \in \mathcal{A}} C_\alpha\right)$ swept by C_α.

Here by C_α we denote the graph of the function φ_α. The notion of a ***pinched domain***, though intuitively clear, see Figure 1, is discussed in details at the beginning of section 2.

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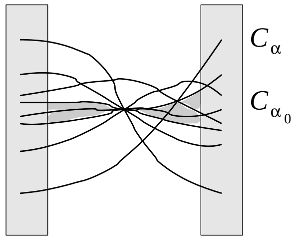

FIGURE 1. The brighter dashed zone on this picture represents the ring domain $R_{1+\varepsilon}$ and curves are the graphs C_α. Around C_{α_0}, the graph of $\varphi_0 = \varphi_{\alpha_0}$, the analytic family $\{C_\alpha\}_{\alpha\in\mathcal{A}}$ fills in an another (darker) dashed zone, a pinched domain $\mathcal{P}$. On this picture there is exactly one pinch, the point at which most of graphs intersect.

Definition 1.1. *Let's say that the graphs $\{C_k\}$ of our functions $\{\varphi_k\}$ are in* general position *if for every point $\lambda_0 \in \Delta$ there exists a subsequence $\{\varphi_{k_p}\}$ such that zeroes of $\varphi_{k_p} - \varphi_0$ do not accumulate to λ_0.*

Theorem 1 implies the following *non-linear* Levi-type extension theorem:

Corollary 1. *If under the conditions of Theorem 1 curves $\{C_k\}$ are in general position then f extends to a meromorphic function in the bidisk $\Delta_{1+\varepsilon} \times \Delta$.*

Remark 1. Let us explain the condition of a general position. Take the sequence $C_k = \{z = \frac{1}{k}\lambda\}$ in $\mathbb{C}^2$. Then the function $f(\lambda, z) = e^{\frac{z}{\lambda}}$ is holomorphic in $R := \mathbb{C}^* \times \mathbb{C}$ and extends holomorphically along every curve C_k. But it is not holomorphic (even not meromorphic) in $\mathbb{C}^2$. It is also holomorphic when restricted to any curve $C = \{z = \varphi(\lambda)\}$ provided $\varphi(0) = 0$. Therefore the subspace H_0 of $\varphi \in \mathsf{Hol}(\Delta_{1+\varepsilon}, \Delta)$ such that f extends along the corresponding curve is of codimension one. In fact this is the general case: the Banach ball $\mathcal{A}$ in Theorem 1 appears as a neighborhood of the limit point α_0 in the subspace of finite codimension of a well chosen Banach space of holomorphic functions.

Remark 2. To explain the condition (1.2) we impose on our test sequences we construct in section 5 for the following *non test* sequence $\varphi_k(\lambda) = \left(\frac{2}{3}\lambda\right)^k$ a holomorphic function f in $\mathbb{C}^* \times \mathbb{C}$ which holomorphically extends along every graph C_k but which is not extendable meromorphically along any one-parameter analytic family $\{\varphi_\alpha\}$, see Example 1 there.

1.2. Meromorphic mappings. The assumption that f is a function in Theorem 1 is not really important. We prove also a non-linear version of an extension theorem for meromorphic mappings with values in general complex spaces putting it into a form suitable for applications. Let's call a family $\{\varphi_t \in \mathsf{Hol}(\Delta_{1+\varepsilon}, \Delta) : t \in T\}$ a *test family* if there exists $N \in \mathbb{N}$ such that for every pair $s \neq t \in T$ there exists a radius $1 - \varepsilon/2 < r < 1 + \varepsilon/2$ such that $(\varphi_s - \varphi_t)|_{\partial\Delta_r}$ doesn't vanish and has winding number $\leqslant N$. As usual by C_t we denote the graph of φ_t.

Corollary 2. *Let X be a reduced, disk-convex complex space and $f : R_{1+\varepsilon} \to X$ a meromorphic mapping. Suppose that there exists an uncountable test family of holomorphic functions $\{\varphi_t : \mathsf{Hol}(\Delta_{1+\varepsilon}, \Delta) : t \in T\}$ such that $f|_{C_t \cap R_{1+\varepsilon}}$ holomorphically extends to C_t for every $t \in T$. Then f extends to a meromorphic mapping from a pinched domain $\mathcal{P}$ to X.*

Moreover, there exists, like in Theorem 1, an infinite dimensional family of graphs C_α parameterized by a Banach ball $\mathcal{A}$ such that $f|_{C_\alpha \cap R_{1+\varepsilon}}$ holomorphically extends to C_α for all $\alpha \in \mathcal{A}$. The condition that $f|_{C_t \cap R_{1+\varepsilon}}$ is assumed to extend *holomorphically* should not be confusing because meromorphic functions on curves are precisely holomorphic mappings to the Riemann sphere $\mathbb{P}^1$. I.e., the meromorphic functions case is the case $X = \mathbb{P}^1$ in this Corollary.

1.3. Structure of the paper, notes, acknowledgement. *1.* Theorem 1 is proved in section 2. The set $\mathcal{A}$ such that $f|_{C_\alpha \cap R_{1+\varepsilon}}$ meromorphically extends to C_α for $\alpha \in \mathcal{A}$ is always a Banach analytic subset of a neighborhood of φ_0 in the Banach space $\mathsf{Hol}(\Delta_{1+\varepsilon}, \Delta)$. In particular the sequence $\{\varphi_k\}$ of Example 1 is a Banach analytic set, namely the zero set of an appropriate singular integral operator, see section 3 for more details. The (known) problem however is that a *every* metrisable compact can be endoved with a structure of a Banach analytic set (in an appropriate complex Banach space), see [Mz]. For the case of a converging sequence of points see Remark 3.3 for a very simple example. Therefore in infinite dimensional case from the fact that our Banach analytic set contains a non-isolated point we cannot deduce that it contains an analytic disk. Our major task here is to overcome this difficulty.

2. In the case when $C_k = \Delta_{1+\varepsilon} \times \{z_k\}$ with $z_k \to 0$, *i.e.,* when C_k are horizontal disks this result is exactly the theorem of E. Levi, see [Lv] (case of holomorphic extension is due to Hartogs, see [Ht]). It should be said that Levi's theorem is usually stated in the form as our Corollary 2: *if f as above extends along an uncountable family of horizontal disks $C_t = \Delta_{1+\varepsilon} \times \{t\}$, then f meromorphically extends to Δ^2.* But the proof goes as follows: one remarks that then there exists a sequence $\{t_k\}$ (in fact an uncountable subfamily) such that extensions along C_{t_k}-s have uniformly bounded number of poles, and then the statement like our Theorem 1 is proved.

3. If the number of poles of extensions $f|_{C_k}$ is not uniformly bounded then the conclusion of Theorem 1 fails to be true even in the case of horizontal disks. This is shown by the Example 2 in section 5.

4. In the case when $\{C_t\}_{t \in T}$ are non-horizontal straight disks, *i.e.,* intersections of lines with Δ^2, Corollary 1 is due to Dinh, see [Dh] Corollaire 1. The proof in [Dh] uses results on the complex Plateau problem in projective space (after an appropriate Segre imbedding) and is essentially equivalent to the solution of this problem. From the point of view of this paper this is a special case when $\{C_k\}$ *ad hoc* belong to a finite dimensional analytic family: in Levi case the family is one-dimensional, in the case of Dinh two-dimensional. In section 3, after recalling the necessary facts about singular integral transforms, we give a very short proof of a non-linear extension theorem, see Theorem 3.1, in the case when $\{C_k\}$ are ad hoc included in an *arbitrary* finite dimensional family. In the straight case, *i.e.,* when $\{C_t\}$ are non-horizontal straight disks, the result of Corollary 2 for Kähler X was proved in [Sk] following the approach of [Dh].

5. It is important to outline that we do not suppose *a priori* that $\{C_k\}$ are included into any finite dimensional family of complex curves (ex. any family of algebraic curves of uniformly bounded degree) and, in fact, it is the main point of this paper to develop techniques for producing analytic disks C_α in families.

6. Corollary 2 is proved in section 4, where also a general position assumption is discussed. Examples 1 and 2 are treated in section 5.

At the end I would like to give my thanks to the Referee of this paper for the valuable remarks and suggestions.

2. Extension to Pinched Domains

2.1. Analytic families and pinched domains. By an *analytic family* of holomorphic mappings from Δ to Δ we understand the quadruple $(\mathcal{X},\pi,\mathcal{A},\Phi)$ where:

- $\mathcal{X}$ is a complex manifold, which is either a finite dimensional or a Banach one;
- a holomorphic submersion $\pi:\mathcal{X}\to\mathcal{A}$, where $\mathcal{A}$ is a positive dimensional complex (Banach) manifold such that for every $\alpha\in\mathcal{A}$ the preimage $\mathcal{X}_\alpha:=\pi^{-1}(\alpha)$ is a disk;
- a holomorphic map $\Phi:\mathcal{X}\to\mathbb{C}^2$ of generic rank 2 such that for every $\alpha\in\mathcal{A}$ the image $\Phi(\mathcal{X}_\alpha)=C_\alpha$ is a graph of a holomorphic function $\varphi_\alpha:\Delta\to\Delta$.

A family $(\mathcal{X},\pi,\mathcal{A},\Phi)$ we shall often call also an analytic family of complex disks in Δ^2. In our applications $\mathcal{A}$ will be always a neighborhood of some α_0 and without loss of generally we may assume for convenience that $\varphi_{\alpha_0}\equiv 0$, *i.e.*, that $C_{\alpha_0}=\Delta\times\{0\}$. In this local case, after shrinking $\mathcal{X}$ and Δ^2 if necessary, we can suppose that $\mathcal{X}=\Delta\times\mathcal{A}$, and we shall regard in this case Φ as a natural universal map

$$\Phi:(\lambda,\alpha)\to(\lambda,\varphi_\alpha(\lambda)) \tag{2.1}$$

from $\Delta\times\mathcal{A}$ to Δ^2, writing $\Phi(\lambda,\alpha)=(\lambda,\varphi(\lambda,\alpha))$ when convenient, meaning $\varphi(\lambda,\alpha)=\varphi_\alpha(\lambda)$. We shall often consider the case when $\mathcal{A}$ is a one-dimensional disk, in that case we say that our family is a *complex one-parameter* analytic family. In this case taking as $\mathcal{A}$ a sufficiently small neighborhood of α_0 and perturbing $\partial\Delta$ in λ-variable slightly we can suppose without loss of generality that φ_α doesn't vanish on $\partial\Delta$ if $\alpha\neq\alpha_0$. In particular the winding number of $\varphi_\alpha|_{\partial\Delta}$ is constant for $\alpha\in\mathcal{A}\setminus\{\alpha_0\}$, see Proposition 3.1.

Denote as $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ the image $\Phi(\mathcal{X})$, where $(\mathcal{X},\pi,\Delta,\Phi)$ is some complex one-parameter analytic family of complex disks in Δ^2. Point λ_0 such that $\varphi(\lambda_0,\alpha)\equiv 0$ as a function of α we call *a pinch* of $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ and say that $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ has a pinch at λ_0. Let us describe the shape of $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ near a pinch λ_0. Since $\varphi(\lambda_0,\alpha)\equiv 0$ we can divide it by $(\lambda-\lambda_0)^{l_0}$ with some (taken to be maximal) $l_0\geqslant 1$. I.e., in a neighborhood of $(\lambda_0,\alpha_0)\in\Delta\times\mathcal{A}$ we can write

$$\varphi(\lambda,\alpha)=(\lambda-\lambda_0)^{l_0}\varphi_1(\lambda,\alpha), \tag{2.2}$$

where $\varphi_1(\lambda_0,\alpha)\not\equiv 0$. Set

$$\Phi_1:(\lambda,\alpha)\to(\lambda,\varphi_1(\lambda,\alpha)). \tag{2.3}$$

The image of Φ_1 contains a bidisk $\Delta_r^2(\lambda_0,0)$ of some radius $r>0$ centered at $(\lambda_0,0)$. Therefore

$$\bar{\mathcal{P}}_{\mathcal{X},\Phi}\supset\Delta_r^2(\lambda_0,0)\cap\{|z|<c|\lambda-\lambda_0|^{l_0}\} \tag{2.4}$$

with some constant $c>0$.

Definition 2.1. *By a* pinched domain *we shall understand an open neighborhood $\mathcal{P}$ of $\bar{\Delta}\setminus\Lambda$, where Λ is a finite set of points in Δ, such that in a neighborhood of every $\lambda_0\in\Lambda$ domain $\mathcal{P}$ contains*

$$\Delta_r^2(\lambda_0,0)\cap\{|z|<c|\lambda-\lambda_0|^{l_0}\}\setminus\{(\lambda_0,0)\}. \tag{2.5}$$

We shall call l_0 the order of the pinch *λ_0.*

After shrinking Δ (in λ-variable) if necessary, we can suppose that the set $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ which corresponds to a complex one-parameter analytic family $(\mathcal{X},\pi,\mathcal{A},\Phi)$ has only finite number of pinches, say at $\lambda_1,...,\lambda_N$ of orders $l_1,...,l_N$ respectively, and therefore $\mathcal{P}_{\mathcal{X},\Phi}:=$

$\bar{\mathcal{P}}_{\mathcal{X},\Phi} \setminus \{\lambda_1, ..., \lambda_N\}$ is a pinched domain. Remark that $\bar{\mathcal{P}}_{\mathcal{X},\Phi}$ obviously contains every curve in a neighborhood $\mathcal{B}$ of $\varphi_0 \equiv 0$ of the subspace

$$\{\varphi \in \mathsf{Hol}(\Delta,\Delta) : \mathsf{ord}_0(\varphi,\lambda_j) \geqslant l_j\} \subset \mathsf{Hol}(\Delta,\Delta), \tag{2.6}$$

which is of finite codimension.

Remark 2.1. (a) Therefore, let us make the following precisions: our pinched domains will be always supposed to have only finitely many pinches and moreover, these pinches *do not belong* to the corresponding pinched domain *by definition*.

(b) Hilbert manifold structure on $\mathcal{B}$ (if needed) can be insured by considering instead of $\mathsf{Hol}(\Delta,\Delta)$ the Hilbert space $H^{1,2}_{+}(\mathbb{S}^1)$ of Sobolev functions on the circle, which holomorphically extend to Δ, for example. This will be done later in section 3. At that point it will be sufficient for us to remark that extension along one-parameter analytic families is equivalent to that of along of infinite dimensional ones, and both imply the extension to pinched domains. More precisely, the following is true:

Proposition 2.1. *Let $(\mathcal{X},\pi,\mathcal{A},\Phi)$ be a complex one-parameter analytic family of complex disks in Δ^2 and let $\mathcal{P}_{\mathcal{X},\Phi}$ be the corresponding pinched domain. Suppose that a holomorphic function f on $R_{1+\varepsilon}$ meromorphically extends along every $C_\alpha, \alpha \in \mathcal{A}$. Let $\mathcal{B}$ be the infinite dimensional analytic family of complex disks in Δ^2 constructed as in (2.6). Then:*

i) Function f meromorphically extends to $\mathcal{P}_{\mathcal{X},\Phi}$ as a function of two variables.

ii) For every $\beta \in \mathcal{B}$ the restriction $f|_{C_\beta \cap R_{1+\varepsilon}}$ extends to a meromorphic function on C_β and the number of poles of these extensions is uniformly bounded.

Proof. Writing $\mathcal{X} = \Delta \times \mathcal{A}$ with $\mathcal{A} = \Delta$ and $\alpha_0 = 0$, and taking the preimage $W := \Phi^{-1}(R_{1+\varepsilon})$ in $\Delta \times \mathcal{A} \equiv \Delta^2$ we find ourselves in the following situation:

i) W contains a ring domain (denote it by W as well), and $g := f \circ \Phi$ is meromorphic (or holomorphic, after shrinking) on W.

ii) For every $\alpha \in \mathcal{A}$ the restriction $g|_{(\Delta\times\{\alpha\})\cap W}$ meromorphically extends to $\Delta \times \{\alpha\}$.

The classical theorem of Levi, [Lv, Si1], implies now that g meromorphically extends to $\mathcal{X} = \Delta \times \mathcal{A}$, and this gives us the extension of f to $\mathcal{P}_{\mathcal{X},\Phi}$.

For the proof of the extendability of $f|_{C_\beta \cap R_{1+\varepsilon}}$ to C_β for every $\beta \in \mathcal{B}$ close enough to zero let us first of all remark that $\Phi^{-1}(C_\beta)$ is contained in a relatively compact part of $\mathcal{X}$. Indeed, take a pinch λ_0 and suppose without loss of generality that $\lambda_0 = 0$. Write

$$\varphi(\lambda,\alpha) = \lambda^{l_0}\varphi_1(\lambda,\alpha) \tag{2.7}$$

as in (2.2). Since $\varphi_1(0,\alpha) \not\equiv 0$ we can use the Weierstrass preparation theorem and present

$$\varphi_1(\lambda,\alpha) = u \cdot \left(\alpha^k + g_1(\lambda)\alpha^{k-1} + ... + g_k(\lambda)\right) \tag{2.8}$$

with $u(0,0) \neq 0$ and $g_1(0) = ... = g_k(0) = 0$. Take the corresponding $\varphi_\beta \in \mathcal{B}$ with graph C_β and write it as $\varphi_\beta(\lambda) = c_0\lambda^{l_0}\tilde{\varphi}(\lambda)$. Consider the equation $\varphi(\lambda,\alpha) = \varphi_\beta(\lambda)$, *i.e.*,

$$\lambda^{l_0} \cdot \left(\alpha^k + g_1(\lambda)\alpha^{k-1} + ... + g_k(\lambda)\right) = u^{-1}c_0\lambda^{l_0}\tilde{\varphi}(\lambda), \tag{2.9}$$

or, equivalently

$$\alpha^k + g_1(\lambda)\alpha^{k-1} + ... + g_k(\lambda) = u^{-1}c_0\tilde{\varphi}(\lambda). \tag{2.10}$$

For $\lambda \sim 0$ all solutions $\alpha_1(\lambda), ..., \alpha_k(\lambda)$ of (2.10) are close to zero, provided c_0 is small enough. This proves our assertion that $\Phi^{-1}(C_\beta) \Subset \mathcal{X}$ and implies that $f|_{C_\beta \cap R_{1+\varepsilon}}$ meromorphically extends to C_β.

The orders of poles of meromorphic function g of two variables (λ,α) is bounded on every relatively compact part of $\mathcal{X} = \Delta\times\mathcal{A}$ and therefore the orders of poles of our extensions are also bounded.

□

Remark 2.2. Remark that f meromorphically extends to the pinched domain $\mathcal{P}_{\mathcal{B}}$ swept by the family $\mathcal{B}$ as well, simply because it is the same domain (up to shrinking).

2.2. Proof of Theorem 1. We start with the proof of item *(ii)* first. Without loss of generality we may assume that $\varphi_0\equiv 0$. Indeed, the condition $\varphi_0\equiv 0$ is not a restriction neither here, nor anywhere else in this paper, because it can be always achieved by the coordinate change

$$\begin{cases}\lambda\to\lambda,\\ z\to z-\varphi_0(\lambda).\end{cases} \tag{2.11}$$

Furthermore, when considering the extension of a meromorphic function f from a ring domain $R_{1+\varepsilon}$ to the bidisk $\Delta_{1+\varepsilon}\times\Delta$ one can suppose that f is holomorphic on $R_{1+\varepsilon}$ (after shrinking $R_{1+\varepsilon}$ if necessary and after multiplying by some power of z), and moreover, decomposing $f=f^+ + f^-$ where f^+ is holomorphic in $\Delta_{1+\varepsilon}\times\Delta$ and f^- in $(\mathbb{P}^1\setminus\bar{\Delta})\times\Delta$, one can subtract f^+ from f and suppose that $f^+\equiv 0$. That means that we can suppose that f has the Taylor decomposition

$$f(\lambda,z)=\sum_{n=0}^{\infty}A_n(\lambda)z^n \tag{2.12}$$

in $R_{1+\varepsilon}$ with

$$A_n(\lambda)=\sum_{l=-\infty}^{-1}a_{n,l}\lambda^l. \tag{2.13}$$

As the result along this proof we may suppose that $f=f^-$ and f^- is holomorphic in $A_{1-\varepsilon,1+\varepsilon}\times\Delta_{1+2\varepsilon}$. Therefore for $|\lambda|$ near 1 the Taylor expansion of f writes as

$$f(\lambda,z)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n f(\lambda,0)}{\partial z^n}z^n=\sum_{n=0}^{\infty}A_n(\lambda)z^n, \tag{2.14}$$

and we have the estimates

$$|A_n(\lambda)|=\frac{1}{n!}\left|\frac{\partial^n f(\lambda,0)}{\partial z^n}\right|\leqslant\frac{C}{(1+\varepsilon)^n}, \tag{2.15}$$

for some constant C, all $k\in\mathbb{N}$ and all $\lambda\in\mathbb{S}^1:=\partial\Delta$. Under the assumptions of the Theorem we see that meromorphic extensions $f_k(\lambda)$ of $f(\lambda,\varphi_k(\lambda))$ have uniformly bounded number of poles counted with multiplicities. As well as the numbers of zeroes of φ_k are uniformly bounded too. Up to taking a subsequence we can suppose that:

a) The number of poles of f_k-s, counted with multiplicities, is constant, say M, and these poles converge to the finite set $b_1,...,b_M\in\Delta_{1-\varepsilon}$ with corresponding multiplicities, *i.e.,* some of $b_1,...,b_M$ may coincide.

b) The number of zeroes of φ_k, counted with multiplicities, is also constant, say N and these zeroes converge to a finite set with corresponding multiplicities. We shall denote it as $a_1,...,a_N$, meaning that some of them can coincide.

Step 1. For every k take a Blaschke product P_k having zeroes exactly at poles of f_k with corresponding multiplicities and subtract from $\{P_k\}$ a converging subsequence with the limit

$$P_0(\lambda) = \prod_{i=1}^{M} \frac{\lambda - b_i}{1 - \bar{b}_i \lambda}. \tag{2.16}$$

Holomorphic functions $g_k := P_k f_k$ have uniformly bounded modulus on Δ and converge to some g_0, with modulus bounded by C (a constant from (2.15)). Therefore f_k converge on compacts of $\Delta \setminus \{b_1, ..., b_M\}$ to a meromorphic function, which is nothing but A_0, and it satisfies the estimate

$$|A_0(\lambda)| \leqslant \frac{CC_1}{|\lambda - b_1| ... |\lambda - b_M|}, \tag{2.17}$$

where $C_1 = \max\{\Pi_{i=1}^{M} |1 - \bar{b}_i \lambda| : |\lambda| \leqslant 1\}$.

Step 2. Consider the function

$$f_1(\lambda, z) := \frac{f(\lambda, z) - A_0(\lambda)}{z}, \tag{2.18}$$

and the following functions

$$f_{1,k}(\lambda) := f_1(\lambda, \varphi_k(\lambda)) = \frac{f(\lambda, \varphi_k(\lambda)) - A_0(\lambda)}{\varphi_k(\lambda)}. \tag{2.19}$$

These functions are well defined and meromorphic on $\Delta_{1+\varepsilon}$, the equality in (2.19) has sense on $A_{1-\varepsilon,1+\varepsilon}$. After taking a subsequence we see that poles of $f_{1,k}$, which are different from zeroes of φ_k, converge to the same points $b_1, ..., b_N$. So the multiplicities do not increase. Let again P_k be the Blaschke product having zeroes at poles of $f_{1,k}$ with corresponding multiplicities. After taking a subsequence P_k uniformly converge to a corresponding Blaschke product P_0 and holomorphic functions $g_k := P_k f_{1,k}$ uniformly converge to some holomorphic function g_0. In $A_{1-\varepsilon,1+\varepsilon}$ it is straightforward to see that $g_0 = P_0 A_1$. This proves that A_1 (if not identically zero), has at most $N + M$ poles counting with multiplicities and these poles are located at $a_1, ..., a_N, b_1, ..., b_M$.

Moreover, for $|\lambda| = 1$ from (2.15) we have the estimate

$$|P_0(\lambda) A_1(\lambda)| \leqslant \frac{C}{1+\varepsilon}, \tag{2.20}$$

which implies the estimate

$$|A_1(\lambda)| \leqslant \frac{1}{|\lambda - a_1| ... |\lambda - a_N| |\lambda - b_1| |\lambda - b_N|} \cdot \frac{CC_1C_2}{1+\varepsilon} \tag{2.21}$$

for $\lambda \in \Delta \setminus \{a_1, ..., a_N, b_1, ..., b_M\}$. Here $C_1 = \max\{\Pi_{i=1}^{N} |1 - \bar{a}_i \lambda| : |\lambda| \leqslant 1\}$. Denote from now CC_1C_2 by C'.

Step 3. Suppose we proved that A_n extends to a meromorphic function in Δ with the estimate

$$|A_n(\lambda)| \leqslant \frac{1}{\prod_{j=1}^{N} |\lambda - a_j|^n \prod_{j=1}^{M} |\lambda - b_j|} \cdot \frac{C'}{(1+\varepsilon)^n} \tag{2.22}$$

for $\lambda \in \Delta \setminus \{a_1, ..., a_N, b_1, ..., b_M\}$. Remark that (2.22) means, in particular, that $A_0, ..., A_n$ have no other poles than $a_1, ..., b_N$ with corresponding multiplicities. Apply considerations as above to

$$f_{n+1}(\lambda,z)=\frac{1}{z^{n+1}}\left(f(\lambda,z)-\sum_{j=0}^{n}A_j(\lambda)z^j\right),$$

i.e., consider

$$f_{n+1,k}(\lambda)=\frac{1}{\varphi_k^{n+1}}\left(f(\lambda,\varphi_k)-\sum_{j=0}^{n}A_j(\lambda)\varphi_k^j\right)$$

and repeat the same demarche with Blaschke products. Remark only that products $A_j(\lambda)\varphi_k^j$ have no poles at zeroes of φ_k. On the boundary $\{|\lambda|=1\}$ functions $|f_{n+1,k}(\lambda)|$ are bounded by $C/(1+\varepsilon)^{n+1}$ due to Cauchy inequalities and therefore we get the conclusion that A_{n+1} meromorphically extends to Δ with the estimate

$$|A_{n+1}(\lambda)|\leqslant\frac{1}{\prod_{j=1}^{N}|\lambda-a_j|^{n+1}\prod_{j=1}^{M}|\lambda-b_j|}\cdot\frac{C'}{(1+\varepsilon)^{n+1}}. \tag{2.23}$$

Estimate (2.23) implies that (2.14) converges in the domain

$$\left\{(\lambda,z)\in\Delta^2:|z|<c|\lambda-a_{j_1}|^{l_1}...|\lambda-a_{N_1}|^{l_{N_1}}\right\}\setminus\bigcup_{i=1}^{M}\{\lambda=b_i\}, \tag{2.24}$$

for an appropriately chosen $c>0$. Here N_1 is the number of *different* a_j-s, which are denoted as $a_{j_1},...,a_{N_1}$ having corresponding multiplicities $l_1,...,l_{N_1}$. In particular we mean here that b_i are different from a_{j_1} for all i,j_1. Estimate (2.23) implies that the extension of $f\cdot\prod_{j=1}^{M}(\lambda-b_j)$ to (2.24) is locally bounded near every vertical disk $\{\lambda=b_{i_1}\}$ and therefore extends across it by Riemann extension theorem. We conclude that f extends as a meromorphic function to the pinched domain

$$\mathcal{P}=\left\{(\lambda,z)\in\Delta^2:|z|<c|\lambda-a_{j_1}|^{l_1}...|\lambda-a_{N_1}|^{l_{N_1}}\right\}, \tag{2.25}$$

and this proves the part *(ii)* of Theorem 1.

(i) Take now any holomorphic function φ in $\Delta_{1+\varepsilon}$ of the form

$$\varphi(\lambda)=(\lambda-a_{j_1})^{l_1}...(\lambda-a_{N_1})^{l_{N_1}}\psi$$

with ψ small enough in order that the graph C_φ is contained in $\mathcal{P}$ (more precisely should be $C_\varphi\cap(\Delta\setminus\{a_{j_1},...,a_{N_1}\})\times\Delta\subset\mathcal{P}$). To prove the part *(i)* of our theorem we need to prove the following

Step 4. $f(\lambda,\varphi(\lambda))$ meromorphically extends from $A_{1-\varepsilon,1+\varepsilon}$ to $\Delta_{1+\varepsilon}$. Indeed, $f(\lambda,\varphi(\lambda))$ is meromorphic on $\Delta\setminus\{a_{j_1},...,a_{N_1}\}$. At the same from the estimate (2.22) we see that the terms in the series

$$f(\lambda,\varphi(\lambda))=\sum_{n=0}^{\infty}A_n(\lambda)\varphi^n(\lambda) \tag{2.26}$$

are, in fact, holomorphic in a neighborhood of every a_j and converge normally there, provided $\|\psi\|_\infty$ was taken small enough. Uniform boundedness of the number of poles follows now from Proposition 2.1. Part *(i)* is proved.

□

Remark 2.3. In order to prove Corollary 1 remark that pinches that appeared along the proof of Theorem 1 are limits of zeroes of φ_k. General position assumption means that for every $\lambda_0\in\Delta$ we can take a subsequence such that the resulting pinched domain will not have a pinch in λ_0. The rest follows.

3. Extension Along Finite Dimensional Families

3.1. Properties of the Singular Integral Transform. By $L^{1,2}(\mathbb{S}^1)$ we denote the Sobolev space of complex valued functions on the unit circle having their first derivative in L^2. This is a complex Hilbert space with the scalar product $(h,g) = \int_0^{2\pi}[h(e^{i\theta})\bar{g}(e^{i\theta}) + h'(e^{i\theta})\bar{g}'(e^{i\theta})]d\theta$. Recall that by Sobolev Imbedding Theorem $L^{1,2}(\mathbb{S}^1) \subset \mathcal{C}^{\frac{1}{2}}(\mathbb{S}^1)$, where $\mathcal{C}^{\frac{1}{2}}(\mathbb{S}^1)$ is the space of Hölder $\frac{1}{2}$-continuous functions on $\mathbb{S}^1$.

For the convenience of the reader we recall few well known facts about the Hilbert Transform in $L^{1,2}(\mathbb{S}^1)$.

Lemma 3.1. *A function $\varphi \in L^{1,2}(\mathbb{S}^1)$ extends holomorphically to Δ if and only if the following condition is satisfied:*

$$P(\varphi)(\tau) := \frac{-1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)-\varphi(\tau)}{t-\tau}dt \equiv 0. \tag{3.1}$$

Proof. The fact that φ extends holomorphically to Δ can be obviously expressed as

$$\lim_{z\to\tau, z\in\Delta}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)}{t-z}dt = \varphi(\tau) \tag{3.2}$$

for all $\tau \in \mathbb{S}^1$. Write then

$$\lim_{z\to\tau, z\in\Delta}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)}{t-z}dt = \lim_{z\to\tau, z\in\Delta}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)-\varphi(\tau)}{t-z}dt + \tag{3.3}$$

$$+\lim_{z\to\tau, z\in\Delta}\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(\tau)}{t-z}dt = -P(\varphi)(\tau) + \varphi(\tau).$$

From (3.2) and (3.3) we immediately get (3.1).

□

Denote by $\mathbb{S}^1_\varepsilon(\tau)$ the circle $\mathbb{S}^1$ without the ε-neighborhood of τ. Consider the following singular integral operator (the Hilbert Transform)

$$S(\varphi)(\tau) := \mathbf{p.v.}\frac{1}{\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)}{t-\tau}dt := \lim_{\varepsilon\to 0}\frac{1}{\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\varphi(t)}{t-\tau}dt. \tag{3.4}$$

In the sequel we shall write simply

$$S(\varphi)(\tau) := \frac{1}{\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)}{t-\tau}dt, \tag{3.5}$$

i.e., the integral in the right hand side will be always understood in the sense of the principal value.

Lemma 3.2. *The following relation between operators S and P holds*

$$S = -2P + \mathsf{Id}. \tag{3.6}$$

Therefore a function $\varphi \in L^{1,2}(\mathbb{S}^1)$ holomorphically extends to the unit disk if an only if

$$S(\varphi)(\tau) \equiv \varphi(\tau). \tag{3.7}$$

Proof. Write

$$\frac{1}{2}S(\varphi)(\tau)=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)}{t-\tau}dt=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\varphi(t)}{t-\tau}dt=$$
$$=\frac{1}{2\pi i}\int_{\mathbb{S}^1}\frac{\varphi(t)-\varphi(\tau)}{t-\tau}+\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\mathbb{S}^1_\varepsilon(\tau)}\frac{\varphi(\tau)}{t-\tau}dt=-P(\varphi)(\tau)+\frac{1}{2}\varphi(\tau).$$

Therefore one has

$$S(\varphi)=-2P(\varphi)+\varphi, \tag{3.8}$$

which is (3.6), and which implies (3.7).

□

Denote by $H^{1,2}_+(\mathbb{S}^1)$ the subspace of $L^{1,2}(\mathbb{S}^1)$ which consists of functions holomorphically extendable to the unit disk Δ. By $H^{1,2}_-(\mathbb{S}^1)$ denote the subspace of functions holomorphically extendable to the complement of the unit disk in the Riemann sphere $\mathbb{P}^1$ and zero at infinity. Observe the following orthogonal decomposition

$$L^{1,2}(\mathbb{S}^1)=H^{1,2}_+(\mathbb{S}^1)\oplus H^{1,2}_-(\mathbb{S}^1). \tag{3.9}$$

We finish this review with the following:

Lemma 3.3. *i) P and S are bounded linear operators on $L^{1,2}(\mathbb{S}^1)$ and*

$$S^2=\mathsf{Id}. \tag{3.10}$$

ii) Moreover, on the space $H^{1,2}_+(\mathbb{S}^1)$ operator S acts as identity and on the space $H^{1,2}_-(\mathbb{S}^1)$ as $-\mathsf{Id}$.

iii) Consequently P is an orthogonal projector onto $H^{1,2}_-(\mathbb{S}^1)$.

For the proof of (3.10) we refer to [MP] pp. 46, 50, 69. In fact, since $S=-2P+\mathsf{Id}$ and because of $\mathsf{Ker}\,P=H^{1,2}_+(\mathbb{S}^1)$, we see that $S=\mathsf{Id}$ on $H^{1,2}_+(\mathbb{S}^1)$. From (3.10) and (3.6) we also see that $P=\mathsf{Id}$ on $H^{1,2}_-(\mathbb{S}^1)$, *i.e.,* P projects $L^{1,2}(\mathbb{S}^1)$ onto $H^{1,2}_-(\mathbb{S}^1)$ parallel to $H^{1,2}_+(\mathbb{S}^1)$.

This lemma clearly implies the following

Corollary 3.1. *Function $\varphi\in L^{1,2}(\mathbb{S}^1)$ extends to a meromorphic function in Δ with not more than N poles if and only if $P(\varphi)$ is rational, zero at infinity and has not more than N poles.*

Indeed, decompose $\varphi=\varphi^++\varphi^-$ according to (3.9). φ is meromorphic with at most N poles, all in Δ, if and only if φ^- is such. Which means that φ^- should be ratioanl with at most N poles. But since, according to (*iii*) of Lemma 3.3 one has $P(\varphi)=\varphi^-$, the last is equivalent to the fact that $P(\varphi)$ is rational with at most N poles.

3.2. Case of finite dimensional families. To clarify the finite vs. infinite dimensional issues in this paper let us give a simple proof of Theorem 1 in the special case when φ_k belong to an analytic family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ parameterized by a finite dimensional complex manifold $\mathcal{A}$. More precisely, as in section 2, we are given a complex manifold $\mathcal{X}$, a holomorphic submersion $\pi:\mathcal{X}\to\mathcal{A}$ such that for every $\alpha\in\mathcal{A}$ the preimage $\mathcal{X}_\alpha:=\pi^{-1}(\alpha)$ is a disk. We are given also a holomorphic map $\Phi:\mathcal{X}\to\mathbb{C}^2$ such that for every $\alpha\in\mathcal{A}$ the image $\Phi(\mathcal{X}_\alpha)=C_\alpha$ is a graph of a holomorphic function $\varphi_\alpha:\Delta_{1+\varepsilon}\to\Delta$. We shall regard $\mathcal{A}$ as a (locally closed) complex submanifold of $H^{1,2}_+(\mathbb{S}^1)$. And, finally, by saying that $\{\varphi_k\}_{k=1}^{\infty}$ belong to $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ we mean that there exist $\alpha_k \in \mathcal{A}, \alpha_k \to \alpha_0 \in \mathcal{A}$, such that $\varphi_k = \varphi_{\alpha_k}$ for $k \geqslant 0$.

After shrinking, if necessary, we can suppose that our function f is holomorphic on $R_{1+\varepsilon} = A_{1-\varepsilon,1+\varepsilon} \times \Delta_{1+\varepsilon}$. Consider the following analytic mapping $F : L^{1,2}(\mathbb{S}^1) \to L^{1,2}(\mathbb{S}^1)$

$$F : \varphi(\lambda) \to f(\lambda, \varphi(\lambda)), \tag{3.11}$$

and consider also the following integral operator $\mathcal{F} : H_+^{1,2}(\mathbb{S}^1) \to H_-^{1,2}(\mathbb{S}^1)$

$$\mathcal{F}(\varphi)(\lambda) = \frac{-1}{2\pi i}\int_{\mathbb{S}^1} \frac{f(\zeta,\varphi(\zeta)) - f(\lambda,\varphi(\lambda))}{\zeta-\lambda} d\zeta = P\left(F(\varphi)\right). \tag{3.12}$$

According to Lemma 3.1 $f(\lambda,\varphi(\lambda))$ extends to a holomorphic function in $\Delta_{1+\varepsilon}$ if and only if $\mathcal{F}(\varphi) = 0$, and according to Corollary 3.1 it extends meromorphically to $\Delta_{1+\varepsilon}$ with at most N poles in $\Delta_{1-\varepsilon}$ if an only if $\mathcal{F}(\varphi)$ is a boundary value of a rational function with at most N poles all in $\Delta_{1-\varepsilon}$.

Theorem 3.1. *Let f be a meromorphic function on $R_{1+\varepsilon}$ and $\{\varphi_k : \Delta_{1+\varepsilon} \to \Delta\}_{k=1}^{\infty}$ be a sequence of holomorphic functions converging to some $\varphi_0 : \Delta_{1+\varepsilon} \to \Delta$, $\varphi_k \not\equiv \varphi_0$ for all k. Suppose that:*

a) $\{\varphi_k\}$ belong to a finite dimensional analytic family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$, i.e., $\varphi_k = \varphi_{\alpha_k}$ for some $\alpha_k \in \mathcal{A}$ and $\alpha_k \to \alpha_0$ in $\mathcal{A}$ with $\varphi_0 = \varphi_{\alpha_0}$;

b) for every k the restriction $f|_{C_k\cap R_{1+\varepsilon}}$ is well defined and extends to a meromorphic function on the curve C_k;

c) the number of poles counting with multiplicities of these extensions is uniformly bounded.

Then there exists a complex disk $\Delta \subset \mathcal{A}$ containing α_0 such that for every $\alpha \in \Delta$ the restriction $f|_{C_\alpha\cap R_{1+\varepsilon}}$ meromorphically extends to C_α, and the number of poles of these extensions counting with multiplicities is uniformly bounded.

Proof. *i)* Consider the holomorphic case first. Restrict $\mathcal{F}$ to $\mathcal{A}$ to obtain a holomorphic map $\mathcal{F}_{\mathcal{A}} : \mathcal{A} \to H_-^{1,2}(\mathbb{S}^1)$. $f|_{C_\alpha\cap R_{1+\varepsilon}}$ holomorphically extends to C_α if and only if $\mathcal{F}(\alpha) = 0$. Therefore we are interested in the zero set $\mathcal{A}^0$ of $\mathcal{F}_{\mathcal{A}}$. But the zero set $\mathcal{A}^0$ of a holomorphic mapping from a finite dimensional manifold is a *finite dimensional* analytic set. Since this set contains a converging sequence $\{\alpha_k\}$ it has positive dimension. Denote (with the same letter) by $\mathcal{A}^0$ a positive dimensional irreducible component of our zero set which contains an infinite number of φ_{α_k}-s. Suppose, up to replacing α_k by a subsequence, that all α_k are in $\mathcal{A}^0$. Let $\mathcal{X}^0$ be the corresponding universal family, *i.e.*, the restriction of $\pi : \mathcal{X} \to \mathcal{A}$ to $\mathcal{A}^0$, and $\Phi^0 : \mathcal{X}^0 \to \Delta_{1+\varepsilon} \times \Delta$ the corresponding evaluation map. Φ^0 should be of generic rank two, otherwise φ_k would be constant. Therefore $\mathcal{A}^0$ contains a complex disk through α_0 with properties as required.

ii) The meromorphic extension in this case is also quite simple. Without loss of generality we suppose that all extensions $f_{\alpha_k}(\lambda)$ have at most N poles counting with multiplicities. Since $f_{\alpha_k}(\lambda) = f(\lambda, \varphi_{\alpha_k}(\lambda))$ for $\lambda \in A_{1-\varepsilon,1+\varepsilon}$, all poles of these extensions are contained in $\bar{\Delta}_{1-\varepsilon}$. Denote by $R^N(1-\varepsilon)$ the subset of $H_-^{1,2}(\mathbb{S}^1)$ which consists of rational functions, holomorphic on $\mathbb{P}^1 \setminus \bar{\Delta}$, zero at infinity and having not more than N poles, all contained in $\bar{\Delta}_{1-\varepsilon}$. $R^N(1-\varepsilon)$ can be explicitly described as the set of the following functions:

$$R^N(1-\varepsilon) = \left\{ \sum_j (z-a_j)^{-m_j} \sum_{k=0}^{m_j-1} c_{jk}(z-a_j)^k : c_{jk} \in \mathbb{C}, a_j \in \bar{\Delta}_{1-\varepsilon}, \sum_j m_j = N \right\}. \tag{3.13}$$

Let us note that $\mathcal{F}_{\mathcal{A}}(\varphi_{\alpha_k}) \in R^N(1-\varepsilon)$ for all k and that the set $\mathcal{A}^N$ of those $\alpha \in \mathcal{A}$ that $f(\lambda, \varphi_\alpha(\lambda))$ is meromorphically extendable to Δ with not more N poles, all in $\bar{\Delta}_{1-\varepsilon}$, is in fact $\mathcal{F}_{\mathcal{A}}^{-1}(R^N(1-\varepsilon))$.

Set $g_0 = \mathcal{F}_{\mathcal{A}}(\varphi_{\alpha_0})$. From (3.13) we see that $R^N(1-\varepsilon)$ is a finite dimensional subspace of $H^{1,2}_-(\mathbb{S}^1)$. Therefore we can take an orthogonal complement $H \subset H^{1,2}_-(\mathbb{S}^1)$ to it at g_0 in such a way that $H^{1,2}_-(\mathbb{S}^1) = R^N(1-\varepsilon) \times H$ locally in a neighborhood of g_0. Denote by Ψ the composition of $\mathcal{F}_{\mathcal{A}}$ with the projection onto H. Now $\mathcal{A}^N$ is the zero set of Ψ and therefore we are done as in the case *(i)* .

$\square$

Let us make a few remarks concerning the finite dimensional case of the last theorem.

Remark 3.1. a) Horizontal disks belong to an one dimensional family, non-horizontal straight disk to two-dimensional. Therefore Theorem 3.1 generalizes Hartogs-Levi theorem and the result of Dinh.

b) We do not claim in Theorem 3.1 that the disks Δ contains α_k for $k >> 1$ and this is certainly not true in general. What is true is that the set $\mathcal{A}^0$ (or $\mathcal{A}^N$) of *all* $\alpha \in \mathcal{A}$ such that f extends along C_α is an analytic set of positive dimension, so contains an analytic disk with center at α_0.

Remark 3.2. At the same time Theorem 3.1 is a particular case of Theorem 1 because of the following observation.

Proposition 3.1. *Let $(\mathcal{X}, \pi, \mathcal{A}, \Phi)$ be a finite dimensional analytic family of holomorphic maps $\Delta_{1+\varepsilon} \to \Delta$ and let $\alpha_0 \in \mathcal{A}$ be a point. Then there exists a neighborhood $V \ni \alpha_0$, a complex hypersurface A in V and a radius $r \sim 1$ such that for $\alpha \in V \setminus A$ the restriction $(\varphi_\alpha - \varphi_0)|_{\partial\Delta_r}$ doesn't vanish and therefore $\mathsf{VarArg}_{\partial\Delta}(\varphi_\alpha - \varphi_{\alpha_0})$ is constant on $V \setminus A$. If, in particular, $(\mathcal{X}, \pi, \mathcal{A}, \Phi)$ is a one-parameter family then $A = \{\alpha_0\}$.*

Proof. After shrinking we can suppose that $\mathcal{X} = \Delta_{1+\varepsilon} \times \Delta^n$, $\alpha_0 = 0$ and $\varphi_0 \equiv 0$. Since $\Phi : \mathcal{X} \to \Delta_{1+\varepsilon} \times \Delta$ writes as $(\lambda, \alpha) \to (\lambda, \varphi(\lambda, \alpha))$ we can consider the zero divisor $\mathcal{Z} = \varphi^{-1}(0)$ of φ. $\mathcal{Z}$ is not empty, because $\varphi(\lambda, 0) \equiv 0$, and is proper, because Φ is of generic rank two. Denote by $\mathcal{Z}_1$ the union of all irreducible components of $\mathcal{Z}$ which contain $\Delta_{1+\varepsilon} \times \{0\}$. Set $A := \mathcal{Z}_1 \cap \{0\}$ and remark that A is a hypersurface in Δ^n.

Denote by $\mathcal{Z}_0$ the union of all irreducible components of $\mathcal{Z}$ which do not contain $\Delta_{1+\varepsilon} \times \{0\}$. Intersection $\mathcal{Z}_0 \cap \Delta_{1+\varepsilon} \times \{0\}$ is a discrete set. Therefore we can find $r \sim 1$ such that $\mathcal{Z}_0 \cap \partial\Delta_r = \varnothing$. Now it is clear that for a sufficiently small neighborhood $V \ni 0$ in Δ^n we have that $\varphi(\cdot, \alpha)$ doesn't vanish on $\partial\Delta_r$ provided $\alpha \in V \setminus A$. Then $\mathsf{VarArg}_{\partial\Delta}(\varphi_\alpha)$ is clearly constant. In one-parameter case A is discrete but contains α_0.

$\square$

c) Let us remark that in general test sequence doesn't belong to any finite dimensional family. Take for example $\varphi_k(\lambda) = \frac{1}{k}\lambda^2 + e^{-k}\lambda^k$. Therefore Theorem 1 properly contains Theorem 3.1.

Remark 3.3. a) If C_k are intersections of $\Delta_{1+\varepsilon} \times \Delta$ with algebraic curves of bounded degree, then they are included in a finite dimensional analytic (even algebraic in this case) family.

b) If $\varphi_k(\partial\Delta) \subset M$, where M is totally real in $\partial\Delta \times \bar{\Delta}$, and have bounded Maslov index then they are included in a finite dimensional analytic family.

c) If we do not suppose *ad hoc* that φ_k belong to some finite dimensional analytic family of holomorphic functions then the argument above is clearly not sufficient. The following example is very instructive. Consider a holomorphic map $\mathcal{F} : l^2 \to l^2 \oplus l^2$ defined as

$$\mathcal{F} : \{z_k\}_{k=1}^{\infty} \to \{\{z_k(z_k - 1/k)\} \oplus \{z_k z_j\}_{j>k}\}. \tag{3.14}$$

The zero set of $\mathcal{F}$ is a sequence $\{Z_k = (0,...,0,1/k,0,...)\}_{k\geqslant 1} \subset l^2$ together with zero. These Z_k-s might well be ours φ_k-s and therefore we cannot conclude the existence of families in the zero set of our $\mathcal{F}$ from (3.12) at this stage.

d) Example 1 has precisely the feature as above with $\mathcal{F}$ being the integral operator (3.12).

4. Mappings to Complex Spaces

In this section we shall prove Corollary 2. The proof consists in making a reduction to the holomorphic function case of Theorem 1. This reduction will follow the lines of arguments developed in [Iv1, Iv2, Iv3, Iv4]. For the convenience of the reader we shall briefly recall the key statements from these papers which are relevant to our present task.

4.1. Continuous families of analytic disks. Analytic disk in a complex space X is a holomorphic map $h : \Delta \to X$ continuous up to the boundary. Recall that a complex space X is called disk-convex if for every compact $K \Subset X$ there exists another compact $\hat{K}$ such that for every analytic disk $h : \bar{\Delta} \to X$ with $h(\partial\Delta) \subset K$ one has $h(\bar{\Delta}) \subset \hat{K}$. $\hat{K}$ is called the *disk envelope* of K. All compact, Stein, 1-convex complex spaces are disk-convex.

Given a meromorphic mapping $f : R_{1+\varepsilon} \to X$, where $R_{1+\varepsilon} = A_{1-\varepsilon,1+\varepsilon} \times \Delta$, we can suppose without loss of generality that f is holomorphic on $R_{1+\varepsilon}$ and that $f(R_{1+\varepsilon})$ is contained in some compact K. We suppose that our space X is reduced and that it is equipped with some Hermitian metric form ω. Denote by $\nu = \nu(\hat{K})$ the minima of areas of rational curves in the disk envelope $\hat{K}$ of K. Remark that ν is achievable by some rational curve and therefore $\nu > 0$. We are given an *uncountable* family of disks $\{C_t : t \in T\}$ which are the graphs of holomorphic functions $\varphi_t : \Delta_{1+\varepsilon} \to \Delta$. Remark that the condition on our family of disks to be *test* (see Introduction) implies, in particular, that they are all distinct. In the sequel when writing $f(C_t)$ we mean more precisely $f|_{C_t}(C_t)$, *i.e.,* the restriction f to C_t. This is an analytic disk in X and since $f|_{C_t}(\partial C_t) \subset K$ we see that for every $t \in T$ one has $f(C_t) \subset \hat{K}$. For every natural k set

$$T_k = \left\{ t \in T : \mathsf{area}\left(\Gamma_{f|_{C_t}}\right) \leqslant k\frac{\nu}{2} \right\}, \tag{4.1}$$

where $\Gamma_{f|_{C_t}} =: \Gamma_t$ is the graph of $f|_{C_t}$ in $\Delta^2_{1+\varepsilon} \times X$ and area is taken with respect to the standard Euclidean form $\omega_e = dd^c(|\lambda|^2 + |z|^2)$ on $\mathbb{C}^2$ and ω on X. For some k the set $T_k \setminus T_{k-1}$ is uncountable, so denote this set as T again.

It will be convenient in the sequel to consider our parameter space T as a subset of the space of 1-cycles in $\Delta^2_{1+\varepsilon} \times X$. Let us say few words about this issue. For general facts about cycle spaces we refer to [Ba], for more details concerning our special situation to §1 of [Iv4]. Recall that a 1-cycle in a complex space Y is a formal sum $Z = \sum_j n_j Z_j$, where $\{Z_j\}$ is a locally finite sequence of irreducible analytic subsets of Y of pure dimension one. The space of analytic 1-cycles in Y will be denoted as $\mathcal{C}^{loc}_1(Y)$. It carries a natural topology, *i.e.,* the topology of currents.

From now on $Y = \Delta^2_{1+\varepsilon} \times X$. Denote by $\mathcal{C}_T$ the subset of $\mathcal{C}^{loc}_1(Y)$ which consits of graphs Γ_t, *i.e.,* $\mathcal{C}_T = \{\Gamma_t : t \in T\}$. We see $\mathcal{C}_T$ as a topological subspace of $\mathcal{C}$ and in the sequel we shall identify T with $\mathcal{C}_T$. Indeed, note that $t \to \Gamma_t$ is injective, because such is already $t \to C_t$. Since T was supposed to be uncountable, therefore so such is also $\{\Gamma_t : t \in T\} = \mathcal{C}_T$.

Denote by $\bar{\mathcal{C}}_T$ the closure of $\mathcal{C}_T$ in our space of 1-cycles $\mathcal{C}_1^{loc}(Y)$ on $\Delta_{1+\varepsilon}^2 \times X$. Cycles Z in $\bar{\mathcal{C}}_T$ are characterized by following two properties:

i) Z has an irreducible component Γ which is a graph of the extension of the restriction $f|_{C \cap R_{1+\varepsilon}}$, where C is a graph of some holomorphic function $\varphi : \Delta_{1+\varepsilon} \to \bar{\Delta}$.

ii) other irreducible components fo Z (if any) are a finite number of rational curves projecting to points in $\Delta \times \Delta_{1+\varepsilon}$.

This directly follows from the theorem of Bishop, because areas of graphs Γ_t are uniformly bounded, and from Lemma 7 in [Iv1], which says that a limit of a sequence of disks is a disk plus a finite number of rational curves. More presicely in (i) we mean that C is a graph of some holomorphic $\varphi : \Delta_{1+\varepsilon} \to \bar{\Delta}$ and $f|_{C \cap R_{1+\varepsilon}}$ holomorphically extends to C with $\Gamma_{f|_C} = \Gamma$, see Lemma 1.3 from [Iv4] for more details. Remark that by the choice we made we have that

$$(k-1)\frac{\nu}{2} \leqslant \mathsf{area}(Z) \leqslant k\frac{\nu}{2} \tag{4.2}$$

for all $Z \in \bar{\mathcal{C}}_T$. Indeed (4.2) is satisfied for $Z = \Gamma_t$ and therefore for their limits.

Remark 4.1. Let us turn attention of the reader that we write Z both for an 1-cycle as an analytic subset of $Y = \Delta_{1+\varepsilon}^2 \times X$ and for a corresponding *point* in the cycle space $\mathcal{C}_1^{loc}(Y)$.

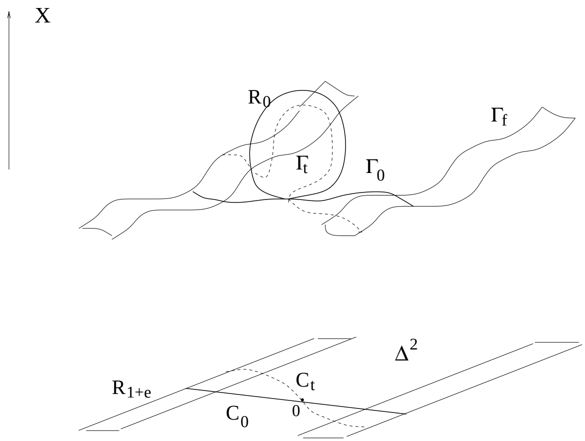

FIGURE 2. When C_t (the punctured curve downstairs) approaching C_0 (the bold line) the graph $\Gamma_t = Z_t$ (the punctured curve upstairs) of $f|_{C_t}$ stays irreducible and approaches Z_0 (the bold curve upstairs). This last is reducible, its irreducible component Γ_0 is a graph over C_0. Its second irreducible component R_0 is a rational curve, which is contained in $\{0\} \times X$. $\Gamma_0 = \Gamma_0' \cup R_0$ is an element of $\bar{\mathcal{C}}_T \setminus \mathcal{C}_T = \mathcal{R}_T$. Γ_f is the graph of f over $R_{1+\varepsilon}$.

Denote by $\mathcal{R}_T$ the subset of reducible cycles in $\bar{\mathcal{C}}_T$. This is a closed subset of $\bar{\mathcal{C}}_T$. Indedd, if Z_n is a converging sequence from $\mathcal{R}_T$ then every Z_n has at least one irreducible component, say R_n, which is a rational curve. Therefore the limit $Z := \lim Z_n$ contains a limit of $R := \lim R_n$ (up to taking a subsequence, if necessary). This R can be only a union of rational curves. I.e., Z is reducible. The difference $\bar{\mathcal{C}}_T \setminus R_{\bar{T}}$ is uncountable since it contains $\mathcal{C}_T$.

From here we get easily that there exists a point $Z_0 \in \bar{\mathcal{C}}_T \setminus \mathcal{R}_T$ having a fundamental system of neighborhoods $\{U_n\}$ in $\mathcal{C}_1^{loc}(Y)$ such that $U_n \cap \bar{\mathcal{C}}_T \subset \bar{\mathcal{C}}_T \setminus \mathcal{R}_T$ for all n and such that all these intersections are uncountable and relatively compact. The last is again by the theorem of Bishop.

Remark now that for every $Z_1, Z_2 \in \bar{\mathcal{C}}_T \cap U_1$ we have

$$|\mathsf{area}\,(Z_1) - \mathsf{area}\,(Z_2)| \leqslant \frac{\nu}{2}. \tag{4.3}$$

This readily follows from (4.2). First step in the proof of Lemma 2.4.1 from [Iv3] states that a family of cycles satisfying (4.3) is continuous in the cycle space topology. Remark that since Z_1 and Z_2 are irreducible we have that $Z_1 = \Gamma_{f|_{C_1}}$ and $Z_2 = \Gamma_{f|_{C_2}}$ for some discs $C_i = \{z = \varphi_(\lambda)\}$, and (4.3) means that $f|_{C_1}$ is close to $f|_{C_2}$.

4.2. The proof. For every radius r close to 1 consider the subfamily T_r of T such that $\forall t \in T_r$ function $\varphi_t - \varphi_{t_0}$ doesn't vanish on $\partial\Delta_r$. Take a sequence $r_m \nearrow 1-\varepsilon/2$. Suppose that T_{r_m} is a most countable for every r_m. Then for all $t \in T \setminus (T_{r_1} \cup ... \cup T_{r_m})$, which is an uncountable set, the function $\varphi_t - \varphi_{t_0}$ vanishes on all $\partial\Delta_{r_i}, i = 1,..,m$. For $m > N$ we get a contradiction (N here bounds the winding numbers of our test family, see discussion before Corollary 2 in Introduction). Therefore for some radius $r \sim 1-\varepsilon/2$ (we can suppose that $r = 1$ after all) we find an uncountable subfamily $T' \subset T$ such that for every $t \in T'$ function $\varphi_t - \varphi_{t_0}$ doesn't vanish on $\partial\Delta$. Take this T' as T and make the reductions of the previous subsection for this T.

Now remark that Z_0 is irreducible, *i.e.*, is an analytic disk, simply because Z_0 was taken from $\bar{\mathcal{C}}_T \setminus \mathcal{R}_T$. According to (*i*) Z_0 is a graph of the extension of $f|_{C_0 \cap R1+\varepsilon}$ to C_0 for some curve $C_0 = \{z = \varphi_0(\lambda)\}$. Take a Stein neighborhood W of the disk $Z_0 = \Gamma_{f|_{C_0}}$, see [Si2] and remark that by continuity of the family $\{Z : Z \in \bar{\mathcal{C}}_T \cap U_1\}$ we have that $Z \subset W$ for all $Z \in U \cap (\bar{\mathcal{C}}_T \setminus \mathcal{R}_T)$ for some neighborhood $U \subset U_1$ of Z_0 in the space of cycles. Every such Z is the graph of the extension to some $C = \{z = \varphi(\lambda)\}$ of the resriction $f|_{C \cap R_{1+\varepsilon}}$. Via an imbedding of W to an appropriate $\mathbb{C}^n$, our f is an n-couple holomorphic functions which holomorphically extend to every corresponding C. We are in position to apply the (holomorphic functions case of) Theorem 1 and get a holomorphic extension of f to an appropriate pinched domain. This finishes the proof.

$\square$

4.3. General position and further assumptions. In practice one looks for extending f to a bidisk Δ^2. As we had seen this depends first of all on whether a test sequence/family is in general position. The last can be expressed in several different ways. One of hem was given in Introduction. Another one was given in [Dh] and used also in [Sk]. It sounds as follows: a family (or, a sequence) $\{C_t\}$ is said to be in general position if for any $t_1 \neq t_1 \neq t_3$ one has

$$C_{t_1} \cap C_{t_2} \cap C_{t_3} = \varnothing, \tag{4.4}$$

i.e., if no three of our curves pass through one point. When C_t *ad hoc* belong to a finite dimensional analytic family this notion is equivalent to ours, simply because the set of α such that f extends along C_α is an analytic set and *a fortiori* forms a pinched domain, to which all but finite of C_t should belong. In general these notions seem to be different. Hoverer let us remark that for an uncountable family condition (4.4) implies ours. Indeed, given $\lambda_0 \in \Delta$ if for every bidisk $\Delta^2((\lambda_0,0),\frac{1}{n})$ the set of $t \in T$ such that $C_t \cap \Delta^2((\lambda_0,0),\frac{1}{n}) = \varnothing$ is at most countable then T would be at most countable, unless almost all C_t pass through $(\lambda_0,0)$. Since this is forbidden by (4.4) we see that there exists an uncountable $T' \subset T$ such that $C_t \cap \Delta^2((\lambda_0,0),\frac{1}{n}) = \varnothing$ for $t \in T'$. Taking a convergent sequence from T' we have that zeroes of this sequence do not accumulate to λ_0. Applying Theorem 1 we extend f to a pinched domain which has no pinch at λ_0. Repeating this argument a finite number of times we extend f to a neighborhood of $\Delta \times \{0\}$.

One can try to define the general position condition as such that it insures the "non-pinching". Again if φ_k a priori belong to a finite dimensional family this condition will be equivalent to the both just discussed. Indeed, after all we know that the set of φ-s such that f extends along its graph is an analytic set in a finite dimensional parameter space. Therefore all φ_k except *finitely many* fit into a positive dimensional families, *i.e.*, all (except finitely many) pass (or not) through some fixed number of points. When φ_k do not belong to a finite dimensional family (but is a test sequence) the situation is unclear. It may happen that the Banach analytic family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ of those φ_α along which f extend doesn't contain any of φ_k. And therefore it is not clear how to "read off" the "non-pinching" property of the family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ from the behavior of φ_k.

For the last point suppose now that our sequence/family is in general position, as in Introduction, and therefore f extends to a neighborhood of $\Delta \times \{0\}$ (or to a neighborhood of the graph C_{φ_0}, but this is the same). The extendability of f further to the whole of Δ^2 depends now on the image space X. More precisely it depends on the fact wether a Hartogs type extension theorem is valid for meromorphic mappings with values in this particular X. If X is projective or, more generally Kähler, then this is true and was proved in [Iv2]. For more general X this is not always the case, see [Iv4] for examples and further statements on this subject.

5. Examples

5.1. Construction of the Example 1.

Example 1. *Let the function f be defined by the following series*

$$\sum_{n=1}^{\infty} 3^{-4n^3} \prod_{j=1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} z^n. \tag{5.1}$$

Then f is holomorphic in the ring domain $R := \mathbb{C}^* \times \mathbb{C}$, holomorphically extends along every $C_k := \{z = \left(\frac{2}{3}\lambda\right)^k\}$, but there doesn't exist an analytic family $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ parameterized by a disk $\mathcal{A} \ni 0$, $\varphi_0 \equiv 0$, such that $f|_{C_\alpha \cap (\mathbb{C}^* \times \mathbb{C})}$ meromorphically extends to C_α for all $\alpha \in \mathcal{A}$.

First of all the terms of this series are holomorphic and converge normally to a holomorphic function in the ring domain $R = \mathbb{C}^* \times \mathbb{C}$. Indeed, fix any $0 < \varepsilon < 1/3$, then for $\varepsilon < |\lambda| < \frac{1}{\varepsilon}$ and $|z| < \frac{1}{3\varepsilon}$ one has

$$\prod_{j=1}^{n} \left| z - \left(\frac{2}{3}\lambda\right)^j \right| \leqslant \prod_{j=1}^{n} \left(\frac{1}{\varepsilon}\right)^j = \left(\frac{1}{\varepsilon}\right)^{\frac{n(n+1)}{2}},$$

and therefore

$$\sum_{n=1}^{\infty} 3^{-4n^3} \prod_{j=1}^{n} \left| z - \left(\frac{2}{3}\lambda\right)^j \right| \cdot \frac{|z|^n}{|\lambda|^{n^2}} \leqslant \sum_{n=1}^{\infty} 3^{-4n^3-n} \left(\frac{1}{\varepsilon}\right)^{\frac{n(n+1)}{2}} \left(\frac{1}{\varepsilon}\right)^{n^2+n} \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} 3^{-4n^3-n} \left(\frac{1}{\varepsilon}\right)^{\frac{3}{2}(n^2+n)}.$$

I.e., the series (5.1) normally converge on compacts in R to a holomorphic function, which will be still denoted as $f(\lambda,z)$. About f let us remark that for

$$z = \varphi_l(\lambda) = \left(\frac{2}{3}\right)^l \lambda^l, \qquad l \geqslant 2, \tag{5.2}$$

the sum in (5.1) is finite and is equal to

$$\sum_{n=1}^{l-1} 3^{-4n^3} \prod_{j=1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \cdot \frac{z^n}{\lambda^{n^2}} = \sum_{n=1}^{l-1} 3^{-4n^3} \prod_{j=1}^{n} \left[\left(\frac{2}{3}\lambda\right)^l - \left(\frac{2}{3}\lambda\right)^j \right] \cdot \left(\frac{2}{3}\right)^{nl} \lambda^{n(l-n)},$$

with all terms being polynomials, because $l > n$ there.

Proposition 5.1. *There doesn't exist a complex one-parameter analytic family $\{\varphi_\alpha\}_{\alpha\in\Delta}$ of holomorphic functions in Δ_2 with values in $\bar{\Delta}$ with $\varphi_0 \equiv 0$ and such that for every $\alpha \in \Delta$ the restriction $f(\lambda,\varphi_\alpha(\lambda))$ extends from Δ_2^* to a meromorphic function in Δ_2.*

Proof. Suppose such family exists and let $\mathcal{P}$ be a corresponding pinched domain. All pinches of $\mathcal{P}$ except at zero can be removed using graphs C_k and Theorem 1. For this it is sufficient to remark that on a small disk Δ_δ around such pinch φ_k never vanishes and therefore our sequence is test on Δ_δ. After that by Proposition 2.1 one can take as our one-parameter family the family

$$\varphi_\alpha(\lambda) = \alpha\lambda^{n_0-1} \tag{5.3}$$

with some $n_0 \geqslant 1$. From (5.3) we see that for λ close to zero the image of $\varphi_\alpha(\lambda)$ as a function of α will contain a disk of radius $\sim c|\lambda|^{n_0}$. Therefore for every $\lambda \in \mathbb{R}^+$ close to zero there exists $\alpha \in \Delta_{1/2}$ such that $\varphi_\alpha(\lambda) \in \mathbb{R}^+$ and $\varphi_\alpha(\lambda) \geqslant c\lambda^{n_0}$ for some constant $c > 0$.

Take some $n_1 > n_0$ such that $\left(\frac{2}{3}\right)^{n_1} < \frac{c}{2}$. First of all represent our function as

$$f(\lambda,z) = f_1(\lambda,z) + \prod_{j=1}^{n_1} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] f_2(\lambda,z), \tag{5.4}$$

where

$$f_1(\lambda,z) = \sum_{n=1}^{n_1} 3^{-4n^3} \prod_{j=1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} z^n$$

and

$$f_2(\lambda,z) = \prod_{j=1}^{n_1} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \cdot \sum_{n=n_1+1}^{\infty} 3^{-4n^3} \prod_{j=n_1+1}^{n} \left[z - \left(\frac{2}{3}\lambda\right)^j \right] \lambda^{-n^2} z^n.$$

Since f_1 is a rational function its restriction $f_1(\lambda,\varphi_\alpha(\lambda))$ will be meromorphic in Δ_2. Therefore would $f(\lambda,\varphi_\alpha(\lambda))$ be meromorphic in Δ_2 we would conclude that $f_2(\lambda,\varphi_\alpha(\lambda))$ is meromorphic in Δ_2 to, unless

$$\prod_{j=1}^{n_1}\left[\varphi_\alpha(\lambda)-\left(\frac{2}{3}\lambda\right)^j\right]$$

is identically zero. The latter is possible only if φ_α is one of φ_l in (5.2). This is not the case and actually by Proposition 3.1 any complex one-parameter family cannot contain a converging sequence with infinitely growing winding numbers. Therefore we have that φ_α is not one of φ_l for all non-zero α small enough. Therefore $f_2(\lambda,\varphi_\alpha(\lambda))$ should be meromorphic in Δ_2 with pole only at zero if we suppose that $f(\lambda,\varphi_\alpha(\lambda))$ is such. This implies that $f_2(\lambda,\varphi_\alpha(\lambda))$ should be meromorphic in $\Delta_2\times\Delta$ as function of two variables (λ,α). But the series

$$f_2(\lambda,\varphi_\alpha(\lambda))=\sum_{n=n_1+1}^{\infty}3^{-4n^3}\prod_{j=n_1+1}^{n}\left[\varphi_\alpha(\lambda)-\left(\frac{2}{3}\lambda\right)^j\right]\lambda^{-n^2}\varphi_\alpha(\lambda)^n \tag{5.5}$$

representing $f_2(\lambda,\varphi_\alpha(\lambda))$ at point $(\lambda,\varphi_\alpha(\lambda))\in\mathbb{R}^+\times\mathbb{R}^+$ can be estimated as follows. Since

$$\prod_{j=n_1+1}^{n}\left[\varphi_\alpha(\lambda)-\left(\frac{2}{3}\lambda\right)^j\right]\geqslant\lambda^{n_0(n-n_1)}\prod_{j=n_1+1}^{n}\left[c-\frac{c}{2}\lambda^{j-n_0}\right]\geqslant\lambda^{n_0(n-n_1)}\left(\frac{c}{2}\right)^{n-n_1},$$

we get that

$$\sum_{n=n_1+1}^{\infty}3^{-4n^3}\prod_{j=n_1+1}^{n}\left[\varphi_\alpha(\lambda)-\left(\frac{2}{3}\lambda\right)^j\right]\lambda^{-n^2}\varphi_\alpha(\lambda)^n\geqslant \tag{5.6}$$

$$\sum_{n=n_1+1}^{\infty}3^{-4n^3}\lambda^{n_0(n-n_1)-n^2}\left(\frac{c}{2}\right)^n c^n\lambda^{n_0n}=\sum_{n=n_1+1}^{\infty}2^{-n}3^{-4n^3}\lambda^{n_0(2n-n_1)-n^2}c^{2n}.$$

The right hand side in (5.6) grows faster than any polynomial of $\frac{1}{\lambda}$ as $\lambda\to 0,\lambda\in\mathbb{R}^+$. Therefore $f_2(\lambda,\varphi_\alpha(\lambda))$ has essential singularity at $\{\lambda=0\}$. Contradiction.

□

5.2. One more example. The following example can be found in [Si1], see p. 16.

Example 2. Let $\{z_k\}_{k=0}^{\infty}$ be a sequence converging to zero, $z_k\neq 0$. Let $P_l(z)$ be a polynomial of degree $l+1$ such that $P_l(z_0)=\ldots=P_l(z_l)=0$ and $P_l(0)\neq 0$ with $\|P_l\|_{L^\infty(\Delta)}=\frac{1}{l!}$. Set

$$f(\lambda,z)=\sum_{l=1}^{\infty}P_l(z)\lambda^{-l}. \tag{5.7}$$

Function f is holomorphic in $\mathbb{C}^*\times\mathbb{C}$ and $\{0\}\times\mathbb{C}$ is its essential singularity. For every z_k the restriction $f|_{C_k}:=f(\cdot,z_k)$, where $C_k:=\Delta\times\{z_k\}$, is rational, having a pole of order k at zero. Moreover disks C_k are test and in general position. Therefore the conclusion of Theorem 1 and of Corollary 1 fails when the orders of poles of restrictions are not uniformly bounded.

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