

EVEN UNIMODULAR LORENTZIAN LATTICES AND HYPERBOLIC VOLUME

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ABSTRACT. We compute the hyperbolic covolume of the automorphism group of each even unimodular Lorentzian lattice. The result is obtained as a consequence of a previous work with Belolipetsky, which uses Prasad's volume to compute the volumes of the smallest hyperbolic arithmetic orbifolds.

1. INTRODUCTION

Let \mathbf{H}^n be the hyperbolic n -space, of constant curvature -1 . We denote by $\text{Isom}(\mathbf{H}^n)$ the group of isometries of \mathbf{H}^n . One way to construct a lattice in $\text{Isom}(\mathbf{H}^n)$ is to consider the automorphism group $O(L)$ of a Lorentzian lattice $L \subset \mathbb{R}^{n,1}$. Of particular interest are the unimodular Lorentzian lattices. There exist two such types of lattices: the odd unimodular Lorentzian lattice $I_{n,1}$ and the even unimodular Lorentzian lattice $II_{n,1}$. Their study appears in connection with the study of Euclidean lattices, as shown in the book of Conway and Sloane [4]. While $I_{n,1}$ exists for every dimension n , the even lattice $II_{n,1}$ exists only when $n \equiv 1 \pmod{8}$.

In [2] (see also [5]) the following theorem was proved.

Theorem 0. *For each odd dimension $n = 2r - 1 \geq 5$, there is a unique orientable non-compact arithmetic hyperbolic n -orbifold $\Delta_n \backslash \mathbf{H}^n$ of the smallest volume (with Δ_n an arithmetic lattice of $\text{Isom}(\mathbf{H}^n)$). Its volume is given by:*

$$\begin{aligned}
 (1) \quad & \frac{1}{2^{r-2}} \zeta(r) \prod_{j=1}^{r-1} \frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) && \text{if } n \equiv 1 \pmod{8}; \\
 (2) \quad & \frac{(2^r - 1)(2^{r-1} - 1)}{3 \cdot 2^{r-1}} \zeta(r) \prod_{j=1}^{r-1} \frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) && \text{if } n \equiv 5 \pmod{8}; \\
 (3) \quad & \frac{3^{r-1/2}}{2^{r-1}} L_{\mathbb{Q}(\sqrt{-3})|\mathbb{Q}}(r) \prod_{j=1}^{r-1} \frac{(2j-1)!}{(2\pi)^{2j}} \zeta(2j) && \text{if } n \equiv 3 \pmod{4}.
 \end{aligned}$$

It is remarkable that the smallest volume has the simplest form (1) exactly for the dimensions n where the even unimodular Lorentzian lattice $II_{n,1}$ exists. The main purpose of this article is to show that for these n the arithmetic group Δ_n of Theorem 0 is actually given by the group $\text{SO}(II_{n,1})$

Date: June 5, 2012.

of special automorphisms of $\mathrm{II}_{n,1}$ (cf. Theorem 1). In particular, this allows to deduce in Corollary 2 the hyperbolic covolume of the automorphism group $\mathrm{O}(\mathrm{II}_{n,1})$. This complements the work of Ratcliffe and Tschantz [8], where the covolume of $\mathrm{O}(\mathrm{I}_{n,1})$ was determined for every n .

In §3–4 we discuss some interesting consequences of our main result. Finally, in §5 we discuss the case of Formulas (2)–(3). In particular, we state in Proposition 4 the exact relation between Δ_n and $\mathrm{O}(\mathrm{I}_{n,1})$ when $n \equiv 5 \pmod{8}$.

Acknowledgements. I would like to thank Curtis McMullen for the interesting discussions that are at the origin of this article. I thank Jiu-Kang Yu for his help concerning Bruhat-Tits theory, Steve Tschantz for the numerical computation mentioned in §3, Anna Felikson and Pavel Tumarkin for helpful discussions, and Ruth Kellerhals, John Ratcliffe and the referee for helpful comments. I am thankful to the MPIM in Bonn for the hospitality and the financial support.

2. MAIN RESULT AND ITS PROOF

For $n \equiv 1 \pmod{8}$, we consider the even unimodular lattice $\mathrm{II}_{n,1}$ embedded in the real quadratic space equipped with the standard rational quadratic form:

$$(4) \quad q(x) = -x_0^2 + x_1^2 + \cdots + x_n^2.$$

The group of automorphisms of this quadratic space acts then isometrically on \mathbf{H}^n , via an identification of \mathbf{H}^n with its projective model. The group $\mathrm{O}(\mathrm{II}_{n,1})$ (resp. $\mathrm{SO}(\mathrm{II}_{n,1})$) of automorphisms (resp. special automorphisms) preserving q and the lattice $\mathrm{II}_{n,1}$ acts discontinuously on \mathbf{H}^n . More precisely, the group $\mathrm{PO}(\mathrm{II}_{n,1}) = \mathrm{O}(\mathrm{II}_{n,1})/\{\pm I\}$ (resp. $\mathrm{PSO}(\mathrm{II}_{n,1}) = \mathrm{SO}(\mathrm{II}_{n,1})/\{\pm I\}$), where I is the identity matrix, can be seen as a discrete subgroup of $\mathrm{Isom}(\mathbf{H}^n)$.

Theorem 1. *For $n \equiv 1 \pmod{8}$, the group Δ_n is conjugate in $\mathrm{Isom}(\mathbf{H}^n)$ to $\mathrm{PSO}(\mathrm{II}_{n,1})$.*

Proof. We denote by V the quadratic space over \mathbb{Q} equipped with quadratic form $\frac{1}{2}q$, where q is given in (4). Let G be the algebraic group defined over \mathbb{Q} with $G(\mathbb{Q}) = \mathrm{Spin}(V)$, the group of spinors of V . Let \overline{G} be the adjoint form of G . Then $\overline{G}(\mathbb{R})$ is isomorphic to $\mathrm{Isom}(\mathbf{H}^n)$. For each prime p we consider the quadratic space $V_p = V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and the Bruhat-Tits building \mathcal{B}_p associated with $\mathrm{Spin}(V_p)$ and $\mathrm{SO}(V_p)$. Note that G and \overline{G} are split over \mathbb{Q}_p , for every prime p (cf. [2, Prop. 3.9]).

Let L be the lattice in V that identifies to $\mathrm{II}_{n,1}$ via the embedding in the quadratic space $(V \otimes_{\mathbb{Q}} \mathbb{R}, q)$. For each prime p , we consider the lattice $L_p = L \otimes \mathbb{Z}_p$, which is a maximal lattice in V_p [3, §5]. Bruhat-Tits theory allows to identify the lattice L_p as an hyperspecial point of the building \mathcal{B}_p (cf. [3, §5]), whose stabilizer in $\mathrm{SO}(V_p)$ is $\mathrm{SO}(L_p)$.

Let us denote by K_p the hyperspecial parahoric subgroup of $\text{Spin}(V_p)$ that stabilizes $L_p \in \mathcal{B}_p$. The set of all these K_p for p prime is a coherent collection of parahoric subgroups, and this defines a principal arithmetic subgroup of $G(\mathbb{Q})$ (see [2, §2.2] for details):

$$(5) \quad \Lambda = G(\mathbb{Q}) \cap \prod_p K_p,$$

which by construction maps into $\text{SO}(L) = \text{SO}(\text{II}_{n,1})$. But Λ corresponds exactly to the group Λ_1 in [2], whose image in $\overline{G}(\mathbb{R})$ gives the group Δ_n . It was proved in [2] that Λ_1 is maximal, and that up to conjugacy its construction does not depend on the choice of a coherent collection of hyperspecial subgroups. It follows that $\text{PSO}(\text{II}_{n,1})$ is conjugate to Δ_n in $\text{Isom}(\mathbf{H}^n)$. \square

From Theorem 0 and 1 we obtain the covolume of the group $\text{PO}(\text{II}_{n,1})$, which contains $\text{PSO}(\text{II}_{n,1})$ as a subgroup of index two. In order to simplify even more the volume formula, we use the well-known expression of $\zeta(2j)$ in terms of the Bernoulli number B_{2j} .

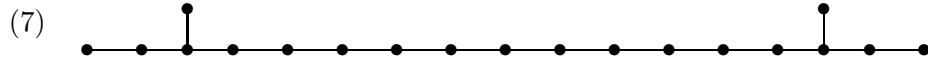
Corollary 2. *The covolume of the action of $\text{PO}(\text{II}_{n,1})$ on \mathbf{H}^n equals*

$$(6) \quad \zeta(r) \prod_{j=1}^{r-1} \frac{|B_{2j}|}{8^j},$$

where B_k is the k -th Bernoulli number.

3. VOLUME OF COXETER POLYTOPES

Corollary 2 was already known in dimension $n = 9$ (see §5), where $\text{PO}(\text{II}_{9,1})$ is the Coxeter group generated by reflections through the faces of a simplex. The only other group $\text{PO}(\text{II}_{n,1})$ that is reflective is $\text{PO}(\text{II}_{17,1})$, as it follows from the work of Conway and Vinberg (cf. [4, Ch. 27] and [11, Part II Ch. 6 §2.1]). It contains as a subgroup of index two the following Coxeter group:



Corollary 3. *Let $P \subset \mathbf{H}^{17}$ be a Coxeter polytope corresponding to the diagram (7). Then*

$$\begin{aligned} \text{vol}_{\mathbf{H}}(P) &= \frac{691 \cdot 3617}{2^{38} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17} \zeta(9) \\ &\approx 2.072451981 \cdot 10^{-18}. \end{aligned}$$

It is not clear how one could compute precisely the volume of such an high- and odd-dimensional hyperbolic polytope without an identification with the fundamental domain of an arithmetic group. Steve Tschantz was able to compute the following numerical approximation, which agrees with the result of Corollary 3.

$$(8) \quad \text{vol}_{\mathbf{H}}(P) = 2.069 \cdot 10^{-18} \pm 2.4 \cdot 10^{-20}.$$

The computation took about 60 hours, showing that for this kind of polytopes even numerical computation is not an easy task.

It is most likely that P realizes the smallest volume among all hyperbolic Coxeter polytopes (non-compact or not), independently of the dimension. The results of [5, 2] (odd dimensions) and [1] (even dimensions) determine the smallest possible arithmetic orientable hyperbolic orbifolds. From them we see that a Coxeter polytope smaller than P and being the fundamental domain of an arithmetic group must necessarily lie in \mathbf{H}^{17} , be commensurable to P , and have exactly half of the volume of P . We don't know if such a Coxeter polytope could exist.

The small size of P can also be explained by Schläfli differential formula for the volume of polytopes (see [11, Part I Ch. 7 §2.2]). According to this formula, the volume of Coxeter polytopes tends to be smaller for polytopes having large dihedral angles. The small size of P results then from the combination of two factors: the only dihedral angles in P are $\pi/2$ and $\pi/3$; and relatively to its dimension, P is determined by few hyperplanes (actually the smallest possible number in \mathbf{H}^{17}). These two conditions are a very rare occurrence in high dimensions.

4. COMPARISON WITH THE MASS FORMULA

The lattice $\text{II}_{25,1}$ plays an important role in connection with the study of even unimodular Euclidean lattices in dimension 24 (see [4, Theorem 5, Ch. 26]). For $n \equiv 1 \pmod{8}$, let \mathcal{L}_{n-1} denotes the set (up to isomorphism) of $(n-1)$ -dimensional even unimodular Euclidean lattices. This is a finite set, and an important invariant is its *mass*, defined as

$$(9) \quad \text{mass}(\mathcal{L}_{n-1}) = \sum_{L \in \mathcal{L}_{n-1}} \frac{1}{|\text{O}(L)|}.$$

For $n = 9, 17$ and 25 each group $\text{O}(L)$ (with $L \in \mathcal{L}_{n-1}$) appears as a subgroup of $\text{O}(\text{II}_{n,1})$ as the stabilizer of a point at infinity of \mathbf{H}^n . Therefore, the groups $\text{O}(L)$ correspond to cusps of the hyperbolic orbifold defined by $\text{O}(\text{II}_{n,1})$, and $\text{mass}(\mathcal{L}_{n-1})$ could be regarded as a measurement of the contribution from these cusps to the volume. It is then quite natural to consider the ratio “covolume of $\text{O}(\text{II}_{n,1})$ divided by $\text{mass}(\mathcal{L}_{n-1})$ ”. From the mass formula [4, Theorem 2, Ch.16] we obtain the rather simple formula:

$$(10) \quad \frac{\text{covolume of } \text{O}(\text{II}_{n,1})}{\text{mass}(\mathcal{L}_{n-1})} = 2^{-r} \frac{|B_{2r-2}|}{|B_{r-1}|} \zeta(r).$$

Note that this ratio goes quickly to ∞ when r grows.

We refer to [10] for more precise results on the behaviour of cusps of arithmetic orbifolds with respect to the dimension.

5. THE CASE OF THE OTHER ODD DIMENSIONS

The covolume of $\mathrm{PO}(\mathbb{I}_{n,1})$ was computed by Ratcliffe and Tschantz in all dimensions $n > 1$ [8]. They obtain the result by evaluating a formula due to Siegel. Note that Prasad's volume formula, the main ingredient to obtain Theorem 0, may be considered as a far-reaching extension of this formula of Siegel. Using the fact that $\mathrm{PO}(\mathbb{II}_{n,1})$ and $\mathrm{PO}(\mathbb{I}_{n,1})$ are commensurable, Ratcliffe and Tschantz could also deduce the covolume of $\mathrm{PO}(\mathbb{II}_{9,1})$ (cf. [6, p. 345]). By the work of Vinberg and Kaplinskaya, the group $\mathrm{PO}(\mathbb{I}_{n,1})$ is known to be reflective for $n \leq 19$, and combining this fact with the work of Ratcliffe and Tschantz one can obtain the volume of several Coxeter polytopes.

By its construction in [2], it is clear that for $n \equiv 5 \pmod{8}$ the arithmetic group Δ_n of Theorem 0 is commensurable to $\mathrm{PSO}(\mathbb{I}_{n,1})$. Moreover, we can see that the ratio of the covolumes of these two groups is equal to 3. In fact, using [3, Prop. 5.9] and the same kind of argument as in the proof of Theorem 1, we get the following result. It agrees with known facts about simplices in dimension 5 (cf. [6, §5]).

Proposition 4. *For $n \equiv 5 \pmod{8}$, the group $\mathrm{PSO}(\mathbb{I}_{n,1})$ is conjugate in $\mathrm{Isom}(\mathbf{H}^n)$ to a subgroup of index 3 in Δ_n .*

For $n \equiv 3 \pmod{4}$, the group Δ_n is not commensurable to $\mathrm{PSO}(\mathbb{I}_{n,1})$. Instead, it is commensurable to the group $\mathrm{PO}(f, \mathbb{Z})$ given by the integral automorphisms of the following quadratic form:

$$(11) \quad f = -3x_0^2 + x_1^2 + \cdots + x_n^2.$$

McLeod showed that the group $\mathrm{PO}(f, \mathbb{Z})$ is reflective when $n \leq 13$ [7]. Recently, elaborating on their earlier work on Siegel's formula (cf. [6, pp. 344–345]), Ratcliffe and Tschantz determined the covolume of $\mathrm{PO}(f, \mathbb{Z})$ (thus obtaining the covolumes of McLeod's polytopes) [9]. For $n \equiv 3 \pmod{4}$, the ratio between the covolumes of $\mathrm{PO}(f, \mathbb{Z})$ and Δ_n is then computed to be equal to $a(n)/4$, where $a(n)$ is some odd integer tending to ∞ when $n \rightarrow \infty$ (see [9, (35)]). An alternative way to obtain the covolume of McLeod's polytopes would be to determine the relation between $\mathrm{PO}(f, \mathbb{Z})$ and Δ_n in terms of subgroup inclusions, using the same kind of arguments as for Theorem 1 and Proposition 4.

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