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# Spectral theory on 3-dimensional hyperbolic space and Hermitian modular forms

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**Abstract.** We study some arithmetics of Hermitian modular forms of degree two by applying the spectral theory on 3-dimensional hyperbolic space. This paper presents three main results: (1) a 3-dimensional analogue of Katok–Sarnak's correspondence, (2) an analytic proof of a Hermitian analogue of the Saito–Kurokawa lift by means of a converse theorem, (3) an explicit formula for the Fourier coefficients of a certain Hermitian Eisenstein series.

**Keywords.** 3-dimensional hyperbolic space, Hermitian modular forms.

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## 1 Introduction

In [17], Imai discovered how one can apply the spectral theory on the upper half-plane to Siegel modular forms of degree two. The purpose of this paper is to generalize this method to Hermitian modular forms with actual applications. More precisely, we present three main results:

(1) a 3-dimensional analogue of Katok–Sarnak's correspondence,

(2) an analytic proof of a Hermitian analogue of the Saito–Kurokawa lift by means of a converse theorem,

(3) an explicit formula for the Fourier coefficients of a certain Hermitian Eisenstein series.

Our main object is to study a unimodular invariant Fourier series $F(Z)$ on the Hermitian upper half-space $H_2 = \{Z \in M_2(\mathbb{C}) : (Z - {}^t\overline{Z})/(2i) > O_2\}$ of degree 2. For $Z \in H_2$, the hermitian imaginary part $Y = (Z - {}^t\overline{Z})/(2i)$ belongs to the set $\mathcal{P}_2$ of all 2 by 2 positive definite hermitian matrices. The set $\mathcal{P}_2$ is parametrized by determinant and 3-dimensional hyperbolic space $\mathbb{H}^3$. In view of

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this fact combined with the unimodular invariance, we can study $F(iY)$ as a function on $\mathrm{SL}_2(\mathcal{O}) \setminus \mathbb{H}^3$ by means of the spectral decomposition. Here $\mathcal{O}$ is the ring of integers of an imaginary quadratic field. Moreover, some properties of $F(iY)$ are also hold for $F(Z)$ by the principle of analytic continuation. Accordingly, the spectral decomposition of $F(iY)$ turns out to be useful in order to study $F(Z)$.

Similarly to the Siegel modular case, a certain integral formula describes the spectral coefficients by the associated Dirichlet series, now called the Koecher–Maass series. Duke–Imamoḡlu [9] observed that the Koecher–Maass series is the Rankin–Selberg convolution of modular forms, whenever the Fourier coefficients of $F(Z)$ satisfy a Maass relation. This fact is important in actual applications. In this point, the key result is Katok–Sarnak's correspondence for Maass forms on the upper half-plane [9, 19]. Our first purpose is to give a precise analogue of this correspondence for automorphic functions on $\mathbb{H}^3$.

Let $K = \mathbb{Q}(i)$ be the Gaussian number field, $\mathcal{O} = \mathbb{Z}[i]$ the ring of all integers, $\mathcal{D}^{-1} = (2i)^{-1}\mathcal{O}$ the inverse different and $\chi_K = (\frac{-4}{\cdot})$ the Kronecker symbol of $K$. Let $\mathbb{H}^3 = \{P = z + rj : z \in \mathbb{C},\ r > 0\}$ be 3-dimensional hyperbolic space. An automorphic function on $\mathbb{H}^3$ for $\mathrm{SL}_2(\mathcal{O})$ is an eigenfunction of the Laplacian

$$\Delta = r^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\right) - r\frac{\partial}{\partial r}$$

which is, in addition, invariant with respect to $\mathrm{SL}_2(\mathcal{O})$ and is of polynomial growth as $r \to \infty$. Denote by

$$L_2 = \left\{T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) : a, d \in \mathbb{Z},\ b \in \mathcal{D}^{-1}\right\}$$

the set of all half-integral hermitian matrices of size two and by $L_2^+ = L_2 \cap \mathcal{P}_2$ the set of all half-integral positive definite hermitian matrices of size two. The group $\mathrm{SL}_2(\mathcal{O})$ acts on each set by $T \to [U]T = UT{}^t\overline{U}$. To any positive definite $T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2^+$, we associate the point

$$P_T = b/d + (\sqrt{\det T}/d)j \in \mathbb{H}^3.$$

While to any indefinite $T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2$, we associate the geodesic hyperplane

$$S_T = \{P = z + rj \in \mathbb{H}^3 : a + b\bar{z} + \bar{b}z + d(|z|^2 + r^2) = 0\}.$$

Moreover, we denote by $E(T) = \{U \in \mathrm{SL}_2(\mathcal{O}) : [U]T = T\}$ the unit group of $T$. Recall that $P_{[\sigma]T} = \sigma P_T$ and $S_{[\sigma]T} = \sigma S_T$ for $\sigma \in \mathrm{SL}_2(\mathbb{C})$ ([11, Propositions 1.2 and 1.4, p. 409]). The following is a 3-dimensional analogue of Katok–Sarnak [19] and Duke–Imamoḡlu [9]. See also [27], [29] and [33].

**Theorem 1.1.** *Let $\mathcal{U}(P)$ be any spectral eigenfunction on $\mathbb{H}^3$ such that $-\Delta\mathcal{U} = (1-\mu^2)\mathcal{U}$ with some complex number $\mu$. In the case of cusp eigenfunctions, there exists a real analytic cusp form $\varphi(\tau)$ on $H_1 = \{\tau = u + iv : v > 0\}$ of weight $-1$, character $\chi_K$ for $\Gamma_0(4)$, namely*

$$\varphi(\gamma\tau) = \chi_K(d)|c\tau + d|(c\tau + d)^{-1}\varphi(\tau), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4),$$

*such that the Fourier expansion*

$$\varphi(\tau) = \sum_{0\neq l\in\mathbb{Z}} b_{\mathcal{U}}(l) W_{-\operatorname{sgn}(l)/2,\, \mu/2}(4\pi|l|v)e^{2\pi i l u}$$

*satisfies*

$$\begin{aligned} b_{\mathcal{U}}(l) &= l^{-1} \sum_{T\in \mathrm{SL}_2(\mathcal{O})\backslash L_2^+,\ 4\det T = l} \mathcal{U}(P_T)/\sharp E(T) && \text{for } l > 0, \\ b_{\mathcal{U}}(l) &= \frac{1}{2\pi}|l|^{-1} \sum_{T\in \mathrm{SL}_2(\mathcal{O})\backslash L_2,\ 4\det T = l} \int_{E(T)\backslash \mathcal{S}_T} \mathcal{U}(P)d\sigma && \text{for } l < 0, \end{aligned} \tag{1.1}$$

*where $d\sigma$ is hyperbolic measure on $\mathcal{S}_T$ (given explicitly in* [13, 27]*) and $W_{\alpha,\beta}(v)$ is the usual Whittaker function. In the case of non-cusp eigenfunctions, there exists a real analytic Eisenstein series $\varphi(\tau)$ of weight $-1$, character $\chi_K$ with respect to $\Gamma_0(4)$ whose Fourier coefficients are given by the same formulas for $l$ such that all of $T \in L_2$ with $4\det T = l$ are not zero-forms. Moreover $\varphi(\tau)$ satisfies the plus condition, that is, if $\chi_K(l) = 1$, then $b_{\mathcal{U}}(l) = 0$ for any integer $l$.*

As discovered by Duke–Imamoḡlu [9] in the Siegel modular case, this allows us to analyze each spectral coefficient of $F(iY)$ by the Rankin–Selberg method. We can reprove a Hermitian analogue of Saito–Kurokawa lift by means of a converse theorem. This lifting was discovered by Kojima [21] and generalized by Krieg [23].

Suppose that a natural number $k$ is divisible by 4. Take a cusp form $g(\tau)$ of weight $k-1$, character $\chi_K$ for $\Gamma_0(4)$ belonging to the plus space in the sense of Kojima [21], that is

$$g(\tau) = \sum_{l\geq 1,\ \chi_K(l)\neq 1} c(l)e^{2\pi i l\tau} \in S_{k-1}(\Gamma_0(4), \chi_K) \quad (\tau \in H_1). \tag{1.2}$$

Put $\alpha^*(l) = c(l)/(\chi_K(-l) + 1)$ and define a function on $H_2$ by

$$F(Z) = \sum_{T\in L_2^+} \left( \sum_{d|e(T)} d^{k-1}\alpha^*((4\det T)/d^2) \right) e^{2\pi i \operatorname{tr}(TZ)}, \tag{1.3}$$

where $e(T) = \max\{q \in \mathbb{N} : q^{-1}T \in L_2^+\}$.

We denote by $\Gamma_2$ the full Hermitian modular group

$$\Gamma_2 = \left\{\gamma \in M_4(\mathcal{O}) : {}^t\overline{\gamma} J \gamma = J\right\}, \quad J = \left(\begin{smallmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{smallmatrix}\right), \tag{1.4}$$

where $O_2$ is the zero matrix and $I_2$ is the identity matrix of size 2.

**Theorem 1.2.** *The function $F(Z)$ is a modular form of weight $k$ for $\Gamma_2$.*

Another example is an application to Hermitian Eisenstein series. Theorem 1.1 makes it possible to determine every spectral coefficients of the non-degenerate part of a certain Hermitian Eisenstein series defined below. Using Maass lift, we can construct a Hermitian modular form with the same spectral coefficients. Consequently, the Hermitian Eisenstein series coincides with this image of the Maass lift. This fact implies an explicit form of the Fourier coefficients of the Hermitian Eisenstein series.

Suppose that $k > 4$ is even and $N$ is a natural number. Let $\omega$ be a character on $\mathcal{O}^\times$ such that $\omega(i) = i^{-k}$ and $\psi$ a character on $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $\psi(-1) = (-1)^k = 1$. Then put $\rho(\epsilon d) = \omega(\epsilon)\psi(d)$ for $\epsilon \in \mathcal{O}^\times, d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Denote by $\Gamma_2$ the full Hermitian modular group (1.4) and put

$$\Gamma_0^{(2)}(N) = \left\{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2 : C \equiv O_2 \pmod{N\mathcal{O}}\right\},$$
$$\Gamma_\infty^{(2)} = \left\{\gamma \in \Gamma_2 : C = O_2\right\}.$$

A Hermitian Eisenstein series of weight $k$, degree two and character $\rho$ for $\Gamma_0^{(2)}(N)$ is then defined by

$$E_{k,\overline{\rho}}^{(2)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty^{(2)} \backslash \Gamma_0^{(2)}(N)} \overline{\rho}(\det D) \det(CZ + D)^{-k}, \quad Z \in H_2.$$

It has a Fourier expansion indexed by positive semi-definite $T \in L_2$. If $N = 1$, some explicit forms of the Fourier coefficients are obtained by Krieg [23] and Nagaoka [35,36]. In fact, we will use their formula to obtain the following theorem for $N > 1$.

For any Dirichlet character $\chi$ mod $M$, we put

$$\tau_M(\chi) = \sum_{r=1}^{M} \chi(r) e^{2\pi i r/M}.$$

**Theorem 1.3.** *Suppose that $N > 1$ is a square-free odd natural number and the above $\psi$ is a primitive Dirichlet character* mod $N$. *The $T$-th Fourier coefficient of the Hermitian Eisenstein series for any positive definite $T \in L_2^+ = L_2 \cap \mathcal{P}_2$ is* *given by*

$$A(T,E^{(2)}_{k,\overline{\rho}}) = \frac{(-2\pi i)^k \tau_N(\overline{\psi})}{N^k \Gamma(k) L(k,\overline{\psi})} \sum_{d|e(T)} \psi(d) d^{k-1} e^{\infty}_{\overline{\rho}}((4\det T)/d^2),$$

*where* $\Gamma(s)$ *is the gamma function,* $L(s,\psi)$ *is the Dirichlet* $L$*-function of* $\psi$*,*

$$e(T) = \max\{q \in \mathbb{N} : q^{-1}T \in L_2^+\}$$

*and* $e^{\infty}_{\overline{\rho}}(t)$ *has the form*

$$\begin{aligned} e^{\infty}_{\overline{\rho}}(t) &= \frac{2^{2-k}\pi^{k-1}}{i^k \Gamma(k-1)} t^{k-2} \frac{\gamma_{2,\overline{\psi}}(t,k-1)}{L(k-1,\chi_K\overline{\psi})} \\ &\qquad \times \prod_{odd\ prime\ p} \gamma_{p,\overline{\psi}}(t,k-1) \prod_{prime\ p|N} C^{\infty}_{\overline{\psi},p}(t), \end{aligned}$$

$$\gamma_{p,\psi}(t,k-1) = \frac{1-(\chi_K(p)\psi(p)p^{2-k})^{l_p+1}}{1-\chi_K(p)\psi(p)p^{2-k}} \quad for\ \ p \neq 2,$$

$$\gamma_{2,\psi}(t,k-1) = \begin{cases} 1, & for\ \ l_2 = 0, \\ 1+\chi_K(-t/2^{l_2})(\psi(2)2^{2-k})^{l_2}, & for\ \ l_2 \geq 1, \end{cases}$$

$$C^{\infty}_{\psi,p}(t) = \psi_p(4) \frac{\psi^*_p(p^{l_p+1})}{p^{(k-1)(l_p+1)}} \chi_K(p)^{l_p+1} \overline{\psi_p}(t/p^{l_p}) p^{l_p} \tau_p(\psi_p).$$

*Here, for any prime* $q$*, we denote by* $l_q$ *the non-negative integer such that* $q^{l_q}$ *is the exact power of* $q$ *dividing* $t$*,* $\psi_p$ *are the primitive Dirichlet characters* mod $p$ *so that*

$$\psi = \prod_{prime\ p|N} \psi_p, \quad \psi^*_p = \prod_{prime\ q|(N/p)} \psi_q.$$

This paper is organized as follows. We prove Theorem 1.1 in Section 2.1, and explain a relation between the form $\varphi(\tau)$ in Theorem 1.1 and a vector valued modular form on $SL_2(\mathbb{Z})$ in Section 2.2. Section 3.1 gives some basic facts on Hermitian Jacobi forms. In Section 3.2, we introduce Hermitian Jacobi Eisenstein series and compute their Fourier coefficients. Some basic facts on Hermitian modular forms and a relation with automorphic functions on 3-dimensional hyperbolic space are given in Section 4. Section 5 gives some analytic preparation. Theorem 1.3 is proved in Section 6 and Theorem 1.2 is proved in Section 7. These are done by studying the associated Koecher–Maass series.

In view of the Siegel modular case (see [3,8,9,17,34]) and the present Hermitian modular case, it seems to be interesting to study modular forms on $O(2,n+1)$ by using the spectral theory on $n$-dimensional hyperbolic space. See also [28].

## 2 Katok–Sarnak type correspondence

We refer to [11] as a basic reference for automorphic functions on 3-dimensional hyperbolic space. A useful summary is also given in [40]. Let

$$\mathbb{H}^3 = \{P = z + rj : z \in \mathbb{C},\ r > 0\}$$

be 3-dimensional hyperbolic space. The action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ on

$$P = z + rj \in \mathbb{H}^3$$

is given by (see [11, p. 3])

$$g \cdot P = \frac{(az+b)\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |c|^2r^2} + \frac{r}{|cz+d|^2 + |c|^2r^2} j.$$

An automorphic function on $\mathbb{H}^3$ for $\mathrm{SL}_2(\mathcal{O})$ is any function $\mathcal{U}(P)$ on $\mathbb{H}^3$ satisfying the following three conditions ([11, Definition 3.5, p. 108]).

(G-i) $\mathcal{U}(\gamma P) = \mathcal{U}(P)$ for all $\gamma \in \mathrm{SL}_2(\mathcal{O})$.

(G-ii) $\mathcal{U}(P)$ is a $C^2$-function on $\mathbb{H}^3$ with respect to $x, y, r$, where

$$P = x + yi + rj \in \mathbb{H}^3.$$

It satisfies a differential equation $-\Delta\mathcal{U} = \lambda\mathcal{U}$ with some $\lambda \in \mathbb{C}$, where $\Delta = r^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}) - r\frac{\partial}{\partial r}$.

(G-iii) $\mathcal{U}(P)$ is of polynomial growth as $r$ tends to $\infty$.

If $\mathcal{U}(P)$ is a cusp eigenfunction such that $\lambda = 1 - \mu^2$ in (G-ii), then $\mathcal{U}(P)$ possesses a Fourier expansion ([11, Theorem 3.1, p. 105])

$$\mathcal{U}(z + rj) = \sum_{0 \neq \lambda \in \mathcal{O}} b_\lambda r K_\mu(2\pi|\lambda|r)e(\Re(\bar{\lambda}z)). \tag{2.1}$$

Here $e(x) = e^{2\pi ix}$ and $K_s$ is the usual $K$-Bessel function. On the other hand, there is an Eisenstein series $E(P,t)$ defined by

$$E(P,t) = \frac{1}{4} \sum_{\substack{c,d \in \mathcal{O} \\ (c,d) = \mathcal{O}}} \left(\frac{r}{|cz+d|^2 + |c|^2r^2}\right)^{1+t}, \quad \Re(t) > 1,$$

where $(c,d)$ is a fractional ideal generated by $c, d$. It has a Fourier expansion of the form (see [11, Definition 1.1, p. 359, (2.21), p. 370])

$$E(z + rj, t) = r^{1+t} + \phi(t)r^{1-t} + \sum_{0 \neq \lambda \in \mathcal{O}} \phi_\lambda(t) r K_t(2\pi|\lambda|r)e(\Re(\bar{\lambda}z)), \tag{2.2}$$

where

$$\phi(t) = \frac{\pi \zeta_K(t)}{t\zeta_K(1+t)}, \quad \phi_\lambda(t) = \frac{2\pi^{1+t}}{\Gamma(1+t)\zeta_K(1+t)}|\lambda|^t \sum_{(\omega)|(\lambda)} |\omega|^{-t}.$$

Here $\zeta_K(t)$ is the Dedekind zeta function of $K$. The Eisenstein series $E(P,t)$ has a meromorphic continuation to the whole complex $t$-plane and it is holomorphic for $\Re(t) > 0$ except for a simple pole at $t = 1$ ([11, Theorem 3.8, p. 377]). See also (5.11) in Section 5.4.

## 2.1 Proof of Theorem 1.1

We prove Theorem 1.1 in this section. In order to describe the spectral decomposition of $L^2(\mathrm{SL}_2(\mathcal{O}) \setminus \mathbb{H}^3)$, we need the constant function $\mathcal{U}_0(P) = \pi/\sqrt{2\zeta_K(2)}$, an orthonormal system of cusp eigenfunctions $\mathcal{U}_m(P)$ (see [11, Proposition 2.2, p. 245, Corollary 3.4, p. 107]) and the Eisenstein series $E(P, it)$, where $t \in \mathbb{R}$. We call any one of them by the spectral eigenfunction in this paper.

If $\mathcal{U}(P)$ is a spectral cusp eigenfunction, Theorem 1.1 follows from [27]. In fact, in [27, Corollary 1.1, p. 485], we take a quadratic form $Q$ of signature $(1, 3)$ and level 4 by

$$Q = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that, if we put

$$v_T = {}^t(a, d, b_1, b_2) \in \mathbb{Z}^4 \quad \text{for } T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2$$

with $b = \frac{i}{2}(b_1 + b_2 i) \in \mathcal{D}^{-1}$, then $4 \det T = {}^t v_T Q v_T$. The Siegel theta function used there is

$$\Theta(\tau, P) = v \sum_{T \in L_2} e((4 \det T)u + iv[P, T]), \ \tau = u + iv \in H_1, P = z + rj \in \mathbb{H}^3,$$

where $[P, T] = 2(\mathrm{tr}(W^{-1}T))^2 - 4 \det T$ with

$$W = \begin{pmatrix} (|z|^2 + r^2)r^{-1} & zr^{-1} \\ \bar{z}r^{-1} & r^{-1} \end{pmatrix}.$$

The theta lifting is defined for any spectral cusp eigenfunction $\mathcal{U}(P)$ by

$$\mathcal{U}^\Theta(\tau) = \int_{\mathcal{F}} \mathcal{U}(P)\Theta(\tau, P)\frac{dxdydr}{r^3},$$

where $\mathscr{F}$ is the fundamental domain of $\mathbb{H}^3$ with respect to $\mathrm{SL}_2(\mathcal{O})$. By [27, Corollary 1.1, p. 485],

$$\varphi(\tau) = 4\pi^{-1/2}\mathcal{U}^{\Theta}(\tau)$$

is a desired Maass form for $\mathcal{U}(P)$. See also [29], where theta lifting including the case of non-cusp eigenfunctions is discussed. If $l$ satisfies $4\det T \neq l$ for all $T$, then the $l$-th Fourier coefficient of $\varphi(\tau) = 4\pi^{-1/2}\mathcal{U}^{\Theta}(\tau)$ does not appear by the construction.

On the other hand, for the Eisenstein series $E(P,t)$ (and the constant function),

$$\varphi(\tau) = 4^{-1}v^{-1/2}F(\tau,t)$$

(and its residue at $t = 1$) is a desired Maass form. Here, $F(\tau,t)$ is the Eisenstein series defined by (see [33, Section 2])

$$\begin{aligned} F(\tau,t) = \frac{\Gamma(t/2)\zeta(t)}{(4\pi)^{(t+1)/2}L(t+1,\chi_K)} & \\ \times v^{t/2+1}\{4^t E_{-1}(4\tau, t/2+1; \chi_0, \chi_K) & \\ + (2i)^{-1}E_{-1}(\tau, t/2+1; \chi_K, \chi_0)\}, & \end{aligned}$$

where $\chi_0$ is the principal character, $\chi_K = (\frac{-4}{\cdot})$ is the Kronecker symbol of $K$ and $E_{-1}(\tau,t;\chi,\psi)$ is the Eisenstein series (see [30, p. 274])

$$E_{-1}(\tau,t;\chi,\psi) = \sum_{(m,n)\neq(0,0)} \chi(m)\psi(n)\frac{(m\tau+n)}{|m\tau+n|^{2t}}, \quad t \in \mathbb{C},\ \Re t > 3/2.$$

In fact, the case $l > 0$ follows from [33, Theorem 5, p. 902]. In the case of $l < 0$, the integral of $E(P,t)$ over $E(T) \setminus S_T$ is the zeta function of representation numbers of binary hermitian forms, if $T$ is not a zero-form ([13, Satz 2.26, p. 19]). In view of [32, Theorem 4, p. 169], their average over all

$$\{T \in \mathrm{SL}_2(\mathcal{O}) \setminus L_2 : 4\det T = l\}$$

is just a sum of divisor functions and it is proportional to the $l$-th Fourier coefficient of $4^{-1}v^{-1/2}F(\tau,t)$ (see the proof of [33, Theorem 5], [13, Korollar 2.27 and Satz 2.28, p. 22]). This completes the proof of Theorem 1.1. □

### 2.2 Plus condition and vector valued modular forms

For the later use, we associate to $\varphi(\tau)$ constructed in Theorem 1.1 a vector valued modular form on $\mathrm{SL}_2(\mathbb{Z})$. Put $f(\tau) = v^{1/2}\varphi(\tau)$. For $\alpha \in \mathcal{D}^{-1}$ and

$$f(\tau) = v^{1/2}\varphi(\tau) = \sum_{l\in\mathbb{Z}} B(l,v)e(lu),$$

put

$$f_\alpha(\tau) = \frac{-2i}{a_4(-4\mathcal{N}(\alpha))} \sum_{l \equiv -4\mathcal{N}(\alpha) \pmod 4} B(l, v/4) e(lu/4), \quad \tau = u + iv, \tag{2.3}$$

and

$$\iota(f)(\tau, z, w) = \sum_{\alpha \in \mathcal{D}^{-1}/\mathcal{O}} f_\alpha(\tau)\theta_\alpha(\tau, z, w), \quad (\tau, z, w) \in H_1 \times \mathbb{C}^2.$$

Here $a_4(l) = \chi_K(-l) + 1$ and $\theta_\alpha(\tau, z, w)$ is the theta function

$$\theta_\alpha(\tau, z, w) = \sum_{\beta \in \alpha + \mathcal{O}} q^{\mathcal{N}(\beta)} \zeta_1^{\overline{\beta}} \zeta_2^{\beta}, \tag{2.4}$$

where $q^\beta = e^{2\pi i \beta\tau}$, $\zeta_1^\alpha = e^{2\pi i \alpha z}$, $\zeta_2^\alpha = e^{2\pi i \alpha w}$.

Fix a representatives of $\mathcal{D}^{-1}/\mathcal{O}$ by

$$\alpha_1 = 0, \quad \alpha_2 = 1/2, \quad \alpha_3 = i/2, \quad \alpha_4 = (1+i)/2.$$

A column vector

$$\Theta^J(\tau, z, w) = {}^t(\theta_{\alpha_1}(\tau, z, w), \dots, \theta_{\alpha_4}(\tau, z, w))$$

of the theta functions satisfies the transformation formula

$$(c\tau+d)^{-1} e^{-2\pi i czw/(c\tau+d)} \Theta^J(\gamma\tau, z/(c\tau+d), w/(c\tau+d)) = U(\gamma)\Theta^J(\tau, z, w) \tag{2.5}$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where $U(\gamma)$ is a certain unitary matrix of size 4. See [23, Lemma, p. 669].

**Proposition 2.1.** *The vector* $(f_{\alpha_1}(\tau), \dots, f_{\alpha_4}(\tau))$ *associate to* $\varphi(\tau)$ *by* (2.3) *satisfies*

$$(f_{\alpha_1}(\tau), \dots, f_{\alpha_4}(\tau)) = (f_{\alpha_1}(\gamma\tau), \dots, f_{\alpha_4}(\gamma\tau))(c\tau + d)U(\gamma)$$

*for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *where* $U(\gamma)$ *is the unitary matrix in* (2.5). *This implies*

$$e^{-2\pi i czw/(c\tau+d)} \iota(f)(\gamma\tau, z/(c\tau+d), w/(c\tau+d)) = \iota(f)(\tau, z, w) \tag{2.6}$$

*for all* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

*Proof.* We follow [23, Section 6] treating holomorphic modular forms. The operators $U_4, V_4$ are defined by

$$g|_{-1}U_4 = \sum_{j \pmod 4} g|_{-1}\left(\begin{smallmatrix} 1 & j \\ 0 & 4 \end{smallmatrix}\right), \quad g|_{-1}V_4 = g|_{-1}U_4|_{-1}Q_4, \quad Q_4 = \left(\begin{smallmatrix} 0 & -1 \\ 4 & 0 \end{smallmatrix}\right),$$

where $g|_{-1}M = (\det M)^{-1/2}(c\tau + d)g(M\tau)$ for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. First of all, we shall prove (cf. [23, Proposition, p. 671])

$$f|_{-1}V_4 = (2i)f. \tag{2.7}$$

In fact this can be shown as follows. By definition,

$$f|_{-1}V_4 = \sum_{j \pmod 4} f|_{-1}\left(\begin{smallmatrix} 4j & -1 \\ 16 & 0 \end{smallmatrix}\right).$$

Decompose as

$$f|_{-1}V_4 = f^+ + f^-.$$

Here we denote by $f^+$ the sum over the terms indexed by even $j$ and by $f^-$ that of odd $j$. The identity $\left(\begin{smallmatrix} 4j & -1 \\ 16 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} j & (j^2-1)/4 \\ 4 & j \end{smallmatrix}\right)\left(\begin{smallmatrix} 4 & -j \\ 0 & 4 \end{smallmatrix}\right)$ for odd $j$ implies

$$f^-(\tau) = 2i \sum_{r=1,3} \chi_K(-r) \sum_{l \equiv r \pmod 4} B(l,v)e(lu).$$

The identity $\left(\begin{smallmatrix} 4j & -1 \\ 16 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1/2 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1-2j & j^2/2 \\ -8 & 2j+1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 4j & -1 \\ 16 & 0 \end{smallmatrix}\right)$ for even $j$ gives

$$f^+(\tau + 1/2) = f^+(\tau).$$

Put $h = f|_{-1}V_4 - (2i)f$ and $F(\tau) = 2^{1/2}h(\tau/2)$. Then

$$F|_{-1}\delta_2 = h, \qquad \delta_m = \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{2.8}$$

$$h|_{-1}\gamma = \chi_K(d)h, \qquad \gamma = \left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right) \in \Gamma_0(4), \tag{2.9}$$

$$F|_{-1}T = F, \qquad T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right). \tag{2.10}$$

Here we used the plus condition $b_{\mathfrak{U}}(l) = 0$ for $\chi_K(l) = 1$ to prove (2.10).

For any $\gamma = \left(\begin{smallmatrix} a & b \\ 4c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, the modularity (2.9) combined with

$$\delta_2\gamma\delta_2^{-1} = \left(\begin{smallmatrix} a & 2b \\ 2c & d \end{smallmatrix}\right)$$

implies

$$F|_{-1}\left(\begin{smallmatrix} a & 2b \\ 2c & d \end{smallmatrix}\right) = \chi_K(d)F. \tag{2.11}$$

By (2.10) and (2.11) with $a = d = 1$,$b = 0$, $c = 1$, we have

$$F|_{-1}\gamma_1 = F, \quad \gamma_1 = T\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)T. \tag{2.12}$$

On the other hand, since $\gamma_1 = \left(\begin{smallmatrix} 3 & 4 \\ 2 & 3 \end{smallmatrix}\right)$, the identity (2.11) with $a = d = 3$, $b = 2$ and $c = 1$ implies $F|_{-1}\gamma_1 = -F$. This combined with (2.12) implies $F = 0$ and $h = 0$. This completes the proof of (2.7).

It is easy to see $f = i\sum_{\beta \in \mathfrak{D}^{-1}/\mathcal{O}} f_\beta|_{-1}\delta_4$ and $f_\beta|_{-1}T^j = e(-j\mathcal{N}(\beta))f_\beta$. Thus the identity $\delta_4\left(\begin{smallmatrix} 1 & j \\ 0 & 4 \end{smallmatrix}\right) = 4T^j$ combined with these two equations implies

$$f|_{-1}\left(\begin{smallmatrix} 1 & j \\ 0 & 4 \end{smallmatrix}\right) = i \sum_{\beta \in \mathfrak{D}^{-1}/\mathcal{O}} e(-j\mathcal{N}(\beta))f_\beta. \tag{2.13}$$

For $\alpha \in \mathcal{D}^{-1}$ and each natural number $\mu$ dividing 4, put $c_\alpha = \{4ia_4(-4\mathcal{N}(\alpha))\}^{-1}$ and set

$$f_\alpha^{(\mu)} = c_\alpha \sum_{r \pmod 4,\,(r,4)=\mu} e(r\mathcal{N}(\alpha)) f|_{-1}\left(\begin{smallmatrix} r & -1 \\ 4 & 0 \end{smallmatrix}\right).$$

In case $\mu = 1$, the identity $\left(\begin{smallmatrix} r & -1 \\ 4 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} r & -1 \\ 4 & -3r \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 3r \\ 0 & 4 \end{smallmatrix}\right)$ for $r = \pm 1$ combined with (2.13) yields

$$\begin{aligned} f_\alpha^{(1)} &= (c_\alpha i) \sum_{\beta \in \mathcal{D}^{-1}/\mathcal{O}} \sum_{r=\pm 1} e(r\mathcal{N}(\alpha) - 3r\mathcal{N}(\beta))\chi_K(-3r) f_\beta \\ &= (-2c_\alpha) \sum_{\beta \in \mathcal{D}^{-1}/\mathcal{O}} \{\chi_K(4\mathcal{N}(\alpha)) + \chi_K(4\mathcal{N}(\beta))\} e(\alpha\overline{\beta} + \beta\overline{\alpha}) f_\beta. \end{aligned}$$

In case $\mu = 2$, we use $f = (2i)^{-1} f|_{-1} V_4$ and

$$\left(\begin{smallmatrix} 4j & -1 \\ 16 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & -1 \\ 4 & 0 \end{smallmatrix}\right) = 4\left(\begin{smallmatrix} 2j-1 & b_j \\ 8 & d_j \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 4b_j - jd_j \\ 0 & 4 \end{smallmatrix}\right),$$

where $b_j, d_j$ are integers such that $(2j-1)d_j - 8b_j = 1$. Then it follows from (2.13) that

$$\begin{aligned} f_\alpha^{(2)} &= (c_\alpha/2) \sum_{\beta \in \mathcal{D}^{-1}/\mathcal{O}} \sum_{j=0}^{3} \chi_K(d_j) e(2\mathcal{N}(\alpha) + jd_j\mathcal{N}(\beta)) f_\beta \\ &= (-2c_\alpha) e(\alpha(1-i)/2 + \overline{\alpha}(1+i)/2) f_{\frac{1+i}{2}}. \end{aligned}$$

In case $\mu = 4$, (2.7) implies $f_\alpha^{(4)} = (ic_\alpha/2) f|_{-1} U_4 = (-2c_\alpha) f_0$.

Because $f_\alpha = c_\alpha \sum_{j \pmod 4} e(\mathcal{N}(\alpha)j) f|_{-1}\left(\begin{smallmatrix} 1 & j \\ 0 & 4 \end{smallmatrix}\right)$, we conclude that

$$f_\alpha|_{-1}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \sum_{\mu=1,2,4} f_\alpha^{(\mu)} = \frac{i}{2} \sum_{\beta \in \mathcal{D}^{-1}/\mathcal{O}} e(\alpha\overline{\beta} + \beta\overline{\alpha}) f_\beta. \tag{2.14}$$

Proposition 2.1 follows from this and $f_\beta|_{-1} T = e(-\mathcal{N}(\beta)) f_\beta$ (cf. [37, proof of Theorem 4]). □

# 3 Hermitian Jacobi forms

## 3.1 Basic facts

In this section we recall some basic facts on Hermitian Jacobi forms. We refer to [14, 23, 37] for more details. Let $H_1 = \{\tau = u + iv : v > 0\}$ be the upper half-plane. The action of $\mathrm{SL}_2(\mathbb{R})$ on $H_1$ is denoted by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\tau = \frac{a\tau+b}{c\tau+d}$. Put

$$U(1,1) = \{\epsilon M : \epsilon \in S^1,\ M \in \mathrm{SL}_2(\mathbb{R})\}.$$

For integers $k$ and $m$, there is an action of the Jacobi group $U(1,1) \ltimes (\mathbb{C}^2 \times S^1)$ for functions $\phi$ on $H_1 \times \mathbb{C}^2$ given by

$$\begin{aligned}\phi|_{k,m}\xi(\tau,z,w) = \epsilon^{-k}(c\tau+d)^{-k} e^m\biggl(&\frac{-c(z+\lambda\tau+\mu)(w+\overline{\lambda}\tau+\overline{\mu})}{c\tau+d} \\ &+\mathcal{N}(\lambda)\tau+\overline{\lambda}z+\lambda w\biggr) \\ &\times s^m\phi\left(M\tau, \frac{\epsilon(z+\lambda\tau+\mu)}{c\tau+d}, \frac{\overline{\epsilon}(w+\overline{\lambda}\tau+\overline{\mu})}{c\tau+d}\right),\end{aligned}$$

where $\xi = \left(\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), (\lambda,\mu), s\right) \in U(1,1) \ltimes (\mathbb{C}^2 \times S^1), (\tau,z,w) \in H_1 \times \mathbb{C}^2$ and $e^m(x) = e^{2\pi i m x}$.

Let $k, m$ and $N$ be natural numbers. We suppose that $k \geq 2$ and $N$ is a square-free odd natural number. Denote by $\Gamma_0^{(1)}(N)$ the congruence subgroup

$$\Gamma_0^{(1)}(N) = \left\{\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) : \epsilon \in \mathcal{O}^\times,\ \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), c \equiv 0 \pmod{N}\right\}.$$

Let $\omega$ be a character on $\mathcal{O}^\times$ such that $\omega(i) = i^{-k}$ and $\psi$ a character on $(\mathbb{Z}/N\mathbb{Z})^\times$ such that $\psi(-1) = (-1)^k$. Put

$$\rho(\epsilon d) = \omega(\epsilon)\psi(d)$$

for $\epsilon \in \mathcal{O}^\times, d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Then $\rho(\epsilon\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)) = \rho(\epsilon d)$ is a character on $\Gamma_0^{(1)}(N)$. We denote by $J_{k,m}(\Gamma_0^{(1)}(N), \rho)$ the space consisting of all holomorphic functions $\phi$ on $H_1 \times \mathbb{C}^2$ satisfying the following two conditions.

(J-i) $\phi|_{k,m}\xi = \rho(\gamma)\phi$ for all $\xi = (\gamma, (\lambda,\mu)) \in \Gamma^J = \Gamma_0^{(1)}(N) \ltimes \mathcal{O}^2$.

(J-ii) For each $M \in \mathrm{SL}_2(\mathbb{Z})$, the function $\phi|_{k,m}M$ has a Fourier expansion of the form

$$\phi|_{k,m}M(\tau,z,w) = \sum_{\substack{n\in\mathbb{Z},\, \alpha\in\mathfrak{D}^{-1} \\ nm-\nu\mathcal{N}(\alpha)\geq 0}} c_M(n,\alpha) q^{n/\nu}\zeta_1^{\alpha}\zeta_2^{\overline{\alpha}}, \tag{3.1}$$

where $q^\beta = e^{2\pi i\beta\tau}, \zeta_1^\alpha = e^{2\pi i\alpha z}, \zeta_2^\alpha = e^{2\pi i\alpha w}$ and $\nu$ is a natural number depending on $M$.

If $m = 1$, then $c_M(n,\alpha)$ depends only on $\alpha \pmod{\mathcal{O}}$ and $t_{(\nu)} = 4(n - \nu\mathcal{N}(\alpha))$ as a consequence of the invariance of $\phi|_{k,m}M$ with respect to $\mathcal{O}^2$. Moreover if $\nu$ is odd, the action of $\epsilon \in \mathcal{O}^\times$ and (J-i) imply that $c_M(n,\alpha)$ depends only on $t_{(\nu)}$ (cf.

[37, Lemma 1, p. 303]). Using the theta functions (2.4), we have a theta expansion of (3.1) such as

$$\phi|_{k,1}M(\tau,z,w) = \sum_{\alpha\in\mathfrak{D}^{-1}/\mathcal{O}} f_\alpha(\tau)\theta_\alpha(\tau,z,w),$$

where

$$f_\alpha(\tau) = \sum_{\substack{l\geq 0\\ l\equiv -\nu 4\mathcal{N}(\alpha) \pmod 4}} c_M(l)q^{l/(4\nu)}, \quad c_M(t_{(\nu)}) = c_M(n,\alpha).$$

In the case of the cusp $\infty$ ($M = I_2$, $\nu = 1$ in (3.1)), one has

$$\phi(\tau,z,w) = \sum_{\substack{l\geq 0,\,\alpha\in\mathfrak{D}^{-1}\\ l\equiv -4\mathcal{N}(\alpha) \pmod 4}} c(l)q^{(l+4\mathcal{N}(\alpha))/4}\zeta_1^{\alpha}\zeta_2^{\overline{\alpha}}.$$

The condition (J-i) combined with (2.5) shows that the coefficient functions in the theta expansion behave like a vector valued modular form on

$$\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N\right\},$$

that is

$$\psi(d)(f_{\alpha_1}(\tau),\ldots,f_{\alpha_4}) = (f_{\alpha_1}(\gamma\tau),\ldots,f_{\alpha_4}(\gamma\tau))(c\tau+d)^{1-k}U(\gamma) \tag{3.2}$$

for all $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \Gamma_0(N)$, where

$$f_\alpha(\tau) = \sum_{\substack{l\geq 0\\ l\equiv -4\mathcal{N}(\alpha) \pmod 4}} c(l)q^{l/4}.$$

By [23, Lemma (i), p. 669], we know $f_\alpha(\tau) \in M_{k-1}(\Gamma(4N))$. Accordingly, one has

$$c(l) = O(l^{k-3/2}) \quad \text{for } l \geq 1.$$

### 3.2 Hermitian Jacobi Eisenstein series

In this section, we compute some Fourier developments of Hermitian Jacobi Eisenstein series on $\Gamma^J = \Gamma_0^{(1)}(N) \ltimes \mathcal{O}^2$ associated with the cusps 0 and $\infty$. With the previous notation, suppose that $N > 1$ is square-free odd, $k > 4$ even and $\psi$ is primitive.

For $G \subset \Gamma_0^{(1)}(1) \ltimes \mathcal{O}^2$, put $G_\infty = \left\{\left(\epsilon\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right),(0,\mu)\right) \in G\right\}$. For any cusp $\kappa$ of $\Gamma_0(N)$, take $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $g(\infty) = \kappa$. The Hermitian Jacobi Eisenstein series of weight $k$ and index 1 associated with $\kappa$ is defined by

$$E^{\kappa}_{k,1,\rho}(\tau,z,w) = \sum_{\gamma\in(g\Gamma^J g^{-1})_\infty\backslash g\Gamma^J} \rho(g^{-1}\gamma)1|_{k,1}\gamma.$$

One easily has

$$E^{\kappa}_{k,1,\rho}|_{k,1}\gamma = \overline{\rho}(\gamma)E^{\kappa}_{k,1,\rho}$$

for all $\gamma \in \Gamma^J$.

The Fourier coefficients of $E^{\kappa}_{k,1,\rho}$ for $\kappa \in \{\infty, 0\}$ can be computed in the same way as in [36, Theorem 2.1, p. 22]. See also the proof of Proposition 3.2 given later. Here we choose $g = I_2$ (resp. $g = \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$) for $\kappa = \infty$ (resp. $\kappa = 0$) so that $g(\infty) = \kappa$.

**Proposition 3.1.** *For $\kappa \in \{\infty, 0\}$, the Fourier development of $E^{\kappa}_{k,1,\rho}$ is given by*

$$E^{\kappa}_{k,1,\rho}(\tau,z,w) = \delta_{\kappa,\infty}\theta_0(\tau,z,w) + \sum_{\substack{t>0,\,\alpha\in\mathfrak{D}^{-1}\\ t\equiv -4\mathcal{N}(\alpha) \pmod 4}} e^{\kappa}_{\rho}(t)q^{(t+4\mathcal{N}(\alpha))/4}\zeta_1^{\alpha}\zeta_2^{\overline{\alpha}},$$

*where*

$$e^{\infty}_{\rho}(t) = \alpha_{k,4}t^{k-2}B_{\psi}(t)\prod_{\textit{prime } p|N} C^{\infty}_{\psi,p}(t), \quad e^{0}_{\rho}(t) = \alpha_{k,4}\psi(-1)t^{k-2}B_{\overline{\psi}}(t).$$

*Here $\alpha_{k,4} = 2^{2-k}\pi^{k-1}i^{-k}\Gamma(k-1)^{-1}$ and $\delta_{i,j}$ is Kronecker's delta,*

$$B_{\psi}(t) = \frac{\gamma_{2,\psi}(t,k-1)}{L(k-1,\chi_K\psi)}\prod_{\textit{odd prime } p}\gamma_{p,\psi}(t,k-1),$$

*where $\gamma_{q,\psi}(t,k-1)$ and $C^{\infty}_{\psi,p}(t)$ are as in Theorem* 1.3. □

Since $N$ is square-free, $\{\infty,0\}\bigcup\{1/\mu : 1<\mu<N,\ \mu \mid N\}$ is a set of representatives of non-equivalent cusps of $\Gamma_0(N)$. We define elements in $\mathrm{SL}_2(\mathbb{Z})$ by

$$\sigma_\infty = I_2, \quad \sigma_0 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right), \quad \sigma_\mu = \left(\begin{smallmatrix}1 & \alpha\\ \mu & N\beta/\mu\end{smallmatrix}\right), \tag{3.3}$$

where integers $\alpha$ and $\beta$ are chosen so that $N\beta/\mu - \alpha\mu = 1$. These transform $\infty$ to each cusps. For the cusp $\kappa$, we will also use symbols $\sigma_\kappa$ instead of (3.3) without any confusion.

**Proposition 3.2.** *The following statements hold:*

(1) *The Fourier development of* $E^0_{k,1,\rho}|_{k,1}\sigma_0$ *is given by*

$$E^0_{k,1,\rho}|_{k,1}\sigma_0(\tau,z,w) = \theta_0(\tau,z,w) + \sum_{\substack{t>0,\,\alpha\in\mathfrak{D}^{-1} \\ t\equiv -4N\mathcal{N}(\alpha) \pmod 4}} a^0_\rho(Nt) q^{(t+4N\mathcal{N}(\alpha))/(4N)} \zeta_1^\alpha \zeta_2^{\overline{\alpha}},$$

*where*

$$a^0_\rho(Nt) = \alpha_{k,4}\chi_K(N) t^{k-2} B_{\overline{\psi}}(Nt) \prod_{\text{prime } p|N} C^\infty_{\overline{\psi},p}(t). \tag{3.4}$$

*Here the notation is the same as in Proposition* 3.1.

(2) *For* $\kappa = 1/\mu$ *with* $1 < \mu < N$, $\mu \mid N$, *the coefficient functions* $H^\kappa_\alpha(\tau)$ *of the theta expansion*

$$E^0_{k,1,\rho}|_{k,1}\sigma_\kappa(\tau,z,w) = \sum_{\alpha\in\mathfrak{D}^{-1}/\mathcal{O}} H^\kappa_\alpha(\tau)\theta_\alpha(\tau,z,w) \tag{3.5}$$

*are of rapid decay as* $\Im\tau$ *tends to* $\infty$.

*Proof.* We prove only (1) here, since the proof of (2) is similar. By definition, we have

$$\begin{aligned} &E^0_{k,1,\rho}|_{k,1}\sigma_0(\tau,z,w) \\ &\quad = \tau^{-k} e(-zw/\tau) E^0_{k,1,\rho}(-1/\tau, z/\tau, w/\tau) \\ &\quad = \theta_0(\tau,z,w) + \sum_{\substack{d>0,\,c\in\mathbb{Z} \\ (c,Nd)=1}} \frac{\overline{\psi}(-c)}{(d\tau-c)^k} \\ &\qquad\qquad \times \sum_{\lambda\in\mathcal{O}} e\biggl(\mathcal{N}(\lambda)\frac{b\tau-aN}{d\tau-c} + \frac{\lambda z+\overline{\lambda}w}{d\tau-c} - \frac{dzw}{d\tau-c}\biggr). \end{aligned}$$

Denote the sum over $c, d, \lambda$ by $B(\tau,z,w)$. In the same manner as in [36, Section 2.2], we get

$$\begin{aligned} B(N\tau,z,w) = &\sum_{\substack{n\in\mathbb{Z},\,\alpha\in\mathfrak{D}^{-1} \\ n-N\mathcal{N}(\alpha)>0}} \alpha_{k,4} N^{1-k}\{4(n-N\mathcal{N}(\alpha))\}^{k-2} \\ &\times \sum_{c\geq 1} c^{-k} \sum_{\substack{d\in(\mathbb{Z}/cN\mathbb{Z})^\times \\ \lambda\in\mathcal{O}/c\mathcal{O}}} \overline{\psi}(d) e_{Nc}(dQ(\lambda)) q^n \zeta_1^\alpha \zeta_2^{\overline{\alpha}}, \end{aligned} \tag{3.6}$$

where $Q(\lambda) = N\mathcal{N}(\lambda) + N\mathcal{T}(\alpha\lambda) + n$, $\mathcal{N}(\lambda) = \lambda\overline{\lambda}$, $\mathcal{T}(\lambda) = \lambda + \overline{\lambda}$.

If $c = a \prod_{\text{prime } p|N} p^e$, $(a, N) = 1$ is a factorization of $c$, the Chinese Remainder Theorem tells us that the $(n, \alpha)$-th coefficient in (3.6) is equal to

$$\alpha_{k,4} N^{1-k} \{4(n - N\mathcal{N}(\alpha))\}^{k-2} B_{n,\alpha} \prod_{\text{prime } p|N} C_{n,\alpha,p}, \tag{3.7}$$

where

$$\begin{aligned} B_{n,\alpha} &= \sum_{a \geq 1} \overline{\psi}(a) a^{-k} \sum_{\substack{d \in \mathbb{Z}/a\mathbb{Z}, (d,a)=1 \\ \lambda \in \mathcal{O}/a\mathcal{O}}} e_a(dQ(\lambda)), \\ C_{n,\alpha,p} &= p^{k-2} \sum_{e \geq 1} \overline{\psi_p^*}(p^e) p^{-ek} \sum_{\substack{d \in (\mathbb{Z}/p^e\mathbb{Z})^\times \\ \lambda \in \mathcal{O}/p^e\mathcal{O}}} \overline{\psi_p}(d) e_{p^e}(dQ(\lambda)). \end{aligned} \tag{3.8}$$

Following [36, Section 2.3.1], one has

$$B_{n,\alpha} = B_{\overline{\psi}}(Nt_{(N)})$$

with $t_{(N)} = 4(n - N\mathcal{N}(\alpha))$.

In order to simplify $C_{n,\alpha,p}$, choose an integer $g$ so that $4g \equiv 1 \pmod{p^e}$. Since $p$ is odd and $2i\alpha \in \mathcal{O}$, it follows from $Q(\lambda) \equiv Ng\mathcal{N}(2i\lambda + 2i\overline{\alpha}) + gt_{(N)} \pmod{p^e}$ that the inner double sum in (3.8) is

$$p^2 \sum_{d \in (\mathbb{Z}/p^e\mathbb{Z})^\times} \overline{\psi_p}(d) e_{p^e}(dgt_{(N)}) \sum_{\lambda \in \mathcal{O}/p^{e-1}\mathcal{O}} e_{p^{e-1}}(d(N/p)g\mathcal{N}(\lambda)). \tag{3.9}$$

We now claim that, for any odd prime $p$ and $e \geq 1$, one has

$$\sum_{\lambda \in \mathcal{O}/p^{e+1}\mathcal{O}} e_{p^{e+1}}(dg\mathcal{N}(\lambda)) = (\chi_K(p)p)^{e+1}$$

where $d$ is any integer relatively prime to $p$ and $g$ is any integer such that $4g \equiv 1 \pmod{p^{e+1}}$. Indeed, the left-hand side can be written as

$$\sum_{l=0}^{e} \sum_{\Delta_1 \in (\mathbb{Z}/p^{e-l+1}\mathbb{Z})^\times} r(\mathcal{O}, p^{e+1}, -p^l\Delta_1) e_{p^{e+1}}(dgp^l\Delta_1) + r(\mathcal{O}, p^{e+1}, -p^{e+1}),$$

where

$$r(\mathcal{O}, k, \Delta) = \sharp\{\lambda \in \mathcal{O}/k\mathcal{O} : \mathcal{N}(\lambda) \equiv -\Delta \pmod{k}\}.$$

This equals $(\chi_K(p)p)^{e+1}$ by the formulas in [10, Propositions 2.6, 2.7].

Applying this result to the inner sum in (3.9), we have

$$C_{n,\alpha,p} = \overline{\psi_p}(4)p^{k-1}\sum_{e\geq 1}\frac{\overline{\psi_p^*}(p^e)}{p^{(k-1)e}}\chi_K(p)^{e-1}\sum_{d\in(\mathbb{Z}/p^e\mathbb{Z})^\times}\overline{\psi_p}(d)e_{p^e}(dt). \quad (3.10)$$

Let $e \geq 1$ and $p^m$ be the exact power of $p$ dividing $t$. The inner sum in (3.10) is equal to $\psi_p(t/p^m)p^m\tau_p(\overline{\psi_p})$ if $e = m+1$, and 0 otherwise (cf. [34, proof of Proposition 4, p. 843]). This completes the proof of Proposition 3.2 (1). □

## 4 Hermitian modular forms

### 4.1 Basic facts

Let $U(2,2) = \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_4(\mathbb{C}) : {}^t\overline{M}JM = J\}$ with $J = \left(\begin{smallmatrix} O_2 & -I_2 \\ I_2 & O_2 \end{smallmatrix}\right)$. This group acts on the Hermitian upper half-space

$$H_2 = \{Z \in M_2(\mathbb{C}) : (Z - {}^t\overline{Z})/(2i) > O_2\}$$

by

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)Z = (AZ+B)(CZ+D)^{-1}.$$

Let $K = \mathbb{Q}(i)$ be the Gaussian number field of discriminant $-4$. Let $\mathcal{O} = \mathbb{Z}[i]$ be the ring of integers in $K$ and $\mathcal{D}^{-1} = (2i)^{-1}\mathcal{O}$ the inverse different. We denote by $\Gamma_2 = U(2,2) \cap M_4(\mathcal{O})$ the full Hermitian modular group of degree two. For any natural number $N$, the congruence subgroup $\Gamma_0^{(2)}(N)$ is defined to be

$$\Gamma_0^{(2)}(N) = \{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2; C \equiv O_2 \pmod{N\mathcal{O}}\}.$$

Using $\rho$ as in Section 3, we define a character on $\Gamma_0^{(2)}(N)$ by $\rho(\gamma) = \rho(\det D)$ for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N)$. Note here that $\det D \in \mathbb{Z} \cup i\mathbb{Z}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$. In fact, by the proof of [39, Lemma 1.1, p. 421] combined with [7, Theorem I, p. 143], we can deduce that $\epsilon \det D \in \mathbb{Z}$ with some $\epsilon \in \mathcal{O}^\times$ (cf. [20, Remark 2.1]).

For an even natural number $k$, denote by $M_k(\Gamma_0^{(2)}(N), \rho)$ the space of all holomorphic functions $f(Z)$ on $H_2$ which satisfy

$$f(\gamma Z) = \rho(\gamma)\det(CZ+D)^k f(Z), \quad \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N).$$

Since $k$ is even, the condition $(A')$ in [24, p. 92] holds, namely the character $\rho$ is trivial on the principal congruence subgroup of level $N$.

Any $f \in M_k(\Gamma_0^{(2)}(N), \rho)$ has a Fourier expansion

$$f(Z) = \sum_{T \in L_2 \geq 0} A(T, f)e(\mathrm{tr}(TZ)),$$

where the sum is extended over all half-integral positive semi-definite hermitian matrices of size two. There exists a constant $C$ such that

$$|A(T, f)| \le C(\det T)^k$$

for all positive definite matrices $T \in L_2^+$. See [24, Theorem 3.1, p. 93], the proofs of [24, Lemma 1.9, p. 80] and [1, Theorem 2.3.4, p. 70].

### 4.2 Relation with 3-dimensional hyperbolic space

For any $Z \in H_2$, the hermitian imaginary part $Y = (Z - {}^t\overline{Z})/(2i)$ is a positive definite hermitian matrix of size two. Denote by $\mathcal{P}_2$ the set of all positive definite hermitian matrices of size two and by $\mathcal{SP}_2$ the determinant one part of $\mathcal{P}_2$. We identify $\mathcal{SP}_2$ with 3-dimensional hyperbolic space $\mathbb{H}^3$ by ([11, Definition 1.1, p. 408])

$$W = \begin{pmatrix} (|z|^2 + r^2)r^{-1} & zr^{-1} \\ \overline{z}r^{-1} & r^{-1} \end{pmatrix} \to P_W = z + rj. \tag{4.1}$$

Any automorphic function $\mathcal{U}(P)$ on $\mathbb{H}^3$ gives a function on $\mathcal{P}_2$ by setting $\mathcal{U}(Y) = \mathcal{U}(P_Y)$, where $P_Y$ corresponds to $(\det Y)^{-1/2}Y$. In other words, $Y \in \mathcal{P}_2$ is identified with $P_Y \in \mathbb{H}^3$ by

$$Y = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \to P_Y = \frac{b}{d} + \frac{\sqrt{\det Y}}{d} j.$$

Put moreover $\hat{\mathcal{U}}(Y) = \mathcal{U}(Y^{-1})$. Recall that ([11, Proposition 1.2, p. 409])

$$P_{[\sigma]Y} = \sigma P_Y \quad \text{for } Y \in \mathcal{P}_2,\ \sigma \in \mathrm{SL}_2(\mathbb{C}).$$

### 4.3 Maass lift

As in [23, 37], the Maass lift $\mathcal{M}$ from the space $J_{k,1}(\Gamma_0^{(1)}(N), \rho)$ to the space $M_k(\Gamma_0^{(2)}(N), \rho)$ is defined as follows. For $\phi \in J_{k,1}(\Gamma_0^{(1)}(N), \rho)$ and any natural number $m$, we define the operator $V_m^J$ by

$$\phi|_{k,1} V_m^J(\tau, z, w) = m^{k-1} \sum_{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)\backslash M_2^*(m)} \psi(a)(c\tau + d)^{-k} \times e\left(-\frac{cmzw}{c\tau + d}\right)\phi\left(M\tau, \frac{mz}{c\tau + d}, \frac{mw}{c\tau + d}\right),$$

where

$$M_2^*(m) = \left\{M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) : \det M = m,\ c \equiv 0 \pmod N,\ (a, N) = 1\right\}.$$

It is easy to see that $\phi|_{k,1} V_m^J$ transforms like an element in $J_{k,m}(\Gamma_0^{(1)}(N), \rho)$ and if

$$\phi(\tau, z, w) = \sum_{\substack{n \in \mathbb{Z}, \alpha \in \mathcal{D}^{-1} \\ n - \mathcal{N}(\alpha) \geq 0}} c(n, \alpha) q^n \zeta_1^{\alpha} \zeta_2^{\overline{\alpha}},$$

then

$$\phi|_{k,1} V_m^J(\tau, z, w) = \sum_{\substack{n \in \mathbb{Z}, \alpha \in \mathcal{D}^{-1} \\ nm - \mathcal{N}(\alpha) \geq 0}} \left( \sum_{d | (n, \alpha, m)} \psi(d) d^{k-1} c\left( \frac{mn}{d^2}, \frac{\alpha}{d} \right) \right) q^n \zeta_1^{\alpha} \zeta_2^{\overline{\alpha}}. \tag{4.2}$$

Here the sum over $d \mid (n, \alpha, m)$ is taken over all $d \in \mathbb{N}$ such that $n/d, m/d \in \mathbb{Z}$, $\alpha/d \in \mathcal{D}^{-1}$. Moreover, put

$$\phi_0(\tau) = \left\{ \frac{N^k \Gamma(k) L(k, \overline{\psi})}{(-2\pi i)^k \tau_N(\overline{\psi})} + \sum_{n \geq 1} \left( \sum_{d | n} \psi(d) d^{k-1} \right) q^n \right\} c(0, 0).$$

This is an Eisenstein series on $\Gamma_0(N)$ for the cusp $\infty$ ([30, Theorem 7.1.3, p. 270]).

The Maass lift $\mathcal{M}\phi$ is then defined for $\left(\begin{smallmatrix} \tau' & z \\ w & \tau \end{smallmatrix}\right) \in H_2$ by

$$\mathcal{M}\phi(\left(\begin{smallmatrix} \tau' & z \\ w & \tau \end{smallmatrix}\right)) = \phi_0(\tau) + \sum_{m \geq 1} \phi|_{k,1} V_m^J(\tau, z, w) e(m\tau'). \tag{4.3}$$

Since $c(n, \alpha) = c(4(n - \mathcal{N}(\alpha)) = O((n - \mathcal{N}(\alpha))^{k-3/2})$ as noted in the end of Section 3.1, the Fourier series (4.3) w.r.t. $e(n\tau + \alpha z + \overline{\alpha} w + m\tau')$ converges absolutely and uniformly in any domain $Y \geq Y_0 > O_2$. See Section 5.1 for the proof.

**Proposition 4.1.** *One has $\mathcal{M}\phi \in M_k(\Gamma_0^{(2)}(N), \rho)$.*

*Proof.* For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we denote by $M_\perp$, $M^\perp$ the matrices

$$M_\perp = \begin{pmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{pmatrix}, \quad M^\perp = \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix}.$$

For simplicity, put $G(Z) = \mathcal{M}\phi(Z)$ and

$$J_2(\gamma, Z) = \det(CZ + D), \quad \rho(\gamma) = \rho(\det D) \quad \text{for } \gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(2)}(N).$$

Since $\phi_0(\tau) \in M_k(\Gamma_0(N), \psi)$, a direct computation using (4.3) implies

$$G(M_\perp Z) = \rho(M_\perp) J_2(M_\perp, Z)^k G(Z).$$

On the other hand, (4.2) implies that

$$G(Z) = \frac{N^k \Gamma(k) L(k, \overline{\psi})}{(-2\pi i)^k \tau_N(\overline{\psi})} c(0,0) + \sum_{\substack{m \geq 0,\, n \geq 0,\, \alpha \in \mathcal{D}^{-1} \\ mn \geq \mathcal{N}(\alpha),\, (n,\alpha,m) \neq (0,0,0)}} \left( \sum_{d | (n,\alpha,m)} \psi(d) d^{k-1} c\left( \frac{mn}{d^2}, \frac{\alpha}{d} \right) \right) \times e(n\tau + \alpha z + \overline{\alpha} w + m\tau').$$

Because $c(n,\alpha)$ depends only on $4(n - \mathcal{N}(\alpha))$ by Section 3.1, this expression tells us

$$G(V^\sharp Z) = G(Z) \quad \text{for } V^\sharp = \begin{pmatrix} V & O_2 \\ O_2 & V \end{pmatrix}, \text{ where } V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, for any $M \in \Gamma_0(N)$, one has

$$\begin{aligned} G(M^\perp Z) &= G(V^\sharp M_\perp V^\sharp Z) = G(M_\perp V^\sharp Z) \\ &= \rho(M_\perp) J_2(M_\perp, V^\sharp Z)^k G(V^\sharp Z) = \rho(M^\perp) J_2(M^\perp, Z)^k G(Z). \end{aligned}$$

By Lemma 4.2 given below, $\Gamma_0^{(2)}(N)$ is generated by the elements consisting of $M^\perp$, $M_\perp$ for $M \in \Gamma_0(N)$, $t^+(S)$ for $S \in \mathrm{Her}_2(\mathcal{O})$ and $U^\sharp$ for $U \in \mathrm{GL}_2(\mathcal{O})$, where

$$t^+(S) = \begin{pmatrix} I_2 & S \\ O_2 & I_2 \end{pmatrix}, \quad U^\sharp = \begin{pmatrix} U & O_2 \\ O_2 & {}^t\overline{U}^{-1} \end{pmatrix}.$$

Hence the desired result follows. □

**Lemma 4.2.** *The group $\Gamma_0^{(2)}(N)$ is generated by the elements consisting of $M^\perp$ and $M_\perp$ for $M \in \Gamma_0(N)$, $t^+(S)$ for $S \in \mathrm{Her}_2(\mathcal{O})$ and $U^\sharp$ for $U \in \mathrm{GL}_2(\mathcal{O})$.*

*Proof.* Put $t^-(S) = \begin{pmatrix} I_2 & O_2 \\ S & I_2 \end{pmatrix}$ for $S \in \mathrm{Her}_2(\mathcal{O})$. By [24, Theorem 4.2, p. 68], the group $\Gamma_0^{(2)}(N)$ is generated by the elements consisting of $(\epsilon M)_\perp$ for $\epsilon \in \mathcal{O}^\times$, $M \in \Gamma_0(N)$, $t^+(S), t^-(NS)$ for $S \in \mathrm{Her}_2(\mathcal{O})$ and $U^\sharp$ for $U \in \mathrm{GL}_2(\mathcal{O})$. In the first one, we can restrict $\epsilon = 1$. In fact

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^\sharp M_\perp = (\epsilon M)_\perp.$$

Hence we have only to show that $t^-(NS)$ can be expressed by a product of matrices in Lemma 4.2. Let us denote by $\tilde{\Gamma}$ the group generated by matrices in Lemma 4.2. First $t^-(\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}_\perp, t^-(\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}^\perp \in \tilde{\Gamma}$ so that

$$t^-(\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}^\perp \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}_\perp \in \tilde{\Gamma}.$$

Using

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{\sharp} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}_{\perp} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{\sharp} = t^{-}\left(\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}\right) t^{-}\left(\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}\right),$$

we have $t^{-}\left(\begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}\right) \in \tilde{\Gamma}$. Similarly, $t^{-}\left(\begin{pmatrix} 0 & iN \\ -iN & 0 \end{pmatrix}\right) \in \tilde{\Gamma}$ follows from

$$\begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}^{\sharp} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}_{\perp} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}^{\sharp} = t^{-}\left(\begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}\right) t^{-}\left(\begin{pmatrix} 0 & iN \\ -iN & 0 \end{pmatrix}\right)$$

suggested by Krieg. Accordingly, we conclude $t^{-}(NS) \in \tilde{\Gamma}$ for $S \in \mathrm{Her}_2(\mathcal{O})$ and so $\tilde{\Gamma} = \Gamma_0^{(2)}(N)$ as desired. □

## 5 Spectral decomposition

In this section, we consider the spectral decomposition of unimodular invariant Fourier series on $H_2$. Take a Fourier series

$$F(Z) = \sum_{T \in L_2^{+}} A(T, F) e(\mathrm{tr}(TZ)), \quad Z \in H_2, \tag{5.1}$$

where $L_2^{+} = L_2 \cap \mathcal{P}_2$. We assume that

$$A(T, F) = A([U]T, F)$$

for any $U \in \mathrm{GL}_2(\mathcal{O})$, and

$$A(T, F) = O((\det T)^{\delta_1})$$

with a positive constant $\delta_1$.

Here we summarize the facts on hermitian matrices, that are needed below.

**Lemma 5.1.** *The following statements hold:*

(1) *For hermitian matrices $T, X, Y$, one has* $\mathrm{tr}(TX), \mathrm{tr}(TY) \in \mathbb{R}$.

(2) *If hermitian matrices $A, B, C$ satisfy $A \geq B$ and $C \geq O_2$, then*

$$\mathrm{tr}(AC) \geq \mathrm{tr}(BC).$$

(3) *For any positive semi-definite hermitian matrix $A = \begin{pmatrix} a_1 & a_{12} \\ \overline{a_{12}} & a_2 \end{pmatrix} \geq O_2$, one has* $a_1 a_2 \geq \det A$.

(4) *Let $t$ be any natural number. If $T = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in L_2^{+}$ satisfies $ad = t$, then*

$$a, d \leq t \quad \textit{and} \quad |b| \leq 2t.$$

*Moreover, one has* $\sharp\{T = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in L_2^{+} : ad = t\} \leq t^2(4t+1)^2$.

(5) *For any* $S \in \mathcal{P}_2$*, there exists* $U \in \mathrm{GL}_2(\mathcal{O})$ *such that* $[U]S \in \mathcal{R}_2$*, where*

$$\mathcal{R}_2 = \left\{ \left(\begin{smallmatrix} s_1 & s_{12} \\ \overline{s_{12}} & s_2 \end{smallmatrix}\right) \in \mathcal{P}_2 : 2|\Im(s_{12})| \le 2\Re(s_{12}) \le s_1 \le s_2 \right\}.$$

*Moreover, there exist constants* $\alpha, \beta > 0$ *such that*

$$s_1 s_2 \le \alpha(\det S), \quad 2\left(\begin{smallmatrix} s_1 & 0 \\ 0 & s_2 \end{smallmatrix}\right) \ge S \ge \beta\left(\begin{smallmatrix} s_1 & 0 \\ 0 & s_2 \end{smallmatrix}\right)$$

*for all* $S = \left(\begin{smallmatrix} s_1 & s_{12} \\ \overline{s_{12}} & s_2 \end{smallmatrix}\right) \in \mathcal{R}_2$. □

See [24, Theorem 4.12, p. 35] for Lemma 5.1 (5). Other statements are elementary. By Lemma 5.1, we have the following inequality, which (or a similar argument) will be used frequently in this section (see [17, p. 912]). If

$$S = \left(\begin{smallmatrix} s_1 & s_{12} \\ \overline{s_{12}} & s_2 \end{smallmatrix}\right) \in \mathcal{R}_2 \quad \text{and} \quad T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2^+,$$

then

$$\begin{aligned} \mathrm{tr}(TS) &\ge \beta \mathrm{tr}\left(T\left(\begin{smallmatrix} s_1 & 0 \\ 0 & s_2 \end{smallmatrix}\right)\right) = \beta(as_1 + ds_2) \\ &\ge 2\beta\sqrt{ads_1s_2} \ge 2\beta\sqrt{ad}\sqrt{\det S}. \end{aligned} \tag{5.2}$$

We also use the Stirling formula frequently.

**Lemma 5.2.** *For given* $c > 0$ *and* $\sigma_1 < \sigma_2$*, there exist positive constants* $N_1, N_2$ *such that*

$$|\Gamma(\sigma + it)| \le N_1 e^{-\frac{\pi}{2}|t|}(1+|t|)^{\sigma-1/2}, \quad |\Gamma(\sigma + it)|^{-1} \le N_2 e^{\frac{\pi}{2}|t|}(1+|t|)^{1/2-\sigma}$$

*for any complex number* $\sigma + it$ *in the region* $|t| \ge c$, $\sigma_1 \le \sigma \le \sigma_2$. □

### 5.1 Convergence and estimate of Fourier series

The Fourier series $F(Z)$ defined in (5.1) converges absolutely and uniformly in any domain $Y \ge Y_0 > O_2$, and $F(Z)$ is bounded on this region. In fact, if we take $\epsilon > 0$ so that $Y \ge Y_0 > \epsilon I_2$, the argument in [17, p. 912] combined with Lemma 5.1 gives

$$|F(Z)| \le \sum_{T \in L_2^+} |A(T,F)| e^{-2\pi\,\mathrm{tr}(TY)} \le K_1 \sum_{t=1}^{\infty} t^{\delta_1} t^2 (4t+1)^2 e^{-4\pi\sqrt{t}\epsilon}$$

with a constant $K_1$. Hence we obtain the claims. Accordingly, we have

$$F(i[U]Y) = F(iY) \quad \text{for } U \in \mathrm{GL}_2(\mathcal{O}). \tag{5.3}$$

Moreover there exist positive constants $C_1, C_2, \delta_3, l$ such that

$$|F(iY)| \le [C_1(\det Y)^{-(l+1)} + C_2(\det Y)^{-l}]e^{-\delta_3(\det Y)^{1/2}} \quad \text{for } Y \in \mathscr{P}_2. \quad (5.4)$$

In order to prove (5.4), we may assume $Y \in \mathscr{R}_2$. Then the argument in [17, p. 913] combined with (5.2) gives

$$\begin{aligned}|F(iY)| &\le K_2 \sum_{t=1}^{\infty} t^{\delta_1} t^2 (4t+1)^2 e^{-4\pi\beta\sqrt{\det Y} t^{1/2}} \\ &\le K_3 \left( \sum_{t=1}^{\infty} t^{\delta_1+4} e^{-2\pi\beta\sqrt{\det Y} t^{1/2}} \right) e^{-2\pi\beta\sqrt{\det Y}} \\ &\le K_3 \left( B_1(\det Y)^{-((\delta_1+4)+1)} + B_2(\det Y)^{-(\delta_1+4)} \right) e^{-2\pi\beta\sqrt{\det Y}} \quad (5.5)\end{aligned}$$

with some positive constants $K_2, K_3, B_1, B_2$. Hence we obtain (5.4) as desired.

Here we show the convergence of the series given in (4.3). For $n, m, \alpha$ such that $nm - \mathscr{N}(\alpha) > 0$, put $T = \left(\begin{smallmatrix} m & \overline{\alpha} \\ \alpha & n \end{smallmatrix}\right) \in L_2^+$ and $e(T) = \max\{q \in \mathbb{N} : q^{-1}T \in L_2^+\}$. The Fourier coefficient w.r.t. $e(n\tau + \alpha z + \overline{\alpha} w + m\tau')$ is

$$A(T) = \sum_{d|e(T)} \psi(d) d^{k-1} c\left(\frac{mn}{d^2}, \frac{\alpha}{d}\right).$$

Since $e(T)$ divides $n, m$, it follows that $e(T)^2 \le nm$. Using the facts that $k \ge 2$, $c(n, \alpha) = O((n - \mathscr{N}(\alpha))^{k-3/2})$ and Lemma 4.2 (5), we have

$$|A(T)| \le C_0(\det T)^{k-3/2} e(T) \le C_0(\det T)^{k-3/2}(nm)^{1/2} \le C_0\alpha(\det T)^{k-1}$$

for $T \in \mathscr{R}_2$. By the $\mathrm{GL}_2(\mathscr{O})$ invariance of the expression, this inequality holds for any $T \in L_2^+$. Hence the convergence of the non-degenerate $T$ parts follows. On the other hand, the convergence of the degenerate $T$ parts can be treated as in the proof of [3, Lemma 5, p. 206]. The uniform convergence on the domain $Y \ge Y_0 > O_2$ follows from the argument of [1, Theorem 2.3.1, p. 65] combined with [24, Proposition 1.3, p. 75].

### 5.2 Mellin transform of Fourier series

Any $Y \in \mathscr{P}_2$ has the form $Y = uW$, where $u = (\det Y)^{1/2} > 0$ and $W \in \mathscr{SP}_2$. Assuming $\Re(s)$ to be sufficiently large, set

$$\tilde{F}_s(P) = \int_0^{\infty} F(iuW) u^{2s-1} du, \quad P = P_W.$$

For $s = \sigma + it$, the estimate (5.4) implies

$$|\tilde{F}_s(P)| \leq \int_0^\infty |F(iuW)| u^{2\sigma-1} du$$
$$\leq C_1 \Gamma(2\sigma - 2l - 2)\delta_3^{-(2\sigma-2l-2)} + C_2 \Gamma(2\sigma - 2l)\delta_3^{-(2\sigma-2l)}.$$

Hence, if $\sigma_1, \sigma_2$ ($\sigma_1 < \sigma_2$) are sufficiently large, the integral converges absolutely and uniformly on $\mathbb{H}^3 \times \{s = \sigma + it : \sigma_1 \leq \sigma \leq \sigma_2\}$. Moreover, $\tilde{F}_s(P)$ is bounded on the same region.

By (5.3), it satisfies

$$\tilde{F}_s(\gamma P) = \tilde{F}_s(P) \quad \text{for } \gamma \in \mathrm{SL}_2(\mathcal{O}).$$

For sufficiently large $\sigma = \Re(s)$, one has

$$\tilde{F}_s(P) = (2\pi)^{-2s}\Gamma(2s)\tilde{f}_s(P), \quad \tilde{f}_s(P) = \sum_{T \in L_2^+} \frac{A(T, F)}{\operatorname{tr}(TW)^{2s}}. \tag{5.6}$$

In fact, if $\sigma_1, \sigma_2$ ($\sigma_1 < \sigma_2$) are sufficiently large, the series $\tilde{f}_s(P)$ converges absolutely and uniformly on $\mathbb{H}^3 \times \{s = \sigma + it : \sigma_1 \leq \sigma \leq \sigma_2\}$. Moreover $\tilde{f}_s(P)$ is bounded on the same region. In order to prove these claims, we may assume that $W \in \mathcal{R}_2$. As $\det W = 1$, (5.2) implies $\operatorname{tr}(TW) \geq 2\beta\sqrt{ad}$ for $T = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in L_2^+$. It follows for sufficiently large $\sigma = \Re(s)$ that

$$|\tilde{f}_s(P)| \leq \sum_{T \in L_2^+} \frac{|A(T, F)|}{\operatorname{tr}(TW)^{2\sigma}} \leq \frac{K_4}{(2\beta)^{2\sigma}} \sum_{T \in L_2^+} \frac{1}{(ad)^{\sigma-\delta_1}}$$
$$\leq \frac{K_4}{(2\beta)^{2\sigma}} \sum_{t=1}^{\infty} \frac{t^2(4t+1)^2}{t^{\sigma-\delta_1}}$$

with a positive constant $K_4$. Hence the claims follow.

By Lemma 5.2, one has

$$F(iy^{1/2}W) = \frac{1}{2\pi i} \int_{\Re(s)=\sigma} 2\tilde{F}_s(P) y^{-s} ds, \quad \sigma \gg 0, \tag{5.7}$$

for $y > 0$ from (5.6).

### 5.3 Estimate of $\Delta^2 \tilde{f}_s(P)$

Let us study $\Delta^2 \tilde{f}_s(P)$, where $\tilde{f}_s(P)$ is as in (5.6) and $\Delta$ is the Laplacian. See Section 2 (G-ii). First, we summarize the facts on the elementary set $\mathcal{E}_2(\alpha)$. See [24, Definition, Theorems 4.9, 4.10 and 4.11, pp. 33–35]. For $\alpha > 0$, put

$$\mathcal{E}_2(\alpha) = \left\{ \left(\begin{smallmatrix} 1 & 0 \\ b_{12} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} d_1 & 0 \\ 0 & d_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b_{12} \\ 0 & 1 \end{smallmatrix}\right) : 0 < d_1 < \alpha d_2, |b_{12}| < \alpha \right\} \subset \mathcal{P}_2.$$

**Lemma 5.3.** *The following statements hold:*

(1) *There is $\alpha > 0$ such that $\mathcal{R}_2 \subset \mathcal{E}_2(\alpha)$.*

(2) *There exists a constant $\beta > 0$ such that*

$$\begin{pmatrix} s_1 & s_{12} \\ \overline{s_{12}} & s_2 \end{pmatrix} \geq \beta \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$$

*for all $\begin{pmatrix} s_1 & s_{12} \\ \overline{s_{12}} & s_2 \end{pmatrix} \in \mathcal{E}_2(\alpha)$.*

(3) *For any compact subset $\mathcal{K} \subset \mathcal{P}_2$, there is $\alpha > 0$ such that $\mathcal{K} \subset \mathcal{E}_2(\alpha)$.* □

We claim

**Proposition 5.4.** *If $\sigma_1, \sigma_2$ ($\sigma_1 < \sigma_2$) are sufficiently large, then there exist positive constants $N_0, \alpha$ such that*

$$|\Delta^2 \tilde{f}_s(P)| \leq N_0(1 + 2|t|)^\alpha \tag{5.8}$$

*on $\mathbb{H}^3 \times \{s = \sigma + it : \sigma_1 \leq \sigma \leq \sigma_2\}$.*

*Proof.* In order to prove this claim, note that on the left-hand side of the equation $\Delta^2 \tilde{F}_s(P) = (2\pi)^{-2s}\Gamma(2s)\Delta^2 \tilde{f}_s(P)$, the differentiations under the integral sign is permissible. In fact, for any compact subset $K \subset \mathbb{H}^3$, let $\mathcal{K} \subset \mathcal{P}_2$ be the subset corresponding to $K$ by means of (4.1). Take $\alpha > 0$ such that $\mathcal{K} \subset \mathcal{E}_2(\alpha)$ by Lemma 5.3 (3). For $P = P_W = x + iy + jr \in K$, the argument used in Sections 5.1 and 5.2 combined with Lemma 5.3 (2) gives the estimate

$$\begin{aligned} |\partial_x F(iuW)| &= \left| \sum_{T = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \in L_2^+} A(T, F)(-2\pi u)(2ax + b + \overline{b}) r^{-1} e^{-2\pi u \operatorname{tr}(TW)} \right| \\ &\leq \sum_{T = \begin{pmatrix} a & b \\ \overline{b} & d \end{pmatrix} \in L_2^+} |A(T, F)| 2\pi u (2|a| \cdot |x| + 2|b|) r^{-1} e^{-4\pi\beta u \sqrt{ad}}. \end{aligned}$$

Now there are constants $N_3, N_4 > 0$ such that $|z| < N_3$ and $r^{-1} < N_4$ for all $P = z + rj \in K$. Thus Lemma 5.1 (4) tells us that

$$|\partial_x F(iuW)| \leq \sum_{t=1}^{\infty} t^2(4t + 1)^2 \cdot K_1 t^{\delta_1} 2\pi u \cdot (2tN_3 + 4t) N_4 e^{-4\pi\beta u\sqrt{t}}.$$

This implies the estimate of $|\partial_x F(iuW)|$ similar to (5.5). The majorant functions is independent of $P = P_W \in K$ and is integrable from 0 to $\infty$ w.r.t. $u$, when it is multiplied by $u^{2\sigma-1}$. Hence, we can differentiate $\tilde{F}_s(P)$ w.r.t. $x$ under the integral sign. Other variables and repeating differentiations can be justified by similar way.

Put $B = \mathrm{tr}(TW)$. We have the identities

$$\Delta B = 3B \quad \text{and} \quad r^2((B_x)^2 + (B_y)^2 + (B_r)^2) = B^2 - 4\det T,$$

where $B_x = \frac{\partial B}{\partial x}$ etc. It follows that

$$\Delta^2 e^{-2\pi u B} = (-2\pi u)e^{-2\pi u B}\{9B - 2\pi u(8(B^2 - 4\det T) + 15B^2) \\ + (2\pi u)^2 10B(B^2 - 4\det T) - (2\pi u)^3(B^2 - 4\det T)^2\},$$

and that $\Delta^2 F(iuW)$ is the finite sum of the functions like

$$F_{\alpha',\beta',\gamma'}(iuW) = \sum_{T \in L_2^+} A(T,F)u^{\alpha'}\mathrm{tr}(TW)^{\beta'}(\det T)^{\gamma'} e^{-2\pi u\,\mathrm{tr}(TW)},$$

where $\alpha', \beta', \gamma'$ are some positive constants. Accordingly, $\Delta^2 \tilde{F}_s(P)$ for $\Re(s) \gg 0$ is the finite sum of the functions of the form

$$\widetilde{F_{\alpha',\beta',\gamma',s}}(P_W) = \int_0^\infty F_{\alpha',\beta',\gamma'}(iuW)u^{2s-1}du.$$

Integrating term by term, we can deduce similarly to Section 5.2 that

$$|\Gamma(2s+\alpha_1)^{-1}\widetilde{F_{\alpha',\beta',\gamma',s}}(P)|$$

is bounded on $\mathbb{H}^3 \times \{s = \sigma + it : \sigma_1 \le \sigma \le \sigma_2\}$, if $\sigma_1, \sigma_2$ $(\sigma_1 < \sigma_2)$ are sufficiently large.

By Lemma 5.2, there is a positive constant $N_5$ such that $|\Gamma(2s+\alpha_1)\Gamma(2s)^{-1}| < N_5(1+2|t|)^{\alpha_1}$ on the region $\sigma_1 \le \sigma \le \sigma_2$. Hence, we get the desired estimate (5.8) for $\Delta^2 \tilde{f}_s(P) = (2\pi)^{2s}\Gamma(2s)^{-1}\Delta^2\tilde{F}_s(P)$. □

### 5.4 Associated Dirichlet series

For any automorphic function $\mathcal{U}(P)$, put

$$D(F,\mathcal{U},s) = \sum_{T \in \mathrm{SL}_2(\mathcal{O})\backslash L_2^+} \frac{A(T,F)\hat{\mathcal{U}}(T)}{\epsilon(T)(\det T)^s}, \tag{5.9}$$

where the summation extends over all $T \in L_2^+$ modulo the action

$$T \to [U]T = UT\,{}^t\overline{U}$$

of the group $\mathrm{SL}_2(\mathcal{O})$,

$$\epsilon(T) = \sharp\{U \in \mathrm{SL}_2(\mathcal{O}) : [U]T = T\}$$

is the order of the unit group of $T$ and

$$\hat{\mathcal{U}}(T) = \mathcal{U}(T^{-1}) = \mathcal{U}(P_{T^{-1}}).$$

This is so-called Koecher–Maass series twisted by $\mathcal{U}(P)$ (cf. [17,25] in the Siegel modular case). Recall that $A(T, F)$ is $\mathrm{GL}_2(\mathcal{O})$ invariant, and

$$A(T, F) = O((\det T)^{\delta_1})$$

with a constant $\delta_1 > 0$.

**Proposition 5.5.** *For any spectral eigenfunction $\mathcal{U}(P)$, the series defining the functions $D(F, \mathcal{U}, s)$ and $D(F, \hat{\mathcal{U}}, s)$ in* (5.9) *converge absolutely and uniformly for $\Re(s) > \frac{9}{4} + \delta_1$.*

*Proof.* In order to prove this claim, we use the following Lemmas 5.6 and 5.7. These are consequence of [33, Proposition 2, p. 902] combined with

$$J \cdot P_{T^{-1}} = P_{[J]T^{-1}} = P_{\overline{T}},$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

**Lemma 5.6.** *The series*

$$\sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{E(P_{T^{-1}}, u)}{\epsilon(T)(\det T)^s}$$

*converges absolutely and uniformly for $\Re(u) > 1$, $\Re(s) > \frac{\Re(u)}{2} + \frac{3}{2}$.*

**Lemma 5.7.** *The series*

$$\sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{1}{\epsilon(T)(\det T)^s}$$

*converges absolutely and uniformly for $\Re(s) > 2$.*

Any cusp eigenfunction $\mathcal{U}_m(P)$ and the constant function $\mathcal{U}_0(P)$ are bounded [11, Corollary 3.3, p. 107]. Hence Lemma 5.7 implies that the series (5.9) converges absolutely and uniformly for $\Re(s) > 2 + \delta_1$. This completes the proof of Proposition 5.5 in these cases.

On the other hand, Lemma 5.6 implies for $\Re(u) > 1$ that

$$D(F, E(P, u), s) = \sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{A(T, F) E(P_{T^{-1}}, u)}{\epsilon(T)(\det T)^s} \tag{5.10}$$

converges absolutely and uniformly for $\Re(s) > \frac{\Re(u)}{2} + \frac{3}{2} + \delta_1$. To extend the result from $\Re(u) > 1$ to $\Re(u) = 0$, we follow the proof of [3, Lemma 2, p. 202].

For $t > 0$, put

$$\theta_{\mathcal{O}}(P,t) = \sum_{c,d\in\mathcal{O}} e^{-\frac{\pi}{r}(|cz+d|^2+|c|^2r^2)t}$$

$$Z_+(P,u) = \int_1^\infty (\theta_{\mathcal{O}}(P,t)-1)t^{1+u}\frac{dt}{t}.$$

By [11, p. 402],

$$\begin{aligned}
&\pi^{-(1+u)}\Gamma(1+u)4\zeta_K(1+u)E(P,u) \\
&\quad = \int_0^\infty (\theta_{\mathcal{O}}(P,t)-1)t^{1+u}\frac{dt}{t} \\
&\quad = \int_1^\infty (\theta_{\mathcal{O}}(P,t)-1)(t^{1+u}+t^{1-u})\frac{dt}{t} + \frac{2}{u^2-1} \\
&\quad = Z_+(P,u) + Z_+(P,-u) + \frac{2}{u^2-1}. \qquad (5.11)
\end{aligned}$$

Note that this is invariant under $u \to -u$, and implies the meromorphic continuation and the functional equation of the Eisenstein series.

Then we claim that

**Lemma 5.8.** *For any complex number $u$, the series*

$$\sum_{T\in \mathrm{SL}_2(\mathcal{O})\backslash L_2^+} \frac{Z_+(P_{T^{-1}}, \Re(u))}{\epsilon(T)(\det T)^s}$$

*converges absolutely and uniformly for $\Re(s) > \frac{\Re(u)+M}{2} + \frac{3}{2}$, where $M$ is any real number such that $\Re(u) + M > 1$.*

*Proof.* For any $u$, we take $M > 0$ such that $v = u + M$ satisfies $\Re(v) > 1$. Then

$$\begin{aligned}
&Z_+(P_{T^{-1}}, \Re(u)) \\
&\quad \le Z_+(P_{T^{-1}}, \Re(v)) \le \int_0^\infty (\theta_{\mathcal{O}}(P_{T^{-1}},t)-1)t^{1+\Re(v)}\frac{dt}{t} \\
&\quad = \pi^{-(1+\Re(v))}\Gamma(1+\Re(v))4\zeta_K(1+\Re(v))E(P_{T^{-1}}, \Re(v)). \qquad (5.12)
\end{aligned}$$

The desired claim follows from Lemma 5.6. □

Let us complete the proof of Proposition 5.5 for the case of Eisenstein series $E(P,it)$. By (5.11),

$$|\pi^{-(1+u)}\Gamma(1+u)4\zeta_K(1+u)| \sum_{T\in\mathrm{SL}_2(\mathcal{O})\backslash L_2^+} \left|\frac{A(T,F)E(P_{T^{-1}},u)}{\epsilon(T)(\det T)^s}\right|$$
$$\le C' \sum_{T\in\mathrm{SL}_2(\mathcal{O})\backslash L_2^+} \frac{1}{\epsilon(T)(\det T)^{\Re(s)-\delta_1}}$$
$$\times \left\{Z_+(P_{T^{-1}},\Re(u)) + Z_+(P_{T^{-1}},-\Re(u)) + \frac{2}{|u^2-1|}\right\}$$

with a constant $C'$. Since $\Gamma(1+it)\neq 0$ and $\zeta_K(1+it)\neq 0$ for any real number $t$, Lemmas 5.7 and 5.8 imply the desired result for $E(P,it)$ for any real number $t\neq 0$ by taking $M=3/2$. Note that $E(P,0)=0$ and use

$$J\cdot P_{T^{-1}} = P_{[J]T^{-1}} = P_{\overline{T}} \quad \text{with } J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

in Lemma 5.6 in order to treat $D(F,\hat{E}(P,u),s)$. We remark here that we have proved the convergence for any fixed $u$ such that $u\neq\pm 1$, $\Gamma(1+u)\zeta_K(1+u)\neq 0$ and sufficiently large $\Re(s)$. □

In order to analyze continuous spectrum, we will use the following two lemmas (see [3, Lemma 4, p. 205]).

**Lemma 5.9.** *Let $\sigma$ be sufficiently large. Then there exist constants $M,\delta$ depending only on $\sigma$ such that*

$$|D(F,E(P,i\eta),\sigma+it)| < M(1+|\eta|^\delta)$$

*for any real numbers $t,\eta$.*

*Proof.* For any $u=\rho+i\eta$ with fixed $1<\rho\notin\mathbb{Z}$, it follows from (5.10) and Lemma 5.6 that there exists a constant $M_1>0$ depending only on $\rho,\sigma$ such that

$$|(u^2-1)\zeta_K(1+u)D(F,E(P,u),\sigma+it)| < M_1(1+|\eta|^2).$$

By the functional equation of $E(P,u)$ and Lemma 5.2 together with this estimate, there exists a constant $M_2>0$ depending only on $\rho,\sigma$ such that

$$|(u^2-1)\zeta_K(1+u)D(F,E(P,u),\sigma+it)| < M_2(1+|\eta|^2)(1+|\eta|)^{2\rho}$$

on the line $u=-\rho+i\eta$. Moreover, by (5.10) and (5.11) combined with Lemmas 5.2, 5.7 and 5.8, there exist constants $c'$ and $M_3>0$ depending only on $\rho,\sigma$ such that

$$|(u^2-1)\zeta_K(1+u)D(F,E(P,u),\sigma+it)| \le M_3 e^{c'|\Im(u)|}$$

on the region $-\rho \leq \Re(u) \leq \rho$. Since $\sigma = \Re(s) \gg 0$, it follows that the series $(u^2-1)\zeta_K(1+u)D(F, E(P,u), s)$ converges uniformly on any compact subset of $-\rho \leq \Re(u) \leq \rho$ by (5.11). Thus as a function of $u \in \mathbb{C}$, it is holomorphic on the region $-\rho \leq \Re(u) \leq \rho$. According to the Phragmén–Lindelöf Theorem, there exist constants $M_4, \delta_4 > 0$ depending only on $\rho, \sigma$ such that

$$|(u^2-1)\zeta_K(1+u)D(F, E(P,u), \sigma+it)| < M_4(1+|\Im(u)|)^{\delta_4} \tag{5.13}$$

on the region $-\rho \leq \Re(u) \leq \rho$.

By Bauer [4, (33), p. 227], we have $\zeta_K(1+i\eta)^{-1} = O(|\eta|^{\epsilon})$ as $|\eta| \to \infty$ with some $\epsilon$. Note that Bauer proved there a better result for any number field. See also [22, (3.12), p. 482] for this estimation. Since $\zeta_K(1+i\eta) \neq 0$ for any real number $\eta$, the desired result follows from the case $u = i\eta$ in (5.13). □

**Lemma 5.10.** *For fixed $P \in \mathbb{H}^3$, there exists a constant $\delta_2$ such that*

$$E(P, i\eta) = O(1+|\eta|^{\delta_2}).$$

*Proof.* Let $u = \rho + i\eta$ with fixed $1 < \rho \notin \mathbb{Z}$. Since $u$ is in the region of convergence of $E(P,u)$ and $\zeta_K(1+u)$, there exists a constant $M_5$ such that

$$|(u^2-1)\zeta_K(1+u)E(P,u)| < M_5(1+|\eta|^2).$$

By the functional equation and Lemma 5.2 together with this estimate, there exists a constant $M_6$ such that

$$|(u^2-1)\zeta_K(1+u)E(P,u)| < M_6(1+|\eta|^2)(1+|\eta|)^{2\rho}$$

on the line $u = -\rho + i\eta$. Moreover, (5.11) combined with Lemma 5.2 implies that there exist constants $c''$ and $M_7$ such that

$$|(u^2-1)\zeta_K(1+u)E(P,u)| \leq M_7 e^{c''|\Im(u)|}$$

on the region $-\rho \leq \Re(u) \leq \rho$. Since $(u^2-1)\zeta_K(1+u)E(P,u)$ is holomorphic on $-\rho \leq \Re(u) \leq \rho$ by (5.11), the Phragmén–Lindelöf Theorem tells us that there exist constants $M_8, \delta_5$ such that

$$|(u^2-1)\zeta_K(1+u)E(P,u)| < M_8(1+|\Im(u)|)^{\delta_5} \tag{5.14}$$

on the region $-\rho \leq \Re(u) \leq \rho$.

By Bauer [4, (33), p. 227] (see also [22, (3.12), p. 482]), we have

$$\zeta_K(1+i\eta)^{-1} = O(|\eta|^{\epsilon}) \quad \text{as } |\eta| \to \infty$$

with some $\epsilon$. Since $\zeta_K(1+i\eta) \neq 0$ for any real number $\eta$, the desired result follows from the case $u = i\eta$ in (5.14). □

Applying the known estimate of $K$-Bessel function ([18, p. 204]) and that of $\phi_\lambda(it)$ ([11, Theorem 4.10 (4), p. 294]) to (2.2), we have

**Lemma 5.11.** *Let $I \subset \mathbb{R}$ be any compact set and $r_0$ be a sufficiently large real number. Then there exists a positive constant $\kappa$ such that*

$$E(z+rj, it) = r^{1+it} + \phi(it)r^{1-it} + O(e^{-\kappa r}) \quad (t \in I, r > r_0),$$

*where $\phi(s)$ is as in* (2.2).

See also [11, (3.21), p. 270] for this estimation.

### 5.5 Spectral decomposition

Put $B = \mathrm{tr}(TW)$ as before. Using

$$\Delta e^{-2\pi u B} = (-2\pi u)\{3B - 2\pi u(B^2 - 4\det T)\}e^{-2\pi u B},$$

we can deduce $\tilde{F}_s(P), \Delta\tilde{F}_s(P) \in L^2(\mathrm{SL}_2(\mathcal{O}) \setminus \mathbb{H}^3)$ for sufficiently large $\Re(s)$. In fact, we can prove this in the same way as in Section 5.3. By Section 5.2 and Lemma 5.11, $\tilde{F}_s(P)\overline{E(P,it)}$ is absolutely integrable over the region $\mathcal{F} \times [0,T]$ for every $T \geq 0$, where $\mathcal{F}$ is a fundamental domain of $\mathrm{SL}_2(\mathcal{O}) \setminus \mathbb{H}^3$. We shall take $\mathcal{F}$ by $\mathcal{F} = \{P = x + yi + rj : 0 \leq x \leq 1/2,\ |y| \leq 1/2,\ x^2 + y^2 + r^2 \geq 1\}$.

According to [11, Theorem 3.4 (3), p. 267], we have the spectral decomposition of $\tilde{F}_s(P)$ ($\Re(s) \gg 0$)

$$\tilde{F}_s(P) = \sum_m D^*(F, \overline{\mathcal{U}}_m, s)\mathcal{U}_m(P) + \frac{1}{2\pi}\int_{-\infty}^{\infty} D^*(F, \overline{E(\ ,it)}, s)E(P,it)dt, \tag{5.15}$$

where $\{\mathcal{U}_m(P)\}$ consists of the constant function $\mathcal{U}_0(P) = \pi/\sqrt{2\zeta_K(2)}$ and an orthonormal system of cusp eigenfunctions (see [11, Proposition 2.2, p. 245, Corollary 3.4, p. 107]) and $E(P,it)$ is the Eisenstein series. We called one of them by the spectral eigenfunction. For any automorphic function $\mathcal{U}(P)$, the function $D^*(F, \overline{\mathcal{U}}, s)$ is defined by the inner product

$$D^*(F, \overline{\mathcal{U}}, s) = \int_{\mathcal{F}} \tilde{F}_s(P)\overline{\mathcal{U}(P)}dv(P),$$

where $dv(P) = r^{-3}dxdydr$ with $P = (x+yi) + rj$. By [40, Proposition 2, p. 397], if $-\Delta\mathcal{U}_m = (1-\mu_m^2)\mathcal{U}_m$ for $m \neq 0$, then $1-\mu_m^2 \in \mathbb{R}$, $1-\mu_m^2 > 1$ and $\mu_m$ is pure imaginary for $m \neq 0$.

Proposition 5.12 below gives the spectral coefficient as Koecher–Maass series (5.9) completed by a certain gamma factor. For any automorphic function $\mathcal{U}(P)$ and any hermitian modular form

$$f(Z) = \sum_{T \in L_2 \geq O} A(T, f) e(\mathrm{tr}(TZ)) \in M_k(\Gamma_0^{(2)}(N), \rho),$$

we define $D^*(f, \mathcal{U}, s)$ by $D^*(F, \mathcal{U}, s)$, where $F(Z)$ is the non-degenerate part of $f(Z)$, that is

$$F(Z) = \sum_{T \in L_2^+} A(T, f) e(\mathrm{tr}(TZ)).$$

**Proposition 5.12.** *The following statements hold:*

(1) *Let $F$ be any Fourier series as in* (5.1) *and a spectral eigenfunction $\mathcal{U}(P)$ has the eigenvalue $1 - \mu^2$ of $-\Delta$. Then, for sufficiently large $\Re(s)$, one has*

$$D^*(F, \overline{\mathcal{U}}, s) = \pi(2\pi)^{-2s} \Gamma(s - 1/2 + \overline{\mu}/2) \Gamma(s - 1/2 - \overline{\mu}/2) D(F, \overline{\mathcal{U}}, s).$$

*Moreover, we have the same formula for $D^*(F, \hat{\overline{\mathcal{U}}}, s)$, if we change $\overline{\mathcal{U}}$ by $\hat{\overline{\mathcal{U}}}$ on the right hand side.*

(2) *For any spectral eigenfunction $\mathcal{U}(P)$ and $f(Z) \in M_k(\Gamma_0^{(2)}(N), \rho)$ $(\rho = \omega\psi)$, the functions $D^*(F, \overline{\mathcal{U}}, s)$ and $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ have a meromorphic continuation to all $s$, and satisfies the functional equation*

$$N^s D^*(f, \overline{\mathcal{U}}, s) = (-1)^k N^{k-s} D^*(g, \hat{\overline{\mathcal{U}}}, k - s),$$

*where $g(Z) = N^{-k} (\det Z)^{-k} f(-(NZ)^{-1}) \in M_k(\Gamma_0^{(2)}(N), \omega\overline{\psi})$.*

*Proof.* For sufficiently large $\sigma = \Re(s)$, one has from (5.6)

$$\tilde{F}_s(P) = \frac{\Gamma(2s)}{(2\pi)^{2s}} \sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{A(T, F)}{\epsilon(T)} \sum_{U \in \mathrm{SL}_2(\mathcal{O})} \mathrm{tr}([U]T \cdot W)^{-2s}.$$

Hence

$$\int_{\mathcal{F}} \tilde{F}_s(Q) \mathcal{U}(Q) dv(Q) = 2 \frac{\Gamma(2s)}{(2\pi)^{2s}} \sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{A(T, F)}{\epsilon(T)} \Omega_{\mathcal{U}}(s, T), \quad (5.16)$$

where we put

$$\Omega_{\mathcal{U}}(s, T) = \int_0^\infty \int_{\mathbb{C}} \mathrm{tr}(TW)^{-2s} \mathcal{U}(Q) dv(Q), \quad dv(Q) = \frac{dx dy dr}{r^3}$$

with $Q = (x + yi) + rj \in \mathbb{H}^3$ and $W = \begin{pmatrix} (|z|^2+r^2)r^{-1} & zr^{-1} \\ \bar{z}r^{-1} & r^{-1} \end{pmatrix}$. According to [11, Proposition 1.6, p. 6], we define

$$\delta(P, P') = \frac{|z - z'|^2 + r^2 + r'^2}{2rr'} \quad \text{for } P = z + rj,\ P' = z' + r'j.$$

Then, for $Q = z + rj$ and $P = \frac{(\det T)^{1/2} j - b}{a}$ with $T = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$, one has

$$2(\det T)^{1/2}\delta(P, Q) = \mathrm{tr}(TW).$$

Hence

$$\Omega_{\mathcal{U}}(s, T) = (\det T)^{-s} \int_{\mathbb{H}^3} (2\delta(P, Q))^{-2s}\,\mathcal{U}(Q)\,dv(Q). \tag{5.17}$$

Take $g \in \mathrm{SL}_2(\mathbb{C})$ so that $P = g\cdot j$. Since $\delta(P, Q) = \delta(g\cdot j, Q) = \delta(j, g^{-1}\cdot Q)$ and $\mathcal{U}(Q) = O(\max\{r^{\alpha}, r^{-\beta}\})$ with some positive $\alpha, \beta$, we have

$$\begin{aligned}
&\int_{\mathbb{H}^3} |(2\delta(P, Q))^{-2s}\,\mathcal{U}(Q)|\,dv(Q) \\
&\quad = \int_{\mathbb{H}^3} |(2\delta(j, Q))^{-2s}\,\mathcal{U}(g\cdot Q)|\,dv(Q) \\
&\quad \le \int_0^{\infty}\int_{\mathbb{C}} \left(\frac{r}{|z|^2 + r^2 + 1}\right)^{2\sigma} \left|\mathcal{U}\left(\frac{(\det T)^{1/2}Q - b}{a}\right)\right| \frac{dx\,dy\,dr}{r^3} \\
&\quad \le M_9 \int_0^{\infty}\int_{\mathbb{C}} \left(\frac{r}{|z|^2 + r^2 + 1}\right)^{2\sigma} \\
&\qquad\qquad \times \max\left\{\left(\frac{(\det T)^{1/2}}{a} r\right)^{\alpha}, \left(\frac{(\det T)^{1/2}}{a} r\right)^{-\beta}\right\} \frac{dx\,dy\,dr}{r^3}.
\end{aligned}$$

Since, for $\sigma \gg 0$,

$$\int_{\mathbb{C}} \frac{1}{(|z|^2 + r^2 + 1)^{2\sigma}}\,dx\,dy = \pi\frac{\Gamma(2\sigma - 1)}{\Gamma(2\sigma)}(r^2 + 1)^{1-2\sigma},$$
$$\int_0^{\infty} r^{\sigma-3+\alpha}(r^2 + 1)^{1-2\sigma}\,dr,\ \int_0^{\infty} r^{\sigma-3-\beta}(r^2 + 1)^{1-2\sigma}\,dr < +\infty,$$

the integral (5.17) is absolutely convergent for sufficiently large $\sigma$.

In (5.16), we take $T \in \mathrm{SL}_2(\mathcal{O}) \setminus L_2^{+}$ so that so that $2T$ is reduced in the sense of [11, Definition 2.3, p. 411]. Then $a \le \det(2T)^2$ by [11, Proposition 2.6 (1), p. 412]. Hence Lemma 5.7 justifies interchanging the summation and the integration to obtain (5.16).

Theorem 5.3 in [11, p. 119] combined with the third formula in [26, p. 6] evaluates the integral in (5.17) as

$$\mathcal{U}(P)\cdot\frac{\pi}{\mu}\int_1^\infty (t+t^{-1})^{-2s}(t^\mu - t^{-\mu})(t-t^{-1})\frac{dt}{t}$$
$$= \mathcal{U}(T^{-1})\cdot\frac{\pi}{2}\frac{\Gamma(s-1/2-\mu/2)\Gamma(s-1/2+\mu/2)}{\Gamma(2s)}.$$

This completes the proof of the statement (1) (cf. [25, (55), (63), pp. 100–102] for the Siegel modular case).

The statement (2) can be shown in the same way as in the proof of [2, Corollary 2.3, p. 271] and [3, Theorem 10, p. 209]. □

We note here that Ibukiyama [16] established a general theory of Koecher–Maass series with Grössencharacter (suitable automorphic forms) associated with modular forms on tube domains. The convergence of the series, determination of the gamma factor, meromorphic continuation and functional equation are given in [16].

If the Fourier coefficients $A(T,F)$ satisfy a Maass type relation, then $D(F,\mathcal{U},s)$ is a convolution product of two Dirichlet series.

**Proposition 5.13.** *Let $\chi$ be a Dirichlet character. Suppose that there exists a function $\alpha$ on the set of all natural numbers satisfying*

$$A(T,F) = \sum_{d\,|\,e(T)} \chi(d)d^{k-1}\alpha((4\det T)/d^2), \tag{5.18}$$

*where*

$$e(T) = \max\{q\in\mathbb{N} : q^{-1}T \in L_2^+\}.$$

*Then for any spectral eigenfunction $\mathcal{U}(P)$ on $\mathbb{H}^3$ whose eigenvalue of $-\Delta$ is $1-\mu^2$, we have*

$$D^*(F,\overline{\mathcal{U}},s) = \pi(2\pi)^{-2s}\Gamma(s-1/2+\overline{\mu}/2)\Gamma(s-1/2-\overline{\mu}/2)$$
$$\times 4^s L(2s-k+1,\chi)\sum_{l\geq 1}\frac{\alpha(l)b_{\hat{\overline{\mathcal{U}}}}(l)l}{l^s},$$

*for $\Re(s)\gg 0$, where $b_{\hat{\mathcal{U}}}(l)$ is defined by (1.1). Moreover, we have the same formula for $D^*(F,\hat{\overline{\mathcal{U}}},s)$, if we change $\overline{\mathcal{U}}$ by $\hat{\overline{\mathcal{U}}}$ on the right hand side.*

*Proof.* Substituting equation (5.18) into (5.9), the result follows without difficulties. See [5, Satz 3] and [9, Lemma 3]. □

## 6 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. First of all, we give an explicit form of the Koecher–Maass series associated with Hermitian Eisenstein series.

For any spectral eigenfunction $\mathcal{U}(P)$ and

$$f(Z) = \sum_{T \in L_2 \geq O} A(T, f) e(\mathrm{tr}(TZ)) \in M_k(\Gamma_0^{(2)}(N), \rho),$$

put

$$D(f, \mathcal{U}, s) = \sum_{T \in \mathrm{SL}_2(\mathcal{O}) \backslash L_2^+} \frac{A(T, f) \hat{\mathcal{U}}(T)}{\epsilon(T)(\det T)^s}.$$

By Proposition 5.12 (1), the spectral coefficient $D^*(f, \overline{\mathcal{U}}, s)$ of the non-degenerate part of $f$ is given by

$$D^*(f, \overline{\mathcal{U}}, s) = \pi(2\pi)^{-2s} \Gamma(s - 1/2 + \overline{\mu}/2) \Gamma(s - 1/2 - \overline{\mu}/2) D(f, \overline{\mathcal{U}}, s),$$

where $-\Delta \mathcal{U} = (1 - \mu^2)\mathcal{U}$.

**Proposition 6.1.** *For any spectral eigenfunction $\mathcal{U}(P)$ corresponding to the eigenvalue $-(1 - \mu^2)$ of $\Delta$, the Koecher–Maass series of $E_{k,\overline{\rho}}^{(2)}$ has the form*

$$\begin{aligned} D^*(E_{k,\overline{\rho}}^{(2)}, \overline{\mathcal{U}}, s) = {} & \frac{\tau_{k,2} 4^{-k+2} N^{-k} \pi^{1-2s} \tau_N(\overline{\psi})}{\alpha_{k,4} L(k, \overline{\psi})} \Gamma(s - 1/2 + \overline{\mu}/2) \\ & \times \Gamma(s - 1/2 - \overline{\mu}/2) L(2s - k + 1, \psi) \sum_{l=1}^{\infty} \frac{e_{\overline{\rho}}^{\infty}(l) b_{\hat{\overline{\mathcal{U}}}}(l) l}{l^s}, \end{aligned}$$

*where*

$$\tau_{k,2} = (-1)^k (2\pi)^{2k-1} \{2\Gamma(k)\Gamma(k-1)\}^{-1}$$

*and $\alpha_{k,4}$ is as in Proposition* 3.1.

*Proof.* Let

$$F_{k,\overline{\rho}}^{(2)}(Z) = N^{-k} (\det Z)^{-k} E_{k,\overline{\rho}}^{(2)}(-(NZ)^{-1})$$

be the involuted Eisenstein series. It has a Fourier expansion

$$\begin{aligned} & F_{k,\overline{\rho}}^{(2)}(Z) = \sum_{T \in L_2^+} C(T) e(\mathrm{tr}(TZ)), \\ & C(T) = \frac{\tau_{k,2} N^{-k}}{4^{k-2} \alpha_{k,4} (-1)^k L(k, \overline{\psi})} \sum_{d \mid e(T)} \overline{\psi}(d) d^{k-1} e_{\rho}^0((4 \det T)/d^2). \end{aligned} \tag{6.1}$$

Indeed similarly to [6], we have

$$C(T) = \tau_{k,2} N^{-k} (\det T)^{k-2} \sum_{R \in \mathrm{Her}_2(K)/\mathrm{Her}_2(\mathcal{O})} \overline{\rho}(\nu(R)) \nu(R)^{-k} e(\mathrm{tr}(TR)),$$

where $\nu(R) = |\det C|$ with $R = C^{-1}D$, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_2$, $\det C \neq 0$. For $T > O$, the Siegel series

$$b(s,T) = \sum_{R \in \mathrm{Her}_2(K)/\mathrm{Her}_2(\mathcal{O})} \nu(R)^{-s} e(\mathrm{tr}(TR)) \quad (\Re(s) \gg 0)$$

has an Euler product, whose Euler $p$-factor is a partial series of $b(s,T)$ consisting of the terms indexed by $R \in \mathrm{Her}_2(K)/\mathrm{Her}_2(\mathcal{O})$ such that $\nu(R)$ is a power of $p$. The explicit formula due to Nagaoka [35] tells us that each Euler $p$-factor is a polynomial of $p^{-s}$. Replacing $p^{-s}$ by $\overline{\rho}(p)p^{-k}$ in his formula, one can deduce a formula for $C(T)$ as an Euler product of the polynomial of $\overline{\psi}(p)p^{-k}$. The result combined with some elementary manipulation implies (6.1).

Since $-\Delta\overline{\mathcal{U}} = (1-\overline{\mu}^2)\overline{\mathcal{U}}$ and $-\Delta\hat{\mathcal{U}} = (1-\mu^2)\hat{\mathcal{U}}$, it follows from (6.1) and Proposition 5.13 that

$$D^*(F^{(2)}_{k,\overline{\rho}}, \hat{\overline{\mathcal{U}}}, s) = \frac{\tau_{k,2} N^{-k} \pi^{1-2s}}{4^{k-2}\alpha_{k,4}(-1)^k L(k,\overline{\psi})} \Gamma(s-1/2-\overline{\mu}/2)\Gamma(s-1/2+\overline{\mu}/2)$$
$$\times L(2s-k+1,\overline{\psi}) \sum_{l=1}^{\infty} \frac{e^0_\rho(l) b_{\overline{\mathcal{U}}}(l) l}{l^s}. \tag{6.2}$$

For each cusp $\kappa \in \{\infty, 0\} \bigcup \{1/\mu : 1 < \mu < N,\ \mu \mid N\}$, we choose elements in $\mathrm{SL}_2(\mathbb{R})$ by

$$g_\infty = \sigma_\infty, \quad g_0 = \sigma_0 A_1, \quad g_\mu = \sigma_\mu A_\mu, \quad A_\mu = \begin{pmatrix} \sqrt{N/\mu} & 0 \\ 0 & \sqrt{\mu/N} \end{pmatrix}$$

with $\sigma_\kappa$ in (3.3). So the assumptions in [31, Section 2.1] are fulfilled. We will also use $g_\kappa$ instead of the above symbols $g_j$.

Let $h_\alpha(\tau)$ be the $\alpha$-th coefficient of the theta expansion of $E^0_{k,1,\rho}$ and put

$$\xi(\tau) = \sum_{\alpha \in \mathcal{D}^{-1}/\mathcal{O}} h_\alpha(\tau)\overline{f_\alpha(\tau)}, \quad \xi_\kappa(\tau) = \nu_{k,2}(g_\kappa, \tau)^{-1} \xi(g_\kappa \tau),$$

where $f_\alpha(\tau)$ is as in (2.3) and $\nu_{k,2}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = (c\tau+d)^k/|c\tau+d|^2$. Since $U(\gamma)$ is unitary, Proposition 2.1 combined with (3.2) implies

$$\xi(\gamma\tau) = \overline{\psi}(\gamma)\nu_{k,2}(\gamma,\tau)\xi(\tau) \quad \text{for all } \gamma \in \Gamma_0(N).$$

Moreover, we claim that

$$\xi_\kappa(\tau) - \delta_{0,\kappa}(2i)N^{k/2-1}\overline{B(0, Nv/4)}$$

is of rapid decay as $\Im\tau$ tends to $\infty$. In order to check this claim, to the theta expansions

$$E^0_{k,1,\rho}|_{k,1}\sigma_\kappa(\tau, z, w) = \sum_{\alpha\in\mathcal{D}^{-1}/\mathcal{O}} H^\kappa_\alpha(\tau)\theta_\alpha(\tau, z, w),$$

we associate the function

$$\eta_\kappa(\tau) = \sum_{\alpha\in\mathcal{D}^{-1}/\mathcal{O}} H^\kappa_\alpha(\tau)\overline{f_\alpha(\tau)}.$$

The theta formula (2.5) gives the relation $\eta_\kappa(\tau) = \nu_{k,2}(\sigma_\kappa, \tau)^{-1}\xi(\sigma_\kappa\tau)$, and this implies $\xi_\kappa(\tau) = (\mu/N)^{1-k}\eta_\kappa(N\tau/\mu)$ for $\kappa = 1/\mu$ and $\xi_0(\tau) = N^{k/2-1}\eta_0(N\tau)$. Hence the desired claim follows from the Fourier expansions of $\eta_\kappa(\tau)$ (Propositions 3.1, 3.2). In particular, $\xi_0(\tau) = N^{k/2-1}\eta_0(N\tau)$ and

$$H^0_\alpha(\tau) = \delta_{0,\alpha} + \sum_{l>0,\ l\equiv -4N\mathcal{N}(\alpha) \pmod 4} a^0_\rho(Nl)q^{\frac{l}{4N}}.$$

Under these observations, we define the Rankin–Selberg transform $R_\kappa(s)$ for $\Re(s)$ sufficiently large by (cf. [31, 41])

$$R_\kappa(s) = \int_0^\infty\int_0^1 (\xi_\kappa(\tau) - \delta_{0,\kappa}(2i)N^{k/2-1}\overline{B(0, Nv/4)})v^{s-2}du\,dv.$$

Theorem 2 of [31] is applicable to the present setting and it gives meromorphic continuation of $R_\kappa(s)$ to all $s$ and the functional equation

$$R_\infty(s - 1/2) = \frac{2^{k+1-2s}\pi i^k N^{-s+k/2-1/2}\Gamma(2s-k)L(2s-k,\overline{\psi})}{\Gamma(s-k+1/2)\Gamma(s+1/2)L(2s-k+1,\overline{\psi})} \times R_0(k-s-1/2).$$

Using [12, formula 11, p. 816] combined with the Fourier expansion of $\xi(\tau)$ and (6.2), it follows that $D^*(F^{(2)}_{k,\overline{\rho}}, \hat{\overline{\mathcal{U}}}, s)$ coincides with $R_\infty(s - 1/2)$ up to a gamma factor. The functional equation of the Dirichlet $L$-function and Proposition 5.12 (2) combined with (6.2) imply

$$D^*(E^{(2)}_{k,\overline{\rho}}, \overline{\mathcal{U}}, s) = \frac{\tau_{k,2}N^{s-3k/2-1/2}\pi^{-s}\tau_N(\overline{\psi})}{i4^{k-2}\alpha_{k,4}L(k,\overline{\psi})} \times \Gamma(s+1/2)L(2s-k+1,\psi)R_0(s-1/2).$$

Taking the relations $e^{\infty}_{\overline{\rho}}(t) = a^0_\rho(N^2 t)$ and $b_{\hat{\mathcal{U}}}(l) = b_{\mathcal{U}}(l)$ into account, a Fourier expansion of $\xi_0(\tau)$ combined with [12, formula 11, p. 816] yields the desired result. □

*Proof of Theorem* 1.3. Set

$$F = E^{(2)}_{k,\overline{\rho}} - \frac{(-2\pi i)^k \tau_N(\overline{\psi})}{N^k \Gamma(k) L(k,\overline{\psi})} \mathcal{M} E^{\infty}_{k,1,\overline{\rho}}.$$

Applying the Siegel operator $\Phi$ defined by

$$\Phi F(\tau) = \lim_{\lambda \to +\infty} F\left(\begin{smallmatrix} i\lambda & 0 \\ 0 & \tau \end{smallmatrix}\right) \in M_k(\Gamma_0(N), \rho),$$

we have $\Phi F = 0$ so that the Fourier expansion of $F(Z)$ has only the terms indexed by $L_2^+$. Moreover, Propositions 6.1 and 5.13 imply $D^*(F, \overline{\mathcal{U}}, s) = 0$ for any spectral eigenfunctions $\mathcal{U} = \mathcal{U}_m, E(\cdot, it)$. Hence we have $\tilde{F}_s(P) = 0$ in (5.15) and so $F(Z) = 0$ by Mellin inversion (5.7) and the principle of analytic continuation ([24, Lemma 1.6, p. 48]). This completes the proof of Theorem 1.3. □

In [34, p. 858], we used the termwise Mellin inversion of the spectral decomposition of a Siegel modular form. This is justified in [15]. See also the proof of Proposition 7.1 below. While, the termwise Mellin inversion is unnecessary, if we proceed the proof as above (cf. [8, proof of Lemma 2.1, p. 155]).

## 7 Proof of Theorem 1.2

The converse theorem for Hermitian modular forms analogous to the Siegel modular case ([9, Theorem 2]) is the following.

**Proposition 7.1.** *Suppose that a natural number $k$ is divisible by* 4 *and take $F(Z)$ as in* (5.1). *If $D^*(F, \overline{\mathcal{U}}, s)$ and $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ can be analytically continued to entire functions of $s$, which are bounded in every vertical strip in $s$ and satisfy*

$$D^*(F, \hat{\overline{\mathcal{U}}}, s) = D^*(F, \overline{\mathcal{U}}, k - s)$$

*for any spectral eigenfunction $\mathcal{U}$ on $\mathbb{H}^3$, then $F(Z)$ is a Hermitian modular form of weight $k$ on $\Gamma_2$.*

*Proof.* We follow Ibukiyama's proof of [9, Theorem 2] (see [15]). The estimate (5.8) implies

$$\Delta^2 \tilde{f}_s(P) \in L^2(\mathrm{SL}_2(\mathcal{O}) \setminus \mathbb{H}^3)$$

for fixed $s$ with sufficiently large $\Re(s)$. Suppose that $-\Delta\mathcal{U}_m = \lambda_m\mathcal{U}_m$. Since $-\Delta$ is symmetric ([11, Theorem 1.7, p. 136]), we have $(\tilde{f}_s, \mathcal{U}_m) = \lambda_m^{-2}(\Delta^2\tilde{f}_s, \mathcal{U}_m)$ for cusp eigenfunctions $\mathcal{U}_m$, where $(\cdot, \cdot)$ denotes the inner product on $\mathrm{SL}_2(\mathcal{O})\backslash\mathbb{H}^3$ defined by

$$(f, g) = \int_{\mathcal{F}} f(P)\overline{g(P)}r^{-3}dxdydr$$

with the fundamental domain $\mathcal{F}$ and $P = (x + yi) + rj \in \mathbb{H}^3$ (see [11, (1.2), p. 133]). This relation combined with an inequality about geometric-arithmetic means and Schwarz' inequality tells us that

$$\begin{aligned}|(\tilde{F}_s, \mathcal{U}_m)\mathcal{U}_m(P)| &\le |(2\pi)^{-2s}\Gamma(2s)|2^{-1}\{\lambda_m^{-2}(\Delta^2\tilde{f}_s, \Delta^2\tilde{f}_s) + \lambda_m^{-2}|\mathcal{U}_m(P)|^2\} \\ &\le (2\pi)^{-2\sigma}|\Gamma(2s)|2^{-1}\{\lambda_m^{-2}\cdot N_0^2(1+2|t|)^{2\alpha}\cdot 2\zeta_K(2)\pi^{-1} \\ &\qquad + \lambda_m^{-2}|\mathcal{U}_m(P)|^2\}.\end{aligned}$$

Here, for the last inequality, we used (5.8) and $\int_{\mathcal{F}} r^{-3}dxdydr = 2\zeta_K(2)\pi^{-1}$. Then, by [11, Corollaries 5.3 and 5.5, p. 182], the sum

$$\sum_{m\neq 0}|D^*(F, \overline{\mathcal{U}}_m, s)\mathcal{U}_m(P)| = \sum_{m\neq 0}|(\tilde{F}_s, \mathcal{U}_m)\mathcal{U}_m(P)|$$

on the right hand side of (5.15) converges uniformly on any compact subset $L$ of $\mathbb{H}^3$. See also [11, Corollary 5.4, p. 182]. By Lemma 5.2, we have

$$\int_{-\infty}^{\infty}\sum_{m\neq 0}|(\tilde{F}_{\sigma+it}, \mathcal{U}_m)\mathcal{U}_m(P)|dt < +\infty \quad \text{for } \sigma \gg 0.$$

On the other hand, according to [11, Theorem 3.4 (3), p. 267],

$$\int_{-\infty}^{\infty}|D^*(F, \overline{E(\ , i\eta)}, s)E(P, i\eta)|d\eta$$

converges uniformly on any compact subset $L$ of $\mathbb{H}^3$ for fixed $s$ with $\Re(s) \gg 0$. Lemmas 5.2, 5.9 and 5.10 imply for any fixed $P \in \mathbb{H}^3$ that

$$\int_{t=-\infty}^{\infty}\int_{\eta=-\infty}^{\infty}|D^*(F, \overline{E(\cdot, i\eta)}, \sigma+it)E(P, i\eta)|d\eta dt < +\infty \quad \text{for } \sigma \gg 0.$$

Hence, we can apply Mellin inversion of (5.15) term by term.

By Section 5.4, $D(F, \overline{\mathcal{U}}, s)$ and $D(F, \hat{\overline{\mathcal{U}}}, s)$ are $O(1)$ on the line $\sigma_1 = \Re(s)$, if $\sigma_1$ is sufficiently large. By the functional equation and Lemma 5.2, there exists a constant $A$ such that $D(F, \overline{\mathcal{U}}, s)$ and $D(F, \hat{\overline{\mathcal{U}}}, s)$ are $O(1 + |\Im(s)|^A)$ on the line

$\sigma_2 = \Re(s)$, if $\sigma_2 < 0$ and $|\sigma_2|$ is sufficiently large. As $D^*(F, \overline{\mathcal{U}}, s)$, $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ are entire and bounded in the strip $\sigma_2 \le \Re(s) \le \sigma_1$, there exists a constant $B$ such that $D(F, \overline{\mathcal{U}}, s)$ and $D(F, \hat{\overline{\mathcal{U}}}, s)$ are $O(e^{B|\Im(s)|})$ on the strip $\sigma_2 \le \Re(s) \le \sigma_1$ by Lemma 5.2. According to the Phragmén–Lindelöf theorem, we deduce that there exists a constant $C$ such that $D(F, \overline{\mathcal{U}}, s)$ and $D(F, \hat{\overline{\mathcal{U}}}, s)$ are $O(1+|\Im(s)|^C)$ on the strip $\sigma_2 \le \Re(s) \le \sigma_1$. Hence, Lemma 5.2 implies that $D^*(F, \overline{\mathcal{U}}, s)$, $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ are of rapid decay as $|\Im(s)| \to \infty$ on the strip $\sigma_2 \le \Re(s) \le \sigma_1$. This combined with the entireness of $D^*(F, \overline{\mathcal{U}}, s)$, $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ and the functional equation gives

$$\int_{(\sigma)} D^*(F, \overline{\mathcal{U}}, s) y^{-s} ds = y^{-k} \int_{(\sigma)} D^*(F, \hat{\overline{\mathcal{U}}}, s) \left(\frac{1}{y}\right)^{-s} ds, \quad \sigma \gg 0,$$

for $y > 0$ by shifting the path of integration. Then it follows from (5.7) and (5.15) that $F(iu^{-1}W^{-1}) = u^{2k} F(iuW)$ and $F(-(iY)^{-1}) = \det(iY)^k F(iY)$. In fact, by

$$\left[\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\right] W^{-1} = \left(\begin{smallmatrix} (|z|^2+r^2)r^{-1} & \bar{z}r^{-1} \\ zr^{-1} & r^{-1} \end{smallmatrix}\right), \tag{7.1}$$

the function $D^*(F, \hat{\overline{\mathcal{U}}}, s)$ is the spectral coefficient of $\tilde{F}_s(P_{W^{-1}})$ with respect to $\mathcal{U}(P_W)$. Note that, similarly to $\tilde{F}_s(P_W)$, the function $\tilde{F}_s(P_{W^{-1}})$ satisfies the assumptions to apply the spectral decomposition [11, Theorem 3.4 (3), p. 267] and the termwise Mellin inversion. In view of [24, Lemma 1.7, p. 79] and [24, Lemma 1.6, p. 48], we complete the proof of Proposition 7.1. □

*Proof of Theorem* 1.2. Let $F(Z)$ be as in (1.3) and suppose that

$$-\Delta \mathcal{U} = (1 - \mu^2)\mathcal{U}.$$

The identity (7.1) tells us $b_{\hat{\mathcal{U}}}(l) = b_{\mathcal{U}}(l)$. This combined with Proposition 5.13 implies

$$D^*(F, \overline{\mathcal{U}}, s) = \pi(2\pi)^{-2s}\Gamma(s - 1/2 + \overline{\mu}/2)\Gamma(s - 1/2 - \overline{\mu}/2)$$
$$\times 4^s \zeta(2s - k + 1) \sum_{l \ge 1} \frac{\alpha^*(l) b_{\overline{\mathcal{U}}}(l) l}{l^s}.$$

For $\alpha \in \mathcal{D}^{-1}$, we associate to $g(\tau)$ in (1.2) the function

$$g_\alpha(\tau) = \frac{-2i}{a_4(-4\mathcal{N}(\alpha))} \sum_{l \ge 1,\ l \equiv -4\mathcal{N}(\alpha) \pmod 4} c(l) e(l\tau/4).$$

Similarly to Proposition 2.1, one has

$$(g_{\alpha_1}(\tau), \ldots, g_{\alpha_4}(\tau)) = (g_{\alpha_1}(\gamma\tau), \ldots, g_{\alpha_4}(\gamma\tau))(c\tau + d)^{1-k} U(\gamma) \tag{7.2}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where $U(\gamma)$ is the unitary matrix in (2.5). While, to

$$f(\tau) = v^{1/2}\varphi(\tau)$$

with $\varphi(\tau)$ corresponding to $\mathcal{U}(P)$ by means of Theorem 1.1, we associate $f_\alpha(\tau)$ as in (2.3). It follows from the transformation formulas (7.2) and (2.5) that the function

$$\xi(\tau) = \sum_{\alpha \in \mathfrak{D}^{-1}/\mathcal{O}} g_\alpha(\tau)\overline{f_\alpha(\tau)}$$

satisfies $\xi(\gamma\tau) = \nu_{k,2}(\gamma,\tau)\xi(\tau)$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Here

$$\nu_{k,2}(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) = (c\tau + d)^k / |c\tau + d|^2.$$

Hence the Rankin–Selberg method tells us that

$$D^*(F, \overline{\mathcal{U}}, s) = 2\pi^{-s}\Gamma(s + 1/2)\zeta(2s - k + 1)\mathcal{R}_\infty(s - 1/2)$$

with

$$\mathcal{R}_\infty(s) = \iint_D \xi(\tau)\mathcal{E}(\tau, s)\frac{du dv}{v^2},$$

where $D = \{\tau = u + iv \in H_1 : |\tau| \geq 1, |u| \leq 1/2\}$ and

$$\mathcal{E}(\tau, s) = \frac{v^s}{2} \sum_{c,d \in \mathbb{Z},\ (c,d)=1} \frac{(c\tau + d)^k}{|c\tau + d|^{2s+2}}.$$

If we use the notation of Shimura [38, p. 461], this has the form

$$D^*(F, \overline{\mathcal{U}}, s) = \pi^{1/2} \iint_D \xi(\tau) v^{-1} H_k(s + 1/2, \tau, id)\frac{du dv}{v^2}.$$

It follows from the proof of [38, Lemma 3.3, p. 461] that $D^*(F, \overline{\mathcal{U}}, s)$ is entire (see also [30, Corollary 7.2.11, p. 286]) and satisfies $D^*(F, \overline{\mathcal{U}}, s) = D^*(F, \overline{\mathcal{U}}, k - s)$. Moreover, it is bounded in every vertical strip in $s$. Using $b_{\hat{\mathcal{U}}}(l) = b_{\mathcal{U}}(l)$, we have $D^*(F, \overline{\mathcal{U}}, s) = D^*(F, \overline{\hat{\mathcal{U}}}, s)$. By Proposition 7.1, we complete the proof of Theorem 1.2. □

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