SYMPLECTIC 4-MANIFOLDS WITH FIXED POINT FREE CIRCLE ACTIONS

JONATHAN BOWDEN

Abstract. We show that recent results of Friedl-Vidussi and Chen imply that a symplectic 4-manifold admits a fixed point free circle action if and only if it admits a symplectic structure that is invariant under the action and we give a complete description of the symplectic cone in this case. This then completes the topological characterisation of symplectic 4-manifolds that admit non-trivial circle actions.

1. Introduction

Recently Friedl and Vidussi, [7] solved the long standing Taubes Conjecture, which classifies which 4-manifolds of the form $M \times S^1$ admit symplectic forms. Moreover, they determined exactly which cohomology classes can be represented by symplectic forms. Using recent results of D. Wise, [13] they have extended their results to the case of non-trivial S^1-bundles in [9]. In this note we observe that their results as well of those of Chen, who obtained partial results in the fixed point free case in [3], imply the analogue of ([9], Theorem 1.3) for all fixed point free circle actions.

Before stating our main result we fix some notation and terminology. Let $X \xrightarrow{p} M$ be an orientable 4-manifold with a fixed point free circle action and quotient space $M = X/S^1$. The quotient space is an orbifold whose underlying topological space $|M|$ is a manifold since all the stabilisers of the S^1-action are necessarily cyclic and the singular locus consists of a collection of branching circles (cf. [1], [6]).

We let M_{reg} denote the complement of an open tubular neighbourhood of the singular locus of M and $X_{reg} = p^{-1}(M_{reg})$, which is an honest S^1-bundle so that the pushforward map p_* is well-defined for cohomology classes in $H^*(X_{reg}, \mathbb{R})$. The manifold M_{reg} has toroidal boundary and thus one may define the Thurston norm on $H^1(M_{reg}, \mathbb{R})$ in the usual fashion. Finally for $\psi \in H^2(X, \mathbb{R})$ we let ψ_{reg} denote the restriction of ψ to X_{reg}.

Theorem 1. *Let $X \xrightarrow{p} M$ be an oriented manifold admitting a fixed point free S^1-action with quotient space M and let $\psi \in H^2(X, \mathbb{R})$. Then the following are equivalent:*

(1) *ψ can be represented by a symplectic form,*
(2) *ψ can be represented by an S^1-invariant symplectic form,*
(3) *$\psi^2 > 0$ and $p_*\psi_{reg} \in H^1(M_{reg}, \mathbb{R})$ lies in the open cone over a fibered face of the Thurston norm ball and is the restriction of a class in $H^1(|M|, \mathbb{R})$.*

Note that if a class $\phi \in H^1(|M|, \mathbb{R})$ is integral, then ϕ can be represented by a fibration over S^1 that is transverse to the singular locus of $|M|$ if and only if its restriction to M_{reg}, which we denote by ϕ_{reg}, is fibered. Recall that a fibration of a manifold with boundary

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is required to be transverse to the boundary. Furthermore, since ϕ_{reg} is the restriction of a class in $H^1(|M|, \mathbb{R})$, it automatically vanishes on the meridian classes in ∂M_{reg} so that if ϕ_{reg} is fibered, then the induced fibration on the boundary is necessarily meridional. Thus the fibration dual to ϕ_{reg} extends to $|M|$ in the desired way by filling in discs near the singular locus. In particular, part (3) of Theorem 1 implies that the underlying manifold $|M|$ is fibered and we obtain a positive answer to the following conjecture, which implies ([3], Conjecture 1.7) as a special case.

Conjecture 1 (Generalised Taubes conjecture)**.** *Let X be a symplectic 4-manifold that admits a non-trivial fixed point free circle action with quotient orbifold M. Then the (possibly empty) singular locus L of M is a meridionally fibered link.*

Furthermore, as noted in ([3], p. 6), Theorem 1 completes the characterisation of which symplectic manifolds admit non-trivial S^1-actions. For Baldridge, [1] showed that if a non-trivial S^1-action on a symplectic 4-manifold has fixed points then X is rational or ruled and thus admits an S^1-invariant symplectic form for *some* non-trivial S^1-action. In view of this we obtain the following corollary.

Corollary 1. *Let X be a symplectic 4-manifold that admits a non-trivial S^1-action. Then either the action is fixed point free and the quotient space fibers over S^1 or X is rational or ruled. In either case, X admits a non-trivial symplectic S^1-action.*

2. Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma, which provides a generalisation of ([4], Theorem 5.2) to include irrational classes. For the proof we assume a certain familiarity with the basic properties of the Thurston norm (cf. [12]).

Lemma 1. *Let $\overline{M}$ be a 3-manifold with an orientation preserving smooth action of a finite group G and quotient orbifold $M = \overline{M}/G$. An element $\overline{\phi}$ in the invariant subspace $H^1(\overline{M}, \mathbb{R})^G$ admits a non-degenerate de Rham representative if and only if it admits a non-degenerate de Rham representative that is G-invariant.*

In particular, the restriction of the associated class $\phi \in H^1(|M|, \mathbb{R}) \cong H^1(\overline{M}, \mathbb{R})^G$ to M_{reg} lies in the open cone over a fibered face of the Thurston norm ball.

Proof. We first assume that $\overline{\phi}$ is rational. Since nothing changes after multiplying with positive constants, we may assume that $\overline{\phi}$ is in fact integral. In this case the first claim is just a restatement of ([4], Theorem 5.2), which can be applied in complete generality in view of ([11], Theorem 8.1). Note that the assumption $H^1(\overline{M}, \mathbb{Q})^G = \mathbb{Q}$ in ([4], Theorem 5.2) can be replaced by the fact that the fibration is given by a fibered class $\overline{\phi}$ that is G-invariant. Moreover, the proof in [4] actually gives a fibration that is transverse to the branching locus in $\overline{M}$. The quotient map π induces an isomorphism $H^1(\overline{M}, \mathbb{R})^G \cong H^1(|M|, \mathbb{R})$ so that there is a unique class ϕ with $\overline{\phi} = \pi^*\phi$ and the fibration dual to $\overline{\phi}$ descends to a fibration of $|M|$ dual to ϕ. Finally since the fibration is transverse to the singular locus it follows that the restriction of ϕ to M_{reg} is fibered.

We next assume that $\overline{\phi}$ is irrational and let ϕ be the unique class with $\overline{\phi} = \pi^*\phi$. We let ι_{reg} denote the natural inclusion $M_{reg} \hookrightarrow |M|$ and set $V = Im(\iota_{reg}^*)$. By the previous case all rational classes in V that are sufficiently close to $\phi_{reg} = \iota_{reg}^*\phi$ are fibered. If ϕ_{reg} itself did not lie in the open cone over a fibered face of the Thurston unit ball, then it must lie in the closed cone over the boundary of a fibered face by the assumption that it can be approximated by fibered elements. Since the Thurston unit ball is rational, the intersection of the closed cone containing ϕ_{reg} with V must contain non-fibered rational points arbitrarily close to ϕ_{reg}, which gives a contradiction. Thus ϕ_{reg} admits a non-degenerate de Rham representative η_{reg}. Since η_{reg} can be approximated by rational classes that are fibered and restrict to meridional fibrations on the boundary of M_{reg}, the foliation induced by η_{reg} on the boundary is also meridional.

We let $(z, \theta) \in D^2 \times S^1$ denote coordinates on a tubular neighbourhood of a component of the branching locus of $|M|$. After applying a suitable isotopy we may assume that η_{reg} has the form $f(\theta)d\theta$ near $\partial D^2 \times S^1$. It follows that η_{reg} extends to a non-degenerate closed form η which is transverse to the branching locus of $|M|$. The pullback $\overline{\eta} = \pi^*\eta$ then gives the desired non-degenerate G-equivariant representative of $\overline{\phi}$.

□

Proof of Theorem 1. The implication (2) $\Longrightarrow$ (1) is trivial.

(1) $\Longrightarrow$ (3): Let (X, ω) be a symplectic manifold with a fixed point free S^1-action and quotient space M. By ([3], Proposition 1.8) there is a manifold $\overline{M}$ and a smooth action by a finite group so that $M = \overline{M}/G$. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc} \pi^*X = \overline{X} & \xrightarrow{\overline{p}} & \overline{M} \\ \Big\downarrow{\scriptstyle \overline{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{p} & M, \end{array}$$

where π is the quotient map, $\overline{\pi}$ is an unramified covering and the induced S^1-action on $\overline{X}$ is free. Moreover, the group G acts naturally on $\overline{X}$ as the group of deck transformations of $\overline{\pi}$.

Thus $\overline{\omega} = \overline{\pi}^*\omega$ is a symplectic form and by ([9], Theorem 1.4) its image under the pushforward map $\overline{p}_*(\overline{\omega}) \in H^1(\overline{M}, \mathbb{R})$ lies in the open cone over a fibered face of the Thurston norm ball. Since $\overline{\omega}$ is G-invariant and the action on $\overline{X}$ is fiber preserving, the class $\overline{\phi} = \overline{p}_*(\overline{\omega})$ is also G-invariant. We let ϕ be the unique class such that $\pi^*\phi = \overline{\phi}$. By Lemma 1 the restriction ϕ_{reg} to M_{reg} lies in the open cone over a fibered face. Finally the naturality of the transfer homomorphism implies that the restriction of ϕ_{reg} agrees with $p_*\omega_{reg}$.

(3) $\Longrightarrow$ (2): By assumption $\phi_{reg} = p_*\psi_{reg}$ lies in the open cone over a fibered face of the Thurston norm ball and ϕ_{reg} is the restriction of a class $\phi \in H^1(|M|, \mathbb{R})$. In particular, $|M|$ fibers over S^1. We first note that M is a *very good* orbifold so that it is a quotient of a manifold $\overline{M}$ by a smooth action of a finite group G. For this it suffices to rule out bad 2-suborbifolds by ([2], Corollary 3.28). However, a bad 2-suborbifold is topologically a sphere that is essential in $H_2(|M|, \mathbb{Z})$ and as in the proof of ([3], Lemma 2.3) this implies that $b_2^+(X) = b_2(|M|) - 1$. Thus $|M| = S^2 \times S^1$ and $b_2^+(X) = 0$, contradicting the assumption that $\psi^2 > 0$.

Thus since M is very good we can proceed as in the proof of the previous implication. In particular, M is a quotient of a manifold $\overline{M}$ by a smooth action of a finite group G, the total space has a finite covering $\overline{X}$ which is a genuine S^1-bundle and these bundles fit into a pullback diagram as above. Since a degree one cohomology class on $\overline{M}$ is determined by its restriction to the complement of the branching locus, we deduce that $\overline{\phi} = \pi^*\phi$ and $\overline{p}_*(\overline{\pi}^*\psi)$ agree as cohomology classes. We then note that the construction of S^1-invariant forms in [5] and its extension to irrational classes ([8], Theorem 1.1) can be done G-equivariantly.

First choose a G-invariant representative γ of $e(\overline{X})$, which can be obtained as the curvature of a G-equivariant angular form. By ([8], Lemma 2.1) we may write $\gamma = \overline{\phi} \wedge \beta$. After averaging over G this equation still holds, so β can be assumed to be G-equivariant. Let η be a G-invariant angular 1-form so that $d\eta = \overline{p}^*\gamma$ and let $\overline{\Omega} \in H^2(\overline{M}, \mathbb{R})$ be the unique class such that the following holds in cohomology

$$\overline{\pi}^*\psi - \eta \wedge \overline{p}^*\,\overline{\phi} = \overline{p}^*\overline{\Omega}.$$

Such an $\overline{\Omega}$ exists in view of the Gysin sequence since the left hand lies in the kernel of $\overline{p}_*$ and since the left hand side is G-equivariant so is $\overline{\Omega}$. The fact that $\overline{\pi}^*\psi^2 > 0$ implies that $\overline{p}^*\,\overline{\phi} \wedge \overline{\Omega} > 0$. Thus by ([8], Lemma 2.2) there is a non-vanishing 2-form representing the class $\overline{\Omega}$ so that $\overline{\phi} \wedge \overline{\Omega} > 0$, again after averaging we may assume that $\overline{\Omega}$ is G-invariant. Thus the S^1-invariant form

$$\overline{\omega}_{inv} = \eta \wedge \overline{p}^*\,\overline{\phi} + \overline{p}^*\,\overline{\Omega}$$

represents $\overline{\pi}^*\psi$ and descends to an S^1-invariant form ω_{inv} on X which is cohomologous to ψ. □

Remark 1. A vital step in the proof of Theorem 1 was Chen's observation that the base orbifold of a symplectic manifold X admitting a fixed point free S^1-action is a quotient of a manifold by a finite group action. The main technical point in the proof of ([3], Proposition 1.8) is to rule out bad 2-orbifolds in the base. This is achieved by results relating the Seiberg-Witten invariants of the base orbifold to those of the underlying manifold.

We sketch a different proof which uses more standard Seiberg-Witten vanishing results. For background on the Seiberg-Witten invariants we refer to [10] and the references therein. First observe that a bad 2-suborbifold Σ in the quotient orbifold $M = X/S^1$ can intersect at most 2 singular curves L_1, L_2 each in at most one point. Taking a neighbourhood N of $|\Sigma| \cup L_1$ gives a topological splitting of the base $|M| = (S^2 \times S^1)\#M'$ so that preimage of the splitting sphere in $|M|$ induces a splitting $X = X_1 \cup_S X_2$, where S is either $S^2 \times S^1$ or S^3 depending on whether L_2 is empty or not. Moreover, as in the proof of ([3], Lemma 2.3) we must have $b_1(M') > 0$ by the assumption that $b_2^+(X) > 0$. If S is a 3-sphere, then $b_2^+(X_2) > 0$ and by taking the covering $\overline{X}$ of X induced by the natural surjection

$$\pi_1(X) \to \pi_1(S^2 \times S^1) \to \mathbb{Z}_n$$

we obtain a splitting of $\overline{X} = \overline{X}_1 \cup_{S^3} \overline{X}_2$, where $b_2^+(\overline{X}_1), b_2^+(\overline{X}_2) \geq 1$. It follows that the Seiberg-Witten invariants of $\overline{X}$ are trivial.

If $S = S^2 \times S^1$, then we take the covering $\overline{X}$ of X induced by a surjection

$$\pi_1(X) \to \pi_1(S^2 \times S^1) * \pi_1(|M'|) \to \mathbb{Z}_n \times \mathbb{Z}_n.$$

The embedded 2-sphere $S^2 \times \{pt\}$ in S then becomes essential in the covering and $b_1(\overline{X})$, and hence $b_2^+(\overline{X})$, may be assumed to be arbitrarily large. Furthermore, the sphere $S^2 \times \{pt\}$ has trivial self-intersection and consequently the Seiberg-Witten invariants of $\overline{X}$ are trivial. Thus in both cases we obtain a contradiction to the non-vanishing results of Taubes for the Seiberg-Witten invariants of a symplectic 4-manifold.

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Mathematisches Institut, Universität Augsburg, Universitätsstr. 14, 86159 Augsburg, Germany

Current address: Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany

E-mail address: jonathan.bowden@math.uni-augsburg.de