CONCAVITY AND RIGIDITY IN NON-NEGATIVE CURVATURE

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Dedicated to D.V. Alekseevsky on his 70th birthday

Abstract. We show that for a manifold with non-negative curvature one obtains a collection of concave functions, special cases of which are the concavity of the length of a Jacobi field in dimension 2, and the concavity of the volume in general. We use these functions to show that there are many cohomogeneity one manifolds which do not carry an analytic invariant metric with non-negative curvature. This implies in particular, that one of the candidates in [GWZ] does not carry an invariant metric with positive curvature.

There are few known examples of manifolds with positive sectional curvature in Riemannian geometry. Until recently, they were all homogeneous spaces [Be, Wa, AW] and biquotients [E1, E2, Ba], i.e., quotients of compact Lie groups G by a free isometric "two sided" action of a subgroup $H \subset G \times G$. See [Zi1] for a survey of the known examples. Recently a new example of a positively curved 7-manifold, homeomorphic but not diffeomorphic to $T_1\mathbb{S}^4$, was constructed in [GVZ], see also [De] for a different approach. A new method has also been proposed in [PW] to construct a metric of positive curvature on the Gromoll-Meyer exotic 7-sphere. The new example in [GVZ] is part of a larger family of "candidates" for positive curvature discovered in [GWZ]. One of the applications of this paper is to exclude one of these candidates.

The obstruction that we use to do this turns out to be of a general nature that does not require the presence of a group action. It comes from a new concavity property of Jacobi fields in positive curvature. The method also gives rise to certain rigidity properties in nonnegative curvature.

Let $c(t)$ be a geodesic in M^{n+1} and $J(t)$ a Jacobi field along c. For a surface it is well known that positive curvature is equivalent to requiring that the length of all Jacobi fields is strictly concave. In higher dimensions, the length $|J|$ satisfies the differential equation

$$\frac{|J|''}{|J|} = -\sec_M(\dot c, J) + \frac{|J'|^2}{|J|^2}\sin^2(\sphericalangle(J', J)).$$

Thus in negative curvature $|J|$ is a strictly convex function. But in positive curvature $|J|$ does not have any distinctive properties. For example, the Hopf action on a round sphere induces a Killing vector field of constant length.

For positive curvature we suggest the concept of a "virtual" Jacobi field. For this it is best to study Jacobi fields via Jacobi tensors. Let A_t be a solution of the differential equation

$$A'' + RA = 0$$

where $E_t = \dot c(t)^\perp \subset T_{c(t)}M$ and, after a choice of a base point t_0, $A_t \colon E_{t_0} \to E_t$ and $R = R(\cdot, \dot c)\dot c \colon E_t \to E_t$. A is uniquely determined by A_{t_0} and A'_{t_0}. Thus for any $v \in E_{t_0}$, $J(t) = A_t v$ is a Jacobi field along c. We denote by A^* the adjoint of A and call a point $c(t^*)$ regular if A_{t^*} is invertible. The Jacobi tensor A is called a Lagrange tensor if A is non-degenerate (i.e. Av is not the 0-Jacobi field for all v) and $S := A'A^{-1}$ is symmetric at regular points. Equivalently, S is the shape operator of a family of parallel hypersurfaces orthogonal to c.

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Theorem A. *Let A be a Lagrange tensor along the geodesic c and $v \in E_{t_0}$ non-zero. Define $Z_t = (A_t^*)^{-1} v$ and let $g = g_v(t) = \dfrac{||v||^2}{||Z_t||}$. Then*

(a) *$g_v(t) \le ||A_t v||$ and at regular points*
$$\frac{g''}{g} = -\sec_M(\dot{c}, Z) - 3\frac{|SZ|^2}{|Z|^2}\sin^2(\sphericalangle(SZ, Z)).$$
(b) *$g_v(t)$ is continuous for all t. Furthermore, it is smooth (and positive) at $t = t^*$ iff $v \perp \ker A_{t^*}$.*
(c) *If $\sec_M \ge 0$ (resp. $\sec_M > 0$), then g_v is concave (resp. strictly concave) on any interval where g_v is positive. If g_v is constant, then the virtual Jacobi field Z is a parallel Jacobi field, and if $A_{t_0} = \mathrm{Id}$, then $Z_t = A_t v$.*

Notice that for a surface $g_v = ||A_t v||$ is simply the length of the Jacobi field.

As an immediate consequence one has the following result by B.Wilking [Wi] which was crucial in proving the smoothness of the Sharafudinov projection in the soul theorem: If M has non-negative sectional curvature and A is a Lagrange tensor defined along c for *all* t, normalized so that $A_{t_0} = \mathrm{Id}$, then one has an orthogonal splitting
$$E_{t_0} = \mathrm{span}\{v \in E_{t_0} \mid A_t v = 0 \text{ for some } t \in \mathbb{R}\} \oplus \{v \in E_{t_0} \mid A_t v \text{ is parallel for all } t \in \mathbb{R}\}.$$

There is another well known concave function in positive curvature given in terms of the volume along the geodesic: if A is Lagrange, then $(\det A_t)^{1/n}$ is concave if $\mathrm{Ric} \ge 0$. One of the advantages of the class of concave functions in Theorem A is that by part (b) and (c), some of them are well defined and concave at singular points of A, whereas $\det A$ vanishes at such points. This property of g_v is crucial in our applications.

There exists a sequence of concave functions interpolating between g_v and the volume. For each p-dimensional subspace $W \subset E_{t_0}$ set
$$g_W(t) = (\det M_t)^{-1/2p} \quad \text{where } \langle M_t e_i, e_j\rangle = \langle (A_t^*)^{-1} e_i, (A_t^*)^{-1} e_j\rangle = \langle\, (A^* A)^{-1} e_i, e_j\rangle$$
and $e_1, \ldots, e_p$ is an orthonormal basis of W. If W is one dimensional, $g_W = g_v$ with v a unit vector in W, and if $W = E_{t_0}$ then $g_W = (\det A_t)^{1/n}$.

Recall that a manifold is said to have p-positive sectional curvature if the sum of the p smallest eigenvalues of $R(\cdot, v)v$ is positive for all v. Thus $p = 1$ is positive sectional curvature and $p = n$ is positive Ricci curvature.

Theorem B. *Let A be a Lagrange tensor along the geodesic c and $W \subset E_{t_0}$ a p-dimensional subspace.*

(a) *If the p-sectional curvature is non-negative (resp. positive), then g_W is concave (resp. strictly concave) on any interval where g_W is positive.*
(b) *g_W is smooth (and positive) at $t = t^*$ iff $W \perp \ker A_{t^*}$.*
(c) *If the p-sectional curvature is non-negative and g_W is constant, then $(A_t^*)^{-1} v$ is a parallel Jacobi field for all $v \in W$.*

The example of positive curvature in [GVZ] arose from a systematic study of *cohomogeneity one* manifolds, i.e., manifolds with an isometric action whose orbit space is one dimensional, or equivalently the principal orbits have codimension one. A classification of positively curved cohomogeneity one manifolds was carried out in even dimensions in [V1, V2] and in odd dimensions an exhaustive description was given in [GWZ] of all simply connected cohomogeneity one manifolds that can possibly support an invariant metric with positive curvature. In addition to some of the known examples of positive curvature which admit isometric cohomogeneity one actions, two infinite families, $P_k^7, Q_k^7, k \ge 1$, and one exceptional manifold R^7, all of dimension seven and admitting a cohomogeneity one action by $\mathrm{SO}(4)$, appeared as the only possible new candidates, see Section 4 (as well as [Zi2]) for a more detailed description. Here P_1^7 is the 7-sphere and Q_1^7 is the normal homogeneous positively curved Aloff-Wallach space. The manifold P_2^7 is the new example of positive curvature in [GVZ].

These candidates belong to two much larger classes of cohomogeneity one manifolds depending on 4 integers, described in terms of the isotropy groups, see Section 4. One is denoted by $P_{(p_-,q_-),(p_+,q_+)}$, a family of cohomogeneity one manifolds with $\pi_1 = \pi_2 = 0$, and a second by $Q_{(p_-,q_-),(p_+,q_+)}$, where $\pi_1 = 0$, $\pi_2 = \mathbb{Z}$. They all admit a cohomogeneity one action by $G = \mathrm{SO}(4)$. In terms of these, the candidates for positive curvature are given by $P_k = P_{(1,1),(1+2k,1-2k)}$, $Q_k = Q_{(1,1),(k,k+1)}$, with $k \geq 1$, and the exceptional manifold $R^7 = Q_{(3,1),(1,2)}$.

Theorem C. *Let M be one of the 7-manifolds $Q_{(p_-,q_-),(p_+,q_+)}$ with its cohomogeneity one action by $G = \mathrm{SO}(4)$ and assume that M is not of type $Q_k, k \geq 0$. Then there exists no analytic metric with non-negative sectional curvature invariant under G, although there exists a smooth one.*

The existence of a smooth metric with non-negative curvature follows from a more general result on cohomogeneity one manifolds in [GZ1]. In particular we obtain:

Corollary. *The exceptional cohomogeneity one manifold R^7 does not admit an invariant metric with positive sectional curvature.*

The method also applies to the family $P_{(p_-,q_-),(p_+,q_+)}$. Here we will show that if the manifold is not one of the candidates P_k or of type $P_{(1,\,q),(p\,,1)}$, then there exists a G-invariant metric with non-negative sectional curvature, but no G-invariant analytic metric with non-negative curvature. On the other hand, the exceptional family $P_{(1,\,q),(p,1)}$ contains several G-invariant analytic metrics with non-negative curvature since $P_{(1,1),(-3,1)}$ is $\mathbb{S}^7$, $P_{(1,-3),(-3,1)}$ is the positively curved Berger space and $P_{(1,1),(1,1)} = \mathbb{S}^3 \times \mathbb{S}^4$. We do not know if any of the other manifolds $P_{(1,\,q),(p\,,1)}$ carry analytic metrics with non-negative curvature.

The proof of Theorem C is obtained as follows. For a cohomogeneity one G-manifold one chooses a geodesic c orthogonal to all orbits. Then the action of G induces Killing vector fields on M, which along c are Jacobi fields. They give rise to a Lagrange tensor A, to which we can apply Theorem A. One then shows that there exists a Jacobi field $A_t v$, and an interval $[a, b]$, such that the corresponding function g_v has derivatives equal to 0 at the endpoints, and is positive on $[a, b]$. Thus, if the curvature is non-negative, Theorem A implies that g_v is constant on $[a, b]$. On the other hand, one shows that g_v must vanish at other singular points along c due to smoothness conditions imposed by the group action. This implies that there exists a Jacobi field which is parallel on $[a, b]$, but is not parallel at all points along c.

We finally discuss an application of Theorem B. There is a third family of 7-dimensional manifolds $N_{p,q}$ on which $G = \mathrm{S}^3 \times \mathrm{S}^3$ acts by cohomogeneity one, see Section 4. We will show:

Theorem D. *The cohomogeneity one manifolds $N_{p,q}$ have no invariant metric with 2-positive sectional curvature, and $N_{1,1}$ has no invariant metric with 3-positive sectional curvature.*

In contrast, it was shown in [GZ2] that every simply connected cohomogeneity one manifold carries an invariant metric with positive Ricci curvature.

The differential equation and its applications also hold if we consider Jacobi fields only in a subbundle invariant under parallel translation. This arises frequently in the presence of an isometric group action. For example, a group action is called polar if there exists a so called section Σ, which is an immersed submanifold orthogonal to all orbits. Such a section must be totally geodesic, and hence the group action gives rise to a self adjoint family of Jacobi fields in the parallel subbundle orthogonal to Σ.

In Section 1 we recall properties of the Riccati equation and prove Theorem A. In Section 2 we prove Theorem B and in Section 3 we discuss rigidity properties. Finally, in Section 4, we prove Theorems C and D.

1. Concavity

In this section we present a new concavity result about Jacobi fields, and first recall some standard notation, see e.g. [E3, EH, EO].

Let c be a geodesic in a Riemannian manifold M^{n+1} defined on an interval $t_1 \leq t \leq t_2$ and let $E_t = \dot{c}^{\perp}$ be the orthogonal complement of $\dot{c}(t) \subset T_{c(t)}M$. For a vector field X along c, orthogonal to $\dot{c}$, we denote by X' the covariant derivative $\nabla_{\dot{c}}X$.

Let V be an n-dimensional vector space of Jacobi fields along c orthogonal to $\dot{c}$. Along the geodesic we have that $\langle X', Y\rangle - \langle X, Y'\rangle$ is constant for any $X, Y \in V$. If this constant is 0, V is called *self adjoint*, i.e.

$$\langle X', Y\rangle = \langle X, Y'\rangle, \quad \text{for all } X, Y \in V. \tag{1.1}$$

We call t regular if $X(t),\ X \in V$ span E and singular otherwise. One easily sees that

$$E_t = \{X(t) \mid X \in V\} \oplus \{X'(t) \mid X \in V \text{ with } X(t) = 0\} =: V_1(t) \oplus V_2(t) \tag{1.2}$$

for all $t \in [t_1, t_2]$. Notice that self adjointness implies that the decomposition is orthogonal. In particular, the singular points are isolated.

We fix a base point $t_0 \in [t_1, t_2]$. We can then describe the set of Jacobi fields V by a (smooth) family of linear maps $A_t \colon E_{t_0} \to E_t$. It is standard to do this by assuming the base point is regular and define $A_t v = X(t)$ for $X \in V$ with $X(t_0) = v$. In this case $A_{t_0} = \mathrm{Id}$. But in the applications it will be useful to allow the base point t_0 to be singular as well.

Definition 1.3. Let V be selfadjoint family of Jacobi fields and fix $t_0 \in [t_1, t_2]$. Decompose $v \in E_{t_0}$ as $v = v_1 + v_2$, $v_i \in V_i(t_0)$, and define:

$$A_t \colon E_{t_0} \to E_t \quad : \quad A_t v = X_1(t) + X_2(t)$$
$$\text{where } X_1, X_2 \in V \text{ with } X_1(t_0) = v_1,\ X_1'(t_0) \in V_1, \text{ and } X_2(t_0) = 0,\ X_2'(t_0) = v_2.$$

For this we observe:

Lemma 1.4. *Let V be selfadjoint family of Jacobi fields and choose a base point t_0.*

(a) *Given $v \in E_{t_0}$, the Jacobi fields X_1 and X_2 in Definition 1.3 are well defined and unique.*
(b) *Given $X \in V$, there exists a unique $v \in E_{t_0}$ such that $X = A_t v$.*
(c) *At the base point t_0 we have, with respect to the orthogonal decomposition $V_1 \oplus V_2$:*

$$A_{t_0} = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix} \qquad A'_{t_0} = \begin{pmatrix} B & 0 \\ 0 & \mathrm{Id} \end{pmatrix}$$

with B self adjoint.

Proof. (a) Existence of X_2 is clear. As for X_1, first choose $Y_1 \in V$ with $Y_1(t_0) = v_1$ and set $Y_1'(t_0) = w_1 + w_2$ with $w_i \in V_i(t_0)$. By (1.2), there exists a $Y_2 \in V$ such that $Y_2(t_0) = 0$ and $Y_2'(t_0) = w_2$. Then set $X_1 = Y_1 - Y_2$. Uniqueness clearly follows from (1.2) as well.

(b) Given $X \in V$, set $v_1 := X(t_0)$ and $X'(t_0) = w_1 + w_2$ with $w_i \in V_i(t_0)$. There exists a unique $X_2 \in V$ with $X_2(t_0) = 0$ and $X_2'(t_0) = w_2$. Setting $X_1 := X - X_2$ we see that $X = A_t v$ with $v = v_1 + w_2$.

Part (c) is clear from the definition and self adjointness. □

Thus V is indeed uniquely described in terms of A_t. Notice though that $A_{t_0}v = v$ for all $v \in E_{t_0}$ if only if t_0 is regular.

A point t is regular for V if and only if A_t is invertible. At regular points t one defines the Riccati operator:

$$S_t\colon E_t \to E_t \;\text{ where }\; S_t v = X'(t) \text{ for } X \in V \text{ with } X(t) = v, \text{ i.e. } A_t' = S_t A_t\,. \tag{1.5}$$

Thus the family of Jacobi fields V is self adjoint iff S_t is self adjoint. A_t satisfies the Jacobi equation and S_t the Riccati equation:

$$A'' + RA = 0 \;\text{ if and only if }\; S' + S^2 + R = 0 \;\text{ and }\; A' = SA \tag{1.6}$$

where $R = R_t\colon E_t \to E_t$ is the self adjoint curvature endomorphism $R(\,\cdot\,, \dot c)\dot c$.

Conversely, let $A_t\colon E_{t_0} \to E_t$ be a solution of (1.6). We say that A_t is *non-degenerate*, if $\ker A_{t_0} \cap \ker A_{t_0}' = 0$. Furthermore, A_t is called a *Lagrange tensor* if A_t is non-degenerate and S_t is self adjoint. A Lagrange tensor defines an n-dimensional family of Jacobi fields $V = \{A_t v \mid v \in E_{t_0}\}$ which is self adjoint.

We point out that if A_t is Lagrange, then $A_t \circ F$, for any fixed linear isomorphism $F\colon E_{t_0} \to E_{t_0}$, is also a Lagrange tensor, in fact with the same tensor S. Furthermore, if S_t is self adjoint at one point, it is self adjoint at all points. Notice also that if two Lagrange tensors A_t and $\tilde A_t$, with base points t_0 and $\tilde t_0$, give rise to the same self adjoint family V, they differ from each other by a linear isomorphism $F : E_{t_0} \to E_{\tilde t_0}$. Indeed, if $v \in E_{t_0}$ and hence $A_t v \in V$, then Lemma 1.4 implies that there exists a unique $w \in E_{\tilde t_0}$ with $A_t v = \tilde A_t w$. Then $F(v) = w$ clearly defines an isomorphism with $\tilde A_t \circ F = A_t$. This applies in particular if we choose a different base point when defining A_t in terms of V. Thus Lagrange tensors, modulo composing with F, are in one to one correspondence with n-dimensional vector spaces of Jacobi fields which are self adjoint.

From now on let A be a Lagrange tensor. Thus for any $v \in E_{t_0}$, $A_t v$ is a Jacobi field, and t is regular if and only if A_t is invertible. Furthermore,

$$\langle A_t' v, A_t w\rangle = \langle A_t v, A_t' w\rangle \text{ for all } t \text{ and } v, w \in E_{t_0}. \tag{1.7}$$

Notice that here we do not assume that A_{t_0} has any special form as is the case when A is associated to V. When clear from context we simply write $A = A_t,\ S = S_t$.

Let A_t^* be defined by $\langle A_t^* v, w\rangle = \langle v, A_t w\rangle$ for all $v \in E_t,\ w \in E_{t_0}$ and for simplicity set $(A_t^*)^{-1} = A_t^{-*}\colon E_{t_0} \to E_t$.

The main purpose of this section is to study the functions

$$g_v(t) = \frac{||v||^2}{||A_t^{-*}v||}\,, \quad v \in E_{t_0}.$$

The scaling guarantees that $g_{\lambda v} = \lambda g_v$. We first discuss smoothness properties.

Proposition 1.8. *Let A_t be a Lagrange tensor and fix a vector $v \in E_{t_0}$. Then*

(a) *The vector field $A_t^{-*}v$, and hence the function g_v, is smooth outside of the singular set. If t^* is a singular point, then $A_t^{-*}v$ has a smooth extension at $t = t^*$ if and only if v is orthogonal to* $\ker A_{t^*}$.

(b) *g_v is continuous for all t and $g_v(t) > 0$ if and only if v is orthogonal to* $\ker A_t$.

Proof. The first claim in part (a) is clear. For simplicity assume that the singular point is $t^* = 0$. Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of E_{t_0} such that $\{e_1, \ldots, e_k\}$ is a basis of $\ker A_0$.

Choose ϵ such that A_t is non-singular for $t \in (0, \epsilon]$. Then A_t has a block form (with respect to a parallel basis)

$$A_t = \begin{pmatrix} tX & Y + tY_2 \\ tZ & W + tW_2 \end{pmatrix} + o(t^2) \text{ and hence } A_t^* = \begin{pmatrix} tX^T & tZ^T \\ Y^T + tY_2^T & W^T + tW_2^T \end{pmatrix} + o(t^2).$$

We first claim that the matrix

$$N = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

is non-singular. This is equivalent to saying that $A'e_1, \ldots, A'e_k, Ae_{k+1}, \ldots, Ae_n$ are linearly independent. If not, there exists a $v \in \ker A_0$ and $w \in (\ker A_0)^\perp$ such that $A'v = Aw$. Using self adjointness, $\langle Aw, Aw \rangle = \langle A'v, Aw \rangle = \langle Av, A'w \rangle = 0$. Thus $Aw = 0$ and hence $A'v = 0$, which contradicts non-degeneracy. In particular, $\det A_t = at^k + o(t^{k+1})$ with a nonzero. It follows that the matrix of minors of A_t^* has the form

$$M = \begin{pmatrix} t^{k-1}\overline{X} & t^{k-1}\overline{Y} \\ t^k\overline{Z} & t^k\overline{W} \end{pmatrix} + o(t^k) \text{ where } \overline{N} = \begin{pmatrix} \overline{X} & \overline{Y} \\ \overline{Z} & \overline{W} \end{pmatrix}$$

is the matrix of minors of N^T, and hence non-singular. Thus

$$A_t^{-*} = \frac{1}{\det A^*} M^T = \frac{1}{\det A} \begin{pmatrix} t^{k-1}\overline{X}^T & t^k\overline{Z}^T \\ t^{k-1}\overline{Y}^T & t^k\overline{W}^T \end{pmatrix} + o(1).$$

Hence $A_t^{-*}v$ is smooth, and non-zero, if v is orthogonal to $\ker A_0$. If $v \in E_{t_0}$ is not orthogonal to $\ker A_0$, we have $\lim_{t\to 0} ||A_t^{-*}v|| = \infty$ since $\overline{R}$ is non-singular. Hence $g_v(0) = 0$ which finishes (b) as well. $\square$

Remark. The function g_v provides a lower bound for the norm of the corresponding Jacobi field, i.e.

$$g_v \leq ||A_t v||$$

since

$$\langle v, v \rangle = \langle A^{-1}Av, v \rangle = \langle Av, A^{-*}v \rangle \leq ||Av|| \cdot ||A^{-*}v||.$$

Our main tool is the following differential equation for $g_v(t)$:

PROPOSITION 1.9. *Let A be a Lagrange tensor and $S = A'A^{-1}$. Then at regular points we have*

$$g_v'' + rg_v = 0 \tag{1.10}$$

where

$$r = \langle Rz, z \rangle + 3(\ ||Sz||^2 - \langle Sz, z \rangle^2\ ) \quad \textit{and} \quad z = \frac{A_t^{-*}v}{||A_t^{-*}v||}\ .$$

Proof. To simplify the notation we assume $||v|| = 1$ (which does not effect the differential equation) and set

$$f_v = \frac{1}{g_v^2} = ||A_t^{-*}v||^2.$$

First observe that

$$(A_t^{-*})' = -A_t^{-*}(A_t^*)'A_t^{-*} = -(A_t'A_t^{-1})^*A_t^{-*} = -S^*A_t^{-*} = -SA_t^{-*}$$

and hence

$$f_v' = -2\langle SA_t^{-*}v, A_t^{-*}v \rangle.$$

Furthermore

$$\begin{aligned} f_v'' &= -2\langle S'A_t^{-*}v, A_t^{-*}v\rangle + 4\langle S^2A_t^{-*}v, A_t^{-*}v\rangle = \\ &= -2\langle(-S^2-R)A_t^{-*}v, A_t^{-*}v\rangle + 4\langle SA_t^{-*}v, SA_t^{-*}v\rangle = \\ &= 2\langle RA_t^{-*}v, A_t^{-*}v\rangle + 6||SA_t^{-*}v||^2. \end{aligned}$$

and thus

$$\begin{aligned} g_v'' &= \left(\frac{3}{4}f_v'^2 - \frac{1}{2}f_v''f_v\right) f^{-5/2} \\ &= \left(3\langle SA_t^{-*}v, A_t^{-*}v\rangle^2 - \langle RA_t^{-*}v, A_t^{-*}v\rangle ||A_t^{-*}v||^2 - 3||SA_t^{-*}v||^2||A_t^{-*}v||^2\right) f^{-5/2} \\ &= -rg_v. \end{aligned}$$

□

Remark. Notice that A_t^{-*} itself satisfies the differential equation

$$(A_t^{-*})'' = (2S^2+R)A_t^{-*}.$$

Proposition 1.9 implies certain concavity properties in non-negative curvature.

Corollary 1.11. *Let A be a Lagrange tensor. If $R \geq 0$ (resp. $R > 0$), then for any $v \in E_{t_0}$, g_v is a concave (resp. strictly concave) function on any interval where $g_v > 0$.*

Remark. The concavity of g_v implies the convexity of $f_v = ||A_t^{-*}v||^2$ (but not conversely). Since $f_v = \langle (A^*A)^{-1}v, v\rangle$, this can also be interpreted as saying the operator $(A^*A)^{-1}$ is convex. The zeros of g_v correspond to vertical asymptotes of f_v.

Example 1. If $\dim M = 2$, then $g_v(t) = ||A_tv||$ and hence the concavity of g_v is indeed a generalization of the concavity of Jacobi fields in dimension two.

Example 2. Let $M = S^3 \subset \mathbb{C}^2$ with the standard metric. The restriction of the action field of the Hopf action of S^1 to a geodesic is a Jacobi field J_1 with unit length. Consider the geodesic $c(t) = (\cos(t), \sin(t))$, then $J_1 = i\,c(t) = (i\cos(t), i\sin(t))$. Let $J_2 = (0, i\sin(t))$ then *span*$\{J_1, J_2\}$ is a self-adjoint family of Jacobi fields V along $c(t)$. The singular points along $c(t)$ are $t = n\frac{\pi}{2}$, $n \in \mathbb{Z}$, since $J_2 = 0$ for $t = n\pi$ and $J_1 - J_2 = 0$ for $t = (2n+1)\frac{\pi}{2}$. Now $t_0 = \frac{\pi}{4}$ is a regular point and, if $v = J_1(t_0)$, one easily sees that $g_v(t) = |\sin(2t)| \leq ||J_1(t)|| = 1$. Notice also that g_w with $w = J_2(t_0)$ is smooth across the singularity at $\frac{\pi}{2}$.

Example 3. If $\sec_M \geq \delta$, then $g_v'' + rg_v = 0$ with $r \geq \delta$. Thus Sturm comparison implies that $g_v \leq f_\delta$ with $f_\delta'' + \delta f = 0$ and $f_\delta(t_0) = g_v(t_0) = |J|(t_0)$, $f_\delta'(t_0) = g_v'(t_0) = |J|'(t_0)$ (see Proposition 1.12 below). This comparison holds up to the first point where g_v vanishes.

In contrast, the usual Rauch comparison theorem implies that $|J| \leq f_\delta$, but only holds up to the first singularity of A_t, i.e. there could be other Jacobi fields A_tw which vanish before $|J|$.

For g_v one obtains an upper bound on $[t_0, t_1]$ as long v is orthogonal to the kernels of A_t, $t \in [t_0, t_1]$, or equivalently $g_v > 0$. Of course a zero of g_v also corresponds to a singularity of A_t. For example, if $\sec_M \geq 1$, this implies that the index of the geodesic is at least $n-1$ after length π.

We remark that for an upper curvature bound $\sec_M \leq \mu$, one can analogously use the differential equation for $|J|$ in the Introduction to get the usual lower bound on $|J|$, without having to prove a Rauch comparison theorem.

It is useful to compare the higher derivatives of g_v with those of $||A_tv||$.

Proposition 1.12. *Let A be the Lagrange tensor defined by a self adjoint family of Jacobi fields V as in* (1.3) *with base point t_0 and $v \perp \ker A_{t_0}$. Then for $t = t_0$ we have:*

$$g_v = ||Av|| \quad , \quad g_v' = ||Av||' \quad , \quad g_v'' = ||Av||'' - 4\left(||A'v||^2 - \langle A'v, v\rangle^2\right) \leq ||Av||''.$$

Proof. The assumption $v \perp \ker A_{t_0}$ implies that g_v is smooth and non-zero at t_0. But to determine its value and derivatives at $t = t_0$ we need to carefully take the limit as $t \to t_0$.

Since the equations are scale invariant, we can assume $||v|| = 1$. Recall that at the base point we have $A_{t_0|V_1} = \mathrm{Id}$, $A_{t_0|V_2} = 0$ and $V_1 \perp V_2$. Thus $v \in V_1$ and hence $A_{t_0}v = v$, as well as $A^*_{t_0}v = v$.

To compute the derivatives of g, recall that Proposition 1.8 also implies that $A_t^{-*}v$ is smooth at $t = t_0$. We begin by showing that:

$$\lim_{t\to t_0} A_t^{-*}v = v, \quad \lim_{t\to t_0}(A_t^{-*}v)' = -A'_{t_0}v\ . \tag{1.13}$$

For the first claim, observe that Lemma 1.4 implies that $A'_{t_0}w = w$ if $w \in V_2$. Thus, in the language of the proof of Proposition 1.8, it follows that $N = \mathrm{Id}$ and hence $\overline{N} = \mathrm{Id}$ as well. Furthermore, $\det A_t = t^k + o(t^{k+1})$ and thus the formula for the inverse implies that $\lim_{t\to t_0} A_t^{-*}v = v$.

For the second claim we first observe that $\lim_{t\to t_0}(A_t^{-*}v)' \in V_1$ since for $w \in V_2$ we have that A'_tw and w have the same limit and

$$\langle w, \lim_{t\to t_0} A_t^{-*}v\rangle = \lim_{t\to t_0}\langle A'_tw, A_t^{-*}v\rangle = -\lim_{t\to t_0}\langle A_tw, (A_t^{-*}v)'\rangle = -\langle A_{t_0}w, \lim_{t\to t_0}(A_t^{-*}v)'\rangle = 0$$

where the second equality follows by differentiating $\langle w, v\rangle = \langle A_t^{-1}A_tw, v\rangle = \langle A_tw, A_t^{-*}v\rangle$. Now, if $w \in V_1$ we have

$$\lim_{t\to t_0}\langle (A_t^{-*}v)', A_tw\rangle = -\lim_{t\to t_0}\langle A_t^{-*}v, A'_tw\rangle = -\lim_{t\to t_0}\langle A_tv, A'_tw\rangle = -\lim_{t\to t_0}\langle A'_tv, A_tw\rangle$$

where we have used the fact that $A_t^{-*}v$ and A_tv have the same limit. This implies the second part of (1.13) since $A_{t_0}w = w$.

We now apply (1.13) to g. First, note that $g_v(t_0) = 1 = ||A_{t_0}v||$. For the derivative, using $f_v(t) = ||A_t^{-*}v||^2$, we see that

$$g_v'(t_0) = -\frac{1}{2}\lim_{t\to t_0}\frac{f_v'(t)}{f_v(t)^{\frac{3}{2}}} = -\frac{\lim_{t\to t_0}\langle (A_t^{-*}v)', A_t^{-*}v\rangle}{\lim_{t\to t_0}||A_t^{-*}v||^3} = -\lim_{t\to t_0}\langle (A_t^{-*}v)', A_t^{-*}v\rangle$$

and thus

$$g_v'(t_0) = \langle A'v, v\rangle = \langle A'v, Av\rangle = ||Av||'_{t_0}.$$

For the second derivative, we use the differential equation from Proposition 1.9 for g_v:

$$g_v''(t_0) = -\lim_{t\to t_0} r g_v = -\lim_{t\to t_0}\left\{3(||Sz||^2 - \langle Sz, z\rangle^2) - \langle Rz, z\rangle\right\}$$

where $z = A_t^{-*}v/||A_t^{-*}v||$. From the proof of Proposition 1.9, recall that at regular points we have $SA_t^{-*}v = -(A_t^{-*}v)'$ and hence (1.13) implies that $\lim_{t\to t_0} Sz = A'_{t_0}v$. Thus

$$g_v''(t_0) = -3(||A'v||^2 - \langle A'v, v\rangle^2) - \langle Rv, v\rangle$$

and since

$$||Av||'' = \frac{-\langle RAv, Av\rangle||Av||^2 + ||A'v||^2||Av||^2 - \langle A'v, Av\rangle^2}{||Av||^3}$$

we have

$$g_v''(t_0) = ||Av||'' - 4(||A'v||^2 - \langle A'v, v\rangle^2).$$

□

2. Concavity of Volumes

We construct a collection of concave functions which contain g_v as a special case. For this we fix a p-dimensional subspace $W \subset E_{t_0}$ and choose an orthonormal basis $e_1, \ldots, e_p$ of W. Define

$$(2.1) \qquad M\colon W \to W \quad \text{with} \quad \langle M_t e_i, e_j\rangle = \langle A_t^{-*} e_i, A_t^{-*} e_j\rangle = \langle (A^*A)^{-1} e_i, e_j\rangle, \quad 1 \le i, j \le p\,.$$

Thus M represents the upper $p \times p$ block of the matrix $(A^*A)^{-1}$. Furthermore, we decompose $S = A'A^{-1}$, where we have set $W_t := A_t^{-*}W$, as

$$S_1\colon W_t \to W_t, \quad S_2\colon W_t \to W_t^{\perp} \quad \text{with} \quad Sw = S_1 w + S_2 w \ \text{ for all } \ w \in W_t.$$

Notice that S_1 is again a symmetric endomorphism. Notice also that since $(A^*A)^{-1}$ is positive definite at regular points, so is the upper $p \times p$ block by Sylvester's theorem and thus $\det M_t > 0$.

Proposition 2.2. *Let A be a Lagrange tensor and $W \subset E_{t_0}$ a p-dimensional subspace. Then at regular points the function*

$$g_W(t) = (\det M_t)^{-1/2p}$$

satisfies the differential equation

$$p\,\frac{g''}{g} = \frac{1}{p}(\operatorname{tr} S_1)^2 - \operatorname{tr}(S_1^2) - 3\operatorname{tr}(S_2^T S_2) - \sum_{i=1}^{i=p} \langle\, R w_i, w_i\rangle$$

where w_i is an orthonormal basis of W_t.

Proof. As in the proof of Proposition 1.9, one easily sees that

$$(2.3) \qquad \langle M' e_i, e_j\rangle = -2\,\langle S A^{-*} e_i, A^{-*} e_j\rangle \quad , \quad \langle M'' e_i, e_j\rangle = \langle\,(6S^2 + 2R) A^{-*} e_i, A^{-*} e_j\rangle.$$

For convenience, set $f = \det M_t$. Differentiating we obtain:

$$(2.4) \qquad f' = (\det M)' = \det M \operatorname{tr}(M^{-1}M'), \quad \text{or} \quad \frac{f'}{f} = \operatorname{tr}(M^{-1}M')$$

and hence

$$\begin{aligned} \frac{f''}{f} &= \big[\operatorname{tr}(M^{-1}\,M')\big]^2 + \operatorname{tr}((M^{-1})'\,M') + \operatorname{tr}(M^{-1}\,M'') \\ &= \big[\operatorname{tr}(M^{-1}\,M')\big]^2 + \operatorname{tr}(-M^{-1}\,M'\,M^{-1}\,M') + \operatorname{tr}(M^{-1}\,M'') \\ &= \big[\operatorname{tr}(M^{-1}\,M')\big]^2 - \operatorname{tr}(\big[M^{-1}\,M'\big]^2) + \operatorname{tr}(M^{-1}\,M''). \end{aligned}$$

We now examine each term separately. For this, fix a regular point t^* and choose an orthonormal basis $e_1, \ldots, e_p$ of W which diagonalizes the symmetric matrix M_{t^*}, i.e. $\langle A_{t^*}^{-*} e_i, A_{t^*}^{-*} e_j\rangle = ||A_{t^*}^{-*} e_i||^2\,\delta_{i,j}$. Thus $Z_i := \frac{A_{t^*}^{-*} e_i}{||A_{t^*}^{-*} e_i||}$ is an orthonormal basis of W_{t^*}.

Dropping the index t^* from now on, the entries of $M^{-1}M'$ are $\frac{-2}{||A^{-*}e_i||^2}\langle SA^{-*}e_i, A^{-*}e_j\rangle$ and thus

$$\operatorname{tr}(M^{-1}\,M') = -2\sum_{i=1}^{i=p} \frac{1}{||A^{-*}e_i||^2}\langle SA^{-*}e_i, A^{-*}e_i\rangle = -2\sum_{i=1}^{i=p}\langle SZ_i, Z_i\rangle = -2\operatorname{tr} S_1.$$

For a general matrix $B = (b_{ij})$ we have $\operatorname{tr} B^2 = \sum_{i,j} b_{ij} b_{ji}$ and hence

$$\begin{aligned} \operatorname{tr}(\big[M^{-1}\,M'\big]^2) &= 4\sum_{i,j} \frac{\langle SA^{-*}e_i, A^{-*}e_j\rangle\langle SA^{-*}e_j, A^{-*}e_i\rangle}{||A^{-*}e_i||^2||A^{-*}e_j||^2} \\ &= 4\sum_{i,j}\langle SZ_i, Z_j\rangle^2 = 4\sum_{i,j}\langle S_1 Z_i, Z_j\rangle^2 = 4\operatorname{tr}(S_1^2). \end{aligned}$$

Finally

$$\begin{aligned}
\operatorname{tr}(M^{-1}M'') &= \sum_i \frac{1}{||A^{-*}e_i||^2}\langle\,(6S^2+2R)A^{-*}e_i, A^{-*}e_i\rangle \\
&= 6\sum_i \langle\, S^2 Z_i, Z_i\rangle + 2\sum_i\langle\, RZ_i, Z_i\rangle = 6\sum_i\langle\, SZ_i, SZ_i\rangle + 2\sum_i\langle\, RZ_i, Z_i\rangle \\
&= 6\sum_i\langle\, S_1 Z_i, S_1 Z_i\rangle + 6\sum_i\langle\, S_2 Z_i, S_2 Z_i\rangle + 2\sum_i\langle\, RZ_i, Z_i\rangle \\
&= 6\operatorname{tr}(S_1^2) + 6\operatorname{tr}(S_2^T S_2) + 2\sum_i\langle\, RZ_i, Z_i\rangle.
\end{aligned}$$

Altogether

$$\frac{f'}{f} = -2\operatorname{tr} S_1 \quad , \quad \frac{f''}{f} = 4(\operatorname{tr} S_1)^2 + 2\operatorname{tr}(S_1^2) + 6\operatorname{tr}(S_2^T S_2) + 2\sum_i\langle\, RZ_i, Z_i\rangle.$$

For the function $g = f^{-1/2p}$ we have

$$2p\,\frac{g''}{g} = -\frac{f''}{f} + \frac{2p+1}{2p}\left(\frac{f'}{f}\right)^2 = \frac{2}{p}(\operatorname{tr} S_1)^2 - 2\operatorname{tr}(S_1)^2 - 6\operatorname{tr}(S_2^T S_2) - 2\sum_i\langle\, RZ_i, Z_i\rangle$$

which proves our claim. □

Remark. If W is one dimensional, clearly $g_W = g_v$ for v a unit vector in W. If $W = E_{t_0}$, we have $\det M = \det(A^*A)^{-1} = 1/(\det A)^2$ and thus $g_W = (\det A)^{1/n}$. The differential equation in this case reduces to $n\,g''/g = \frac{1}{n}(\operatorname{tr} S)^2 - \operatorname{tr}(S^2) - \operatorname{Ric}(\dot c, \dot c)$ giving rise to the well known concavity of the volume in positive Ricci curvature. Notice also that the concavity of g_W already holds under the assumption that the curvature is p -positive, i.e. the sum of the p smallest eigenvalues of R are positive.

Proof of Theorem B: We first prove part (b). If $W \perp \ker A_{t^*}$, then Proposition 1.8 implies that $A^{-*}v$ is smooth at t^* for any $v \in W$, and hence M_t is smooth at t^* as well. The proof of Proposition 1.8 also shows that if $e_1, \ldots, e_p$ is a basis of W, then $A^{-*}e_1, \ldots, A^{-*}e_p$ are linearly independent at $t = t^*$ and hence $g_W(t^*) > 0$. It also follows that if W is not orthogonal to $\ker A_{t^*}$, then $g_W(t^*) = 0$.

To prove part (a), first recall that $(x_1 + \cdots + x_p)^2 \le p(x_1^2 + \cdots + x_p^2)$ with equality if and only if all x_i are equal to each other. Thus $(\operatorname{tr} S_1)^2 - p\,\operatorname{tr}(S_1^2) \le 0$ with equality iff $S_1 = \lambda\,\mathrm{Id}$. Furthermore, if the sum of the p smallest eigenvalues of R are non-negative, one easily sees that $\sum_{i=1}^{i=p}\langle\, Rw_i, w_i\rangle \ge 0$ if $w_1, \ldots, w_p$ is an orthonormal basis of any p dimensional subspace of E_{t_o}. Finally, $S_2^T S_2$ is clearly positive semi-definite. Altogether, Proposition 2.2 implies that g_W is concave.

If g is constant, the differential equation implies that for any $v \in W_t$ we have $S_1 v = \lambda v$ for some function $\lambda(t)$. Furthermore, $0 = \langle S_2^T S_2 v, v\rangle = \langle S_2 v, S_2 v\rangle$ and hence $S_2 v = 0$. In other words, $Sv = \lambda v$ for all $v \in W_t$. But if g is constant f is constant as well and $f' = 0$ implies that $\operatorname{tr} S_1 = 0$ and hence $\lambda = 0$. Thus $(A^{-*}v)' = -SA^{-*}v = 0$, for all $v \in W$, which implies that the function g_v is constant, and hence by Proposition 3.1 below, $A^{-*}v$ is a parallel Jacobi field. This proves part (c). □

3. Rigidity

We now use the results in Section 1 to prove the existence of parallel Jacobi fields in non-negative curvature, i.e. vectors $v \in E_{t_0}$ with $A_t' v = 0$. We allow endpoints and interior points of the geodesic to be singular.

Proposition 3.1. *Let A be a Lagrange tensor along the geodesic $c\colon [t_0, t_1] \to M$. If $R \geq 0$ and if there exists a non-zero vector $v \in E_{t_0}$ such that*

(a) $g_v'(t_0) = g_v'(t_1) = 0$,

(b) *v is orthogonal to $\ker A_t$ for all $t_0 \leq t \leq t_1$,*

then $w = A_t^{-1} A_t^{-*} v \in E_{t_0}$ is constant and $A_t' w = 0$ for all t. Thus $A_t^{-*} v = A_t w$ is a parallel Jacobi field.

Proof. By Proposition 1.8, assumption (b) implies that $g_v(t)$ is smooth and positive for all $t_0 \leq t \leq t_1$, and by Corollary 1.11, g_v is concave and hence constant. Thus $f_v = ||A_t^{-*} v||$ is constant as well. At regular points we thus have

$$0 = f_v'' = 2\langle RA^{-*}v, A^{-*}v\rangle + 6||SA^{-*}v||^2$$

and hence $SA^{-*}v = 0$. Thus $(A^{-*}v)' = -SA^{-*}v = 0$ and hence

$$(A^{-1}A^{-*}v)' = -A^{-1}A'A^{-1}A^{-*}v = -A^{-1}SA^{-*}v = 0.$$

Therefore, on any connected component of the regular points $A^{-1}A^{-*}v = w$ is constant and $Aw = A^{-*}v$ is parallel. Since $A^{-*}v$ is continuous, Aw is parallel for all t. ☐

Here is one possibility to translate Proposition 3.1 into a statement about Jacobi fields only, which is what we will use for the obstruction in Section 4.

Proposition 3.2. *Let M^{n+1} be a manifold with non-negative sectional curvature and V a self adjoint family of Jacobi fields along the geodesic $c : [t_0, t_1] \to M$. Assume there exists $X \in V$ such that*

(a) *$||X||_t \neq 0$, $||X||_t' = 0$ for $t = t_0$ and $t = t_1$,*

(b) *If $Y \in V$ and $\langle X(t_1), Y(t_1)\rangle = 0$ then $\langle X(t_0), Y(t_0)\rangle = 0$,*

(c) *If $Y \in V$ and $Y(t) = 0$ for some $t \in (t_0, t_1)$ then $\langle X(t_0), Y(t_0)\rangle = 0$,*

(d) *If $Y(t_0) = 0$, then $\langle X'(t_0), Y'(t_0)\rangle = 0$,*

Then X is a parallel Jacobi field along c.

Proof. We choose as a base point $t = t_0$. Then V defines Lagrange tensor A_t as in (1.3) with $A_{t_0|V_1} = \mathrm{Id}$, $A_{t_0|V_2} = 0$ and $V_1 \perp V_2$. By (a) we have that $X(t_0) \neq 0$ and we set $v := X(t_0) \in V_1$. If $Y \in V$ and $Y(t_0) = 0$ then $Y'(t_0) \in V_2$ and V_2 is spanned by such vectors. Thus (d) implies $X'(t_0) \in V_1$ and hence by the definition (1.3) we have $X(t) = A_t v$, and $A_{t_0} v = v$.

We now want to show that the assumptions of Proposition 3.1 are satisfied by A_t. We start with the second part.

Let $w \in \ker(A_t)$, i.e. $A_t w = 0$ for $t \in (t_0, t_1)$. Set $w = w_1 + w_2$ with $w_i \in V_i(t_0)$ and hence $A_{t_0} w = w_1$. Assumption (c) implies that $\langle A_{t_0} v, A_{t_0} w\rangle = \langle v, w_1\rangle = 0$. Since $\langle v, V_2\rangle = 0$ as well, we have $\langle v, w\rangle = 0$ and hence $v \perp \ker A_t$. The same argument shows that $v \perp \ker A_{t_1}$ by using (b). If $A_{t_0} w = 0$, then $w \in V_2$ and hence $\langle v, w\rangle = 0$. Thus Proposition 1.8 implies that $A_t{}^{-*} v$ and hence g_v is smooth for all $t \in [t_0, t_1]$.

We now show that g_v' vanishes at the endpoints. By Proposition 1.12, $g_v'(t_0) = ||Av||' = ||X||'(t_0) = 0$. For $t = t_1$ the proof is similar to the proof of Proposition 1.12. We first claim that

$$\lim_{t \to t_1} A_t^{-*} v = \lambda A_{t_1} v \;\text{ for some } \lambda \in \mathbb{R}. \tag{3.3}$$

To see this, we begin by showing that $\lim_{t\to t_1} A_t^{-*}v \in V_1(t_1)$. But $V_1(t_1) \perp V_2(t_1)$ and $V_2(t_1)$ is spanned by $A'_{t_1}w$ for some $w \in E_{t_0}$ with $A_{t_1}w = 0$. By differentiating $\langle w, v\rangle = \langle A_t^{-*}v, A_t w\rangle$ we obtain

$$\langle \lim_{t\to t_1} A_t^{-*}v, A'_{t_1}w\rangle = \lim_{t\to t_1} \langle A_t^{-*}v, A'_t w\rangle = -\lim_{t\to t_1} \langle (A_t^{-*})'v, A_t w\rangle = -\langle \lim_{t\to t_1} (A_t^{-*})'v, A_{t_1}w\rangle = 0.$$

Next, we show that $\langle \lim_{t\to t_1} A_t^{-*}v, A_{t_1}w\rangle = 0$ whenever $\langle A_{t_1}v, A_{t_1}w\rangle = 0$, which clearly implies (3.3) since $\operatorname{Im} A_{t_1} = V_1(t_1)$. To see this, we observe that (b) implies $0 = \langle A_{t_0}w, A_{t_0}v\rangle = \langle w_1, v\rangle = \langle w_1 + w_2, v\rangle = \langle w, v\rangle$ and hence

$$\langle \lim_{t\to t_1} A_t^{-*}v, A_{t_1}w\rangle = \lim_{t\to t_1} \langle A_t^{-*}v, A_t w\rangle = \langle v, w\rangle = 0.$$

We now use (3.3) to show that $g'_v(t_1) = 0$. Since $g_v(t_1) \neq 0$ by (a), this is equivalent to $f'_v(t_1) = 0$. By (3.3), $A_t^{-*}v$ and $\lambda A_t v$ have the same limit and thus

$$\begin{aligned} f'_v(t_1) =& 2\lim_{t\to t_1} \langle (A_t^{-*}v)', A_t^{-*}v\rangle = 2\lim_{t\to t_1} \langle (A_t^{-*}v)', \lambda A_t v\rangle \\ &= -2\lambda \lim_{t\to t_1} \langle A_t^{-*}v, A'_t v, \rangle = -2\lambda^2 \langle A_{t_1}v, A'_{t_1}v\rangle = -\lambda^2 (||Av||^2)'_{t=t_1}. \end{aligned}$$

which is 0 since $||Av||'(t_1) = ||X||'(t_1) = 0$.

Proposition 3.1 now implies that $A_t{}^{-*}v = Aw$, for some $w \in E_{t_0}$, is a parallel Jacobi field in V and $A_{t_0}^{-*}v = v = A_{t_0}w$ by (1.13). Since $A'_{t_0}w = 0$, (1.3) implies that $A_{t_0}w = w$, and hence $w = v$ and thus $A_t w = A_t v = X$ is a parallel Jacobi field. □

Remark. (a) Notice that the first three conditions are necessary for X to be parallel, using, for (b) and (c) that in a self adjoint family of Jacobi fields, $\langle X, Y\rangle' = \langle X, Y'\rangle = \langle X', Y\rangle = 0$ for all $X, Y \in V$ with X parallel. If there are no interior singular points, (b) is the only global condition and relates the Jacobi fields at t_0 and t_1. Some global condition is clearly necessary since there are Jacobi fields of constant length (restricted to a geodesic with no singularities) which are not parallel.

Also notice that assumption (d) is necessary since on $M = \mathbb{S}^1 \times \mathbb{S}^2$ with the product metric we can take the geodesic $c(t) = (1, \gamma(t))$ with γ a great circle from north pole to south pole. Then $V = \operatorname{span}\{Z_1, Z_2\}$ with $Z_1 = (1, 0)$, $Z_2 = (0, Y(t))$ and Y a Jacobi field vanishing at north and south pole is a self adjoint family along c. Setting $X = Z_1 + Z_2$ one sees that all conditions in Proposition 3.2, except for (d), are satisfied, but X is not parallel.

(b) The fact that assumption (d) makes the Proposition asymmetric is due to the fact that the definition of g_v involves the choice of a base point. This turns out to be quite useful since for the manifolds in Section 3, (d) is sometimes satisfied at one endpoint, but not necessarily at the other. Of course, if t_0 is regular, condition (d) is empty.

Proposition 3.4. *Let V and $X \in V$ satisfy the conditions in Proposition 3.2 and assume that V is defined on a larger interval $[t_0, t_2] \supset [t_0, t_1]$. If there exists a Jacobi field $Y \in V$ such that $Y(t^*) = 0$ for some $t^* \in (t_1, t_2]$ and $\langle X(t_0), Y(t_0)\rangle \neq 0$, then X is not parallel on $[t_0, t_2]$.*

Proof. Let A_t be the Lagrange tensor associated to V with base point t_0. Recall that in the proof of Proposition 3.2 we showed that $X(t) = A_t v$ with $v = X(t_0)$. The assumption that $\langle X(t_0), Y(t_0)\rangle \neq 0$ means that v is not orthogonal to $\ker A_{t^*}$ and hence $g_v(t^*) = 0$ by Proposition 1.8 (b). Now assume that X is parallel on $[t_0, t_2]$. We claim that in that case $g_v(t)$ would be constant on $[t_0, t_2]$, contradicting that fact that $g_v(t^*) = 0$.

To see this, we show that $A'_t v = 0$ with $A_{t_0}v = v$ implies $g_v(t) = ||A_t v||$. First observe that by self adjointness $\langle A_t v, A_t w\rangle' = \langle A'_t v, A_t w\rangle + \langle A_t v, A'_t w\rangle = 2\langle A'_t v, A_t w\rangle = 0$. Thus if $\langle v, w\rangle = 0$, we have $\langle A_t v, A_t w\rangle = \langle A_{t_0}v, A_{t_0}w\rangle = \langle v, w_1\rangle = \langle v, w\rangle = 0$. Furthermore, at regular points $\langle v, w\rangle = \langle A_t^{-*}v, A_t w\rangle$ and hence $A_t^{-*}v = \lambda A_t v$ for some function λ. But then $\langle v, v\rangle =$

$\langle A_t^{-*}v, A_t v\rangle = \lambda\langle A_t v, A_t v\rangle = \lambda\langle v, v\rangle$ and thus $\lambda = 1$, i.e. $A_t^{-*}v = A_t v$ for all regular t. Thus $g_v(t) = ||v||^2/||A_t^{-*}v|| = ||v||^2/||A_t v|| = ||v|| = ||A_t v||$ for all regular t and hence for all t. □

4. Proof of Theorem C and D

We now use Proposition 3.2 and Proposition 3.4 to prove Theorem C and D.

A simply connected compact cohomogeneity one manifold is the union of two homogeneous disc bundles. Given compact Lie groups H, K^-, K^+ and G with inclusions $H \subset K^\pm \subset G$ satisfying $K^\pm/H = \mathbb{S}^{\ell_\pm}$, the transitive action of $K^\pm$ on $\mathbb{S}^{\ell_\pm}$ extends to a linear action on the disc $\mathbb{D}^{\ell_\pm+1}$. We can thus define $M = G \times_{K^-} \mathbb{D}^{\ell_-+1} \cup G \times_{K^+} \mathbb{D}^{\ell_++1}$ glued along the boundary $\partial(G \times_{K^\pm} \mathbb{D}^{\ell_\pm+1}) = G \times_{K^\pm} K^\pm/H = G/H$ via the identity. G acts on M on each half via left action in the first component. This action has principal isotropy group H and singular isotropy groups $K^\pm$. One possible description of a cohomogeneity one manifold is thus simply in terms of the Lie groups $H \subset \{K^-, K^+\} \subset G$ (see e.g. [AA]).

The first family of cohomogeneity one manifolds we denote by $P_{(p_-,q_-),(p_+,q_+)}$ and is given by the group diagram

$$H = \{\pm(1,1), \pm(i,i), \pm(j,j), \pm(k,k)\} \subset \{(e^{ip_-t}, e^{iq_-t})\cdot H\ ,\ (e^{jp_+t}, e^{jq_+t})\cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3\,.$$

where $\gcd(p_-, q_-) = \gcd(p_+, q_+) = 1$ and all 4 integers are congruent to 1 mod 4.

The second family $Q_{(p_-,q_-),(p_+,q_+)}$ is given by the group diagram

$$H = \{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{ip_-t}, e^{iq_-t})\cdot H\ ,\ (e^{jp_+t}, e^{jq_+t})\cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

where $\gcd(p_-, q_-) = \gcd(p_+, q_+) = 1$, q_+ is even, and p_-, q_-, p_+ are congruent to 1 mod 4.

The candidates for positive curvature in [GWZ] are $P_k = P_{(1,1),(1+2k,1-2k)}$, $Q_k = Q_{(1,1),(k,k+1)}$ with $k \geq 1$, and the exceptional manifold $R^7 = Q_{(-3,1),(1,2)}$.

We now describe the geometry of a general cohomogeneity one action. A G invariant metric is determined by its restriction to a geodesic c normal to all orbits. At the points $c(t)$ which are regular with respect to the action of G, the isotropy is constant and we denote it by H. In terms of a fixed biinvariant inner product Q on the Lie algebra $\mathfrak{g}$ and corresponding Q-orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$ we identify, at regular points, $\dot{c}^\perp \subset T_{c(t)}M$ with $\mathfrak{h}^\perp$ via action fields: $X \in \mathfrak{h}^\perp \to X^*(c(t))$. H acts on $\mathfrak{h}^\perp$ via the adjoint representation and a G invariant metric on G/H is described by an $\mathrm{Ad}(H)$ invariant inner product on $\mathfrak{h}^\perp$. Along c the metric on M is thus described by a collection of functions, which at the endpoint must satisfy certain smoothness conditions.

Since G acts by isometries, X^*, $X \in \mathfrak{g}$, are Killing vector fields and hence the restriction to a geodesic is a Jacobi field. This gives rise to an $(n-1)$-dimensional family of Jacobi fields along c defined by $V := \{X^*(c(t)) \mid X \in \mathfrak{h}^\perp\}$. The self adjoint shape operator S_t of the regular hypersurface orbit G/H at $c(t)$ satisfies $\nabla_{\dot{c}(t)}X^* = \nabla_{X^*}\dot{c} = S_t(X^*(c(t)))$, i.e. $X' = S_t(X)$, $X \in \mathfrak{h}^\perp$. Hence V is self adjoint.

A singular point of V is a point $c(t_0)$ such that there exists an $X^* \in V$ with $X^*(c(t_0)) = 0$, i.e. the isotropy group $G_{c(t_0)}$ satisfies $\dim G_{c(t_0)} > \dim H$ and is thus a singular isotropy group of the action. For simplicity set $K := G_{c(t_0)}$ and define a Q-orthogonal decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{p} \ \text{ and thus } \ \mathfrak{h}^\perp = \mathfrak{p} \oplus \mathfrak{m}.$$

Here $\mathfrak{m}$ can be viewed as the tangent space to the singular orbit G/K at $c(t_0)$. The slice D, i.e. the vector space normal to G/K at $c(t_0)$, can be identified with $D := \dot{c}(t_0) \oplus \mathfrak{p}$ where $\mathfrak{p} \subset D$ via $X \in \mathfrak{p} \to (X^*)'(c(t_0))$. Notice that $X^*(c(t_0)) = 0$. Since the slice is orthogonal to the orbit, we have $\langle (X^*)', Y^*\rangle_{c(t_0)} = 0$ for $X \in \mathfrak{p}$ and $Y \in \mathfrak{m}$. K acts via the isotropy action $\mathrm{Ad}(K)_{|\mathfrak{m}}$ of G/K on $\mathfrak{m}$ and via the slice representation on D. The second fundamental form of the singular orbit can be viewed as a linear map $B\colon D \to S^2(\mathfrak{m})$, $N \to \{(X,Y) \to \langle S_N(X), Y\rangle\}$. Since K acts by isometries, B is equivariant with respect to the slice representation of K on D and the action on $S^2(\mathfrak{m})$ induced by its isotropy representation on $\mathfrak{m}$. An $\mathrm{Ad}(K)$ invariant irreducible splitting $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ induces a splitting of $S^2(\mathfrak{m})$ into irreducible summands. If for some i, the slice representation (which is irreducible) is not a subrepresentation of $S^2(\mathfrak{m}_i)$, this implies that $\langle S_{\dot c(t_0)}X, Y\rangle = \langle X', Y\rangle_{c(t_0)} = 0$ for $X, Y \in \mathfrak{m}_i$. In particular, $||X||'_{c(t_0)} = 0$. This describes some of the smoothness conditions that must be satisfied at the endpoints.

We now apply this to the P family and show:

Proposition 4.1. *Let M be one of the 7-manifolds $P_{(p_-,q_-),(p_+,q_+)}$ with its cohomogeneity one action by $G = \mathrm{S}^3 \times \mathrm{S}^3$. Assume that M is not one of the candidates for positive curvature P_k or $P_{(1,q),(p,1)}$. Furthermore, let $c\colon (-\infty,\infty) \to M$ be a geodesic orthogonal to all orbits. Then for any invariant metric with non-negative curvature there exists a Jacobi field along c, given by the restriction of a Killing vector field X^*, $X \in \mathfrak{g}$, such that X^* is parallel on some interval but not for all t. In particular, the metric is not analytic.*

Proof. Since H is finite, we have $\mathfrak{h}^\perp = \mathfrak{p} \oplus \mathfrak{m} = \mathfrak{g}$. Regarding S^3 as the unit quaternions, we choose the basis of $\mathfrak{g}$ given by the left invariant vector fields X_i and Y_i on $G = \mathrm{S}^3 \times \mathrm{S}^3$ corresponding to i, j and k in the Lie algebras of the first and second S^3 factor of G. Then the action fields X_i^*, Y_i^* are Jacobi fields along the geodesic $c(t)$, $\infty < t < \infty$ and are a basis of a self adjoint family V.

We start with three general observations.

Observation 1. Non-trivial irreducible representations of the identity component $K_0 = \mathrm{S}^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ consist of two dimensional representations given by multiplication by $e^{in\theta}$ on $\mathbb{C}$, called a weight n representation. If $K_0 = (e^{ip\theta}, e^{iq\theta}) \subset \mathrm{S}^3 \times \mathrm{S}^3$ has slope (p,q) with $\gcd(p,q) = 1$, and H is finite, the vector space $\mathfrak{p}$ is given by $\mathfrak{p} = \mathrm{span}\{pX_1 + qY_1\}$. The tangent space $\mathfrak{m}$ to the singular orbit G/K (which is spanned by the action fields X^*) splits up into K irreducible subspaces $W_0 = \mathrm{span}\{X_1, Y_1\}$, $W_1 = \mathrm{span}\{X_2, X_3\}$ and $W_2 = \mathrm{span}\{Y_2, Y_3\}$. Notice that W_0 is one dimensional since $pX_1 + qY_1 = 0$. Thus we can also write $W_0 = \mathrm{span}\{-qX_1 + pY_1\}$. The isotropy action on $\mathfrak{m}$, which is given by conjugation on imaginary quaternions in each component, is trivial on W_0 and has weight $2p$ on W_1 and $2q$ on W_2 since e.g. $e^{ip\theta} j e^{-ip\theta} = e^{2ip\theta} j$. If $p \neq q \neq 0$, all representations in $\mathfrak{m}$ are inequivalent and hence orthogonal by Schur's Lemma. Furthermore, the metric on W_i is a multiple of the Killing form, again by Schur's Lemma, and since X_i, Y_i are orthogonal in the Killing form, they are orthogonal in the metric as well. Thus, unless $(p,q) = (1,1)$, the vector fields $-qX_1 + pY_1, X_2, X_3, Y_2, Y_3$ are orthogonal and $pX_1 + qY_1$ vanishes.

Observation 2. In order to determine the derivatives $||X||'(0)$, we will use equivariance of the second fundamental form $B\colon S^2\mathfrak{m} \to D$ under K_0, where $D = \mathbb{R}^2$ is the slice. If $H \cap K_0 = \mathbb{Z}_k$, then the action of K_0 on the slice has $\mathbb{Z}_k$ as its ineffective kernel since it acts via rotation of a circle and if it fixes one point, as does H, then it acts trivially on D. Hence the slice representation has weight $k = |H \cap K_0|$. The vector space $S^2\mathfrak{m}$ splits as $S^2W_0 \oplus S^2W_1 \oplus S^2W_2 \oplus W_1 \otimes W_2 \oplus W_0 \otimes W_1 \oplus W_0 \otimes W_2$. The action of K_0 on $S^2\mathfrak{m}$ has weight 0 on S^2W_0, $4p$ on S^2W_1, $4q$ on S^2W_2, and $2p \pm 2q$ on $W_1 \otimes W_2$, $2p$ on $W_0 \otimes W_1$ and $2q$ on $W_0 \otimes W_2$. Thus the second fundamental form vanishes on W_0, on W_1 if $4|p| \neq k$, on W_2 if $4|q| \neq k$, on $W_1 \otimes W_2$ if $|2p \pm 2q| \neq k$ and on $W_0 \otimes W_1$ if $2|p|$ resp. $2|q| \neq k$. This will be used to show that in some cases $B(X,Y) = \langle X', Y\rangle = 0$ for $X \in W_i, Y \in W_j$.

Observation 3. We will also use the the Weyl group $W \subset N(H)/H$ of the cohomogeneity one action (see e.g. [AA], [Zi2]), which is defined as the subgroup of G which preserves the geodesic c. One easily sees that there exists a so called Weyl group element $w_- \in W$ in the normalizer of H in $K^- = G_{c(0)}$, unique modulo H, which, via the action of $G_{c(0)}$ on the slice D, satisfies

$w_-(c'(0)) = -c'(0)$ and hence reverses the geodesic at $t = 0$. Similarly, there exists a w_+ in the normalizer of H in $K^+ = G_{c(L)}$, unique modulo H, which reverses the geodesic at $t = L$. This implies that conjugation by w_- takes the isotropy group $G_{c(rL)}$ to $G_{c(-rL)}$, $r \in \mathbb{Z}$, and w_+ takes $G_{c(rL)}$ to $G_{c(2L-rL)}$. Furthermore, W is the dihedral group generated by w_- and w_+. The geodesic c is closed iff the Weyl group is finite, in which case the length of c is kL where k is the order of W. Finally, since K_0 acts via rotation on the 2-dimensional slice, the Weyl group element w_- can be represented by a rotation by π and hence can also be characterized as the unique element in K_0^- which does not lie in H, but whose square lies in H.

We now apply these observations to the manifold $P_{(p_-,q_-),(p_+,q_+)}$. The Weyl group elements are given by

$$w_- = (e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}) \in K_0^- \mod H, \quad \text{and } w_+ = (e^{j\frac{\pi}{4}}, e^{j\frac{\pi}{4}}) \in K_0^+ \mod H$$

since e.g. $w_-^2 = (i, i) \in H$, but $w_- \notin H$. Notice that conjugation by $e^{i\frac{\pi}{4}}$ interchanges j and k and fixes i, and conjugation by $e^{j\frac{\pi}{4}}$ interchanges i and k and fixes j. Thus w_- fixes X_1 and Y_1 but interchanges X_2 with X_3 and Y_2 with Y_3. One easily sees that W, which is generated by w_- and w_+, has order 12 since $(w_-w_+)^6 \in H$ but $(w_-w_+)^3 \notin H$. Thus c has length $12L$. This easily implies that

$$\begin{aligned} G_{c(0)} &= (e^{ip_-t}, e^{iq_-t}) \cdot H\ , \quad G_{c(L)} = (e^{jp_+t}, e^{jq_+t}) \cdot H\ , \quad G_{c(2L)} = (e^{kp_-t}, e^{kq_-t}) \cdot H \\ G_{c(3L)} &= (e^{ip_+t}, e^{iq_+t}) \cdot H\ , \quad G_{c(4L)} = (e^{jp_-t}, e^{jq_-t}) \cdot H\ , \quad G_{c(5L)} = (e^{kp_+t}, e^{kq_+t}) \cdot H \end{aligned}$$

and $G_{c(rL)} = G_{c((r-6)L)}$ for $r = 6, \dots, 11$.

At $t = 0$ we have $H \cap K_0^- = \{\pm(1,1), \pm(i,i)\}$ and hence $k = 4$. The tangent space to G/K^- is the direct sum of $W_0 = \operatorname{span}\{X_1, Y_1\} = \operatorname{span}\{X_1\} = \operatorname{span}\{Y_1\}$ (since $p_-, q_- \neq 0$), and $W_1 = \operatorname{span}\{X_2, X_3\}$ and $W_2 = \operatorname{span}\{Y_2, Y_3\}$. Observation 2 implies that the second fundamental form vanishes on $S^2(W_1)$ if $p_- \neq 1$, on $S^2(W_2)$ if $q_- \neq 1$, and on $W_1 \otimes W_2$ if $2p_- + 2q_- \neq \pm 4$, i.e. $p_- + q_- \neq \pm 2$. Notice that $p_- - q_- = \pm 2$ is not possible since $p_-, q_- \equiv 1 \mod 4$ and that $p_- \neq -1$ and $q_- \neq -1$ as well. Similarly at $t = rL$, $r \in \mathbb{Z}$ since in all cases $k = 4$.

Claim 1: If $p_- \neq 1$ and $p_+ \neq 1$, then X_3^* is a parallel Jacobi field on $[0, L]$, but is not parallel on $[0, 2L]$. Similarly, if $q_- \neq 1$ and $q_+ \neq 1$ for Y_3^*.

For this we will show that X_3^* satisfies all properties of Proposition 3.2 on the interval $[t_0, t_1] = [0, L]$. At $t = L$ the tangent space of G/K^+ is the direct sum of $\overline{W}_0 = \operatorname{span}\{-q_+X_2 + p_+Y_2\}$, $\overline{W}_1 = \operatorname{span}\{X_1, X_3\}$ and $\overline{W}_2 = \operatorname{span}\{Y_1, Y_3\}$. Since $X_3 \in W_1 \cap \overline{W}_1$, we have $X_3(t) \neq 0$ for $t = 0, L$, and by Observation 2, the assumptions imply that $||X_3||_t' = 0$ at $t = 0, L$ as well. Thus condition (a) is satisfied. For condition (b), observe that $p_- \neq q_-$ since $p_- = q_-$ implies that $(p_-, q_-) = (1, 1)$. Thus by Observation 1, the vectors $-q_-X_1 + p_-Y_1, X_2, X_3, Y_2, Y_3$ are orthogonal at $t = 0$ and $p_-X_1 + q_-Y_1$ vanishes. Similarly, $p_+ \neq q_+$ and hence at $t = L$, the vectors $-q_+X_2 + p_+Y_2, X_1, X_3, Y_1, Y_3$ are orthogonal and $p_+X_2 + q_+Y_2$ vanishes. Thus any $Z \in V$ orthogonal to X_3 at $t = 0$ is also orthogonal to X_3 at $t = L$. Condition (c) holds since there are no interior singular points.

Finally, we come to condition (d). Here we use the action of the principal isotropy group $H = \Delta Q$ on the tangent space of the regular orbits G/H. It acts via conjugation and thus (i, i) acts via Id on $\operatorname{span}\{X_1, Y_1\}$ and as $-\mathrm{Id}$ on $\operatorname{span}\{X_2, X_3, Y_2, Y_3\}$. Similarly for (j, j) and (k, k). Hence the representation of H on $\operatorname{span}\{X_1, Y_1\}$, $\operatorname{span}\{X_2, Y_2\}$, and $\operatorname{span}\{X_3, Y_3\}$ are inequivalent and thus by Schur's Lemma these subspaces are orthogonal to each other for all t. Furthermore, they are invariant under parallel translation since parallel translation commutes with isometries and hence with the action of H. This implies condition (d) at $t = 0$ since $Y = p_-X_1 + q_-Y_1$ is the only element in V with $Y(0) = 0$ and thus $\langle X_3'(0), Y'(0)\rangle = 0$.

Altogether, Proposition 3.2 now implies that X_3^* is parallel on $[0, L]$. On the other hand, $Z := p_- X_3 + q_- Y_3$ vanishes at $2L$, but $X_3^*(0)$ is not orthogonal to $Z(0)$ since $X_3(0)$ and $Y_3(0)$ are orthogonal and $p_- \neq 0$. Hence Proposition 3.4 implies that X_3^* is not parallel on $[0, 2L]$.

Claim 2: If $p_- \neq 1, q_- \neq 1$ and $p_- + q_- \neq \pm 2$ and $(p_-, q_-) \neq (p_+, q_+)$, then a certain linear combination of X_3^* and Y_3^* is a parallel Jacobi field on $[0, 2L]$, but not on $[0, 3L]$. Similarly, for p_+, q_+.

The only Jacobi field that vanishes at $2L$ is $Z = p_- X_3 + q_- Y_3$. In order to satisfy condition (b), we choose $X = aX_3 + bY_3$ such that $\langle X(0), Z(0)\rangle = 0$. We will show that X^* satisfies all properties of Proposition 3.2 on the interval $[t_0, t_1] = [0, 2L]$. Notice that at 0 and $2L$ the slopes are both (p_-, q_-).

We start with condition (a). At $t = 0$ we have $X \in W_1 \oplus W_2$ and hence $X \neq 0$. The assumptions on the slopes imply that the second fundamental form vanishes on $S^2(W_0 \oplus W_1 \oplus W_2)$, i.e. the orbit G/K^- is totally geodesic. This in particular implies that $||X||'(0) = 0$. Similarly, $||X||'(2L) = 0$ since the slopes are the same. We also have $X(2L) \neq 0$ since the only Jacobi field vanishing at $2L$ is Z. Thus $X(2L) = 0$ would contradict the orthogonality assumption at $t = 0$.

Condition (b) again follows from Observation 1 since $(p_-, q_-) \neq (1, 1)$ implies that $p_- \neq q_-$. Hence the vectors $-q_- X_1 + p_- Y_1, X_2, X_3, Y_2, Y_3$ are orthogonal at $t = 0$ and $p_- X_1 + q_- Y_1$ vanishes, and at $t = 2L$, the vectors $-q_- X_3 + p_- Y_3, X_1, X_2, Y_1, Y_2$ are orthogonal and $p_- X_3 + q_- Y_3$ vanishes. Since we have $\langle X(2L), Z(2L)\rangle = 0$, we chose X such that $\langle X(0), Z(0)\rangle = 0$ as well. Notice also that $\langle X(2L), -q_- X_3(2L) + p_- Y_3(2L)\rangle = 0$ is not possible, since then $Z(2L)$ would be orthogonal to $X_3(2L)$ or $Y_3(2L)$ or both, but this is not possible since a, b, p_-, q_- are all non-zero.

Condition (c) holds since the only interior singularity is at $t = L$, and $p_+ X_2 + q_+ Y_2$ is the only vector that vanishes there. But this vector is clearly orthogonal to X at $t = 0$.

For condition (d) we can argue as in Claim 1.

Thus X^* is parallel on $[0, 2L]$. Finally, observe that $X(0)$ is not orthogonal to the kernel at $t = 3L$, which is spanned by $p_+ X_3 + q_+ Y_3$, unless $\langle aX_3 + bY_3, p_+ X_3 + q_+ Y_3\rangle_{t=0} = ap_+ ||X_3||^2 + bq_+ ||Y_3||^2 = 0$. Since we also have $\langle X, p_- X_3 + q_- Y_3\rangle = 0$, this would imply that $(p_-, q_-) = (p_+, q_+)$. This was excluded, and thus X^* is not parallel on $[0, 3L]$.

Now we combine Claim 1 and Claim 2. Claim 1 implies that, up to possibly switching the two S^3 factors or interchanging 0 and L, we have the desired Jacobi field, unless the slopes are $(1, q_-), (p_-, 1)$ or $(p_-, q_-), (1, 1)$. The first family was excluded by assumption. In the second family we can assume that $p_- \neq 1$, $q_- \neq 1$ and $(p_-, q_-) \neq (p_+, q_+)$, since otherwise we are in the first family. Thus Claim 2 implies that in the second family we have the desired Jacobi field unless $p_- + q_- = \pm 2$. Reversing the orientation of the circle, we can assume $p_- + q_- = 2$. This leaves only the candidates with slopes $(1 + 2k, 1 - 2k), (1, 1)$. □

Remark. The exceptional family $P_{(1,\,q),(p,1)}$ contains several G-invariant analytic metrics with non-negative curvature. Indeed, $P_{(1,1),(-3,1)}$ is $\mathbb{S}^7$, and $P_{(1,-3),(-3,1)}$ is the positively curved Berger space (see e.g. [GWZ] or [Zi2]). It also contains $P_{(1,1),(1,1)}$. This manifold is not primitive, and hence does not admit positive curvature. But it does admit an analytic metric with non-negative curvature. Indeed, we claim that the manifold is $\mathbb{S}^3 \times \mathbb{S}^4$ and that the product metric of round sphere metrics is invariant. For this we identify the action of $\mathrm{S}^3 \times \mathrm{S}^3$ on $\mathbb{S}^3 \times \mathbb{S}^4$ as $(r_1, r_2) \in \mathrm{S}^3 \times \mathrm{S}^3$ acting as $(p, q) \to (r_1 p\ r_2^{-1}, \phi(r_2) q)$ where $\phi(r_2)$ acts via the well known cohomogeneity one action of S^3 on $\mathbb{S}^4$ (effectively an SO(3) action) with group diagram $H = \{\pm 1, \pm i, \pm j, \pm k\} \subset \{e^{it} \cdot H\ ,\ e^{jt} \cdot H\} \subset \mathrm{S}^3$. One now easily identifies the isotropy groups of this action to be those of $P_{(1,1),(1,1)}$.

We now prove Theorem C in the Introduction.

Proposition 4.2. *Let M be one of the 7-manifolds $Q_{(p_-,q_-),(p_+,q_+)}$ with its cohomogeneity one action by $G = \mathrm{S}^3 \times \mathrm{S}^3$. Assume that M is not of type $Q_k = Q_{(1,1),(k,k+1)}, k \geq 0$. Furthermore, let $c\colon (-\infty, \infty) \to M$ be a geodesic orthogonal to all orbits. Then for any invariant metric with non-negative curvature there exists a Jacobi field along c, given by the restriction of a Killing vector field X^*, $X \in \mathfrak{g}$, such that X^* is parallel on some interval but not for all t. In particular, the metric is not analytic.*

Proof. We indicate the changes that are necessary. The first difference is the Weyl group since the Weyl group elements are now

$$w_- = (e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{4}}) \in K_0^- \mod H, \text{ and } w_+ = (j, \pm 1) \in K_0^+ \mod H$$

and hence $|W| = 8$, i.e. the closed geodesic has length $8L$. The isotropy groups are given by

$$\begin{aligned} G_{c(0)} &= (e^{ip_-t}, e^{iq_-t}) \cdot H\ , \quad & G_{c(L)} &= (e^{jp_+t}, e^{jq_+t}) \cdot H\ , \quad & G_{c(2L)} &= (e^{-ip_-t}, e^{iq_-t}) \cdot H \\ G_{c(3L)} &= (e^{-kp_+t}, e^{kq_+t}) \cdot H\ , \quad & G_{c(4L)} &= (e^{ip_-t}, e^{iq_-t}) \cdot H\ , \quad & G_{c(5L)} &= (e^{-jp_+t}, e^{jq_+t}) \cdot H \\ G_{c(6L)} &= (e^{-ip_-t}, e^{iq_-t}) \cdot H\ , \quad & G_{c(7L)} &= (e^{kp_+t}, e^{kq_+t}) \cdot H\ , \quad & G_{c(8L)} &= (e^{ip_-t}, e^{iq_-t}) \cdot H \end{aligned}$$

A second difference is the normal weights. At $t = 0$ we still have $H \cap K_0^- = \{\pm(1,1), \pm(i,i)\}$ and hence $k = 4$. But at $t = L$ we have $H \cap K_0^+ = \{(\pm 1, 1)\}$ and hence $k = 2$. Similarly, $k = 4$ at $t = 2L, 4L$ and $k = 2$ at $t = 3L, 5L$. In particular, Observation 2 implies that $||X_3||' = ||Y_3||' = 0$ at $t = L$ and $t = 3L$.

We first claim that $(p_-, q_-) = (1,1)$. Indeed, if e.g. $p_- \neq 1$, then we can apply Proposition 3.2 to X_3 on the interval $[t_0, t_1] = [0, L]$ as in the proof of Claim 1 in Proposition 4.1, since $k = 2$ at L. For condition (b) notice that $p_+ \neq q_+$ since p_+ is odd, and q_+ even. Furthermore, notice that if $q_+ = 0$, the vectors $Y_1, Y_3, -q_+X_2 + p_+Y_2$ do not need to be orthogonal to each other since K_0^+ acts trivially on $\overline{W}_0 \oplus \overline{W}_2$, but they are orthogonal to $X_3 \in \overline{W}_1$ which is sufficient for condition (b).

For condition (d) we again use the action of the principal isotropy group $H = \{(\pm 1, \pm 1), (\pm i, \pm i)\}$ on the tangent space of the regular orbits G/H. H acts via Id on $\mathrm{span}\{X_1, Y_1\}$ and as $-\mathrm{Id}$ on $\mathrm{span}\{X_2, X_3, Y_2, Y_3\}$. Thus by Schur's Lemma these two subspaces are orthogonal for all t and are also invariant under parallel translation. This implies condition (d) since $Y = p_-X_1 + q_-Y_1$ is the only element in V with $Y(0) = 0$ and thus $\langle X_3', Y' \rangle_{t=0} = 0$. Finally, notice that $Z = -p_+X_3 + q_+Y_3$ satisfies $Z(3L) = 0$, but $\langle X_3(0), Z(0) \rangle = p_+||X_3(0)||^2 \neq 0$ and hence by Proposition 3.4 X_3^* is not parallel on $[0, 3L]$.

Next, we claim that if $p_+ \pm q_+ \neq \pm 1$, then we can argue as in the proof of Claim 2 in Proposition 4.1. Indeed, we choose $X = aX_3 + bY_3$ so that $\langle X, -p_+X_3 + q_+Y_3 \rangle = 0$ at $t = L$ and apply Proposition 3.2 to X^* on the interval $[L, 3L]$. At the endpoints, the second fundamental form vanishes on S^2W_i and $W_0 \otimes W_i$ since $k = 2$, and on $W_1 \otimes W_2$ since $p_+ \pm q_+ \neq \pm 1$. Thus the singular orbits at $t = L$ and $t = 3L$ are totally geodesic, which implies $||X^*||' = 0$ at $t = L, 3L$. The orthogonality condition on X again implies condition (b), and for (c) we use the action of H to conclude that $-p_-X_1 + q_-Y_1$, the only vanishing Jacobi field at $t = 2L$, is orthogonal to X at $t = L$. For condition (d) we argue as in the previous case. Finally, notice that $Z = p_+X_3 + q_+Y_3$ satisfies $Z(7L) = 0$, but $\langle X(L), Z(L) \rangle \neq 0$ since otherwise $ap_+||X_3(L)|| + bq_+||Y_3(L)||^2 = 0$, which contradicts $\langle X(L), -p_+X_3 + q_+Y_3 \rangle = -ap_+||X_3||^2 + bq_+||Y_3||^2 = 0$ since $p_+ \neq 0$ and $a \neq 0$. Thus X_3^* is not parallel on $[L, 7L]$.

Altogether, we can now assume that $(p_-, q_-) = (1,1)$ and $p_+ + q_+ = \pm 1$ or $p_+ - q_+ = \pm 1$. We can changes the sign of p_+ by conjugating all groups with $(1, j)$ and both signs by reversing the orientation of the circle. Thus it is sufficient to assume $q_+ - p_+ = 1$. But this is precisely the family Q_k with slopes $(1,1), (k, k+1)$, $k \geq 0$, after possibly switching the two S^3 factors. $\square$

Remark. Q_1 is the positively curved Aloff Wallach space which admits an invariant analytic metric with positive curvature. It is not known if Q_k with $k > 1$ admit such metrics, not even if they admit analytic metrics with non-negative curvature.

The manifold Q_0 is special. In the language of our paper, any linear combination of Y_2 and Y_3 is orthogonal to all kernels, and hence a parallel Jacobi field for all t. But there is no Jacobi field which is necessarily parallel for some t but not for all t. In [GWZ] it was shown that Q_0 has the cohomology of $\mathbb{S}^2 \times \mathbb{S}^5$, but we do not know if it is diffeomorphic to it. Furthermore, in [GZ3] it was shown that it is also the total space of the SO(3) principle bundle over $\mathbb{CP}^2$ with $w_2 \neq 0$ and $p_1 = 1$.

We finally come to the proof of Theorem D. Here we consider the cohomogeneity one manifolds with group diagram

$$H = \{e\} \subset \{\Delta\,\mathrm{S}^3,\ (e^{ipt}, e^{iqt})\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

where $\Delta\,\mathrm{S}^3$ is embedded diagonally and p, q are arbitrary relatively prime integers. Here we have $w_- = (-1,-1)$ and w_+ is one of $(\pm 1, \pm 1)$ and thus the normal geodesic has length $4L$. This implies that $G_{c(2L)} = G_{c(0)}$ and $G_{c(3L)} = G_{c(L)}$. Here it is convenient to choose the base point t_0 to be regular in which case the Lagrange tensor satisfies $A_{t_0} = \mathrm{Id}$ and thus $X = A_t v$ with $v = X(t_0)$. A_t has two kernels, at $t = 0$ and at $t = L$ (which agree with the kernels at $2L$ and $3L$ resp): $\ker A_0 = \mathrm{span}\{X_1 + Y_1, X_2 + Y_2, X_3 + Y_3\}$ and $\ker A_L = \mathrm{span}\{pX_1 + qY_1\}$, all evaluated at t_0. If $(p, q) = (1, 1)$, clearly $\ker A_L \subset \ker A_0$. There exists a 2-dimensional subspace $W \subset E_{t_0}$ (3-dimensional if $(p, q) = (1, 1)$) which is orthogonal to both kernels. Thus g_W is concave for all t, and hence constant. By Theorem B, this implies that the Jacobi fields $X \in V$ with $X(t_0) \in W$ are parallel, and hence R vanishes on this subspace. In particular, R cannot be 2-positive. This finishes the proof of Theorem D

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