MULTIPLE EXPONENTIAL AND CHARACTER SUMS WITH MONOMIALS

IGOR E. SHPARLINSKI

ABSTRACT. We obtain new bounds of multivariate exponential sums with monomials, when the variables run over rather short intervals. Furthermore, we use the same method to derive estimates on similar sums with multiplicative characters to which previously known methods do not apply. In particular, in the multiplicative characters modulo a prime p we break the barrier of $p^{1/4}$ for ranges of individual variables.

1. Background

Let p be a prime number and let $n \ge 2$ be a fixed integer.

We also assume that we are given a system $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ of n complex-valued weights with

$$|\rho_j(x)| \le 1, \ x \in \mathbb{R}, \qquad j = 1, \ldots, n, \tag{1}$$

and an integer vector $\mathbf{e}$ with non-zero components:

$$\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n \qquad \text{and} \qquad e_1 \ldots e_n \ne 0. \tag{2}$$

We define the multiple exponential sums with monomials

$$S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B}) = \sum_{\substack{(x_1,\ldots,x_n)\in\mathcal{B} \\ x_1\ldots x_n \not\equiv 0 \pmod p}} \rho_1(x_1)\ldots\rho_n(x_n)\mathbf{e}_p(\lambda x_1^{e_1}\ldots x_n^{e_n}),$$

where $\lambda \in \mathbb{Z}$, over an n-dimensional cube

$$\mathcal{B} = [k_1+1, k_1+h] \times \ldots \times [k_n+1, k_n+h] \tag{3}$$

with a side length $h < p$ and some integers $k_1, \ldots, k_n$, where $\mathbf{e}_p(z) = \exp(2\pi z/p)$ and for a negative exponent e the inversion in x^e is taken modulo p.

The case when

$$\rho_j(x) = \exp(2\pi i \lambda_j x/p) \qquad \text{and} \qquad e_j = -1, \qquad j = 1, \ldots, n, \tag{4}$$

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for some integers $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$, corresponds to multivariate incomplete Kloosterman sums. In this case we denote these sums as

$$K_p(\boldsymbol{\lambda}; \mathcal{B}) = \sum_{\substack{(x_1,\ldots,x_n)\in\mathcal{B}\\ x_1\ldots x_n\not\equiv 0 \pmod p}} \mathbf{e}_p(\lambda x_1^{-1}\ldots x_n^{-1} + \lambda_1 x_1 + \ldots + \lambda_n x_n)$$

where $\boldsymbol{\lambda} = (\lambda, \lambda_1, \ldots, \lambda_n)$.

Using the Deligne bound [10], Luo [13] has shown that for (4) we have

$$K_p(\boldsymbol{\lambda}; \mathcal{B}) = O(p^{n-1} + p^{n/2}(\log p)^n), \qquad \gcd(\lambda, p) = 1.$$

Furthermore, using the Burgess bound of character sums (see [12, Theorem 12.6]), Luo [13, Theorem 1] has also given a bound on $K_p(\boldsymbol{\lambda}; \mathcal{B})$ which is nontrivial when $h \ge p^{1/4+\varepsilon}$, provided that n is large enough (see also [13, Theorem 2] that applies to the sums $S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$). This bound (but not the range of h) has been improved in [14] by bringing in an additional argument based on a result of Ayyad, Cochrane and Zheng [1, Theorem 2]. Unfortunately, the improvement claimed in [15] has never been justified (as the proof of [15, Lemma 1] is invalid). Note that the method of [15] may still work for the cube at the origin, that is, for $k_1 = \ldots = k_n = 0$. Bourgain and Garaev [6] have recently obtained a series of estimates of Kloosterman sums over much smaller boxes, for example, for boxes with the side length $h \ge p^{c/n^2}$ for some absolute constant $c > 0$.

Finally, we note that the sums $S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$ can be treated as general multilinear sums using the results of Bourgain [2, Theorems 3 and 5] and [3, Theorem 3] and of Garaev [11, Theorem 4.1]. For example, if $\rho_j(x) = 1$ for $x \in \mathbb{R}$, $j = 1, \ldots, n$, then [11, Theorem 4.1] implies a nontrivial estimate

$$\sum_{\substack{(x_1,\ldots,x_n)\in\mathcal{B}\\ x_1\ldots x_n\not\equiv 0 \pmod p}} \mathbf{e}_p(\lambda x_1^{e_1}\ldots x_n^{e_n}) = O\left(h^n p^{-\delta}\right)$$

if $h > p^{81/(n+160)+\varepsilon}$ for some fixed $\varepsilon > 0$, where $\delta > 0$ depends on ε and n. It is quite possible that this bound can be extended to the sums $K_p(\boldsymbol{\lambda}; \mathcal{B})$ and even to $S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$.

Here we show that the results of [8] can be used to get better and fully explicit estimates for the sums $S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$ for almost all primes p. Moreover, for $4 \le n \le 7$, we obtain new bounds that hold for all primes p. These bounds enable us to estimate sums over a large number of variables, however unfortunately this larger number of variables does not bring any additional gains, see, however, a discussion in Section 5.

Using the same approach, we also derive similar bounds for the sums

$$T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B}) = \sum_{\substack{(x_1,\ldots,x_n)\in\mathcal{B}\\ x_1\ldots x_n\not\equiv 0 \pmod p}} \rho_1(x_1)\ldots\rho_n(x_n)\chi(x_1^{e_1}\ldots x_n^{e_n}+\lambda),$$

with a multiplicative character χ modulo p and $\lambda \in \mathbb{Z}$. Note that the methods of [2, 3, 11] do not seem to apply to such sums.

It is interesting to note that in many cases our results go beyond the restriction $h \ge p^{1/4}$, which is associated with the Burgess bound (see [12, Theorem 12.6]). For exponential sums $S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$, in the case of positive exponents $e_1, \ldots, e_n$, one can apply general bounds for exponential sums with polynomials obtained via the method of Vinogradov, see [16], which allows to study rather short sums with general polynomials. Furthermore, even shorter exponential sums (with positive and negative exponents $e_1, \ldots, e_n$) can be estimated via the methods of additive combinatorics, see [2, 3, 11]. However, for the sums of multiplicative characters, these approaches do not apply. Thus this work seems to be the first example where the Burgess barrier of $p^{1/4}$ for the range of individual variables in multiplicative character sums has been broken, see [4].

Throughout, any implied constants in the symbols O and $\ll$ may occasionally depend, where obvious, on the integer n, the vector $\mathbf{e}$, and the integer parameters r and ν, but are absolute otherwise. We recall that the notations $A \ll B$ and $A = O(B)$ are both equivalent to the statement that the inequality $|A| \le c\,B$ holds with some constant $c > 0$.

2. Congruences with products and character sums

For a prime p and integers $h \ge 3$, $\nu \ge 1$ and k, we denote by $I_{p,\nu}(h,k)$ the number of solutions of the congruence

$$\begin{aligned}(x_1+k)\ldots(x_\nu+k) \equiv &(y_1+k)\ldots(y_\nu+k) \not\equiv 0 \pmod p,\\ 1 \le x_j, y_j &\le h_j, \qquad j = 1,\ldots,\nu.\end{aligned}$$

We now define $d_2 = 2$ and

$$d_\nu = \max\{\nu^2 - 2\nu - 2, \nu^2 - 3\nu + 4\}, \qquad \nu = 3, 4, \ldots.$$

We need the following estimate which for $\nu = 2$ is a special case of [1, Theorem 1] and for $\nu \ge 3$ follows from [7, Theorem 17].

Lemma 1. *Let $\nu \ge 2$ be a fixed integer. Then for any integers k and $h < p$ we have*

$$I_{p,\nu}(h,k) \le \left(h^\nu + h^{2\nu}p^{-\nu/d_\nu}\right) h^{o(1)}.$$

We also need the following estimate from [8]:

Lemma 2. *Let $\nu \ge 1$ be a fixed integer. Then for a sufficiently large positive integers $T \ge h \ge 3$, for all but $o(T/\log T)$ primes $p \le T$ and any integers k and $h < p$, we have the bound*

$$I_{p,\nu}(h,k) \le \left(h^{\nu} + h^{2\nu-1/2}p^{-1/2}\right) h^{o(1)}.$$

Furthermore, sometimes we need to estimate the number of solutions to a more general congruence. For a prime p, an integer $\nu \ge 1$, and vectors

$$\begin{aligned} \mathbf{h} &= (h_1, \ldots, h_\nu) \in \mathbb{N}^\nu, \\ \mathbf{k} &= (k_1, \ldots, k_\nu) \in \mathbb{Z}^\nu, \\ \mathbf{e} &= (e_1, \ldots, e_\nu) \in Z^\nu, \end{aligned}$$

we denote by $J_{p,\nu}(\mathbf{e},\mathbf{h},\mathbf{k})$ the number of solutions of the congruence

$$\begin{gathered} (x_1+k_1)^{e_1}\ldots(x_\nu+k_\nu)^{e_\nu} \equiv (y_1+k_1)^{e_1}\ldots(y_\nu+k_\nu)^{e_\nu} \not\equiv 0 \pmod p, \\ 1 \le x_j, y_j \le h_j, \qquad j = 1, \ldots, \nu. \end{gathered}$$

The following result, which is a slight generalisation of a similar statement from [7], relates $J_{p,\nu}(\mathbf{e},\mathbf{h},\mathbf{s})$ and $I_{p,\nu}(h,k_j)$, $j = 1,\ldots,\nu$. Here we always have $h_1 = \ldots = h_\nu = h$, however we record this simple inequality in full generality.

Lemma 3. *We have*

$$J_{p,\nu}(\mathbf{e},\mathbf{h},\mathbf{k}) \le \prod_{j=1}^{\nu} I_{p,\nu}(h_j,k_j)^{1/\nu}.$$

Proof. Using the orthogonality of multiplicative characters, we write

$$J_{p,\nu}(\mathbf{e},\mathbf{h},\mathbf{s}) = \frac{1}{p-1} \sum_{\substack{1\le x_1,y_1\le h_1 \\ x_1,y_1\not\equiv -k_1 \pmod p}} \ldots \sum_{\substack{1\le x_\nu,y_\nu\le h_\nu \\ x_\nu,y_\nu\not\equiv -k_\nu \pmod p}} \sum_{\chi} \chi\left(\prod_{j=1}^{\nu} \frac{(x_j+k_j)^{e_j}}{(y_j+k_j)^{e_j}}\right),$$

where χ runs through all multiplicative characters modulo p.

Since $\chi(z^{-1}) = \overline{\chi}(z)$, we obtain

$$J_{p,\nu}(\mathbf{e},\mathbf{h},\mathbf{s}) = \frac{1}{p-1}\sum_{\chi}\prod_{j=1}^{\nu}\left|\sum_{1\le x_j\le h_j}\chi^{e_j}(x_j+k_j)\right|^2, \tag{5}$$

(we also define $\chi(0) = \chi^{-1}(0) = 0$). Using the Hölder inequality, we obtain

$$J_{p,\nu}(\mathbf{e}, \mathbf{h}, \mathbf{s})^{\nu} \le \frac{1}{p-1} \prod_{j=1}^{\nu} \sum_{\chi} \left| \sum_{1 \le x_j \le h_j} \chi^{e_j}(x_j + k_j) \right|^{2\nu}.$$

For any integer $e \neq 0$, when χ runs runs through all multiplicative characters modulo p, the character χ^e takes the same value no more than $\gcd(e, p-1) \le |e|$ times. Therefore

$$J_{p,\nu}(\mathbf{e}, \mathbf{h}, \mathbf{s})^{\nu} \le \frac{1}{p-1} \prod_{j=1}^{\nu} |e_j| \sum_{\chi} \left| \sum_{1 \le x_j \le h_j} \chi(x_j + k_j) \right|^{2\nu}. \tag{6}$$

Similarly to (5), we also obtain

$$I_{p,\nu}(h, k) = \frac{1}{p-1} \sum_{\chi} \left| \sum_{1 \le x \le h} \chi(x + k) \right|^{2\nu},$$

which together with (6) implies the desired result. □

We also need the following well-known result which is slight generalisation of the classical result of Davenport and Erdős [9], which in turn follows from the Weil bound of multiplicative character sums, see [12, Theorem 11.23].

Lemma 4. *For any non-trivial multiplicative character χ modulo p, integers k and $h < p$, any complex-valued weights with*

$$|\rho(x)| \le 1, \quad x \in [k+1, k+h],$$

and integer λ with $\gcd(\lambda, p) = 1$, we have

$$\sum_{u=1}^{p-1} \left| \sum_{x=k+1}^{k+h} \rho(x)\chi(ux + \lambda) \right|^{2r} \ll \begin{cases} hp & \text{if } r = 1, \\ h^r p + h^{2r} p^{1/2} & \text{if } r = 2, 3, \ldots. \end{cases}$$

3. Bounds for all primes

We now derive a nontrivial bound on the sums $|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})|$ for $4 \le n \le 7$. The proof is based on Lemma 1, which unfortunately is not strong enough to produce a nontrivial result for $n \ge 8$.

Theorem 5. *For any prime p, integer $h \le p$, cube $\mathcal{B}$ of the form (3), weights $\boldsymbol{\rho}$ with (1), vector $\mathbf{e}$ with (2) and integer λ with $\gcd(\lambda, p) = 1$,* *we have*

$$|S_p(\lambda,\boldsymbol{\rho},\mathbf{e};\mathcal{B})| \le h^{o(1)}\cdot \begin{cases} h^2p^{1/2}+h^4p^{-1/2} & \text{if } n=4,\\ h^{5/2}p^{1/2}+h^4p^{1/8}+h^5p^{-3/8} & \text{if } n=5,\\ h^3p^{1/2}+h^6p^{-1/4} & \text{if } n=6,\\ h^{11/2}p^{1/4}+h^7p^{-1/8} & \text{if } n=7.\end{cases}$$

Proof. Let $s=\lfloor n/2\rfloor$ and $t=n-s=\lceil n/2\rceil$. We define

$$\mathcal{B}_1=[k_1+1,k_1+h]\times\ldots\times[k_s+1,k_s+h],$$
$$\mathcal{B}_2=[k_{s+1}+1,k_{s+1}+h]\times\ldots\times[k_n+1,k_n+h],$$

and write

$$\eta_1(u)=\sum_{\substack{(x_1,\ldots,x_s)\in\mathcal{B}_1\\ x_1^{e_1}\ldots x_s^{e_s}\equiv u \pmod p}}\rho_1(x_1)\ldots\rho_s(x_s),$$
$$\eta_2(v)=\sum_{\substack{(x_{s+1},\ldots,x_n)\in\mathcal{B}_2\\ x_{s+1}^{e_{s+1}}\ldots x_n^{e_n}\equiv v \pmod p}}\rho_{s+1}(x_{s+1})\ldots\rho_n(x_n).$$

Therefore

$$S_p(\lambda,\boldsymbol{\rho},\mathbf{e};\mathcal{B})=\sum_{u,v=1}^{p-1}\eta_1(u)\eta_2(v)\mathbf{e}_p(\lambda uv). \tag{7}$$

Recalling (1) and using Lemmas 1 and 3, we obtain

$$\sum_{u=1}^{p-1}\eta_1(u)^2\le\left(h^s+h^{2s}p^{-s/d_s}\right)h^{o(1)},$$
$$\sum_{v=1}^{p-1}\eta_2(v)^2\le\left(h^t+h^{2t}p^{-t/d_t}\right)h^{o(1)},$$

where the integers d_ν are defined in Section 2.

We now use the standard method of estimating bilinear sums via the Cauchy inequality, see, for example, [11, Lemma 4.1] and derive from (7) that

$$\begin{aligned}|S_p(\lambda,\boldsymbol{\rho},\mathbf{e};\mathcal{B})| &\le \left(p\sum_{u=1}^{p-1}\eta_1(u)^2\sum_{v=1}^{p-1}\eta_2(v)^2\right)^{1/2}\\ &\le p^{1/2}\left(h^s+h^{2s}p^{-s/d_s}\right)^{1/2}\left(h^t+h^{2t}p^{-t/d_t}\right)^{1/2}h^{o(1)}\\ &= p^{1/2}\left(h^n+h^{n+s}p^{-s/d_s}+h^{n+t}p^{-t/d_t}+h^{2n}p^{-s/d_s-t/d_t}\right)^{1/2}h^{o(1)}.\end{aligned} \tag{8}$$

Note that for $n \le 7$ we have $s \le t \le 4$ and also

$$\frac{2}{d_2} = 1 > \frac{3}{d_3} = \frac{3}{4} > \frac{4}{d_4} = \frac{1}{2}.$$

Thus for $n \le 7$ we have $h^s p^{-s/d_s} \le h^t p^{-t/d_t}$ and the bound (8) simplifies as

$$|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le p^{1/2} \left(h^n + h^{n+t} p^{-t/d_t} + h^{2n} p^{-s/d_s - t/d_t}\right)^{1/2} h^{o(1)}.$$

Now, for $n = 4$ we obtain

$$\begin{aligned} |S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| &\le p^{1/2} \left(h^4 + h^6 p^{-1} + h^8 p^{-2}\right)^{1/2} h^{o(1)} \\ &\le p^{1/2} \left(h^4 + h^8 p^{-2}\right)^{1/2} h^{o(1)} = h^{2+o(1)} p^{1/2} + h^{4+o(1)} p^{-1/2}, \end{aligned} \tag{9}$$

as the middle term never dominates.

Similarly, for $n = 5$, we obtain

$$\begin{aligned} |S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| &\le p^{1/2} \left(h^5 + h^8 p^{-3/4} + h^{10} p^{-7/4}\right)^{1/2} h^{o(1)} \\ &\le h^{5/2+o(1)} p^{1/2} + h^{4+o(1)} p^{1/8} + h^{5+o(1)} p^{-3/8}. \end{aligned} \tag{10}$$

For $n = 6$, we see again that the middle term never dominates, so

$$\begin{aligned} |S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| &\le p^{1/2} \left(h^6 + h^{12} p^{-3/2}\right)^{1/2} h^{o(1)} \\ &= h^{3+o(1)} p^{1/2} + h^{6+o(1)} p^{-1/4}. \end{aligned} \tag{11}$$

Finally, for $n = 7$, we derive

$$\begin{aligned} |S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| &\le p^{1/2} \left(h^7 + h^{11} p^{-1/2} + h^{14} p^{-5/4}\right)^{1/2} h^{o(1)} \\ &\le h^{7/2+o(1)} p^{1/2} + h^{11/2+o(1)} p^{1/4} + h^{7+o(1)} p^{-1/8}. \end{aligned}$$

Note that for $h \le p^{1/6}$ we have $h^{11/2} p^{1/4} \le h^7$, thus the above bound is trivial. On the other hand, for $h > p^{1/6}$ we have $h^{7/2} p^{1/2} < h^{11/2} p^{1/4}$. Hence, we derive

$$|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le h^{11/2+o(1)} p^{1/4} + h^{7+o(1)} p^{-1/8}. \tag{12}$$

Collecting the bounds (9), (10), (11) and (12) we obtain the desired result. □

It is easy to see that Theorem 5 is nontrivial for any $\varepsilon > 0$ and

$$h \ge \begin{cases} p^{1/4+\varepsilon} & \text{if } n = 4, \\ p^{1/5+\varepsilon} & \text{if } n = 5, \\ p^{1/6+\varepsilon} & \text{if } n = 6, \\ p^{1/6+\varepsilon} & \text{if } n = 7. \end{cases} \tag{13}$$

It is easy to see that a full analogue of Theorem 5 also holds for the sums $T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})$.

Theorem 6. *For any prime p, integer $h \le p$, cube $\mathcal{B}$ of the form* (3)*, weights $\boldsymbol{\rho}$ with* (1)*, vector $\mathbf{e}$ with* (2)*, nontrivial multiplicative characters χ modulo p, and integer λ with* $\gcd(\lambda, p) = 1$*, we have*

$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le h^{o(1)} \cdot \begin{cases} h^2 p^{1/2} + h^4 p^{-1/2} & \text{if } n = 4, \\ h^{5/2} p^{1/2} + h^4 p^{1/8} + h^5 p^{-3/8} & \text{if } n = 5, \\ h^3 p^{1/2} + h^6 p^{-1/4} & \text{if } n = 6, \\ h^{11/2} p^{1/4} + h^7 p^{-1/8} & \text{if } n = 7. \end{cases}$$

We now show that for sums of multiplicative characters one can derive yet another estimate.

Theorem 7. *For any prime p, integer $h \le p$, cube $\mathcal{B}$ of the form* (3)*, weights $\boldsymbol{\rho}$ with* (1)*, vector $\mathbf{e}$ with* (2)*, nontrivial multiplicative characters χ modulo p, and integer λ with* $\gcd(\lambda, p) = 1$*,*

- *if $n = 3$ then*

$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le h^{o(1)} \cdot \begin{cases} h^{5/2} & \text{if } p > h \ge p^{1/2}, \\ h^{3/2} p^{1/2} & \text{if } p^{1/2} > h \ge p^{3/8}, \\ h^{5/2} p^{1/8} & \text{if } p^{3/8} > h \ge p^{1/4}, \end{cases}$$

- *if $n = 4$ then*

$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le h^{o(1)} \cdot \begin{cases} h^4 p^{-1/4} & \text{if } p > h \ge p^{1/2}, \\ h^2 p^{1/2} & \text{if } p^{1/2} > h \ge p^{9/32}, \\ h^4 p^{-1/16} & \text{if } p^{9/32} > h \ge p^{1/4}, \\ h^{11/4} p^{1/4} & \text{if } p^{1/4} > h \ge p^{2/9}, \\ h^{7/2} p^{1/12} & \text{if } p^{2/9} > h \ge p^{1/6}. \end{cases}$$

Proof. We define

$$\mathcal{B}_0 = [k_1 + 1, k_1 + h] \times \ldots \times [k_{n-1} + 1, k_{n-1} + h]$$

and write

$$\eta_0(u) = \sum_{\substack{(x_1, \ldots, x_{n-1}) \in \mathcal{B}_0 \\ x_1^{e_1} \ldots x_{n-1}^{e_{n-1}} \equiv u \pmod p}} \rho_1(x_1) \ldots \rho_{n-1}(x_{n-1}).$$

Therefore

$$\begin{aligned} T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B}) &= \sum_{u=1}^{p-1} \eta_0(u) \sum_{x=k_n+1}^{k_n+h} \rho_n(x_n) \chi(ux + \lambda) \\ &= \sum_{u=1}^{p-1} \left(\eta_0(u)^2\right)^{1/2r} (\eta_0(u))^{(r-1)/r} \sum_{x=k_n+1}^{k_n+h} \rho_n(x) \chi(ux + \lambda). \end{aligned}$$

Thus, by the Hölder inequality

$$
\begin{aligned}
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2r} = &\sum_{u=1}^{p-1}\eta_0(u)^2\left(\sum_{u=1}^{p-1}\eta_0(u)\right)^{2r-2} \\
&\sum_{u=1}^{p-1}\left|\sum_{x=k_n+1}^{k_n+h}\rho_n(x_n)\chi(ux+\lambda)\right|^{2r}.
\end{aligned} \tag{14}
$$

Clearly

$$
\sum_{u=1}^{p-1}\eta_0(u) = h^{n-1}.
$$

Recalling (1) and using Lemmas 1 and 3, we obtain

$$
\sum_{u=1}^{p-1}\eta_0(u) \le \left(h^{n-1} + h^{2n-2}p^{-(n-1)/d_{n-1}}\right)h^{o(1)}.
$$

Substituting the above bounds in (14), and using Lemma 4 with $r=1$ we obtain

$$
\begin{aligned}
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2} &\le \left(h^{n-1} + h^{2n-2}p^{-(n-1)/d_{n-1}}\right)h^{1+o(1)}p \\
&\le \left(h^{n}p + h^{2n-1}p^{1-(n-1)/d_{n-1}}\right)h^{o(1)}.
\end{aligned}
$$

Similarly, we derive from Lemma 4 that for $r=2,3,\ldots$ we have

$$
\begin{aligned}
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2r} &\le \left(h^{n-1} + h^{2n-2}p^{-(n-1)/d_{n-1}}\right)h^{(n-1)(2r-2)+o(1)} \\
&\qquad\qquad \left(h^{r}p + h^{2r}p^{1/2}\right) \\
&= \left(h^{(n-1)(2r-1)} + h^{2(n-1)r}p^{-(n-1)/d_{n-1}}\right)\left(h^{r}p + h^{2r}p^{1/2}\right)h^{o(1)} \\
&= (h^{2nr-n-r+1}p + h^{(2n-1)r}p^{1-(n-1)/d_{n-1}} \\
&\qquad + h^{2nr-n+1}p^{1/2} + h^{2nr}p^{1/2-(n-1)/d_{n-1}})h^{o(1)}.
\end{aligned}
$$

Now, for $n=3$ we obtain

$$
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2} \le \left(h^{3}p + h^{5}\right)h^{o(1)} \tag{15}
$$

if $r=1$, while for $r=2,3,\ldots$ we have

$$
\begin{aligned}
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2r} &\le \left(h^{5r-2}p + h^{5r} + h^{6r-2}p^{1/2} + h^{6r}p^{-1/2}\right)h^{o(1)} \\
&\le \left(h^{5r-2}p + h^{6r-2}p^{1/2} + h^{6r}p^{-1/2}\right)h^{o(1)},
\end{aligned} \tag{16}
$$

as the second term never dominates. We now use (15) for $h \ge p^{3/8}$ and use (16) with $r=2$ for $h \ge p^{3/8}$, getting the desired result for $n=3$.

Similarly, for $n=4$ we obtain

$$
|T_p(\lambda,\chi,\boldsymbol{\rho},\mathbf{e};\mathcal{B})|^{2} \le \left(h^{4}p + h^{7}p^{1/4}\right)h^{o(1)}h^{o(1)} \tag{17}
$$

if $r = 1$, while for $r = 2, 3, \ldots$ we obtain
(18)
$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})|^{2r} \le \left(h^{7r-3}p + h^{7r}p^{1/4} + h^{8r-3}p^{1/2} + h^{8r}p^{-1/4}\right) h^{o(1)}.$$

We now notice that (17) is always weaker that the bound of Theorem 6 (for $n = 4$), which we use for $h \ge p^{9/32}$. We also use (18) with $r = 3$ for $h < p^{2/9}$ we also use (18) with $r = 2$ for $p^{2/9} \le h < p^{9/32}$, getting the desired result for $n = 4$. $\square$

Clearly, Theorem 6 is nontrivial under the condition (13), while Theorem 7 is nontrivial for

$$h \ge \begin{cases} p^{1/4+\varepsilon} & \text{if } n = 3, \\ p^{1/6+\varepsilon} & \text{if } n = 4. \end{cases}$$

4. Bounds for almost all primes

Theorem 8. *For a sufficiently large positive integer T, $h \ge 3$, for all but $o(T/\log T)$ primes $p \le T$, uniformly over positive integers $h \le p$, cubes $\mathcal{B}$ of the form* (3)*, weights $\boldsymbol{\rho}$ with* (1)*, vectors $\mathbf{e}$ with* (2) *and integers λ with $\gcd(\lambda, p) = 1$, we have*

$$|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le \left(h^{n/2}p^{1/2} + h^{n/2+\lceil n/2\rceil/2-1/4}p^{1/4} + h^{n-1/2}\right) h^{o(1)}$$

Proof. Let p be one of the non-exceptional primes for which the bound of Lemma 2 holds.

We set $s = \lfloor n/2 \rfloor$ and $t = n - s = \lceil n \rceil$ and define $\eta_1(u)$ and $\eta_2(v)$ as in the proof of Theorem 5. Recalling (1) and using Lemmas 2 and 3, we obtain

$$\sum_{u=1}^{p-1} \eta_1(u)^2 \le \left(h^s + h^{2s-1/2}p^{-1/2}\right) h^{o(1)},$$
$$\sum_{v=1}^{p-1} \eta_2(v)^2 \le \left(h^t + h^{2t-1/2}p^{-1/2}\right) h^{o(1)}.$$

We now apply the standard method of estimating bilinear sums via the Cauchy inequality, see, for example, [11, Lemma 4.1] and derive from (7) that

$$\begin{aligned}|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le & \left(p \sum_{u=1}^{p-1} \eta_1(u)^2 \sum_{v=1}^{p-1} \eta_2(v)^2 \right)^{1/2} \\ \le & p^{1/2} \left(h^s + h^{2s-1/2}p^{-1/2}\right)^{1/2} \left(h^t + h^{2t-1/2}p^{-1/2}\right)^{1/2} h^{o(1)} \\ = & p^{1/2} \left(h^n + h^{n+t-1/2}p^{-1/2} + h^{n+s-1/2}p^{-1/2} + h^{2n-1}p^{-1}\right)^{1/2} h^{o(1)} \\ = & p^{1/2} \left(h^n + h^{n+t-1/2}p^{-1/2} + h^{2n-1}p^{-1}\right)^{1/2} h^{o(1)},\end{aligned}$$

and the result now follows. □

It is easy to see that for an even n the middle term in the bound of Theorem 8 never dominates and the bound simplifies as

$$|S_p(\lambda, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le \left(h^{n/2}p^{1/2} + h^{n-1/2}\right) h^{o(1)}.$$

Clearly, Theorem 8 is nontrivial provided that $h \ge p^{1/n+\varepsilon}$ for some fixed $\varepsilon > 0$.

Furthermore, as before, we also have an analogue of Theorem 8 for the sums of multiplicative characters.

Theorem 9. *For a sufficiently large positive integer T, $h \ge 3$, for all but $o(T/\log T)$ primes $p \le T$, uniformly over positive integers $h \le p$, cubes $\mathcal{B}$ of the form* (3)*, weights $\boldsymbol{\rho}$ with* (1)*, vectors $\mathbf{e}$ with* (2)*, nontrivial multiplicative characters χ modulo p, and integers λ with* $\gcd(\lambda, p) = 1$*, we have*

$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le \left(h^{n/2}p^{1/2} + h^{n/2+\lceil n/2\rceil/2-1/4}p^{1/4} + h^{n-1/2}\right) h^{o(1)}.$$

Finally, using Lemma 2 instead of Lemma 1 in the proof of Theorem 7, we obtain:

Theorem 10. *For a sufficiently large positive integer T, $h \ge 3$, for all but $o(T/\log T)$ primes $p \le T$, uniformly over positive integers $h \le p$, cubes $\mathcal{B}$ of the form* (3)*, weights $\boldsymbol{\rho}$ with* (1)*, vectors $\mathbf{e}$ with* (2)*, nontrivial multiplicative characters χ modulo p, and integers λ with* $\gcd(\lambda, p) = 1$*, we have*

$$\begin{aligned}|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le (& h^{n-1/2-(n-1)/2r}p^{1/2r} + h^{n-1/2-1/4r}p^{1/4r} \\ & + h^{n-(n-1)/2r}p^{1/4r} + h^{n-1/4r})h^{o(1)}\end{aligned}$$

for $r = 2, 3 \ldots$.

Clearly one can also obtain a version of Theorem 10 with $r = 1$, that is,

$$|T_p(\lambda, \chi, \boldsymbol{\rho}, \mathbf{e}; \mathcal{B})| \le (h^{n/2}p^{1/2} + h^{n-3/4}p^{1/4})h^{o(1)}.$$

However it is always weaker than the bound of Theorem 9.

One easily verifies that Theorems 9 and 10 are nontrivial provided that $h \ge p^{1/n+\varepsilon}$ and $h \ge p^{1/2(n-1)+\varepsilon}$, respectively, for some fixed $\varepsilon > 0$.

5. Comments

It is easy to see that the implied constants depend only on $\gcd(p-1, e_i)$ rather on e_i, $i = 1, \ldots, n$. Thus our results remain nontrivial even for very large values of $e_1, \ldots, e_n$, provided that the corresponding greatest common divisors are small.

Furthermore, the method of this work can certainly be adjusted to apply to sums over more general boxes

$$\mathcal{B} = [k_1 + 1, k_1 + h_1] \times \ldots \times [k_n + 1, k_n + h_n]$$

with distinct sides. However finding an optimal choice of parameters can be quite technically cluttered in this case.

Clear the bounds of Theorems 5 and 6 and of Theorem 7 also apply to sums with $n \ge 8$ and $n \ge 5$ variables, respectively. Simply, for each choice of "unused" variables, we use one of these bounds. However, it is natural to expect that using more variables may lead to stronger bounds over smaller cubes $\mathcal{B}$. For example, one can try to split all variables into three groups and the replace the estimate (8) with an appropriate variant of the bound of Bourgain and Garaev [5, Theorem 1.2] of trilinear sums. More precisely, one needs a bound on trilinear sums with weights bounded in L_2-norm rather than in L_∞-norm as in [5], see also [2, 3].

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Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052, Australia

E-mail address: `igor.shparlinski@unsw.edu.au`