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Lie-algebraic symmetries of generalized Davey-Stewartson equations

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Abstract. We identify the full Lie-algebraic structure of the generalized Davey-Stewartson (GDS) system of equations with symmetries of a specific of continual Lie algebras. In particular, we show that they are related to two copies of the Poisson bracket continual Lie algebra.

1. Generalized Davey-Stewartson equations

A system of nonlinear partial differential equations in 2+1 dimensions as a model of wave propagation in a bulk medium of an elastic material with couple stresses has recently been derived in [5], namely

$$\begin{aligned} & i\psi_t + \delta\psi_{xx} + \psi_{yy} = \chi|\psi|^2\psi + \gamma(w_x + \phi_y)\psi \\ & w_{xx} + n\phi_{xy} + m_2 w_{yy} = (|\psi|^2)_x \\ & n w_{xy} + \lambda\phi_{xx} + m_1\phi_{yy} = (|\psi|^2)_y \end{aligned} \tag{1.1}$$

with the condition $(\lambda - 1)(m_1 - m_2) = n^2$. The subscripts in (1.1) denote corresponding partial derivatives. Here $\psi(t, x, y)$ is a complex function, $w(t, x, y)$ and $\phi(t, x, y)$ are real functions, and δ, n, m_1,m_2, λ, χ, γ are real constants. The authors of [5] demonstrated that when the parameters are related by $n = 1-\lambda = m_1-m_2$, then (1.1) can be reduced to the standard Davey-Stewartson (DS) equations (in general not integrable) by a non-invertible point transformation of dependent variables. Therefore, they called (1.1) the generalized Davey-Stewartson (GDS) equations. In [5] some travelling type solutions of (1.1) in terms of elementary and elliptic functions were obtained. Global existence results were studied in [6]. In another recent work [7], under some constraints on the physical parameters, the so-called hyperbolic-elliptic-elliptic case of the system (1.1) (in [7] the system is classified into different types according to the signs of parameters $(\delta, m_1, m_2, \lambda)$) was shown to admit singular solutions that blow up in a finite time.

In [2] authors work with a differential form of the equations (1.1). Differentiating the last two equations of (1.1) by x and y respectively, performing the substitutions $w_x \to w$, $\phi_y \to \phi$, and rewriting the corresponding system in a real form by separating $\psi = u + iv$ into real and imaginary parts, one obtains a system of four real partial differential equations

$$\begin{aligned} u_t + \delta v_{xx} + v_{yy} &= \chi v(u^2 + v^2) + \gamma v(w + \phi) \\ -v_t + \delta u_{xx} + u_{yy} &= \chi u(u^2 + v^2) + \gamma u(w + \phi) \\ w_{xx} + n\phi_{xx} + m_2 w_{yy} &= 2(u_x^2 + uu_{xx} + v_x^2 + vv_{xx}) \\ n w_{yy} + \lambda\phi_{xx} + m_1\phi_{yy} &= 2(u_y^2 + uu_{yy} + v_y^2 + vv_{yy}) \end{aligned} \tag{1.2}$$

In the sequel, following [2], we call (1.2) the GDS (generalized Davey-Stewartson) equations.

In [2] the Lie symmetry algebra of the generalized Davey-Stewartson (GDS) equations (1.2) was computed. In particular, it was shown that for special choice of parameters it is a centerless Kac-Moody-Virasoro algebra. It was also shown that under certain conditions imposed on parameters in the system is infinite-dimensional and isomorphic to that of the standard integrable Davey-Stewartson equations which is known to have a very specific Kac-Moody-Virasoro loop algebra structure. The main result of the paper [2] was to show that, when some conditions on physical parameters δ, n, m_1, m_2, λ are imposed, the Lie algebra of the symmetry group of the GDS system has a Kac-Moody-Virasoro (KMV) loop structure (same as for the usual integrable DS equations [8]). In particular, they show in that this algebra is isomorphic to that of DS equations:

$$i\psi_t + \delta_1\psi_{xx} + \psi_{yy} = \delta_2|\psi|^2\psi + w\psi, \quad \varepsilon_1 w_{xx} + w_{yy} = \varepsilon_2(|\psi|^2)_{yy} \tag{1.3}$$

with $\delta_1 = \pm 1$, $\delta_2 = \pm 1$. The Lie algebra of the symmetry group of the integrable DS system is referred to as the DS algebra.

The purpose of this note is some further study of GDS equations from group theoretical point of view. We show that the system (1.2) possesses a conitnual Lie algebra symmetry. In particular, the systems of generators of corresponding Lie algebras is a special case of a continual Lie algebra [18]. Finally, let us also mention that the GDS is related to the generalized Alber's equation discissed in [1, 22].

2. Continual Lie algebra symmetries of GDS system

In appendix we recall the definition of a continual Lie algebra. Two basic examples relevant to the Lie algebra of symmetries describing the GDS equations are the Witt and Poisson bracket continual Lie algebras [18] are as follows.

2.1. Witt algebra

The Witt algebra [9] is the centerless Virasoro algebra. The commutation relation on the single generator $X(\phi)$, for two ϕ, $\psi \in \mathcal{E}$, are

$$[X(\phi), X(\psi)] = X(\phi\partial\psi - \psi\partial\phi) \equiv X\left(\left[\phi,^{\partial} \psi\right]\right), \quad K(\phi,\psi) = \phi\partial\psi - \psi\partial\phi \tag{2.1}$$

where ∂ denotes the differentiation with respect to a real parameter with obvious notation after the last equality in (2.1). The only condition that the mapping $K(\phi,\psi)$ satisfies is (A.2) with $k = m = n = 0$.

2.2. Poisson bracket algebra

As the second example we consider in this section is a continual Lie algebra defined by the mappings

$$K_{0,0}(\phi,\psi) = 0, \quad K_{0,\pm1}(\phi,\psi) = \mp i\partial\phi\cdot\psi, \quad K_{1,-1}(\phi,\psi) = -i\partial(\phi\cdot\psi) \tag{2.2}$$

and $K_{n,m}(\phi,\psi) = i(n\ \partial\psi\cdot\phi - m\ \partial\phi\cdot\psi)$, n, $m \in \mathbb{Z}$. In [16] it was proved that this continual Lie algebra is isomorphic to the Poisson bracket algebra under an appropriate substitution of variables.

2.3. Generators of the GDS continual Lie algebra
In [2] the commutation relations of the symmetry algebra of the system of equations of (1.2) were obtain. They have the following form:

$$[T(f_1), T(f_2)] = T(f_1 f_2' - f_1' f_2) \tag{2.3a}$$
$$[T(f), X(g)] = X(fg' - \frac{1}{2}f'g) \tag{2.3b}$$
$$[T(f), Y(h)] = Y(fh' - \frac{1}{2}f'h) \tag{2.3c}$$
$$[T(f), W(m)] = W(fm') \tag{2.3d}$$
$$[X(g_1), X(g_2)] = -\frac{1}{2\delta}W(g_1 g_2' - g_1' g_2) \tag{2.3e}$$
$$[Y(h_1), Y(h_2)] = -\frac{1}{2}W(h_1 h_2' - h_1' h_2) \tag{2.3f}$$
$$[X(g), Y(h)] = [X(g), W(m)] = [Y(h), W(m)] = [W(m_1), W(m_2)] = 0 \tag{2.3g}$$

Using the generators $T(f)$, $X(g)$, $Y(h)$, $W(m)$, let us take

$$X_0 = T(f), \quad X_1(g) = X(g), \quad X_{-1} = Y(h), \quad X_{-2} = W(m), \quad X_2 = \frac{1}{\delta}W(m) \tag{2.4}$$

Then it is easy to see that (2.4) form a a continual Lie algebra [16] with a non-abelian Cartan subalgebra. Then the corresponding kernels in the commutation relations (A.1) have the form

$$K_{0,0}(f_1, f_2) = f_1 f_2' - f_1' f_2 \tag{2.5a}$$
$$K_{0,1}(f, g) = fg' - \frac{1}{2}f'g \tag{2.5b}$$
$$K_{0,-1}(f, h) = fh' - \frac{1}{2}f'h \tag{2.5c}$$
$$K_{0,2}(f, m) = fm' \tag{2.5d}$$
$$K_{0,-2}(f, m) = \frac{1}{\delta}fm' \tag{2.5e}$$
$$K_{1,1}(g_1, g_2) = -\frac{1}{2\delta}(g_1 g_2' - g_1' g_2) \tag{2.5f}$$
$$K_{-1,-1}(h_1, h_2) = -\frac{1}{2}W(h_1 h_2' - h_1' h_2) \tag{2.5g}$$
$$K_{1,-1}(g, h) = K_{1,\pm 2}(g, h) = K_{-1,\pm 2}(g, h) = K_{2,\pm 2}(g, h) = 0 \tag{2.5h}$$

By substitution one checks that the mappings (2.5e) indeed satisfy to the consequences (A.2) of Jacobi identity. The Cartan part of this algebra is isomorphic to the Witt algebra (2.1).

A partial differential operator representation in variables (x, y, t) for the Lie algebra are given by (2.5e) was obtained in [2]:

$$X_0(f) = f(t)\partial_t + \frac{1}{2}f'(t)\left(x\partial_x + y\partial_y - u\partial_u - v\partial_v - 2w\partial_w - 2\phi\partial_\phi\right) - \frac{(x^2 + \delta y^2)}{8\delta}\left[f''(t)(v\partial_u - u\partial_v) + \frac{f'''(t)}{2\gamma}(\partial_w + \partial_\phi)\right] \tag{2.6a}$$
$$X(g) = g(t)\partial_x - \frac{x}{2\delta}\left[g'(t)(v\partial_u - u\partial_v) + \frac{g''(t)}{2\gamma}(\partial_w + \partial_\phi)\right] \tag{2.6b}$$
$$Y(h) = h(t)\partial_y - \frac{y}{2}\left[h'(t)(v\partial_u - u\partial_v) + \frac{h''(t)}{2\gamma}(\partial_w + \partial_\phi)\right] \tag{2.6c}$$

$$W(m) = m(t)(v\partial_u - u\partial_v) + \frac{m'(t)}{2\gamma}(\partial_w + \partial_\phi) \tag{2.6d}$$

The functions $g(t)$, $h(t)$, and $m(t)$ are arbitrary functions of class $C^\infty(I)$, $I \subseteq \mathbb{R}$.

In [2] the standard infinitesimal procedure [14] to find the symmetry algebra L and hence the symmetry group G of (1.2) was applied. The article [2] is mainly focused on the case when $f(t)$ is allowed to be arbitrary. The symmetry algebra realized by (2.6a) is then infinite-dimensional and has the structure of a Kac-Moody-Virasoro algebra [2]. This is called in [2] the GDS symmetry algebra and the corresponding system.

On the other hand, it is easy to see that the algebra defined by the mappings (2.5e) represents $m = 2$ generalized local part of two copies of the Poisson bracket continual Lie algebra (see appendix for the definitions). Let us identify

$$x_1(f) = iX_1(f), \quad x_{-1}(f) = iX_{-1}(f), \quad x_2(g) = \frac{1}{2\delta}W(g), \quad x_{-2}(g) = -W(g) \tag{2.7}$$

where x_i, $|i| \leq 2$, denote generators of the Poisson bracket algebra (2.2). Then we put

$$T(f) = X_0(f) = x_0(f) + \frac{1}{2}\tilde{x}_0(f) \tag{2.8}$$

where the generator $\widetilde{x}_0$ belongs to the second copy of the Poisson bracket continual Lie algebra defined by the Poisson bracket continual algebra mappings with inverted grading indices, i.e., $\widetilde{K}_{i,j} = K_{j,i}$. Then in commutation relations (A.1) one has, for example,

$$[X_0(f), X_1(g)] = \left[x_0(f) + \frac{1}{2}\tilde{x}_0(f),\ x_1(g)\right] = x_1\left(K_{0,1}(f,g) + \frac{1}{2}K_{1,0}(f,g)\right) \tag{2.9a}$$

$$= x_1\left(K_{0,1}(f,g) - \frac{1}{2}K_{0,1}(g,f)\right) = x_1\left(fg' - \frac{1}{2}gf'\right) \tag{2.9b}$$

and similarly for generators $x_{-1} = Y$, $x_2 = -\frac{1}{2\delta}W$ and $x_{-2} = W$, i.e., one can check that the relations (A.2) are also fulfilled.

Thus we can see that the full algebra describing the GDS system of equations is given by two copies of the Poisson bracket continual Lie algebra with "dual" mappings.

3. Conclusions

In this paper we have derived the continual Lie symmetries of the generalized Davey-Stewartson equations. Provided with the continual Lie algebraic formulation of the generalized DS system, we are able, in principle, to formulate the general algebraic procedure of solving such a system according to [12]. This will be a topic of a separate paper.

Appendix A. Continual Lie algebras

Let $\mathcal{E}$ be a vector space. A continual Lie algebra [11, 15, 16, 17] is generated by the generalized local part $\mathcal{G}^{m_0} = \oplus_{|n|\leq m_0}\mathcal{G}_n$, $\mathcal{G}_n = \{X_n(\phi), \phi \in \mathcal{E}\}$, $n \in \mathbb{Z}$, satisfying the defining relations for all $\phi, \psi \in \mathcal{E}$, and $|n|, |m|, |n+m| \leq m_0$,

$$[X_n(\phi), X_m(\psi)] = X_{n+m}(K_{n,m}(\phi,\psi)) \tag{A.1}$$

where $K_{n,m} : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, $n,m \in \mathbb{Z}$, are bilinear mappings. In analogy with classical case of discrete root space Lie algebra, we call $\mathcal{E}$ *the root space*. Ordinary Jacobi identity applied to elements $X_i(\phi)$ imply the following conditions on $K_{n,m}$:

$$K_{k,m+n}(\phi, K_{m,n}(\psi,\chi)) + K_{m,n+k}(\psi, K_{n,k}(\chi,\phi)) + K_{n,k+m}(\chi, K_{k,m}(\phi,\psi)) = 0 \tag{A.2}$$

$$K_{n,m}(\phi,\psi) = -K_{m,n}(\psi,\phi) \tag{A.3}$$

for all $\phi, \psi, \chi \in \mathcal{E}$, and $|l| \leq m_0$, where l denotes an index (or a sum of indeces) in (A.2). Then an infinite dimensional algebra $\mathcal{G}(\mathcal{E};K) = \mathcal{G}'(\mathcal{E};K)/J$ is called a continual contragredient Lie algebra where $\mathcal{G}'(\mathcal{E};K)$ is a Lie algebra freely generated by the minimal (in accordance with m_0) generalized local part $\mathcal{G}^{m_0}$, and J is the largest homogeneous ideal with trivial intersection with $\mathcal{G}_0$ (consideration of the quotient is equivalent to imposing the Serre relations in an ordinary Lie algebra case) [16, 17]. When $|m_0| \leq 1$, the commutation relations (A.1) have the form

$$[X_0(\phi), X_0(\psi)] = X_0(K_{0,0}(\phi,\psi)), \quad [X_0(\phi), X_{\pm1}(\psi)] = X_{\pm1}(K_{0,\pm1}(\phi,\psi)) \tag{A.4}$$
$$[X_1(\phi), X_{-1}(\psi)] = X_0(K_{1,-1}(\phi,\psi)) \tag{A.5}$$

for all ϕ, and $\psi \in \mathcal{E}$. Then conditions (A.2-A.3) reduce to

$$K_{0,0}(\phi,\psi) = -K_{0,0}(\psi,\phi) \tag{A.6a}$$
$$K_{\pm1}(K_{0,0}(\phi,\psi),\chi) = K_{\pm1}(\phi, K_{\pm1}(\psi,\chi)) - K_{\pm1}(\psi, K_{\pm}(\phi,\chi)) \tag{A.6b}$$
$$K_{0,0}(\psi, K_0(\phi,\chi)) = K_{1,-1}(K_1(\psi,\phi),\chi) + K_{1,-1}(\phi, K_{-1}(\psi,\chi)) \tag{A.6c}$$
$$K_{0,0}(\phi, K_{0,0}(\psi,\chi)) + K_{0,0}(\psi, K_{0,0}(\chi,\phi)) + K_{0,0}(\chi, K_{0,0}(\phi,\psi)) = 0 \tag{A.6d}$$

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