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THE ALGEBRAIC FUNDAMENTAL GROUP OF A REDUCTIVE GROUP SCHEME OVER AN ARBITRARY BASE SCHEME

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Abstract. We define the algebraic fundamental group $\pi_1(G)$ of a reductive group scheme G over an arbitrary non-empty base scheme and show that the resulting functor $G \mapsto \pi_1(G)$ is exact.

1. Introduction

If G is a (connected) reductive algebraic group over a field k of characteristic 0 and T is a maximal k-torus of G, the algebraic fundamental group $\pi_1(G,T)$ of the pair (G,T) was defined by the first-named author [1] and shown there to be independent (up to a canonical isomorphism) of the choice of T and useful in the study of the first Galois cohomology set of G. See Definition 3.11 below for a generalization of the original definition of $\pi_1(G,T)$. Independently, and at about the same time, Merkurjev [12, §10.1] defined the algebraic fundamental group of G over an arbitrary field. Later, Colliot-Thélène [4, Proposition-Definition 6.1] defined the algebraic fundamental group $\pi_1(G)$ of G in terms of a flasque resolution of G, showed that his definition was independent (up to a canonical isomorphism) of the choice of the resolution, and established the existence of a canonical isomorphism $\pi_1(G) \simeq \pi_1(G,T)$, see [4, Proposition A.2]. Recall that a *flasque resolution of* G is a central extension

$$1 \to F \to H \to G \to 1$$

where the derived group H^{der} of H is simply connected, $H^{\text{tor}} := H/H^{\text{der}}$ is a quasi-trivial k-torus, and F is a *flasque* k-torus, i.e., the group of cocharacters of F is an H^1-trivial Galois module. It turns out that flasque resolutions of reductive group schemes exist over bases that are more general than spectra of fields, and the second-named author has used such resolutions to generalize Colliot-Thélène's definition of $\pi_1(G)$ to reductive group schemes G over any non-empty, reduced, connected, locally Noetherian and geometrically unibranch scheme. See [9, Definition 3.7].

In the present paper we extend the definition of [9] to reductive group schemes G over an *arbitrary* non-empty scheme. Since flasque resolutions are not available in this general setting (see [9, Remark 2.3]), we shall use

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instead t-resolutions, which exist over any non-empty base scheme S. A *t-resolution* of G is a central extension

$$1 \to T \to H \to G \to 1,$$

where T is an S-torus and H is a reductive S-group scheme such that the derived group H^{der} is simply connected. Since a flasque resolution is a particular type of t-resolution, the definition of $\pi_1(G)$ given here (Definition 2.11) does indeed extend the definition of the second-named author [9]. Further, since the choice of a maximal S-torus of G (when one exists) canonically determines a t-resolution of G (see Lemma 3.9), our Definition 2.11 turns out to be a common generalization of the definitions of [1] and of [4] and [9].

Once the general definition of $\pi_1(G)$ is in place, we proceed to study some of the basic properties of the resulting functor $G \mapsto \pi_1(G)$, culminating in a proof of its exactness (Theorem 3.8). We give, in fact, two proofs of Theorem 3.8, the second of which makes use of the étale-local existence of maximal tori in reductive S-group schemes and generalizes [3, proof of Lemma 3.7]. In the final section of the paper we use t-resolutions to relate the (flat) abelian cohomology of G over S introduced in [8] to the cohomology of S-tori, thereby generalizing [9, §4].

Remark 1.1. Let G be a (connected) reductive group over the field of complex numbers $\mathbb{C}$. Here we comment on the interrelation between the algebraic fundamental group $\pi_1(G)$, the topological fundamental group $\pi_1^{\mathrm{top}}(G(\mathbb{C}))$, and the étale fundamental group $\pi_1^{\text{ét}}(G)$. By [1, Prop. 1.11] the algebraic fundamental group $\pi_1(G)$ is canonically isomorphic to the group

$$\pi_1^{\mathrm{top}}(G(\mathbb{C}))(-1) := \mathrm{Hom}(\pi_1^{\mathrm{top}}(\mathbb{C}^\times), \pi_1^{\mathrm{top}}(G(\mathbb{C})).$$

It follows that if G is a reductive k-group G over an algebraically closed field k of characteristic zero, then the profinite completion of $\pi_1(G)$ is canonically isomorphic to the group

$$\pi_1^{\text{ét}}(G)(-1) := \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\text{ét}}(\mathbb{G}_{m,k}), \pi_1^{\text{ét}}(G)).$$

where $\mathrm{Hom}_{\mathrm{cont}}$ denotes the group of continuous homomorphisms and $\mathbb{G}_{m,k}$ denotes the multiplicative group over k. See [2] for details and for a generalization of the algebraic fundamental group $\pi_1(G)$ to arbitrary homogeneous spaces of connected linear algebraic groups.

Notation and terminology. Throughout this paper, S denotes a non-empty scheme. An *S-torus* is an S-group scheme which is fpqc-locally isomorphic to a group of the form $\mathbb{G}_{m,S}^n$ for some integer $n \geq 0$ [6, Exp. IX, Definition 1.3]. An S-torus is affine, smooth and of finite presentation over S [6, Exp. IX, Proposition 2.1(a), (b) and (e)]. An S-group scheme G is called *reductive* (respectively, *semisimple*, *simply connected*) if it is affine and smooth over S and its geometric fibers are *connected* reductive (respectively, semisimple, simply connected) algebraic groups [6, Exp. XIX, Definition 2.7].

An S-torus is reductive, and any reductive S-group scheme is of finite presentation over S [6, Exp. XIX, 2.1]. Now, if G is a reductive group scheme over S, $\mathrm{rad}(G)$ will denote the radical of G, i.e., the identity component of the center $Z(G)$ of G. Further, G^{der} will denote the derived group of G. Thus G^{der} is a normal semisimple subgroup scheme of G and $G^{\mathrm{tor}} := G/G^{\mathrm{der}}$ is the largest quotient of G which is an S-torus. We shall write $\widetilde{G}$ for the simply connected central cover of G^{der} and $\mu := \mathrm{Ker}\big[\widetilde{G} \to G^{\mathrm{der}}\big]$ for the fundamental group of G^{der}. See [9, §2] for the existence and basic properties of $\widetilde{G}$. There exists a canonical homomorphism $\partial\colon \widetilde{G} \to G$ which factors as $\widetilde{G} \twoheadrightarrow G^{\mathrm{der}} \hookrightarrow G$. In particular, $\mathrm{Ker}\,\partial = \mu$ and $\mathrm{Coker}\,\partial = G^{\mathrm{tor}}$.

If X is a (commutative) finitely generated twisted constant S-group scheme [6, Exp. X, Definition 5.1], then X is quasi-isotrivial, i.e., there exists a surjective étale morphism $S' \to S$ such that $X \times_S S'$ is constant. Further, the functors

$$X \mapsto X^* := \underline{\mathrm{Hom}}_{\,S\text{-gr}}(X, \mathbb{G}_{m,S}) \quad \text{and} \quad M \mapsto M^* := \underline{\mathrm{Hom}}_{\,S\text{-gr}}(M, \mathbb{G}_{m,S})$$

are mutually quasi-inverse anti-equivalences between the categories of finitely generated twisted constant S-group schemes and S-group schemes of finite type and of multiplicative type[1] [6, Exp. X, Corollary 5.9]. Further, $M \to M^*$ and $X \to X^*$ are exact functors (see [6, Exp. VIII, Theorem 3.1] and use faithfully flat descent). If G is a reductive S-group scheme, its group of characters G^* equals $(G^{\mathrm{tor}})^*$ (see [6, Exp. XXII, proof of Theorem 6.2.1(i)]). Now, if T is an S-torus, the functor $\underline{\mathrm{Hom}}_{\,S\text{-gr}}(\mathbb{G}_{m,S}, T)$ is represented by a (free and finitely generated) twisted constant S-group scheme which is denoted by T_* and called the *group of cocharacters of* T (see [6, Exp. X, Corollary 4.5 and Theorem 5.6]). There exists a canonical isomorphism of free and finitely generated twisted constant S-group schemes

$$T^* \simeq (T_*)^{\vee} := \mathrm{Hom}_{\,S\text{-gr}}(T_*, \mathbb{Z}_S). \tag{1}$$

A sequence

$$0 \to T \to H \to G \to 0 \tag{2}$$

of reductive S-group schemes and S-homomorphisms is called *exact* if it is exact as a sequence of sheaves for the fppf topology on S. In this case the sequence (2) will be called *an extension of* G *by* T.

If G is a reductive S-group scheme, the identity homomorphism $G \to G$ will be denoted id_G. Further, if T is an S-torus, the inversion automorphism $T \to T$ will be denoted inv_T.

[1]Although [6, Exp. IX, Definition 1.4] allows for groups of multiplicative type which may not be of finite type over S, such groups will play no role in this paper.

2. Definition of π_1

Definition 2.1. Let G be a reductive S-group scheme. A *t-resolution of G* is a central extension

$$1 \to T \to H \to G \to 1, \tag{3}$$

where T is an S-torus and H is a reductive S-group scheme such that H^{der} is simply connected.

Proposition 2.2. *Every reductive S-group scheme admits a t-resolution.*

Proof. By [6, Exp. XXII, 6.2.3], the product in G defines a faithfully flat homomorphism $\mathrm{rad}(G) \times_S G^{\mathrm{der}} \to G$ which induces a faithfully flat homomorphism $\mathrm{rad}(G) \times_S \widetilde{G} \to G$. Let $\mu_1 = \ker[\mathrm{rad}(G) \times_S \widetilde{G} \to G]$, which is a finite S-group scheme of multiplicative type contained in the center of $\mathrm{rad}(G) \times_S \widetilde{G}$ (see [9, proof of Proposition 3.2, p. 9]). By [5, Proposition B.3.8], there exist an S-torus T and a closed immersion $\psi\colon \mu_1 \hookrightarrow T$. Let H be the pushout of $\varphi\colon \mu_1 \hookrightarrow \mathrm{rad}(G) \times_S \widetilde{G}$ and $\psi\colon \mu_1 \hookrightarrow T$, i.e., the cokernel of the central embedding

$$(\varphi, \mathrm{inv}_T \circ \psi)_S \colon \mu_1 \hookrightarrow \big(\mathrm{rad}(G) \times_S \widetilde{G}\big) \times_S T. \tag{4}$$

Then H is a reductive S-group scheme, cf. [6, Exp. XXII, Corollary 4.3.2], which fits into an exact sequence

$$1 \to T \to H \to G \to 1,$$

where T is central in H. Now, as in [4, proof of Proposition-Definition 3.1] and [9, proof of Proposition 3.2, p. 10], there exists an embedding of $\widetilde{G}$ into H which identifies $\widetilde{G}$ with H^{der}. Thus H^{der} is simply connected, which completes the proof. □

As in [4, p. 93] and [9, (3.3)], a t-resolution

$$1 \to T \to H \to G \to 1 \tag{$\mathscr{R}$}$$

induces a "fundamental diagram"

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu & \longrightarrow & \widetilde{G} & \longrightarrow & G^{\mathrm{der}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & M & \longrightarrow & R & \longrightarrow & G^{\mathrm{tor}} & \longrightarrow & 1, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where $M = T/\mu$ and $R = H^{\mathrm{tor}}$. This diagram induces, in turn, a canonical isomorphism in the derived category

$$\big(Z(\widetilde{G}) \xrightarrow{\partial_Z} Z(G)\big) \approx (T \to R) \tag{5}$$

(cf. [9, Proposition 3.4]) and a canonical exact sequence

$$1 \to \mu \to T \to R \to G^{\mathrm{tor}} \to 1, \tag{6}$$

where μ is the fundamental group of G^{der}. Since μ is finite, (6) shows that the induced homomorphism $T_* \to R_*$ is injective. Set

$$\pi_1(\mathscr{R}) = \mathrm{Coker}\big[T_* \to R_*\big]. \tag{7}$$

Thus there exists an exact sequence of (étale, finitely generated) twisted constant S-group schemes

$$1 \to T_* \to R_* \to \pi_1(\mathscr{R}) \to 1. \tag{8}$$

Set

$$\mu(-1) := \mathrm{Hom}_{S\text{-gr}}(\mu^*, (\mathbb{Q}/\mathbb{Z})_S).$$

Proposition 2.3. *A t-resolution $\mathscr{R}$ of a reductive S-group scheme G induces an exact sequence of finitely generated twisted constant S-group schemes*

$$1 \to \mu(-1) \to \pi_1(\mathscr{R}) \to (G^{\mathrm{tor}})_* \to 1.$$

Proof. The proof is similar to that of [4, Proposition 6.4], using (6). □

Definition 2.4. Let G be a reductive S-group scheme and let

$$1 \to T' \to H' \to G \to 1 \tag{$\mathscr{R}'$}$$

$$1 \to T \to H \to G \to 1 \tag{$\mathscr{R}$}$$

be two t-resolutions of G. A *morphism from $\mathscr{R}'$ to $\mathscr{R}$*, written $\phi\colon \mathscr{R}' \to \mathscr{R}$, is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T' & \longrightarrow & H' & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_T} & & \downarrow{\scriptstyle \phi_H} & & \downarrow{\scriptstyle \mathrm{id}_G} & & \\ 1 & \longrightarrow & T & \longrightarrow & H & \longrightarrow & G & \longrightarrow & 1, \end{array} \tag{9}$$

where ϕ_T and ϕ_H are S-homomorphisms. Note that, if $R' = (H')^{\mathrm{tor}}$ and $R = H^{\mathrm{tor}}$, then ϕ_H induces an S-homomorphism $\phi_R\colon R' \to R$.

We shall say that a t-resolution $\mathscr{R}'$ of G *dominates* another t-resolution $\mathscr{R}$ of G if there exists a morphism $\mathscr{R}' \to \mathscr{R}$.

The following lemma is well-known.

Lemma 2.5. *A morphism of complexes $f\colon P \to Q$ in an abelian category is a quasi-isomorphism if and only its cone $C(f)$ is acyclic (i.e., has trivial cohomology).*

Proof. By [7, Lemma III.3.3] there exists a short exact sequence of complexes

$$0 \to P \to \mathrm{Cyl}(f) \to C(f) \to 0, \tag{10}$$

where $\mathrm{Cyl}(f)$ is the cylinder of f. Further, the complex $\mathrm{Cyl}(f)$ is canonically isomorphic to Q in the derived category. Now the short exact sequence (10) induces a cohomology exact sequence

$$\dots \to H^i(P) \to H^i(Q) \to H^i(C(f)) \to H^{i+1}(P) \to \dots$$

from which the lemma is immediate. □

Lemma 2.6. *Let $g\colon C \to D$ be a quasi-isomorphism of bounded complexes of split S-tori. Then the induced morphism of complexes of cocharacter S-group schemes $g_*\colon C_* \to D_*$ is a quasi-isomorphism.*

Proof. Since the assertion is local in the étale topology, we may and do assume that S is connected. The given quasi-isomorphism induces a quasi-isomorphism $g^*\colon D^* \to C^*$ of bounded complexes of free and finitely generated constant S-group schemes. Thus, by (1), it suffices to check that the functor $X \mapsto X^\vee$ on the category of bounded complexes of free and finitely generated constant S-group schemes preserves quasi-isomorphisms. We thank Joseph Bernstein for the following argument. By Lemma 2.5 a morphism $f\colon P \to Q$ of bounded complexes in the (abelian) category of finitely generated constant S-group schemes is a quasi-isomorphism if and only if its cone $C(f)$ is acyclic. Now, if $f\colon P \to Q$ is a quasi-isomorphism and P and Q are bounded complexes of *free* and finitely generated constant S-group schemes, then $C(f)$ is an acyclic complex of free and finitely generated constant S-group schemes. We see immediately that the dual complex

$$C(f)^\vee = C\big(f^\vee\big)\big[-1\big]$$

is acyclic, whence $f^\vee$ is a quasi-isomorphism by Lemma 2.5. □

Lemma 2.7. *Let G be a reductive S-group scheme and let $\mathscr{R}'$ be a t-resolution of G which dominates another t-resolution $\mathscr{R}$ of G. Then a morphism of t-resolutions $\phi\colon \mathscr{R}' \to \mathscr{R}$ induces an isomorphism of finitely generated twisted constant S-group schemes $\pi_1(\phi)\colon \pi_1(\mathscr{R}') \xrightarrow{\sim} \pi_1(\mathscr{R})$ which is independent of the choice of ϕ.*

Proof. Let $\mathscr{R}'\colon 1 \to T' \to H' \to G \to 1$ and $\mathscr{R}\colon 1 \to T \to H \to G \to 1$ be the given t-resolutions of G, as in Definition 2.4, and set $R = H^{\mathrm{tor}}$ and $R' = (H')^{\mathrm{tor}}$. Since the assertion is local in the étale topology, we may and do assume that the tori T, T', R and R' are split and that S is connected. From (6) we see that the morphism of complexes of split tori (in degrees 0 and 1)

$$(\phi_T, \phi_R)\colon \big(T' \to R'\big) \to (T \to R)$$

is a quasi-isomorphism. Now by Lemma 2.6,

$$\pi_1(\phi) := H^1((\phi_T, \phi_R)_*)\colon \pi_1(\mathscr{R}') \xrightarrow{\sim} \pi_1(\mathscr{R})$$

is an isomorphism. In order to show that this isomorphism does not depend on the choice of ϕ, assume that $\psi\colon \mathscr{R}' \to \mathscr{R}$ is another morphism of t-resolutions. It is clear from diagram (9) that ψ_H differs from ϕ_H by some homomorphism $H' \to T$ which factors through $R' = (H')^{\mathrm{tor}}$. It follows that the induced homomorphisms $(\psi_R)_*, (\phi_R)_*\colon R'_* \to R_*$ differ by a homomorphism which factors through T_*. Consequently, the induced homomorphisms

$$\pi_1(\phi), \pi_1(\psi)\colon \mathrm{Coker}\,[T'_* \to R'_*] \to \mathrm{Coker}\,[T_* \to R_*]$$

coincide. □

Proposition 2.8. *Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let*

$$(\mathscr{R}_1) \qquad 1 \to T_1 \to H_1 \to G_1 \to 1$$

$$(\mathscr{R}_2) \qquad 1 \to T_2 \to H_2 \to G_2 \to 1$$

be t-resolutions of G_1 and G_2, respectively. Then there exists an exact commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}_G} & & \\ 1 & \longrightarrow & T'_1 & \longrightarrow & H'_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \varkappa} & & \\ 1 & \longrightarrow & T_2 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & 1, \end{array} \tag{11}$$

where the middle row is a t-resolution of G_1.

Proof. We follow an idea of Kottwitz [11, Proof of Lemma 2.4.4]. Let $H'_1 = H_1 \times_{G_2} H_2$, where the morphism $H_1 \to G_2$ is the composition $H_1 \to G_1 \xrightarrow{\varkappa} G_2$. Clearly, there are canonical morphisms $H'_1 \to H_1$ and $H'_1 \to H_2$. Now, since $H_2 \to G_2$ is faithfully flat, so also is $H'_1 \to H_1$. Consequently the composition $H'_1 \to H_1 \to G_1$ is faithfully flat as well. Let T'_1 denote its kernel, i.e., $T'_1 = S \times_{G_1} H'_1$. Then

$$T'_1 = (S \times_{G_1} H_1) \times_{G_2} H_2 = T_1 \times_S (S \times_{G_2} H_2) = T_1 \times_S T_2,$$

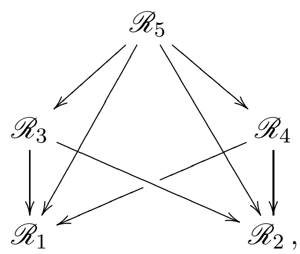
which is an S-torus. The existence of diagram (11) is now clear. Further, since T_i is central in H_i $(i = 1, 2)$, $T'_1 = T_1 \times_S T_2$ is central in $H'_1 = H_1 \times_{G_2} H_2$. The S-group scheme H'_1 is affine and smooth over S and has connected reductive fibers, i.e., is a reductive S-group scheme. Further, the faithfully flat morphism $H'_1 \to G_1$ induces a surjection $(H'_1)^{\mathrm{der}} \to G_1^{\mathrm{der}}$ with (central) kernel $T'_1 \cap (H'_1)^{\mathrm{der}}$. Since $(H'_1)^{\mathrm{der}}$ is semisimple, the last map is in fact a central isogeny. Consequently, $(H'_1)^{\mathrm{der}} \to H_1^{\mathrm{der}} = \widetilde{G}_1$ is a central isogeny as well, whence $(H'_1)^{\mathrm{der}} = \widetilde{G}_1$ is simply connected. Thus the middle row of (11) is indeed a t-resolution of G_1. □

Corollary 2.9. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two t-resolutions of a reductive S-group scheme G. Then there exists a t-resolution $\mathscr{R}_3$ of G which dominates both $\mathscr{R}_1$ and $\mathscr{R}_2$.*

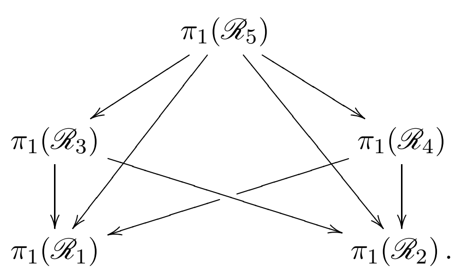
Proof. This is immediate from Proposition 2.8 (with $G_1 = G_2 = G$ and $\varkappa = \mathrm{id}_G$ there). □

Lemma 2.10. *Let $\mathscr{R}_1$ and $\mathscr{R}_2$ be two t-resolutions of a reductive S-group scheme G. Then there exists a canonical isomorphism of finitely generated twisted constant S-group schemes $\pi_1(\mathscr{R}_1) \cong \pi_1(\mathscr{R}_2)$.*

Proof. By Corollary 2.9, there exists a t-resolution $\mathscr{R}_3$ of G and morphisms of resolutions $\mathscr{R}_3 \to \mathscr{R}_1$ and $\mathscr{R}_3 \to \mathscr{R}_2$. Thus, Lemma 2.7 gives a composite isomorphism $\psi_{\mathscr{R}_3} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_3) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. Let $\mathscr{R}_4$ be another t-resolution of G which dominates both $\mathscr{R}_1$ and $\mathscr{R}_2$ and let $\psi_{\mathscr{R}_4} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_4) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$ be the corresponding composite isomorphism. There exists a t-resolution $\mathscr{R}_5$ which dominates both $\mathscr{R}_3$ and $\mathscr{R}_4$. Then $\mathscr{R}_5$ dominates $\mathscr{R}_1$ and $\mathscr{R}_2$ and we obtain a composite isomorphism $\psi_{\mathscr{R}_5} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_5) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. We have a diagram of t-resolutions



which may not commute. However, by Lemma 2.7, this diagram induces a *commutative* diagram of twisted constant S-group schemes and their isomorphisms



We conclude that

$$\psi_{\mathscr{R}_3} = \psi_{\mathscr{R}_5} = \psi_{\mathscr{R}_4} \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_2),$$

from which we deduce the existence of a *canonical* isomorphism $\psi \colon \pi_1(\mathscr{R}_1) \xrightarrow{\sim} \pi_1(\mathscr{R}_2)$. □

Definition 2.11. Let G be a reductive S-group scheme. Using the preceding lemma, we shall henceforth identify the S-group schemes $\pi_1(\mathscr{R})$ as $\mathscr{R}$ ranges over the family of all t-resolutions of G. Their common value will be denoted by $\pi_1(G)$ and called the *algebraic fundamental group of* G. Thus

$$\pi_1(G)=\pi_1(\mathscr{R})$$

for any t-resolution $\mathscr{R}$ of G.

Note that, by (8), a t-resolution $1\to T\to H\to G\to 1$ of G induces an exact sequence

$$1\to T_*\to (H^{\mathrm{tor}})_*\to \pi_1(G)\to 1. \tag{12}$$

Further, by Proposition 2.3, there exists a canonical exact sequence

$$1\to \mu(-1)\to \pi_1(G)\to (G^{\mathrm{tor}})_*\to 1. \tag{13}$$

Remark 2.12. One can also define $\pi_1(G)$ using m-resolutions. By an *m-resolution of* G we mean a short exact sequence

$$1\to M\to H\to G\to 1, \tag{$\mathscr{R}$}$$

where H is a reductive S-group scheme such that H^{der} is simply connected, and M is an S-group scheme of multiplicative type. Clearly, a t-resolution of G is in particular an m-resolution of G. It is very easy to see that any reductive S-group scheme G admits an m-resolution: we can take $H:=\mathrm{rad}(G)\times_S \widetilde{G}$, with the homomorphism $H\to G$ from the beginning of the proof of Proposition 2.2, and set $M:=\mu_1=\mathrm{Ker}\,[H\to G]$, which is a finite S-group scheme of multiplicative type.

Now let $\mathscr{R}$ be an m-resolution of G and consider the induced homomorphism $M\to H^{\mathrm{tor}}$. We claim that there exists a complex of S-tori $T\to R$ which is isomorphic to $M\to H^{\mathrm{tor}}$ in the derived category. Indeed, by [5, Proposition B.3.8] there exists an embedding $M\hookrightarrow T$ of M into an S-torus T. Denote by R the pushout of the homomorphisms $M\to H^{\mathrm{tor}}$ and $M\to T$. Then the complex of S-tori $T\to R$ is quasi-isomorphic to the complex $M\to H^{\mathrm{tor}}$, as claimed.

Now we choose an m-resolution $\mathscr{R}$ of G, a complex of S-tori $T\to R$ which is isomorphic to $M\to H^{\mathrm{tor}}$ in the derived category, and set $\pi_1(G)=\pi_1(\mathscr{R}):=\mathrm{Coker}\,[T_*\to R_*]$.

3. Functoriality and exactness of π_1

In this section we show that π_1 is an exact covariant functor from the category of reductive S-group schemes to the category of finitely generated twisted constant S-group schemes.

Definition 3.1. Let $\varkappa\colon G_1\to G_2$ be a homomorphism of reductive S-group schemes. A *t-resolution of* $\varkappa$, written $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1\to\mathscr{R}_2$, is an exact commutative diagram

$$\begin{array}{lccccccccc} (\mathscr{R}_1) & 1 & \longrightarrow & T_1 & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & & \downarrow{\scriptstyle \varkappa_T} & & \downarrow{\scriptstyle \varkappa_H} & & \downarrow{\scriptstyle \varkappa} & & \\ (\mathscr{R}_2) & 1 & \longrightarrow & T_2 & \longrightarrow & H_2 & \longrightarrow & G_2 & \longrightarrow & 1, \end{array}$$

where $\mathscr{R}_1$ and $\mathscr{R}_2$ are t-resolutions of G_1 and G_2, respectively.

Thus, if G is a reductive S-group scheme and $\mathscr{R}'$ and $\mathscr{R}$ are two t-resolutions of G, then a morphism from $\mathscr{R}'$ to $\mathscr{R}$ (as in Definition 2.4) is a t-resolution of $\mathrm{id}_G\colon G \to G$.

Remark 3.2. A t-resolution $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $\varkappa\colon G_1 \to G_2$ induces a homomorphism of finitely generated twisted constant S-group schemes

$$\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(\mathscr{R}_1) \to \pi_1(\mathscr{R}_2).$$

If G_3 is a third reductive S-group scheme, $\lambda\colon G_2 \to G_3$ is an S-homomorphism and $\lambda_{\mathscr{R}}\colon \mathscr{R}_2 \to \mathscr{R}_3$ is a t-resolution of λ, then $\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_3$ is a t-resolution of $\lambda \circ \varkappa$ and

$$\pi_1(\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}) = \pi_1(\lambda_{\mathscr{R}}) \circ \pi_1(\varkappa_{\mathscr{R}}).$$

Lemma 3.3. *Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let $\mathscr{R}_2$ be a t-resolution of G_2. Then there exists a t-resolution $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $\varkappa$ for a suitable choice of t-resolution $\mathscr{R}_1$ of G_1. In particular, every homomorphism of reductive S-group schemes admits a t-resolution.*

Proof. Choose any t-resolution $\mathscr{R}_1'$ of G_1 and apply Proposition 2.8 to $\varkappa$, $\mathscr{R}_1'$ and $\mathscr{R}_2$. ☐

Definition 3.4. Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes and let $\varkappa'_{\mathscr{R}}\colon \mathscr{R}_1' \to \mathscr{R}_2'$ and $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ be two t-resolutions of $\varkappa$. A *morphism from $\varkappa'_{\mathscr{R}}$ to $\varkappa_{\mathscr{R}}$*, written $\varkappa'_{\mathscr{R}} \to \varkappa_{\mathscr{R}}$, is a commutative diagram

$$\begin{array}{ccc} \mathscr{R}_1' & \xrightarrow{\varkappa'_{\mathscr{R}}} & \mathscr{R}_2' \\ \downarrow & & \downarrow \\ \mathscr{R}_1 & \xrightarrow{\varkappa_{\mathscr{R}}} & \mathscr{R}_2, \end{array}$$

where the left-hand vertical arrow is a t-resolution of id_{G_1} and the right-hand vertical arrow is a t-resolution of id_{G_2}. By a t-resolution *dominating* a t-resolution $\varkappa_{\mathscr{R}}$ of $\varkappa$ we mean a t-resolution $\varkappa'_{\mathscr{R}}$ of $\varkappa$ admitting a morphism $\varkappa'_{\mathscr{R}} \to \varkappa_{\mathscr{R}}$.

Lemma 3.5. *If $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ and $\varkappa'_{\mathscr{R}}\colon \mathscr{R}_1' \to \mathscr{R}_2'$ are two t-resolutions of a morphism $\varkappa\colon G_1 \to G_2$, then there exists a third t-resolution $\varkappa''_{\mathscr{R}}$ of $\varkappa$ which dominates both $\varkappa_{\mathscr{R}}$ and $\varkappa'_{\mathscr{R}}$.*

Proof. By Corollary 2.9, there exists a t-resolution $\mathscr{R}_2''$ of G_2 which dominates both $\mathscr{R}_2$ and $\mathscr{R}_2'$. On the other hand, by Lemma 3.3, there exists a t-resolution $\widetilde{\varkappa}_{\mathscr{R}}\colon \mathscr{R}_1''' \to \mathscr{R}_2''$ of $\varkappa$ for a suitable choice of t-resolution $\mathscr{R}_1'''$ of G_1. Now a second application of Corollary 2.9 yields a t-resolution $\mathscr{R}_1''$ of G_1 which dominates $\mathscr{R}_1$, $\mathscr{R}_1'$ and $\mathscr{R}_1'''$. Let $\phi\colon \mathscr{R}_1'' \to \mathscr{R}_1'''$ be the corresponding morphism, which is a t-resolution of id_{G_1}. Then $\varkappa_{\mathscr{R}}'' = \widetilde{\varkappa}_{\mathscr{R}} \circ \phi\colon \mathscr{R}_1'' \to \mathscr{R}_2''$ is a t-resolution of $\varkappa$ which dominates both $\varkappa_{\mathscr{R}}$ and $\varkappa_{\mathscr{R}}'$. □

Construction 3.6. Let $\varkappa\colon G_1 \to G_2$ be a homomorphism of reductive S-group schemes. By Lemma 3.3, there exists a t-resolution $\varkappa_{\mathscr{R}}\colon \mathscr{R}_1 \to \mathscr{R}_2$ of $\varkappa$, which induces a homomorphism $\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(\mathscr{R}_1) \to \pi_1(\mathscr{R}_2)$ of finitely generated twisted constant S-group schemes. Thus, if we identify $\pi_1(G_i)$ with $\pi_1(\mathscr{R}_i)$ for $i = 1, 2$, we obtain an S-homomorphism $\pi_1(\varkappa_{\mathscr{R}})\colon \pi_1(G_1) \to \pi_1(G_2)$ which, by Lemma 3.5, can be shown to be independent of the chosen t-resolution $\varkappa_{\mathscr{R}}$ of $\varkappa$. We denote it by

$$\pi_1(\varkappa)\colon \pi_1(G_1) \to \pi_1(G_2).$$

Lemma 3.7. *Let $G_1 \xrightarrow{\varkappa} G_2 \xrightarrow{\lambda} G_3$ be homomorphisms of reductive S-group schemes. Then*

$$\pi_1(\lambda \circ \varkappa) = \pi_1(\lambda) \circ \pi_1(\varkappa).$$

Proof. Choose a t-resolution $\mathscr{R}_3$ of G_3. Applying Lemma 3.3 first to λ and then to $\varkappa$, we obtain t-resolutions $\mathscr{R}_1 \xrightarrow{\varkappa_{\mathscr{R}}} \mathscr{R}_2 \xrightarrow{\lambda_{\mathscr{R}}} \mathscr{R}_3$ of $\varkappa$ and λ, and the composition $\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}$ is a t-resolution of $\lambda \circ \varkappa$. Thus, by Remark 3.2,

$$\pi_1(\lambda \circ \varkappa) = \pi_1(\lambda_{\mathscr{R}} \circ \varkappa_{\mathscr{R}}) = \pi_1(\lambda_{\mathscr{R}}) \circ \pi_1(\varkappa_{\mathscr{R}}) = \pi_1(\lambda) \circ \pi_1(\varkappa),$$

as claimed. □

Summarizing, for any non-empty scheme S, we have constructed a covariant functor π_1 from the category of reductive S-group schemes to the category of finitely generated twisted constant S-group schemes. Now assume that S is admissible in the sense of [9, Definition 2.1] (i.e., reduced, connected, locally Noetherian and geometrically unibranch), so that every reductive S-group scheme admits a flasque resolution [9, Proposition 3.2]. In this case the functor π_1 defined here in terms of t-resolutions coincides with the functor π_1 defined in [9, Definition 3.7] in terms of flasque resolutions, because a flasque resolution is a particular case of a t-resolution. A basic example of a non-admissible scheme S to which the constructions of the present paper apply, but not those of [9], is an algebraic curve over a field having an ordinary double point. See [9, Remark 2.3].

The following result generalizes [3, Lemma 3.7], [4, Proposition 6.8] and [9, Theorem 3.14].

Theorem 3.8. *Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive S-group schemes. Then the induced sequence of finitely generated twisted constant S-group schemes*

$$0 \to \pi_1(G_1) \to \pi_1(G_2) \to \pi_1(G_3) \to 0$$

is exact.

Proof. The proof is similar to that of [9, Theorem 3.14] using the exact sequence (13). Namely, one first proves the theorem when G_1 is semisimple using the same arguments as in the proof of [9, Lemma 3.12] (those arguments rely on [9, Proposition 2.8], which is valid over any non-empty base scheme S). Secondly, one proves the theorem when G_1 is an S-torus using the same arguments as in the proof of [9, Lemma 3.13] (which rely on [9, Proposition 2.9], which again holds over any non-empty base scheme S). Finally, the theorem is obtained by combining these two particular cases as in the proof of [9, Theorem 3.14]. □

We shall now present a second proof of Theorem 3.8 which relies on the étale-local existence of maximal tori in reductive S-group schemes. To this end, we shall first show that if G is a reductive S-group scheme which contains a maximal torus T, then T canonically determines a t-resolution of G.

Lemma 3.9. *Let G be a reductive S-group scheme having a maximal S-torus T, and set $\widetilde{T} := \widetilde{G} \times_G T$, it is a maximal S-torus of $\widetilde{G}$. Then there exists a t-resolution of G*

$$1 \to \widetilde{T} \to H \to G \to 1 \qquad (\mathscr{R}_T)$$

such that H^{tor} is canonically isomorphic to T.

Proof. By [9, proof of Proposition 3.2], the product in G and the canonical epimorphism $\widetilde{G} \to G^{\text{der}}$ induce a faithfully flat homomorphism $\text{rad}(G) \times_S \widetilde{G} \to G$ whose (central) kernel μ_1 embeds into $Z(\widetilde{G})$ via the canonical projection $\text{rad}(G) \times_S \widetilde{G} \to \widetilde{G}$. In particular, we have a central extension

$$1 \to \mu_1 \xrightarrow{\varphi} \text{rad}(G) \times_S \widetilde{G} \to G \to 1. \qquad (14)$$

Since $Z(\widetilde{G}) \subset \widetilde{T}$ by [6, Exp. XXII, Corollary 4.1.7], we obtain an embedding $\psi\colon \mu_1 \hookrightarrow \widetilde{T}$. Let H be the pushout of $\varphi\colon \mu_1 \hookrightarrow \text{rad}(G) \times_S \widetilde{G}$ and $\psi\colon \mu_1 \hookrightarrow \widetilde{T}$, i.e., the cokernel of the central embedding

$$(\varphi, \text{inv}_{\widetilde{T}} \circ \psi)_S\colon \mu_1 \hookrightarrow \big(\text{rad}(G) \times_S \widetilde{G}\big) \times_S \widetilde{T}. \qquad (15)$$

Now let $\varepsilon\colon S \to \text{rad}(G) \times_S \widetilde{G}$ be the unit section of $\text{rad}(G) \times_S \widetilde{G}$ and set

$$j = (\varepsilon, \text{id}_{\widetilde{T}})_S\colon S \times_S \widetilde{T} \to \big(\text{rad}(G) \times_S \widetilde{G}\big) \times_S \widetilde{T}.$$

Composing j with the canonical isomorphism $\widetilde{T} \simeq S \times_S \widetilde{T}$, we obtain an S-morphism $\widetilde{T} \to \text{rad}(G) \times_S \widetilde{G} \times_S \widetilde{T}$ which induces an embedding $\iota_T\colon \widetilde{T} \hookrightarrow H$. Further, let $\pi_T\colon H \to G$ be the homomorphism which is induced by the projection

$$\text{rad}(G) \times_S \widetilde{G} \times_S \widetilde{T} \to \text{rad}(G) \times_S \widetilde{G}.$$

Then we obtain a t-resolution of G

$$1 \longrightarrow \widetilde{T} \xrightarrow{\iota_T} H \xrightarrow{\pi_T} G \longrightarrow 1$$

which is canonically determined by T (cf. the proof of Proposition 2.2). It remains to show that $H^{\,\rm tor}$ is canonically isomorphic to T. Let $\varepsilon_{\rm rad}\colon S\to{\rm rad}(G)$ and $\varepsilon_{\widetilde{T}}\colon S\to\widetilde{T}$ be the unit sections of ${\rm rad}(G)$ and $\widetilde{T}$, respectively, and consider the homomorphism

$$(\varepsilon_{\rm rad},{\rm id}_{\widetilde{G}},\varepsilon_{\widetilde{T}})_S\colon S\times_S\widetilde{G}\times_S S\to{\rm rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}.$$

Composing this homomorphism with the canonical isomorphism $\widetilde{G}\simeq S\times_S\widetilde{G}\times_S S$, we obtain a canonical embedding $\widetilde{G}\hookrightarrow{\rm rad}(G)\times_S\widetilde{G}\times_S\widetilde{T}$. The latter map induces a homomorphism $\widetilde{G}\to H$ which identifies $\widetilde{G}$ with $H^{\rm der}$. Now consider the composite homomorphism

$$\varphi_{\rm rad}\colon\mu_1\xrightarrow{\varphi}{\rm rad}(G)\times_S\widetilde{G}\xrightarrow{{\rm pr}_1}{\rm rad}(G).$$

Then $H^{\,\rm tor}:=H/H^{\rm der}=H/\widetilde{G}$ is isomorphic to the cokernel of the central embedding

$$(\varphi_{\rm rad},{\rm inv}_{\widetilde{T}}\circ\psi)_S\colon\mu_1\hookrightarrow{\rm rad}(G)\times_S\widetilde{T}. \qquad (16)$$

Compare (15). Finally, the canonical embedding $\widetilde{T}\hookrightarrow\widetilde{G}$ induces an embedding $H^{\,\rm tor}\hookrightarrow G$ (see (14) and (16)) whose image is ${\rm rad}(G)\cdot(T\cap G^{\rm der})=T$ [6, Exp. XXII, proof of Proposition 6.2.8(i)]. This completes the proof. □

Remark 3.10. It is clear from the above proof that the homomorphism $\widetilde{T}\to H^{\,\rm tor}=T$ induced by the t-resolution $\mathscr{R}_T$ of Lemma 3.9 is the canonical homomorphism $\partial\colon\widetilde{T}\to T$.

Definition 3.11. Let G be a reductive S-group scheme containing a maximal S-torus T. The *algebraic fundamental group of the pair* (G,T) is the S-group scheme $\pi_1(G,T):={\rm Coker}\,[\partial_*\colon\widetilde{T}_*\to T_*]$.

By Lemma 3.9 and Definition 2.11 we have a canonical isomorphism

$$\vartheta_T\colon\pi_1(G,T)\xrightarrow{\sim}\pi_1(\mathscr{R}_T)=\pi_1(G). \qquad (17)$$

Further, any morphism of pairs $\varkappa\colon(G_1,T_1)\to(G_2,T_2)$ (in the obvious sense) induces an S-homomorphism $\varkappa_*\colon\pi_1(G_1,T_1)\to\pi_1(G_2,T_2)$. It can be shown that the following diagram commutes:

$$\begin{array}{ccc}\pi_1(G_1,T_1) & \xrightarrow{\varkappa_*} & \pi_1(G_2,T_2)\\ {\scriptstyle\vartheta_{T_1}}\downarrow & & \downarrow{\scriptstyle\vartheta_{T_2}}\\ \pi_1(G_1) & \xrightarrow{\varkappa_*} & \pi_1(G_2)\,.\end{array} \qquad (18)$$

This is immediate in the case where $\varkappa$ is a *normal* homomorphism, i.e. $\varkappa(G_1)$ is normal in G_2 (this is the only case needed in this paper). Indeed, in this case we have $\varkappa({\rm rad}(G_1))\subset{\rm rad}(G_2)$ and therefore $\varkappa$ induces a morphism of t-resolutions $\varkappa_{\mathscr{R}}\colon\mathscr{R}_{T_1}\to\mathscr{R}_{T_2}$. See the proof of Lemma 3.9.

Remark 3.12. The preceding considerations and Lemma 2.10 show that, if S is an admissible scheme in the sense of [9, Definition 2.1], so that every reductive S-group scheme G admits a flasque resolution $\mathscr{F}$, and G contains a maximal S-torus T, then there exists a canonical isomorphism $\pi_1(\mathscr{F}) \cong \mathrm{Coker}\,[\partial_*\colon \widetilde{T}_* \to T_*]$. This fact generalizes [4, Proposition A.2], which is the case $S = \mathrm{Spec}\, k$, where k is a field, of the present remark.

Lemma 3.13. *Let*

$$1 \to (G_1, T_1) \xrightarrow{\varkappa} (G_2, T_2) \xrightarrow{\lambda} (G_3, T_3) \to 1$$

be an exact sequence of reductive S-group schemes with maximal tori. Then the sequence of étale, finitely generated twisted constant S-group schemes

$$0 \to \pi_1(G_1, T_1) \xrightarrow{\varkappa_*} \pi_1(G_2, T_2) \xrightarrow{\lambda_*} \pi_1(G_3, T_3) \to 0$$

is exact.

Proof. The assertion of the lemma is local for the étale topology, so we may and do assume that T_1, T_2, and T_3 are split. By [9, Proposition 2.10], there exists an exact commutative diagram of reductive S-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{G}_1 & \longrightarrow & \widetilde{G}_2 & \longrightarrow & \widetilde{G}_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_3} & & \\ 1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 & \longrightarrow & 1, \end{array}$$

which induces an exact commutative diagram of split S-tori

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{T}_1 & \longrightarrow & \widetilde{T}_2 & \longrightarrow & \widetilde{T}_3 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle \partial_3} & & \\ 1 & \longrightarrow & T_1 & \longrightarrow & T_2 & \longrightarrow & T_3 & \longrightarrow & 1, \end{array} \tag{19}$$

where $\widetilde{T}_i := \widetilde{G}_i \times_{G_i} T_i$ $(i = 1, 2, 3)$. Now, as in [3, Proof of Lemma 3.7], diagram (19) induces an exact commutative diagram of constant S-group schemes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{T}_{1*} & \longrightarrow & \widetilde{T}_{2*} & \longrightarrow & \widetilde{T}_{3*} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \partial_{1*}} & & \downarrow{\scriptstyle \partial_{2*}} & & \downarrow{\scriptstyle \partial_{3*}} & & \\ 1 & \longrightarrow & T_{1*} & \longrightarrow & T_{2*} & \longrightarrow & T_{3*} & \longrightarrow & 1 \end{array}$$

with injective vertical arrows. An application of the snake lemma to the last diagram now yields the exact sequence

$$0 \to \mathrm{Coker}\ \partial_{1*} \to \mathrm{Coker}\ \partial_{2*} \to \mathrm{Coker}\ \partial_{3*} \to 0,$$

which is the assertion of the lemma. □

Second proof of Theorem 3.8. Let $1 \to G_1 \to G_2 \to G_3 \to 1$ be an exact sequence of reductive S-group schemes. By [6, Exp. XIX, Proposition 6.1], for any reductive S-group scheme G there exists an étale covering $\{S_\alpha \to S\}_{\alpha\in A}$ such that each $G_{S_\alpha} := G \times_S S_\alpha$ contains a split maximal S_α-torus T_α. Thus, since the assertion of the theorem is local for the étale topology, we may and do assume that G_2 contains a split maximal S-torus T_2. Let $T_1 = G_1 \times_{G_2} T_2$ and let T_3 be the cokernel of $T_1 \to T_2$. Then T_i is a split maximal S-torus of G_i for $i = 1, 2, 3$ and we have an exact sequence of pairs

$$1 \to (G_1, T_1) \to (G_2, T_2) \to (G_3, T_3) \to 1.$$

Now the theorem follows from Lemma 3.13, (17) and (18). □

4. Abelian cohomology and t-resolutions

Let S_{fl} (respectively, $S_{\text{ét}}$) be the small fppf (respectively, étale) site over S. If F_1 and F_2 are abelian sheaves on S_{fl} (regarded as complexes concentrated in degree 0), $F_1 \otimes^{\mathbf{L}} F_2$ (respectively, $\text{RHom}\,(F_1, F_2)$) will denote the total tensor product (respectively, right derived Hom functor) of F_1 and F_2 in the derived category of the category of abelian sheaves on S_{fl}.

Let G be a reductive group scheme over S. For any integer $i \geq -1$, the i-th *abelian (flat) cohomology group of* G is by definition the hypercohomology group

$$H^i_{\text{ab}}(S_{\text{fl}}, G) = \mathbb{H}^i\big(S_{\text{fl}}, Z\big(\widetilde{G}\,\big) \xrightarrow{\partial_Z} Z(G)\big).$$

On the other hand, the i-th *dual abelian cohomology group of* G is the group

$$H^i_{\text{ab}}(S_{\text{ét}}, G^*) = \mathbb{H}^i\big(S_{\text{ét}}, Z(G)^* \xrightarrow{\partial_Z^*} Z\big(\widetilde{G}\,\big)^*\big).$$

Here all the complexes of length 2 are in degrees $(-1, 0)$. See [9, beginning of §4] for basic properties of these cohomology groups and [1, 8, 10] for (some of) their arithmetical applications.

The following result is an immediate consequence of (5).

Proposition 4.1. *Let G be a reductive S-group scheme and let $1 \to T \to H \to G \to 1$ be a t-resolution of G. Then the given t-resolution defines isomorphisms $H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i\big(S_{\text{fl}}, T \to R\big)$ and $H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i\big(S_{\text{ét}}, R^* \to T^*\big)$, where $R = H^{\text{tor}}$. Further, there exist exact sequences*

$$\ldots \to H^i(S_{\text{ét}}, T) \to H^i(S_{\text{ét}}, R) \to H^i_{\text{ab}}(S_{\text{fl}}, G) \to H^{i+1}(S_{\text{ét}}, T) \to \ldots$$

and

$$\ldots \to H^i(S_{\text{ét}}, R^*) \to H^i(S_{\text{ét}}, T^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G^*) \to H^{i+1}(S_{\text{ét}}, R^*) \to \ldots. \quad \square$$

Corollary 4.2. *Let G be a reductive S-group scheme. Then, for every integer $i \geq -1$, there exist isomorphisms*

$$H^i_{\text{ab}}(S_{\text{fl}}, G) \simeq \mathbb{H}^i(S_{\text{fl}}, \pi_1(G) \otimes^{\mathbf{L}} \mathbb{G}_{m,S})$$

and

$$H^i_{\text{ab}}(S_{\text{ét}}, G^*) \simeq \mathbb{H}^i(S_{\text{ét}}, \text{RHom}\,(\pi_1(G), \mathbb{Z}_S)).$$

Proof. This follows from Proposition 4.1 in the same way as [9, Corollary 4.3] follows from [9, Proposition 4.2]. □

Proposition 4.3. *Let* $1 \to G_1 \to G_2 \to G_3 \to 1$ *be an exact sequence of reductive* S*-group schemes. Then there exist exact sequences of abelian groups*

$$\dots \to H^i_{\text{ab}}(S_{\text{fl}}, G_1) \to H^i_{\text{ab}}(S_{\text{fl}}, G_2) \to H^i_{\text{ab}}(S_{\text{fl}}, G_3) \to H^{i+1}_{\text{ab}}(S_{\text{fl}}, G_1) \to \dots$$

and

$$\dots \to H^i_{\text{ab}}(S_{\text{ét}}, G_3^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G_2^*) \to H^i_{\text{ab}}(S_{\text{ét}}, G_1^*) \to H^{i+1}_{\text{ab}}(S_{\text{ét}}, G_3^*) \to \dots .$$

Proof. This follows from Corollary 4.2 and Theorem 3.8. □

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