Proc. London Math. Soc. (3) 109 (2014) 946–974
doi:10.1112/plms/pdu026

Boundary value problems for noncompact boundaries of Spin^c manifolds and spectral estimates

Nadine Grosse and Roger Nakad

Abstract

We study boundary value problems for the Dirac operator on Riemannian Spin^c manifolds of bounded geometry and with noncompact boundary. This generalizes a part of the theory of boundary value problems by Bär and Ballmann for complete manifolds with closed boundary. As an application, we derive the lower bound of Hijazi–Montiel–Zhang, involving the mean curvature of the boundary, for the spectrum of the Dirac operator on the noncompact boundary of a Spin^c manifold, and the limiting case is studied.

1. *Introduction*

In the last years, the spectrum of the Dirac operator on hypersurfaces of Spin manifolds has been intensively studied. Indeed, many extrinsic upper bounds have been obtained (see [**1**, **2**, **4**, **6**, **7**, **10**] and references therein) and more recently in [**16**–**20**, **30**], extrinsic lower bounds for the hypersurface Dirac operator are established. From these spectral estimates and their limiting cases, many topological and geometric informations on the hypersurface are derived.

In [**16**], Hijazi, Montiel and Zhang investigated the spectral properties of the Dirac operator on a compact manifold with boundary for the Atiyah–Patodi–Singer type boundary condition (or shortly APS-boundary condition) corresponding to the spectral resolution of the classical Dirac operator of the boundary hypersurface. They proved that, on the compact boundary $\Sigma = \partial M$ of a compact Riemannian Spin manifold (M^{n+1}, g) of nonnegative scalar curvature scal^M, the first nonnegative eigenvalue of the Dirac operator on the boundary satisfies

$$\lambda_1 \geqslant \frac{n}{2} \inf_\Sigma H, \tag{1}$$

where the mean curvature of the boundary H is calculated with respect to the inner normal and assumed to be nonnegative. Equality holds in (1) if and only if H is constant and every eigenspinor associated with the eigenvalue λ_1 is the restriction to Σ of a parallel spinor field on M (and hence M is Ricci-flat). As application of the limiting case, they gave an elementary Spin proof of the famous Alexandrov theorem: *The only closed embedded hypersurface in* $\mathbb{R}^{n+1}$ *of constant mean curvature is the sphere of dimension* n.

Furthermore, Inequality (1) does not only give an extrinsic lower bound on the first nonnegative eigenvalue, but can also be seen as an obstruction to positive scalar curvature of the interior given only in terms of a neighbourhood of the boundary. More precisely, let a neighbourhood of the boundary Σ be equipped with a metric of nonnegative scalar curvature and such that the boundary has nonnegative mean curvature. If the lowest positive eigenvalue of the Dirac operator on the boundary is smaller than $(n/2)\inf_\Sigma H$, then the metric cannot be extended to all of M such that the scalar curvature remains nonnegative.

Received 23 August 2012; revised 17 July 2013; published online 27 June 2014.

2010 *Mathematics Subject Classification* 30E25, 58C40, 53C27, 53C42 (primary).

In this paper, we extend the lower bound (1) to noncompact boundaries of Riemannian Spin^c manifolds under suitable geometric assumptions, see Theorem 1.2. When shifting from the compact case to the noncompact case, many obstacles occur. Moreover, when shifting from the classical Spin geometry to Spin^c geometry, the situation is more general since the spectrum of the Dirac operator will not only depend on the geometry of the manifold but also on the connection of the auxiliary line bundle associated with the fixed Spin^c structure.

When we consider a Riemannian Spin or Spin^c manifold with noncompact boundary, the main technical difference to the compact case is that we cannot restrict all our computations to smooth spinors. For compact manifolds, this is possible by using the spectral decomposition of L^2 by an eigenbasis. For complete manifolds, eigenspinors do not have to exist or even if they do, in general, they do not form an orthonormal basis of L^2 since continuous spectrum can occur. Additionally, the proof of Inequality (1) in the closed case uses the existence of a solution of a boundary value problem defined under the APS-boundary condition. While for noncompact boundaries the idea of APS-boundary conditions can be carried over to noncompact boundaries by using the spectral theorem, it is not clear to us whether they actually define an actual boundary condition, see Example 4.16.

To circumvent all these problems, a large part of the paper is devoted to give a generalization of the theory of boundary value problems for noncompact boundaries, see Section 4. We stick to the part of the theory that gives existence of solutions of such boundary value problem, cf. Remark 4.15. For complete manifolds with closed boundary, the theory of boundary value problems is given by Bär and Ballmann [**9**]. They did not only restrict to the classical Dirac operator but they generalized the traditional theory of elliptic boundary value problems to Dirac-type operators. Additionally, they proved a decomposition theorem for the essential spectrum, a general version of Gromov and Lawson's relative index theorem and a generalization of the cobordism theorem.

In Section 4, we will classify boundary conditions for a Riemannian Spin^c manifold (M^{n+1}, g) with noncompact boundary $\Sigma := \partial M$ and of bounded geometry, see Definition 2.2. Indeed, we prove in Section 4 that the trace map or the restriction map $R : \varphi \mapsto \varphi|_\Sigma$, where φ is a compactly supported smooth spinor on M can be extended to a bounded operator

$$R : \operatorname{dom} D_{\max} \longrightarrow H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma).$$

Here, $\operatorname{dom} D_{\max}$ is the maximal domain of the Dirac operator on M, $\mathbb{S}_M|_\Sigma$ is the restriction of the Spin^c bundle $\mathbb{S}_M$ to Σ and for $H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$, see Definition 3.5. The map R is not surjective. But in Theorem 3.13, we show that there is an extension map $\tilde{\mathcal{E}}$, a right inverse to the restriction map $R : \Gamma_c^\infty(M, \mathbb{S}_M) \to \Gamma_c^\infty(M, \mathbb{S}_M)$, such that $\tilde{\mathcal{E}}R$ is a bounded linear operator from $\operatorname{dom} D_{\max}$ to itself. The definition of $\tilde{\mathcal{E}}$ uses the extension map for closed boundaries introduced by Bär and Ballmann [**9**] as local building blocks. This will allow one to equip $R(\operatorname{dom} D_{\max})$ with a norm $\|.\|_{\check{R}}$ that turns it into a Hilbert space. With these ingredients, we can then classify the closed extensions of the Dirac operator D_{cc} acting on smooth compactly supported spinors on M: For every closed extension of the Dirac operator acting on smooth compactly supported spinors on M the set $B := R(\operatorname{dom} D) \subset H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$ is closed in $(R(\operatorname{dom} D_{\max}), \|.\|_{\check{R}})$. Conversely, every closed linear subset $B \subset (R(\operatorname{dom} D_{\max}), \|.\|_{\check{R}})$ gives the domain $\operatorname{dom} D_B$ of a closed extension. Such subsets B are called a boundary conditions.

Then, we generalize the existence result for boundary value problems to our noncompact setting. For this, we need the notion of *B-coercivity at infinity*, see Definition 4.17. This notion generalizes the notion of *coercivity at infinity* for closed boundaries as used in [**9**], where this assumption is also needed when characterizing the Fredholmness of the Dirac operator. The *B-coercivity at infinity* condition will, in general, depend on the boundary condition B and under some additional assumptions, it coincides with the *coercivity at infinity* condition used in [**9**].

Theorem 1.1. *Let M be a Riemannian* Spin^c *manifold with boundary N. Let (M,N) and the auxiliary line bundle L over M be of bounded geometry, cp. Definitions 2.2 and 2.3. Let $B \subset R(\mathrm{dom}\, D_{\max})$ be a boundary condition, and let the Dirac operator*

$$D_B\colon \mathrm{dom} D_B \subset L^2(M,\mathbb{S}_M) \longrightarrow L^2(M,\mathbb{S}_M)$$

be B-coercive at infinity. Let P_B be a projection from $R(\mathrm{dom}\, D_{\max})$ to B. Then, for all $\psi \in L^2(M,\mathbb{S}_M)$ and $\tilde\rho \in \mathrm{dom}\, D_{\max}$, where $\psi - D\tilde\rho \in (\ker(D_B)^*)^\perp$ the boundary value problem

$$\begin{cases} D\varphi = \psi & \text{on } M,\\ (\mathrm{Id} - P_B)R\varphi = (\mathrm{Id} - P_B)R\tilde\rho & \text{on } \Sigma \end{cases}$$

has a unique solution $\varphi \in \mathrm{dom}\, D_{\max}$, up to elements of the kernel $\ker D_B$.

Note that projection just means a linear operator that restricted to B acts as identity operator.

Theorem 1.1 will be one of the main ingredients to generalize Inequality (1) to our noncompact setting. As boundary condition B we will not take the APS-boundary condition as in the closed case but another one: $B_\pm$, cf. Section 5. For closed boundaries, the $B_\pm$ boundary condition was introduced in [**18**] to prove a conformal version of (1). Using Theorem 1.1 for the boundary condition $B_\pm$ and the Spin^c Reilly inequality on possibly open boundary domains, we obtain the following theorem.

Theorem 1.2. *Let (M^{n+1},g) be a complete Riemannian* Spin^c *manifold with boundary Σ and L be the auxiliary line bundle associated to the* Spin^c*-structure. Assume that (M,Σ) and L are of bounded geometry. Moreover, we assume that Σ has nonnegative mean curvature H with respect to its inner unit normal field of Σ, the Dirac operator D is (B_+)- or (B_-)-coercive at infinity and that $\mathrm{scal}^M + 2i\Omega\cdot$ is a nonnegative operator where $i\Omega$ denotes the curvature 2-form of L. Then, the infimum λ_1 of the nonnegative part of the spectrum of the Dirac operator on Σ satisfies*

$$\lambda_1 \geqslant \frac{n}{2}\inf_\Sigma H.$$

If $\lambda_1 \geqslant 0$ is an eigenvalue, then equality holds if and only if H is constant and any eigenspinor corresponding to λ_1 is the restriction of a parallel Spin^c *spinor φ on M.*

The paper is structured as follows: In Section 2, we give all the preliminaries as, for example, the Spin^c Dirac operator and the assumption on the bounded geometry. In Section 3, we review the trace and extension theorem for Sobolev spaces on manifolds of bounded geometry and appropriate noncompact boundary, the spectral decomposition of the Dirac operator on the boundary and analyse an extension map for the maximal domain of the Dirac operator. The theory of boundary values will be generalized to our noncompact setting in Section 4. The special boundary condition $B_\pm$ needed to proof the desired inequality is examined in Section 5. In Section 6, we study the coercivity condition for the Dirac operator. Then, we review the spinorial Reilly inequality in order to ready to proof the inequality in Section 8.

2. *Notations and preliminaries*

In this section, we briefly review some basic facts about Spin^c geometry. Then, we give the necessary preliminaries on Sobolev spaces on manifolds with boundary, the Trace Theorem and its implications, some basics of spectral theory, and we recall the closed range theorem.

The Spin^c *Dirac operator*. Let (M^{n+1}, g) be an $(n+1)$-dimensional Riemannian Spin^c manifold with boundary. On such a manifold, we have a Hermitian complex vector bundle $\mathbb{S}_M$ endowed with a natural scalar product $\langle .,. \rangle$ and with a connection ∇ that parallelizes the metric. Moreover, the bundle $\mathbb{S}_M$, called the Spin^c bundle, is endowed with a Clifford multiplication denoted by '$\cdot$', $\cdot : TM \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_M)$, such that at every point $x \in M$, '$\cdot$' defines an irreducible representation of the corresponding Clifford algebra. Hence, the complex rank of $\mathbb{S}_M$ is $2^{[(n+1)/2]}$. Given a Spin^c structure on (M^{n+1}, g), one can prove that the determinant line bundle $\det \mathbb{S}_M$ has a root of index $2^{[(n+1)/2]-1}$, see [**13**, Section 2.5]. We denote by L this root line bundle over M and call it the auxiliary line bundle associated with the Spin^c structure.

Locally, a Spin structure always exists. We denote by $\mathbb{S}'_M$ the (possibly globally non-existent) spinor bundle. Moreover, the square root of the auxiliary line bundle L always exists locally. But, $\mathbb{S}_M = \mathbb{S}'_M \otimes L^{1/2}$, see [**13**, Appendix D; **23**]. This essentially means that, while the spinor bundle and $L^{1/2}$ may not exist globally, their tensor product (the Spin^c bundle) is defined globally. Thus, the connection ∇ on $\mathbb{S}_M$ is the twisted connection of the one on the spinor bundle (coming from the Levi-Civita connection) and a fixed connection on L.

We denote by $\Gamma_c^\infty(M, \mathbb{S}_M)$ the set of all compactly supported smooth spinors on M. This allows boundary values if $\partial M \neq \emptyset$. The set of smooth spinors that are compactly supported in the interior of M is denoted by $\Gamma_{cc}^\infty(M, \mathbb{S}_M)$. For abbreviation, we set $L^2 = L^2(M) = L^2(M, \mathbb{S}_M)$ and $L^2(\Sigma) = L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ and analogously for other function spaces. Moreover, $(.,.)$ shall always denote the L^2-scalar product on M and $(.,.)_\Sigma$ the one on Σ.

With these ingredients, we may define the Dirac operator D acting on the space of smooth sections of $\mathbb{S}_M$, denoted by $\Gamma^\infty(M, \mathbb{S}_M)$, by the composition of the metric connection and the Clifford multiplication. In local coordinates, this reads as

$$D = \sum_{j=1}^{n+1} e_j \cdot \nabla_{e_j},$$

where $\{e_j\}_{j=1,\ldots,n+1}$ is an orthonormal basis of TM. It is a first-order elliptic operator satisfying for all smooth spinors φ, ψ on M at least one of them being compactly supported

$$(D\psi, \varphi) - (\psi, D\varphi) = -\int_{\partial M} \langle \nu \cdot \psi|_{\partial M}, \varphi|_{\partial M} \rangle \, ds, \tag{2}$$

where $(.,.)$ is the L^2-scalar product given by $(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle \, dv$, ∂M is the boundary of M, $|_{\partial M}$ denotes the restriction to the boundary, ν the inner unit normal vector of the embedding $\partial M \hookrightarrow M$, and dv (respectively, ds) is the Riemannian volume form of M (respectively, of ∂M). Hence, if $\partial M = \emptyset$, then the Dirac operator is formally self-adjoint with respect to the L^2-scalar product.

An important tool when examining the Dirac operator on Spin^c manifolds is the Schrödinger–Lichnerowicz formula:

$$D^2 = \nabla^* \nabla + \frac{1}{4} \mathrm{scal}^M \, \mathrm{Id}_{\Gamma(\mathbb{S}_M)} + \frac{i}{2} \Omega \cdot, \tag{3}$$

where ∇^* is the adjoint of ∇ with respect to the L^2-scalar product, $i\Omega$ is the curvature of the auxiliary line bundle L associated with a fixed connection (Ω is a real 2-form on M) and $\Omega\cdot$ is the extension of the Clifford multiplication to differential forms.

EXAMPLE 2.1. (i) A Spin structure can be seen as a Spin^c structure with trivial auxiliary line bundle L and trivial connection (and so $i\Omega = 0$).

(ii) Every almost complex manifold $(M^{2m=n+1}, g, J)$ of complex dimension m has a canonical Spin^c structure. In fact, the complexified cotangent bundle $T^*M \otimes \mathbb{C} = \Lambda^{1,0} M \oplus \Lambda^{0,1} M$ decomposes into the $\pm i$-eigenbundles of the complex linear extension of the complex structure J. Thus, the spinor bundle of the canonical Spin^c structure is given by

$$\mathbb{S}_M = \Lambda^{0,*}M = \bigoplus_{r=0}^{m} \Lambda^{0,r}M,$$

where $\Lambda^{0,r}M = \Lambda^r(\Lambda^{0,1}M)$ is the bundle of r-forms of type $(0,1)$. The auxiliary line bundle of this canonical Spin^c structure is given by $L = (K_M)^{-1} = \Lambda^m(\Lambda^{0,1}M)$, where K_M is the canonical bundle of M (see [**13**, **15**, **21**, **23**]). Let α be the Kähler form defined by the complex structure J, that is, $\alpha(X, Y) = g(X, JY)$ for all vector fields $X, Y \in \Gamma(TM)$. The auxiliary line bundle $L = (K_M)^{-1}$ has a canonical holomorphic connection induced from the Levi-Civita connection whose curvature form is given by $i\Omega = i\rho$, where ρ is the Ricci 2-form given by $\rho(X, Y) = \text{Ric}(X, JY)$. Here, Ric denotes the Ricci tensor of M. For any other Spin^c structure on M^{2m}, the spinorial bundle can be written as [**13**, **15**]

$$\mathbb{S}_M = \Lambda^{0,*}M \otimes \mathcal{L},$$

where $\mathcal{L}^2 = K_M \otimes L$ and L is the auxiliary bundle associated with this Spin^c structure. In this case, the 2-form α can be considered as an endomorphism of $\mathbb{S}_M$ via Clifford multiplication and we have the well-known orthogonal splitting $\mathbb{S}_M = \bigoplus_{r=0}^m \mathbb{S}_M^r$, where $\mathbb{S}_M^r$ denotes the eigensubbundle corresponding to the eigenvalue $i(m - 2r)$ of α, with complex rank $\binom{m}{k}$. The bundle $\mathbb{S}_M^r$ corresponds to $\Lambda^{0,r}M \otimes \mathcal{L}$. For the canonical Spin^c structure, the subbundle $\mathbb{S}_M^0$ is trivial. Hence and when M is a Kähler manifold, this Spin^c structure admits parallel spinors (constant functions) lying in $\mathbb{S}_M^0$ (see [**21**]). Of course, we can define another Spin^c structure for which the spinor bundle is given by $\Lambda^{*,0}M = \bigoplus_{r=0}^m \Lambda^r(T^*_{1,0}M)$ and the auxiliary line bundle by K_M. This Spin^c structure is called the anti-canonical Spin^c structure.

Any Spin^c structure on (M^{n+1}, g) induces a Spin^c structure on its boundary $\Sigma = \partial M$ and we have

$$\begin{cases} \mathbb{S}_M|_\Sigma \simeq \mathbb{S}_\Sigma & \text{if } n \text{ is even,} \\ \mathbb{S}^+_M|_\Sigma \simeq \mathbb{S}_\Sigma & \text{if } n \text{ is odd.} \end{cases}$$

We recall that if n is odd, the spinor bundle $\mathbb{S}_M$ splits into

$$\mathbb{S}_M = \mathbb{S}_M^+ \oplus \mathbb{S}_M^-,$$

by the action of the complex volume element. Moreover, Clifford multiplication with a vector field X tangent to Σ is given by

$$X \bullet \varphi = (X \cdot \nu \cdot \psi)|_\Sigma,$$

where $\psi \in \Gamma^\infty(M, \mathbb{S}_M)$ (or $\psi \in \Gamma^\infty(\mathbb{S}_M^+)$ if n is odd), φ is the restriction of ψ to Σ, '$\bullet$' is the Clifford multiplication on M. When n is odd we also obtain $\mathbb{S}_M^- \simeq \mathbb{S}_\Sigma$. In this case, the Clifford multiplication by a vector field X tangent to Σ is given by $X \bullet \varphi = -(X \cdot \nu \cdot \psi)|_\Sigma$ and hence we have $\mathbb{S}_M|_\Sigma \simeq \mathbb{S}_\Sigma \oplus \mathbb{S}_\Sigma$. Moreover, the corresponding auxiliary line bundle L^Σ on Σ is the restriction to Σ of the auxiliary line bundle L and $i\Omega^\Sigma = i\Omega|_\Sigma$. We denote by ∇^Σ the spinorial Levi-Civita connection on $\mathbb{S}_\Sigma$. For all smooth vector fields $X \in \Gamma^\infty(T\Sigma)$ and for every smooth spinor field $\psi \in \Gamma^\infty(M, \mathbb{S}_M)$, we consider $\varphi = \psi|_\Sigma$ and we have the following Spin^c Gauss formula [**15**, **22**, **23**]:

$$(\nabla_X \psi)|_\Sigma = \nabla_X^\Sigma \varphi + \tfrac{1}{2} II(X) \bullet \varphi,$$

where II denotes the Weingarten map with respect to ν. Moreover, let D and D^Σ be the Dirac operators on M and Σ. After denoting any smooth spinor and its restriction to Σ by the same symbol, we have on Σ (see [**15**, **22**, **23**]) that

$$\tilde{D}^\Sigma\varphi = \frac{n}{2}H\varphi - \nu\cdot D\varphi - \nabla_\nu\varphi, \tag{4}$$

$$\tilde{D}^\Sigma(\nu\cdot\varphi) = -\nu\cdot\tilde{D}^\Sigma\varphi, \tag{5}$$

where $H = (1/n)\,\mathrm{tr}(II)$ denotes the mean curvature and $\tilde{D}^\Sigma = D^\Sigma$ if n is even and $\tilde{D}^\Sigma = D^\Sigma \oplus (-D^\Sigma)$ if n is odd. Note that $\sigma(\tilde{D}^\Sigma) = \{\pm\lambda \mid \lambda\in\sigma(D^\Sigma)\}$, where $\sigma(A)$ denotes the spectrum of an operator A.

Bounded geometry. In this paragraph, we recall the definition of manifolds of bounded geometry.

Definition 2.2 [**25**, Definition 2.2]. Let (M^{n+1}, g) be a complete Riemannian manifold with boundary Σ. We say that (M,Σ) is of bounded geometry if the following is fulfilled.

(i) The curvature tensor of M and all its covariant derivatives are bounded.
(ii) The injectivity radius of Σ is positive.
(iii) There is a collar around Σ, that is: there is $r_\partial > 0$ such that the geodesic collar

$$F : U_\Sigma = [0, r_\partial)\times\Sigma \longrightarrow M,\quad (t,x)\longmapsto \exp_x(t\nu)$$

is a diffeomorphism onto its image where ν is the inner unit normal field on Σ. We equip U_Σ with the induced metric and will identify U_Σ with its image.

(iv) There exists $\varepsilon > 0$ such that the injectivity radius of each point $x\in M\setminus U_\Sigma$ is greater or equal than ε.

(v) The mean curvature of Σ and all its covariant derivatives are bounded.

Definition 2.3 (cp. [**26**, A.1.1] together with [**12**, Theorem B]). Let E be a hermitian vector bundle over M, where (M,Σ) is of bounded geometry. Then, E is said to be of bounded geometry if its curvature and all its covariant derivatives are bounded.

Remark 2.4. (1) Note that the above definition contains the usual definition of manifold of bounded geometry without boundary. Moreover, if (M,g) is of bounded geometry, then $(\Sigma, g|_\Sigma)$ is also of bounded geometry [**25**, Corollary 2.24].

(2) For the spinor bundle $\mathbb{S}'_M$ associated with a Spin structure, the bounded geometry follows automatically from the bounded geometry of M (see [**3**, Section 3.1.3]). For a Spin^c manifold the situation is more general since the Spin^c bundle $\mathbb{S}_M$ does not only depend on the geometry of the underlying manifold, but also on the geometry of the auxiliary line bundle L. But, $\mathbb{S}_M = \mathbb{S}'_M\otimes L^{1/2}$, where $\mathbb{S}'_M$ is the locally defined spinor bundle, $L^{1/2}$ is locally defined too and $\mathbb{S}_M$ is globally defined. Thus, the assumption that L is of bounded geometry assures that $\mathbb{S}_M$ is also of bounded geometry.

Assumption for the rest of the paper: (M,Σ) and L are of bounded geometry.

The Sobolev space H_1 on manifolds with boundary. We define the $H_1 = H_1(M,\mathbb{S}_M)$-norm on $\Gamma_c^\infty(M,\mathbb{S}_M)$ by

$$\|\varphi\|^2_{H_1(M,\mathbb{S}_M)} = \|\varphi\|^2_{L^2(M,\mathbb{S}_M)} + \|\nabla\varphi\|^2_{L^2(M,\mathbb{S}_M)}.$$

Finally, we define $H_1 = H_1(M,\mathbb{S}_M)$ as the closure of $\Gamma_c^\infty(M,\mathbb{S}_M)$ with respect to the H_1-norm defined above.

Using the Lichnerowicz formula (3), the Gauß theorem $(\nabla^*\nabla\varphi,\varphi) = \|\nabla\varphi\|^2_{L^2} + \int_\Sigma\langle\nabla_\nu\varphi,\varphi\rangle\,ds$, (2) and (4), we obtain another description of the H_1-norm: For all $\varphi\in$

$\Gamma_c^\infty(M, \mathbb{S}_M)$, we have

$$\|\varphi\|_{H_1}^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2 - \int_M \frac{\mathrm{scal}^M}{4}|\varphi|^2\, dv - \int_M \frac{i}{2}\langle \Omega\cdot\varphi, \varphi\rangle\, dv + \int_\Sigma \langle \varphi|_\Sigma, D^W(\varphi|_\Sigma)\rangle\, ds, \tag{6}$$

where $D^W = \tilde{D}^\Sigma - (n/2)H$ is the so-called Dirac–Witten operator. Note that due to the local expression of D and the Cauchy–Schwarz inequality, we always have

$$\|D\varphi\|_{L^2}^2 \leqslant \int_M \left(\sum_{i=1}^{n+1} |\nabla_{e_i}\varphi|\right)^2 dv \leqslant (n+1)\|\nabla\varphi\|_{L^2}^2, \tag{7}$$

for all $\varphi \in H_1(M, \mathbb{S}_M)$.

Spectral theory. Most of the following can be found in [8]. In this paragraph, we shortly review the spectral theory of the Dirac operator $D\colon H_1(N, \mathbb{S}_N) \subset L^2(N, \mathbb{S}_N) \to L^2(N, \mathbb{S}_N)$ on a complete Riemannian Spin^c manifold N without boundary. Note that we assume that N is of bounded geometry, and hence the graph norm of D, $\|.\|_D$, and the H_1-norm are equivalent. Then, D is self-adjoint and the spectrum is real. A real number λ is an eigenvalue of D if there exists a non-zero spinor $\varphi \in H_1$ with $D\varphi = \lambda\varphi$. Then, φ is called an eigenspinor to the eigenvalue λ. Standard local elliptic regularity theory gives that an eigenspinor is always smooth. The set of all eigenvalues is denoted by $\sigma_p(D^\Sigma)$, the point spectrum. If N is closed, then the Dirac operator has a pure point spectrum. But on open manifolds, the spectrum might have a continuous part. In general, the spectrum, denoted by $\sigma(D)$, is composed of the point, the continuous and the residual spectrum. In case of a self-adjoint operator, as we have, there is no residual spectrum. Often another decomposition of the spectrum is used, the one into discrete spectrum $\sigma_d(D)$ and essential spectrum $\sigma_{\mathrm{ess}}(D)$. A real number λ lies in the essential spectrum of D if there exists a sequence of smooth compactly supported spinors φ_i which $\|\varphi_i\|_{L^2} = 1$, φ_i converge weakly to zero and

$$\|(D-\lambda)\varphi_i\|_{L^2} \longrightarrow 0.$$

The essential spectrum contains amongst other elements all eigenvalues of infinite multiplicity. In contrast, the discrete spectrum $\sigma_d(D) := \sigma_p(D) \setminus \sigma_{\mathrm{ess}}(D)$ consists of all eigenvalues of finite multiplicity.

Closed Range Theorem. Next, we want to recall briefly (a part of) the Closed Range Theorem for later use.

THEOREM 2.5 [29, p.205]. *Let $T : X \to Y$ be a closed linear operator between Banach spaces X, Y. Then, the range $\mathrm{ran}(T)$ of T is closed in Y if and only if $\mathrm{ran}(T) = \ker(T^*)^\perp$, where T^* is the adjoint operator of T and $\ker(T^*)$ is the kernel of T^*.*

A linear operator $T : X \to Y$ between Banach spaces is called Fredholm if its kernel is finite-dimensional and its image has finite codimension.

3. *Trace theorems and extensions*

We consider the restriction operator

$$R : \Gamma_c^\infty(M, \mathbb{S}_M) \longrightarrow \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma),$$
$$\varphi \longmapsto \varphi|_\Sigma.$$

If it is clear from the context that $R\varphi$ is considered instead of φ, then we will sometimes abbreviate by using φ only. The first part of this section will be devoted to see how the restriction operator R extends to a bounded linear operator between the Sobolev spaces $H_1(M, \mathbb{S}_M)$ and $H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. This theorem is known as Trace Theorem and is a very classical result for $\mathbb{R}^n_+$ and compact manifolds with boundary. After reviewing the Euclidean result and basic definitions, we will shortly review how this result extends to manifolds (M, Σ) of bounded geometry. In particular, the restriction operator will have a bounded linear right inverse, that is called extension operator $\mathcal{E}$.

For more details on the definition of bounded geometry on manifolds with boundary, see [**25**]. For the equivalence of all those different definitions of Sobolev-norms involved here and the corresponding theorems for submanifolds (not necessarily hypersurfaces), see [**14**].

For our purpose, Sobolev spaces will not be sufficient later on. The maximal domain of the Dirac operator is bigger than $H_1(\mathbb{S}_M)$. The rest of this section is devoted to define an extension operator $\tilde{\mathcal{E}}$ such that $\tilde{\mathcal{E}}R : \Gamma_c^\infty(M, \mathbb{S}_M) \to \Gamma_c^\infty(M, \mathbb{S}_M)$ extends to a bounded operator with respect to the graph norm of D. For the definition of $\tilde{\mathcal{E}}$ we will use the special extension map introduced by Bär and Ballmann [**9**] for closed boundaries.

3.1. *Trace and extension for Sobolev spaces*

Trace Theorem for functions on $\mathbb{R}^{n+1}_+ = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 \geqslant 0\}$.

We identify the boundary of $\mathbb{R}^{n+1}_+$ with $\mathbb{R}^n$. First, we repeat the definition of the Sobolev spaces $H_s(\mathbb{R}^n, \mathbb{C}^r)$:

Definition 3.1 [**27**, Definition 3.1]. Let $s \in \mathbb{R}$. The $H_s := H_s^2$-norm of a compactly supported function $f : \mathbb{R}^n \mapsto \mathbb{C}^r$ is defined as

$$\|f\|^2_{H_s(\mathbb{R}^n,\mathbb{C}^r)} := \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 (1 + |\xi|)^s d\xi,$$

where $\hat{f}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(\xi)\, d\xi$ denotes the Fourier transform of f. The space $H_s(\mathbb{R}^n, \mathbb{C}^r)$ is then defined as the completion of $\Gamma_c^\infty(\mathbb{R}^n, \mathbb{C}^r)$, the space of smooth compactly supported functions on $\mathbb{R}^n$ with values in $\mathbb{C}^r$, with respect to the H_s-norm.

The spaces $H_s(\mathbb{R}^{n+1}_+, \mathbb{C}^r)$ are defined analogously.

Theorem 3.2 [**24**, Theorems 7.34 and 7.36; **27**, Theorem I.3.4; **28**, p.138, Remark 1]. *Let $s > \frac{1}{2}$. The restriction map for complex-valued smooth functions $R : \Gamma_c^\infty(\mathbb{R}^{n+1}_+) \to \Gamma_c^\infty(\mathbb{R}^n)$, $f \to f|_{\mathbb{R}^n}$ extends to a bounded linear operator from $H_s(\mathbb{R}^{n+1}_+)$ to $H_{(s-1/2)}(\mathbb{R}^n)$. Moreover, there is an extension operator $\mathcal{E} : H_{(s-1/2)}(\mathbb{R}^n) \to H_s(\mathbb{R}^{n+1}_+)$ that is a bounded linear operator and a right inverse to R.*

The generalization of this theorem to vector-valued Sobolev spaces follows immediately by the following: Let $f = (f_1, \dots, f_r) : \mathbb{R}^n \to \mathbb{C}^r$. Then, the norms $\|f\|_{H_s(\mathbb{R}^n,\mathbb{C}^r)}$ and $\sum_{i=1}^r \|f_i\|_{H_s(\mathbb{R}^n,\mathbb{C})}$ are equivalent.

Trace theorem on manifolds of bounded geometry.

From now on, let M be a Riemannian manifold possibly with boundary and of bounded geometry, as in Definition 2.2. Moreover, let E be a hermitian vector bundle over M. We assume that E is also of bounded geometry, see Definition 2.3. To obtain a trace theorem for sections in E we need coordinates of the manifold that are adapted to the structure of the boundary. Those will be Fermi coordinates and there will be a adapted synchronous trivialization of E. This will allow that we can use the trace theorem on $\mathbb{R}^n$ on the individual charts to obtain the trace theorem on (M, Σ).

In the following, we restrict to trace theorems for Sobolev spaces over L^2, for more general domains as Sobolev spaces over L^p or Triebel–Lizorkin spaces see [**14**].

Before we define Sobolev spaces for sections of E, we introduce Fermi coordinates adapted to the boundary and a corresponding synchronous trivialization of the vector bundle.

Definition 3.3 [**14**, Definition 4.3 and Lemma 4.4; **25**, Definition 2.3]. Let (M^n, Σ) be of bounded geometry, see Definition 2.2 and the notions defined therein.

Let $r = \min\{\frac{1}{2}r_\Sigma, \frac{1}{4}r_M, \frac{1}{2}r_\partial\}$, where r_Σ is the injectivity radius of Σ and r_M the one of M. Let $p_\alpha^\Sigma \in \Sigma$ and $p_\beta \in M$ be points such that

(i) the metric balls $B_r^\Sigma(p_\alpha^\Sigma)$ in Σ (that is, with respect to the metric $g|_\Sigma$) give a uniformly locally finite cover of Σ;
(ii) the metric balls $B_r(p_\beta)$ in M cover $M \setminus U_r(\Sigma)$, where $U_r(\Sigma) := F([0,r) \times \Sigma)$ and those balls are uniformly locally finite on all of M.

Let $(U_\gamma)_\gamma$ be a locally finite covering of M where each U_γ is of the form $B_r(p_\beta)$ or $U_{p_\alpha^\Sigma}^\Sigma = F([0,2r) \times B_{2r}^\Sigma(p_\alpha^\Sigma))$. By construction, the covering $(U_\gamma)_\gamma$ is locally finite. Coordinates on U_γ are chosen to be geodesic normal coordinates around p_β in case $U_\gamma = B_r(p_\beta)$. Otherwise, coordinates are given by Fermi coordinates

$$\kappa_\alpha : U_{p_\alpha^\Sigma}^\Sigma := [0,2r) \times B_{2r}(0) \subset \mathbb{R}^n \longrightarrow U_{p_\alpha^\Sigma}^\Sigma, \quad (t,x) \longmapsto \exp_{\exp_{p_\alpha^\Sigma}^\Sigma(x)}(t\nu),$$

where ν is the inner normal field of Σ and $\exp^\Sigma$ is the exponential map on Σ with respect to the induced metric. We call such coordinates $(U_\gamma, \kappa_\gamma)_\gamma$ Fermi coordinates for (M, Σ). If $U_\gamma = B_r(p_\gamma)$, then $E|_{U_\gamma}$ is trivialized via parallel transport along radial geodesic and we identify $E|_{U_\gamma}$ with the trivial $\mathbb{C}^r$-bundle over U_γ. Otherwise, $E|_{U_\gamma}$ is trivialized via parallel transport along radial geodesic of the boundary and along the normal direction. The obtained trivialization is denoted by $(\xi_\gamma)_\gamma$.

In case of manifolds without boundary, the Definition of ξ_γ in 3.3 is the usual definition of synchronous trivialization as found in [**3**, Section 3.1.3]. Note that by construction the restriction of a synchronous trivialization of E over a manifold M to its boundary Σ gives a synchronous trivialization of $E|_\Sigma$.

Lemma 3.4 [**14**, Lemma 4.8]. *There is a partition of unity h_γ subordinated to the Fermi coordinates introduced above fulfilling: For all $k \in \mathbb{N}$ there is $c_k > 0$ such that for all γ and all multi-indices $\mathfrak{a} = (\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$ with $|\mathfrak{a}| \leqslant k$*

$$|D^\mathfrak{a}(h_\gamma \circ \kappa_\gamma)| \leqslant c_k.$$

Here, $D^\mathfrak{a} = \partial^{\mathfrak{a}_1}/(\partial x_1)^{\mathfrak{a}_1} \cdots \partial^{\mathfrak{a}_n}/(\partial x_n)^{\mathfrak{a}_n}$, where x_i are the coordinates.

Now, we have all the ingredients to define Sobolev spaces on E via local pullback to vector-valued functions over $\mathbb{R}^n$.

Definition 3.5 [**14**, Definition 5.9]. Let $s \in \mathbb{R}$. Let $(U_\gamma)_\gamma$ be a covering of M together with a synchronous trivialization ξ_γ of E as defined above. Moreover, let the covering be locally finite, and let h_γ be a partition of unity subordinated to U_γ as in Lemma 3.4. Then,

$$\|\varphi\|_{H_s(M,E)}^2 := \sum_\alpha \|\xi_{\alpha*}(h_\alpha \varphi)\|_{H_s(\mathbb{R}_+^n, \mathbb{C}^r)}^2.$$

Note that up to equivalence the H_s-norm does not depend on the choices of $(U_\gamma, h_\gamma, \xi_\gamma)$, cp. [**14**, Theorem 4.9, 5.11 and Lemma 5.13].

REMARK 3.6. (i) For $s \in \mathbb{N}$, the definition of $H_s(M,E)$ from above is equivalent to the usual definition given by

$$\|\varphi\|_{H_s(M,E)} := \sum_{i=0}^{s} \|\underbrace{\nabla^E \cdots \nabla^E}_{i \text{ times}} \varphi\|_{L^2(M,E)},$$

cp. [**14**, Theorem 5.7; **25**].

(ii) For $s \leqslant t$, we have $\|\varphi\|_{H_s(M,E)} \leqslant \|\varphi\|_{H_t(M,E)}$. That is seen for $M = \mathbb{R}^n_+$ immediately using $(1+|\xi|)^s \leqslant (1+|\xi|)^t$. For general M, one just lifts this result by using a partition of unity and a synchronous trivialization.

(iii) Let $D^\Sigma : \Gamma_c^\infty(\Sigma, \mathbb{S}_\Sigma) \to \Gamma_c^\infty(\Sigma, \mathbb{S}_\Sigma)$ be the Dirac operator on Σ. For any $s \in \mathbb{R}$, there is a unique closed extension of D^Σ from $H_s(\Sigma, \mathbb{S}_\Sigma) \to H_{s-1}(\Sigma, \mathbb{S}_\Sigma)$.

THEOREM 3.7. *Let M^n be a Riemannian manifold with boundary Σ. Assume that (M, Σ) is of bounded geometry and that E is a hermitian vector bundle over M that is also of bounded geometry. Then, for all $s \in \mathbb{R}$ with $s > \frac{1}{2}$, the operator $R : \Gamma_c^\infty(M,E) \to \Gamma_c^\infty(\Sigma, E|_\Sigma)$ with $\varphi \mapsto \varphi|_\Sigma$ extends to a bounded linear operator from $H_s(M,E)$ to $H_{s-1/2}(\Sigma, E|_\Sigma)$. Moreover, there is a bounded right inverse $\mathcal{E} : H_{s-1/2}(\Sigma, E_\Sigma) \to H_s(M,E)$ of the trace map $R : H_s(M,E) \to H_{s-1/2}(\Sigma, E|_\Sigma)$. In particular, $\mathcal{E}(\Gamma_c^\infty(\Sigma, E|_\Sigma)) \subset \Gamma_c^\infty(M, E_M)$.*

Proof. This theorem is a special case of [**14**, Theorem 5.14]. We shortly sketch the basic idea: We choose a covering U_γ together with a synchronous trivialization ξ_γ of E and a subordinated partition of unity h_γ as in Definition 3.3 and Lemma 3.4. The restrictions $U_\gamma \cap \Sigma$ then cover Σ. Let $\varphi \in H_s(M,E)$. Then, for all α, we have $\xi_{\alpha *}(h_\alpha \varphi) \in H_s(\mathbb{R}^n_+, \mathbb{C}^r)$. Thus, there exists a $C > 0$ with $\|R(\xi_{\gamma *}(h_\gamma \varphi))\|_{H_{s-1/2}(\mathbb{R}^{n-1}, \mathbb{C}^r)} \leqslant C \|\xi_{\gamma *}(h_\gamma \varphi)\|_{H_s(\mathbb{R}^n_+, \mathbb{C}^r)}$.

With $R(\xi_{\alpha *}(h_\alpha \varphi)) = \xi_{\alpha *}(h_\alpha R\varphi)$ we get after summing up that $\|R\varphi\|_{H_{s-1/2}(\Sigma, E|_\Sigma)} \leqslant C\|\varphi\|_{H_s(M,E)}$ since ξ_α is still a synchronous trivialization for $E|_\Sigma$.

The rest is proved analogously as the Trace Theorem using the original Euclidean version of the extension map $\mathcal{E} : H_{s-1/2}(\mathbb{R}^{n-1}) \to H_s(\mathbb{R}^n)$. The last inclusion follows immediately from $\mathcal{E}(\Gamma_c^\infty(\mathbb{R}^{n-1})) \subset \Gamma_c^\infty(\mathbb{R}^n)$. □

The last theorem gives immediately the following corollary.

COROLLARY 3.8. *The map $\mathcal{E}R : \Gamma_c^\infty(M,E) \to \Gamma_c^\infty(M,E)$ extends to a bounded linear map $\mathcal{E}R : H_s(M,E) \to H_s(M,E)$ for all $s > \frac{1}{2}$.*

LEMMA 3.9. *The L^2-product $(\varphi, \psi) = \int_\Sigma \langle \varphi, \psi \rangle \, dv$ for $\varphi, \psi \in \Gamma_c^\infty(\Sigma, E|_\Sigma)$ extends to a perfect pairing $H_s(\Sigma, E|_\Sigma) \times H_{-s}(\Sigma, E|_\Sigma) \to \mathbb{C}$ for all $s \in \mathbb{R}$.*

Proof. This is also proved in the same way as above, by lifting the corresponding result from the Euclidean case [**27**, Section I.3]. □

The Trace Theorem now allows one to extend the allowed domain for the spinors in the Equalities (6) and (2).

Lemma 3.10. *For all $\varphi, \psi \in H_1(M, \mathbb{S}_M)$, Equalities (6) and (2) hold.*

Proof. The proof is a more or less straightforward usage of the Trace Theorem 3.7 and the corresponding equalities on $\Gamma_c^\infty(M, \mathbb{S}_M)$. Indeed, let φ_i be a sequence in $\Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ in $H_1(M, \mathbb{S}_M)$. The Trace Theorem 3.7 gives $R\varphi_i \to R\varphi$ in $H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$ and, hence, $\tilde{D}^\Sigma R\varphi_i \to \tilde{D}^\Sigma R\varphi$ in $H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$, cf. Remark 3.6(iii). Clearly, $\|\varphi_i - \varphi\|_{H_1} \to 0$ and with (7), this implies $\|\varphi_i - \varphi\|_D \to 0$. Moreover, the bounded geometry of (M, Σ) implies

$$\int_M \mathrm{scal}^M |\varphi_i|^2 \, dv \longrightarrow \int_M \mathrm{scal}^M |\varphi|^2 \, dv, \quad \int_\Sigma H|\varphi_i|^2 \, ds \longrightarrow \int_\Sigma H|\varphi|^2 \, ds,$$

and

$$\left| \int_M \langle \Omega \cdot \varphi_i, \varphi_i \rangle \, dv - \int_M \langle \Omega \cdot \varphi, \varphi \rangle \, dv \right| \leqslant (\|\varphi_i - \varphi\|_{L^2} \|\varphi\|_{L^2} + \|\varphi_i\|_{L^2} \|\varphi_i - \varphi\|_{L^2}) \sup_M |\Omega| \longrightarrow 0.$$

Note that due to the bounded geometry of L, $\sup_M |\Omega|$ is finite. It remains to consider the term $\int_\Sigma \langle R\varphi, \tilde{D}^\Sigma R\varphi \rangle \, ds$. First, we note that due to the pairing in Lemma 3.9, the Trace Theorem 3.7, and $\tilde{D}^\Sigma : H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma) \to H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$, this expression is finite for all $\varphi \in H_1(M, \mathbb{S}_M)$. Abbreviating $R\varphi$ by φ, we have

$$\begin{aligned} |(\tilde{D}^\Sigma \varphi_i, \varphi_i)_\Sigma - (\tilde{D}^\Sigma \varphi, \varphi)_\Sigma| &\leqslant |(\tilde{D}^\Sigma \varphi_i, \varphi - \varphi_i)_\Sigma| + |(\tilde{D}^\Sigma \varphi - \tilde{D}^\Sigma \varphi_i, \varphi)_\Sigma| \\ &\leqslant \|\tilde{D}^\Sigma \varphi_i\|_{H_{-1/2}} \|\varphi - \varphi_i\|_{H_{1/2}} + \|\tilde{D}^\Sigma \varphi - \tilde{D}^\Sigma \varphi_i\|_{H_{-1/2}} \|\varphi\|_{H_{1/2}}, \end{aligned}$$

which gives the convergence of the last term. This proves Equality (6) for all $\varphi \in H_1(M, \mathbb{S}_M)$. Now, let φ_i, ψ_j be sequences in $\Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ and $\psi_j \to \psi$ in $H_1(M, \mathbb{S}_M)$. Then,

$$\begin{aligned} |(D\psi_j, \varphi_i) - (D\psi, \varphi)| &\leqslant |(D\psi_j, \varphi_i) - (D\psi_j, \varphi)| + |(D\psi_j, \varphi) - (D\psi, \varphi)| \\ &\leqslant \|D\psi_j\|_{L^2} \|\varphi_i - \varphi\|_{L^2} + \|D(\psi_j - \psi)\|_{L^2} \|\varphi\|_{L^2}. \end{aligned}$$

Using (7) and that φ_i and ψ_j are uniformly bounded in H_1, we get for a certain constant $C > 0$ that

$$|(D\psi_j, \varphi_i) - (D\psi, \varphi)| \leqslant C\|\varphi_i - \varphi\|_{L^2} + C\|\psi_j - \psi\|_{H_1} \longrightarrow 0.$$

Analogously, one obtains $(\psi_j, D\varphi_i) \to (\psi, D\varphi)$. Moreover, using again the Trace Theorem 3.7, we obtain

$$\begin{aligned} \left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i \rangle - \langle \nu \cdot R\psi_j, R\varphi \rangle \, ds \right| &\leqslant \|R\psi_j\|_{L^2(\Sigma)} \|R(\varphi_i - \varphi)\|_{L^2(\Sigma)} \\ &\leqslant C\|\psi_j\|_{H_1} \|\varphi_i - \varphi\|_{H_1} \longrightarrow 0. \end{aligned}$$

In the same way, $|\int_\Sigma \langle \nu \cdot R\psi_j, R\varphi \rangle - \langle \nu \cdot R\psi, R\varphi \rangle \, ds| \to 0$. Hence,

$$\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i \rangle - \langle \nu \cdot R\psi, R\varphi \rangle \, ds \right| \longrightarrow 0.$$

This proves Equality (2) for all $\varphi, \psi \in H_1(M, \mathbb{S}_M)$. □

3.2. *Extension and the graph norm*

Spectral decomposition of the boundary. Let (M, Σ) be of bounded geometry. Then, $(\Sigma, g|_\Sigma)$ is complete and, thus, the Dirac operator D^Σ on $\mathbb{S}_\Sigma$, and thus also $\tilde{D}^\Sigma$ on $\mathbb{S}_M|_\Sigma$, is self-adjoint.

Let $\{E_I\}_{I\subset\mathbb{R}}$ be the family of projector-valued measures belonging to the self-adjoint operator

$$\tilde{D}^\Sigma : H_1(\Sigma, \mathbb{S}_M|_\Sigma) \subset L^2(\Sigma, \mathbb{S}_M|_\Sigma) \longrightarrow L^2(\Sigma, \mathbb{S}_M|_\Sigma).$$

We define for a connected (not necessarily bounded) interval $I \in \mathbb{R}$ the spectral projection

$$\pi_I : L^2(\Sigma, \mathbb{S}_M|_\Sigma) \longrightarrow L^2(\Sigma, \mathbb{S}_M|_\Sigma), \quad \varphi \longmapsto E_I\varphi$$

and the spaces

$$\Gamma_I^{\mathrm{APS}} = \{\varphi \in L^2(\Sigma, \mathbb{S}_M|_\Sigma) \,|\, \varphi = \pi_I\varphi\}.$$

Next, we will show that for every $s \in \mathbb{R}$ the spectral projections extend to bounded linear maps from $H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ to itself: First, we note that the spectral projections commute with $\tilde{D}^\Sigma$. Moreover, since $(\Sigma, g|_\Sigma)$ has bounded geometry, the norm $\|\varphi\|_{L^2}^2 + \|D^k\varphi\|_{L^2}^2$ and the H_k-norm are equivalent on $\Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$ for $k \in \mathbb{N}_0$, cp. [**3**, Lemma 3.1.6]. Hence, π_I restricts to a bounded linear map from $H_k(\Sigma, \mathbb{S}_M|_\Sigma)$ to itself for $k \in \mathbb{N}_0$. Let now k be a negative integer, $\varphi \in \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$ and $\psi \in H_{-k}(\Sigma, \mathbb{S}_M|_\Sigma)$. Using that π_I is symmetric with respect to L^2-product on $(\Sigma, \mathbb{S}_M|_\Sigma)$ and Lemma 3.9, we obtain

$$|(\pi_I\varphi, \psi)_\Sigma| = |(\varphi, \pi_I\psi)_\Sigma| \leqslant C\|\varphi\|_{H_{-k}(\Sigma)}\|\pi_I\psi\|_{H_k(\Sigma)} \leqslant C'\|\varphi\|_{H_{-k}(\Sigma)}\|\psi\|_{H_k(\Sigma)}.$$

Thus, π_I extends to a bounded linear map from $H_k(\Sigma, \mathbb{S}_M|_\Sigma)$ to itself for all nonnegative integers k. Then, by Riesz–Thorin Interpolation Theorem we obtain that $\pi_I : H_s(\Sigma, \mathbb{S}_M|_\Sigma) \to H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ for all $s \in \mathbb{R}$.

We abbreviate $\pi_> = \pi_{(0,\infty)}$ and $\pi_\leqslant = \pi_{(-\infty,0]}$. As in [**9**, Section 5], we define, for $\varphi \in \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$,

$$\|\varphi\|_{\check{H}}^2 = \|\pi_\leqslant\varphi\|_{H_{1/2}(\Sigma)}^2 + \|\pi_>\varphi\|_{H_{-1/2}(\Sigma)}^2 \quad \text{and} \quad \|\varphi\|_{\hat{H}}^2 = \|\pi_\leqslant\varphi\|_{H_{-1/2}(\Sigma)}^2 + \|\pi_>\varphi\|_{H_{1/2}(\Sigma)}^2$$

and the spaces

$$\check{H} := \overline{\Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)}^{\|.\|_{\check{H}}} \quad \text{and} \quad \hat{H} := \overline{\Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)}^{\|.\|_{\hat{H}}}. \tag{8}$$

Local description of the graph norm on (M, Σ). Let (M, g) be a manifold with boundary Σ. Let $(U_\gamma, \kappa_\gamma, \xi_\gamma, h_\gamma)_\gamma$ be Fermi coordinates on (M, g) together with a synchronous trivialization ξ_γ and a partition of unity h_γ.

LEMMA 3.11. *On $\Gamma_c^\infty(M, \mathbb{S}_M)$ the norms $\|.\|_D$ and $(\sum_\gamma \|h_\gamma.\|_D^2)^{1/2}$ are equivalent.*

Proof. All the constants c_i involved here are positive. Let $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$. Since the cover U_γ is uniformly locally finite the norms $\|.\|_{L^2}$ and $(\sum_\gamma \|h_\gamma.\|_{L^2}^2)^{1/2}$ are equivalent. Thus,

$$\begin{aligned}
\|D\varphi\|_{L^2}^2 &\leqslant c_1 \sum_\gamma \|h_\gamma D\varphi\|_{L^2}^2 = c_1 \sum_\gamma \|D(h_\gamma\varphi) - \nabla h_\gamma \cdot \varphi\|_{L^2}^2 \\
&\leqslant c_2 \sum_\gamma (\|D(h_\gamma\varphi)\|_{L^2}^2 + \|\nabla h_\gamma \cdot \varphi\|_{L^2}^2) \leqslant c_3 \sum_\gamma (\|D(h_\gamma\varphi)\|_{L^2}^2 + \|\varphi|_{U_\gamma}\|_{L^2}^2) \\
&\leqslant c_3 \sum_\gamma \|D(h_\gamma\varphi)\|_{L^2}^2 + c_4\|\varphi\|_{L^2}^2,
\end{aligned}$$

where the end of the second line follows by Lemma 3.4, and the last inequality follows since the cover U_γ is uniformly locally finite. Hence, $\|\varphi\|_D^2 \leqslant c_5 \sum_\gamma \|h_\gamma\varphi\|_D^2$.

Conversely, we obtain analogously

$$\sum_\gamma \|D(h_\gamma\varphi)\|_{L^2}^2 = \sum_\gamma \|h_\gamma D\varphi + \nabla h_\gamma \cdot \varphi\|_{L^2}^2 \leqslant c_6\|\varphi\|_D^2. \qquad \square$$

LEMMA 3.12. *Let $(\Sigma, g|_\Sigma)$ be a manifold of bounded geometry. Then, there is an $\varepsilon > 0$ smaller than the injectivity radius of Σ and a constant $c > 0$ such that for all $x \in \Sigma$ and $\varphi \in \Gamma_c^\infty(B_\varepsilon(x) \subset N, \mathbb{S}_N)$, we have $\|D^N\varphi\|_{L^2} > c\|\varphi\|_{L^2}$.*

Proof. Let $\exp_x^\Sigma : B_\varepsilon(0) \subset \mathbb{R}^n \to B_\varepsilon(x) \subset \Sigma$ be the exponential map. Set $\tilde{g} := (\exp_x^\Sigma)^* g|_{B_\varepsilon(x)}$. We will compare the Dirac operator $D^{\tilde{g}}$ with D^E, [**5**, Proposition 3.2]:

$$D^{\tilde{g}}\varphi = D^E\varphi + \sum_{ij}(b_i^j - \delta_i^j)\partial_i \cdot \nabla_{\partial_j}\varphi + \frac{1}{4}\sum_{ijk}\tilde{\Gamma}_{ij}^k e_i \cdot e_j \cdot e_k \cdot \varphi,$$

where φ is a smooth spinor over $B_\varepsilon(0)$, ∂_i and e_i form an orthonormal basis with respect to the Euclidean metric and $\tilde{g}$, respectively. Moreover, $e_i = b_i^j\partial_j$, ∇ is the Levi-Civita connection with respect to the Euclidean metric, and $\tilde{\Gamma}_{ij}^k$ are the Christoffel symbols with respect to the metric $\tilde{g}$. By [**5**, (11)–(13) and below], $|b_i^j - \delta_i^j| \leqslant Cr^2$ and $|\tilde{\Gamma}_{ij}^k| \leqslant Cr$, where r is the Euclidean distance to the origin and C can be chosen to depend only on the global curvature bounds of g. Moreover, note that there is a positive constant C also depending only on the global curvature bounds of g such that $C^{-1} \leqslant f \leqslant C$ where $\mathrm{dvol}_{\tilde{g}} = f\mathrm{dvol}_{g_E}$. Thus, for ε small enough we can estimate for all smooth spinors φ compactly supported in $B_\varepsilon(0)$ that

$$\begin{aligned}\frac{\|D^{\tilde{g}}\varphi\|_{L^2(\tilde{g})}^2}{\|\varphi\|_{L^2(\tilde{g})}^2} &\geqslant C_1\frac{\|D^E\varphi\|_{L^2(g_E)}^2}{\|\varphi\|_{L^2(g_E)}^2} - C_2\varepsilon^2\frac{\|\nabla\varphi\|_{L^2(g_E)}^2}{\|\varphi\|_{L^2(g_E)}^2} - C_3\varepsilon \\ &\geqslant C_4\frac{\|D^E\varphi\|_{L^2(g_E)}^2}{\|\varphi\|_{L^2(g_E)}^2} - C_5\varepsilon,\end{aligned}$$

where the last step uses the equivalence of the graph norm and the H_1-norm. Let A be such that $\|D^E\psi\|_{L^2(g_E)}^2 \geqslant A\|\psi\|_{L^2(g_E)}^2$ for smooth spinors compactly supported in $B_\varepsilon(0)$. Then, one can always choose ε small enough that $C_4A - C_5\varepsilon \geqslant 2^{-1}C_4A =: c$ Thus, the same is true for D^g on $B_\varepsilon(x) \subset \Sigma$. $\square$

Let $(\hat{M}, \hat{N} = \partial\hat{M})$ be manifold of bounded geometry with closed boundary. Let $\mathcal{E}_{\mathrm{BB}}$ be an extension map as defined in [**9**, (43)]. Let D and $D^{\hat{N}}$ be the Dirac operators on $\hat{M}$ and $\hat{N}$, respectively. By [**9**, Lemma 6.1, 6.2, (41) and below], we have, for all $\varphi \in \Gamma_c^\infty(\hat{M}, \mathbb{S}_{\hat{M}}|_{\hat{N}})$,

$$\|\mathcal{E}_{\mathrm{BB}}R\varphi\|_D \leqslant C\|\varphi\|_D. \tag{9}$$

Note that C can be chosen to depend only on curvature bounds of $(\hat{M}, \hat{N})$ including mean curvature, the injectivity radii of $\hat{M}$ and $\hat{N}$, respectively, and the spectral gap of $D^{\hat{N}}$.

We now come back to our pair (M, N): Let $\varepsilon, c > 0$ be constants such that Lemma 3.12 is fulfilled. Let $(U_\gamma, \kappa_\gamma, \xi_\gamma, h_\gamma)$ be Fermi coordinates together with a subordinated partition of unity such that there are $x_\gamma \in \Sigma$ with $U_\gamma \cap \Sigma \subset B_\varepsilon(x_\gamma)$. Let $\hat{U}_\gamma$ be a manifold with closed boundary $\hat{U}'_\gamma := \partial\hat{U}_\gamma$ such that $\tilde{U}_\gamma := U_\gamma \cup (\bigcup_{\alpha; U_\alpha\cap U_\gamma\neq\varnothing} U_\alpha)$ can be isometrically embedded in $\hat{U}_\gamma$, $\tilde{U}_\gamma \cap \Sigma \subset \hat{U}'_\gamma$, such that the spectral gap of the Dirac operator on $\hat{U}'_\gamma$ is at least $[-c, c]$ and such that the curvature, mean curvature of the boundary and the injectivity radii are still uniformly bounded in γ.

Define the map $\tilde{\mathcal{E}} : \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma) \to \Gamma_c^\infty(M, \mathbb{S}_M)$ by

$$\tilde{\mathcal{E}}\psi = \sum_{\gamma,\alpha;\ U'_\gamma \neq \varnothing, U_\gamma \cap U_\alpha \neq \varnothing} h_\alpha \mathcal{E}_{\mathrm{BB}}(h_\gamma|_\Sigma \psi),$$

where $h_\gamma|_\Sigma \varphi$ is understood to be a spinor on $U_\gamma \cap N \subset \hat{U}'_\gamma$ and then $\mathcal{E}_{\mathrm{BB}}(h_\gamma|_\Sigma\psi)$ is a spinor on $\hat{U}_\gamma$. The only reason why $\sum_\alpha h_\alpha$ appears in the definition is to assure that each summand can be seen to live on M and that $R\tilde{\mathcal{E}} = \mathrm{Id}$. Note that just using h_γ in front of $\mathcal{E}_{\mathrm{BB}}$ would be enough to first requirement but not the second.

Proposition 3.13. *Using the notations from above, there is a positive constant C such that, for all $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$,*

$$\|\tilde{\mathcal{E}} R\varphi\|_D \leqslant C\|\varphi\|_D.$$

Proof. We abbreviate $h'_\gamma := h_\gamma|_\Sigma$. Using (in this order) the definition of $\tilde{\mathcal{E}}$, Lemma 3.11 the uniform local finiteness of the cover U_γ, (9), and again Lemma 3.11 we estimate

$$\|\tilde{\mathcal{E}} R\varphi\|_D^2 \leqslant C_1 \Bigg\| \sum_{\gamma, U'_\gamma \neq \varnothing} \mathcal{E}_{\mathrm{BB}} R(h_\gamma\varphi) \Bigg\|_D^2 \leqslant C_2 \sum_{\gamma, U'_\gamma \neq \varnothing} \|\mathcal{E}_{\mathrm{BB}} R(h_\gamma\varphi)\|_D^2$$
$$\leqslant C_3 \sum_{\gamma, U'_\gamma \neq \varnothing} \|h_\gamma\varphi\|_D^2 \leqslant C\|\varphi\|_D^2. \qquad \square$$

4. *Boundary value problems*

The general theory of boundary value problems for elliptic differential operators of order 1 on complete manifolds with closed boundary can be found in [**9**]. The aim of this section is to generalize a part of this theory to noncompact boundaries on manifolds of bounded geometry. We restrict to the part that gives existence of solutions of boundary value problems as in Theorem 1.2. The property needed to assure a solution to such a problem is the closedness of the range. For that, we introduce a type of coercivity condition which, in general, can depend on the boundary values (that is not the case for closed boundaries). Moreover, we restrict to the classical Spin^c Dirac operator.

In the first part, we first give some generalities on domains of the Dirac operator and introduce a coercivity condition that implies closed range of the Dirac operator. Then, we extend the trace map R to the whole maximal domain of the Dirac operator and give some examples and properties of boundary conditions. In particular, we will introduce two boundary conditions $B_\pm$ which will be used to prove Theorem 1.2 in Section 8. At the end, we give an existence result for boundary value problems in our context.

General domains and closed range. Let D be the Dirac operator acting on $\Gamma_{cc}^\infty(M, \mathbb{S})$ on a manifold M with boundary Σ. If we want to emphasize that D acts on the domain $\Gamma_{cc}^\infty(M, \mathbb{S})$, then we shortly write D_{cc}. We denote the graph norm of D by

$$\|\varphi\|_D^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

By $D_{\max} := (D_{cc})^*$ we denote the maximal extension of D. Here, A^* denotes the adjoint operator of A in the sense of functional analysis. Note that $H_1(M, \mathbb{S}_M) \subset \mathrm{dom}\, D_{\max}$ and

$$\mathrm{dom}\, D_{\max} = \{\varphi \in L^2(M, \mathbb{S}_M) \,|\, \exists \tilde{\varphi} \in L^2(M, \mathbb{S}_M) \forall \psi \in \Gamma_{cc}^\infty(M, \mathbb{S}_M) : (\tilde{\varphi}, \psi) = (\varphi, D\psi)\},$$

and together with $\|.\|_D$, the space $\mathrm{dom}\, D_{\max}$ is a Hilbert space. Moreover, we denote by $D_{\min} := (D_{cc})^{**} = \bar{D}_{cc}^{\|.\|_D}$ the minimal extension of D. Here, $\bar{A}^{\|\cdot\|_D}$ denotes the closure of the set A with respect to the graph norm. Any closed linear subset of $\operatorname{dom} D_{\max}$ between $\operatorname{dom} D_{\min}$ and $\operatorname{dom} D_{\max}$ gives the domain of a closed extension of $D\colon \Gamma_{cc}^\infty(M,\mathbb{S}_M)\to\Gamma_{cc}^\infty(M,\mathbb{S}_M)$. Before examining those domains, let us extend the trace map to $\operatorname{dom} D_{\max}$:

Extension of the trace map. The Trace Theorem 3.7 extends the trace map

$$R:\Gamma_c^\infty(M,\mathbb{S}_M)\to\Gamma_c^\infty(\Sigma,\mathbb{S}_M|_\Sigma),\\ \varphi\mapsto\varphi|_\Sigma$$

to a bounded map $R: H_1(M,\mathbb{S}_M)\to H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)$. Here, we will extend R further to $\operatorname{dom} D_{\max}$. This will generalize the corresponding result [**9**, Theorem 6.7(ii)] for closed boundaries to noncompact boundaries. Moreover, we give some auxiliary lemmata which are found in [**9**] for closed boundaries. Some of the proofs and the order of obtaining them will be a little bit different since we do not use (and cannot use, cf. Example 4.16(iv)) the projection to the negative spectrum. Note that in this part we could use an arbitrary extension map as given by Theorem 3.7 and are not restricted to the explicit one defined via the eigenvalue decomposition of $\tilde{D}^\Sigma$ on closed boundaries used in [**9**].

Lemma 4.1. *The space* $\Gamma_c^\infty(M,\mathbb{S}_M)$ *is dense in* $\operatorname{dom} D_{\max}$ *with respect to the graph norm.*

Proof. For a closed boundary, this is done in [**9**, Theorem 6.7(i)]. We use a different proof here. Let $\varphi\in\operatorname{dom} D_{\max}$. Let K_i be a compact exhaustion of M that comes together with smooth cut-off functions $\eta_i: M\to[0,1]$ such that $\eta_i=1$ on K_i, $\eta_i=0$ on $M\setminus K_{i+1}$ and $\max|d\eta_i|\leqslant 2/i$. Then, $\varphi_i=\eta_i\varphi$ are compactly supported sections in $\operatorname{dom} D_{\max}$ fulfilling

$$\begin{aligned}\|\varphi_i-\varphi\|_D^2&=\|\varphi_i-\varphi\|_{L^2}^2+\|D\varphi_i-D\varphi\|_{L^2}^2\\ &\leqslant\|(1-\eta_i)\varphi\|_{L^2}^2+\left(\|(1-\eta_i)D\varphi\|_{L^2}+\frac{2}{i}\|\varphi\|_{L^2}\right)^2\to 0.\end{aligned}$$

Each φ_i has now compact support in K_{i+1}. Thus, there is a sequence $\varphi_{ij}\in\Gamma_c^\infty(K_{i+1},\mathbb{S}_M)$ with $\varphi_{ij}\to\varphi_i$ in the graph norm on K_{i+1}. Choose $j=j(i)\geqslant i$ such that $\|\varphi_{ij}-\varphi_i\|_D\to 0$ as $i\to\infty$. Then, $\|\varphi_{ij}-\varphi\|_D\leqslant\|\varphi_{ij}-\varphi_i\|_D+\|\varphi_i-\varphi\|_D\to 0$, too. Then

$$\begin{aligned}\|\eta_j\varphi_{ij}-\varphi_{ij}\|_D^2&\leqslant\|(1-\eta_j)\varphi_{ij}\|_{L^2}^2+(\|(1-\eta_j)D\varphi_{ij}\|_{L^2}+\|d\eta_j\cdot\varphi_{ij}\|_{L^2})^2\\ &\leqslant(\|\varphi_{ij}-\varphi_i\|_{L^2}+\|(1-\eta_j)\eta_i\varphi\|_{L^2})^2+\Big(\|D(\varphi_{ij}-\varphi_i)\|_{L^2}\\ &\quad+\|(1-\eta_j)(\eta_iD\varphi+d\eta_i\cdot\varphi)\|_{L^2}+\frac{2}{j}\|\varphi_{ij}-\varphi_i\|_{L^2}+\frac{2}{j}\|\varphi\|_{L^2}\Big)^2\longrightarrow 0\end{aligned}$$

for $i\to\infty$. Thus, we have a sequence $\hat\varphi_i:=\eta_{j(i)}\varphi_{ij(i)}\in\Gamma_c^\infty(M,\mathbb{S}_M)$ such that $\hat\varphi_i\to\varphi$ in the graph norm as $i\to\infty$. □

Note that the proof of Lemma 4.1 only uses the completeness of M and not the bounded geometry.

Theorem 4.2. *The trace map* $R:\Gamma_c^\infty(M,\mathbb{S}_M)\to\Gamma_c^\infty(\Sigma,\mathbb{S}_M|_\Sigma)$ *can be extended to a bounded operator*

$$R:\operatorname{dom} D_{\max}\longrightarrow H_{-1/2}(\Sigma,\mathbb{S}_M|_\Sigma).$$

Proof. Let $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$ and $\psi \in H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. Then, by Theorem 3.7, the spinor $\mathcal{E}\psi \in H_1(M, \mathbb{S}_M)$. Thus, we can use Lemma 3.10, (7) and Theorem 3.7 to obtain

$$\begin{aligned}
|(\varphi|_\Sigma, \nu \cdot \psi)_\Sigma| &= |(D\varphi, \mathcal{E}(\nu \cdot \psi)) - (\varphi, D\mathcal{E}(\nu \cdot \psi))| \\
&\leqslant \|D\varphi\|_{L^2} \|\mathcal{E}(\nu \cdot \psi)\|_{L^2} + \|\varphi\|_{L^2} \|D\mathcal{E}(\nu \cdot \psi)\|_{L^2} \\
&\leqslant 2\|\varphi\|_D \|\mathcal{E}(\nu \cdot \psi)\|_D \leqslant C\|\varphi\|_D \|\mathcal{E}(\nu \cdot \psi)\|_{H_1} \leqslant C' \|\varphi\|_D \|\nu \cdot \psi\|_{H_{1/2}(\Sigma)}.
\end{aligned}$$

Together with Lemma 3.9, this implies

$$\|\varphi|_\Sigma\|_{H_{-1/2}(\Sigma)} \leqslant C' \|\varphi\|_D.$$

Since $\Gamma_c^\infty(M, \mathbb{S}_M)$ is dense in $\operatorname{dom} D_{\max}$ with respect to the graph norm, cf. Lemma 4.1, the claim follows. □

Remark 4.3. Note that R is not surjective here. For closed boundaries the image was specified in [**9**, Theorems 1.7 and 6.7(ii)]. For noncompact boundaries the image will be further considered in Lemma 4.8 and below.

Lemma 4.4. *Equality (2) holds for all $\varphi \in \operatorname{dom} D_{\max}$ and $\psi \in H_1(M, \mathbb{S}_M)$.*

Proof. The proof is done as the one of Lemma 3.10 starting with $\psi_j, \varphi_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$, where $\psi_j \to \psi$ in H_1 and $\varphi_i \to \varphi$ in the graph norm of D and using the (extended) Trace Theorem 4.2. The only difference is seen in the estimate of the boundary integrals which now read, for example,

$$\left| \int_\Sigma \langle \nu \cdot R\psi_j, R\varphi_i - R\varphi \rangle \, ds \right| \leqslant \|R\psi_j\|_{H_{1/2}(\Sigma)} \|R(\varphi_i - \varphi)\|_{H_{-1/2}(\Sigma)} \leqslant C\|\psi_j\|_{H_1} \|\varphi_i - \varphi\|_D \longrightarrow 0,$$

where the last inequality uses both versions of the Trace Theorems 3.7 and 4.2. □

The next lemma gives a full description of $\operatorname{dom} D_{\min}$.

Lemma 4.5. *The H_1-norm and the graph norm $\|.\|_D$ are equivalent on*

$$\{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\}.$$

In particular,

$$\begin{aligned}
\operatorname{dom} D_{\min} = \overline{\Gamma_{cc}^\infty(M, \mathbb{S}_M)}^{\|\cdot\|_D} &= \overline{\Gamma_{cc}^\infty(M, \mathbb{S}_M)}^{\|\cdot\|_{H_1}} = \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\} \\
&= \{\varphi \in H_1(M, \mathbb{S}_M) \mid R\varphi = 0\}.
\end{aligned}$$

Proof. First, we show the equivalence on $\{\psi \in \Gamma_c^\infty(M, \mathbb{S}_M) \mid R\psi = 0\}$: Let φ be in this set. Then, by (6) we have

$$\|\varphi\|_{H_1}^2 = \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2 - \int_M \frac{\operatorname{scal}^M}{4} |\varphi|^2 \, dv - \int_M \frac{i}{2} \langle \Omega \cdot \varphi, \varphi \rangle \, dv \leqslant C\|\varphi\|_D^2,$$

where we used that M and L are of bounded geometry and, hence, $|\operatorname{scal}^M|$ and $|\Omega|$ are uniformly bounded on all of M. The reverse inequality was seen in (7).

From the definition of $\operatorname{dom} D_{\min}$ and the equivalence of the norms from above, we already have $\operatorname{dom} D_{\min} = \overline{\Gamma_{cc}^\infty}^{\|\cdot\|_D} = \overline{\Gamma_{cc}^\infty}^{\|\cdot\|_{H_1}}$. From the Trace Theorem 4.2, we obtain

$$\overline{\Gamma_{cc}^\infty}^{\|\cdot\|_D} \subset \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi = 0\}.$$

Next, we show that $D\colon \{\varphi\in \operatorname{dom} D_{\max}\,|\, R\varphi=0\}\to L^2(M,\mathbb{S}_M)$ already equals $D_{\min}$. First, we note that by the Trace Theorem 4.2, D is a closed extension of D_{cc}. Hence, it suffices to show that $D^*=D_{\max}$. By definition, we have

$$\operatorname{dom} D^*=\{\vartheta\in L^2(M,\mathbb{S}_M)\,|\,\exists \chi\in L^2(M,\mathbb{S}_M)\,\forall\psi\in \operatorname{dom} D_{\max}, R\psi=0:(\vartheta,D\psi)=(\chi,\psi)\}.$$

Let $\vartheta\in\operatorname{dom} D_{\max}$. By Lemma 4.1, there exists a sequence $\vartheta_i\in\Gamma_c^\infty(M,\mathbb{S}_M)$ with $\vartheta_i\to\vartheta$ in the graph norm. Hence, for all $\psi\in\operatorname{dom} D_{\max}$ with $R\psi=0$, we have $(\vartheta,D\psi)=\lim_{i\to\infty}(\vartheta_i,D\psi)$. Then, by Lemma 4.4 and $R\psi=0$, we obtain

$$(\vartheta,D\psi)=\lim_{i\to\infty}(D\vartheta_i,\psi)=(D\vartheta,\psi),$$

which implies that $\vartheta\in\operatorname{dom} D^*$. Thus, $D^*=D_{\max}$ and $D=D_{\min}$. Together with

$$\operatorname{dom} D_{\min}=\overline{\Gamma_{cc}^\infty}^{\|.\|_{H_1}}\subset\{\varphi\in H_1(M,\mathbb{S}_M)\,|\,R\varphi=0\}\subset\{\varphi\in\operatorname{dom} D_{\max}\,|\,R\varphi=0\}=\operatorname{dom} D_{\min},$$

the rest of the lemma follows. □

Now, we can describe H_1 in terms of its image under the trace map.

LEMMA 4.6. *We have* $H_1(M,\mathbb{S}_M)=\{\varphi\in\operatorname{dom} D_{\max}\,|\,R\varphi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)\}$.

Proof. The inclusion '$\subset$' is clear from the Trace Theorem 3.7. It remains to prove '$\supset$': Let $\varphi\in\operatorname{dom} D_{\max}$ with $R\varphi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)$. Then, Theorem 3.7 implies that $\psi:=\mathcal{E}R\varphi\in H_1(M,\mathbb{S}_M)$. Thus, $\varphi-\psi\in\operatorname{dom} D_{\max}$ and $R(\varphi-\psi)=0$. But due to Lemma 4.5, $\varphi-\psi\in H_1(M,\mathbb{S}_M)$ and, hence, $\varphi\in H_1(M,\mathbb{S}_M)$. □

In Proposition 3.13, we have shown that there is a linear map $\tilde{\mathcal{E}}$ such that $\tilde{\mathcal{E}}R:\Gamma_c^\infty(M,\mathbb{S}_M)\to\Gamma_c^\infty(M,\mathbb{S}_M)$ fulfills for all $\varphi\in\Gamma_c^\infty(M,\mathbb{S}_M)$

$$\|\tilde{\mathcal{E}}R\varphi\|_D^2\leqslant C\|\varphi\|_D^2. \tag{10}$$

Thus, $\tilde{\mathcal{E}}R$ extends uniquely to a bounded linear map

$$\tilde{\mathcal{E}}R:\operatorname{dom} D_{\max}\longrightarrow\operatorname{dom} D_{\max}. \tag{11}$$

Note that $\tilde{\mathcal{E}}|_{H_{1/2}}$ is an extension map in the sense of Theorem 3.7 as can be seen in the following: Let $\psi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)$. By Lemma 4.6, there is a $\varphi\in H_1(M,\mathbb{S}_M)$ with $R\varphi=\psi$. Thus, by Lemma 4.5 $\tilde{\mathcal{E}}\psi-\varphi\in\operatorname{dom} D_{\min}\subset H_1(M,\mathbb{S}_M)$. In particular, $\tilde{\mathcal{E}}|_{H_{1/2}}:H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)\to H_1(M,\mathbb{S}_M)$.

From now, we choose any extension map $\mathcal{E}$ fulfilling (10). Obviously, all those maps lead to equivalent norms $\|\mathcal{E}R.\|_D$.

CONJECTURE 4.7. Every extension map in the sense of Theorem 3.7 fulfills (10) with an appropriate constant C.

On $R(\operatorname{dom} D_{\max})$, we set

$$\|\psi\|_{\check{R}}:=\|\mathcal{E}R\varphi\|_D,$$

where $R\varphi=\psi$. By Theorem 3.13 and (11), this is well defined.

LEMMA 4.8. *The space* $\check{R}:=(R(\operatorname{dom} D_{\max}),\|.\|_{\check{R}})$ *is a Hilbert space with* $\check{R}=\overline{\Gamma_c^\infty(\Sigma,\mathbb{S}_M|_\Sigma)}^{\|.\|_{\check{R}}}$.

Proof. From the definition of $\|.\|_{\check{R}}$, the linearity of the maps $\mathcal{E}$ and R, and the fact that $(\operatorname{dom} D_{\max}, \|.\|_D)$ is a Hilbert space, we get immediately that $\|.\|_{\check{R}}$ is a norm on $R(\operatorname{dom} D_{\max})$. Moreover, $\|.\|_{\check{R}}$ comes from a scalar product $(\varphi, \psi)_{\check{R}} := (\mathcal{E}\varphi, \mathcal{E}\psi)_D :\, = (\mathcal{E}\varphi, \mathcal{E}\psi) + (D\mathcal{E}\varphi, D\mathcal{E}\psi)$. To show that $\check{R}$ is a Hilbert space it remains to show completeness: For that, we consider a Cauchy sequence ψ_i in $\check{R}$. Then, there is a sequence $\varphi_i \in \operatorname{dom} D_{\max}$ with $R\varphi_i = \psi_i$. With the definition of the $\check{R}$-norm, we get that $\mathcal{E}R\varphi_i$ is a Cauchy sequence in $(\operatorname{dom} D_{\max}, \|.\|_D)$ and, hence, there is a $\varphi \in \operatorname{dom} D_{\max}$ with $\mathcal{E}R\varphi_i \to \varphi$ with respect to the graph norm. By Theorem 3.13, we obtain

$$\|\mathcal{E}R(\varphi_i - \varphi)\|_D = \|\mathcal{E}R(\mathcal{E}R\varphi_i - \varphi)\|_D \leqslant C\|\mathcal{E}R\varphi_i - \varphi\|_D \longrightarrow 0.$$

Thus, $\mathcal{E}R\varphi = \varphi$ and $\|\psi_i - R\varphi\|_{\check{R}} = \|\mathcal{E}(R\varphi_i - R\varphi)\|_D \to 0$. Hence, $\psi_i \to \psi$ in the $\check{R}$-norm.

Clearly, $\overline{\Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)}^{\|.\|_{\check{R}}} \subset R(\operatorname{dom} D_{\max})$. Let now $\psi \in R(\operatorname{dom} D_{\max})$. Then, there is a $\varphi \in \operatorname{dom} D_{\max}$ with $R\varphi = \psi$. By Lemma 4.1, there is a sequence $\varphi_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$ with $\|\varphi_i - \varphi\|_D \to 0$ as $i \to \infty$. Thus, by Theorem 3.13 the sequence $\psi_i := R\varphi_i \in \Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$ converges to ψ in the $\check{R}$-norm. □

Remark 4.9. (i) The proof of Proposition 3.13 and [9, Lemma 6.1] implies

$$\|\tilde{\mathcal{E}}R\varphi\|_D^2 \leqslant C' \sum_{\gamma, \hat{U}'_\gamma \neq \varnothing} \|R(h_\gamma\varphi)\|^2_{\check{H}(\hat{U}'_\gamma)} =: C'\|R\varphi\|^2_{\check{H}_\gamma}.$$

On the other hand, by [9, Lemma 6.2, (41) and below] $\|R(h_\gamma\varphi)\|^2_{\check{H}(\hat{U}'_\gamma)} \leqslant C\|h_\gamma\varphi\|_D^2$, where C again only depends on the curvature bounds of (M, Σ) and the spectral gap c on $\hat{U}'_\gamma$. Thus, together with Lemma 3.11 the norms $\|.\|_{\check{R}}$ and $\|.\|_{\check{H}_\gamma}$ are equivalent.

(ii) Using (i) and [9, Lemma 6.3], we see

$$\|\tilde{\mathcal{E}}(\nu \cdot R\varphi)\|_D^2 \leqslant C' \sum_{\gamma, U'_\gamma \neq \varnothing} \|\nu \cdot R(h_\gamma\varphi)\|^2_{\check{H}(\hat{U}'_\gamma)} = C' \sum_{\gamma, U'_\gamma \neq \varnothing} \|R(h_\gamma\varphi)\|^2_{\hat{H}(\hat{U}'_\gamma)} =: \|R\varphi\|^2_{\hat{H}_\gamma}.$$

Together with [9, Lemma 6.1], we obtain for all $\varphi \in \Gamma_c^\infty(M, \mathbb{S}_M)$

$$\|\tilde{\mathcal{E}}(\nu \cdot R\varphi)\|_D^2 \leqslant C\|\varphi\|_D^2$$

and, thus, $\|\psi\|_{\hat{R}} := \|\mathcal{E}(\nu \cdot R\varphi)\|_D$ also gives rise to a norm on $R(\operatorname{dom} D_{\max})$. Moreover, the analogous statement of Lemma 4.8 holds for $\hat{R} := (R(\operatorname{dom} D_{\max}), \|.\|_{\hat{R}})$, and we have $\|\psi\|_{\check{R}} = \|\nu \cdot \psi\|_{\hat{R}}$. In particular, we get as in (i) that the norms $\|\tilde{\mathcal{E}}(\nu \cdot .)\|_D$ and $\|.\|_{\hat{H}_\gamma}$ are equivalent.

Remark 4.10. Note that by Theorems 4.2 and 4.6

$$H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma) \subset (R(\operatorname{dom} D_{\max}), \|.\|_{\check{R}\,(\text{resp. } \hat{R})}) \subset H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma).$$

Moreover, the perfect pairing of $\hat{H}_\gamma$ and $\check{H}_\gamma$, induced by the pairing of $H_{1/2}$ and $H_{-1/2}$, gives immediately the following lemma.

Lemma 4.11. *The L^2-product on $\Gamma_c^\infty(\Sigma, \mathbb{S}_M|_\Sigma)$ extends uniquely to a perfect pairing $\check{R} \times \hat{R} \to \mathbb{C}$.*

So for now, we have seen that the $\check{R}$-norm is equivalent to the norm $\|.\|_{\check{H}_\gamma}$, cp. Remark 4.9(i) where the second norm comes with an appropriate trivialization of the manifold near the boundary, see before Proposition 3.13. But we also think that as in the closed case there should be a ‘more intrinsic’ equivalent norm:

CONJECTURE 4.12. The $\check{R}$-norm on $R(\operatorname{dom} D_{\max})$ is equivalent to the $\check{H}$-norm as defined in (8). Moreover, $\check{H} = R(\operatorname{dom} D_{\max})$ as vector spaces.

Boundary conditions. In this part, we show that each closed extension of D_{cc} can be realized by a closed linear subset of $\check{R}$, and we give some examples.

LEMMA 4.13. *Let D be a closed extension of D_{cc} with $B := R(\operatorname{dom} D) \subset H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. Then, its domain* $\operatorname{dom} D$ *equals* $\operatorname{dom} D_B := \{\varphi \in \operatorname{dom} D_{\max} \mid R\varphi \in B\}$, *and B is a closed linear subset of $\check{R}$. Conversely, for every closed linear subset $B \subset \check{R}$ the operator $D_B \colon \operatorname{dom} D_B \to L^2(M, \mathbb{S}_M)$ is a closed extension of D_{cc}.*

Owing to this lemma, a closed subspace B of $\check{R}$ is called *boundary condition*.

Proof. Let D be a closed extension of D_{cc} with domain $\operatorname{dom} D$ and $B := R(\operatorname{dom} D)$. Clearly, $\operatorname{dom} D \subset \operatorname{dom} D_B$. We have to show that also the converse is true: Let $\varphi \in \operatorname{dom} D_B$. Then, there exists $\psi \in \operatorname{dom} D$ with $R\varphi = R\psi$. By Lemma 4.5, $\varphi - \psi \in \operatorname{dom} D_{\min} \subset \operatorname{dom} D$ and, hence, $\varphi \in \operatorname{dom} D$. This implies that $\operatorname{dom} D = \operatorname{dom} D_B$. Moreover, from (11) and the definition of the $\check{R}$-norm the maps $R : \operatorname{dom} D_{\max} \to \check{R}$ and $\mathcal{E} : \check{R} \to \operatorname{dom} D_{\max}$ are continuous. Hence, if $\operatorname{dom} D$ is closed in $\operatorname{dom} D_{\max}$, then the set $B = \mathcal{E}^{-1}(\operatorname{dom} D)$ is closed in $R(\operatorname{dom} D_{\max})$. Conversely, if B is closed in $\check{R}$, then $\operatorname{dom} D = R^{-1}(B)$ is closed in $\operatorname{dom} D_{\max}$. □

LEMMA 4.14. *Let B be a boundary condition such that $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. Then, the H_1-norm and the graph norm $\|.\|_D$ are equivalent on* $\operatorname{dom} D_B$.

Proof. Since B is a boundary condition, $\operatorname{dom} D_B$ is closed in $(\operatorname{dom} D_{\max}, \|.\|_D)$. Moreover, by $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$, Lemma 4.6 and (7), $\operatorname{dom} D_B$ is closed in $(H_1(M, \mathbb{S}_M), \|.\|_{H_1})$. Thus, $(\operatorname{dom} D_B, \|.\|_D)$ and $(\operatorname{dom} D_B, \|.\|_{H_1})$ are both Hilbert spaces. By (7), the identity map $\mathrm{Id} \colon (\operatorname{dom} D_B, \|.\|_{H_1}) \to (\operatorname{dom} D_B, \|.\|_D)$ is a bijective bounded linear map. From the bounded inverse theorem, we know that also the inverse is bounded. Hence, the H_1- and the graph norm are equivalent on $\operatorname{dom} D_B$. □

REMARK 4.15. The definition of $\operatorname{dom} D_B$ in [**9**, Section 7] uses $H_1^D := \overline{\Gamma_c^\infty(M, \mathbb{S}_M)}^{\|.\|_{H_1^D}}$ instead of H_1 where the H_1^D-norm is given by

$$\|\varphi\|_{H_1^D}^2 = \|\chi\varphi\|_{H_1}^2 + \|\varphi\|_{L^2}^2 + \|D\varphi\|_{L^2}^2.$$

Here, χ denotes an appropriate cut-off function such that $\chi\varphi$ only lives on a small collar of the boundary. Since we work with the classical Dirac operator on Spin^c manifolds and assume (M, Σ) and L being of bounded geometry, the H_1- and the H_1^D-norm coincide. Bär and Ballmann consider a more general situation where it suffices that M is only complete but not necessarily of bounded geometry. Then, the H_1^D-norm is needed. We could also switch to this more general setup when dropping the condition (i) and (iii) in Definition 2.2 while still assuming that $(\Sigma, g|_\Sigma)$ is of bounded geometry and that the curvature tensor and its derivatives are bounded on U_Σ. For that situation, we would also obtain Theorem 1.2. But in order to simplify notation we stick to the bounded geometry of (M, Σ).

EXAMPLE 4.16. (i) *Minimal and maximal extension.* $B = 0$ gives the minimal extension $D_{B=0} = D_{\min}$, cf. Lemma 4.5. The maximal extension is obtained with $B = R(\operatorname{dom} D_{\max})$.

(ii) $D_{B=H_{1/2}}\colon H_1(M,\mathbb{S}_M)\to L^2(M,\mathbb{S}_M)$ is an extension of D_{cc} but not closed (if the boundary is non-empty): Since $\Gamma_c^\infty(M,\mathbb{S}_M)\subset H_1$ and $\Gamma_c^\infty(M,\mathbb{S}_M)$ dense in $\operatorname{dom} D_{\max}$, the closure of $D_{B=H_{1/2}}$ is $D_{\max}$.

(iii) [**18**, Section 6] Let $P_\pm : L^2(\Sigma,\mathbb{S}_M|_\Sigma)\to L^2(\Sigma,\mathbb{S}_M|_\Sigma)$, $\varphi\mapsto \frac{1}{2}(\varphi\pm i\nu\cdot\varphi)$ and

$$D_\pm\colon \operatorname{dom} D_\pm := \{\varphi\in\operatorname{dom} D_{\max}\,|\, P_\pm R\varphi=0\}\to L^2(M,\mathbb{S}_M).$$

In Section 5, we will show that $D_\pm$ is a closed extension and that $D_\pm = D_{B_\pm}$, where

$$B_\pm=\{\varphi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)\,|\,P_\pm\varphi=0\}.$$

Each φ decomposes uniquely into $\varphi=P_+\varphi+P_-\varphi$, and if $\varphi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)$, then $P_\pm\varphi\in H_{1/2}(\Sigma,\mathbb{S}_M|_\Sigma)$, too. This assures that the boundary condition $B_\pm$ is honestly larger than the trivial boundary condition $B=\{0\}$. More properties of this boundary condition can be found in Section 5.

(iv) *APS-boundary conditions*. An obvious way to generalize the APS-boundary conditions for a closed boundary to our situation is given by the following: Let (M,Σ) be of bounded geometry. We use the notations introduced in Section 3.7.
We set $B^{\mathrm{APS}}_{\geqslant a}=R(\operatorname{dom} D_{\max})\cap\Gamma^{\mathrm{APS}}_{[a,\infty)}$ and $B^{\mathrm{APS}}_{<a}=R(\operatorname{dom} D_{\max})\cap\Gamma^{\mathrm{APS}}_{(-\infty,a]}$, respectively. In the same ways, let $B^{\mathrm{APS}}_{\leqslant a}$ and $B^{\mathrm{APS}}_{>a}$ be defined. If a neighbourhood of a is in the spectrum of D^Σ, $B^{\mathrm{APS}}_{<a}$ and $B^{\mathrm{APS}}_{>a}$ will not be closed. We conjecture that for (M,Σ) of bounded geometry the sets $B^{\mathrm{APS}}_{\geqslant a}$ and $B^{\mathrm{APS}}_{\leqslant a}$ define boundary conditions. But actually we do not know.

Boundary value problems. In this part, we want to prove Theorem 1.1. For that, we need to define first the notion coercivity at infinity:

Definition 4.17. A closed linear operator $D\colon \operatorname{dom} D\subset L^2(M,\mathbb{S}_M)\to L^2(M,\mathbb{S}_M)$ is said to be $(\operatorname{dom} D)$-coercive at infinity if there is a $c>0$ such that

$$\forall\varphi\in\operatorname{dom} D\cap(\ker D)^\perp :\ \|D\varphi\|_{L^2}\geqslant c\|\varphi\|_{L^2},$$

where $^\perp$ denotes the orthogonal complement in L^2.

Note that in case that D is the Dirac operator on a complete manifold without boundary, coercivity at infinity follows immediately if 0 is not the essential spectrum. Conversely, if the Dirac operator is coercive at infinity, then either 0 is not in the essential spectrum or the kernel is infinite-dimensional. For manifolds with boundary, D is, in general, no longer self-adjoint. Thus, the spectrum is, in general, complex and this translation to the essential spectrum is not possible.

In Section 6, we will compare this coercivity condition with the originally one used in [**9**, Defintion 8.2] for closed boundaries. But first, we will see how this condition forces the range of the operator to be closed which is crucial in order to apply the Closed Range Theorem 2.5 and show existence of preimages for linear operator as we will need in Theorem 1.1.

Lemma 4.18. *If the closed linear operator* $D\colon \operatorname{dom} D\subset L^2(M,\mathbb{S}_M)\to L^2(M,\mathbb{S}_M)$ *is* $(\operatorname{dom} D)$*-coercive at infinity, then the range is closed.*

Proof. Let φ_i be a sequence in $\operatorname{dom} D$ with $D\varphi_i\to\psi$ in L^2. We have to show that ψ is in the image of D. Without loss of generality, we can assume that $\varphi_i\perp\ker D$. Then, $(\operatorname{dom} D)$-coercivity at infinity gives that φ_i is bounded in L^2 and, thus, also in the graph norm of D. Thus, $\varphi_i\to\varphi$ weakly in $\|.\|_D$. Let $\eta\in\operatorname{dom} D^*$. Then, $(D\varphi,\eta)=\lim_{i\to\infty}$

$(D\varphi_i, \eta) = \lim_{i\to\infty}(\varphi_i, D^*\eta) = (\varphi, D^*\eta)$. Thus, $\varphi \in \operatorname{dom} D$ and closedness of $\operatorname{dom} D$ then implies that $D\varphi = \psi$. □

We are now ready to prove the following theorem.

THEOREM (Theorem 1.1). *Let B be a boundary condition, and let the Dirac operator*

$$D_B : \operatorname{dom} D_B \subset L^2(M, \mathbb{S}_M) \longrightarrow L^2(M, \mathbb{S}_M)$$

be B-coercive at infinity. Let $P_B : R(\operatorname{dom} D_{\max}) \to B$ be a projection. Then, for all $\psi \in L^2(M, \mathbb{S}_M)$ and $\tilde{\rho} \in \operatorname{dom} D_{\max}$ where $\psi - D\tilde{\rho} \in (\ker(D_B)^*)^\perp$, the boundary value problem

$$\begin{cases} D\varphi = \psi & \text{on } M, \\ (\mathrm{Id} - P_B)R\varphi = (\mathrm{Id} - P_B)R\tilde{\rho} & \text{on } \Sigma \end{cases}$$

has a solution $\varphi \in \operatorname{dom} D_{\max}$ that is unique up to elements of the kernel $\ker D_B$.

Projection only means here that P_B is linear and $P_B|_B = \mathrm{Id}$.

Proof. Since D is B-coercive at infinity, its range is closed by Lemma 4.18. Thus, due to the Closed Range Theorem 2.5, the spinor $\psi - D\tilde{\rho} \in \operatorname{ran} D_B$. Hence, there exists $\hat{\varphi} \in \operatorname{dom} D_B$ with $D\hat{\varphi} = \psi - D\tilde{\rho}$. Setting $\varphi = \hat{\varphi} + \tilde{\rho}$, we obtain $\varphi \in \operatorname{dom} D_{\max}$, $D\varphi = \psi$ and $(\mathrm{Id} - P_B)R\varphi = (\mathrm{Id} - P_B)R\hat{\varphi} + (\mathrm{Id} - P_B)R\tilde{\rho} = (\mathrm{Id} - P_B)R\tilde{\rho}$. □

COROLLARY 4.19. *Let B be a boundary condition such that $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. We assume that the Dirac operator $D : \operatorname{dom} D_B \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is B-coercive at infinity. Let $P_B : H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma) \to B$ be a projection. Moreover, assume that $\psi \in L^2(M, \mathbb{S}_M)$ and $\rho \in H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$ satisfy*

$$(\psi, \chi) + (\nu \cdot \rho, R\chi)_\Sigma = 0 \tag{12}$$

for all $\chi \in \ker(D_B)^*$. Then, the boundary value problem

$$\begin{cases} D\varphi = \psi & \text{on } M, \\ (\mathrm{Id} - P_B)R\varphi = (\mathrm{Id} - P_B)\rho & \text{on } \Sigma \end{cases}$$

has a solution $\varphi \in H_1(M, \mathbb{S}_M)$ that is unique up to elements of the kernel $\ker D_B$.

Proof. By Lemma 4.6, $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$ implies $\operatorname{dom} D_B \subset H_1(M, \mathbb{S}_M)$. We set $\tilde{\rho} = \mathcal{E}\rho$. By the Trace Theorem 3.7, $\tilde{\rho} \in H_1(M, \mathbb{S}_M)$. Moreover, by Lemma 4.4 the integrability condition (12) implies that $\psi - D\tilde{\rho} \in (\ker(D_B)^*)^\perp$. Hence, together with the Closed Range Theorem there is $\hat{\varphi} \in \operatorname{dom} D_B \subset H_1(M, \mathbb{S}_M)$ with $D\hat{\varphi} = \psi - D\tilde{\rho}$. Thus, as in the proof of Theorem 1.1 $\varphi = \hat{\varphi} + \tilde{\rho}$ gives a solution which is now in $H_1(M, \mathbb{S}_M)$. □

REMARK 4.20. To give a full generalization of the theory given in [9] it would be interesting to examine the following questions.

(i) Consider general boundary conditions, in particular we would like to identify the image of the extended trace map in Theorem 4.2.

(ii) Give a generalization of the definition for elliptic boundary conditions for noncompact boundaries (of bounded geometry) and study them.

(iii) Consider, more generally, complete Dirac-type operators as in [9].

5. *On the boundary condition $B_\pm$*

In this section, we briefly recall and give some basic facts on $P_\pm$. Some of them can be found in [**18**, Section 6]. Moreover, we prove the claims of Example 4.16(iii).

LEMMA 5.1. *Let $P_\pm\colon L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ be the map $\varphi \mapsto \frac{1}{2}(\varphi \pm i\nu\cdot\varphi)$ and consider $B_\pm := \{\varphi \in H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma) \,|\, P_\pm\varphi = 0\}$. Then, the following hold:*

(i) *$P_\pm$ are self-adjoint projections, orthogonal to each other and $\nu P_\pm = P_\pm \nu = \mp i P_\pm$;*
(ii) *for all $s\in\mathbb{R}$, $P_\pm(\varphi) = \frac{1}{2}(\varphi \pm i\nu\cdot\varphi)$ gives an operator from $H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ to itself such that for all $\varphi \in H_s(\Sigma, \mathbb{S}_M|_\Sigma)$ and $\psi \in H_{-s}(\Sigma, \mathbb{S}_M|_\Sigma)$ we have $(P_+\varphi, P_-\psi)_\Sigma = 0$ and $(P_\pm\varphi, \psi)_\Sigma = (\varphi, P_\pm\psi)_\Sigma$;*
(iii) $\tilde{D}^\Sigma P_\pm = P_\mp \tilde{D}^\Sigma$;
(iv) $D_\pm$ *(see Example* 4.16(iii) *for the definition) is a closed extension of D_{cc};*
(v) $D_\pm = D_{B_\pm}$;
(vi) $(D_{B_\pm})^* = D_{B_\mp}$;
(vii) *let each connected component of M have a non-empty boundary. Then,* $\ker D_{B_\pm} = \{0\}$.

Proof. Assertions (i) and (ii) follow directly by simple calculations, and (iii) follows directly from (5). For (iv), we have by definition of $D_\pm$ (see Example 4.16(iii)) that $D_\pm = D_{\tilde{B}_\pm}$, where $\tilde{B}_\pm = \{\varphi \in R(\operatorname{dom} D_{\max}) \,|\, P_\pm\varphi = 0\}$.

To show the closedness of $D_\pm$ we want to apply Lemma 4.13. For that, we have to show that $\tilde{B}_\pm$ is closed in $\check{R}$: Let $\varphi_i \in \tilde{B}_\pm$ with $\varphi_i \to \varphi$ in $\check{R}$. Then, we get, together with Remark 4.9(ii), that

$$\begin{aligned}\|P_\pm\varphi\|_{\check{R}} &= \|P_\pm(\varphi-\varphi_i)\|_{\check{R}} = \|\mathcal{E}P_\pm(\varphi-\varphi_i)\|_D \leqslant \tfrac{1}{2}(\|\mathcal{E}(\varphi-\varphi_i)\|_D + \|\mathcal{E}\nu\cdot(\varphi-\varphi_i)\|_D)\\ &\leqslant C\|\mathcal{E}(\varphi-\varphi_i)\|_D = \|\varphi-\varphi_i\|_{\check{R}} \longrightarrow 0.\end{aligned}$$

Hence, $P_\pm\varphi = 0$ and $\varphi \in \tilde{B}_\pm$.

For (v), we have clearly that $\operatorname{dom} D_{B_\pm} \subset \operatorname{dom} D_\pm$. It remains to show that any $\varphi \in \operatorname{dom} D_\pm$ is already in $H_1(M, \mathbb{S}_M)$. By Lemma 4.1, there is a sequence $\varphi_i \in \Gamma_c^\infty(M, \mathbb{S}_M)$ with $\varphi_i \to \varphi$ in the graph norm. Consider $\mathcal{E}P_\pm R\varphi_i$. By the linearity of $\mathcal{E}$, (11) and Remark 4.9(ii), we obtain

$$\begin{aligned}\|\mathcal{E}P_\pm R\varphi_i\|_D &= \|\mathcal{E}P_\pm R(\varphi_i-\varphi)\|_D\\ &\leqslant \tfrac{1}{2}(\|\mathcal{E}R(\varphi_i-\varphi)\|_D + \|\mathcal{E}(\nu\cdot R(\varphi_i-\varphi)\|_D)) \leqslant C\|\varphi_i-\varphi\|_D \longrightarrow 0.\end{aligned}$$

Hence, $\psi_i := \varphi_i - \mathcal{E}P_\pm R\varphi_i \to \varphi$ in the graph norm. Since $\psi_i \in \operatorname{dom} D_{B_\pm}$, this implies that $\operatorname{dom} D_{B_\pm}$ is dense in $\operatorname{dom} D_\pm$. Moreover, note that with (iii) and (i) we have

$$\int_\Sigma \langle R\psi_i, \tilde{D}^\Sigma R\psi_i\rangle\, ds = \int_\Sigma \langle P_\mp R\psi_i, \tilde{D}^\Sigma P_\mp R\psi_i\rangle\, ds = \int_\Sigma \langle P_\mp R\psi_i, P_\pm\tilde{D}^\Sigma R\psi_i\rangle\, ds = 0.$$

Hence, together with the Lichnerowicz formula in Lemma 3.10, the bounded geometry, (i) and Lemma 3.10, we obtain

$$\begin{aligned}\|\psi_i-\psi_j\|_{H_1}^2 =& \|\psi_i-\psi_j\|_D^2 - \frac{1}{4}\int_M \langle(\mathrm{scal}^M + 2i\Omega\cdot)(\psi_i-\psi_j), (\psi_i-\psi_j)\rangle\, dv\\ & - \frac{n}{2}\int_\Sigma H|R(\psi_i-\psi_j)|^2\, ds\\ \leqslant& C\|\psi_i-\psi_j\|_D^2 \mp i\frac{n}{2}\int_\Sigma \langle \nu\cdot R(\psi_i-\psi_j), HR(\psi_i-\psi_j)\rangle\\ \leqslant& C\|\psi_i-\psi_j\|_D^2.\end{aligned}$$

Thus, ψ_i is even a Cauchy sequence in H_1 which implies that φ is already in $H_1(M, \mathbb{S}_M)$. Note that this implies in particular that $B_\pm = \tilde{B}_\pm$. For (vi), the domain of the adjoint is defined by

$$\mathrm{dom}\,(D_+)^* = \{\vartheta \in L^2(M, \mathbb{S}_M) \,|\, \exists \chi \in L^2(M, \mathbb{S}_M)\, \forall \psi \in \mathrm{dom}\, D_+ : (\chi, \psi) = (\eta, D\psi)\}.$$

Since, $\Gamma^\infty_{cc}(M, \mathbb{S}_M) \subset \mathrm{dom}\, D_+$, we get $\mathrm{dom}\,(D_+)^* \subset \mathrm{dom}\, D_{\max}$. Thus,

$$\mathrm{dom}\,(D_+)^* = \{\vartheta \in \mathrm{dom}\, D_{\max} \,|\, \forall \psi \in \mathrm{dom}\, D_+ : (D\vartheta, \psi) = (\vartheta, D\psi)\}.$$

Due to Lemma 4.4, the definition of $\mathrm{dom}\, D_+$ and (v), we obtain

$$\mathrm{dom}\,(D_+)^* = \left\{ \vartheta \in \mathrm{dom}\, D_{\max} \,\middle|\, \forall \psi \in H_1(M, \mathbb{S}_M) : \int_\Sigma \langle \nu \cdot R\vartheta, P_- R\psi \rangle \, ds = 0 \right\}.$$

By (i) and (ii), we have

$$-\int_\Sigma \langle R\vartheta, \nu \cdot P_- R\psi \rangle \, ds = i \int_\Sigma \langle R\vartheta, P_- R\psi \rangle \, ds = i \int_\Sigma \langle P_- R\vartheta, R\psi \rangle \, ds$$

and $P_- R\vartheta \in H_{-1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. Hence, together with Lemmas 3.9 and 4.6,

$$\begin{aligned} \mathrm{dom}\,(D_+)^* &= \left\{ \vartheta \in \mathrm{dom}\, D_{\max} \,\middle|\, \forall \hat{\psi} \in H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma) : \int_\Sigma \langle P_- R\vartheta, \hat{\psi} \rangle \, ds = 0 \right\} \\ &= \{\vartheta \in \mathrm{dom}\, D_{\max} \,|\, P_- R\vartheta = 0\} = \mathrm{dom}\ D_-. \end{aligned}$$

The assertion (vii) is proved as in the closed case [**18**, Proof of Corollary 6]: Let $\varphi \in \ker D_\pm$, that is, $\varphi \in \mathrm{dom}\, D_{\max}$, $D\varphi = 0$ on M, and $P_\pm R\varphi = 0$ on Σ. Using this, (2), Lemma 4.4 and (i), we compute

$$\begin{aligned} 0 &= \int_M \langle \varphi, iD\varphi \rangle \, dv - \int_M \langle D\varphi, i\varphi \rangle \, dv = \int_\Sigma \langle \nu \cdot R\varphi, iR\varphi \rangle \, ds \\ &= \int_\Sigma \langle \nu \cdot P_\mp R\varphi, iP_\mp R\varphi \rangle \, ds = \pm \int_\Sigma |R\varphi|^2 \, ds. \end{aligned}$$

Hence, $R\varphi = 0$ and $\varphi \in \mathrm{dom}\, D_{\min}$, cf. Lemma 4.5. But due to the strong unique continuation property of the Dirac operator [**11**, Section 1.2], $D_{\min}\varphi = 0$ implies $\varphi = 0$. □

6. *Examples and the coercivity condition*

In Definition 4.17, we defined when an operator D_B is $(\mathrm{dom}\, D_B)$-coercive at infinity. When working with B, we will also use the short version, B-coercive at infinity. In this passage, we will compare this notion with the one of coercivity at infinity given in [**9**, Definition 8.2] as cited below and give some examples.

Definition 6.1 [**9**, Definition 8.2]. $D\colon \mathrm{dom}\, D_{\max} \subset L^2(M, \mathbb{S}_M) \to L^2(M, \mathbb{S}_M)$ is coercive at infinity if there is a compact subset $K \subset M$ and a constant $c > 0$ such that

$$\|D\varphi\|_{L^2} \geqslant c\|\varphi\|_{L^2},$$

for all $\varphi \in \Gamma^\infty_c(M \setminus K, \mathbb{S}_M)$.

By [**9**, Lemma 8.4], D is coercive at infinity for a closed boundary Σ if and only if there is a compact subset $K \subset M$ and a constant $c > 0$ such that for all $\varphi \in \Gamma^\infty_{cc}(M \setminus K, \mathbb{S}_M)$ we have $\|D\varphi\|_{L^2} \geqslant c\|\varphi\|_{L^2}$. For noncompact boundaries, just the 'only if'-direction survives since in contrast to closed boundaries there is no compact K such that $\Gamma^\infty_c(M \setminus K, \mathbb{S}_M) \subset \Gamma^\infty_{cc}(M, \mathbb{S}_M)$.

Before we compare those different coercivity conditions we give some examples:

EXAMPLE 6.2. (i) By the unique continuation property, the kernel of $D_{\min}$ is trivial. Thus, together with Lemma 4.5, we have that D is $(B = 0)$-coercive at infinity if and only if there is a constant $c > 0$ such that for all $\varphi \in \Gamma^\infty_{cc}(M, \mathbb{S}_M)$

$$\|D\varphi\|_{L^2} \geqslant c\|\varphi\|_{L^2}.$$

For closed boundaries, this implies coercivity at infinity by [**9**, Lemma 8.4] which was cited above. We will see that for closed boundaries also the converse is true, cf. Corollary 6.7.

(ii) By Lemma 5.1, $\ker D_{B_\pm} = \{0\}$. Thus, D is $B_\pm$-coercive at infinity if and only if there is a constant $c > 0$ such that

$$\|D\psi\|_{L^2} \geqslant c\|\psi\|_{L^2}$$

for all $\psi \in H_1(M, \mathbb{S}_M)$ with $P_\pm R\psi = 0$. In particular, this implies $(B = 0)$-coercivity at infinity. More generally, if $B_1 \subset B_2$ and $\ker D_{B_1} = \ker D_{B_2}$, then B_2-coercivity at infinity implies B_1-coercivity at infinity.

LEMMA 6.3. *Let D be coercive at infinity, and let B be a boundary condition. Assume that* $\operatorname{dom} D_B \cap (\ker D_B)^\perp \subset H_1(M, \mathbb{S}_M)$ *and that the H_1-norm and the graph norm are equivalent on* $\operatorname{dom} D_B \cap (\ker D_B)^\perp$*. Then, D is B-coercive at infinity.*

Proof. Since D is coercive at infinity, there is a compact subset $K \subset M$ and a constant $c > 0$ such that $\|D\varphi\|_{L^2} \geqslant c\|\varphi\|_{L^2}$ for all $\varphi \in \Gamma^\infty_c(M \setminus K, \mathbb{S}_M)$. Assume that D is not B-coercive at infinity. Then, there is a sequence $\varphi_i \in \operatorname{dom} D_B \cap (\ker D_B)^\perp$ with $\|\varphi_i\|_{L^2} = 1$ and $\|D\varphi_i\|_{L^2} \to 0$. By equivalence of the norms, φ_i is also bounded in H_1. This implies $\varphi_i \to \varphi$ weakly in H_1 and, thus, locally strongly in L^2. Moreover, $D\varphi = 0$. Together with $\varphi_i \perp \ker D_B$, this implies $\varphi = 0$. Thus, for each compact subset $K' \subset M$ we have $\int_{K'} |\varphi_i|^2\, dv \to 0$ as $i \to \infty$. Let $\eta\colon M \to [0, 1]$ be a cut-off function and K' be a compact subset such that $K \subset K' \subset M$ and $\eta = 0$ on K, $\eta = 1$ on $M \setminus K'$ and $|d\eta| \leqslant a$ for a constant $a > 0$ big enough. Then, $\operatorname{supp}(\eta\varphi_i) \subset M \setminus K$, $\|D(\eta\varphi_i)\|_{L^2} \leqslant a\|\varphi_i\|_{L^2(K')} + \|D\varphi_i\|_{L^2} \to 0$ and

$$1 \geqslant \|\eta\varphi_i\|_{L^2} \geqslant \|\varphi_i\|_{L^2} - \|(1-\eta)\varphi_i\|_{L^2} \geqslant 1 - \|\varphi_i\|_{L^2(K')} \longrightarrow 1.$$

By Lemma 4.1, we can choose a sequence $(\varphi_{ij})_j \subset \Gamma^\infty_c(M, \mathbb{S}_M)$ with $\varphi_{ij} \to \varphi_i$ in the graph norm as $j \to \infty$. Then, $\eta\varphi_{ij} \to \eta\varphi_i$ in the graph norm and $\operatorname{supp}(\eta\varphi_{ij}) \in M \setminus K$. Thus, we can find $j = j(i)$ such that $\|D(\eta\varphi_{ij(i)})\|_{L^2} \to 0$ and $\|\eta\varphi_{ij(i)}\|_{L^2} \to 1$ as $i \to \infty$. But this contradicts the assumption that D is coercive at infinity. □

From the last lemma and Lemma 4.14, we obtain immediately the following corollary.

COROLLARY 6.4. *If D is coercive at infinity and $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$, then D is B-coercive at infinity.*

Next, we give some (very restrictive) conditions that are sufficient to prove that B-coercivity at infinity implies coercivity at infinity. Those additional assumptions are needed to make sure that the φ_i appearing in Definition 6.1 are in $\operatorname{dom} D_B$.

LEMMA 6.5. *Let B be a boundary condition with $B \subset H_{1/2}(\Sigma, \mathbb{S}_M|_\Sigma)$. Assume that there exists a compact subset $K' \subset M$ with $\Gamma^\infty_c(M \setminus K', \mathbb{S}_M) \subset \operatorname{dom} D_B$. If $D\colon \operatorname{dom} D_B \subset L^2(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ has a finite-dimensional kernel and D is B-coercive at infinity, then D is coercive at infinity.*

Proof. Assume that D is not coercive at infinity. Then, for all compact subsets $K \subset M$ there exists a sequence $\varphi_i \in \Gamma_c^\infty(M \setminus K, \mathbb{S})$ with $\|\varphi_i\|_{L^2} = 1$ and $\|D\varphi_i\|_{L^2} \to 0$. We choose K such that $K' \subset K$. Then, all those $\varphi_i \in \operatorname{dom} D_B$. Thus, $\varphi_i \to \varphi \in \operatorname{dom} D_B$ weakly in the graph norm of D, $\varphi \in \ker D_B$ and $\varphi = 0$ on K. We decompose $\varphi_i = \varphi_i^k + \varphi_i^\perp$, where $\varphi_i^k \in \ker D_B$ and $\varphi_i^\perp \in (\ker D_B)^\perp$. Then $\|D\varphi_i^\perp\|_{L^2} \to 0$. Moreover, we assume that the kernel is finite-dimensional, that is, $\varphi_i^k = \sum_{j=1}^l a_{ij}\psi_j$, where the ψ_j's form an orthonormal basis of $\ker D_B$. Thus, $\|\varphi_i^k\|_{L^2}^2 = \sum_{j=1}^l |a_{ij}|^2$. Assume now that $\|\varphi_i^\perp\|_{L^2} \to 0$. Then $\varphi_i^\perp \to 0$ in the graph norm. But $\|\varphi_i\|_{L^2} = 1$. This implies that there is at least one $j \in \{1, \ldots, l\}$ with $|a_{ij}|$ is bounded away from zero for almost all i, that is, φ cannot be zero everywhere. Since φ is zero on K, this is a contradiction to the unique continuation principle. Thus, the assumption was wrong and there exists $c > 0$ with $\|\varphi_i^\perp\|_{L^2} > c$ and D is not B-coercive at infinity. □

Note that the assumption on the existence of K' is very restrictive. If the boundary is closed, then it is automatically satisfied and we get the corollary below. If the boundary is noncompact, for a general $\operatorname{dom} D$, for example, for the minimal domain of D, then it is not true. But there are also examples for manifolds with noncompact boundary and closed extension of D_{cc} where the assumptions of the last lemma are satisfied:

Example 6.6. Let (Σ, h) be a complete Riemannian Spin manifold. Let $M_\infty = \Sigma \times \mathbb{R}$ and $M = \Sigma \times [0, \infty)$ be equipped with product metric $h + dt^2$. Both manifolds are of bounded geometry. Since M_∞ is complete with no boundary, the Dirac operator on M_∞ is essentially self-adjoint. Assume that the Dirac operator on M_∞ is invertible.

Let $K' \subset M_\infty$ be a compact subset that intersects $\Sigma \times \{0\}$ in a subset of non-zero measure. Define $\mathcal{L}$ to be the linear span of $\Gamma_c^\infty(M \setminus K', \mathbb{S}_M) \cup \Gamma_{cc}^\infty(M, \mathbb{S}_M)$ and $\operatorname{dom} D_B \colon = \bar{\mathcal{L}}^{\|\cdot\|_D}$. Then, $B = \overline{\Gamma_c^\infty(\Sigma \setminus K', \mathbb{S}_M|_\Sigma)}^{\|\cdot\|_{\check{R}}}$. Note that by construction $\operatorname{dom} D_B$ is the domain of a closed extension of D_{cc}. But it is honestly smaller than $\operatorname{dom} D_{\max}$ since all $\varphi \in B$ have to vanish on $\Sigma \cap K'$. In particular, by the strong unique continuation property of D (see [**11**, Section 1.2]) $D_B \colon \operatorname{dom} D_B \to L^2(M, \mathbb{S}_M)$ has trivial kernel.

It remains to show that D_B is B-coercive at infinity, that is, there is $c > 0$ such that for all $\varphi \in \mathcal{L}$ we have $\|D\varphi\|_{L^2} \geqslant c\|\varphi\|_{L^2}$. We will show this by contradiction, that is, we assume that there is a sequence $\varphi_i \in \mathcal{L}$ with $\|\varphi_i\|_{L^2} = 1$ and $\|D\varphi_i\|_{L^2} \to 0$. We will construct a sequence of spinors on M_∞. Let $\tilde{\varphi}_i$ be obtained from φ_i by reflection along Σ. Clearly, $\tilde{\varphi}_i \in L^2(M_\infty, \mathbb{S}_{M_\infty})$. Moreover, note that $\tilde{\varphi}_i$ is everywhere continuous. Let ν be the inward normal vector field of M. For $\psi \in \Gamma_c^\infty(M_\infty, \mathbb{S}_{M_\infty})$, we can estimate using (2)

$$\begin{aligned}
|(\tilde{\varphi}_i, D\psi)_{L^2(M_\infty)}| &= \left| \int_{\Sigma \times (0,\infty)} \langle \tilde{\varphi}_i, D\psi \rangle + \int_{\Sigma \times (-\infty,0)} \langle \tilde{\varphi}_i, D\psi \rangle \right| \\
&= \left| \int_{\Sigma \times (0,\infty)} \langle D\tilde{\varphi}_i, \psi \rangle + \int_\Sigma \langle \nu \cdot \tilde{\varphi}_i|_\Sigma, \psi|_\Sigma \rangle + \int_{\Sigma \times (-\infty,0)} \langle D\tilde{\varphi}_i, \psi \rangle \right. \\
&\quad \left. + \int_\Sigma \langle -\nu \cdot \tilde{\varphi}_i|_\Sigma, \psi|_\Sigma \rangle \right| \\
&\leqslant 2\|D\varphi_i\|_{L^2(M)} \|\psi\|_{L^2(M_\infty)} \longrightarrow 0.
\end{aligned}$$

In particular, this means that $\tilde{\varphi}_i \in H_1(M_\infty, \mathbb{S}_{M_\infty})$ and that $\|D\tilde{\varphi}_i\|_{L^2(M_\infty)} \to 0$ while $\|\tilde{\varphi}_i\|_{L^2(M_\infty)} = 2$. This gives a contradiction to the invertibility of the Dirac operator on M_∞.

Corollary 6.7. *Let the boundary Σ be closed. If B is an elliptic boundary condition as defined in* [**9**, Definition 7.5], *then B-coercivity at infinity implies coercivity at infinity. In particular, D is $(B = 0)$-coercive at infinity if and only if it is coercive at infinity.*

Proof. If the boundary is closed and B is elliptic, then D_B has a finite kernel [**9**, Theorem 8.5]. The rest of the assumption in Lemma 6.5 is trivially fulfilled which gives the first claim. The rest follows with Corollary 6.4. □

For closed boundaries and spin manifolds, assuming uniformly positive scalar curvature at infinity is a sufficient condition to have that D is coercive at infinity, see [**9**, Example 8.3]. For noncompact boundaries, we obtain the following lemma.

LEMMA 6.8. (i) *If* $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot$ *is a positive operator, then the Dirac operator* D *is* $(B=0)$*-coercive at infinity.*

(ii) *If* $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot$ *is a positive operator and* $H \geqslant 0$, *then the Dirac operator* D *is* $B_\pm$*-coercive at infinity.*

Proof. Let $c > 0$ such that $\frac{1}{2}\mathrm{scal}^M + i\Omega\cdot \geqslant 2c$. The Lichnerowicz formula (6) and Lemma 3.10 give

$$\begin{aligned}\|D\varphi\|^2_{L^2} &= \|\nabla\varphi\|^2_{L^2} + \int_M \frac{\mathrm{scal}^M}{4}|\varphi|^2\,dv + \int_M \frac{i}{2} < \Omega\cdot\varphi,\varphi > \,dv - \int_\Sigma \langle R\varphi, \tilde{D}^\Sigma(R\varphi)\rangle\,ds\\ &\quad + \frac{n}{2}\int_\Sigma H|R\varphi|^2\,ds \geqslant c\|\varphi\|^2_{L^2} - \int_\Sigma \langle R\varphi, \tilde{D}^\Sigma(R\varphi)\rangle\,ds + \frac{n}{2}\int_\Sigma H|R\varphi|^2\,ds,\end{aligned}$$

for all $\varphi \in H_1(M, \mathbb{S}_M)$. Then, (i) follows directly with Lemma 4.5. For (ii), let now $H \geqslant 0$ and $R\varphi \in B_\pm$. Then, together with Lemma 5.1, it implies

$$\begin{aligned}\|D\varphi\|^2_{L^2} &\geqslant c\|\varphi\|^2_{L^2} - \int_\Sigma \langle R\varphi, \tilde{D}^\Sigma(R\varphi)\rangle\,ds = c\|\varphi\|^2_{L^2} - \int_\Sigma \langle P_\mp R\varphi, \tilde{D}^\Sigma(P_\mp R\varphi)\rangle\\ &= c\|\varphi\|^2_{L^2} - \int_\Sigma \langle P_\mp R\varphi, P_\pm \tilde{D}^\Sigma(R\varphi)\rangle = c\|\varphi\|^2_{L^2}.\end{aligned}$$

□

7. Spin^c *Reilly inequality on possibly open boundary domains*

In this section, we shortly review the spinorial Reilly inequality. This inequality together with those boundary value problems discussed in Section 4 will be the main ingredient in the proof of Theorem 1.2.

THEOREM 7.1 (Spin^c Reilly inequality). *For all* $\psi \in H_1(M, \mathbb{S}_M)$, *we have*

$$\int_\Sigma \left(\langle \tilde{D}^\Sigma\psi,\psi\rangle - \frac{n}{2}H|\psi|^2\right)ds \geqslant \int_M \left(\frac{1}{4}\mathrm{scal}^M|\psi|^2 + \frac{1}{2}\langle i\Omega\cdot\psi,\psi\rangle - \frac{n}{n+1}|D\psi|^2\right)dv, \tag{13}$$

where dv (*respectively,* ds) *is the Riemannian volume form of* M (*respectively,* Σ). *Moreover, equality occurs if and only if the spinor field* ψ *is a twistor–spinor, that is, if and only if* $P\psi = 0$, *where* P *is the twistor operator acting on* $\mathbb{S}_M$ *and is locally given by* $P_X\psi = \nabla_X\psi + (1/(n+1))X\cdot D\psi$ *for all* $X \in \Gamma(TM)$.

Proof. The inequality is proved for $\psi \in \Gamma^\infty_c(M, \mathbb{S}_M)$ analogously as in the compact Spin case [**16**, (17)]. For the convenience of the reader, we will shortly recall it here. Then, for all $\psi \in H_1(M, \mathbb{S}_M)$, the claim follows using the Trace Theorem 3.7 in the same way as in Lemma 3.10: We define 1-forms α and β on M by $\alpha(X) = \langle X\cdot D\psi, \psi\rangle$ and $\beta(X) = \langle \nabla_X\psi, \psi\rangle$

for all $X \in \Gamma^\infty(TM)$. Then, α and β satisfy

$$\delta\alpha = \langle D^2\psi, \psi\rangle - |D\psi|^2, \quad \delta\beta = -\langle \nabla^*\nabla\psi, \psi\rangle + |\nabla\psi|^2.$$

Applying the divergence theorem with (3) and (4), we obtain

$$\int_\Sigma \left(\langle \tilde{D}^\Sigma\psi, \psi\rangle - \frac{n}{2}H|\psi|^2\right) ds = \int_M \left(|\nabla\psi|^2 - |D\psi|^2 + \frac{1}{4}\mathrm{scal}^M\ |\psi|^2 + \frac{i}{2}\langle \Omega\cdot\psi, \psi\rangle\right) dv. \quad (14)$$

On the other hand, for any spinor field ψ, we have

$$|\nabla\psi|^2 = |P\psi|^2 + \frac{1}{n+1}|D\psi|^2. \quad (15)$$

Combining the identities (15), and (14) and $|P\psi|^2 \geqslant 0$, the result follows. Equality holds if and only if $|P\psi|^2 = 0$, that is, the spinor ψ is a twistor–spinor. □

8. *A lower bound for the first nonnegative eigenvalue of the Dirac operator on the boundary*

In this section, we prove Theorem 1.2. For that we will not follow the original proof given in [**16**] due to our problems concerning the APS-boundary conditions as remarked at the end of Example 4.16(iv). But we will use $B_\pm$ as given in Example 4.16(iii).

Proof of Theorem 1.2. Since Σ is of bounded geometry, $\tilde{D}^\Sigma : H_1(\Sigma, \mathbb{S}_M|_\Sigma) \to L^2(\Sigma, \mathbb{S}_M|_\Sigma)$ is self-adjoint and, hence, λ_1 is an eigenvalue or in the essential spectrum of $\tilde{D}^\Sigma$. In both cases, there is a sequence $\varphi_i \in H_1(\Sigma, \mathbb{S}_M|_\Sigma)$ with $\|\varphi_i\|_{L^2(\Sigma)} = 1$ and $\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2(\Sigma)} \to 0$. Then, $\varphi_i \to \varphi$ weakly in $L^2(\Sigma, \mathbb{S}_M|_\Sigma)$. (In case that $\varphi \neq 0$, then φ is an eigenspinor of $\tilde{D}^\Sigma$ to the eigenvalue λ_1 otherwise λ_1 is in the essential spectrum of $\tilde{D}^\Sigma$). We assumed that D is B_--coercive at infinity (everything which follows is also true when assuming B_+-coercivity at infinity when switching the signs). Then, by Lemma 4.18, the range of D_{B_-} is closed. Moreover, from Lemma 5.1 we have $\ker (D_{B_-})^* = \ker D_{B_+} = \{0\}$. Thus, due to Corollary 4.19 for each i there exists a unique $\Psi_i \in H_1(M, \mathbb{S}_M)$ with $D\Psi_i = 0$ and $P_+R\Psi_i = P_+\varphi_i$. Using Theorem 7.1 and $\mathrm{scal}^M + 2i\Omega\cdot \geqslant 0$, we obtain

$$0 \leqslant \int_\Sigma \left(\langle \tilde{D}^\Sigma R\Psi_i, R\Psi_i\rangle - \frac{n}{2}H|R\Psi_i|^2\right) ds.$$

Moreover,

$$\begin{aligned}
&(\tilde{D}^\Sigma(P_+R\Psi_i + P_-R\Psi_i), P_+R\Psi_i + P_-R\Psi_i)_\Sigma \\
&\quad = (\tilde{D}^\Sigma P_+R\Psi_i, P_-R\Psi_i)_\Sigma + (\tilde{D}^\Sigma P_-R\Psi_i, P_+R\Psi_i)_\Sigma \\
&\quad = (\tilde{D}^\Sigma P_+R\Psi_i, P_-R\Psi_i)_\Sigma + (P_-R\Psi_i, \tilde{D}^\Sigma RP_+\Psi_i)_\Sigma,
\end{aligned}$$

where we used Lemma 5.1 and that $\tilde{D}^\Sigma$ is self-adjoint on $H_1(\Sigma, \mathbb{S}_M|_\Sigma)$. Hence, summarizing we get that

$$\begin{aligned}
\frac{n}{2}\int_\Sigma H|R\Psi_i|^2\, ds &\leqslant 2\Re\int_\Sigma \langle \tilde{D}^\Sigma P_+R\Psi_i, P_-R\Psi_i\rangle\, ds = 2\Re\int_\Sigma \langle P_-\tilde{D}\varphi_i, P_-R\Psi_i\rangle\, ds \\
&\leqslant 2\Re\int_\Sigma \langle P_-(\tilde{D}^\Sigma - \lambda_1)\varphi_i, P_-R\Psi_i\rangle\, ds + 2\lambda_1\Re\int_\Sigma \langle P_-\varphi_i, P_-R\Psi_i\rangle\, ds.
\end{aligned}$$

Using $2\Re\int_\Sigma\langle P_-\varphi_i, P_-R\Psi_i\rangle ds \leqslant \|P_-\varphi_i\|^2_{L^2(\Sigma)} + \|P_-R\Psi_i\|^2_{L^2(\Sigma)}$ and $\lambda_1 \geqslant 0$, we obtain

$$\frac{n}{2}\inf_\Sigma H\|R\Psi_i\|^2_{L^2(\Sigma)} \leqslant 2\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_-\varphi_i\|^2_{L^2(\Sigma)} + \|P_-R\Psi_i\|^2_{L^2(\Sigma)}).$$

Moreover, $(\tilde{D}^\Sigma P_\pm\varphi_i, P_\mp\varphi_i) = (P_\mp(\tilde{D}^\Sigma - \lambda_1)\varphi_i, P_\mp\varphi_i) + \lambda_1\|P_\mp\varphi_i\|^2_{L^2}$. Since $\tilde{D}^\Sigma$ is self-adjoint, $\Re(\tilde{D}^\Sigma P_+\varphi_i, P_-\varphi_i) = \Re(\tilde{D}^\Sigma P_-\varphi_i, P_+\varphi_i)$. Thus, together with

$$|(P_\mp(\tilde{D}^\Sigma - \lambda_1)\varphi_i, P_\mp\varphi_i)| \leqslant \|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|\varphi_i\|_{L^2} \longrightarrow 0$$

as $i \to \infty$, this implies that $\lim_{i\to\infty}\|P_-\varphi_i\|_{L^2} = \lim_{i\to\infty}\|P_+\varphi_i\|_{L^2} = \frac{1}{2}$ for $\lambda_1 \neq 0$. Hence, for certain ε_i with $\varepsilon_i \to 0$ as $i \to \infty$

$$\begin{aligned}\frac{n}{2}\inf_\Sigma H\|R\Psi_i\|^2_{L^2(\Sigma)} &\leqslant 2\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_+\varphi_i\|^2_{L^2(\Sigma)} + \varepsilon_i + \|P_-R\Psi_i\|^2_{L^2(\Sigma)})\\ &\leqslant 2\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|P_+R\Psi_i\|^2_{L^2(\Sigma)} + \varepsilon_i + \|P_-R\Psi_i\|^2_{L^2(\Sigma)})\\ &\leqslant 2\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|_{L^2} + \lambda_1(\|R\Psi_i\|^2_{L^2(\Sigma)} + \varepsilon_i).\end{aligned}$$

Hence,

$$\frac{n}{2}\inf_\Sigma H \leqslant 2\|(\tilde{D}^\Sigma - \lambda_1)\varphi_i\|_{L^2}\|R\Psi_i\|^{-1}_{L^2} + \lambda_1(1 + \varepsilon_i\|R\Psi_i\|^{-2}_{L^2}).$$

With $\|R\Psi_i\|_{L^2} \geqslant \|P_+R\Psi_i\|_{L^2} = \|P_+\varphi_i\|_{L^2} \to \frac{1}{2}$, we finally get for $i \to \infty$

$$\frac{n}{2}\inf_\Sigma H \leqslant \lambda_1.$$

Next, we collect all conditions that have to be fulfilled to obtain the equality $\frac{n}{2}\inf_\Sigma H = \lambda_1$:

(1) from the spinorial Reilly Inequality (13), $\int_M |P\Psi_i|^2\,dv \to 0$ which implies together with $D\Psi_i = 0$ that $\int_M |\nabla\Psi_i|^2\,dv \to 0$;
(2) $\int_M \mathrm{scal}^M|\Psi_i|^2 + 2i\langle\Omega\cdot\Psi_i, \Psi_i\rangle\,dv \to 0$;
(3) $\|\varphi_i - R\Psi_i\|_{L^2(\Sigma)} \to 0$;
(4) $\int_\Sigma (H - \inf_\Sigma H)|R\Psi_i|^2\,ds \to 0$.

In case that λ_1 is an eigenvalue of $\tilde{D}^\Sigma$ with eigenspinor φ, one can choose $\varphi_i = \varphi$ for all i. Then, $\Psi_i =: \Psi$ for all i and those equality conditions reduce to $\varphi = R\Psi$, Ψ is a parallel spinor on M, H is constant and $\int_M \mathrm{scal}^M|\Psi|^2 + 2i\langle\Omega\cdot\Psi, \Psi\rangle\,dv = 0$. □

Acknowledgements. The work was initiated during the Junior Trimester Program in Differential Geometry at the Hausdorff Research Institute for Mathematics (HIM) in Bonn where the first author participated. The authors gratefully acknowledge the support of the HIM. The second author thanks the Institute of Mathematics of the University of Leipzig for its support and hospitality. The first author is very much indebted to Bernd Ammann, Ulrich Bunke and Cornelia Schneider for many enlightening discussions. We also want to thank the referee for very helpful comments.

References

1. I. Agricola and T. Friedrich, 'Upper bounds for the first eigenvalue of the Dirac operator on surfaces', *J. Geom. Phys.* 30 (1999) 1–22.
2. B. Ammann, 'The Willmore conjecture for immersed tori with small curvature integral', *Manuscripta Math.* 101 (2000) 1–22.
3. B. Ammann, *A variational problem in Conformal Spin Geometry* (Universität Hamburg, Habilitation, Hamburg, Germany, 2003).
4. B. Ammann, 'Ambient Dirac eigenvalue estimates and the Willmore functional', *Dirac operators: yesterday and today* (eds Bourguignon, Branson, Chamseddine, Hijazi, Stanton; International Press, Boston, MA, 2005) 221–228.
5. B. Ammann, J.-F. Grosjean, E. Humbert and B. Morel, 'A spinorial analogue of Aubin's inequality', *Math. Z.* 260 (2008) 127–151.
6. N. Anghel, 'Extrinsic upper bounds for eigenvalues of Dirac-type operators', *Proc. AMS* 117 (1993) 501–509.
7. C. Bär, 'Extrinsic bounds of the Dirac operator', *Ann. Global Anal. Geom.* 16 (1998) 573–596.
8. C. Bär, 'The spectrum of the Dirac operator', *Dirac operators: yesterday and today* (Int. Press, Somerville, MA, 2005) 145–162.

9. C. Bär and W. Ballmann, *Boundary value problems for elliptic differential operators of first order*, Surveys in Differential Geometry 17 (Int. Press, Boston, MA, 2012) 1–78.
10. H. Baum, 'An upper bound for the first eigenvalue of the Dirac operator on compact spin manifolds', *Math. Z.* 206 (1991) 409–422.
11. B. Booß-Bavnbek and M. Lesch, 'The invertible double of elliptic operators', *Lett. Math. Phys.* 87 (2009) 19–46.
12. J. Eichhorn, 'The boundedness of connection coefficients and their derivatives', *Math. Nachr.* (1991) 145–158.
13. T. Friedrich, *Dirac operators in Riemannian geometry*, Graduate Studies in Mathematics 25 (American Mathematical Society, Providence, RI, 2000).
14. N. Große and C. Schneider, 'Sobolev spaces on Riemannian manifolds with bounded geometry: General coordinates and traces', *Math. Z.* 286 (2013) 1586–1613.
15. O. Hijazi, S. Montiel and F. Urbano, 'Spin^c geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds', *Math. Z.* 253 (2006) 821–853.
16. O. Hijazi, S. Montiel and X. Zhang, 'Dirac operator on embedded hypersurfaces', *Math. Res. Lett.* 8 (2001) 20–36.
17. O. Hijazi, S. Montiel and X. Zhang, 'Eigenvalues of the Dirac operator on manifolds with boundary', *Comm. Math. Phys.* 221 (2001) 255–265.
18. O. Hijazi, S. Montiel and X. Zhang, 'Conformal lower bounds for the Dirac operator on embedded hypersurfaces', *Asian J. Math.* 6 (2002) 23–36.
19. O. Hijazi and X. Zhang, 'Lower bounds for the eigenvalues of the Dirac operator. Part I: the hypersurface Dirac operator', *Ann. Global Anal. Geom.* 19 (2001) 355–376.
20. O. Hijazi and X. Zhang, 'Lower bounds for the eigenvalues of the Dirac operator. Part II: the submanifold Dirac operator', *Ann. Global Anal. Geom.* 19 (2001) 163–181.
21. A. Moroianu, 'Parallel and killing spinors on Spin^c manifolds', *Commun. Math. Phys.* 187 (1997) 417–428.
22. R. Nakad, 'The Energy-Momentum tensor on Spin^c manifolds', *Int. J. Geom. Methods Mod. Phys.,* 8 (2011) 345–365.
23. R. Nakad, 'Special submanifolds of Spin^c manifolds', PhD Thesis, Institut Élie Cartan, 2011.
24. M. Renardy and R. Rogers, *An introduction to partial differential equations*, 2nd edn, Texts in Applied Mathematics (Springer, New-York, 2004) 445.
25. T. Schick, 'Manifolds with boundary and of bounded geometry', *Math. Nachr.* 223 (2001) 103–120.
26. M. A. Shubin, *Spectral theory of elliptic operators on noncompact manifolds*, Méthodes semi-classiques, Vol. 1 (Nantes, 1991), Astérisque (207), 1992, 35–108.
27. M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series 34 (Princeton University Press, 1981) xi+452.
28. H. Triebel, *Theory of function spaces*, Monographs in Mathematics 78 (Birkhäuser, Basel, 1983) 284.
29. K. Yosida, *Functional analysis*, Classics in Mathematics, Reprint of the sixth (1980) edition (Springer, Berlin, 1995) xii+501.
30. X. Zhang, 'Lower bounds for eigenvalues of hypersurface Dirac operators', *Math. Res. Lett.* 5 (1998) 199–210; 'A remark: Lower bounds for eigenvalues of hypersurface Dirac operators', *Math. Res. Lett.* 6 (1999) 465–466.

Nadine Grosse
Mathematisches Institut
Universität Leipzig
04009 Leipzig
Germany

grosse@math.uni-leipzig.de

Roger Nakad
Department of Mathematics and Statistics
Faculty of Natural and Applied Sciences
Notre Dame University-Louaizé
P.O. Box 72, Zouk Mikael
Lebanon

rnakad@ndu.edu.lb