Baxter operator formalism for Macdonald polynomials

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Abstract. We develop basic constructions of the Baxter operator formalism for the Macdonald polynomials associated with root systems of type A. Precisely we construct a dual pair of mutually commuting Baxter operators such that the Macdonald polynomials are their common eigenfunctions. The dual pair of Baxter operators is closely related to the dual pair of recursive operators for Macdonald polynomials leading to various families of their integral representations. We also construct the Baxter operator formalism for the q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions and the Jack polynomials obtained by degenerations of the Macdonald polynomials associated with the type A_ℓ root system. This note provides a generalization of our previous results on the Baxter operator formalism for the Whittaker functions. It was demonstrated previously that Baxter operator formalism for the Whittaker functions has deep connections with representation theory. In particular the Baxter operators should be considered as elements of appropriate spherical Hecke algebras and their eigenvalues are identified with local Archimedean L-factors associated with admissible representations of reductive groups over $\mathbb{R}$. We expect that the Baxter operator formalism for the Macdonald polynomials has an interpretation in representation theory of higher-dimensional arithmetic fields.

Introduction

A new class of operators acting on eigenfunctions of quantum integrable systems was introduced by Baxter to provide a solution of a class of integrable models [Ba]. These operators commute with quantum Hamiltonians of a quantum integrable system and satisfy difference/differential equations with coefficients expressed through quantum Hamiltonians. The Baxter operators were constructed for many integrable models including periodic Toda chains [PG].

In [GLO1] we introduce Baxter operators for non-periodic $\mathfrak{gl}_{\ell+1}$-Toda chains given by one-parameter families of integral operators. We also define dual Baxter operators acting on the spectral variables of Toda chain eigenfunctions. The dual pair of Baxter operators enters a canonical construction of a pair of recursive operators relating eigenfunctions of the $\mathfrak{gl}_{\ell+1}$- and $\mathfrak{gl}_\ell$- Toda chains. This gives rise to various families of integral representations for eigenfunctions. Hence the Baxter operator formalism consisting of a pair of dual Baxter operators, a pair of dual recursive operators provides a complete solution of the Toda chains.

One can expect that the Baxter operator formalism can be constructed for a wide class of quantum integrable systems. Note that $\mathfrak{gl}_{\ell+1}$-Toda chains can be considered as a degeneration of the quantum integrable system constructed by Ruijsenaars [Ru] and Macdonald [M]. The corresponding quantum Hamiltonians are given by mutually commuting difference operators and their common polynomial eigenfunctions are given by the Macdonald polynomials. In this note we construct Baxter operator formalism for the Macdonald-Ruijsenaars integrable system. This includes a dual

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pair of Baxter operators, a dual pair of recursive operators and various families of explicit iterative expressions for the Macdonald polynomials, known and new ones. We also describe Baxter operator formalism for specializations of the Macdonald polynomials given by class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions [GLO2] and Jack's polynomials. Due to the results of [GLO8] the Baxter operator formalism for the standard Whittaker functions [GLO1] can be recovered from the Baxter operator formalism for q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker polynomials in the limit $q \to 1$.

One should stress that the Baxter operators associated with the Whittaker functions have a surprising relation with number theory and representation theory [GLO1]. Recall that the eigenfunctions of $\mathfrak{gl}_{\ell+1}$-Toda chains can be identified with particular matrix elements in the principal series representations of $GL_{\ell+1}(\mathbb{R})$ and thus providing generalizations of the classical Whittaker functions corresponding to $SL_2(\mathbb{R})$. In [GLO1] we argue that for $\mathfrak{gl}_{\ell+1}$-Toda chain the Baxter operator should be considered as a generating function of elements of spherical Hecke algebra associated with the maximal compact subgroup of $GL_{\ell+1}(\mathbb{R})$. Furthermore the $\mathfrak{gl}_{\ell+1}$-Whittaker functions are eigenfunctions of the Baxter operators with the eigenvalues given by local Archimedean L-factors of the corresponding principle series representations of $GL_{\ell+1}(\mathbb{R})$.

This interpretation leads to establishing a deep relation between topological field theories and Archimedean algebraic geometry [GLO5], [GLO6], [GLO7]. The construction of the Baxter operator formalism for the Macdonald-Ruijsenaars integrable system allows to define a (q,t)-generalization of the local Archimedean L-factors associated with principle series representations of $GL_{\ell+1}$ (q-generalization of local L-factors was introduced previously in [GLO2]). One can expect that these generalized L-factors shall be related with principal series representations of loop groups associated with $GL_{\ell+1}$. Taking into account the results of [GLO5], [GLO6], [GLO7] one should look for a higher dimensional topological field theory interpretation of the Macdonald polynomials and the associated Baxter operator formalism. We are going to discuss this interpretation in the future publications.

Finally we would like to point out that many of the constructions of this note are simple reformulations of the results of Macdonald [M]. However we feel that establishing the direct relation of the results of [M] with Baxter operator formalism might be useful.

1 Preliminaries on symmetric polynomials

In this Section we collect basic facts on the Macdonald symmetric polynomials and their degenerate versions given by Jack polynomials and class one q-Whittaker functions. For details on Macdonald and Jack's polynomials see [M]; for class one q-deformed Whittaker functions see [GLO2], [GLO3], [GLO4].

1.1 Macdonald symmetric polynomials

Let $\mathbb{Q}(q, t)$ be a field of rational functions in variables q and t. Define the following (q, t)-analog of the classical Γ-function (see Appendix for its basic properties):

$$\Gamma_{q,t}(x) = \frac{(tx; q)_\infty}{(x; q)_\infty}, \qquad (x; q)_\infty = \prod_{j=0}^\infty (1 - xq^j). \tag{1.1}$$

Let $\Lambda_{q,t}$ be the graded $\mathbb{Q}(q, t)$-algebra of symmetric polynomials of variables $x_1, x_2, \ldots$ of degree one

$$\Lambda_{q,t} = \bigoplus_{n \geq 0} \Lambda_{q,t}^{(n)}, \tag{1.2}$$

where $\Lambda_{q,t}^{(n)}$ is the homogeneous component of $\Lambda_{q,t}$ of degree n. There are various convenient bases in the space of symmetric polynomials in variables $x_1, \ldots, x_{\ell+1}$ enumerated by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{\ell+1})$, $\lambda_i \in \mathbb{Z}_+$. Particularly, the elements of the bases of monomial symmetric functions $m_\lambda(x)$ are given by sums of all distinct monomials obtained from $x^\lambda = x_1^{\lambda_1} \ldots x_{\ell+1}^{\lambda_{\ell+1}}$ by permutations of $x_1, \ldots, x_{\ell+1}$. Let us denote $p_n(x) := m_{(n)}$ the symmetric polynomial for the partition $(n) = (n, 0, \ldots, 0)$. The bases of power series symmetric polynomials consists of the polynomials $p_\lambda(x) = p_{\lambda_1}(x) \cdot \ldots \cdot p_{\lambda_{\ell+1}}(x)$. Equip the space $\Lambda_{q,t}^{(\ell+1)}$ with a scalar product $\langle \, , \, \rangle$ defined by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell+1} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \qquad z_\lambda = \prod_{n \geq 1} n^{m_n} m_n!, \qquad m_n = \left| \{k : \lambda_k = n\} \right|. \tag{1.3}$$

Macdonald introduced a bases $\{P_\lambda(x) = P_\lambda(x; q, t)\}$ of symmetric polynomials over $\mathbb{Q}(q, t)$ enumerated by partitions λ such that

$$P_\lambda(x) = \sum_{\mu \leq \lambda} u_{\lambda\mu} \, m_\mu, \qquad u_{\lambda\lambda} = 1, \tag{1.4}$$

and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0, \qquad \lambda \neq \mu. \tag{1.5}$$

In the above formula $\leq$ denotes the natural ordering:

$$\lambda \leq \mu \iff \lambda_1 + \ldots + \lambda_i \leq \mu_1 + \ldots + \mu_i, \quad i \geq 0.$$

The relation (1.4) is invertible and thus the Macdonald polynomials $P_\lambda(x)$ provide a bases in $\Lambda_{q,t}^{(\ell+1)}$. The inverse norms of the Macdonald polynomials are given by

$$b_\lambda := \langle P_\lambda, P_\lambda \rangle_{q,t}^{-1} = \prod_{(i,j) \in \lambda} \frac{1 - t^{\lambda_j^\top + 1 - i} q^{\lambda_i - j}}{1 - t^{\lambda_j^\top - i} q^{\lambda_i + 1 - j}}, \tag{1.6}$$

where the product is over the boxes (i, j) in the Young diagram attached to partition λ, and $\lambda^\top$ denotes the conjugate partition. In particular, one has

$$b_{(n)} = \prod_{i=1}^n \frac{1 - tq^{n-i}}{1 - q^{n+1-i}} = \frac{\Gamma_{q,tq^{-1}}(q)}{\Gamma_{q,tq^{-1}}(q^{n+1})}. \tag{1.7}$$

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The Macdonald polynomials $P_\lambda(x)$ can be also characterized as common eigenfunctions of the following set of mutually commuting difference operators [M], [Ru] acting in $\Lambda_{q,t}^{(\ell+1)}$:

$$M_r = t^{r(r-1)/2} \sum_{I_r} \prod_{\substack{i \in I_r \\ j \notin I_r}} \frac{tx_i - x_j}{x_i - x_j} T_{I_r}, \qquad T_{I_r} = \prod_{i \in I_r} T_{q,x_i}, \tag{1.8}$$

where the sum goes over all r-element subsets I_r of $(1, 2, \ldots, \ell+1)$ and

$$T_{q,x_i} \cdot f(x_1, \ldots, x_{\ell+1}) = f(x_1, \ldots, qx_i, \ldots, x_{\ell+1}).$$

The operators M_r, $r = 1, \ldots, \ell+1$ are self-adjoint with respect to the scalar product (1.3) and the eigenvalues of M_r acting on $P_\lambda(x)$ are given by the elementary symmetric functions

$$\chi_r(y_1, \ldots, y_{\ell+1}) = \sum_{1 \le i_1 < \ldots < i_r \le \ell+1} y_{i_1} \cdot \ldots \cdot y_{i_r}, \qquad y_i = t^{\varrho_i} q^{\lambda_i}, \tag{1.9}$$

where $\varrho_i = \ell + 1 - i$. The eigenfunction property of the Macdonald polynomials can be succinctly described by the relation

$$\mathcal{M}_{\ell+1}(X) \cdot P_\lambda(x) = c_{\ell+1}(\lambda; X) \, P_\lambda(x),$$

$$c_{\ell+1}(\lambda; X) = (1-t)^{-(\ell+1)} \prod_{i=1}^{\ell+1} \left(1 + t^{\varrho_i} q^{\lambda_i} X\right), \tag{1.10}$$

where $\mathcal{M}_{\ell+1}(X)$ is a generating function of the difference operators (1.8):

$$\mathcal{M}_{\ell+1}(X) = (1-t)^{-(\ell+1)} \left(1 + \sum_{r=1}^{\ell+1} X^r M_r\right). \tag{1.11}$$

Assume now that $q \in \mathbb{C}$ and $|q| < 1$, so that infinite product $(z; q)_\infty$ converges for all $z \in \mathbb{C}$. Following Macdonald define a new scalar product

$$\langle a, b \rangle'_{q,t} = \frac{1}{(\ell+1)!} \int_T d^\times z \, a(z) \, b(z^{-1}) \, \Delta(z; q, t),$$

$$\Delta(z; q, t) = \prod_{\substack{i,j=1 \\ i \ne j}}^{\ell+1} \frac{1}{\Gamma_{q,t}(z_i z_j^{-1})}, \tag{1.12}$$

where $a(z)$ and $b(z)$ are Laurent polynomials and

$$T = \left\{ z = (z_1, \ldots, z_{\ell+1}) \in \mathbb{C}^{\ell+1} : |z_i| = 1, \, i = 1, \ldots, \ell+1 \right\}$$

is the $(\ell+1)$-dimensional torus, with the Haar measure $d^\times z = \prod_{i=1}^{\ell+1} (2\pi\imath)^{-1} d\log z_i$. The polynomials P_λ are pairwise orthogonal with respect to the new scalar product $\langle , \rangle'$ with the norms given by

$$\langle P_\lambda, P_\lambda \rangle'_{q,t} = \prod_{\substack{i,j=1 \\ i<j}}^{\ell+1} \frac{\Gamma_{q,tq^{-1}}(t^{j-i-1} q^{\lambda_i - \lambda_j + 1})}{\Gamma_{q,tq^{-1}}(t^{j-i} q^{\lambda_i - \lambda_j + 1})}. \tag{1.13}$$

Let us recall the properties of the Macdonald polynomials that will play essential role in the following (see [M] for the proofs).

Theorem 1.1 *Consider two sets $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ of variables. Let*

$$\Pi_{n,m}(x,y) := \sum_{\lambda \in Y_{n,m}} b_\lambda \, P_\lambda(x) \, P_\lambda(y),$$
(1.14)

where summation goes over a set $Y_{m,n}$ of the partitions of length $\min(m,n)$ and b_λ are given by (1.6). Then the following identity holds

$$\Pi_{n,m}(x,y) = \prod_{i=1}^{n} \prod_{j=1}^{m} \Gamma_{q,t}(x_i y_j).$$
(1.15)

We will say that partitions μ, λ are interlaced if $\mu_1 \geq \lambda_1 \geq \ldots \geq \mu_{\ell+1} \geq \lambda_{\ell+1}$. In the sequel we shall use the following abbreviation for interlaced partitions: $\mu_i \geq \lambda_i \geq \mu_{i+1}$.

Theorem 1.2 *Let $P_{(n)}(x)$ be the Macdonald polynomial corresponding to the partition $(n) = (n, 0, \ldots, 0)$. Then the following product decomposition holds:*

$$P_{(n)}(x) \times P_\lambda(x) = b_{(n)}^{-1} \sum_{\substack{\mu_i \geq \lambda_i \geq \mu_{i+1} \\ |\mu| - |\lambda| = n}} \varphi_{\mu/\lambda} \, P_\mu(x),$$
(1.16)

where

$$\varphi_{\mu/\lambda} = \prod_{\substack{i,j=1 \\ i \leq j}}^{\ell+1} \frac{\Gamma_{q,tq^{-1}}\left(t^{j-i}q^{\mu_i - \mu_j + 1}\right)}{\Gamma_{q,tq^{-1}}\left(t^{j-i}q^{\mu_i - \lambda_j + 1}\right)} \frac{\Gamma_{q,tq^{-1}}\left(t^{j-i}q^{\lambda_i - \lambda_{j+1} + 1}\right)}{\Gamma_{q,tq^{-1}}\left(t^{j-i}q^{\lambda_i - \mu_{j+1} + 1}\right)},$$
(1.17)

and we omit in the product (1.17) the factors depending on $\lambda_{\ell+2}$ and $\mu_{\ell+2}$.

The Macdonald polynomials possess a remarkable self-duality property discovered by Koornwinder (see [M] and references therein). Let us introduce modified Macdonald polynomials:

$$\Phi_\lambda(x; q, t) := t^{\rho(\lambda)} \prod_{\substack{a,b=1 \\ a < b}}^{\ell+1} \Gamma_{q,t}(t^{b-a} q^{\lambda_a - \lambda_b}) \times P_\lambda(x; q, t),$$
(1.18)

where $\rho(x) = \sum_{i=1}^{\ell+1} \rho_i x_i$, $\rho_i = \varrho_i - \ell/2$. Then for any partitions λ and μ the following duality relation holds:

$$\Phi_\lambda\left(q^{\mu - k\rho}; q, q^{-k}\right) = \Phi_\mu\left(q^{\lambda - k\rho}; q, q^{-k}\right).$$
(1.19)

This duality naturally leads to the set of mutually commuting operators acting in the space of functions on the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) \in \mathbb{Z}_+^{\ell+1}$. The following Theorem was proved in [GLO4].

Theorem 1.3 ([GLO4]) *A set of mutually commuting difference operators*

$$M_r^\vee(\lambda) = t^{r\ell/2} \sum_{I_r} \prod_{\substack{i \in I_r \\ j \notin I_r \\ j < i}} \frac{1 - t^{i-j+1} q^{\lambda_j - \lambda_i - 1}}{1 - t^{i-j} q^{\lambda_j - \lambda_i - 1}} \frac{1 - t^{i-j-1} q^{\lambda_j - \lambda_i}}{1 - t^{i-j} q^{\lambda_j - \lambda_i}} T_{I_r}^\vee,$$
(1.20)

$$T_{q,q^{\lambda_i}} \cdot f_{(\lambda_1, \ldots, \lambda_{\ell+1})} = f_{(\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_{\ell+1})}, \qquad T_{I_r}^\vee = \prod_{i \in I_r} T_{q, q^{\lambda_i}},$$

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acts in the space of functions f_λ labeled by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{\ell+1}) \in \mathbb{Z}_+^{\ell+1}$. The Macdonald polynomials $P_\lambda(x)$ as functions of the variables λ are common eigenfunctions of the difference operators (1.20):

$$\mathcal{M}_{\ell+1}^\vee(X) \cdot P_\lambda(x; q, t) = c_{\ell+1}^\vee(t^{\ell/2}x; X) P_\lambda(x; q, t),$$

$$c_{\ell+1}^\vee(x; X) = \prod_{i=1}^{\ell+1} (1 + X x_i), \tag{1.21}$$

where

$$\mathcal{M}_{\ell+1}^\vee(X) = \sum_{r=0}^{\ell+1} X^r M_r^\vee, \qquad M_0^\vee := 1, \tag{1.22}$$

1.2 Class one q-deformed Whittaker functions

Let Λ_q be the algebra of symmetric functions in variables $x_1, x_2, \ldots$ over the field of rational functions in variable q convergent in the domain $|q| < 1$. Let $\Lambda_q^{(\ell+1)}$ be the homogeneous component Λ_q of degree $\ell + 1$. Consider a pair of the scalar products on $\Lambda_q^{(\ell+1)}$. The first one is defined in terms of power series symmetric polynomials p_λ as follows:

$$\langle p_\lambda, \, p_\mu \rangle_q = \delta_{\lambda\mu} \, z_\lambda \prod_{i=1}^{\ell+1} (1 - q^{\lambda_i}). \tag{1.23}$$

The second scalar product on the space Λ_q is defined by

$$\langle a(z), \, b(z) \rangle_q' = \frac{1}{(\ell+1)!} \int_T d^\times z \, a(z) \, b(z^{-1}) \, \Delta_q^\vee(z),$$

$$\Delta_q^\vee(z) = \prod_{\substack{i,j=1 \\ i \neq j}}^{\ell+1} \frac{1}{\Gamma_q(z_i^{-1} z_j)}, \qquad \Gamma_q(x) = \prod_{j=0}^\infty \frac{1}{1 - q^j x}, \tag{1.24}$$

and the notations of (1.12) are used.

Lemma 1.1 *The polynomials $P_\lambda(x; q) := P_\lambda(x; q, t = 0)$ satisfy the relations*

$$P_\lambda(x) = \sum_{\mu \leq \lambda} u_{\lambda\mu} \, m_\mu, \qquad u_{\lambda\lambda} = 1, \tag{1.25}$$

$$\langle P_\lambda, \, P_\mu \rangle_{q, t} = 0, \qquad \lambda \neq \mu. \tag{1.26}$$

and thus define a bases in Λ_q.

Proof. Directly follows by specialization $t = 0$ from the properties of the Macdonald polynomials. $\square$

Let us define the normalized symmetric polynomials in variables $x_1, \ldots, x_{\ell+1}$, labeled by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{\ell+1}) \in \mathbb{Z}_+^{\ell+1}$ as follows

$$P_\lambda^{qW}(x) = \Delta_q^{-1}(\lambda) P_\lambda(x; q, t = 0), \qquad \Delta_q(\lambda) = \prod_{i=1}^\ell (\lambda_i - \lambda_{i+1})_q!, \tag{1.27}$$

where $(n)_q! = \prod_{i=1}^n (1 - q^{n+1-i})$. In the following we will call $P_\lambda^{qW}(x)$ the q-Whittaker polynomials. These polynomials were introduced in [GLO2] as class one q-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker functions $\Psi_x^{(q)}(\lambda) = P_\lambda^{qW}(x)$.

The q-Whittaker polynomials $P_\lambda^{qW}(x)$ are orthogonal with respect to both scalar products (1.23), (1.24) and are normalized as follows:

$$\langle P_\lambda^{qW}, P_\lambda^{qW} \rangle_q = \frac{(\lambda_{\ell+1})_q!}{\Delta_q(\lambda)}, \qquad \langle P_\lambda^{qW}, P_\lambda^{qW} \rangle_q' = \frac{1}{\Delta_q(\lambda)} \left(\frac{1}{\Gamma_q(q)} \right)^\ell. \tag{1.28}$$

Theorem 1.4 ([GLO4]) *Let $H_1, \ldots, H_{\ell+1}$, and $H_1^\vee, \ldots, H_{\ell+1}^\vee$ be difference operators acting in the space of functions on $\mathbb{R}^{\ell+1} \times \mathbb{Z}_+^{\ell+1}$*

$$H_r^\vee = \sum_{I_r} \prod_{\substack{i \in I_r \\ j \notin I_r}} \frac{x_j}{x_j - x_i} T_{I_r}, \qquad T_{I_r} = \prod_{i \in I_r} T_{q, x_i}, \qquad r = 1, \ldots, \ell+1, \tag{1.29}$$

$$H_r = \sum_{I_r} \prod_{k=1}^r \left(1 - q^{\lambda_{i_k} - \lambda_{i_k+1} + 1} \right)^{1 - \delta_{i_{k+1} - i_k, 1}} T_{I_r}^\vee, \qquad T_{I_r}^\vee = \prod_{i \in I_r} T_{q, q^{\lambda_i}}, \tag{1.30}$$

for $r = 1, \ldots, \ell+1$, where $i_{r+1} := \ell + 2$ is assumed.

These operators are mutually commutative and the q-Whittaker polynomials solve the following dual pair of eigenfunction problems:

$$H_r^\vee \cdot P_\lambda^{qW}(x) = q^{\lambda_{\ell+2-r} + \ldots + \lambda_{\ell+1}} P_\lambda^{qW}(x), \qquad r = 1, \ldots, \ell+1. \tag{1.31}$$

and

$$H_r \cdot P_\lambda^{qW}(x) = \chi_r(x) P_\lambda^{qW}(x), \qquad \chi_r(x) = \sum_{I_r} x_{i_1} \cdots x_{i_r}, \tag{1.32}$$

for $r = 1, \ldots, \ell+1$.

For the generating function $D_{\ell+1}$ of the operators H_r

$$D_{\ell+1}(X) = \sum_{r=0}^{\ell+1} X^r H_r, \qquad H_0 := 1, \tag{1.33}$$

the following relation holds

$$D_{\ell+1}(X) \cdot P_\lambda^{qW}(x) = \mathbf{c}_{\ell+1}^q(x; X) P_\lambda^{qW}(x),$$

$$\mathbf{c}_{\ell+1}^q(x; X) = \prod_{i=1}^{\ell+1} (1 + X x_i). \tag{1.34}$$

The set of operators (1.30) define q-deformed Toda chain Hamiltonians and (1.29) provide a set of mutually commuting difference dual Toda chain Hamiltonians introduced in [GLO4].

Using the relation between q-Whittaker polynomials $P_\lambda^{qW}(x)$ and Macdonald polynomials $P_\lambda(x)$ one can infer an analog of the Pieri formula (1.16)

$$P_{(m,0,\ldots,0)}^{qW}(x) \times P_\lambda^{qW}(x) = \sum_{\substack{\mu_i \geq \lambda_i \geq \mu_{i+1} \\ |\mu|-|\lambda|=n}} \varphi_{\mu/\lambda}^q \, P_\mu^{qW}(x)\,,$$

$$\varphi_{\mu/\lambda}^q = \Delta_q(\mu) \frac{\Theta(\mu_1 - \lambda_1)}{(\mu_1 - \lambda_1)_q!} \prod_{i=1}^{\ell} \frac{\Theta(\lambda_i - \mu_{i+1})}{(\lambda_i - \mu_{i+1})_q!} \frac{\Theta(\mu_{i+1} - \lambda_{i+1})}{(\mu_{i+1} - \lambda_{i+1})_q!}\,.$$

(1.35)

The analog of the Cauchy-Littlewood identity (1.14) is given by

$$\prod_{i=1}^{n}\prod_{j=1}^{m} \Gamma_q(x_i y_j) = \sum_{\lambda \in Y_{n,m}} b_\lambda^q \, P_\lambda^{qW}(x)\, P_\lambda^{qW}(y)\,,$$

$$b_\lambda^q = \langle P_\lambda^{qW}, P_\lambda^{qW} \rangle_q^{-1} = \frac{\Delta_q(\lambda)}{(\lambda_{\ell+1})_q!}\,,$$

(1.36)

summed over $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m)$ with $m \leq n$.

Remark 1.1 *The q-Whittaker polynomials $P_\lambda^{qW}(x)$ can be also obtained from the Macdonald polynomials $P_\lambda(x)$ under the limit $t = q^{-k}$, $k \to +\infty$. Let*

$$D(x) = \prod_{i=1}^{\ell+1} x_i^{k \varrho_{\ell+2-i}}\,,$$

$$D^\vee(\lambda) = \prod_{i=1}^{\ell+1} q^{\lambda(\ell\,k+\varrho)} \times \prod_{\substack{i,j=1 \\ i<j}}^{\ell+1} \frac{1}{\Gamma_{q,\,q^{-k}}(q^{\lambda_i - \lambda_j - k(j-i-1)})} \frac{\Gamma_{q,\,q^{-k}}(q^{\lambda_i - \lambda_j})}{\Gamma_{q^{-1},\,q^{-(k+1)}}(q^{\lambda_j - \lambda_i})}\,.$$

Then we have

$$P_{\underline{p}}^{qW}(q^\lambda) = \lim_{k \to +\infty} \left[D^\vee(\lambda)^{-1} D(q^{\underline{p}+k\varrho})^{-1} \times P_{\lambda+k\rho}(q^{\underline{p}+\varrho(k+1)}) \right].$$

(1.37)

for $\underline{p} = (p_1,\ldots,p_{\ell+1})$ be a partition $(p_1 \geq \ldots \geq p_{\ell+1})$, and $P_{\underline{p}}^{qW}(q^\lambda) = 0$ otherwise.

1.3 Jack's symmetric polynomials

Now we consider a bases of symmetric polynomials consisting of the Jack polynomials, obtained from the Macdonald polynomials by a specialization (see [M] and references therein).

Let κ be a positive integer, and let $\Lambda^{(\ell+1)}$ be the homogeneous component of degree $\ell+1$ in the algebra of symmetric functions in variables $x_1, x_2 \ldots$ over $\mathbb{Q}(\kappa) = \mathbb{Q}$. Define a pair of scalar products on $\Lambda^{(\ell+1)}$ using the standard bases $\{p_\lambda\}$ of power series symmetric polynomials:

$$\langle p_\lambda, p_\mu \rangle_\kappa = \kappa^{-l(\lambda)} \delta_{\lambda\mu}\, z_\lambda\,, \qquad l(\lambda) = \big|\{m|\, \lambda_m \neq 0\}\big|\,,$$

(1.38)

where $z_\lambda = \prod_{n \geq 1} n^{m_n} m_n!$ and $m_n = \big|\{k : \lambda_k = n\}\big|$,

$$\langle p_\lambda, p_\mu \rangle_\kappa' = \frac{1}{(\ell+1)!} \int_T d^\times z \, p_\lambda(z)\, p_\mu(z^{-1})\, \Delta_{(\kappa)}(z)\,,$$

(1.39)

where

$$\Delta_{(\kappa)}(z) \;=\; \prod_{\substack{i,j=1 \\ i \neq j}}^{\ell+1} \left(1 - z_i^{-1} z_j\right)^{\kappa}. \tag{1.40}$$

Definition 1.1 *Jack's symmetric functions $P_\lambda^{(\kappa)}$ are the elements of $\Lambda^{(\ell+1)}$ such that*

$$\left\langle P_\lambda^{(\kappa)}, P_\mu^{(\kappa)} \right\rangle_\kappa \;=\; 0$$

whenever $\lambda \neq \mu$, and

$$P_\lambda^{(\kappa)} \;=\; m_\lambda \;+\; \sum_{\mu < \lambda} u_{\lambda\mu}^{(\kappa)} \, m_\mu \,.$$

The Jack polynomials $P_\lambda^{(\kappa)}(x)$ are orthogonal with respect to both scalar products and the following normalization condition holds:

$$\langle P_\lambda^{(\kappa)}, P_\lambda^{(\kappa)} \rangle_\kappa' \;=\; \prod_{\substack{i,j=1 \\ i<j}}^{\ell+1} \frac{\Gamma\big(\lambda_i - \lambda_j + \kappa(j-i+1)\big)}{\Gamma\big(\lambda_i - \lambda_j + \kappa(j-i)\big)} \, \frac{\Gamma\big(\lambda_i - \lambda_j + 1 + \kappa(j-i-1)\big)}{\Gamma\big(\lambda_i - \lambda_j + 1 + \kappa(j-i)\big)}\,. \tag{1.41}$$

Similarly to the cases of the Macdonald polynomials and q-Whittaker polynomials, Jack's polynomials are eigenfunction of dual families of mutually commuting differential/difference operators.

Theorem 1.5 *(i) The Jack symmetric polynomials are eigenfunctions of a set of mutually commuting Sekiguchi differential operators:*

$$\mathcal{D}_{\ell+1}(X) \cdot P_\lambda^{(\kappa)}(x) \;=\; \prod_{i=1}^{\ell+1} \big(X + (\lambda_i + \varrho_i \kappa)\big) P_\lambda^{(\kappa)}(x)\,, \tag{1.42}$$

where

$$\mathcal{D}_{\ell+1}(X) \;=\; \sum_{r=1}^{\ell+1} X^{\ell+1-r} \mathcal{H}_r$$

$$= \prod_{\substack{i,j=1 \\ i<j}}^{\ell+1} (x_i - x_j)^{-1} \times \sum_{\sigma \in \mathfrak{S}_{\ell+1}} (-1)^\sigma \prod_{i=1}^{\ell+1} x_i^{\sigma(\varrho_i)} \Big\{ X + \sigma(\varrho_i)\kappa + x_i \frac{\partial}{\partial x_i} \Big\}\,, \tag{1.43}$$

with $\mathcal{H}_0 := 1$.

(ii) The Jack polynomials are eigenfunctions of a set of mutually commuting difference operators

$$\mathcal{D}_{(\kappa)}^\vee(X) \cdot P_\lambda^{(\kappa)}(x) \;=\; c_{\ell+1}^\vee(x; X) \, P_\lambda^{(\kappa)}(x)\,,$$

$$c_{\ell+1}^\vee(x; X) \;=\; \prod_{i=1}^{\ell+1} (1 + X x_i)\,, \tag{1.44}$$

where

$$\mathcal{D}_{(\kappa)}^\vee(X) \;=\; \sum_{r=0}^{\ell+1} X^r \mathcal{H}_r^\vee\,, \qquad \mathcal{H}_0^\vee := 1\,,$$

and

$$\mathcal{H}_r^\vee(\lambda) = \sum_{I_r} \prod_{\substack{i \in I_r \\ j \notin I_r}} \frac{(i-j+1)\kappa + \lambda_j - \lambda_i - 1}{(i-j)\kappa + \lambda_j - \lambda_i - 1} \, T_{I_r}^\vee. \tag{1.45}$$

Here the summation goes over all r-element subsets I_r of $(1, 2, \ldots, \ell+1)$ and the operators $T^\vee$ are defined as follows

$$T_{I_r}^\vee = \prod_{i \in I_r} T_{q, q^\lambda}, \qquad T_{q, q^\lambda} \cdot f_\lambda = \left[\begin{array}{ll} f_{\lambda + \delta_{ij}}, & \lambda_{i-1} \geq \lambda_i + 1 \\ 0 & otherwise \end{array} \right. ,$$

where $\lambda + \delta_{ij} := \lambda + (0, \ldots, \underbrace{1}_{i}, \ldots, 0).$

Remark 1.2 *The generating function (1.43) can be considered as an appropriately defined non-commutative determinant:*

$$\mathcal{D}_{\ell+1}(X) = \frac{1}{\det \|x_i^{\varrho_j}\|} \det \left\| x_i^{\varrho_j} \left(X + \varrho_j \kappa + x_i \frac{\partial}{\partial x_i} \right) \right\|.$$

In particular, the first statement implies:

$$\mathcal{H}_r \cdot P_\lambda^{(\kappa)}(x) = \chi_r(\lambda + \varrho\kappa) \, P_\lambda^{(\kappa)}(x), \qquad r = 1, \ldots, \ell+1, \tag{1.46}$$

where $\varrho_i = \ell + 1 - i$. The first two Hamiltonians are given by

$$\mathcal{H}_1 = \sum_{i=1}^{\ell+1} \left\{ x_i \frac{\partial}{\partial x_i} + \varrho_i \kappa \right\},$$

$$\mathcal{H}_2 = \sum_{\substack{i,j=1 \\ i<j}}^{\ell+1} \left(x_i \frac{\partial}{\partial x_i} + \varrho_i \kappa \right) \left(x_j \frac{\partial}{\partial x_j} + \varrho_j \kappa \right) + \kappa \sum_{i=1}^{\ell+1} \left(\varrho_i + \sum_{\substack{j=1 \\ j \neq i}} \frac{x_i}{x_j - x_i} \right) x_i \frac{\partial}{\partial x_i}. \tag{1.47}$$

The Jack symmetric functions can be obtained from Macdonald polynomials by taking the limit $\hbar \to 0$ for $t = e^{\kappa\hbar}$, $q = e^\hbar$ [M]

$$\lim_{\hbar \to 0} \Gamma_{q,t}(x) = \frac{1}{(1-x)^\kappa} \qquad \lim_{\hbar \to 0} b_{(n)} = \frac{\Gamma(n+\kappa)}{\Gamma(n)\,\Gamma(\kappa)}, \qquad t = e^{\kappa\hbar}, \;\; q = e^\hbar. \tag{1.48}$$

It is easy to infer analogs of (1.16) and (1.14) for Jack polynomials. In particular, the Pieri rules for the Jack polynomials are given by (see [S]):

$$P_{(m)}^{(\kappa)} \times P_\lambda^{(\kappa)} = \frac{1}{b_{(m)}^{(\kappa)}} \sum_{\substack{\mu_i \geq \lambda_i \geq \mu_{i+1} \\ |\mu| - |\lambda| = n}} \varphi_{\mu/\lambda}^{(\kappa)} \, P_\mu^{(\kappa)},$$

$$\varphi_{\mu/\lambda}^{(\kappa)} = \prod_{\substack{i,j=1 \\ i \leq j}}^{\ell+1} \left[\frac{\Gamma\big(\mu_i - \mu_j + 1 + (j-i)\kappa\big)}{\Gamma\big(\mu_i - \mu_j + (j-i+1)\kappa\big)} \frac{\Gamma\big(\mu_i - \lambda_j + (j-i+1)\kappa\big)}{\Gamma\big(\mu_i - \lambda_j + 1 + (j-i)\kappa\big)} \right.$$

$$\left. \times \frac{\Gamma\big(\lambda_i - \lambda_{j+1} + 1 + (j-i)\kappa\big)}{\Gamma\big(\lambda_i - \lambda_{j+1} + (j-i+1)\kappa\big)} \frac{\Gamma\big(\lambda_i - \mu_{j+1} + (j-i+1)\kappa\big)}{\Gamma\big(\lambda_i - \mu_{j+1} + 1 + (j-i)\kappa\big)} \right], \tag{1.49}$$

and in the product (1.49) we omit the terms containing $\lambda_{\ell+2}$ and $\mu_{\ell+2}$. The analog of the Cauchy-Littlewood identity (1.14) is given by

$$\Pi_{n,m}^{(\kappa)} := \prod_{i,j} \frac{1}{(1 - x_i y_j)^\kappa} = \sum_{\lambda \in Y_{n,m}} b_\lambda^{(\kappa)} P_\lambda^{(\kappa)}(x) P_\lambda^{(\kappa)}(y), \qquad (1.50)$$

where the summation goes over partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_{\min(m,n)})$ and

$$b_\lambda^{(\kappa)} = \lim_{\hbar \to 0} b_\lambda = \prod_{(i,j) \in \lambda} \frac{\kappa(\lambda_j^\top + 1 - i) + \lambda_i - j}{\kappa(\lambda_j^\top - i) + \lambda_i + 1 - j}, \qquad t = e^{\kappa \hbar}, \quad q = e^\hbar. \qquad (1.51)$$

2 Baxter operator formalism for symmetric polynomials

In the previous Section we describe various bases in the space of symmetric polynomials defined as common eigenfunctions of two sets of mutually commuting operators called (dual) quantum Hamiltonians. In this Section we define a dual pair of the Baxter operators acting in the space of symmetric polynomials, commuting with dual pairs of quantum Hamiltonians. The constructed bases in the space of polynomials is also a bases of eigenfunctions of the dual pair of Baxter operators.

2.1 Baxter operator formalism for Macdonald symmetric polynomials

In this Section we develop the Baxter operator formalism for the Macdonald polynomials. We construct a dual pair of Baxter operators and a dual pair of recursive operators. This results in various families of integral/sum representations for the Macdonald polynomials.

Definition 2.1 *Baxter operator $\mathcal{Q}_\gamma = \mathcal{Q}_\gamma(q,t)$ associated with Macdonald integrable system is a family of operators acting on the space $\Lambda_{q,t}^{(\ell+1)}$ of symmetric polynomials as follows:*

$$\mathcal{Q}_\gamma \cdot P(x) = \int_T d^\times y \, Q_\gamma(x,y) \, \Delta(y) \, P(y^{-1}), \qquad \gamma \in \mathbb{Z}, \qquad (2.1)$$

where integral kernel is given by

$$Q_\gamma(x,y) = \prod_{i=1}^{\ell+1} (x_i y_i)^\gamma \prod_{i,j=1}^{\ell+1} \Gamma_{q,t}(x_i y_j). \qquad (2.2)$$

Theorem 2.1 *The Baxter operator (2.1) acts on the Macdonald polynomials $P_\lambda(x)$ as follows:*

$$\mathcal{Q}_\gamma \cdot P_\lambda(x) = L_\gamma(\lambda) \, P_\lambda(x), \qquad \lambda_{\ell+1} \geq \gamma \qquad (2.3)$$

$$\mathcal{Q}_\gamma \cdot P_\lambda(x) = 0, \qquad \lambda_{\ell+1} < \gamma \qquad (2.4)$$

where

$$L_\gamma(\lambda) := L_\gamma(\lambda, q, t) = \prod_{i=1}^{\ell+1} \frac{\Gamma_{q,\,tq^{-1}}(q)}{\Gamma_{q,\,tq^{-1}}(t^{\ell+1-i} q^{\lambda_i - \gamma + 1})}. \qquad (2.5)$$

Proof. The Baxter operator (2.1) can be represented in the following form

$$\mathcal{Q}_\gamma \; := \; D_\gamma^{\mathfrak{gl}_{\ell+1}} \circ \mathcal{C}^{\mathfrak{gl}_{\ell+1}} \circ D_{-\gamma}^{\mathfrak{gl}_{\ell+1}} \,, \tag{2.6}$$

where the operator $\mathcal{C}^{\mathfrak{gl}_{\ell+1}}$ acts as:

$$\mathcal{C}^{\mathfrak{gl}_{\ell+1}} \cdot P_\lambda(x) \; := \; \langle \Pi_{\ell+1,\,\ell+1}, \, P_\lambda \rangle'_{q,\,t} \,, \tag{2.7}$$

and the operator $D_\gamma^{\mathfrak{gl}_{\ell+1}}$ in (2.6) acts on $P_\lambda(x)$ according the following rule:

$$
\begin{aligned}
D_\gamma^{\mathfrak{gl}_{\ell+1}} \cdot P_\lambda(x_1,\dots,x_{\ell+1}) \; &:= \; P_{\lambda+(\ell+1)\gamma}(x_1,\dots,x_{\ell+1}) \\
&= \; (x_1 \cdot \ldots \cdot x_{\ell+1})^\gamma \, P_\lambda(x_1,\dots,x_{\ell+1}) \,,
\end{aligned}
\tag{2.8}
$$

$P_{\lambda+(\ell+1)\gamma}(x) := P_{\lambda_1+\gamma,\dots,\lambda_{\ell+1}+\gamma}(x)$. The eigenvalue of the operator $\mathcal{C}^{\mathfrak{gl}_{\ell+1}}$ can be found explicitly using the Cauchy-Littlewood identity (1.14) and orthogonality of the Macdonald polynomials with respect to the two scalar products, (1.3) and (1.12)

$$
\begin{aligned}
\mathcal{C}^{\mathfrak{gl}_{\ell+1}} \cdot P_\lambda^{\mathfrak{gl}_{\ell+1}}(x) \; &= \; \int_T d^\times y \, \Pi_{\ell+1,\ell+1}(x,y) P_\lambda^{\mathfrak{gl}_{\ell+1}}(y^{-1}) \Delta(y) \\
&= \; \int_T d^\times y \, \Big(\sum_{\mu_1 \ge \cdots \ge \mu_{\ell+1} \ge 0} b_\mu P_\mu^{\mathfrak{gl}_{\ell+1}}(x) P_\mu^{\mathfrak{gl}_{\ell+1}}(y) \Big) P_\lambda^{\mathfrak{gl}_{\ell+1}}(y^{-1}) \Delta(y) \\
&= \; \sum_{\mu_1 \ge \cdots \ge \mu_{\ell+1} \ge 0} b_\mu \Big(\int_T d^\times y \, P_\mu^{\mathfrak{gl}_{\ell+1}}(y) P_\lambda^{\mathfrak{gl}_{\ell+1}}(y^{-1}) \Delta(y) \Big) P_\mu^{\mathfrak{gl}_{\ell+1}}(x) \\
&= \; \sum_{\mu \in \mathbb{Z}^{\ell+1}} \Big(\prod_{i=1}^\ell \Theta(\mu_i - \mu_{i+1}) \Big) \Theta(\mu_{\ell+1}) \, b_\mu \delta_{\lambda,\mu} \, \langle P_\mu^{\mathfrak{gl}_{\ell+1}}, \, P_\lambda^{\mathfrak{gl}_{\ell+1}} \rangle'_{q,\,t} P_\mu^{\mathfrak{gl}_{\ell+1}}(x) \\
&= \; \prod_{i=1}^\ell \Theta(\lambda_i - \lambda_{i+1}) \Theta(\lambda_{\ell+1}) \frac{\langle P_\lambda^{\mathfrak{gl}_{\ell+1}}, \, P_\lambda^{\mathfrak{gl}_{\ell+1}} \rangle'_{q,\,t}}{\langle P_\lambda^{\mathfrak{gl}_{\ell+1}}, \, P_\lambda^{\mathfrak{gl}_{\ell+1}} \rangle_{q,\,t}} P_\lambda^{\mathfrak{gl}_{\ell+1}}(x) \,.
\end{aligned}
\tag{2.9}
$$

Therefore the eigenvalue of the dual Baxter operator $\mathcal{Q}_\gamma$ on $P_\lambda(x)$ is given by

$$
\Theta(\lambda_{\ell+1} - \gamma) \frac{\langle P_{\lambda-(\ell+1)\gamma}, \, P_{\lambda-(\ell+1)\gamma} \rangle'_{q,\,t}}{\langle P_{\lambda-(\ell+1)\gamma}, \, P_{\lambda-(\ell+1)\gamma} \rangle_{q,\,t}} \; = \; \Theta(\lambda_{\ell+1} - \gamma) \frac{\langle P_\lambda, \, P_\lambda \rangle'_{q,\,t}}{\langle P_{\lambda-(\ell+1)\gamma}, \, P_{\lambda-(\ell+1)\gamma} \rangle_{q,\,t}}
$$

$$
= \; b_{\lambda-(\ell+1)\gamma} \Theta(\lambda_{\ell+1} - \gamma) \langle P_\lambda, P_\lambda \rangle'_{q,\,t} \,.
$$

One can rewrite the right hand side of (1.6) in the following form:

$$
b_\lambda \; = \; \prod_{i=1}^{\ell+1} b_{(\lambda_i - \lambda_{i+1})} \times \prod_{\substack{i,j=1 \\ i<j}}^{\ell+1} \frac{\Gamma_{q,\,tq^{-1}}(t^{j-i} q^{\lambda_i-\lambda_j+1})}{\Gamma_{q,\,tq^{-1}}(t^{j-i} q^{\lambda_i-\lambda_{j+1}+1})} \,, \qquad \lambda_{\ell+2} := 0 \,, \tag{2.10}
$$

where

$$
b_{(n)} \; = \; \prod_{i=1}^n \frac{1 - tq^{n-i}}{1 - q^{n+1-i}} \,.
$$

Then combining (2.10) with (1.13) one readily arrives at (2.3). □

There is an analog of the classical Baxter equation (see Theorem 2.3 in [GLO1]) relating Baxter operator and quantum Hamiltonian operators.

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Proposition 2.1 *The operators $\mathcal{Q}_\gamma$ given by (2.1) and the generating function $\mathcal{M}_{\ell+1}(X)$ from (1.11) commute and satisfy the following relation:*

$$\mathcal{M}_{\ell+1}(-q^{-\gamma}) \circ \mathcal{Q}_\gamma(q, q^{-k}) \;=\; \mathcal{Q}_{\gamma+1}(q, q^{1-k}). \tag{2.11}$$

Proof. Recall that the Macdonald polynomials are common eigenfunctions of $\mathcal{Q}_\gamma$ and $\mathcal{M}_{\ell+1}(X)$. Thus it is enough to check (2.11) on common eigenvalues of $\mathcal{M}_{\ell+1}$ and $\mathcal{Q}_\gamma$ acting on $P_\lambda(x)$. Denoting $L_\gamma(\lambda)$ and $c_{\ell+1}(\lambda)$ the corresponding eigenvalues:

$$L_\gamma(\lambda + k\varrho;\, q,\, q^{-k}) \;=\; \prod_{i=1}^{\ell+1} b_{(\lambda_i - \gamma)}, \qquad c_{\ell+1}(\lambda + k\varrho;\, -q^{-\gamma}) \;=\; (1-t)^{-(\ell+1)} \prod_{i=1}^{\ell+1}\bigl(1 - q^{\lambda_i - \gamma}\bigr),$$

we easily check the following relation:

$$c_{\ell+1}(\lambda + k\varrho;\, -q^{\gamma}) \times L_\gamma(\lambda + k\varrho;\, q, t) \;=\; L_{\gamma+1}(\lambda + k\varrho;\, q, tq). \tag{2.12}$$

This entails the operator relation (2.11). $\square$

Now we define the dual Baxter operator.

Definition 2.2 *The dual Baxter operator $\check{\mathcal{Q}}_z = \check{\mathcal{Q}}_z(q, t)$ is a family of operators acting in $\Lambda_{q,t}^{(\ell+1)}$*

$$\check{\mathcal{Q}}_z \cdot P_\lambda(x) \;=\; \sum_{\mu \in \mathbb{Z}^{\ell+1}} \check{Q}_z(\lambda,\, \mu)\, P_\mu(x) \tag{2.13}$$

with the kernel function

$$\check{Q}_z(\lambda,\, \mu) \;=\; z^{|\mu| - |\lambda|}\, \varphi_{\mu/\lambda}, \tag{2.14}$$

and

$$\varphi_{\mu/\lambda} \;=\; \prod_{\substack{i,j=1 \\ i \leq j}}^{\ell+1} \frac{\Gamma_{q,tq^{-1}}\bigl(t^{j-i}q^{\mu_i - \mu_j + 1}\bigr)}{\Gamma_{q,tq^{-1}}\bigl(t^{j-i}q^{\mu_i - \lambda_j + 1}\bigr)} \frac{\Gamma_{q,tq^{-1}}\bigl(t^{j-i}q^{\lambda_i - \lambda_{j+1} + 1}\bigr)}{\Gamma_{q,tq^{-1}}\bigl(t^{j-i}q^{\lambda_i - \mu_{j+1} + 1}\bigr)}$$

$$\times \Theta(\mu_1 - \lambda_1) \prod_{i=1}^{\ell} \Theta(\lambda_i - \mu_{i+1})\Theta(\mu_{i+1} - \lambda_{i+1}), \tag{2.15}$$

where in the product one should omit the factors depending on $\lambda_{\ell+2}$ and $\mu_{\ell+2}$.

Theorem 2.2 *The action of the dual Baxter operator on the Macdonald polynomials reads*

$$\check{\mathcal{Q}}_z \cdot P_\lambda(x) \;=\; L_z^\vee(x)\, P_\lambda(x), \tag{2.16}$$

where the eigenvalue is given by

$$L_z^\vee(x) \;=\; \prod_{i=1}^{\ell+1} \Gamma_{q,t}(zx_i). \tag{2.17}$$

Proof. The statement of the Theorem directly follows from the Pieri formula for the Macdonald polynomials (1.16), (1.17):

$$\breve{\mathcal{Q}}_z \cdot P_\lambda(x) = \sum_{\mu_i \geq \lambda_i \geq \mu_{i+1}} z^{|\mu|-|\lambda|}\, \varphi_{\mu/\lambda}\, P_\mu(x)$$

$$= \sum_{m=0}^{\infty} z^m \sum_{\substack{\mu_i \geq \lambda_i \geq \mu_{i+1} \\ |\mu|-|\lambda|=m}} \varphi_{\mu/\lambda}\, P_\mu(x) = \Big(\sum_{m=0}^{\infty} b_{(m)} z^m P_m(x) \Big) P_\lambda(x) \tag{2.18}$$

$$= \Pi_{\ell+1,1}(x,z)\, P_\lambda(x),$$

and identification $L_z^\vee(x) = \Pi_{\ell+1,1}(x,z)$. □

Proposition 2.2 *The operator $\breve{\mathcal{Q}}_z$ satisfies the following difference relation:*

$$\mathcal{M}_{\ell+1}^\vee(-q^{k\ell/2}z) \circ \breve{\mathcal{Q}}_z(q,q^{-k}) = \breve{\mathcal{Q}}_{qz}(q,q^{-k-1}). \tag{2.19}$$

where $\mathcal{M}_{\ell+1}^\vee(X)$ is the generating function (1.22) of the dual quantum Hamiltonians.

Proof. It is enough to check (2.19) on common eigenfunctions $P_\lambda(x)$ of $\mathcal{M}_{\ell+1}^\vee$ and $\mathcal{Q}^\vee$. Denoting $L_\mu^\vee$ and $c_{\ell+1}^\vee$ the corresponding eigenvalues

$$L_z^\vee(x) = \prod_{i=1}^{\ell+1} \Gamma_{q,t}(zx_i), \qquad c_{\ell+1}^\vee(x;\, -t^{-\ell/2}z) = \prod_{i=1}^{\ell+1}(1-zx_i),$$

we easily check the following relation:

$$c_{\ell+1}^\vee(x;\, -t^{-\ell/2}z) \times L_z^\vee(x\,;\, q,t) = L_{qz}^\vee(x\,;\, q, tq^{-1}). \tag{2.20}$$

This entails the operator relation (2.19). □

Let us introduce the following notation

$$\underline{a}_n = (a_{n,1}, \ldots, a_{n,n}), \qquad \underline{a}_n' = (a_{n,1}, \ldots, a_{n,n-1}).$$

The following recursive relations hold; the first one (see [AOS])

$$P_\lambda(\underline{x}_{\ell+1}) = \int_T d^\times \underline{x}_\ell\, Q_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1};\, \underline{x}_\ell|\, \lambda_{\ell+1})\, \Delta(\underline{x}_\ell)\, P_{\lambda'}(\underline{x}_\ell^{-1}),$$

$$Q_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1};\, \underline{x}_\ell|\, \lambda_{\ell+1}) = x_{\ell+1,\ell+1}^{\lambda_{\ell+1}} \prod_{i=1}^{\ell}(x_{\ell+1,i}x_{\ell,i})^{\lambda_{\ell+1}} \times \Pi_{\ell+1,\ell}(x_{\ell+1,i},\, x_{\ell,i}), \tag{2.21}$$

and the dual recursive relation (see [M]):

$$P_{\underline{\lambda}_{\ell+1}}(x) = \sum_{\underline{\lambda}_\ell} \breve{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{\lambda}_{\ell+1}, \underline{\lambda}_\ell|\, x_{\ell+1})\, P_{\underline{\lambda}_\ell}(x'),$$

$$Q_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{\lambda}_{\ell+1}, \underline{\lambda}_\ell|\, x_{\ell+1}) = x_{\ell+1}^{|\underline{\lambda}_{\ell+1}|-|\underline{\lambda}_\ell|} \psi_{\underline{\lambda}_{\ell+1}/\underline{\lambda}_\ell}, \tag{2.22}$$

where

$$\psi_{\lambda/\mu} = \prod_{1 \leq i \leq j \leq \ell} \frac{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\mu_i-\mu_j+1})}{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\lambda_i-\mu_j+1})} \frac{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\lambda_i-\lambda_{j+1}+1})}{\Gamma_{q,tq^{-1}}(t^{j-i}q^{\mu_i-\lambda_{j+1}+1})} \tag{2.23}$$

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when λ and μ are interlaced (i.e. $\lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_\ell \geq \mu_\ell \geq \lambda_{\ell+1} \geq 0$), and $\psi_{\lambda/\mu} = 0$ otherwise.

These recursive relations allow to introduce the corresponding recursive operators $\mathcal{Q}_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n+1}}(\lambda_{n+1})$ and $\overset{\vee}{\mathcal{Q}}_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n+1}}(x_{n+1})$:

$$\mathcal{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\lambda_{\ell+1}) \cdot f(\underline{x}_{\ell+1}) \;=\; \int_T d^\times \underline{x}_\ell \, Q_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}; \underline{x}_\ell | \lambda_{\ell+1}) \, \Delta(\underline{x}_\ell) \, f(\underline{x}_\ell^{-1}) \,, \tag{2.24}$$

and

$$\overset{\vee}{\mathcal{Q}}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(x_{\ell+1}) \cdot f(\underline{\lambda}_{\ell+1}) \;=\; \sum_{\underline{\lambda}_\ell} \overset{\vee}{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\underline{\lambda}_{\ell+1}, \underline{\lambda}_\ell | x_{\ell+1}) \, f(\underline{\lambda}_\ell) \,, \tag{2.25}$$

The existence of the two dual recursive representations, (2.21) and (2.22), provide a family of 2^ℓ integral representations for the Macdonald polynomials. Namely, let us change our notations as follows:

$$R_{n+1,n}^I \;:=\; \mathcal{Q}_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n+1}}(\lambda_{n+1}) \,, \qquad R_{n+1,n}^{II} \;:=\; \overset{\vee}{\mathcal{Q}}_{\mathfrak{gl}_n}^{\mathfrak{gl}_{n+1}}(x_{n+1}) \,, \qquad n = 1, \ldots, \ell \,;$$

then for every array $\epsilon = (\epsilon_1, \ldots, \epsilon_\ell)$ of $\epsilon_n \in \{I, II\}$, $n = 1, \ldots, \ell$ the following holds:

$$P_{\underline{\lambda}_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{x}_{\ell+1}) \;=\; \left\{ R_{\ell+1,\ell}^{\epsilon_\ell} \circ \ldots \circ R_{2,1}^{\epsilon_1} \circ R_{1,0} \right\} \cdot 1 \,, \tag{2.26}$$

where $R_{1,0}$ is the $\mathfrak{gl}_1$-Macdonald polynomial $P^{\mathfrak{gl}_1}$.

Let us remark that the recursive operators can be factorized into Baxter operators, similarly to recursive operators for $\mathfrak{gl}_{\ell+1}$-Whittaker function (see Proposition 3.3 in [GLO1]). This reveals the fundamental role of the Baxter operators in the description of various bases of symmetric polynomials.

2.2 Baxter operator formalism for class one q-Whittaker functions

Now we provide similar results for q-Whittaker polynomials $P_\lambda^{qW}(x)$.

Definition 2.3 *The Baxter operator acting in $\Lambda_q^{(\ell+1)}$ is a family of integral operators*

$$\mathbf{Q}_z \cdot f(\lambda) \;=\; \sum_{\mu \in \mathbb{Z}^{\ell+1}} \Delta_q(\mu) \, \mathbf{Q}_{\ell+1,\ell+1}(\mu; \lambda | z) \, f(\mu) \,, \tag{2.27}$$

with the kernel

$$\mathbf{Q}_{\ell+1,\ell+1}(\mu, \lambda | z) \;=\; z^{|\mu-\lambda|} \, \varphi_{\mu/\lambda}^q \;=\; z^{|\mu-\lambda|} \, \varphi_{\mu/\lambda}(q, t=0) \times \Delta_q(\lambda)^{-1}$$
$$= z^{|\mu-\lambda|} \, \frac{\Theta(\mu_1 - \lambda_1)}{(\mu_1 - \lambda_1)_q!} \prod_{i=1}^\ell \frac{\Theta(\lambda_i - \mu_{i+1})}{(\lambda_i - \mu_{i+1})_q!} \frac{\Theta(\mu_{i+1} - \lambda_{i+1})}{(\mu_{i+1} - \lambda_{i+1})_q!} \,. \tag{2.28}$$

Theorem 2.3 *(i) The action of the Baxter operator $\mathbf{Q}_z$ on q-Whittaker polynomials (1.37) is given by*

$$\mathbf{Q}_z \cdot P_\lambda^{qW}(x) \;=\; \mathbf{L}_z(x) \, P_\lambda^{qW}(x) \,, \tag{2.29}$$

where

$$\mathbf{L}_z(x) = \prod_{i=1}^{\ell+1} \Gamma_q(zx_i). \tag{2.30}$$

(ii) The operators $\mathbf{Q}_z$ and $D_{\ell+1}^q(X)$ (1.34) satisfy the following relation:

$$D_{\ell+1}(-z) \circ \mathbf{Q}_z = \mathbf{Q}_{qz}. \tag{2.31}$$

Proof. The relation (2.29) is a direct consequence of the Pieri formula (1.35). The relation (2.31) follows from the relation between the corresponding eigenvalues: $\mathbf{c}_{\ell+1}^q(x; -z) \mathbf{L}_z(x) = \mathbf{L}_{qz}(x)$. □

Definition 2.4 *The dual Baxter operator acting in $\Lambda_q^{(\ell+1)}$ is a family of integral operators*

$$\breve{\mathbf{Q}}_\gamma \cdot P(x) = \int_T d^\times y \, \mathbf{Q}_{\ell+1,\,\ell+1}^\vee(x,\,y;\,\gamma) \Delta_q^\vee(y) \, P(y^{-1}), \qquad \gamma \in \mathbb{Z}, \tag{2.32}$$

with the kernel

$$\breve{\mathbf{Q}}_{\ell+1,\,\ell+1}(x,\,y;\,\gamma) = \prod_{i,j=1}^{\ell+1} (x_i y_i)^\gamma \Gamma_q(x_i y_j). \tag{2.33}$$

Theorem 2.4 *(i) The action of the dual Baxter operator $\breve{\mathbf{Q}}_\gamma$ on q-Whittaker polynomials reads as follows:*

$$\breve{\mathbf{Q}}_\gamma \cdot P_\lambda^{qW}(x) = \mathbf{L}_\gamma^\vee(\lambda) P_\lambda^{qW}(x), \qquad \mathbf{L}_\gamma^\vee(\lambda_1, \ldots, \lambda_{\ell+1}) = \frac{1}{(\lambda_{\ell+1} - \gamma)_q!}, \tag{2.34}$$

when $\gamma \leq \lambda_{\ell+1}$, and

$$\breve{\mathbf{Q}}_\gamma \cdot P_\lambda^{qW}(x) = 0, \qquad \gamma > \lambda_{\ell+1}. \tag{2.35}$$

(ii) The dual Baxter operator satisfies the following difference equation:

$$\left\{ 1 - q^{-\gamma} \mathbf{H}_1^\vee \right\} \circ \breve{\mathbf{Q}}_\gamma = \breve{\mathbf{Q}}_{\gamma+1}. \tag{2.36}$$

Proof. The first statement follows from (1.36) and the orthogonality of the q-Whittaker polynomials (see [GLO1]). The second statement follows from the relation between the corresponding eigenvalues: $(1 - q^{\lambda_{\ell+1}-\gamma}) \times \mathbf{L}_\gamma^\vee(\lambda) = \mathbf{L}_{\gamma+1}^\vee(\lambda)$. □

In [GLO2] and [GLO4] the following recursive relations for the q-deformed Whittaker functions were established.

Proposition 2.3 ([GLO2],[GLO4]) *The following recursive relations hold:*

$$P_\lambda^{qW}(x) = \left(\mathbf{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(x_{\ell+1}) \cdot P^{qW}(x') \right)_\lambda$$

$$= \sum_{\lambda_i \geq \mu_i \geq \lambda_{i+1}} \mathbf{Q}_{\ell+1,\,\ell}(\lambda;\,\mu | \, x_{\ell+1}) \Delta_q(\mu) P_\mu^{qW}(x), \tag{2.37}$$

where

$$\mathbf{Q}_{\ell+1,\,\ell}(\lambda;\,\mu\,|\,z) \;=\; z^{|\lambda|-|\mu|} \prod_{i=1}^{\ell} \frac{\Theta(\lambda_i - \mu_i)}{(\lambda_i - \mu_i)_q!} \frac{\Theta(\mu_i - \lambda_{i+1})}{(\mu_i - \lambda_{i+1})_q!}\,. \tag{2.38}$$

and

$$P_\lambda^{qW}(x) \;=\; \overset{\vee}{\mathbf{Q}}{}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\lambda_{\ell+1}) \cdot P_{\lambda'}^{qW}(x) \;=\; \int_T d^\times y\, \overset{\vee}{\mathbf{Q}}{}_{\ell+1,\,\ell}(x;\,y\,|\,\lambda_{\ell+1})\, \Delta_q^\vee(y)\, P_{\lambda'}^{qW}(y^{-1})\,. \tag{2.39}$$

where

$$\overset{\vee}{\mathbf{Q}}{}_{\ell+1,\,\ell}(x_1,\ldots,x_{\ell+1};\,y_1,\ldots,y_\ell\,|\,\gamma) \;=\; \prod_{i=1}^{\ell+1} x_i^\gamma \prod_{j=1}^{\ell} y_j^\gamma \prod_{\substack{1 \le i \le \ell+1 \\ 1 \le j \le \ell}} \Gamma_q(x_i y_j)\,. \tag{2.40}$$

The action of recursive operators (2.37) and (2.39) provide a pair of dual integral/sum representations of the q-deformed Whittaker functions (see [GLO2] and [GLO4]). Combining the recursive operators of different types one can obtain 2^ℓ explicit formulas for q-Whittaker functions similarly to (2.26).

Remark 2.1 *The recursive operators $\mathbf{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(x_{\ell+1})$ and $\overset{\vee}{\mathbf{Q}}{}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\lambda_{\ell+1})$ can be factorized into the Baxter operators (2.27) and (2.32), similarly to Proposition 3.3 from [GLO1].*

2.3 Baxter operator formalism for Jack's symmetric polynomials

Now we consider Baxter operator formalism associated with the Jack symmetric polynomials.

Definition 2.5 *Baxter operator is a family of integral operators acting in $\Lambda^{(\ell+1)}$ by*

$$\mathcal{Q}_\gamma^{(\kappa)} \cdot P_\lambda^{(\kappa)}(x) \;=\; \int_T d^\times y\, Q^{(\kappa)}(x,y;\,\gamma)\, \Delta_{(\kappa)}(y)\, P_\lambda^{(\kappa)}(y^{-1})\,, \qquad \gamma \in \mathbb{Z}\,, \tag{2.41}$$

with the kernel

$$Q_\gamma^{(\kappa)}(x,y;\,\gamma) \;=\; \prod_{i=1}^{\ell+1} (x_i y_i)^\gamma\, \Pi_{\ell+1,\,\ell+1}^{(\kappa)}(x,y)\,. \tag{2.42}$$

Theorem 2.5 *(i) The action of the Baxter operator $\mathcal{Q}_\gamma^{(\kappa)}$ on the Jack polynomials is given by*

$$\mathcal{Q}_\gamma^{(\kappa)} \cdot P_\lambda^{(\kappa)}(y) \;=\; \mathcal{L}_\gamma(\lambda)\, P_\lambda^{(\kappa)}(x)\,, \qquad \mathcal{L}_\gamma(\lambda) \;=\; \prod_{i=1}^{\ell+1} \frac{\Gamma(\lambda_i - \gamma + (\varrho_i + 1)\kappa)}{\Gamma(\lambda_i - \gamma + \varrho_i \kappa + 1)}\,, \tag{2.43}$$

when $\gamma \le \lambda_{\ell+1} + \kappa$, with $\varrho_i = \ell + 1 - i\,, i = 1,\ldots,\ell+1$, and

$$\mathcal{Q}_\gamma^{(\kappa)} \cdot P_\lambda^{(\kappa)}(y) \;=\; 0\,, \qquad \gamma > \lambda_{\ell+1} + \kappa\,. \tag{2.44}$$

(ii) The Baxter operator $\mathcal{Q}_\gamma^{(\kappa)}$ commutes with the generating function $\mathcal{D}_{\ell+1}$ (1.43) and satisfies the following difference equation:

$$\mathcal{D}_{\ell+1}(\kappa - \gamma) \circ \mathcal{D}_{\ell+1}(1 - \gamma)^{-1} \circ \mathcal{Q}_\gamma^{(\kappa)} \;=\; \mathcal{Q}_{\gamma-1}^{(\kappa)}\,. \tag{2.45}$$

Proof. (2.43) follows from (1.50) and the orthogonality of the Jack polynomials [M]. □

Definition 2.6 *The dual Baxter operator* $\breve{\mathcal{Q}}_z^{(\kappa)}$ *is a family of operators acting in* $\Lambda^{(\ell+1)}$ *by*

$$\breve{\mathcal{Q}}_z^{(\kappa)} \cdot P_\lambda^{(\kappa)}(x) \;=\; \sum_{\mu \in \mathbb{Z}^{\ell+1}} \breve{Q}^{(\kappa)}(\mu, \lambda; z)\, P_\mu^{(\kappa)}(x), \tag{2.46}$$

with the kernel

$$\breve{Q}^{(\kappa)}(\mu, \lambda; z) \;=\; z^{|\mu|-|\lambda|} \prod_{\substack{i,j=1 \\ i \le j}}^{\ell+1} \left[\frac{\Gamma\big(\mu_i - \mu_j + 1 + (j-i)\kappa\big)}{\Gamma\big(\mu_i - \mu_j + (j-i+1)\kappa\big)} \frac{\Gamma\big(\mu_i - \lambda_j + (j-i+1)\kappa\big)}{\Gamma\big(\mu_i - \lambda_j + 1 + (j-i)\kappa\big)} \right.$$

$$\left. \times \frac{\Gamma\big(\lambda_i - \lambda_{j+1} + 1 + (j-i)\kappa\big)}{\Gamma\big(\lambda_i - \lambda_{j+1} + (j-i+1)\kappa\big)} \frac{\Gamma\big(\lambda_i - \mu_{j+1} + (j-i+1)\kappa\big)}{\Gamma\big(\lambda_i - \mu_{j+1} + 1 + (j-i)\kappa\big)} \right] \tag{2.47}$$

$$\times \Theta(\mu_1 - \lambda_1) \prod_{i=1}^{\ell} \Theta(\lambda_i - \mu_{i+1})\, \Theta(\mu_{i+1} - \lambda_{i+1}).$$

Theorem 2.6 *(i) The action of the dual Baxter operator on the Jack polynomials is given by*

$$\breve{\mathcal{Q}}_z^{(\alpha)} \cdot P_\lambda^{(\kappa)}(x) \;=\; \breve{\mathcal{L}}_z(x)\, P_\lambda^{(\kappa)}(x), \tag{2.48}$$

where

$$\breve{\mathcal{L}}_z(x) \;=\; \prod_{i=1}^{\ell+1} \frac{1}{(1 - zx_i)^\kappa}. \tag{2.49}$$

(ii) The dual Baxter operator commutes with $\mathcal{D}_{(\kappa)}^\vee$ *given by (1.44) and satisfies the following difference equation:*

$$\mathcal{D}_{(\kappa)}^\vee(-z) \circ \breve{\mathcal{Q}}_z^{(\kappa)} \;=\; \breve{\mathcal{Q}}_z^{(\kappa-1)}. \tag{2.50}$$

Proof. The first relation (2.48) follows from the Pieri formula (1.49), and the Cauchy-Littlewood identity (1.50) is implied by the relation between the eigenvalues: $c_{\ell+1}^\vee(x; -z) \times \breve{\mathcal{L}}_z^{(\kappa)}(x) \;=\; \breve{\mathcal{L}}_z^{(\kappa-1)}(x)$. □

The recursive relations (2.21) and (2.22) imply similar recursive relations for Jack's symmetric functions. Namely, the following recursive relation hold (see [AMOS1], [AMOS2], [AOS]):

$$P_{\underline{\lambda}_{\ell+1}}^{(\kappa)}(\underline{x}_{\ell+1}) \;=\; \mathcal{Q}_{\mathfrak{gl}_\ell}^{\mathfrak{gl}_{\ell+1}}(\lambda_{\ell+1,\ell+1}) \cdot P_{\underline{\lambda}_{\ell+1}'}^{(\kappa)}$$

$$= \int_T d^\times x_\ell\, Q_{\ell+1,\ell}^{(\kappa)}(\underline{x}_{\ell+1}; \underline{x}_\ell|\, \lambda_{\ell+1,\ell+1})\, \Delta_{(\kappa)}(\underline{x}_\ell)\, P_{\underline{\lambda}_{\ell+1}'}^{(\kappa)}(\underline{x}_\ell^{-1}), \tag{2.51}$$

where

$$Q_{\ell+1,\ell}^{(\kappa)}(\underline{x}_{\ell+1}; \underline{x}_\ell|\, \lambda_{\ell+1,\ell+1})$$

$$= x_{\ell+1,\ell+1}^{\lambda_{\ell+1,\ell+1}} \prod_{1 \le i \le \ell} (x_{\ell+1,i} x_{\ell,i})^{\lambda_{\ell+1,\ell+1}} \times \prod_{i=1}^{\ell+1} \prod_{j=1}^{\ell} \frac{1}{\big(1 - x_{\ell+1,i} x_{\ell,j}\big)^\kappa}. \tag{2.52}$$

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We also have the dual recursive relations

$$
\begin{aligned}
P^{(\kappa)}_{\underline{\lambda}_{\ell+1}}(\underline{x}_{\ell+1}) &= \big(\overset{\vee}{\mathcal{Q}}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(x_{\ell+1,\,\ell+1}) \cdot P^{(\kappa)}(\underline{x}'_{\ell+1}) \big)_{\underline{\lambda}_\ell} \\
&= \sum_{\lambda_{\ell+1,\,i} \geq \lambda_{\ell,\,i} \geq \lambda_{\ell+1,\,i+1}} \overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}\big(\underline{\lambda}_{\ell+1};\,\underline{\lambda}_\ell \,|\, x_{\ell+1,\,\ell+1}\big)\, P^{(\kappa)}_{\underline{\lambda}_\ell}(\underline{x}'_{\ell+1}).
\end{aligned}
\tag{2.53}
$$

Here

$$
\begin{aligned}
&\overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}\big(\mu;\,\lambda \,|\, z\big) \\
&= z^{|\mu|-|\lambda|} \prod_{\substack{i,j=1 \\ i \leq j}}^{\ell} \Big[\, \frac{\Gamma\big(\mu_i - \mu_j + 1 + (j-i)\kappa\big)}{\Gamma\big(\mu_i - \mu_j + (j-i+1)\kappa\big)} \frac{\Gamma\big(\mu_i - \lambda_j + (j-i+1)\kappa\big)}{\Gamma\big(\mu_i - \lambda_j + 1 + (j-i)\kappa\big)} \\
&\qquad\qquad \times \frac{\Gamma\big(\lambda_i - \lambda_{j+1} + 1 + (j-i)\kappa\big)}{\Gamma\big(\lambda_i - \lambda_{j+1} + (j-i+1)\kappa\big)} \frac{\Gamma\big(\lambda_i - \mu_{j+1} + (j-i+1)\kappa\big)}{\Gamma\big(\lambda_i - \mu_{j+1} + 1 + (j-i)\kappa\big)} \,\Big],
\end{aligned}
\tag{2.54}
$$

when $(\mu_1,\ldots,\mu_{\ell+1})$ and $(\lambda_1,\ldots,\lambda_\ell,0)$ are interlaced, and $\overset{\vee}{Q}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\mu;\,\lambda \,|\, z) = 0$ otherwise. Obviously for Jack polynomials one has the proper analogs of the mixed integral/sum representations (2.26) of the Macdonald polynomials.

Example 2.1 *The simplest dual recursive operator intertwining $\mathfrak{gl}_2$ and $\mathfrak{gl}_1$ Jack's symmetric functions reads as follows:*

$$
\begin{aligned}
&\overset{\vee}{Q}{}^{\mathfrak{gl}_2}_{\mathfrak{gl}_1}(\lambda_{21},\,\lambda_{22};\,\lambda_{11} \,|\, x_2) \\
&= x_2^{\lambda_{21}+\lambda_{22}-\lambda_{11}} \frac{\Gamma(\kappa + \lambda_{21} - \lambda_{11})\,\Gamma(\kappa + \lambda_{11} - \lambda_{22})}{\Gamma(\kappa + \lambda_{21} - \lambda_{22})} \frac{(\lambda_{21} - \lambda_{22})!}{(\lambda_{21} - \lambda_{11})!\,(\lambda_{11} - \lambda_{22})!}\,.
\end{aligned}
\tag{2.55}
$$

This leads to the following representation of the $\mathfrak{gl}_2$-Jack's polynomial:

$$
P^{(\kappa)}_{\lambda_1,\lambda_2}(x_1,\,x_2) = \sum_{\mu=\lambda_2}^{\lambda_1} \frac{\Gamma(\kappa + \lambda_1 - \mu)\,\Gamma(\kappa + \mu - \lambda_2)}{\Gamma(\kappa + \lambda_1 - \lambda_2)} \frac{(\lambda_1 - \lambda_2)!}{(\lambda_1 - \mu)!\,(\mu - \lambda_2)!}\, x_1^\mu x_2^{\lambda_1+\lambda_2-\mu}\,.
$$

Remark 2.2 *The recursive operators $\mathcal{Q}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(\lambda_{\ell+1,\,\ell+1})$ and $\overset{\vee}{\mathcal{Q}}{}^{\mathfrak{gl}_{\ell+1}}_{\mathfrak{gl}_\ell}(x_{\ell+1,\,\ell+1})$ can be factorized into the Baxter operators (2.41), (2.46), similarly to Proposition 3.3 from [GLO1].*

3 Appendix: Various analogs of classical Γ-function

In this Appendix we provide basic facts on the analogs of classical Γ-function arising in the Baxter operator formalism for Macdonald, q-Whittaker and Jack polynomials.

Classical Γ-function can be defined by analytic continuation of the function defined by Euler's integral representation:

$$
\Gamma(s) = \int_{\mathbb{R}} dt\; e^{st} e^{-e^t}\,, \qquad \operatorname{Re}(s) > 0.
\tag{3.1}
$$

Equivalently Γ-function is defined as a solution of the functional equation

$$
\Gamma(s + 1) = s\Gamma(s), \qquad \Gamma(1) = 1,
$$

such that $\frac{1}{\Gamma(s)}$ is an entire function on the complex plane. Γ-function allows a representation as the Weierstrass product

$$\Gamma(1+s) = e^{-\gamma s} \prod_{n=1}^{\infty} e^{\frac{s}{n}} \left(1 + \frac{s}{n}\right)^{-1}, \tag{3.2}$$

where $\gamma = -\Gamma'(1)$ is the Euler constant. Also the following reflection property holds

$$\Gamma(s) \times \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{3.3}$$

Note that the integral representation (3.1) can be inverted via the Mellin transform

$$e^{-e^{\tau}} = \frac{1}{2\pi\iota} \int_{\iota\mathbb{R}+\epsilon} ds \ e^{-s\tau} \Gamma(s). \tag{3.4}$$

Define (q,t)-analog of the classical Γ-function as the following infinite product

$$\Gamma_{q,t}(x) = \prod_{n=0}^{\infty} \frac{1 - txq^n}{1 - xq^n}, \tag{3.5}$$

where we imply that q is variable taking values in $|q| < 1$. This function has poles at $x = q^{-m}$, $m \in \mathbb{Z}_{\geq 0}$ and zeroes at $x = t^{-1}q^{-m}$, $m \in \mathbb{Z}_{\geq 0}$.

The function $\Gamma_{q,t}$ defined by (3.5) possesses all the basic properties of the classical Γ-function outlined above. The analog of the Weierstrass product is given by (3.5). The analog of the relation (3.3) is given by:

$$\Gamma_{q,t}(z) \times \Gamma_{q,t^{-1}}(qz^{-1}) = t^{1/2} \frac{\theta_1\big((tz)^{1/2}; q\big)}{\theta_1\big(z^{1/2}; q\big)}, \tag{3.6}$$

where we take into account the product representation

$$\theta_1(z; q) = q^{1/4} \frac{z - z^{-1}}{\iota} \prod_{j \geq 1} (1 - q^j)(1 - z^2 q^j)(1 - z^{-2} q^j). \tag{3.7}$$

of the standard elliptic theta-function $\theta_1(z; q)$.

Finally, the analog of the Euler integral representation (3.1) and its inverse (3.4) are given by

$$\Gamma_{q,t}(x) = \sum_{\lambda \geq 0} x^{\lambda} \frac{\Gamma_{q,tq^{-1}}(q)}{\Gamma_{q,tq^{-1}}(q^{\lambda+1})}, \qquad \frac{\Gamma_{q,tq^{-1}}(q)}{\Gamma_{q,tq^{-1}}(q^{\lambda+1})} = \int_T d^{\times}x \ x^{-\lambda} \Gamma_{q,t}(x). \tag{3.8}$$

Consider now a specialization of the $\Gamma_{q,t}$ at $t = 0$ given by

$$\Gamma_q(z) := \frac{1}{(z; q)_{\infty}} = \prod_{j=0}^{\infty} \frac{1}{1 - zq^j}. \tag{3.9}$$

The q-Gamma function $\Gamma_q(z)$ has poles at $z = q^{-m}$, $m \in \mathbb{Z}_+$ and satisfy proper analogs of (3.1)-(3.3). The q-analog of the Weierstrass product formula (3.2) is given by (3.9). The q-analog of the Euler integral formula (3.1) is given by

$$\Gamma_q(z) = \sum_{\lambda \geq 0} z^{\lambda} \frac{\Gamma_q(q)}{\Gamma_q(q^{\lambda+1})}, \qquad \frac{\Gamma_q(q)}{\Gamma_q(q^{\lambda+1})} = \int_T d^{\times}z \ z^{-\lambda} \Gamma_q(z), \tag{3.10}$$

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and the q-analog of the functional equation (3.3) has the following form:

$$\Gamma_q(z) \times \Gamma_q(qz^{-1}) = \frac{q^{1/4}}{\Gamma_q(q)} \frac{\imath z^{-1/2}}{\theta_1(z^{1/2}; q)} , \qquad (3.11)$$

which can be deduced from (3.7).

Now consider the following analog of the Γ-function

$$\Gamma^{(\kappa)}(z) = \lim_{\hbar \to 0} \Gamma_{q,t}(x) = \left(\frac{1}{1-z}\right)^{\kappa} , \qquad t = e^{\kappa\hbar}, \quad q = e^{\hbar}, \qquad (3.12)$$

depending on a positive integer parameter κ. The analog of the functional equation (3.3) reads

$$\Gamma^{(\kappa)}(z) \times \Gamma^{(-\kappa)}(z^{-1}) = -\frac{1}{z^{\kappa}} , \qquad (3.13)$$

and the binomial formula for $\Gamma^{(\kappa)}(z) = (1-z)^{-\kappa}$ implies the following analogs of the Euler's integral formula (3.1) and its inverse:

$$\Gamma^{(\kappa)}(z) = \sum_{n \geq 0} \frac{z^n}{n!} \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} , \qquad \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = n! \int_T d^{\times}z \; z^n \; \Gamma^{(\kappa)}(z) . \qquad (3.14)$$

The functions $\Gamma_{q,t}(x)$, $\Gamma_q(x)$ and $\Gamma^{(\kappa)}(x)$ play an important role in the Baxter operator formalism for Macdonald, q-Whittaker and Jack polynomials correspondingly.

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