# HOMOTOPY REPRESENTATIONS OVER THE ORBIT CATEGORY

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Abstract. Let $G$ be a finite group. The unit sphere in a finite-dimensional orthogonal $G$-representation motivates the definition of homotopy representations, due to tom Dieck. We introduce an algebraic analogue and establish its basic properties, including the Borel-Smith conditions and realization by finite $G$-CW-complexes.

## 1. Introduction

Let $G$ be a finite group. The unit spheres $S(V)$ in finite-dimensional orthogonal representations of $G$ provide the basic examples of smooth $G$-actions on spheres. Moreover, character theory reveals intricate relations between the dimensions of the fixed subspheres $S(V)^H$, for subgroups $H \leq G$, and the structure of the isotopy subgroups $\{G_x \mid x \in S(V)\}$. Our goal is to better understand the constraints on these basic invariants, in order to construct new smooth *non-linear* finite group actions on spheres (see [8], [9]).

In order to put this problem in a more general setting, tom Dieck [12, II.10.1] introduced *geometric homotopy representations*, as finite $G$-CW-complexes $X$ with the property that each fixed set $X^H$ is homotopy equivalent to a sphere. In this paper, we study an algebraic version of this notion for $R$-module chain complexes over the orbit category $\varGamma_G = \mathrm{Or}_{\mathcal{F}} G$, with respect to a ring $R$ and a family $\mathcal{F}$ of subgroups of $G$. We usually work with $R = \mathbb{Z}_{(p)}$, for some prime $p$, or $R = \mathbb{Z}$. This theory was developed by Lück [10, §9, §17] and tom Dieck [12, §10-11].

The homological dimensions of the various fixed sets are encoded in a conjugation-invariant function $\underline{n}\colon \mathcal{S}(G) \to \mathbb{Z}$, where $\mathcal{S}(G)$ denotes the set of subgroups of $G$. The function $\underline{n}$ is *supported on the family* $\mathcal{F}$, if $\underline{n}(H) = -1$ for $H \notin \mathcal{F}$ (see Definition 2.4). We say that a finite projective chain complex $\mathbf{C}$ over $R\varGamma_G$ is an *$R$-homology $\underline{n}$-sphere* if the reduced homology of $\mathbf{C}(K)$ is the same as the reduced homology of an $\underline{n}(K)$-sphere (with coefficients in $R$) for all $K \in \mathcal{F}$.

If $\mathbf{C}$ is an $R$-homology $\underline{n}$-sphere, which satisfies the internal homological conditions observed for representation spheres (see Definition 2.8), then we say that $\mathbf{C}$ is an *algebraic homotopy representation*. By [12, II.10], these conditions are all necessary for $\mathbf{C}$ to be chain homotopy equivalent to a geometric homotopy representation. In Proposition 2.10, we show more generally that these conditions hold for $\mathbf{C}$ an $R$-homology $\underline{n}$-sphere, whenever $\underline{n} = \mathrm{Dim}\, \mathbf{C}$, where $\mathrm{Dim}\, \mathbf{C}$ denotes the chain dimension function of $\mathbf{C}$. When this equality holds, we say that $\mathbf{C}$ is a *tight* complex.

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In general, $\underline{n}(K) \leq \operatorname{Dim} \mathbf{C}(K)$ for each $K \in \mathcal{F}$, and one would expect obstructions to finding a tight complex which is chain homotopy equivalent to a given $R$-homology $\underline{n}$-sphere. Our first main result shows the relevance of the internal homological conditions for this question.

**Theorem A.** *Let $G$ be a finite group, and $\mathcal{F}$ be a family of subgroups of $G$. Suppose that*

(i) *$R$ is a principal ideal domain,*
(ii) *$\underline{n}\colon \mathcal{S}(G) \to \mathbb{Z}$ is a conjugation-invariant function supported on $\mathcal{F}$, and*
(iii) *$\mathbf{C}$ is a finite chain complex of free $R\Gamma_G$-modules which is an $R$-homology $\underline{n}$-sphere.*

*Then $\mathbf{C}$ is chain homotopy equivalent to a finite free chain complex $\mathbf{D}$ satisfying $\underline{n} = \operatorname{Dim} \mathbf{D}$ if and only if $\mathbf{C}$ is an algebraic homotopy representation.*

Theorem A was motivated by [8, Theorem 8.10], which states that a finite chain complex of free $\mathbb{Z}\Gamma_G$-modules can be realized by a geometric $G$-CW-complex if it is a tight homology $\underline{n}$-sphere such that $\underline{n}(H) \geq 3$ for all $H \in \mathcal{F}$. Upon combining these two statements, we get the following geometric realization result.

**Corollary B.** *Let $\mathbf{C}$ be a finite chain complex of free $\mathbb{Z}\Gamma_G$-modules which is a homology $\underline{n}$-sphere. If $\mathbf{C}$ is an algebraic homotopy representation, and in addition, if $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, then there is a finite $G$-CW-complex $X$, with isotropy in $\mathcal{F}$, such that $\mathbf{C}(X^?; \mathbb{Z})$ is chain homotopy equivalent to $\mathbf{C}$ as chain complexes of $\mathbb{Z}\Gamma_G$-modules.*

We are interested in constructing finite $G$-CW-complexes with some restrictions on the family of isotropy subgroups. We say a $G$-CW-complex $X$ has *rank one isotropy* if for every $x \in X$, the isotropy subgroup $G_x$ has $\operatorname{rank} G_x \leq 1$. Recall that *rank* of a finite group $G$ is defined as the largest integer $k$ such that $(\mathbb{Z}/p)^k \leq G$ for some prime $p$. We will use Theorem A and Corollary B to study the following:

**Question.** Which finite groups $G$ admit a finite $G$-CW-complex $X$ with rank one isotropy, such that $X$ is homotopy equivalent to a sphere ?

One motivation for this work is that rank one isotropy examples lead to free $G$-CW-complex actions of finite groups on *products* of spheres (see Adem and Smith [1]).

In [8] we gave the first non-trivial example, by constructing a finite $G$-CW-complex $X \simeq S^n$ for the symmetric group $G = S_5$, with cyclic 2-group isotropy. However, the arguments used special features of the isotropy family. Corollary B now provides an effective general method for the geometric realization of algebraic models. The algebraic homotopy representation conditions are easy to check locally over $R = \mathbb{Z}_{(p)}$ at each prime, and fit well with the local-to-global procedure for constructing chain complexes $\mathbf{C}$ over $\mathbb{Z}\Gamma_G$. In a sequel [9] to this paper, we apply Corollary B to construct infinitely many new examples with rank one isotropy, for certain interesting families of rank two groups.

In Section 5 we consider the algebraic version of a well-known theorem in transformation groups: the dimension function of a homotopy representation satisfies certain conditions called the Borel-Smith conditions (see Definition 5.1).

**Theorem C.** *Let $G$ be a finite group, $R = \mathbb{Z}/p$, and $\mathcal{F}$ be a given family of subgroups of $G$. If $\mathbf{C}$ is a finite projective chain complex over $R\Gamma_G$, which is an $R$-homology $\underline{n}$-sphere, then the function $\underline{n}$ satisfies the Borel-Smith conditions at the prime $p$.*

As an application, we show that such a finite projective chain complex over $R\Gamma_G$ does not exist for the group $G = \mathrm{Qd}(p)$ with respect to the family $\mathcal{F}$ of rank 1 subgroups (see Example 5.13 and Proposition 5.14). This is an important group theoretic constraint on the existence question for geometric homotopy representations with rank one isotropy.

One of the main ideas in the proof of Theorem C is the reduction of a given chain complex of $R\Gamma_G$-module $\mathbf{C}$ to a chain complex over $R\Gamma_{K/N}$ for a subquotient $K/N$ appearing in the Borel-Smith conditions. For this reduction, we introduce *inflation* and *deflation* of modules over the orbit category, via restriction and induction associated to a certain functor $F$ (see Section 4). Then we use spectral sequence arguments to conclude that the conditions given in the Borel-Smith conditions hold for these reduced chain complexes over $R\Gamma_{K/N}$.

Here is a brief outline of the paper. In Section 2 we give the precise setting and background definitions for the concepts just presented (see Definition 2.8) and prove the "only if" direction of Theorem A. The "if" direction of Theorem A is proved in Section 3, together with Corollary B. In Section 5 we discuss the Borel-Smith conditions and prove Theorem C.

Our methods involve the study of finite-dimensional chain complexes of finitely generated projective modules over the orbit category, called *finite projective chain complexes*, for short. Such chain complexes are the algebraic analogue of finitely-dominated $G$-CW complexes.

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## 2. Algebraic homotopy representations

Let $G$ be a finite group and $\mathcal{F}$ be a family of subgroups of $G$ which is closed under conjugations and taking subgroups. The orbit category $\mathrm{Or}_{\mathcal{F}}\, G$ is defined as the category whose objects are orbits of type $G/K$, with $K \in \mathcal{F}$, and where the morphisms from $G/K$ to $G/L$ are given by $G$-maps:

$$\mathrm{Mor}_{\mathrm{Or}_{\mathcal{F}}\, G}(G/K, G/L) = \mathrm{Map}_G(G/K, G/L).$$

The category $\Gamma_G = \mathrm{Or}_{\mathcal{F}}\, G$ is a small category, and we can consider the module category over $\Gamma_G$ in the following sense. Let $R$ be a commutative ring with 1. A *(right) $R\Gamma_G$-module* $M$ is a contravariant functor from $\Gamma_G$ to the category of $R$-modules. We denote the $R$-module $M(G/K)$ simply by $M(K)$ and write $M(f)\colon M(L) \to M(K)$ for a $G$-map $f\colon G/K \to G/L$.

The category of $R\Gamma_G$-modules is an abelian category, so the usual concepts of homological algebra, such as kernel, direct sum, exactness, projective module, etc., exist for $R\Gamma_G$-modules. A sequence of $R\Gamma_G$-modules $0 \to A \to B \to C \to 0$ is *exact* if and only if

$$0 \to A(K) \to B(K) \to C(K) \to 0$$

is an exact sequence of $R$-modules for every $K \in \mathcal{F}$. For an $R\Gamma_G$-module $M$ the $R$-module $M(K)$ can also be considered as an $RW_G(K)$-module in an obvious way where $W_G(K) = N_G(K)/K$. We will follow the convention in [10] and consider $M(K)$ as a right $RW_G(K)$-module. In particular, we will consider the sequence above as an exact sequence of right $RW_G(K)$-modules.

For each $H \in \mathcal{F}$, let $F_H := R[G/H^?]$ denote the $R\Gamma_G$-module with values $F_H(K) = R[(G/H)^K]$ for every $K \in \mathcal{F}$, and where for every $G$-map $f : G/L \to G/K$, the induced map $F_H(f) : R[(G/H)^K] \to R[(G/H)^L]$ is defined in the obvious way. By the Yoneda lemma, there is an isomorphism

$$\mathrm{Hom}_{R\Gamma_G}(R[G/H^?], M) \cong M(H)$$

for every $R\Gamma_G$-module $M$. From this it is easy to show that the module $R[G/H^?]$ is a projective module in the usual sense, for each $H \in \mathcal{F}$. An $R\Gamma_G$-module is called *free* if it is isomorphic to a direct sum of $R\Gamma_G$-modules of the form $R[G/H^?]$. It can be shown that an $R\Gamma_G$-module is projective if and only if it is a direct summand of a free module. The further details about the properties of modules over the orbit category can be found in [8] (see also Lück [10, §9,§17] and tom Dieck [12, §10-11]).

In this section we consider chain complexes $\mathbf{C}$ of $R\Gamma_G$-modules, with respect to a given family $\mathcal{F}$. When we say a chain complex we always mean a non-negative complex, so $\mathbf{C}_i = 0$ for $i < 0$. We call a chain complex $\mathbf{C}$ *projective* (resp. *free*) if for all $i \geq 0$, the modules $\mathbf{C}_i$ are projective (resp. free). We say that a chain complex $\mathbf{C}$ is *finite* if $\mathbf{C}_i = 0$ for all $i > n$, for some $n \geq 0$, and the chain modules $\mathbf{C}_i$ are all finitely generated $R\Gamma_G$-modules.

**Remark 2.1.** Up to chain homotopy equivalence, there is no difference between finite projective chain complexes and finite-dimensional projective chain complexes with finitely generated homology (see [9, 3.6]). For this reason, our definitions and results are mostly stated for *finite* chain complexes.

We define the *support* of a chain complex $\mathbf{C}$ over $R\Gamma_G$ as the family of subgroups

$$\mathrm{Supp}(\mathbf{C}) = \{H \in \mathcal{F} \mid \mathbf{C}(H) \neq 0\}.$$

It is sometimes convenient to vary the family of subgroups.

**Definition 2.2.** If $\mathcal{F} \subset \mathcal{G}$ are two families, the orbit category $\Gamma_{G,\mathcal{F}} = \mathrm{Or}_{\mathcal{F}}\, G$ is a full-subcategory of $\Gamma_{G,\mathcal{G}} = \mathrm{Or}_{\mathcal{G}}\, G$. If $M$ is a module over $R\Gamma_{G,\mathcal{F}}$, then we define $\mathrm{Inc}_{\mathcal{F}}^{\mathcal{G}}(M)(H) = M(H)$, if $H \in \mathcal{F}$, and zero otherwise. Similarly, for a module $N$ over $R\Gamma_{G,\mathcal{G}}$, define $\mathrm{Res}_{\mathcal{F}}^{\mathcal{G}}(N)(H) = N(H)$, for $H \in \mathcal{F}$. We extend to maps and chain complexes similarly. Note that $\mathrm{Supp}(\mathrm{Inc}_{\mathcal{F}}^{\mathcal{G}}(\mathbf{C})) = \mathrm{Supp}(\mathbf{C})$, and $\mathrm{Supp}(\mathrm{Res}_{\mathcal{F}}^{\mathcal{G}}(\mathbf{D})) = \mathrm{Supp}(\mathbf{D}) \cap \mathcal{F}$.

Given a $G$-CW-complex $X$, there is an associated chain complex of $R\Gamma_G$-modules over the family of all subgroups

$$\mathbf{C}(X^{\mathbf{?}}; R) : \quad \cdots \to R[X_n{}^{\mathbf{?}}] \xrightarrow{\partial_n} R[X_{n-1}{}^{\mathbf{?}}] \to \cdots \xrightarrow{\partial_1} R[X_0{}^{\mathbf{?}}] \to 0$$

where $X_i$ denotes the set of (oriented) $i$-dimensional cells in $X$ and $R[X_i{}^{\mathbf{?}}]$ is the $R\Gamma_G$-module defined by $R[X_i{}^{\mathbf{?}}](H) = R[X_i^H]$ for every $H \leq G$. We denote the homology of this complex by $H_*(X^{\mathbf{?}}; R)$. The chain complex $\mathbf{C}(X^H; R)$ is actually defined for all subgroups $H \leq G$, but for a given family of subgroups $\mathcal{F}$, we can restrict its values from $\mathrm{Or}(G)$ to the full sub-category $\mathrm{Or}_{\mathcal{F}}\, G$.

The smallest family containing all the isotropy subgroups $\{G_x \mid x \in X\}$ is

$$\mathfrak{Iso}(X) = \{H \leq G \mid X^H \neq \emptyset\}$$

and this motivates our notion of support for algebraic chain complexes. In particular, we have

$$\operatorname{Supp}(\operatorname{Res}_{\mathcal{F}}(\mathbf{C}(X^{?}; R))) = \mathcal{F} \cap \mathfrak{Iso}(X).$$

If the family $\mathcal{F}$ includes all the isotropy subgroups of $X$, then the complex $\mathbf{C}(X^{?}; R)$ is a chain complex of free $R\Gamma_G$-modules, hence projective $R\Gamma_G$-modules, but otherwise the chain modules $R[X_n{}^{?}]$ may not be projective over $R\Gamma_G$.

Given a finite-dimensional $G$-CW-complex $X$, there is a *dimension function*

$$\operatorname{Dim} X \colon \mathcal{S}(G) \to \mathbb{Z},$$

given by $(\operatorname{Dim} X)(H) = \dim X^H$ for all $H \in \mathcal{S}(G)$ where $\mathcal{S}(G)$ denote the set of all subgroups of $G$. By convention, we set $\dim \emptyset = -1$ for the dimension of the empty set. In a similar way, we define the following.

**Definition 2.3.** The (chain) *dimension function* of a finite-dimensional chain complex $\mathbf{C}$ over $R\Gamma_G$ is defined as the function $\operatorname{Dim} \mathbf{C} \colon \mathcal{S}(G) \to \mathbb{Z}$ which has the value

$$(\operatorname{Dim} \mathbf{C})(H) = \dim \mathbf{C}(H)$$

for all $H \in \mathcal{F}$, where the *dimension* of a chain complex of $R$-modules is defined as the largest integer $d$ such $C_d \neq 0$ (hence the zero complex has dimension $-1$). If $H \notin \mathcal{F}$, then we set $(\operatorname{Dim} \mathbf{C})(H) = -1$.

The dimension function $\operatorname{Dim} \mathbf{C} \colon \mathcal{S}(G) \to \mathbb{Z}$ is *conjugation-invariant*, meaning that it takes the same value on conjugate subgroups of $G$. The term *super class function* is often used for such functions.

**Definition 2.4.** The *support* of a super class function $\underline{n}$ is defined as the set

$$\operatorname{Supp}(\underline{n}) = \{H \leq G \colon \underline{n}(H) \neq -1\}.$$

We say that a super class function $\underline{n} \colon \mathcal{S}(G) \to \mathbb{Z}$ is *supported on* $\mathcal{F}$, if $\operatorname{Supp}(\underline{n}) \subseteq \mathcal{F}$. Note that $\operatorname{Supp}(\mathbf{C}) \subseteq \mathcal{F}$ is the support of the dimension function $\operatorname{Dim} \mathbf{C}$ of a chain complex $\mathbf{C}$ over $R\Gamma_G$.

In a similar way, we can define the *homological dimension function* of a chain complex $\mathbf{C}$ of $R\Gamma_G$-modules as the function $\operatorname{HomDim} \mathbf{C} \colon \mathcal{S}(G) \to \mathbb{Z}$ where for each $H \in \mathcal{F}$, the integer

$$(\operatorname{HomDim} \mathbf{C})(H) = \operatorname{hdim} \mathbf{C}(H)$$

is defined as the largest integer $d$ such that $H_d(\mathbf{C}(H)) \neq 0$. If $H \notin \mathcal{F}$, then we set $\underline{n}(H) = -1$, as before.

Let us write $(H) \leq (K)$ whenever $g^{-1}Hg \leq K$ for some $g \in G$. Here $(H)$ denotes the set of subgroups conjugate to $H$ in $G$. The notation $(H) < (K)$ means that $(H) \leq (K)$ but $(H) \neq (K)$.

**Definition 2.5.** We call a function $\underline{n} \colon \mathcal{S}(G) \to \mathbb{Z}$ *monotone* if it satisfies the property that $\underline{n}(H) \geqslant \underline{n}(K)$ whenever $(H) \leq (K)$. We say that a monotone function $\underline{n}$ is *strictly monotone* if $\underline{n}(H) > \underline{n}(K)$, whenever $(H) < (K)$. □

We have the following:

**Lemma 2.6.** *The (chain) dimension function of every finite-dimensional projective chain complex* $\mathbf{C}$ *of* $R\Gamma_G$*-modules is monotone.*

*Proof.* Let $(L) \leq (K)$. If $\underline{n}(K) = -1$, then the inequality $\underline{n}(L) \geq \underline{n}(K)$ is clear. So assume $\underline{n}(K) = n \neq -1$. Then $\mathbf{C}_n(K) \neq 0$. By the decomposition theorem for projective $R\Gamma_G$-modules [12, Chap. I, Theorem 11.18], every projective $R\Gamma_G$-module $P$ is of the form $P \cong \oplus_H E_H P_H$, where $H \in \mathcal{F}$ and $P_H$ is a projective $N_G(H)/H$-module. Here the $R\Gamma_G$-module $E_H P_H$ is defined by

$$E_H P_H(?) = P_H \otimes_{RN_G(H)/H} RMap_G(G/?, G/H).$$

Applying this decomposition theorem to $\mathbf{C}_n$, we observe that $\mathbf{C}_n$ must have a summand $E_H P_H$ with $(K) \leq (H)$. But then $\mathbf{C}_n(L) \neq 0$, and hence $\underline{n}(L) \geq \underline{n}(K)$. □

We are particularly interested in chain complexes which have the homology of a sphere when evaluated at every $K \in \mathcal{F}$. To specify the restriction maps in dimension zero, we will consider chain complexes $\mathbf{C}$ which are equipped with an augmentation map $\varepsilon\colon \mathbf{C}_0 \to \underline{R}$ such that $\varepsilon \circ \partial_1 = 0$. Here $\underline{R}$ denotes the constant functor, and we assume that $\varepsilon(H)$ is surjective for $H \in \mathrm{Supp}(\mathbf{C})$. We often consider $\varepsilon$ as a chain map $\mathbf{C} \to \underline{R}$ by considering $\underline{R}$ as a chain complex over $R\Gamma_G$ which is concentrated at zero. We denote a chain complex with an augmentation as a pair $(\mathbf{C}, \varepsilon)$.

By the *reduced homology* of a complex $(\mathbf{C}, \varepsilon)$, we always mean the homology of the augmented chain complex

$$\widetilde{\mathbf{C}} = \{\cdots \to \mathbf{C}_n \xrightarrow{\partial_n} \cdots \to \mathbf{C}_2 \xrightarrow{\partial_2} \mathbf{C}_1 \xrightarrow{\partial_1} \mathbf{C}_0 \xrightarrow{\varepsilon} \underline{R} \to 0\}$$

where $\underline{R}$ is considered to be at dimension $-1$. Note that the complex $\widetilde{\mathbf{C}}$ is the $-1$ shift of the mapping cone of the chain map $\varepsilon\colon \mathbf{C} \to \underline{R}$.

**Definition 2.7.** Let $\underline{n}$ be a super class function supported on $\mathcal{F}$, and let $\mathbf{C}$ be a chain complex over $R\Gamma_G$ with respect to a family $\mathcal{F}$ of subgroups.

(i) We say that $\mathbf{C}$ is an *$R$-homology $\underline{n}$-sphere* if there is an augmentation map $\varepsilon\colon \mathbf{C} \to \underline{R}$ such that the reduced homology of $\mathbf{C}(K)$ is the same as the homology of an $\underline{n}(K)$-sphere (with coefficients in $R$) for all $K \in \mathcal{F}$.

(ii) We say that $\mathbf{C}$ is *oriented* if the $W_G(K)$-action on the homology of $\mathbf{C}(K)$ is trivial for all $K \in \mathcal{F}$.

Note that we do not assume that the dimension function is strictly monotone as in Definition II.10.1 in [12].

In transformation group theory, a $G$-CW-complex $X$ is called a *homotopy representation* if $X^H$ is homotopy equivalent to the sphere $S^{\underline{n}(H)}$ where $\underline{n}(H) = \dim X^H$ for every $H \leq G$ (see tom Dieck [12, Section II.10]). We now introduce an algebraic analogue of this useful notion for chain complexes over the orbit category.

In [12, II.10], there is a list of properties that are satisfied by homotopy representations. We will use algebraic versions of these properties to define an analogous notion for chain complexes.

**Definition 2.8.** Let $\mathbf{C}$ be a finite projective chain complex over $R\Gamma_G$, which is an $R$-homology $\underline{n}$-sphere. We say $\mathbf{C}$ is an *algebraic homotopy representation* (over $R$) if

(i) The function $\underline{n}$ is a monotone function.
(ii) If $H, K \in \mathcal{F}$ are such that $n = \underline{n}(K) = \underline{n}(H)$, then for every $G$-map $f\colon G/H \to G/K$ the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an $R$-homology isomorphism.
(iii) Suppose $H, K, L \in \mathcal{F}$ are such that $H \leq K, L$ and let $M = \langle K, L\rangle$ be the subgroup of $G$ generated by $K$ and $L$. If $n = \underline{n}(H) = \underline{n}(K) = \underline{n}(L) > -1$, then $M \in \mathcal{F}$ and $n = \underline{n}(M)$.

Note that conditions (ii) and (iii) of Definition 2.8 are automatic if the dimension function $\underline{n}$ is strictly monotone. Under condition (iii), the isotropy family $\mathcal{F}$ has an important maximality property.

**Proposition 2.9.** *Let $\underline{n}$ be a super class function and let $\mathbf{C}$ be a projective chain complex of $R\Gamma_G$-modules, which is an $R$-homology $\underline{n}$-sphere. If condition* (iii) *holds, then for each $H \in \mathcal{F}$, the set of subgroups $\mathcal{F}_H = \{K \in \mathcal{F} \mid (H) \leq (K),\ \underline{n}(K) = \underline{n}(H) > -1\}$ has a unique maximal element, up to conjugation.*

*Proof.* Clear by induction from the statement of condition (iii). □

In the remainder of this section we will assume that $R$ is a principal ideal domain. The important examples for us are $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{Z}$. The main result of this section is the following proposition.

**Proposition 2.10.** *Let $\underline{n}$ be a super class function and $\mathbf{C}$ be a finite projective chain complex over $R\Gamma_G$, which is an $R$-homology $\underline{n}$-sphere. Assume that $R$ is a principal ideal domain. If the equality $\underline{n} = \operatorname{Dim} \mathbf{C}$ holds, then $\mathbf{C}$ is an algebraic homotopy representation.*

Before we prove Proposition 2.10, we make some observations and give some definitions for projective chain complexes.

**Lemma 2.11.** *Let $\mathbf{C}$ be a projective chain complex over $R\Gamma_G$. Then, for every $G$-map $f\colon G/H \to G/K$, the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an injective map with an $R$-torsion free cokernel.*

*Proof.* It is enough to show that if $P$ a projective $R\Gamma_G$-module, then for every $G$-map $f\colon G/H \to G/K$, the induced map $P(f)\colon P(K) \to P(H)$ is an injective map with a torsion free cokernel. Since every projective module is a direct summand of a free module, it is enough to prove this for a free module $P = R[X^?]$, where $X$ is a finite $G$-set. Let $f\colon G/H \to G/K$ be the $G$-map defined by $f(H) = gK$. Then the induced map $P(f)\colon R[X^K] \to R[X^H]$ is the linearization of the map $X^K \to X^H$ given by $x \mapsto gx$. Since this map is one-to-one, we can conclude that $P(f)$ is injective with torsion free cokernel. □

When $H \leq K$ and $f\colon G/H \to G/K$ is the $G$-map defined by $f(gH) = gK$ for each $g \in G$, then we denote the induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ by $r_H^K$ and call it the *restriction* map. When $H$ and $K$ are conjugate, so that $K = x^{-1}Hx$ for some $x \in G$, then the map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ induced by the $G$-map $f\colon G/H \to G/K$ defined by $f(gH) = gxK$ for each $g \in G$, is called the *conjugation* map and usually denoted by $c_K^g$. Every $G$-map can be written as a composition of two $G$-maps of the above two types, so every induced map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ can be written as a composition of restriction and conjugation maps.

Since conjugation maps have inverses, they are always isomorphisms. So, the condition (ii) of Definition 2.8 is actually a statement only about restriction maps. To study the restriction maps more closely, we consider the image of $r_H^K\colon \mathbf{C}(K) \to \mathbf{C}(H)$ for a pair $H \leq K$ and denote it by $\mathbf{C}_H^K$. Note that $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ as a chain complex of $R$-modules. We also remark that $\mathbf{C}_H^K$ is isomorphic to $\mathbf{C}(K)$, as a chain complex of $R$-modules, by Lemma 2.11, whenever $\mathbf{C}$ is a projective chain complex.

**Lemma 2.12.** *Let $\mathbf{C}$ be a projective chain complex over $R\varGamma_G$. Suppose that $K, L \in \mathcal{F}$ are such that $H \leq K$ and $H \leq L$, and let $M = \langle K, L\rangle$ be the subgroup generated by $K$ and $L$. If $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$ then $M \in \mathcal{F}$, and hence we have $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$.*

*Proof.* As before it is enough to prove this for a free $R\varGamma_G$-module $P = R[X^?]$ where $X$ is a finite $G$-set whose isotropy subgroups lie in $\mathcal{F}$. The restriction maps $r_H^K$ and $r_H^L$ are linearizations of the maps $X^K \to X^H$ and $X^L \to X^H$, respectively, which are defined by inclusion of subsets. Then it is clear that the intersection of images of $r_H^K$ and $r_H^L$ (if non-zero) would be $R[X^K \cap X^L]$, considered as an $R$-submodule of $R[X^H]$. We have $X^K \cap X^L = X^M$ where $M = \langle K, L\rangle$. Therefore, if $\mathbf{C}_H^K \cap \mathbf{C}_H^L \neq 0$, then we must have $X^M \neq \emptyset$ which implies that $M \in \mathcal{F}$. Thus $\mathbf{C}_H^M$ is defined and we can write $\mathbf{C}_H^K \cap \mathbf{C}_H^L = \mathbf{C}_H^M$ by the above fixed point formula. □

Now, we are ready to prove Proposition 2.10.

*Proof of Proposition 2.10.* The first condition in Definition 2.8 follows from Lemma 2.6. For (ii) and (iii), we use the arguments similar to the arguments given in II.10.12 and II.10.13 in [12].

To prove (ii), let $f\colon G/H \to G/K$ be a $G$-map. By Lemma 2.11, the induced map $\mathbf{C}(f) : \mathbf{C}(K) \to \mathbf{C}(H)$ is injective with torsion free cokernel. Let $\mathbf{D}$ denote the cokernel of $\mathbf{C}(f)$. Then we have a short exact sequence of $R$-modules

$$0 \to \mathbf{C}(K) \to \mathbf{C}(H) \to \mathbf{D} \to 0$$

where both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have dimension $n$. Now consider the long exact *reduced* homology sequence (with coefficients in $R$) associated to this short exact sequence:

$$\cdots \to 0 \to H_{n+1}(\mathbf{D}) \to H_n(\mathbf{C}(K)) \xrightarrow{f^*} H_n(\mathbf{C}(H)) \to H_n(\mathbf{D}) \to \cdots$$

Note that $\mathbf{D}$ has dimension less than or equal to $n$, so $H_{n+1}(\mathbf{D}) = 0$ and $H_n(\mathbf{D})$ is torsion free. Since $H_n(\mathbf{C}(K)) = H_n(\mathbf{C}(H)) = R$, it follows that $f^*$ is an isomorphism. Since both $\mathbf{C}(K)$ and $\mathbf{C}(H)$ have no other reduced homology, we conclude that $\mathbf{C}(f)$ induces an $R$-homology isomorphism between associated augmented complexes. Since the induced map $\underline{R}(f) : \underline{R}(K) \to \underline{R}(H)$ is the identity map $\mathrm{id}\colon R \to R$, the chain map $\mathbf{C}(f)\colon \mathbf{C}(K) \to \mathbf{C}(H)$ is an $R$-homology isomorphism.

To prove (iii), observe that there is a Mayer-Vietoris type exact sequence associated to the pair of complexes $\mathbf{C}_H^K$ and $\mathbf{C}_H^L$ which gives an exact sequence of the form

$$0 \to H_n(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K) \oplus H_n(\mathbf{C}_H^L) \to H_n(\mathbf{C}_H^K + \mathbf{C}_H^L) \to H_{n-1}(\mathbf{C}_H^K \cap \mathbf{C}_H^L) \to 0.$$

Here we again take the homology sequence as the reduced homology sequence.

Let $i^K\colon \mathbf{C}^K_H \to \mathbf{C}(H)$, $i^L_H\colon \mathbf{C}^L_H \to \mathbf{C}(H)$, and $j\colon \mathbf{C}^K_H + \mathbf{C}^L_H \to \mathbf{C}(H)$ denote the inclusion maps. We have zero on the left-most term since $\mathbf{C}^K_H + \mathbf{C}^L_H$ is an $n$-dimensional complex. To see the zero on the right-most term, note that by Lemma 2.11, $\mathbf{C}^K_H \cong \mathbf{C}(K)$ and $\mathbf{C}^L_H \cong \mathbf{C}(L)$ as chain complexes of $R$-modules, so they have the same homology. This gives that $H_i(\mathbf{C}^K_H) = H_i(\mathbf{C}^L_H) = 0$ for $i \leq n-1$.

Also note that by part (ii), the composition

$$H_n(\mathbf{C}(K)) \cong H_n(\mathbf{C}^K_H) \xrightarrow{i^K_*} H_n(\mathbf{C}^K_H + \mathbf{C}^L_H) \xrightarrow{j_*} H_n(\mathbf{C}(H))$$

is an isomorphism. So, $j_*$ is surjective. Since $H_{n+1}(\mathbf{C}(H)/(\mathbf{C}^K_H + \mathbf{C}^L_H)) = 0$, we see that $j_*$ is also injective. Therefore, $j_*$ is an isomorphism. This implies that $i^K_*$ is an isomorphism. Similarly one can show that $i^L_*\colon H_n(\mathbf{C}^L_H) \to H_n(\mathbf{C}^K_H + \mathbf{C}^L_H)$ is also an isomorphism. Using these isomorphisms and looking at the exact sequence above, we conclude that $H_n(\mathbf{C}^K_H \cap \mathbf{C}^L_H) \cong R$ and $H_i(\mathbf{C}^K_H \cap \mathbf{C}^L_H) = 0$ for $i \leq n-1$. So, $\mathbf{C}^K_H \cap \mathbf{C}^L_H$ is an $R$-homology $n$-sphere.

Since $n > -1$, this implies that $\mathbf{C}^K_H \cap \mathbf{C}^L_H \neq 0$, and hence $M = \langle K, L\rangle \in \mathcal{F}$ by Lemma 2.12. Moreover, $\mathbf{C}^K_H \cap \mathbf{C}^L_H = \mathbf{C}^M_H$. This proves that $\underline{n}(M) = n$ as desired. □

## 3. The Proof of Theorem A

In this section we will again assume that $R$ is a principal ideal domain. The main examples for us are $R = \mathbb{Z}_{(p)}$ or $R = \mathbb{Z}$, as before.

**Definition 3.1.** We say a chain complex $\mathbf{C}$ of $R\varGamma_G$-modules is *tight at* $H \in \mathcal{F}$ if

$$\dim \mathbf{C}(H) = \operatorname{hdim} \mathbf{C}(H).$$

We call a chain complex of $R\varGamma_G$-modules *tight* if it is tight at every $H \in \mathcal{F}$.

Suppose that $\mathbf{C}$ is a finite projective complex over $R\varGamma_G$ which is an $R$-homology $\underline{n}$-sphere. If $\mathbf{C}$ is chain homotopy equivalent to a tight complex, then Proposition 2.10 shows that $\mathbf{C}$ is an algebraic homotopy representation. This establishes one direction of Theorem A. The other direction uses the assumption that the chain modules of $\mathbf{C}$ are free over $R\varGamma_G$.

**Theorem 3.2.** *Let* $\mathbf{C}$ *be a finite chain complex of free* $R\varGamma_G$*-modules which is a homology* $\underline{n}$*-sphere. If* $\mathbf{C}$ *is an algebraic homotopy representation over* $R$*, then* $\mathbf{C}$ *is chain homotopy equivalent to a finite free chain complex* $\mathbf{D}$ *which is tight.*

**Remark 3.3.** If $\mathbf{C}$ is a finite projective chain complex, then the analogous result holds for a sufficiently large $k$-fold join tensor product $\mathbf{C}' = \circledast_k \mathbf{C}$, by [8, Theorem 7.6].

We need to show that the complex $\mathbf{C}$ can be made *tight* at each $H \in \mathcal{F}$ by replacing it with a chain complex homotopic to $\mathbf{C}$. The proof is given in several steps.

**3A. Tightness at maximal isotropy subgroups.** Let $H$ be a *maximal element* in $\mathcal{F}$. Consider the subcomplex $\mathbf{C}^{(H)}$ of $\mathbf{C}$ formed by free summands of $\mathbf{C}$ isomorphic to $R[G/H^{?}]$. Note that $\mathbf{C}^{(H)}$ is a submodule because $\operatorname{Hom}_{R\varGamma_G}(R[G/H^?], R[G/K^?]) \neq 0$ only if $(H) \leq (K)$, and since $H$ is maximal, we have $\partial_i(\mathbf{C}^{(H)}_i) \subseteq \mathbf{C}^{(H)}_{i-1}$ for all $i$. The complex $\mathbf{C}^{(H)}$ is a complex of isotypic modules of type $R[G/H^?]$. Recall that free $R\varGamma_G$-module $F$ is called *isotypic* of type $G/H$ if it is isomorphic to a direct sum of copies of a free module $R[G/H^?]$, for some $H \in \mathcal{F}$. For extensions involving isotypic modules we have the following:

**Lemma 3.4.** *Let*

$$\mathcal{E}\colon 0 \to F \to F' \to M \to 0$$

*be a short exact sequence of $R\varGamma_G$-modules such that both $F$ and $F'$ are isotypic free modules of the same type $G/H$. If $M(H)$ is $R$-torsion free, then $\mathcal{E}$ splits and $M$ is stably free.*

*Proof.* This is Lemma 8.6 of [8]. The assumption that $R$ is a principal ideal domain ensures that finitely generated $R$-torsion free modules are free. □

Note that $\mathbf{C}^{(H)}(H) = \mathbf{C}(H)$, since $H$ is maximal in $\mathcal{F}$. This means that $\mathbf{C}^{(H)}$ is a finite free chain complex over $R\varGamma_G$ of the form

$$\mathbf{C}^{(H)}\colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0$$

which is a $R$-homology $\underline{n}(H)$-sphere, with $\underline{n}(H) \leq d$.

**Lemma 3.5.** *Let $\mathbf{C}$ be a finite chain complex of free $R\varGamma_G$-modules. Then $\mathbf{C}$ is chain homotopy equivalent to a finite free chain complex $\mathbf{D}$ which is tight at every maximal element $H \in \mathcal{F}$.*

*Proof.* We apply [8, Proposition 8.7] to the subcomplex $\mathbf{C}^{(H)}$, for each maximal element $H \in \mathcal{F}$. The key step is provided by Lemma 3.4. □

**3B. The inductive step.** To make the complex $\mathbf{C}$ tight at every $H \in \mathcal{F}$ we use a downward induction, but the situation at an intermediate step is more complicated than the first step considered above.

Suppose that $H \in \mathcal{F}$ is such that $\mathbf{C}$ is tight at every $K \in \mathcal{F}$ such that $(K) > (H)$. Let $\mathbf{C}^{(H)}$ denote the subcomplex of $\mathbf{C}$ with free summands of type $R[G/K^?]$ satisfying $(H) \leq (K)$. In a similar way, we can define the subcomplex $\mathbf{C}^{>(H)}$ of $\mathbf{C}$ whose free summands are of type $R[G/K^?]$ with $(H) < (K)$. The complex $\mathbf{C}^{>(H)}$ is a subcomplex of $\mathbf{C}^{(H)}$. Let us denote the quotient complex $\mathbf{C}^{(H)}/\mathbf{C}^{>(H)}$ by $\mathbf{C}_{(H)}$. The complex $\mathbf{C}_{(H)}$ is isotypic with isotropy type $R[G/H^?]$. We have a short exact sequence of chain complexes of free $R\varGamma_G$-modules

$$0 \to \mathbf{C}^{>(H)} \to \mathbf{C}^{(H)} \to \mathbf{C}_{(H)} \to 0.$$

By evaluating at $H$, we obtain an exact sequence of chain complexes

$$0 \to \mathbf{C}^{>(H)}(H) \to \mathbf{C}^{(H)}(H) \to \mathbf{C}_{(H)}(H) \to 0.$$

Since $\mathbf{C}^{(H)}(H) = \mathbf{C}(H)$ and the image of the map on the left is generated by summands of the form $R[G/K^?]$ with $(H) < (K)$, the complex $\mathbf{C}_{(H)}(H)$ is isomorphic to $S_H\mathbf{C}$ as an $R[N_G(H)/H]$-module. Here $S_H$ denotes is splitting functor defined more generally for any module over an EI-category (see [10, Definition 9.26]).

We also have an exact sequence

$$0 \to \mathbf{C}^{(H)} \to \mathbf{C} \to \mathbf{C}/\mathbf{C}^{(H)} \to 0.$$

If we can show that $\mathbf{C}^{(H)}$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at $H$, then by taking the push-out of $\mathbf{D}'$ along the injective map $\mathbf{C}^{(H)} \to \mathbf{C}$, we can find a complex $\mathbf{D}$ homotopy equivalent to $\mathbf{C}$ which is tight at every $K \in \mathcal{F}$ with $(K) \geq (H)$. So it is enough to show that $\mathbf{C}^{(H)}$ is homotopy equivalent to a complex $\mathbf{D}'$ which is tight at $H$.

**Lemma 3.6.** *Let $\mathbf{C}$ be a finite free chain complex of $R\Gamma_G$-modules, such that $\mathbf{C}$ is tight at every $K \in \mathcal{F}$ with $(K) > (H)$, for some $H \in \mathcal{F}$. Suppose*

(i) $n = \operatorname{hdim} \mathbf{C}(H) \geq \dim \mathbf{C}(K)$*, for all* $(K) > (H)$*, and that*
(ii) $H_{n+1}(S_H \mathbf{C}) = 0$.

*Then $\mathbf{C}^{(H)}$ is homotopy equivalent to a finite free chain complex $\mathbf{D}'$ which is tight at every $K \in \mathcal{F}$ with $(K) \geq (H)$.*

*Proof.* Let us fix $H \in \mathcal{F}$ and assume that $\mathbf{C}$ is tight at every $K \in \mathcal{F}$ with $(K) > (H)$. We first observe that $\mathbf{C}^{>(H)}$ has dimension $\leq n = \operatorname{hdim} \mathbf{C}(H)$, since $\mathbf{C}^{>(H)}(K) = \mathbf{C}(K)$ for $(K) > (H)$, and $\dim \mathbf{C}(K) \leq n$. Let $d = \dim \mathbf{C}(H)$. If $d = n$, then we are done, so assume that $d > n$. Then $\dim \mathbf{C}_{(H)} = d$, and $\mathbf{C}_{(H)}$ is a complex of the form

$$\mathbf{C}_{(H)} \colon 0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to 0.$$

We claim that the map $\partial_d \colon F_d \to F_{d-1}$ in the above chain complex is injective. Since $\mathbf{C}_{(H)}$ is isotypic of type $(H)$, it is enough to show that this map is injective when it is calculated at $H$. To see this observe that the map $\partial_d$ is the same as the map obtained by applying the functor $E_H$ to the $N_G(H)/H$-homomorphism $\partial_d(H) \colon F_d(H) \to F_{d-1}(H)$ (see [10, Lemma 9.31]). Since the functor $E_H$ is exact, we have $\ker \partial_d = E_H(\ker \partial_d(H))$. Hence, if $\partial_d(H)$ is injective, then $\partial_n$ is injective.

We will show that $H_d(\mathbf{C}_{(H)}(H)) = H_d(S_H\mathbf{C}) = 0$. To see this consider the short exact sequence $0 \to \mathbf{C}^{>(H)}(H) \to \mathbf{C}(H) \to S_H\mathbf{C} \to 0$. Since the complex $\mathbf{C}^{>(H)}$ has dimension $\leq n$, the corresponding long exact sequence gives that $H_d(S_H\mathbf{C}) \cong H_d(\mathbf{C}(H)) = 0$ when $d > n+1$. If $d = n+1$, then this is true by assumption (ii) in the lemma. Now we apply [8, Proposition 8.7] to $\mathbf{C}_{(H)}$ to obtain a tight complex $\mathbf{D}'' \simeq \mathbf{C}_{(H)}$, and then let $\mathbf{D}' \simeq \mathbf{C}^{(H)}$ denote the pullback of $\mathbf{D}''$ along the surjection $\mathbf{C}^{(H)} \to \mathbf{C}_{(H)}$. □

**3C. Verifying the hypothesis for the inductive step.** To complete the proof of Theorem 3.2, we need to show that the assumptions in Lemma 3.6 hold at an intermediate step of the downward induction. We will make detailed use of the internal homological conditions (i), (ii), and (iii) in Definition 2.8, satisfied by an algebraic homotopy representation $\mathbf{C}$. We proceed as follows:

(1) The dimension assumptions in Lemma 3.6 follow from the condition (i), since when $\underline{n}$ is monotone, we have
$$n := \operatorname{hdim} \mathbf{C}(H) = \underline{n}(H) \geq \underline{n}(K) = \operatorname{hdim} \mathbf{C}(K) = \dim \mathbf{C}(K)$$
for all $K \in \mathcal{F}$ with $(K) > (H)$.
(2) The assumption that $H_{n+1}(S_H\mathbf{C}) = 0$ is established in Corollary 3.9. It follows from the conditions (ii) and (iii) and the Mayer-Vietoris argument given below.

In the rest of the section, we assume that $\mathbf{C}$ is a finite projective chain complex of $R\varGamma_G$-modules, which is an $R$-homology $\underline{n}$-sphere, and satisfies the conditions (i), (ii), and (iii) in Definition 2.8. Assume also that $\mathbf{C}$ is tight for all $K \in \mathcal{F}$ with $(K) > (H)$ for some fixed subgroup $H \in \mathcal{F}$. We will say $\mathbf{C}$ is *tight above* $H$, for short. Let $\mathcal{K}_H$ denote the set of all subgroups

$$\mathcal{K}_H = \{K \in \mathcal{F} \,|\, K > H \text{ and } n := \underline{n}(K) = \underline{n}(H)\}.$$

Let $\mathbf{C}$ be an algebraic homotopy representation, which is tight above $H$. Let $\mathbf{C}_H^K$ denote the image of the restriction map

$$r_H^K \colon \mathbf{C}(K) \to \mathbf{C}(H),$$

for every $K \in \mathcal{F}$ with $K \geq H$. Then $\mathbf{C}_H^K$ is a subcomplex of $\mathbf{C}(H)$ and by Lemma 2.11, it is isomorphic to $\mathbf{C}(K)$. By condition (iii) of Definition 2.8, the collection $\mathcal{K}_H$ has a unique maximal element $M$. In addition, we have the equality

$$\mathbf{C}^{>(H)}(H) = \sum_{K \in \mathcal{K}_H} \mathbf{C}_H^K,$$

since $(G/K)^H$ is the union of the subspaces $(G/K)^L$, with $L > H$ and $(L) = (K)$.

Moreover, if $K \in \mathcal{K}_H$, then by condition (ii), the subcomplex $\mathbf{C}_H^K$ is an $R$-homology $n$-sphere and the map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{C}_H^K)$$

induced by the inclusion map $\mathbf{C}_H^M \hookrightarrow \mathbf{C}_H^K$ is an isomorphism. More generally, the following also holds.

**Lemma 3.7.** *Let $\mathbf{C}$ be an algebraic homotopy representation which is tight above $H$, for some fixed $H \in \mathcal{F}$, and let $K_1, \dots, K_m$ be a set of subgroups in $\mathcal{K}_H$. Then the subcomplex $\sum_{i=1}^m \mathbf{C}_H^{K_i}$ is an $R$-homology $n$-sphere and the map*

$$H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^m \mathbf{C}_H^{K_i}) \tag{3.8}$$

*induced by the inclusion maps is an isomorphism.*

*Proof.* This follows from the Mayer-Vietoris spectral sequence in algebraic topology (see [4, pp. 166-168]), which computes the homology of a union of spaces $X = \bigcup X_i$ in terms of the homology of the subspaces and their intersections. We include a direct argument for the reader's convenience.

The case $m = 1$ follows from the remarks above. For $m > 1$, we have the following Mayer-Vietoris type long exact sequence

$$0 \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) \to H_n(\mathbf{D}_m) \to H_{n-1}(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) \to$$

where $\mathbf{D}_j = \sum_{i=1}^j \mathbf{C}_H^{K_i}$ for $j = m-1, m$. By the inductive assumption, we know that $\mathbf{D}_{m-1}$ is an $R$-homology $n$-sphere and the map $H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1})$ induced by inclusion is an isomorphism.

We have

$$\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m} = (\sum_{i=1}^{m-1} \mathbf{C}_H^{K_i}) \cap \mathbf{C}_H^{K_m} = \sum_{i=1}^{m-1} (\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m}) = \sum_{i=1}^{m-1} \mathbf{C}_H^{\langle K_i, K_m \rangle}$$

where the last equality follows from Lemma 2.12. We can apply Lemma 2.12 here because $\mathbf{C}_H^M \subseteq \mathbf{C}_H^K$ for all $K \in \mathcal{K}_H$ gives that $\mathbf{C}_H^{K_i} \cap \mathbf{C}_H^{K_m} \neq 0$ for every $i = 1, \dots, m-1$. We also obtain $\langle K_i, K_m \rangle \in \mathcal{K}_H$ for all $i$. Applying our inductive assumption again to these subgroups, we conclude that $\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}$ is an $R$-homology $n$-sphere and that the map

$$H_n(\mathbf{C}_H^M) \to H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m})$$

induced by inclusion is an isomorphism. This gives that $H_i(\mathbf{D}_m) = 0$ for $i \leq n-1$. We also obtain a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(\mathbf{C}_H^M) & \longrightarrow & H_n(\mathbf{C}_H^M) \oplus H_n(\mathbf{C}_H^M) & \longrightarrow & H_n(\mathbf{C}_H^M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \varphi & & \\ 0 & \longrightarrow & H_n(\mathbf{D}_{m-1} \cap \mathbf{C}_H^{K_m}) & \longrightarrow & H_n(\mathbf{D}_{m-1}) \oplus H_n(\mathbf{C}_H^{K_m}) & \longrightarrow & H_n(\mathbf{D}_m) & \longrightarrow & 0 \end{array}$$

Since all the vertical maps except the map $\varphi$ are known to be isomorphisms, we obtain that $\varphi$ is also an isomorphism by the five lemma. This completes the proof. □

**Corollary 3.9.** *Let* $\mathbf{C}$ *be an algebraic homotopy representation which is tight above* $H$*, for some fixed* $H \in \mathcal{F}$*. Then* $H_{n+1}(S_H \mathbf{C}) = 0$*.*

*Proof.* Let $\mathcal{K}_H = \{K_1, \dots, K_m\}$. By condition (ii), we know that the composition

$$H_n(\mathbf{C}(M)) \xrightarrow{\cong} H_n(\mathbf{C}_H^M) \to H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. However, we have just proved that the middle map is an isomorphism, and that all the modules involved in the composition are isomorphic to $R$. Therefore, the map induced by inclusion

$$H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \to H_n(\mathbf{C}(H))$$

is an isomorphism. Since $\mathbf{C}$ is tight above $H$, we have $\dim \mathbf{C}(K) < n$ whenever $(H) \leq (K)$ and $\underline{n}(K) < n$, for some $K \in \mathcal{F}$. This implies the relation

$$H_n(\mathbf{C}^{>(H)}(H)) = H_n(\sum_{i=1}^{m} \mathbf{C}_H^{K_i}) \cong H_n(\mathbf{C}(H))$$

where the isomorphism is induced by the the inclusion of chain complexes. From the exact sequence $0 \to \mathbf{C}^{>(H)}(H) \to \mathbf{C}(H) \to S_H \mathbf{C} \to 0$, and the fact that $\operatorname{hdim} \mathbf{C}(H) = n$, we conclude that $H_{n+1}(S_H \mathbf{C}) = 0$, as required. □

This completes the proof of Theorem 3.2 and hence the proof of Theorem A. In [8], we proved the following realization theorem for free $\mathbb{Z}\varGamma_G$-module chain complexes, with respect to any family $\mathcal{F}$, which are $\mathbb{Z}$-homology $\underline{n}$-spheres satisfying certain extra conditions.

**Theorem 3.10** ([8, Theorem 8.10], [11])**.** *Let* $\mathbf{C}$ *be a finite chain complex of free* $\mathbb{Z}\Gamma_G$*-modules which is a* $\mathbb{Z}$*-homology* $\underline{n}$*-sphere. Suppose that* $\underline{n}(K) \geq 3$ *for all* $K \in \mathcal{F}$*. If* $\mathbf{C}_i(H) = 0$ *for all* $i > \underline{n}(H) + 1$*, and all* $H \in \mathcal{F}$*, then there is a finite G-CW-complex* $X$ *with isotropy in* $\mathcal{F}$*, such that* $\mathbf{C}(X^?; \mathbb{Z})$ *is chain homotopy equivalent to* $\mathbf{C}$ *as chain complexes of* $\mathbb{Z}\Gamma_G$*-modules.*

Note that a $\mathbb{Z}$-homology $\underline{n}$-sphere $\mathbf{C}$ with $\operatorname{Dim} \mathbf{C} = \underline{n}$, and $\underline{n}(K) \geq 3$ for all $K \in \mathcal{F}$, will automatically satisfy these conditions. So Corollary B follows immediately from Theorem A and Theorem 3.10.

**Remark 3.11.** The construction actually produces a finite $G$-CW-complex $X$ with the additional property that all the non-empty fixed sets $X^H$ are simply-connected. Moreover, by construction, $W_G(H) = N_G(H)/H$ will act trivially on the homology of $X^H$. Therefore $X$ will be an *oriented* geometric homotopy representation (in the sense of tom Dieck). From the perspective of Theorem A, since we don't specify any dimension function, a $G$-CW-complex $X$ with all fixed sets $X^H$ integral homology spheres will lead (by three-fold join) to a homotopy representation. The same necessary and sufficient conditions for existence apply.

## 4. Inflation and deflation of chain complexes

In this section we define two general operations on chain complexes in preparation for the proof of Theorem C. For a finite $G$-CW complex $X$ which is a mod-$p$ homology sphere, the Borel-Smith conditions can be proved using a reduction argument to certain $p$-group subquotients (compare [12, III.4]). For a subquotient $K/L$, the reduction comes from considering the fixed point space $X^L$ as a $K$-space. To do a similar reduction for chain complexes over $R\Gamma_G$, we first introduce a new functor for $R\Gamma_G$-modules, called the *deflation* functor. We will introduce this functor as a restriction functor between corresponding module categories. For this discussion $R$ can be taken as any commutative ring with 1 and $\mathcal{F}_G$ is any family subject to the extra conditions we assume during the construction.

Let $N$ be a normal subgroup of $G$. We define a functor

$$F \colon \Gamma_{G/N} \to \Gamma_G$$

by considering a $G/N$-set (or $G/N$-map) as a $G$-set (or $G$-map) via composition with the quotient map $G \to G/N$. For this definition to make sense, the families $\mathcal{F}_{G/N}$ and $\mathcal{F}_G$ should satisfy the property that if $K \geq N$ is such that $(K/N) \in \mathcal{F}_{G/N}$, then $K \in \mathcal{F}_G$. Since we always assume the families are nonempty, the above assumption also implies that $N \in \mathcal{F}_G$. For notational simplicity from now on, let us denote $K/N$ by $\overline{K}$ for every $K \geq N$.

If a family $\mathcal{F}_G$ is already given, we will always take $\mathcal{F}_{G/N} = \{\overline{K} \mid K \geq N \text{ and } K \in \mathcal{F}_G\}$ and the condition above will be automatically satisfied. We also assume that $N \in \mathcal{F}_G$ to have a nonempty family for $\mathcal{F}_{G/N}$.

The functor $F$ gives rise to two functors (see [10, 9.15]):

$$\operatorname{Res}_F \colon \operatorname{Mod-}R\Gamma_G \to \operatorname{Mod-}R\Gamma_{G/N}$$

and

$$\mathrm{Ind}_F \colon \mathrm{Mod}\text{-}R\Gamma_{G/N} \to \mathrm{Mod}\text{-}R\Gamma_G \ .$$

The first functor $\mathrm{Res}_F$ takes a $R\Gamma_G$-module $M$ to the $R\Gamma_{G/N}$-module

$$\mathrm{Def}^G_{G/N}(M) := M \circ F \colon \Gamma_{G/N} \to R\text{-Mod}.$$

We call this functor the *deflation functor*. Note that

$$(\mathrm{Def}^G_{G/N} M)(\overline{K}) = M(K).$$

The induction functor $\mathrm{Inf}^G_{G/N} := \mathrm{Ind}_F$ associated to $F$ is called the *inflation functor*. For every $H \in \mathfrak{F}_G$, we have

$$\mathrm{Inf}^G_{G/N}(M)(H) = \Big( \bigoplus\nolimits_{\overline{K} \in \mathfrak{F}_{G/N}} M(\overline{K}) \otimes_{RW_{\overline{G}}(\overline{K})} R\,\mathrm{Map}_G(G/H, G/K) \Big) / \sim$$

where the relations come from the tensor product over $R\Gamma_{G/N}$ (see [10, Definition 9.12]). In general, it can be difficult to calculate $\mathrm{Inf}^G_{G/N} M$ for an arbitrary $R\Gamma_{G/N}$-module $M$. In the case where $M$ is a free $R\Gamma_{G/N}$-module we have the following lemma.

**Lemma 4.1.** *Let $X$ be a finite $G/N$-set. Then, we have*

$$\mathrm{Inf}^G_{G/N} R[X^{\,?}] = R[(\mathrm{Inf}^G_{G/N} X)^{?}].$$

*Proof.* It is enough to show this when $X = \overline{G}/\overline{K}$ for some $K \leq G$ such that $K \geq N$. In this case, $R[(\overline{G}/\overline{K})^{?}]$ is isomorphic to $E_{\overline{K}} P_{\overline{K}}$ where $P_{\overline{K}} = R[W_{\overline{G}}(\overline{K})]$. Since $E_{\overline{K}}(-)$ is defined as induction $\mathrm{Ind}_{F'}(-)$ for the functor $F' : R[W_{\overline{G}}(\overline{K})] \to R\Gamma_{G/N}$ (see [10, 9.30]), we have

$$\mathrm{Inf}^G_{G/N} R[(\overline{G}/\overline{K})^{?}] = \mathrm{Inf}^G_{G/N} E_{\overline{K}} P_{\overline{K}} = \mathrm{Ind}_F \mathrm{Ind}_{F'} P_{\overline{K}} = \mathrm{Ind}_{F \circ F'} P_{\overline{K}}$$

where $F : \Gamma_{G/N} \to \Gamma_G$ is the functor defined above. Since $W_{\overline{G}}(\overline{K}) \cong W_G(K)$, after suitable identification, the composition $F \circ F'$ becomes the same as the inclusion functor $i : W_G(K) \to \Gamma_G$, so we have

$$\mathrm{Ind}_{F \circ F'} P_{\overline{K}} = E_K R W_G(K) = R[G/K^{?}]$$

as desired. □

By general properties of restriction and induction functors associated to a functor $F$, the functor $\mathrm{Def}^G_{G/N}$ is exact and $\mathrm{Inf}^G_{G/N}$ respects projectives (see [10, 9.24]). The deflation functor has the following formula for free modules.

**Lemma 4.2.** *Let $X$ be a $G$-set. Then, we have*

$$\mathrm{Def}^G_{G/N} R[X^{\,?}] = R[(X^N)^{?}]$$

*In particular, if $H \in \mathfrak{F}_G$ implies $HN \in \mathfrak{F}_G$, then the functor $\mathrm{Def}^G_{G/N}$ respects projectives.*

*Proof.* For every $K \in \mathfrak{F}_G$ such that $K \geq N$, we have

$$(\mathrm{Def}^G_{G/N} R[X^{?}])(\overline{K}) = R[X^{?}](K) = R[X^K] = R[(X^N)^{K/N}] = R[(X^N)^{?}](\overline{K}).$$

Note that $(G/H)^N = G/HN$ as a $G/N$-set. If $H \in \mathcal{F}_G$ implies $HN \in \mathcal{F}_G$, then by assumption $\overline{HN} \in \mathcal{F}_{G/N}$. Hence $R[((G/H)^N)^?]$ is free as an $R\Gamma_{G/N}$-module and $\mathrm{Def}^G_{G/N}$ respects projectives. $\square$

## 5. The Borel-Smith conditions for chain complexes

Let $G$ be a finite group, and let $X$ be a finite $G$-CW-complex which is a mod-$p$ homology sphere for some prime $p$. Then by Smith theory, the fixed point space $X^H$ is also a mod-$p$ homology sphere (or empty), for every $p$-subgroup $H \leq G$. So if we take $R = \mathbb{Z}/p$ and $\Gamma_G$ as the orbit category over the family $\mathcal{F}_p$ of all $p$-subgroups of $G$, then the chain complex $\mathbf{C}(X^?; \mathbb{Z})$ over $R\Gamma_G$ is a finite free chain complex which is an $R$-homology $\underline{n}$-sphere. Here, as before, we take $\underline{n}(H) = -1$ when $X^H = \emptyset$. In this case, it is known that the super class function $\underline{n}$ satisfies certain conditions called the Borel-Smith conditions (see [3, Thm. 2.3 in Chapter XIII] or [12, III.5]). These conditions are given as follows:

**Definition 5.1.** Let $G$ be a finite group and let $f\colon \mathcal{S}(G) \to \mathbb{Z}$ be super class function, where $\mathcal{S}(G)$ denotes the family of all subgroups of $G$. We say the function $f$ satisfies the Borel-Smith conditions at a prime $p$, if it has the following properties:

(i) If $L \lhd K \leq G$ are such that $K/L \cong \mathbb{Z}/p$, and $p$ is odd, then $f(L) - f(K)$ is even.

(ii) If $L \lhd K \leq G$ are such that $K/L \cong \mathbb{Z}/p \times \mathbb{Z}/p$, and if $L_i/L$ denote the subgroups of order $p$ in $K/L$, then

$$f(L) - f(K) = \sum_{i=0}^{p} (f(L_i) - f(K)).$$

(iii) If $p = 2$, and $L \lhd K \lhd N \leq G$ are such that $L \lhd N$, $K/L \cong \mathbb{Z}/2$, and $N/L \cong \mathbb{Z}/4$, then $f(L) - f(K)$ is even.

(iv) If $p = 2$, and $L \lhd K \lhd N \leq G$ are such that $L \lhd N$, $K/L \cong \mathbb{Z}/2$, and $N/L = Q_8$ is the quaternion group of order 8, then $f(L) - f(K)$ is divisible by 4.

We will show that these conditions are satisfied by the homological dimension function $\underline{n}$ of a finite projective complex $\mathbf{C}$ over $R\Gamma_G$ which is an $R$-homology $\underline{n}$-sphere. Recall that $\underline{n}(H) = -1$ whenever $H \notin \mathcal{F}$, by Definition 2.7.

**Theorem C.** *Let $G$ be a finite group, $R = \mathbb{Z}/p$, and let $\mathcal{F}$ be a given family of subgroups of $G$. If $\mathbf{C}$ is a finite projective chain complex over $R\Gamma_G$, which is an $R$-homology $\underline{n}$-sphere, then the function $\underline{n}$ satisfies the Borel-Smith conditions at the prime $p$.*

The rest of the section is devoted to the proof of Theorem C. As a first step of the proof we extend the given family $\mathcal{F}$ to the family $\mathcal{S}(G)$ of all subgroups of $G$ by taking $\mathbf{C}(H) = 0$ for every $H \notin \mathcal{F}$. Over the extended family, $\mathbf{C}$ is still a finite projective chain complex over $R\Gamma_G$ and an $R$-homology $\underline{n}$-sphere.

The Borel-Smith conditions are conditions on subquotients $K/L$ where $L \lhd K \leq G$. To show that a Borel-Smith condition holds for a particular subquotient group $K/L$, we consider the complex $\mathrm{Def}^K_{K/L} \mathrm{Res}^G_K \mathbf{C}$ (see Section 4). This is a finite projective complex over $R\Gamma_{K/L}$ because both restriction and deflation functors preserve projectives (the condition in Lemma 4.2 is satisfied because we extended our family $\mathcal{F}$ to the family of all subgroups of $G$).

Our first observation is the following:

**Lemma 5.2.** *Let $G$ be a finite group and let $R = \mathbb{Z}/p$. If $\mathbf{C}$ is a finite projective chain complex over $R\Gamma_G$, which is an $R$-homology $\underline{n}$-sphere, then whenever $L \lhd K \leq G$ and $K/L$ is a $p$-group, we have $\underline{n}(L) \geq \underline{n}(K)$.*

*Proof.* By the discussion above, it is enough to show that if $G = \mathbb{Z}/p$ and $\mathbf{C}$ is a finite projective $R\Gamma_G$-complex which is an $R$-homology $\underline{n}$-sphere, then the inequality $\underline{n}(1) \geq \underline{n}(G)$ holds. Assume that $\underline{n}(1) \neq \underline{n}(G)$. Since $\underline{R}$ is projective, we can add $\mathbf{C}_{-1} = \underline{R}$ and consider the homology of the augmented complex $\widetilde{\mathbf{C}}$. The complex $\widetilde{\mathbf{C}}$ has nontrivial homology only at two dimensions, say $m$ and $k$ with $m > k$, so we get an extension of the form

$$0 \to H_m(\widetilde{\mathbf{C}}) \to \widetilde{\mathbf{C}}_m / \operatorname{im} \partial_{m+1} \to \cdots \to \widetilde{\mathbf{C}}_{k+1} \to \ker \partial_k \to H_k(\widetilde{\mathbf{C}}) \to 0.$$

where the homology modules are $I_1 R$ and $I_G R$ in some order.

For $H \in \mathcal{F}$, the module $I_H M$ denotes the atomic module concentrated at $H$ with the value $(I_H M)(H) = M$ (see [10, 9.29]). We claim that $H_m(\widetilde{\mathbf{C}}) = I_1 R$ and $H_k(\widetilde{\mathbf{C}}) = I_G R$, meaning that the module $I_G R$ appears before $I_1 R$ in the homology. Once we show this, it will imply that $\underline{n}(1) > \underline{n}(G)$ as desired.

Let $\mathbf{D}$ denote the chain complex obtained by erasing the homology groups $H_m(\widetilde{\mathbf{C}})$ and $H_k(\widetilde{\mathbf{C}})$ from the above exact sequence. Since $\ker \partial_k$ is projective and $\operatorname{im} \partial_{m+1}$ has a finite projective resolution, the Ext-group $\operatorname{Ext}^*_{R\Gamma_G}(\mathbf{D}, M)$ is zero after some fixed dimension, for every $R\Gamma_G$-module $M$. We will take $M = I_1 R$ for simplicity.

There is a two-line spectral sequence $E_2^{s,t} = \operatorname{Ext}^s_{R\Gamma_G}(H_t(\mathbf{D}), M)$ which converges to $\operatorname{Ext}^*_{R\Gamma_G}(\mathbf{D}, M)$. Suppose, if possible, that $H_k(\widetilde{\mathbf{C}}) = I_1 R$. The module $I_1 R$ is concentrated at 1, so its projective resolution is of the form $E_1 P_*$ for some projective resolution $P_*$ of $R$ as an $RG$-module. Then the bottom line of this spectral sequence $E_2^{*,0}$ would be isomorphic to

$$\operatorname{Ext}^*_{R\Gamma_G}(H_k(\widetilde{\mathbf{C}}), M) = \operatorname{Ext}^*_{R\Gamma_G}(I_1 R, I_1 R) = H^i(\operatorname{Hom}_{R\Gamma_G}(E_1 P_*, I_1 R)) = H^*(G; R).$$

Since this cohomology ring is not finitely generated, there must be a non-trivial differential from the top line

$$\operatorname{Ext}^*_{R\Gamma_G}(H_m(\widetilde{\mathbf{C}}), M) = \operatorname{Ext}^*_{R\Gamma_G}(I_G R, I_1 R)$$

in order for the spectral sequence to converge to a finite dimensional limit.

The differential of this spectral sequence is given by multiplication with an extension class in $\operatorname{Ext}^{m-k+1}_{R\Gamma_G}(I_1 R, I_G R)$. But, by a similar calculation as above, we see that

$$\operatorname{Ext}^i_{R\Gamma_G}(I_1 R, I_G R) = H^i(G, (I_G R)(1))) = 0$$

for all $i \geq 0$, because $(I_G R)(1) = 0$. This contradiction shows that $H_k(\widetilde{\mathbf{C}}) = I_G R$ and $H_m(\widetilde{\mathbf{C}}) = I_1 R$, as required. □

The above lemma shows that under the conditions of Theorem C, the dimension function $\underline{n}$ is monotone in the sense defined in [12, p. 211]. Now we verify (in separate steps) that the dimension function satisfies the conditions of Definition 5.1. These conditions come from the period of the cohomology of the corresponding subquotient groups.

**Lemma 5.3** (Borel-Smith, part (i)). *Let $G = \mathbb{Z}/p$, for $p$ an odd prime, let $R = \mathbb{Z}/p$, and $\mathbf{C}$ be a finite projective $R\varGamma_G$-complex which is an $R$-homology $\underline{n}$-sphere. Then $\underline{n}(1) - \underline{n}(G)$ is even.*

*Proof.* Consider the subcomplex $\widetilde{\mathbf{C}}^{(G)}$ of $\widetilde{\mathbf{C}}$ consisting of all projectives of type $R[G/G^{?}]$, and let $\mathbf{D} = \widetilde{\mathbf{C}}/\widetilde{\mathbf{C}}^{(G)}$ denote the quotient complex. The complex $\mathbf{D}$ has nontrivial homology only in dimensions $m$ and $k+1$, where $m = \underline{n}(1)$ and $k = \underline{n}(G)$. Moreover, all the $R\varGamma_G$-modules in the complex $\mathbf{D}$ are of the form $R[G/1^{?}]$. Evaluating at the subgroup 1, we obtain a chain complex of free $RG$-modules

$$0 \to Q_d \to \cdots \to Q_{m+1} \xrightarrow{\partial_{m+1}} Q_m \to \cdots \to Q_{k+1} \xrightarrow{\partial_{k+1}} Q_k \to \cdots \to Q_0 \to 0.$$

whose homology is $R$ at dimensions $m$ and $k+1$. This gives an exact sequence of the form

$$0 \to R \to Q_m/\operatorname{im}\partial_{m+1} \to \cdots \to Q_{k+2} \to \ker\partial_{k+1} \to R \to 0.$$

Using the fact that free $RG$-modules are both projective and injective, we conclude that all the modules in the above sequence, except the two $R$'s on the both ends, are projective as $RG$-modules, so we have a periodic resolution. Since the group $G = \mathbb{Z}/p$ has periodic $R$-cohomology with period 2, we have $m - k = \underline{n}(1) - \underline{n}(G) \equiv 0 \pmod 2$. □

**Remark 5.4.** The $R$-cohomology of the group $G = \mathbb{Z}/2$ is periodic of period 1.

For condition (ii), the argument is more involved. As before, after the subquotient reduction we may assume that $G = K/L = \mathbb{Z}/p \times \mathbb{Z}/p$, and that $\mathcal{F}$ is the family of all subgroups of $G$. Since the complex $\mathbf{C}$ is a finite complex of projective modules, for any $R\varGamma_G$-module $M$, we have

$$H^n(\operatorname{Hom}_{R\varGamma_G}(\mathbf{C}, M)) = 0$$

for $n > d$, where $d$ is the dimension of the chain complex $\mathbf{C}$. Consider the hypercohomology spectral sequence for the complex $\mathbf{C}$. This is a spectral-sequence with $E_2$-term given by

$$E_2^{s,t} = \operatorname{Ext}^s_{R\varGamma_G}(H_t(\mathbf{C}), M) \tag{5.5}$$

which converges to $H^{s+t}(\operatorname{Hom}_{R\varGamma_G}(\mathbf{C}, M))$. Since $\underline{R}$ is a projective $R\varGamma_G$-module, we can replace $H_t(\mathbf{C})$ with the reduced homology $\widetilde{H}_t(\mathbf{C})$. So, we have nonzero terms for $E_2^{s,t}$ only when $t$ is equal to $n_1 = \underline{n}(1)$, $n_G = \underline{n}(G)$, or $n_{H_i} = \underline{n}(H_i)$ where $H_i$ are the subgroups of $G$ of order $p$. Since $\underline{n}$ is monotone, we have $n_1 \geq n_{H_i} \geq n_G$ for all $i \in \{0, \dots, p\}$. The required formula is

$$n_1 - n_G = \sum_{i=0}^{p} (n_{H_i} - n_G).$$

**Remark 5.6.** In the proof below we assume $n_1 > n_{H_i} > n_G$ for all $i$, to make the argument easy to follow. If for some $i$, we have $n_{H_i} = n_1$ or $n_{H_i} = n_G$, then the argument below can be adjusted easily to include these cases as well.

By adding free summands to the complex $\mathbf{C}$, we can assume that all the cohomology between dimensions $n_1$ and $n_G$ is concentrated at the dimension $n_M = \max_i\{n_{H_i}\}$. Then the homology at this dimension will be an $R\varGamma_G$-module which is filtered by Heller shifts of homology groups $H_t(\mathbf{C})$ at dimensions $t = n_{H_i}$ for $i = 0, \dots, p$. The homology of the complex $\mathbf{C}$ at dimension $n_{H_i}$ is $I_{H_i}R$, where $I_{H_i}R$ denotes the $R\varGamma_G$ module with value $R$ at $H_i$ and zero at all the other subgroups. We have the following lemma.

**Lemma 5.7.** *If $i, j \in \{0, \dots, p\}$ are such that $i \neq j$, then*

$$\operatorname{Ext}^m_{R\varGamma_G}(I_{H_i}R, I_{H_j}R) = 0$$

*for every $m \geq 0$.*

*Proof.* The projective resolution of $I_{H_i}$ is formed by projective modules of type $E_H P$ with $H = 1$ or $H_i$. Since

$$\operatorname{Hom}_{R\varGamma_G}(E_H P, I_{H_j}R) \cong \operatorname{Hom}_{RW_G(H)}(P, I_{H_j}(H)) = 0$$

when $i \neq j$, we obtain the desired result. □

As a consequence of Lemma 5.7, we conclude that all the extensions in this filtration of $H_{n_M}(\mathbf{C})$ are split extensions. So, the homology module $H_{n_M}(\mathbf{C})$ is isomorphic to a direct sum of Heller shifts of modules $I_{H_i}R$. In particular, we obtain that, for any $R\varGamma_G$-module $M$,

$$\operatorname{Ext}^s_{R\varGamma_G}(H_{n_M}(\mathbf{C}), M) \cong \oplus_i \operatorname{Ext}^{s+n_M-n_{H_i}}_{R\varGamma_G}(I_{H_i}R, M)$$

for every $s \geq 0$.

The spectral sequence given in (5.5) converges to zero for total dimension $> d$. It has only three non-zero horizontal lines, so it gives a long exact sequence of the form

$$\cdots \to \operatorname{Ext}^{k+n_1-n_G+1}_{R\varGamma_G}(I_G R, M) \xrightarrow{\delta} \operatorname{Ext}^k_{R\varGamma_G}(I_1R, M) \xrightarrow{\gamma} \oplus_{i=0}^p \operatorname{Ext}^{k+n_1-n_{H_i}+1}_{R\varGamma_G}(I_{H_i}R, M)$$
$$\to \operatorname{Ext}^{k+n_1-n_G+2}_{R\varGamma_G}(I_G R, M) \xrightarrow{\delta} \operatorname{Ext}^{k+1}_{R\varGamma_G}(I_1R, M) \to \cdots$$

where $k$ is an integer such that $k > d - n_1$ and $M$ is any $R\varGamma_G$-module. If we take $M = I_1R$, then $\operatorname{Ext}^k_{R\varGamma_G}(I_1R, M) \cong H^k(G, R)$. When $M = I_1R$, the other Ext-groups in the above exact sequence also reduce to the cohomology of the group $G$ with some dimension shifts.

**Lemma 5.8.** *For every $i \in \{0, \dots, p\}$, we have*

$$\operatorname{Ext}^m_{R\varGamma_G}(I_{H_i}R, I_1R) \cong \operatorname{Ext}^{m-1}_{R\varGamma_G}(I_1R, I_1R) \cong H^{m-1}(G; R)$$

*for every $m \geq 1$. We also have*

$$\operatorname{Ext}^m_{R\varGamma_G}(I_G R, I_1R) \cong \oplus_p \operatorname{Ext}^{m-2}_{R\varGamma_G}(I_1R, I_1R) \cong \oplus_p H^{m-2}(G; R)$$

*for every $m \geq 2$. Here $\oplus_p$ denotes the direct sum of $p$-copies of the same $R$-module.*

*Proof.* Since we already observed that $\operatorname{Ext}^k_{R\varGamma_G}(I_1R, I_1R) \cong H^k(G, R)$ for every $k \geq 0$, it is enough to show the first isomorphisms. Let $i \in \{0, \dots, p\}$ and $J_{H_i}R$ denote the $R\varGamma_G$ module with value $R$ at subgroups 1 and $H_i$ and zero at every other subgroup. We assume that the restriction map is an isomorphism. So we have a non-split exact sequence of $R\varGamma_G$-modules of the form

$$0 \to I_1R \to J_{H_i}R \to I_{H_i}R \to 0.$$

Since the projective resolution of $J_{H_i}R$ will only include projective modules of the form $E_{H_i}P$, we have $\operatorname{Ext}^m_{R\varGamma_G}(J_{H_i}R, I_1R) = 0$ for all $m \geq 0$. The long exact Ext-group sequence associated to the above short exact sequence will give the desired isomorphism for the module $I_{H_i}R$.

For the second statement in the lemma, we again only need to show that the isomorphism

$$\operatorname{Ext}^m_{R\varGamma_G}(I_GR, I_1R) \cong \oplus_p \operatorname{Ext}^{m-2}_{R\varGamma_G}(I_1R, I_1R)$$

holds for all $m \geq 2$. Let $N$ denote the $R\varGamma_G$-module defined as the kernel of the map $\underline{R} \to I_GR$ which induces the identity homomorphism at $G$. Since the constant module $\underline{R}$ is projective as a $R\varGamma_G$-module, we have

$$\operatorname{Ext}^m_{R\varGamma}(I_GR, I_1R) \cong \operatorname{Ext}^{m-1}_{R\varGamma}(N, I_1R)$$

for $m \geq 2$. In addition, there is an exact sequence of the form

$$0 \to \oplus_p I_1R \to \oplus_{i=0}^p J_{H_i}R \to N \to 0.$$

Since $\operatorname{Ext}^m_{R\varGamma_G}(J_{H_i}R, I_1R) = 0$ for all $m \geq 0$, we obtain

$$\operatorname{Ext}^m_{R\varGamma}(I_GR, I_1R) \cong \operatorname{Ext}^{m-1}_{R\varGamma}(N, I_1R) \cong \oplus_p \operatorname{Ext}^{m-2}_{R\varGamma}(I_1R, I_1R) \cong \oplus_p H^{m-2}(G; R)$$

for every $m \geq 2$. This completes the proof of the lemma. □

**Lemma 5.9** (Borel-Smith, part (ii)). *Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$, let $R = \mathbb{Z}/p$, and let $\mathbf{C}$ be a finite projective $R\varGamma_G$-complex which is an $R$-homology $\underline{n}$-sphere. Then*

$$\underline{n}(1) - \underline{n}(G) = \sum_{i=0}^{p} (\underline{n}(H_i) - \underline{n}(G))$$

*where $H_0, H_1, \dots, H_p$ denote the distinct subgroups of $G$ of order $p$.*

*Proof.* Using the Ext-group calculations given in Lemma 5.8, we obtain a long exact sequence of the form

$$\cdots \to \oplus_p H^{k+n_1-n_G-1}(G; R) \xrightarrow{\delta} H^k(G; R) \xrightarrow{\gamma} \oplus_{i=0}^p H^{k+n_1-n_{H_i}}(G; R)$$
$$\to \oplus_p H^{k+n_1-n_G}(G; R) \xrightarrow{\delta} H^{k+1}(G; R) \to \cdots$$

where $k > d - n_1$. We claim that the map $\gamma$ is injective. Observe that if $\gamma = \oplus \gamma_i$, then for each $i$, the map $\gamma_i$ can be defined as multiplication with some cohomology class $u_i$. To see this observe that $\gamma$ is the map induced by the differential

$$d_{n_1-n_M+1} \colon \operatorname{Ext}^k_{R\varGamma_G}(H_{n_1}(\mathbf{C}), I_1R) \to \operatorname{Ext}^{k+n_1-n_M+1}_{R\varGamma_G}(H_{n_M}(\mathbf{C}), I_1R)$$

on the hypercohomology spectral sequence given at (5.5). This spectral sequence has an $\operatorname{Ext}^*_{R\varGamma}(I_1R, I_1R)$-module structure, where the multiplication is given by the Yoneda product, defined by splicing the corresponding extensions (see [2, Section 4]).

Under the isomorphisms given in Lemma 5.8, the differential $d_{n_1-n_M+1}$ becomes a map $H^k(G, R) \to \oplus_i H^{k+n_1-n_{H_i}}(G, R)$ and the Yoneda product of Ext-groups is the same as the usual cup product multiplication in group cohomology under the canonical isomorphism $\operatorname{Ext}^m_{R\varGamma_G}(I_1R, I_1R) \cong H^m(G, R)$ (for comparison of different products on group cohomology see [5, Proposition 4.3.5]). So we can conclude that $\gamma_i$ is the map defined by multiplication (the usual cup product) with a cohomology class $u_i \in H^{n_1 - n_{H_i}}(G, R)$.

For $p = 2$, the cohomology ring $H^*(G, R)$ is isomorphic to a polynomial algebra $R[t_1, t_2]$ with $\deg t_i = 1$ for $i = 1, 2$. Since there are no nonzero divisors in a polynomial algebra, the map $\gamma$ is either injective or the zero map.

For $p$ an odd prime, the cohomology ring $H^*(G, R)$ is isomorphic to the tensor product of an exterior algebra with a polynomial algebra

$$\Lambda_R(a_1, a_2) \otimes R[x_1, x_2]$$

where $\deg a_i = 1$ and $\deg x_i = 2$, and the nonzero divisors of this ring are multiples of $a_i$ or $a_j$. By Lemma 5.3, $\deg u_i = n_1 - n_{H_i} \equiv 0 \pmod 2$, so the map $\gamma$ is either injective, or each $u_i$ must be a multiple of $a_1 a_2$.

The assumption that $\gamma$ is not injective implies that the entire spectral sequence restricted to some $H_i \cong \mathbb{Z}/p$, with $\mathrm{Res}^G_{H_i}(a_1 a_2) = 0$, will result in a spectral sequence which collapses. This is because $\mathrm{Res}^G_{H_i} I_G R = 0$ and $\mathrm{Res}^G_{H_i} I_{H_j} R = 0$ if $i \neq j$. But the cohomology $H^*(\mathbb{Z}/p; R)$ is not finite-dimensional, and the restriction of $\mathbf{C}$ to a proper subgroup is still a finite projective chain complex, so this gives a contradiction. Hence, we can conclude that $\gamma$ is injective.

The fact that $\gamma$ is injective gives a short exact sequence of the form

$$0 \to H^k(G; R) \xrightarrow{\gamma} \oplus_{i=0}^p H^{k+n_1-n_{H_i}}(G; R) \to \oplus_p H^{k+n_1-n_G}(G; R) \to 0,$$

for every $k > d - n_1$. Since $\dim_R H^m(G; R) = m + 1$, we obtain

$$(k+1) + p(k + n_1 - n_G + 1) = \sum_{i=0}^{p} (k + n_1 - n_{H_i} + 1).$$

Cancelling the $(k + 1)$'s and grouping the terms in a different way gives the desired equality. □

The next part uses the same spectral sequence, but the details are much simpler.

**Lemma 5.10** (Borel-Smith, part (iii))**.** *Let $G = \mathbb{Z}/4$, let $R = \mathbb{Z}/2$, and let $\mathbf{C}$ be a finite projective $R\Gamma_G$-complex which is an $R$-homology $\underline{n}$-sphere. If $1 \lhd K \lhd G$ with $K \cong \mathbb{Z}/2$, then $\underline{n}(1) - \underline{n}(K)$ is even.*

*Proof.* We consider the spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{R\Gamma_G}(H_t(\mathbf{C}), M),$$

with $M = I_1 R$, which converges to $H^{s+t}(\mathrm{Hom}_{R\Gamma_G}(\mathbf{C}, M))$. Write $n_1 = \underline{n}(1)$, $n_K = \underline{n}(K)$, and $n_G = \underline{n}(G)$. Once again, the fact that $H^k(\mathbf{C}; M)$ is zero in large dimensions $k > d = \dim \mathbf{C}(1)$ gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^{k+n_1-n_G+1}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^k_{R\Gamma_G}(I_1 R, M) \xrightarrow{\gamma} \mathrm{Ext}^{k+n_1-n_K+1}_{R\Gamma_G}(I_K R, M)$$
$$\to \mathrm{Ext}^{k+n_1-n_G+2}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^{k+1}_{R\Gamma_G}(I_1 R, M) \to \cdots$$

The analogue of Lemma 5.8 is easier in this case. We obtain

$$\mathrm{Ext}^m_{R\Gamma_G}(I_K R, I_1 R) \cong \mathrm{Ext}^{m-1}_{R\Gamma_G}(I_1 R, I_1 R) \cong H^{m-1}(G; R)$$

for every $m \geq 1$, and $\mathrm{Ext}^m_{R\Gamma_G}(I_G R, I_1 R) = 0$ for every $m \geq 0$. The vanishing result follows from the short exact sequence

$$0 \to J_K R \to \underline{R} \to I_G R \to 0$$

and the fact that $\mathrm{Ext}^m_{R\Gamma_G}(J_K R, I_1 R) = 0$, for $m \geq 0$, since $J_K R$ has a projective resolution consisting of modules of the form $E_K P$. On substituting these values into the long exact sequence, we obtain an isomorphism

$$\gamma\colon H^k(G; R) \cong H^{k+n_1-n_K}(G; R)$$

induced by cup product, for all large $k$. Since the cohomology ring $H^*(G; R)$ modulo nilpotent elements is generated by a 2-dimensional class, it follows that $n_1 - n_K$ must be even. □

**Lemma 5.11** (Borel-Smith, part (iv))**.** *Let $G = Q_8$, let $R = \mathbb{Z}/2$, and let $\mathbf{C}$ be a finite projective $R\Gamma_G$-complex which is an $R$-homology $\underline{n}$-sphere. If $1 \lhd K \lhd G$ with $K \cong \mathbb{Z}/2$, then $\underline{n}(1) - \underline{n}(K)$ is divisible by 4.*

*Proof.* This time we have three index 2 normal subgroups $H_1, H_2, H_3$, each isomorphic to $\mathbb{Z}/4$. Write $n_1 = \underline{n}(1)$, $n_K = \underline{n}(K)$, $n_{H_i} = \underline{n}(H_i)$, for $1 \leq i \leq 3$, and $n_G = \underline{n}(G)$. We again consider the spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{R\Gamma_G}(H_t(\mathbf{C}), M),$$

with $M = I_1 R$, which converges to $H^{s+t}(\mathrm{Hom}_{R\Gamma_G}(\mathbf{C}, M))$. The exact sequences

$$0 \to N \to \underline{R} \to I_G R \to 0$$

and

$$0 \to (J_K R)^2 \to \oplus_i J_{H_i} R \to N \to 0$$

lead to the calculation

$$\mathrm{Ext}^m_{R\Gamma_G}(I_G R, I_1 R) = 0$$

for every $m \geq 0$. The exact sequence

$$0 \to I_1 R \to J_K R \to I_K R \to 0$$

implies that $\mathrm{Ext}^m_{R\Gamma_G}(I_K R, I_1 R) = H^{m-1}(G; R)$, for $m \geq 1$. Finally, the exact sequences

$$0 \to J_K R \to J_{H_i} R \to I_{H_i} R \to 0$$

show that $\mathrm{Ext}^m_{R\Gamma_G}(I_{H_i} R, I_1 R) = 0$, for $m \geq 0$ and $1 \leq i \leq 3$.

As a result of these calculations, we again obtain a 3-line spectral sequence with corresponding long exact sequence

$$\cdots \to \mathrm{Ext}^{k+n_1-n_G+1}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^k_{R\Gamma_G}(I_1 R, M) \xrightarrow{\gamma} \mathrm{Ext}^{k+n_1-n_K+1}_{R\Gamma_G}(I_K R, M)$$

$$\to \mathrm{Ext}^{k+n_1-n_G+2}_{R\Gamma_G}(I_G R, M) \xrightarrow{\delta} \mathrm{Ext}^{k+1}_{R\Gamma_G}(I_1 R, M) \to \cdots$$

in all large dimensions $k > d$. By the vanishing result above, the map

$$\gamma\colon H^k(G; R) \cong H^{k+n_1-n_K}(G; R)$$

is an isomorphism induced by cup product. Since the cohomology ring $H^*(G; R)$ modulo nilpotent elements is generated by a 4-dimensional class, it follows that $n_1 - n_K$ is divisible by 4. □

**Remark 5.12.** The fact that the dimension function of an algebraic $\underline{n}$-homology sphere satisfies the Borel-Smith conditions suggests that more of the classical results on finite group actions on spheres might hold for finite projective chain complexes over a suitable orbit category. For example, one could ask for an algebraic version of the results of Dotzel-Hamrick [6] on $p$-groups. Other potential applications of algebraic models to finite group actions are outlined in [7].

**Example 5.13.** An important test case for groups acting on spheres, or on products of spheres [1], is the rank two group $\mathrm{Qd}(p) = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes SL_2(p)$. At present, it is not known whether $\mathrm{Qd}(p)$ can act freely on a product of two spheres, but Ünlü [13] showed that $\mathrm{Qd}(p)$ does not act on a finite complex homotopy equivalent to a sphere with rank one isotropy.

We can apply the Borel-Smith conditions prove an algebraic version of this result.

**Proposition 5.14.** *Let $p$ be an odd prime, $G = \mathrm{Qd}(p)$, $R = \mathbb{Z}/p$, and $\mathcal{F}$ be the family of all subgroups $H \leq G$ such that $\mathrm{rank}_p(H) \leq 1$. Let $\underline{n}$ be a super class function with $\underline{n}(1) \geq 0$. Then, there exists no finite projective chain complex $\mathbf{C}$ over $R\Gamma_G$ which is an $R$-homology $\underline{n}$-sphere.*

*Proof.* We can extend the family $\mathcal{F}$ to the family $\mathcal{S}(G)$ of all subgroups of $G$ by taking $\mathbf{C}(H) = 0$ for all subgroups such that $H \notin \mathcal{F}$. For these subgroups, we take $\underline{n}(H) = -1$. Observe that by Theorem C, the dimension function $\underline{n} : \mathcal{S}(G) \to \mathbb{Z}$ satisfies the Borel-Smith conditions at the prime $p$.

Now the rest of the argument follows as in Ünlü [13, Theorem 3.3]. Let $P$ be a Sylow $p$-subgroup of $\mathrm{Qd}(p)$. The group $P$ is isomorphic to the extra-special $p$-group of order $p^3$ and exponent $p$. If $Z(P)$ is the center of $P$, then the quotient group $P/Z(P)$ is isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. Applying the Borel-Smith condition (ii) for this quotient, we get $\underline{n}(Z(P)) = -1$. In $G$, it is possible to find two Sylow $p$-subgroups $P_1$ and $P_2$ such that $E = P_1 \cap P_2 \cong \mathbb{Z}/p \times \mathbb{Z}/p$ and $Z(P_1)$ and $Z(P_2)$ are distinct subgroups of order $p$ in $E$. Two such Sylow $p$-subgroups can be given as $P_i = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \langle A_i \rangle$ for $i = 1, 2$ where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

By the above argument, $\underline{n}(Z(P_i)) = -1$, and non-central $p$-subgroups in $E$ are conjugate to each other. So, we obtain that $\underline{n}(K) = -1$ for every subgroup $K$ of order $p$ in $E$. By the Borel-Smith conditions applied to $E$, we get $\underline{n}(1) = -1$, contradicting our assumption on $\underline{n}$. □

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