KORN'S SECOND INEQUALITY AND GEOMETRIC RIGIDITY WITH MIXED GROWTH CONDITIONS

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ABSTRACT. Geometric rigidity states that a gradient field which is L^p-close to the set of proper rotations is necessarily L^p-close to a fixed rotation, and is one key estimate in nonlinear elasticity. In several applications, as for example in the theory of plasticity, energy densities with mixed growth appear. We show here that geometric rigidity holds also in $L^p + L^q$ and in $L^{p,q}$ interpolation spaces. As a first step we prove the corresponding linear inequality, which generalizes Korn's inequality to these spaces.

1. INTRODUCTION

Since Korn's original contributions [16, 17, 18], Korn's inequality has played a central role in the analysis of boundary value problems in linear elasticity. In its basic form, Korn's inequality asserts the following. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded, connected, Lipschitz domain and that $u \in H^1(\Omega; \mathbb{R}^n)$. Then there exists a skew-symmetric matrix S such that $\|Du - S\|_2 \le c\|Eu\|_2$. That is, the L^2-norm of the skew-symmetric part of Du is dominated by the L^2-norm of the symmetric part, after a suitable constant S has been subtracted. Numerous generalizations to different boundary conditions, growth conditions and unbounded domains have been given in the literature, see, e.g., [7, 10, 15] and the references therein.

In view of the fundamental importance of Korn's inequality in linear elasticity, it is not surprising that a suitable nonlinear version, which is often referred to as geometric rigidity, plays a central role in models in nonlinear elasticity. In their basic form, these estimates assert that for a deformation $u \in H^1(\Omega; \mathbb{R}^n)$ the distance of Du to a suitably chosen proper rotation $Q \in \mathrm{SO}(n)$ is dominated in L^2 by the distance function of Du to $\mathrm{SO}(n)$. The proof [8] is based on the fact that the nonlinear estimate can be related to the linear one since the tangent space to the smooth manifold $\mathrm{SO}(n)$ at the identity matrix is given by the linear space of all skew-symmetric matrices.

In fact, geometric rigidity results are the cornerstone of rigorous derivations of two-dimensional plate and shell theories from three-dimensional models in the framework of nonlinear elasticity theory. The quantitative version by Friesecke, James and Müller [8] generalized previous work [12, 13, 22, 14] and allowed for the first time the derivation of limiting theories as the thickness of the three-dimensional structure tends to zero without a priori assumptions on the deformations in various scaling regimes [8, 9].

More recently, the analysis of variational models for the elastic and plastic behavior of single crystals has led to the question of whether analogous estimates can be established under mixed growth conditions. In this paper we generalize both Korn's inequality and the corresponding nonlinear estimate to this setting. Our main result is the following.

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Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with Lipschitz boundary. Suppose that $1 < p < q < \infty$ and that $u \in W^{1,1}(\Omega;\mathbb{R}^n)$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ are given with*

$$\operatorname{dist}(Du, \mathrm{SO}(n)) = f + g \qquad \text{a.e. in } \Omega\,. \tag{1.1}$$

Then there exist a constant c, matrix fields $F \in L^p(\Omega;\mathbb{R}^{n\times n})$, $G \in L^q(\Omega;\mathbb{R}^{n\times n})$, and a proper rotation $Q \in \mathrm{SO}(n)$ such that

$$Du = Q + F + G \qquad \text{a.e. in } \Omega\,,$$

and

$$\|F\|_{L^p(\Omega;\mathbb{R}^{n\times n})} \le c\|f\|_{L^p(\Omega)}\,, \qquad \|G\|_{L^q(\Omega;\mathbb{R}^{n\times n})} \le c\|g\|_{L^q(\Omega)}\,. \tag{1.2}$$

The constant c depends only on n, p, q, and Ω but not on u, f, g.

The case $p = 2$ and $g = 0$ was established in [8, Th. 3.1], the generalization to $p \in (1,\infty)$ follows from the same proof with minor changes, see [3, Sect. 2.4]. This version with mixed growth conditions was first stated without proof in [9, Prop. 5] and has already been used in [1] to study nonlinear models with weak coerciveness and in [23] to study models of geometrically necessary dislocations in finite elastoplasticity. Our result implies a statement on equiintegrability (see Corollary 4.2), which has been used in [9] to show strong convergence of minimizing sequences. We believe that the generalization of Korn's inequality which is the basis for the proof presented here and is stated in Theorem 2.1 below, is of independent interest. In Section 4 we briefly discuss how the present results imply estimates in Lorentz $L^{p,q}$ spaces, present the statement on equiintegrability of sequences and generalize to more than two exponents.

Notation. We use standard notation for Lebesgue and Sobolev spaces and omit in the notation of their norms the domain and the range if they are clear from the context. We use $|E|$ for the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. For $u\colon \Omega \to \mathbb{R}^n$ we define the symmetric part of the deformation gradient as $Eu = (Du + Du^T)/2$. We denote the trace of a matrix $A \in \mathbb{R}^{n\times n}$ by $\operatorname{Tr} A$ and the inner product between to vectors a, $b \in \mathbb{R}^n$ and two matrices A, $B \in \mathbb{R}^{\times n}$ by $a \cdot b$ and $A : B = \operatorname{Tr} A^T B$, respectively. The distance $\operatorname{dist}(\cdot, \mathrm{SO}(n))$ is the usual Euclidean distance. We use the convention that constants may change from line to line as long as they depend only on n, p, q and Ω. Finally we use the fact that an estimate of the norm of a matrix field implies a decomposition of the matrix field with estimates. More precisely, if $A\colon \Omega \to \mathbb{R}^{n\times n}$ satisfies $|A| \le f + g$ with $f \in L^p(\Omega)$, $g \in L^q(\Omega)$ and $f, g \ge 0$, then

$$A = \frac{f}{f+g}\,\chi_{\{f+g\neq 0\}}\,A + \frac{g}{f+g}\,\chi_{\{f+g\neq 0\}}\,A = F + G \tag{1.3}$$

with $\|F\|_p \le \|f\|_p$ and $\|G\|_q \le \|g\|_q$. If f and g are not nonnegative, we replace them first by their absolute values.

2. Linear estimate: Korn's inequality

We start by the generalization of Korn's inequality to the case of mixed growth. This result will also be the key ingredient into the proof of Theorem 1.1.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with Lipschitz boundary. Suppose that $1 < p < q < \infty$ and that $u \in W^{1,1}(\Omega;\mathbb{R}^n)$, $f \in L^p(\Omega;\mathbb{R}^{n\times n})$, $g \in L^q(\Omega;\mathbb{R}^{n\times n})$ are given with*

$$Eu = \frac{1}{2}\big(Du + Du^T\big) = f + g \qquad \text{a.e. in } \Omega\,. \tag{2.1}$$

Then there exists a constant c, matrix fields $F \in L^p(\Omega;\mathbb{R}^{n\times n})$, $G \in L^q(\Omega;\mathbb{R}^{n\times n})$, and a skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$, that is, $S + S^T = 0$, such that

$$Du = S + F + G \qquad \text{a.e. in } \Omega\,, \tag{2.2}$$

and

$$\|F\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}\,, \qquad \|G\|_{L^q(\Omega)} \le c\|g\|_{L^q(\Omega)}\,. \tag{2.3}$$

The constant depends only on n, p, q, and Ω.

Proof. Korn's second inequality in L^p states that for every bounded connected Lipschitz set Ω and every $p \in (1,\infty)$ there is constant $c(\Omega,p)$ such that for every $u \in W^{1,p}(\Omega;\mathbb{R}^n)$ there exists a skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$ with

$$\|Du - S\|_p \le c(\Omega,p)\|Eu\|_p\,, \tag{2.4}$$

see, e.g., [26], [20] or [15, Theorem 8] for a proof.

From this we can easily prove the assertion in the case $\|g\|_{L^q(\Omega)} \le \|f\|_{L^p(\Omega)}$. Indeed, Hölder's inequality implies

$$\|Eu\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + c\|g\|_{L^q(\Omega)} \le (c+1)\|f\|_{L^p(\Omega)}\,,$$

and the assertion holds with $F = Du - S$ and $G = 0$. We may thus assume that

$$\|f\|_{L^p(\Omega)} \le \|g\|_{L^q(\Omega)}\,. \tag{2.5}$$

The proof relies on a covering argument together with a local estimate which is based on splitting u into a harmonic part and a remainder.

Step 1: A representation of Δu. We begin with an expression for Δu in terms of Eu which is frequently used in proofs of Korn's inequality, see, e.g., [15]. Suppose that $U \subset \mathbb{R}^n$ is open and that $u \in W^{1,1}_{\text{loc}}(U;\mathbb{R}^n)$. Let $\phi \in C^\infty_c(U;\mathbb{R}^n)$. Multiplication of the identity $Du : D\phi + Du : (D\phi)^T = 2Eu : D\phi$ with u and integration yields

$$-\int_U u\Delta\phi\,\mathrm{d}y = \int_U Du : D\phi\,\mathrm{d}y = \int_U \big(2Eu : D\phi - Du : (D\phi)^T\big)\,\mathrm{d}y\,.$$

Partial integration transforms the last term into $\int_U u\cdot \operatorname{div}((D\phi)^T)\,\mathrm{d}y = \int_U u\cdot D\operatorname{div}\phi\,\mathrm{d}y = -\int_U \operatorname{div} u \operatorname{div}\phi\,\mathrm{d}y$, and since $\operatorname{div} u = \operatorname{Tr} Du = \operatorname{Tr} Eu$ we conclude that

$$-\int_U u\Delta\phi\,\mathrm{d}y = \int_U \big(2Eu : D\phi - (\operatorname{Tr} Eu)\operatorname{div}\phi\big)\,\mathrm{d}y\,, \tag{2.6}$$

that is, $\Delta u = 2\operatorname{div} Eu - D(\operatorname{Tr} Eu)$ in U the sense of distributions.

Step 2: Construction of a finite cover of $\overline{\Omega}$. We choose for every $x \in \Omega$ an $r_x > 0$ such that $B(x,r_x) \subset \Omega$. For every $x \in \partial\Omega$ we fix an $r_x > 0$ with the following properties. There exist orthonormal vectors $v_1,\dots,v_n \in \mathbb{R}^n$ which determine a coordinate system in $\mathbb{R}^n$ and a Lipschitz function $\phi_x\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\phi_x(0) = 0$ and

$$B(x,r_x)\cap\partial\Omega = \Big\{x + \sum_{i=1}^n \xi_i v_i \colon \xi \in B(0,r_x), \xi_n = \phi_x(\xi_1,\dots,\xi_{n-1})\Big\}\,,$$

$$B(x,r_x)\cap\Omega = \Big\{x + \sum_{i=1}^n \xi_i v_i \colon \xi \in B(0,r_x), \xi_n < \phi_x(\xi_1,\dots,\xi_{n-1})\Big\}\,,$$

that is, the boundary is a Lipschitz graph and the domain lies on one side of the graph. Such a choice of a coordinate system and a Lipschitz function is possible since Ω has a Lipschitz boundary.

We denote by L a uniform Lipschitz constant for all the functions ϕ_x, $x \in \partial\Omega$ (this exists since $\partial\Omega$ is compact). Let $\gamma = 1/(2\sqrt{1+L^2})$. By construction, $\{B(x,\gamma r_x/2)\}_{x\in\overline{\Omega}}$ is an open cover of $\overline{\Omega}$. Since $\overline{\Omega}$ is compact we may choose a finite subcover $(B(x_\ell,\gamma r_\ell/2))_{\ell=0,\dots,M}$ of $\overline{\Omega}$. Moreover, the finitely many balls $B_\ell = B(x_\ell,\gamma r_\ell/2)$ satisfy

$$\alpha = \min\big\{|B_i\cap B_j\cap\Omega| : i,\, j \in \{0,\dots,M\}\,, B_i\cap B_j\cap\Omega \ne \emptyset\big\} > 0\,. \tag{2.7}$$

All constants are allowed to depend on the smallest radius of the balls in the covering.

Step 3: Interior estimate. Let $N = N(z)$ denote the fundamental solution for the Laplace operator $-\Delta$. For any $\psi \in C_c^\infty(\mathbb{R}^n)$ and $i,\ j \in \{1,\dots,n\}$ the function $D_i N * \psi = N * D_i\psi$ satisfies $N * D_i\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the partial derivative with respect to x_j can be represented by

$$D_j\big(N * D_i\psi\big)(x) = \big(T_{ij}\psi\big)(x) - \frac{\delta_{ij}}{n}\,\psi(x)$$

where

$$\big(T_{ij}\psi\big)(x) = \lim_{\varepsilon\to 0}\Big[\int_{\mathbb{R}^n\setminus B(0,\varepsilon)} D_iD_jN(y)\psi(x-y)\,\mathrm{d}y\Big]\,.$$

Classical results on singular integrals [24, Theorem 2 in Section 3.2] ensure that the limit on the right-hand side exists in L^p for all $p \in (1,\infty)$ and that the operator T_{ij} can be extended to a bounded operator from $L^p(\mathbb{R}^n)$ to itself. In particular there exists a constant A_p which depends only on p such that

$$\|T_{ij}f\|_p \le A_p\|f\|_p \qquad \text{for all } f \in L^p(\mathbb{R}^n)\,.$$

Analogously we define for a vector field $\psi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ the function $u_\psi = \sum_{i=1}^n D_iN * \psi^{(i)}$ which is a weak solution of the equation $-\Delta u = \operatorname{div}\psi$ in $\mathbb{R}^n$. Again, this definition can be extended by approximation to vector fields $f \in L^p(\mathbb{R}^n;\mathbb{R}^n)$ and one obtains the corresponding function $u_f = \sum_{i=1}^n D_iN * f^{(i)}$ which satisfies $u_f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $\|Du_f\|_p \le A_p\|f\|_p$ and which is a weak solution of $-\Delta u = \operatorname{div} f$ in the sense that

$$\int_{\mathbb{R}^n} u_f\Delta\phi\,\mathrm{d}x = \int_{\mathbb{R}^n} f\cdot D\phi\,\mathrm{d}x \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n)\,. \tag{2.8}$$

After these preparations we proceed with the local estimate. Fix a ball $B(x_\ell, r_\ell)$, $\ell \in \{0,\dots,M\}$, with $x_\ell \in \Omega$ and define in view of (2.1) and (2.6) the vector field $u_f = (u_f^{(1)}\ \dots, u_f^{(n)})$ by

$$u_f^{(i)} = -\sum_{j=1}^n D_jN * \big((2f_{ij} - \delta_{ij}(\operatorname{Tr} f))\chi_{B(x_\ell,r_\ell)}\big)\,. \tag{2.9}$$

Analogously we set

$$u_g^{(i)} = -\sum_{j=1}^n D_jN * \big((2g_{ij} - \delta_{ij}(\operatorname{Tr} g))\chi_{B(x_\ell,r_\ell)}\big)\,.$$

Then $\|Du_f\|_p \le A_p\|f\|_p$ and $\|Du_g\|_p \le A_p\|g\|_p$. Moreover, u_f and u_g are locally weak solutions of (2.8) in the sense that we have for all $\phi \in C_c^\infty(B(x_\ell, r_\ell))$ the identities

$$-\int_{\mathbb{R}^n} u_f^{(i)}\Delta\phi\,\mathrm{d}x = \int_{\mathbb{R}^n}\sum_{j=1}^n(2f_{ij} - \delta_{ij}(\operatorname{Tr} f))D_j\phi\,\mathrm{d}x$$

and

$$-\int_{\mathbb{R}^n} u_g^{(i)}\Delta\phi\,\mathrm{d}x = \int_{\mathbb{R}^n}\sum_{j=1}^n(2g_{ij} - \delta_{ij}(\operatorname{Tr} g))D_j\phi\,\mathrm{d}x\,.$$

In view of (2.1) and (2.6) we see that the function $w = u - u_f - u_g$ defines a harmonic distribution on $B(x_\ell, r_\ell)$ which can be identified with a smooth harmonic function by Weyl's lemma. By Sobolev's embedding theorem and Caccioppoli estimates for harmonic functions we infer for the harmonic function Ew that

$$\begin{aligned}\|Ew\|_{L^q(B(x_\ell,r_\ell/2))} &\le c\|Ew\|_{L^p(B(x_\ell,r_\ell))} = c\|Eu - Eu_f - Eu_g\|_{L^p(B(x_\ell,r_\ell))}\\ &\le c\|f+g\|_{L^p(B(x_\ell,r_\ell))} + c\|Du_f\|_{L^p(B(x_\ell,r_\ell))} + c\|Du_g\|_{L^p(B(x_\ell,r_\ell))}\\ &\le c\|f\|_{L^p(B(x_\ell,r_\ell))} + c\|g\|_{L^q(B(x_\ell,r_\ell))}\,.\end{aligned}$$

In this estimate the constant may depend on r and hence on Ω. By Korn's inequality in L^q (see (2.4)) applied to the set $B(x_\ell, r_\ell/2)$ there is a skew-symmetric matrix $S_\ell \in \mathbb{R}^{n\times n}$ such that

$$\|Dw - S_\ell\|_{L^q(B(x_\ell,r_\ell/2))} \le c\|Ew\|_{L^q(B(x_\ell,r_\ell/2))}\,.$$

Setting $F = Du_f$ and $G = Du_g + Dw - S_\ell$ we conclude

$$Du = F + G + S_\ell \qquad a.e. \text{ in } B(x_\ell, r_\ell/2)$$

with

$$\|F\|_{L^p(B(x_\ell,r_\ell/2))} \le c\|f\|_{L^p(B(x_\ell,r_\ell))}\,, \quad \|G\|_{L^q(B(x_\ell,r_\ell/2))} \le c\|g\|_{L^q(B(x_\ell,r_\ell))} + c\|f\|_{L^p(B(x_\ell,r_\ell))}\,.$$

The additional term in the estimate for G will be replaced in the global estimate by (2.5).

Step 4: Local estimate at the boundary. Fix a ball $B(x_\ell, r_\ell)$, $\ell \in \{0,\dots,M\}$, with $x_\ell \in \partial\Omega$. By construction, $B(x_\ell, r_\ell)$ satisfies the hypotheses of the extension theorem (Theorem A.1 below) and therefore u, f and g have extensions $\widetilde{u}$, $\widetilde{f}$ and $\widetilde{g}$ which are defined on $\Omega \cup B(x_\ell, \gamma r_\ell)$ with $E\widetilde{u} = \widetilde{f} + \widetilde{g}$ a.e. on $B(x_\ell, \gamma r_\ell)$. Moreover, these functions satisfy the estimates (A.2). We define $\widetilde{u}_f$ and $\widetilde{u}_g$ as in Step 3 and proceed as before to obtain S_ℓ, $\widetilde{F}$, $\widetilde{G}$ such that

$$D\widetilde{u} = \widetilde{F} + \widetilde{G} + S_\ell\,, \qquad a.e. \text{ in } B(x_\ell, \gamma r_\ell/2)$$

and

$$\begin{aligned}
\|\widetilde{F}\|_{L^p(B(x_\ell,\gamma r_\ell/2))} &\le c\|\widetilde{f}\|_{L^p(B(x_\ell,\gamma r_\ell))} \le c\|f\|_{L^p(B(x_\ell,r_\ell)\cap\Omega)}\,,\\
\|\widetilde{G}\|_{L^q(B(x_\ell,\gamma r_\ell/2))} &\le c\|\widetilde{g}\|_{L^q(B(x_\ell,\gamma r_\ell))} + c\|\widetilde{f}\|_{L^p(B(x_\ell,\gamma r_\ell))}\\
&\le c\|g\|_{L^q(B(x_\ell,r_\ell)\cap\Omega)} + c\|f\|_{L^p(B(x_\ell,r_\ell)\cap\Omega)}\,.
\end{aligned}$$

We define F and G to be the restrictions of $\widetilde{F}$ and $\widetilde{G}$ to $B(x_\ell, \gamma r_\ell/2) \cap \Omega$.

Step 5: Global estimate. Let S_i, F_i, G_i, $i = 0,\dots,M$, be the matrices and fields in the balls $B(x_i, \gamma r_i/2)$ which were constructed in Steps 3 and 4, respectively. In order to prove the assertion, we need to verify that we may choose S_0 globally in Ω. Therefore we estimate $|S_i - S_j|$. If $B_i \cap B_j \cap \Omega \ne \emptyset$, then recalling (2.7) we obtain

$$\begin{aligned}
\alpha|S_i - S_j| &\le \int_{B_i\cap B_j\cap\Omega} |S_i - S_j|\,\mathrm{d}y\\
&\le \int_{B_i\cap\Omega} |S_i - Du|\,\mathrm{d}y + \int_{B_j\cap\Omega} |S_j - Du|\,\mathrm{d}y \le c\|f\|_{L^p(\Omega)} + c\|g\|_{L^q(\Omega)}\,.
\end{aligned}$$

Since Ω is connected and the subcover consists of finite number of balls, we infer

$$|S_i - S_0| \le c\|f\|_{L^p(\Omega)} + c\|g\|_{L^q(\Omega)}$$

for all $i = 1,\dots,M$ where c depends only on p, q, n and Ω. We define inductively a family A_i, $i = 0,\dots,M$, of pairwise disjoint and measurable sets by $A_0 = B_0 \cap \Omega$ and

$$A_i = \Big(B_i \setminus \bigcup_{j=0}^{i-1} A_j\Big) \cap \Omega\,, \quad i \ge 1\,,$$

which covers Ω up to a set of measure zero and set

$$F = \sum_{i=0}^{M} F_i \chi_{A_i}\,, \qquad G = \sum_{i=0}^{M} (G_i + S_i - S_0)\chi_{A_i}\,.$$

We obtain the decomposition (2.2) and the corresponding estimates, i.e.,

$$Du - S_0 = F + G\,, \quad \|F\|_{L^p(\Omega)} \le c\|f\|_{L^p(\Omega)}\,, \quad \|G\|_{L^q(\Omega)} \le c\|g\|_{L^q(\Omega)} + c\|f\|_{L^p(\Omega)}\,.$$

The assertion follows from (2.5). The proof is now complete. ☐

3. Nonlinear estimate: geometric rigidity and proof of Theorem 1.1

In this section we prove Theorem 1.1. We start from the case that u is globally Lipschitz continuous. The general case will be reduced to this situation via a truncation argument which provides a Lipschitz constant which depends only on n and Ω.

Lemma 3.1. *Theorem* 1.1 *holds under the additional assumption that there exits a constant* $M > n$ *so that* $|Du| \le M$ *a.e. The constant in the estimates (*1.2*) depends on* M.

Proof. The proof is based on a suitable linearization at the identity. In order to control the higher order terms we first assume that q is not much larger than p. In the following, all constants may depend on n, p, q, M, and Ω.

We may assume that $0 \le f, g \le 2M$. Indeed, if this is not the case we replace f and g by $f' = \min\{|f|, 2M\}$ and $g' = \min\{|g|, 2M\}$, respectively, and estimate $\mathrm{dist}(Du, \mathrm{SO}(n)) \le \min\{f + g, |Du| + \sqrt{n}\} \le \min\{|f| + |g|, M + \sqrt{n}\} \le f' + g'$.

Step 1. Small q. Assume first that $q \le 2p$. We observe that the assumption $|f| \le 2M$ implies $\|f\|_q \le c\|f\|_p^{p/q}$. By the rigidity estimate in L^q, see [8, Th. 3.1] and [3, Sect. 2.4], there exists a $Q \in \mathrm{SO}(n)$ such that in view of (1.1)

$$\|Du - Q\|_q \le c\|\mathrm{dist}(Du, \mathrm{SO}(n))\|_q \le c\|f\|_q + c\|g\|_q \le c\|f\|_p^{p/q} + c\|g\|_q\,. \tag{3.1}$$

If $\|f\|_p^p \le \|g\|_q^q$, then the assertion follows with $F = 0$ and $G = Du - Q$. We may thus assume that

$$\|f\|_p^p \ge \|g\|_q^q \tag{3.2}$$

and compose u with a rotation so that $Q = \mathrm{Id}$. We expand the distance to $\mathrm{SO}(n)$ in the identity matrix and obtain the pointwise estimate

$$|Eu - \mathrm{Id}| \le c\,\mathrm{dist}(Du, \mathrm{SO}(n)) + c|Du - \mathrm{Id}|^2 \qquad a.e. \text{ in } \Omega\,. \tag{3.3}$$

Observe that in view of the L^∞ bound on $|Du|$ and the condition $q \le 2p$

$$\big\||Du - \mathrm{Id}|^2\big\|_p^p = \int_\Omega |Du - \mathrm{Id}|^{2p}\,\mathrm{d}x \le c\int_\Omega |Du - \mathrm{Id}|^q\,\mathrm{d}x\,, \tag{3.4}$$

and that (3.4), (3.1), and (3.2) imply

$$\big\||Du - \mathrm{Id}|^2\big\|_p^p \le c\|f\|_p^p + c\|g\|_q^q \le c\|f\|_p^p\,.$$

Let $\widetilde{f} = f + |Du - \mathrm{Id}|^2$, $\widetilde{g} = g$. By the foregoing estimate, $\|\widetilde{f}\|_p \le c\|f\|_p$, and (3.3) gives

$$|Eu - \mathrm{Id}| \le c\widetilde{f} + \widetilde{g}\,.$$

The assertion follows now from Theorem 2.1. Indeed, there exists a skew-symmetric matrix S and matrix fields $\widetilde{F}$ and $\widetilde{G}$ such that $Du - \mathrm{Id} = S + \widetilde{F} + \widetilde{G}$ with $\|\widetilde{F}\|_p \le c\|\widetilde{f}\|_p \le c\|f\|_p$ and $\|\widetilde{G}\|_q \le c\|\widetilde{g}\|_q = c\|g\|_q$. Let $Q \in \mathrm{SO}(n)$ be a proper rotation such that $|\mathrm{Id} + S - Q| = \mathrm{dist}(\mathrm{Id} + S, \mathrm{SO}(n))$. Then

$$|\mathrm{Id} + S - Q| \le |\mathrm{Id} + S - Du| + \mathrm{dist}(Du, \mathrm{SO}(n)) \le |\widetilde{F}| + |\widetilde{G}| + |f| + |g|\,.$$

Combining with the previous estimates we obtain $|\mathrm{Id} + S - Q| \le c\|f\|_p + c\|g\|_q$. If $\|f\|_p \le \|g\|_q$ we set $F = \widetilde{F}$, $G = \widetilde{G} + Q - \mathrm{Id} - S$, otherwise $F = \widetilde{F} + Q - \mathrm{Id} - S$, $G = \widetilde{G}$.

Step 2: Large q. Let

$$\Lambda_k = \{(p, q) \in (1, \infty)^2 : 2^k p < q \le 2^{k+1}p\}\,.$$

We shall prove by induction on $k \in \mathbb{N}$ the following assertion. For every $(p, q) \in \Lambda_k$ and every u as in the statement there exist a rotation $Q \in \mathrm{SO}(n)$ and matrix fields $F \in L^p(\Omega; \mathbb{R}^{n\times n})$ and $G \in L^q(\Omega; \mathbb{R}^{n\times n})$ such that $Du = Q + F + G$ a.e.,

$$\|F\|_{L^p(\Omega)} \le c_k\|f\|_{L^p(\Omega)}\,, \qquad \|G\|_{L^q(\Omega)} \le c_k\|g\|_{L^q(\Omega)}\,, \tag{3.5}$$

and

$$|F| \leq M + \sqrt{n}\,, \qquad |G| \leq M + \sqrt{n}\,, \tag{3.6}$$

where the constant c_k depends only on n, p, q, M, k and Ω. Notice that it suffices to prove (3.5), then (3.6) follows. Indeed, let $A = \{x \in \Omega\colon |F| > M + \sqrt{n}\}$ and $B = \{x \in \Omega\colon |G| > M + \sqrt{n}\}$. Then it suffices to replace F and G by

$$F' = \chi_{\Omega\setminus A} F + \chi_A (Du - Q)\,, \qquad G' = \chi_{\Omega\setminus(A\cup B)} G + \chi_{B\setminus A}(Du - Q)\,,$$

respectively.

The case $k = 0$ has been verified in Step 1. Suppose thus that the assertion has been proven for $k \geq 0$ and that $(p, q) \in \Lambda_{k+1}$. Then $(2p, q) \in \Lambda_k$ and by assumption there exist a rotation Q and matrix fields F and G which satisfy $Du = Q + F + G$ and the estimate (3.5), that is,

$$\|F\|_{2p} \leq c\|f\|_{2p} \leq c\|f\|_p^{1/2}\,, \qquad \|G\|_q \leq c\|g\|_q\,. \tag{3.7}$$

As above, we may assume that $Q = \mathrm{Id}$ and use the Taylor series (3.3) to obtain the estimates

$$|Eu - \mathrm{Id}| \leq cf + cg + c|F|^2 + c|G|^2 \qquad a.e. \text{ in } \Omega$$

and in view of (3.6) and (3.7)

$$\|\,|F|^2\,\|_p = \|F\|_{2p}^2 \leq c\|f\|_p\,, \qquad \|\,|G|^2\,\|_q \leq c\|G\|_q \leq c\|g\|_q\,.$$

Therefore the assertion follows from Theorem 2.1 with $\widetilde{f} = f + |F|^2$, $\widetilde{g} = g + |G|^2$. □

In order to prove Theorem 1.1 we make use of a well-known truncation result.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $m \geq 1$. Then there is a constant $c_1 = c_1(\Omega) \geq 1$ such that for all $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ and all $\lambda > 0$ there exists a measurable set $E \subset \Omega$ such that:*

(i) *u is $c_1\lambda$-Lipschitz on E;*

(ii) $|\Omega \setminus E| \leq \dfrac{c_1}{\lambda} \displaystyle\int_{\{|Du|>\lambda\}} |Du|\,\mathrm{d}x.$

Proof. This result corresponds essentially to Proposition A.1 in [8] and follows from the same proof. The techniques are analogous to the proof in the case $\Omega = \mathbb{R}^n$ in Section 6.6.3 in [4]. □

Proof of Theorem 1.1. Suppose first that $\|g\|_q \leq \|f\|_p$. Then by (1.1)

$$\|\mathrm{dist}(Du, \mathrm{SO}(n))\|_p \leq \|g\|_p + \|f\|_p \leq c\|g\|_q + \|f\|_p \leq (c+1)\|f\|_p$$

and the result follows from the geometric rigidity estimates in L^p with $F = Du - Q$ and $G = 0$ for a suitable $Q \in \mathrm{SO}(n)$.

We may thus assume that $\|g\|_q \geq \|f\|_p$. In order to apply Lemma 3.1 we choose in Theorem 3.2 $\lambda = 2n$ and obtain a measurable set E such that u is Lipschitz continuous on E with Lipschitz constant $M = 2nc_1$. Let u_M be a Lipschitz-extension of $u|_E$ to Ω with the same Lipschitz constant which exists according to Kirszbraun's Theorem [5, Section 2.10.43]. Then u_M is M-Lipschitz and $u_M = u$ on E. We define

$$f_M = f \text{ and } g_M = g + 2M\chi_{\Omega\setminus E} \text{ if } \|f\|_p^p \leq \|g\|_q^q$$

and

$$f_M = f + 2M\chi_{\Omega\setminus E} \text{ and } g_M = g \text{ if } \|g\|_q^q < \|f\|_p^p$$

and assert that

$$\mathrm{dist}(Du_M, \mathrm{SO}(n)) \leq f_M + g_M \text{ a.e. in } \Omega\,. \tag{3.8}$$

For almost every $x \in E$ we have $Du_M = Du$, $f_M = f$ and $g_M = g$, hence (3.8) holds. For almost every $x \in \Omega \setminus E$ we have

$$\mathrm{dist}(Du_M, \mathrm{SO}(n))(x) \leq |M| + \sqrt{n} \leq 2M \leq 2M\chi_{\Omega\setminus E}(x) \leq (f_M + g_M)(x)\,.$$

This proves (3.8).

We now assert that

$$\|f_M\|_p \le C(\Omega,p)\|f\|_p\,, \text{ and } \|g_M\|_q \le C(\Omega,q)\|g\|_q\,. \tag{3.9}$$

To see this we first estimate $|\Omega\setminus E|$. We have for $A\in\mathbb{R}^{n\times n}$ with $|A|\ge 2n$

$$|A| \le \sqrt{n} + \operatorname{dist}(A,\mathrm{SO}(n)) \le 2\operatorname{dist}(A,\mathrm{SO}(n))\,.$$

Therefore

$$|Du| \le \max\{4f,4g\} \text{ if } |Du|\ge\lambda = 2n\,.$$

Together with Theorem 3.2(ii) this yields

$$\begin{aligned}|\Omega\setminus E| &\le \frac{c_1}{\lambda}\int_{\{|Du|>\lambda\}}|Du|\,\mathrm{d}x\\ &\le \frac{c_1}{\lambda}\int_{\{4f\ge\lambda\}}4f\,\mathrm{d}x + \frac{c_1}{\lambda}\int_{\{4g\ge\lambda\}}4g\,\mathrm{d}x\\ &\le \frac{4^p c_1}{\lambda^p}\int_\Omega f^p\,\mathrm{d}x + \frac{4^q c_1}{\lambda^q}\int_\Omega g^q\,\mathrm{d}x \le c\left(\|f\|_p^p + \|g\|_q^q\right)\,.\end{aligned}$$

If $\|f\|_p^p\le\|g\|_q^q$ then $\|2M\chi_{\Omega\setminus E}\|_q^q \le (2M)^q 2c\|g\|_q^q$ and therefore $\|g_M\|_q\le c\|g\|_q$. The other case is analogous. This concludes the proof of (3.9).

It follows from (3.8), (3.9) and Lemma 3.1 that there exist an $R\in\mathrm{SO}(n)$ and matrix fields F_M, G_M such that

$$Du_M - R = F_M + G_M,\ \ \|F_M\|_p\le c\|f_M\|_p\le c\|f\|_p\,,\ \ \|G_M\|_q\le c\|g_M\|_q\le c\|g\|_q\,. \tag{3.10}$$

Now $|Du-Du_M|\le|Du|+M\le\operatorname{dist}(Du,\mathrm{SO}(n))+\sqrt{n}+M$ almost everywhere on Ω and $Du=Du_M$ almost everywhere on E. Thus $|Du-Du_M|\le f_M+g_M$ and $|Du-R|\le f_M+|F_M|+|G_M|+g_M$ and the assertion follows from (3.9), (3.10) and (1.3). □

4. Applications and extensions

4.1. **Estimates in Lorentz spaces.** As an application of our estimates in Section 3 we present rigidity results in the Lorentz spaces $L^{p,q}$ for $p\in(1,\infty)$, $q\in[1,\infty]$, see [21, 11, 2, 19, 25]. For $q=\infty$ they coincide with the weak L^p spaces. The Lorentz space $L^{p,q}(\Omega)$ is equal to the real interpolation space which is constructed with the K-functional,

$$L^{p,q}(\Omega) = \left(L^{p_1}(\Omega), L^{p_2}(\Omega)\right)_{\theta,q}$$

where the K functional is given by

$$K(w,t) = \inf\left\{\|f\|_{p_1} + t\|g\|_{p_2} : w = f+g,\ f\in L^{p_1},\ g\in L^{p_2}\right\}$$

and

$$1\le p_1<p_2\le\infty,\ 1\le q\le\infty,\ \theta\in(0,1),\ \frac{1}{p}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}\,. \tag{4.1}$$

The norm is given for $q<\infty$ by

$$\|w\|_{p,q} = \left(\int_{(0,\infty)}(t^{-\theta}K(w,t))^q\frac{\mathrm{d}t}{t}\right)^{1/q},$$

in the special case $q=\infty$ by

$$\|w\|_{\theta,\infty} = \sup_{t>0} t^{-\theta}K(w,t)\,.$$

We remark that different choices of θ, p_1, p_2 which satisfy (4.1) give equivalent norms. In this framework we obtain the following result.

Corollary 4.1. *Suppose that $p \in (1,\infty)$, $q \in [1,\infty]$. Then there exists a constant c which only depends on p, n and Ω such that the following assertion is true. If $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ with $\operatorname{dist}(Du,\mathrm{SO}(n)) \in L^{p,q}(\Omega;\mathbb{R}^{n\times n})$, then there exists a rotation $Q \in \mathrm{SO}(n)$ such that*

$$\|Du - Q\|_{p,q} \le c\|\operatorname{dist}(Du,\mathrm{SO}(n))\|_{p,q}\,.$$

Proof. Fix any triple θ, p_1, p_2 with $1 < p_1 < p_2 < \infty$ which satisfies (4.1) and set $w = \operatorname{dist}\big(Du,\mathrm{SO}(n)\big)$. By assumption, $K(w,t) < \infty$ for almost all t. Hence there exists for almost all $t > 0$ a decomposition $w = f_t + g_t$ with $f_t \in L^{p_1}(\Omega)$, $g_t \in L^{p_2}(\Omega)$ and

$$K(w,t) = \|f_t\|_{p_1} + t\|g_t\|_{p_2}\,.$$

In view of the rigidity estimate in $L^{p_1} + L^{p_2}$, Theorem 1.1, there exist a rotation $Q_t \in \mathrm{SO}(n)$ and matrix fields $F_t \in L^{p_1}(\Omega;\mathbb{R}^{n\times n})$, $G_t \in L^{p_2}(\Omega;\mathbb{R}^{n\times n})$ with

$$Du = Q_t + F_t + G_t\,,\quad \|F_t\|_{p_1} \le c\|f_t\|_{p_1}\,,\quad \|G_t\|_{p_2} \le c\|g_t\|_{p_2}\,.$$

We define

$$F_t' = F_t - \frac{1}{|\Omega|}\int_\Omega F_t\,\mathrm{d}x\,,\quad G_t' = G_t - \frac{1}{|\Omega|}\int_\Omega G_t\,\mathrm{d}x\,,\quad R = \frac{1}{|\Omega|}\int_\Omega Du\,\mathrm{d}x$$

and obtain

$$Du = R + F_t' + G_t'\,,\quad \|F_t'\|_{p_1} \le 2\|F_t\|_{p_1}\,,\quad \|G_t'\|_{p_2} \le 2\|G_t\|_{p_2}\,.$$

Therefore for almost all $t > 0$ one has

$$Du = R + F_t' + G_t'\,,\quad \|F_t'\|_{p_1} + t\|G_t'\|_{p_2} \le c\|f_t\|_{p_1} + ct\|g_t\|_{p_2} = cK(w,t)\,,$$

which implies $K(Du - R,t) \le cK(w,t)$ for almost all t. We stress that the constant does not depend on t but only on p_1, p_2, Ω. We conclude that

$$\|Du - R\|_{p,q} \le c\|w\|_{p,q}\,.$$

From $\operatorname{dist}(R,\mathrm{SO}(n)) \le \operatorname{dist}(Du,\mathrm{SO}(n)) + |Du - R|$ we obtain that there is a rotation Q with $|Q - R| \le c\|w\|_{p,q}$, and conclude

$$\|Du - Q\|_{p,q} \le \|Du - R\|_{p,q} + \|Q - R\|_{p,q} \le c\|\operatorname{dist}(Du,\mathrm{SO}(n))\|_{p,q}\,.$$

This concludes the proof. □

4.2. **Equiintegrability.** As a second application of our work we show that equiintegrability of the distance from the set of rotations implies equiintegrability of the distance from a fixed rotation. A sequence $f_k \in L^p(\Omega)$, $k \in \mathbb{N}$, is L^p-equiintegrable if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all measurable sets $E \subset \Omega$ with $|E| < \delta$ one has $\int_E |f_k|^p\,\mathrm{d}x \le \varepsilon$ for all $k \in \mathbb{N}$. For bounded sets $\Omega \subset \mathbb{R}^n$ this is equivalent to the fact that

$$\lim_{T\to\infty}\sup_{k\in\mathbb{N}}\int_{\{|f_k|>T\}} |f_k|^p\,\mathrm{d}x = 0\,, \tag{4.2}$$

see for example [6, Theorem 2.29, page 151] for a proof.

The following statement generalizes the assertion in [9, page 221] concerning the interplay of equiintegrability and rigidity.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a connected and bounded Lipschitz set. Consider a sequence of positive numbers $\eta_k \in (0,\infty)$ and a sequence of functions $u_k \in W^{1,p}(\Omega;\mathbb{R}^n)$. Assume that the sequence*

$$d_k = \eta_k\operatorname{dist}(Du_k,\mathrm{SO}(n))$$

is L^p-equiintegrable. Then there are rotations $Q_k \in \mathrm{SO}(n)$ such that

$$z_k = \eta_k\,(Du_k - Q_k)$$

is L^p-equiintegrable.

The corresponding linear result can be proven exactly in the same way, using Theorem 2.1 instead of Theorem 1.1. We further remark that for bounded sequences (η_k) the statement follows immediately from the boundedness of $\mathrm{SO}(n)$, the interesting case is $\eta_k \to \infty$.

Proof. We shall use the characterization of equiintegrability given in (4.2). Pick some $\varepsilon > 0$. Then there is T_ε such that

$$\int_{\{d_k > T_\varepsilon\}} d_k^p \,\mathrm{d}x \le \varepsilon$$

for all k. We define

$$f_k = \mathrm{dist}(Du_k, \mathrm{SO}(n))\chi_{\{d_k > T_\varepsilon\}} \text{ and } g_k = \mathrm{dist}(Du_k, \mathrm{SO}(n))\chi_{\{d_k \le T_\varepsilon\}}$$

and observe that $\|\eta_k f_k\|_p^p \le \varepsilon$ and $\|\eta_k g_k\|_\infty \le T_\varepsilon$. We fix some $q \in (p, \infty)$, for example $q = p+1$, and estimate

$$\|\eta_k g_k\|_q^q \le T_\varepsilon^{q-p} \|d_k\|_p^p \le M T_\varepsilon^{q-p}\,,$$

where $M = \sup_k \|d_k\|_p^p$. Notice that $M < \infty$, since L^p-equiintegrable sequences are bounded in L^p.

By Theorem 1.1 applied to each u_k there are rotations Q_k and fields F_k, G_k such that

$$Du_k = Q_k + F_k + G_k\,, \qquad \|\eta_k F_k\|_p^p \le c\|\eta_k f_k\|_p^p \le c\varepsilon\,, \qquad \|\eta_k G_k\|_q^q \le c\|\eta_k g_k\|_q^q\,.$$

We set $E_k = \{\eta_k |G_k| > L_\varepsilon\}$ for some $L_\varepsilon > 0$ to be chosen later, estimate its measure

$$|E_k| \le \frac{1}{L_\varepsilon^q} \int_{E_k} |\eta_k G_k|^q \,\mathrm{d}x \le \frac{c}{L_\varepsilon^q} \|\eta_k g_k\|_q^q$$

and the L^p norm on E_k via

$$\|\eta_k G_k\|_{L^p(E_k)} \le |E_k|^{1/p - 1/q} \|\eta_k G_k\|_{L^q(E_k)}\,.$$

We conclude that

$$\int_{E_k} |\eta_k G_k|^p \,\mathrm{d}x \le |E_k|^{(q-p)/q} \|\eta_k g_k\|_q^p \le \frac{c}{L_\varepsilon^{q-p}} \|\eta_k g_k\|_q^q \le cM \frac{T_\varepsilon^{q-p}}{L_\varepsilon^{q-p}}\,.$$

We finally choose $L_\varepsilon = T_\varepsilon / \varepsilon^{1/(q-p)}$, so that the last fraction is smaller than ε and obtain for $z_k = \eta_k(Du_k - Q_k) = \eta_k F_k + \eta_k G_k$, setting $E_k' = \{|z_k| > 2L_\varepsilon\}$,

$$\begin{aligned}\int_{E_k'} |z_k|^p \,\mathrm{d}x &= \int_{E_k' \cap \{|F_k| \ge |G_k|\}} |z_k|^p \,\mathrm{d}x + \int_{E_k' \cap \{|F_k| < |G_k|\}} |z_k|^p \,\mathrm{d}x \\ &\le 2^p \|\eta_k F_k\|_p^p + 2^p \int_{E_k} |\eta_k G_k|^p \,\mathrm{d}x \le c\varepsilon + cM\varepsilon\,.\end{aligned}$$

Therefore for all $t > 2L_\varepsilon$ and all k we have $\int_{\{|z_k| > t\}} |z_k|^p \,\mathrm{d}x \le c(1+M)\varepsilon$, and (4.2) is proven. ☐

4.3. **Multiple exponents.** In this section we generalize our results to the case of more than two exponents. We first establish the linear estimate which is parallel to Theorem 2.1 and then indicate how this estimate implies the corresponding nonlinear estimate.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with Lipschitz boundary. Suppose that $1 < p_1 < p_2 < \ldots < p_N < \infty$, $N \ge 1$, and that $u \in W^{1,1}(\Omega; \mathbb{R}^n)$, $f_\alpha \in L^{p_\alpha}(\Omega; \mathbb{R}^{n\times n})$, $\alpha = 1, \ldots, N$, are given with*

$$Eu = \frac{1}{2}\big(Du + Du^T\big) = \sum_{\alpha=1}^N f_\alpha \qquad \text{a.e. in } \Omega\,.$$

Then there exists a constant c, matrix fields $F_\alpha \in L^{p_\alpha}(\Omega; \mathbb{R}^{n\times n})$, $\alpha = 1, \ldots, N$, and a skew-symmetric matrix $S \in \mathbb{R}^{n\times n}$ such that

$$Du = S + \sum_{\alpha=1}^N F_\alpha \qquad \text{a.e. in } \Omega\,,$$

and

$$\|F_\alpha\|_{L^{p_\alpha}(\Omega)} \le c\|f_\alpha\|_{L^{p_\alpha}(\Omega)}\,, \quad \alpha = 1,\dots,N\,.$$

The constant depends only on n, Ω, $p_1, p_2, \dots, p_N$.

Proof. We follow the scheme of the proof of Theorem 2.1. By induction we can assume that the statement has been proven for $N-1$ exponents with $N \ge 3$, the assertion for $N=2$ is the statement in Theorem 2.1. If $\|f_\alpha\|_{p_\alpha} \ge \|f_{\alpha+1}\|_{p_{\alpha+1}}$ for some $\alpha \in \{1,\dots,N-1\}$, then we eliminate the exponent $\alpha+1$, define $\widetilde{f_\alpha} = f_\alpha + f_{\alpha+1}$, observe $\|\widetilde{f_\alpha}\|_{p_\alpha} \le \|f_\alpha\|_{p_\alpha} + c\|f_{\alpha+1}\|_{p_{\alpha+1}} \le c\|f_\alpha\|_{p_\alpha}$ and apply the statement to the remaining $N-1$ exponents and the corresponding $N-1$ functions $f_1, f_2, \dots, f_{\alpha-1}, \widetilde{f_\alpha}, f_{\alpha+2}, \dots, f_N$.

Therefore we may assume that

$$\|f_\alpha\|_{p_\alpha} \le \|f_{\alpha+1}\|_{p_{\alpha+1}} \text{ for all } \alpha \in \{1,\dots,N\}\,.$$

Steps 1 and 2 in the proof of Theorem 2.1 are unchanged. In Step 3 we define vector fields u_α associated to the matrix fields f_α as in (2.9). Then $\|Du_\alpha\|_{p_\alpha} \le A_{p_\alpha}\|f_\alpha\|_{p_\alpha}$, the function $w = u - \sum_{\alpha=1}^N u_\alpha$ is harmonic, and satisfies

$$\begin{aligned}\|Ew\|_{L^{p_N}(B(x_\ell,r_\ell/2))} &\le c\|Ew\|_{L^{p_1}(B(x_\ell,r_\ell))} = c\Big\|Eu - \sum_{\alpha=1}^N Eu_\alpha\Big\|_{L^{p_1}(B(x_\ell,r_\ell))}\\ &\le c\Big\|\sum_{\alpha=1}^N f_\alpha\Big\|_{L^{p_\alpha}(B(x_\ell,r_\ell))} + c\sum_{\alpha=1}^N \|Du_\alpha\|_{L^{p_\alpha}(B(x_\ell,r_\ell))} \le c\sum_{\alpha=1}^N \|f_\alpha\|_{L^{p_\alpha}(B(x_\ell,r_\ell))}\,.\end{aligned}$$

We apply Korn's inequality in L^{p_N} to the ball $B(x_\ell, r_\ell)$, and obtain the analogous decomposition of Du together with the estimates. An extension theorem corresponding to Theorem A.1 holds with N terms and with the same proof, Steps 4 and 5 can be concluded as before with minor notational changes. □

We now turn to the nonlinear estimate.

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with Lipschitz boundary. Suppose that $1 < p_1 < p_2 < \ldots < p_N < \infty$ and that $u \in W^{1,1}(\Omega;\mathbb{R}^n)$, $f_\alpha \in L^{p_\alpha}(\Omega)$, $\alpha = 1,2,\dots,N$ are given with*

$$\mathrm{dist}(Du, \mathrm{SO}(n)) = \sum_{\alpha=1}^N f_\alpha \qquad \text{a.e. in } \Omega\,.$$

Then there exist a constant c, matrix fields $F_\alpha \in L^{p_\alpha}(\Omega;\mathbb{R}^{n\times n})$ and a rotation $Q \in \mathrm{SO}(n)$ such that

$$Du = Q + \sum_{\alpha=1}^N F_\alpha \qquad \text{a.e. in } \Omega\,,$$

and

$$\|F_\alpha\|_{L^{p_\alpha}(\Omega;\mathbb{R}^{n\times n})} \le c\|f_\alpha\|_{L^{p_\alpha}(\Omega)}\,, \quad \alpha = 1,2,\dots N\,. \tag{4.3}$$

The constant c depends only on n, $p_1, p_2, \dots, p_N$ and Ω but not on u and $f_1, f_2, \dots, f_N$.

As above, we start from the case that u is Lipschitz.

Lemma 4.5. *Theorem* 4.4 *holds under the additional assumption that there exists a constant $M > n$ so that $|Du| \le M$ a.e. The constant in the estimate* (4.3) *depends on M.*

Proof. As above, we may assume $0 \le f_\alpha \le 2M$, and proceed by induction on N.

Step 1: $p_N \le 2p_1$. We first reduce to the case

$$\|f_\alpha\|_{p_\alpha}^{p_\alpha} \ge \|f_{\alpha+1}\|_{p_{\alpha+1}}^{p_{\alpha+1}}\,, \quad \alpha = 1,2,\dots,N-1\,. \tag{4.4}$$

If this does not hold for some α, we set $\widetilde{f}_{\alpha+1} = f_\alpha + f_{\alpha+1}$, estimate $\|\widetilde{f}_{\alpha+1}\|_{p_{\alpha+1}}^{p_{\alpha+1}} \le c\|f_{\alpha+1}\|_{p_{\alpha+1}}^{p_{\alpha+1}} + c\|f_\alpha\|_{p_\alpha}^{p_\alpha} \le c\|f_{\alpha+1}\|_{p_{\alpha+1}}^{p_{\alpha+1}}$, and apply the result with the remaining $N-1$ exponents.

By the rigidity in L^{p_N} there is $Q \in \mathrm{SO}(n)$ with

$$\|Du - Q\|_{p_N}^{p_N} \le c\|\mathrm{dist}(Du, \mathrm{SO}(n))\|_{p_N}^{p_N} \le c\sum_{\alpha=1}^{N} \|f_\alpha\|_{p_N}^{p_N} \le c\|f_1\|_{p_1}^{p_1},$$

where we used (4.4) and $\|f_\alpha\|_{p_N}^{p_N} \le (2M)^{p_N - p_\alpha}\|f_\alpha\|_{p_\alpha}^{p_\alpha}$. We reduce to the case $Q = \mathrm{Id}$, expand and estimate

$$\big\||Du - \mathrm{Id}|^2\big\|_{p_1}^{p_1} = \int_\Omega |Du - \mathrm{Id}|^{2p_1}\,\mathrm{d}x \le (2M)^{2p_1 - p_N}\int_\Omega |Du - \mathrm{Id}|^{p_N}\,\mathrm{d}x \le c\|f_1\|_{p_1}^{p_1}.$$

We set $\widetilde{f}_1 = f_1 + |Du - \mathrm{Id}|^2$, $\widetilde{f}_\alpha = f_\alpha$ for $\alpha \ge 2$. Since $|Eu - \mathrm{Id}| \le \sum_\alpha \widetilde{f}_\alpha$ we can conclude using the linear estimate.

Step 2. We define

$$\Lambda_k = \{(p_1, p_2, \ldots, p_N) \in (1,\infty)^N : p_\alpha \le p_{\alpha+1} \text{ for all } \alpha \text{ and } 2^k p_1 < p_N \le 2^{k+1}p_1\}$$

and proceed by induction on k. Assume $(p_1, \ldots, p_N) \in \Lambda_{k+1}$. We define $q_1 = 2p_1$, $q_\alpha = \min\{2p_\alpha, p_N\}$ for $2 \le \alpha < N$, $q_N = p_N$. Then $(q_1, \ldots, q_N) \in \Lambda_k$. Notice that, for all α, $q_\alpha \ge p_\alpha$ hence $\|f_\alpha\|_{q_\alpha}^{q_\alpha} \le (2M)^{q_\alpha - p_\alpha}\|f_\alpha\|_{p_\alpha}^{p_\alpha}$. We apply the estimate for the exponents $(q_1, \ldots, q_N)$, and the same functions $f_1, \ldots, f_N$, and obtain $Du = Q + \sum_\alpha F_\alpha$, with

$$\|F_\alpha\|_{q_\alpha}^{q_\alpha} \le c\|f_\alpha\|_{q_\alpha}^{q_\alpha} \le c\|f_\alpha\|_{p_\alpha}^{p_\alpha}.$$

We reduce to the case $Q = \mathrm{Id}$ and estimate

$$|Eu - \mathrm{Id}| \le c\sum_{\alpha=1}^{N} f_\alpha + c\sum_{\alpha=1}^{N} |F_\alpha|^2.$$

We estimate, since $q_\alpha \le 2p_\alpha$ for all α,

$$\|\,|F_\alpha|^2\|_{p_\alpha}^{p_\alpha} = \|F_\alpha\|_{2p_\alpha}^{2p_\alpha} \le c\|F_\alpha\|_{q_\alpha}^{q_\alpha} \le c\|f_\alpha\|_{p_\alpha}^{p_\alpha}$$

and conclude with the linear estimate as above. □

Proof of Theorem 4.4. As in the proof of Theorem 4.3, we may assume $\|f_\alpha\|_{p_\alpha} \le \|f_{\alpha+1}\|_{p_{\alpha+1}}$. We choose $\lambda = 2n$ and define u_M as in the proof of Theorem 1.1. Let $\beta \in \{1, \ldots, N\}$ be such that $\|f_\beta\|_{p_\beta}^{p_\beta} \ge \|f_\alpha\|_{p_\alpha}^{p_\alpha}$ for all α. We define

$$f_\beta^M = f_\beta + 2M\chi_{\Omega\setminus E} \text{ and } f_\alpha^M = f_\alpha \text{ for } \alpha \ne \beta.$$

As in the proof of Theorem 1.1, we show that $\mathrm{dist}(Du_M, \mathrm{SO}(n)) \le \sum_\alpha f_\alpha^M$ and that $\|f_\beta^M\|_{p_\beta} \le c\|f_\beta\|_{p_\beta}$. We apply Lemma 4.5 and conclude as in the proof of Theorem 1.1. □

Appendix A. Extension

The subsequent extension theorem for functions with mixed growth follows immediately from the L^2-version in [20]. We include a sketch of the proof for the convenience of the reader.

Theorem A.1. *Let $\varphi \in \mathrm{Lip}(\mathbb{R}^{n-1};\mathbb{R})$ be a Lipschitz function with $\varphi(0) = 0$ and Lipschitz constant L, let $R > 0$ and set $\Omega = B(0,R) \cap \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon x_n < \varphi(x')\}$. Suppose that $1 < p < q < \infty$ and that $u \in W^{1,1}(\Omega;\mathbb{R}^n)$ with*

$$Du + Du^T = f + g\,, \tag{A.1}$$

where $f \in L^p(\Omega;\mathbb{R}^{n\times n})$ and $g \in L^q(\Omega;\mathbb{R}^{n\times n})$. Then there exists for $r = R/(2\sqrt{1+L^2})$ a function $w \in W^{1,1}(B(0,r);\mathbb{R}^n)$, and matrix fields $\widetilde{f}$, $\widetilde{g}$ such that $w = u$, $\widetilde{f} = f$, $\widetilde{g} = g$ on $\Omega \cap B(0,r)$ and

$$Dw + Dw^T = \widetilde{f} + \widetilde{g} \qquad \text{on } B(0,r)$$

with

$$\|\widetilde{f}\|_{L^p(B(0,r))} \le c\|f\|_{L^p(B(0,R)\cap\Omega)}\,, \qquad \|\widetilde{g}\|_{L^q(B(0,r))} \le c\|g\|_{L^q(B(0,R)\cap\Omega)}\,. \tag{A.2}$$

The constant c depends only on n, p, q, Ω but not on u, f, g.

Proof. Let $\delta \in C^2(B(0,R)\setminus\Omega)$ be a function such that

$$2\mathrm{dist}(x,\Omega) \le \delta(x) \le C\mathrm{dist}(x,\Omega)$$

and

$$|D^\alpha\delta(x)| \le C\delta^{1-|\alpha|}(x)\,, \quad \alpha\in\mathbb{N}^n\,, \tag{A.3}$$

see, e.g., [24]. Fix a function $\psi\in C^1(\mathbb{R})$ with

$$\int_1^2 \psi(\lambda)\,\mathrm{d}\lambda = 1\,, \quad \int_1^2 \lambda\psi(\lambda)\,\mathrm{d}\lambda = 0\,. \tag{A.4}$$

We set $w=u$ on Ω and for $x\in B(0,r)\setminus\Omega$ we define

$$w(x) = \int_1^2 \psi(\lambda)\,[u(x-\lambda\delta(x)e_n) - \lambda D\delta(x)u_n(x-\lambda\delta(x)e_n)]\,\mathrm{d}\lambda\,.$$

For ease of notation we omit the arguments in the following calculations and write $\delta=\delta(x)$ and $u = u(x-\lambda\delta(x)e_n)$ with the same convention for their derivatives. By the chain rule

$$\begin{aligned} Dw(x) &= \int_1^2 \psi(\lambda)\left[Du(\mathrm{Id}-\lambda e_n\otimes D\delta) - \lambda D\delta\otimes Du_n(\mathrm{Id}-\lambda e_n\otimes D\delta) - \lambda u_n D^2\delta\right]\mathrm{d}\lambda \\ &= \int_1^2 \psi(\lambda)\left[Du - \lambda D_n u\otimes D\delta - \lambda D\delta\otimes Du_n + \lambda^2 D_n u_n D\delta\otimes D\delta - \lambda u_n D^2\delta\right]\mathrm{d}\lambda\,. \end{aligned}$$

Then the symmetric part of the gradient is given by

$$Ew(x) = \int_1^2 \psi(\lambda)\left[Eu - \lambda(Eue_n)\otimes D\delta - \lambda D\delta\otimes(Eue_n) + \lambda^2(Eu)_{nn}D\delta\otimes D\delta - \lambda u_n D^2\delta\right]\mathrm{d}\lambda\,.$$

In the last term we write

$$u_n(x-\lambda\delta(x)e_n) = u_n(x-\delta(x)e_n) + \int_1^\lambda D_n u_n(x-s\delta(x)e_n)\delta(x)\,\mathrm{d}s\,.$$

In view of the second property in (A.4) the weighted integral of $u_n(x-\delta(x)e_n)$ is equal to zero, and the other term only depends on $(Eu)_{nn}$. We recall (A.1) and define for $x\in B(0,r)\setminus\Omega$

$$\begin{aligned} \widetilde{f}(x) &= \int_1^2 \psi(\lambda)\,[f(x-\lambda\delta(x)e_n) - \lambda(f(x-\lambda\delta(x)e_n)e_n)\otimes D\delta(x)]\,\mathrm{d}\lambda \\ &\quad - \int_1^2 \psi(\lambda)\,[\lambda D\delta(x)\otimes(f(x-\lambda\delta(x)e_n)e_n)]\,\mathrm{d}\lambda \\ &\quad + \int_1^2 \psi(\lambda)\left[\lambda^2 f_{nn}(x-\lambda\delta(x)e_n)D\delta(x)\otimes D\delta(x)\right]\mathrm{d}\lambda \\ &\quad - \int_1^2 \psi(\lambda)\lambda\int_1^\lambda f_{nn}(x-s\delta(x)e_n)\delta(x)\,\mathrm{d}s\,D^2\delta(x)\,\mathrm{d}\lambda\,, \end{aligned}$$

and use the analogous definition for $\widetilde{g}$ in $x\in B(0,r)\setminus\Omega$. On $B(0,r)\cap\Omega$ we set $\widetilde{f}=f$ and $\widetilde{g}=g$. It remains to show that

$$\|\widetilde{f}\|_{L^p(B(0,r)\setminus\Omega)} \le c\|f\|_{L^p(\Omega)}\,, \quad \|\widetilde{g}\|_{L^q(B(0,r)\setminus\Omega)} \le c\|g\|_{L^q(\Omega)}$$

with a constant which only depends on n, p, q and Ω. The calculation is identical to the proof of the estimate for the extension in [20, Lemma 4] . □

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