Almost conical deformations of thin sheets with rotational symmetry

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Abstract

It has been found in numerical experiments [15] that when one removes a sector from an elastic sheet and glues the edges of the sector back together, the resulting configuration is radially symmetric and nearly conical. We make a rigorous analysis of this setting under two simplyfying assumptions: Firstly, we only consider radially symmetric configurations. Secondly, we consider the so-called von-Kármán limit, where the size of the removed region as well as the deformations are small. We choose free boundary conditions for a sheet of infinite size. We show existence of minimizers of the suitably renormalized free energy functional. As a by-product, we obtain a lower bound for the elastic energy that has been conjectured in the related context of *d-cones* [17]. Moreover, we determine the shape of minimizers at infinity up to terms that decay like $\exp(-c\sqrt{r})$.

1 Introduction

The folding of paper or other thin elastic sheets is one of the many examples of energy focusing in the physical world. Starting in the late 90's, there has been a lot of interest in this problem in the physics community [14, 4, 6, 19, 3, 10, 13, 16, 5]. In particular the crumpling of paper (i.e. the crushing of a thin elastic sheet into a container whose diameter is smaller than the size of the sheet) which results in complex folding patterns has drawn a lot of attention. It has been conjectured that the energy density per thickness h of such a folding pattern scales with $h^{5/3}$. One major contribution in the rigorous analysis of this problem is [9], building on ideas from [18].

Here we focus on approximately conical deformations of thin elastic sheets, that can be viewed as (one kind of) building blocks of crumpled deformations. One example for this is a sheet that is pushed into a hollow cylinder, such that the indentation of the sheet is small. The resulting structure is called a *d-cone* (developable cone). In the physics literature, it has been discussed e.g. in [4, 6, 14, 19]. There are several remarkable features of the d-cone, one of which is that the tip of the d-cone consists of a crescent-shaped ridge where curvature and elastic stress focus. In numerical simulations it was found that the radius of the crescent $R_{\text{cres.}}$ scales with the thickness of the sheet h and the radius of the container $R_{\text{cont.}}$ as $R_{\text{cres.}} \sim h^{1/3}R_{\text{cont.}}^{2/3}$. This dependence on the container radius of the shape of the region near the tip is not fully understood [19]. As argued in this latter reference, it cannot be explained by an analysis of the dominant contributions to the elastic energy, which are: The bending energy from the region far away from the center, which is well captured by modeling the d-cone as a developable surface there; and the bending and stretching energy part from a core region of size $O(h)$ where elastic strain is not negligible. The result of this (non-rigorous) argument is an energy scaling $E \sim h^2(C_1|\log h| + C_2)$. This is

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a natural guess – the situation here bears some resemblance to vortices in the Ginzburg-Landau model, where this is the right energy scaling [1].
In [17, 2] the scaling of the elastic energy of a d-cone with respect to its thickness h has been analyzed in a rigorous setting. The result from [17] is

$$h^2(C_1|\log h| - C_2 \log|\log h|) \leq \mathcal{E}_h \leq h^2(C_1|\log h| + C_3)\,. \tag{1}$$

The lower bound does not achieve the conjectured scaling behaviour, and it seems that this claim can not be proved with the methods used in [17].

Here we consider another situation which involves the regularization of an isometric cone through the higher order bending energy. The main difference with the (general) d-cone is that here the underlying cone is a surface of revolution. Hence it is meaningful to study the problem of the competition between bending and stretching energies in a radially symmetric setting. This makes it possible to use ODE methods in addition to energy methods. We will show that a scaling result analogous to the one above without the $\log\log h$ terms on the left hand side holds in this simpler setting.

The setting is the following: to create an approximately conical deformation of an elastic sheet, we cut out a sector of angle β and glue the edges of this sector back together. This situation has been investigated numerically in [15], where it is called "regular cone". In this situation, radially symmetric deformations are admissible – in contrast to the case of the d-cone, where the boundary conditions are not radially symmetric. To the best of our knowledge, it is not known whether the global minimizers of the "regular cone" are radially symmetric. We nonetheless believe that a careful study of minimizers within the class of radially symmetric deformations will help to understand the structure of local and global minimizers as well as the local and global stability of possible radially symmetric minimizers. Since we are interested in the asymptotic behaviour, we consider a sheet of infinite radius with free boundaries (after suitable renormalization of the energy, see below).

Apart from the restriction to radially symmetric configurations, we make one more simplification in comparison to the situation in [17]: We use the so-called von-Kármán approximation of non-linear elasticity [7, 11]. This means that the out-of-plane component of the deformation is supposed to be of the order $\varepsilon \ll 1$, and the size of the removed sector as well as the in-plane deformation are of order ε^2. All terms in the elastic energy of order ε^k, $k > 4$, and of order $h^2\varepsilon^k$, $k > 2$ are discarded.

As we will explain in Section 2, these considerations lead to the definition of the free elastic energy density

$$\rho_\lambda^{\text{el.}} = (\hat{w}^2 - 1 + \hat{u}')^2 + \left(\frac{\hat{u}}{r}\right)^2 + \lambda^2\left(\hat{w}'^2 + \frac{\hat{w}^2}{r^2}\right)$$

where $\lambda = h/\varepsilon$ and the deformation of the sheet is given as a map from spherical to cylindrical coordinates by

$$(r,\varphi) \mapsto \left(r + \frac{\varepsilon^2}{2}(\hat{u} - r), \sqrt{1+\varepsilon^2}\varphi, \varepsilon W\right) \tag{2}$$

with $W' = \hat{w}$. The renormalized energy functional is

$$\begin{array}{rll} \hat{E}_\lambda : & \mathcal{W} & \to \mathbb{R}\cup\{+\infty\} \\ & (\hat{u},\hat{w}) & \mapsto \lim_{R\to\infty}\int_0^R r\mathrm{d}r\left(\rho_\lambda^{\text{el.}}(r) - \lambda^2\frac{\psi(r/\lambda)^2}{r^2}\right) \end{array} \tag{3}$$

where

$$\mathcal{W} = \left\{(\hat{u},\hat{w}) \in W^{1,2}_{\text{loc}}((0,\infty),\mathbb{R}^2) : \int_0^1 r\mathrm{d}r\rho_\lambda^{\text{el.}}(r) < \infty\right\},$$

and ψ is some cutoff function with $\psi(r) = 0$ for r close to 0 and $\psi(r) = 1$ for $r \geq 1$. We will show in Lemma 1 that the condition $\int_0^1 r\mathrm{d}r\rho^{\mathrm{el.}}_\lambda(r) < \infty$ implies $\hat{u}(0) = \hat{w}(0) = 0$ and thus the deformation (2) is continuous at the origin for all $(\hat{u}, \hat{w}) \in \mathcal{W}$.

The aim of the present contribution is to prove

Theorem 1. *The functional $\hat{E}_\lambda$ from eq. (3) is well defined and bounded from below. It possesses minimizers $(\hat{u}, \hat{w})$ in $\mathcal{W}$ with $\hat{w} \geq 0$ and $\hat{E}_\lambda(\hat{u}, \hat{w}) < \infty$. Furthermore, each minimizer $(\hat{u}, \hat{w})$ with $\hat{w} \geq 0$ satisfies*

$$\begin{aligned} \hat{u}(r) =& \frac{\lambda}{2r} + o(\exp(-\sigma\sqrt{r/\lambda})) \\ \hat{w}(r) =& 1 + o(\exp(-\sigma\sqrt{r/\lambda})) \end{aligned}$$

as $r \to \infty$ for any $\sigma < 2$.

As a side product of the proof of Theorem 1, we will get a lower bound for the elastic energy when the radius of the elastic sheet in the reference configuration is assumed to be finite. This lower bound is better than the analogous one from eq. (1) in that the log log-terms are not present. To give an idea how this "improved" lower bound comes about, let

$$I_\lambda = \int_0^1 r\mathrm{d}r\rho^{\mathrm{el.}}_\lambda \,. \tag{4}$$

The first step to establish the lower bound in the present setting is the right renormalization of the elastic energy density. We expect a logarithmic divergence in λ of $\lambda^{-2}I_\lambda$ as $\lambda \to 0$. Thus we make the replacement

$$\rho^{\mathrm{el.}}_\lambda(r) \to \rho^{\mathrm{el.}}_\lambda(r) - \lambda^2\frac{\psi(r/\lambda)^2}{r^2}$$

The key step is now to find a change of variables that makes it obvious that

$$\int_0^1 r\mathrm{d}r \left(\rho^{\mathrm{el.}}_\lambda(r) - \lambda^2\frac{\psi(r/\lambda)^2}{r^2}\right) \tag{5}$$

is bounded from below by some constant times λ^2. As we will see in Section 3, such a change of variables does exist, and will leave us only with manifestly positive terms in the renormalized energy eq. (5) plus some divergence-like term that will be estimated in a suitable manner in Proposition 4. Thus we get the sought-for lower bound

$$\lambda^{-2}I_\lambda \geq |\log\lambda| - C\,. \tag{6}$$

This paper is organised as follows: In Section 2, we motivate and define our model. In Section 3 we establish a lower bound for the renormalized energy and prove the existence of minimizers of the elastic free energy functional. In a remark at the end of that section, we will discuss a pathology of the model presented here. In section 4, we use stable manifold theory to show that minimizers converge to the conical configuration at infinity.

Notation. In this paper, the letter C stands for numerical constants that are independent of all the other variables. Its value may change within the same equation. In section 2, we will choose a cutoff function $\psi \in C^\infty([0,\infty))$, that we have already mentioned above. The cutoff function ψ will then be fixed for the rest of the paper. We will not indicate the dependence of constants on this choice of ψ. Whenever we speak of functions $f \in W^{1,2}_{\mathrm{loc.}}(I)$ for some $I \subset \mathbb{R}$, it will be tacitly understood that we mean its continuous representative.

2 The model

Cutting out a sector, glueing the edges back together. For small $\varepsilon > 0$ let $\beta^{(\varepsilon)}$ be defined by $2\pi/(2\pi - \beta^{(\varepsilon)}) = \sqrt{1+\varepsilon^2}$, and let

$$B^{(\varepsilon)} = \mathbb{R}^2 \setminus \Big\{(x_1, x_2) : x_2 < 0 < x_1,\ -\beta^{(\varepsilon)} < \arctan x_2/x_1 < 0\Big\}$$

We define a deformation $y : B^{(\varepsilon)} \to \mathbb{R}^3$ whose image has rotational symmetry. We are going to use cylindrical coordinates (r, φ),

$$y(r,\varphi) = U(r) e_r^{(\varepsilon)} + V(r) e_z \tag{7}$$

where

$$\begin{aligned}
U &: [0,\infty) \to [0,\infty)\\
V &: [0,\infty) \to (-\infty,\infty)\\
e_r^{(\varepsilon)} &= (\cos\varphi^{(\varepsilon)}, \sin\varphi^{(\varepsilon)}, 0)\\
\varphi^{(\varepsilon)} &= \frac{2\pi}{2\pi - \beta^{(\varepsilon)}}\,\varphi
\end{aligned}$$

We calculate

$$\begin{aligned}
\nabla y &= \Big(U' e_r^{(\varepsilon)} + V' e_z\Big) \otimes e_r + \frac{\sqrt{1+\varepsilon^2}}{r} U\, e_\varphi^{(\varepsilon)} \otimes e_\varphi\\
\nabla y^T \nabla y - \mathrm{Id} &= \big((U')^2 + (V')^2 - 1\big)\, e_r \otimes e_r + \left(\frac{1+\varepsilon^2}{r^2} U^2 - 1\right) e_\varphi \otimes e_\varphi\\
\nabla^2 y &= \Big(U'' e_r^{(\varepsilon)} + V'' e_z\Big) \otimes e_r \otimes e_r + \sqrt{1+\varepsilon^2}\left(\frac{U}{r}\right)' e_\varphi^{(\varepsilon)} \otimes (e_\varphi \otimes e_r + e_r \otimes e_\varphi)\\
&\quad + \left(\frac{U'}{r} - \frac{(1+\varepsilon^2)U}{r^2}\right) e_r^{(\varepsilon)} \otimes e_\varphi \otimes e_\varphi + \frac{V'}{r} e_z \otimes e_\varphi \otimes e_\varphi
\end{aligned}$$

Definition of the elastic energy. As a starting point, we choose the elastic energy density to be

$$\bar\rho_{\mathrm{el.}} = \left|\nabla y^T \nabla y - \mathrm{Id}\right|^2 + h^2 |\nabla^2 y|^2$$

where h is a parameter for the thickness of the sheet under consideration. The first term on the right hand side represents the stretching energy density, the second one the bending energy density. This is a standard ansatz for the elastic energy, for a justification see e.g. [9].

Von-Kármán ansatz, change of variables. Now we make the ansatz

$$\begin{aligned}
U(r) =& r + O(\varepsilon^2)\\
V(r) =& O(\varepsilon)\,,
\end{aligned} \tag{8}$$

where ε is some small parameter. The following changes of variables will be convenient,

$$\begin{aligned}
U(r) =& r + \frac{\varepsilon^2}{2}(\hat u(r) - r)\\
V'(r) =& \varepsilon \hat w(r)
\end{aligned} \tag{9}$$

and, alternatively,

$$\begin{aligned}
U(r) &= r + \frac{\varepsilon^2}{2}\left(u(r) + \lambda^2 \frac{\psi(r/\lambda)}{2r} - r\right)\\
V'(r) &= \varepsilon(\psi(r/\lambda) + w(r))\,,
\end{aligned}$$

where $\lambda = h/\varepsilon$, $\psi \in C^\infty([0,\infty)$ with $\psi(r) = 0$ near $r = 0$ and $\psi(r) = 1$ for $r \geq 1$.
A short computation shows that the elastic energy density is given by

$$\begin{aligned}\bar{\rho}_{\mathrm{el.}} =& \varepsilon^4 \left((\hat{w}^2 - 1 + \hat{u}')^2 + \left(\frac{\hat{u}}{r}\right)^2 + \lambda^2 \left(\hat{w}'^2 + \frac{\hat{w}^2}{r^2} \right) \right) \\ &+ O(\varepsilon^6) + O(\varepsilon^6 \lambda^2) \,.\end{aligned}$$

We build our model only considering the leading terms in ε for fixed λ, defining

$$\begin{aligned}\rho^{\mathrm{el.}}_\lambda =& \lim_{\varepsilon\to 0} \varepsilon^{-4} \bar{\rho}_{\mathrm{el.}} \\ =& \left((\hat{w}^2 - 1 + \hat{u}')^2 + \left(\frac{\hat{u}}{r}\right)^2 + \lambda^2 \left(\hat{w}'^2 + \frac{\hat{w}^2}{r^2} \right) \right) \,.\end{aligned} \tag{10}$$

Renormalization and rescaling. The (u, w) variables have been chosen such that we expect u, u', w, w' to vanish as $r \to \infty$. By an inspection of the energy density $\rho^{\mathrm{el.}}_\lambda(r)$ we expect that the integral

$$\int_0^R r\mathrm{d}r \rho^{\mathrm{el.}}_\lambda(r)$$

diverges logarithmically as $R \to \infty$. To have some hope of a meaningful limit for $R \to \infty$, we introduce the renormalized functional

$$\begin{array}{rccl} \hat{E}^R_\lambda : & \mathcal{W} & \to & \mathbb{R} \\ & (\hat{u}, \hat{w}) & \mapsto & \int_0^R r\mathrm{d}r \left(\rho^{\mathrm{el.}}_\lambda(r) - \lambda^2 \frac{\psi(r/\lambda)^2}{r^2} \right) \,. \end{array}$$

In the sequel, we will set $\lambda \equiv 1$, and derive all results for this value of λ. The general case can be recovered by the change of variables $r \to r/\lambda$, which we will do at the very end. In fact the change of variable formula yields

$$\hat{E}^R_\lambda(\hat{u}, \hat{w}) = \lambda^2 \hat{E}^{R/\lambda}_1 \left(\lambda^{-1} \hat{u}(\lambda \cdot), \hat{w}(\lambda \cdot) \right) \tag{11}$$

for any $\hat{u}, \hat{w} \in W^{1,2}_{\mathrm{loc}}$.
We will use the following notation:

$$\begin{aligned}\hat{E}^R_1 =& \hat{E}^R \\ E^R(u, w) =& \hat{E}^R(\hat{u}, \hat{w})\end{aligned}$$

For the reader's convenience, we summarize some of the notation for future reference (with $\lambda \equiv 1$):

$$\boxed{\begin{aligned}\mathcal{W} =& \Big\{ (\hat{u}, \hat{w}) \in W^{1,2}_{\mathrm{loc}}((0,\infty), \mathbb{R}^2) : E^1(u, w) < \infty \Big\} \\ \psi \in& C^\infty([0,\infty)) \text{ with} \\ & \quad \psi(r) = 0 \text{ near } r = 0 \text{ and } \psi(r) = 1 \text{ for } r \geq 1 \\ u(r) =& \hat{u}(r) - \frac{\psi(r)}{2r} \\ w(r) =& \hat{w}(r) - \psi(r) \\ \rho^{\mathrm{el.}} =& (\hat{w}^2 - 1 + \hat{u}')^2 + (\hat{u}/r)^2 + \hat{w}'^2 + r^{-2} \hat{w}^2 \\ \hat{E}^R, E^R :& \mathcal{W} \to \mathbb{R} \\ \hat{E}^R(\hat{u}, \hat{w}) =& \int_0^R r\mathrm{d}r \left(\rho^{\mathrm{el.}}(r) - r^{-2} \psi^2 \right) \\ E^R(u, w) =& \hat{E}^R(\hat{u}, \hat{w})\end{aligned}} \tag{12}$$

Proposition 1. *If $(u, w) \in \mathcal{W}$ then*

$$\lim_{r\to 0} u(r) = 0, \quad \lim_{r\to 0} w(r) = 0. \tag{13}$$

To show this we recall the following result.

Lemma 1. *(i) Let $a \in \mathbb{R}$ and let $I = (-\infty, a)$ or $I = (a, \infty)$. If $g \in W^{1,2}(I)$ then*

$$\sup_I g^2 \le 2\|g\|_{L^2} \|g'\|_{L^2} \le \int_I \mathrm{d}t(g^2 + g'^2) \tag{14}$$

and

$$\lim_{t\to -\infty} g(t) = 0 \quad \textit{or} \quad \lim_{t\to\infty} g(t) = 0, \quad \textit{respectively.} \tag{15}$$

(ii) If $r_0 \in (0, \infty)$ let $J = (0, r_0)$ or $J = (r_0, \infty)$ and assume that $h \in W^{1,2}_{loc}(J)$ and

$$\int_J r\mathrm{d}r \left[h'^2 + \frac{h^2}{r^2} \right] < \infty \tag{16}$$

then

$$\sup_I h^2 \le 2\|h\|_{L^2(I;\mathrm{d}r/r)} \|h'\|_{L^2(I;r\mathrm{d}r)} \le \int_J r\mathrm{d}r \left[h'^2 + \frac{h^2}{r^2} \right] \tag{17}$$

and

$$\lim_{r\to 0} h(r) = 0 \quad \textit{or} \quad \lim_{r\to\infty} h(r) = 0, \quad \textit{respectively.} \tag{18}$$

Proof. Assertion (ii) follows from assertion (i) and the change of variables $r = e^t$. To prove (i) note that $(g^2)' = 2gg'$ and thus by the fundamental theorem of calculus

$$\sup g^2 - \inf g^2 = \int_I \mathrm{d}t\, 2|gg'| \le \int_I \mathrm{d}t(g^2 + g'^2). \tag{19}$$

Moreover $\inf g^2 = 0$ since $g^2 \in L^1(I)$ and I is unbounded. This proves (14). Assume that $I = (-\infty, a)$.Then for any $t < a$ we also have

$$\sup_{(-\infty,t)} g^2 \le \int_{-\infty}^t \mathrm{d}t(g^2 + g'^2) \tag{20}$$

Now the right hand side goes to 0 as $t \to -\infty$. Thus $\lim_{t\to-\infty} g(t) = 0$. The case $I = (a, \infty)$ is analogous. □

Proof of Proposition 1. It follows from the condition $E^1(u, w) < \infty$ and Lemma 1 that $\sup_{(0,1)} |\hat{w}| < \infty$ and $\lim_{r\to 0} \hat{w}(r) = 0$. This implies that $\int_0^1 r\mathrm{d}r\hat{u}'^2 < \infty$. Hence another application of Lemma 1 shows that $\lim_{r\to 0} \hat{u}(r) = 0$. Since $u = \hat{u}$ and $w = \hat{w}$ in some interval $(0, r_0)$ the assertion of the proposition follows. □

3 Existence of minimizers

We will show that for $(u, w) \in \mathcal{W}$ the limit $E(u, w) = \lim_{R\to\infty} E^R(u, w)$ exists in $(-\infty, \infty]$ and that E has a minimizer in $\mathcal{W}$.

The main difficulty is that the renormalized energy density $\rho^{\mathrm{el.}}(r) - r^{-2}\psi^2$ is not pointwise positive and therefore it is not clear that $E^R(u, w)$ is bounded from below as $R \to \infty$. We thus proceed in several steps.

1. We first show that $\rho^{\text{el.}}(r) - r^{-2}\psi^2$ can be rewritten as a pointwise positive term plus $-\frac{1}{r}\left(\frac{u}{r}\right)'$ (plus a harmless explicit term with rapid decay at infinity). The point is that $-\frac{1}{r}\left(\frac{u}{r}\right)'$ reduces to a boundary term when integrated against the measure $r\mathrm{d}r$. Thus $E^R(u,w)$ is bounded from below by the positive functional

$$E^{+,R}(u,w) := \int_0^1 r\mathrm{d}r \left[\left(\hat{w}^2 - 1 + \hat{u}'\right)^2 + \frac{\hat{u}^2}{r^2} + \hat{w}'^2 + \frac{\hat{w}^2}{r^2}\right] \tag{21}$$

$$+ \int_1^R r\mathrm{d}r \left[(2w + w^2 + u')^2 + \frac{u^2}{r^2} + w'^2\right]. \tag{22}$$

up to the terms $u(1)$ and $u(R)/R$.

2. The integrand in E^+ is a sum of positive terms but does not directly give bounds for w and u'. We derive an interpolation inequality which allows us to estimate g in a (weighted) L^2 space if we control $(g + f')$ and f and g' in suitably weighted L^2 spaces (this is essentially and interpolation between H^1 and H^{-1}, see Lemma 2).

3. We would like to apply the interpolation inequality with $g = 2w + w^2$ but to control g' in L^2 we need to control the L^∞ norm of w. On the other hand if we control g and g' in L^2 then we control g, and hence w, in L^∞. In Lemma 3 we show that one can simultaneously bound the (weigthed) L^2 norm of g and the L^∞ norm of w. This also gives enough control of u' to deduce that $u(R)/R^{1/2}$ is controlled by $E^{+,R}(u,w)$.

4. We finally bound $u(1)$ by a sublinear expression in $E^{+,R}(u,w)$. This allows us to absorb the boundary terms and to obtain a lower bound

$$E^R(u,w) \geq \frac{1}{2} E^{+,R}(u,w) - C, \tag{23}$$

for $R \geq R_0$. From this it easily follows that the limit $\lim_{R\to\infty} E^R(u,w)$ exists and moreover that this limit is finite if and only if $E^+(u,w)$ is finite, see Lemma 4.

5. With these preparations we deduce the existence of minimizers in the usual way by the direct method of the calculus of variations.

We begin by rewriting the renormalized energy. We have $\hat{w} = w + 1$ and $\hat{u} = u + \frac{1}{2r}$ for $r \geq 1$ and thus the renormalized energy (cf. eq. (12)) simplifies for $r \geq 1$ to

$$\begin{aligned}
\rho^{\text{el.}}(r) - \frac{1}{r^2} &= \left(2w + w^2 + u' - \frac{1}{2r^2}\right)^2 + \left(\frac{u + 1/(2r)}{r}\right)^2 + w'^2 + \frac{2w + w^2}{r^2} \\
&= \left(2w + w^2 + u'\right)^2 - \frac{1}{r^2}(2w + w^2 + u') + \frac{1}{4r^4} + \left(\frac{u}{r}\right)^2 + \frac{u}{r^3} + \frac{1}{4r^4} + w'^2 + \frac{2w + w^2}{r^2} \\
&= \left(2w + w^2 + u'\right)^2 + \left(\frac{u}{r}\right)^2 + w'^2 - \frac{u'}{r^2} + \frac{u}{r^3} + \frac{1}{2r^4} \\
&= \left(2w + w^2 + u'\right)^2 + \left(\frac{u}{r}\right)^2 + w'^2 - \frac{1}{r}\left(\frac{u}{r}\right)' + \frac{1}{2r^4}\,.
\end{aligned} \tag{24}$$

The first three terms in (24) are positive, the last term is harmless (since it is integrable with respect to the measure $r\mathrm{d}r$) and the last but one term produces a boundary term $-u(R)/R + u(1)$ when integrated against $r\mathrm{d}r$.

Definition 1. For $0 \leq a < 1 < R \leq \infty$ define

$$E^+(u,w;(a,1)) := \int_a^1 r\mathrm{d}r \left[(\hat{w}^2 - 1 + \hat{u}')^2 + \frac{\hat{u}^2}{r^2} + \hat{w}'^2 + \frac{\hat{w}^2}{r^2}\right], \tag{25}$$

$$E^+(u,w;(1,R)) := \int_1^R r\mathrm{d}r \left[(2w + w^2 + u')^2 + \frac{u^2}{r^2} + w'^2\right] \tag{26}$$

$$E^{+,R}(u,w) := E^+(u,w;(0,1)) + E^+(u,w;(1,R)) \tag{27}$$

$$E^+(u,w) := E^{+,\infty} \tag{28}$$

where $\hat{w}(r) = w(r) + \psi(r)$, $\hat{u}(r) = u(r) + \frac{\psi(r)}{2r}$.

Note that by positivity of the integrand and the monotone convergence theorem the limit $\lim_{R\to\infty} E^{+,R}(u,w)$ exists in $\mathbb{R} \cup \{\infty\}$ for all $(u,w) \in \mathcal{W}$ and agrees with $E^+(u,w)$.

From the formula (24) for the renormalized energy we get

$$E^1(u,w) = E^+(u,w;(0,1)) - \int_0^1 r\mathrm{d}r\, \frac{\psi^2}{r^2} \tag{29}$$

$$E^R(u,w) = E^{+,R}(u,w) + u(1) - \frac{u(R)}{R} + \frac{1}{4}(1-R^{-2}) - \int_0^1 r\mathrm{d}r\, \frac{\psi^2}{r^2} \tag{30}$$

We want to show that $E^R \geq \frac{1}{2}E^{+,R} - C$ for $R \geq R_0$ and that $\lim_{R\to\infty} u(R)/R = 0$ if $E^+(u,w) < \infty$. One difficulty is that the functional $E^+(u,w)$ is not obviously coercive. We get immediately bounds for u and w', but there are no direct bounds for u' and w. We will obtain bounds on u' and w from an interpolation result. To state it, it is more convenient to make the change of variables $R = e^T$, $\tilde{u}(t) = u(e^t)$ and $\tilde{w}(t) = w(e^t)$. Then

$$\begin{aligned} E^+(u,w;(1,R)) &= \int_1^R r\mathrm{d}r \left[(2w + w^2 + u')^2 + \frac{u^2}{r^2} + w'^2 \right] \\ &= \int_0^T \mathrm{d}t \left[\left(e^t(2\tilde{w} + \tilde{w}^2) + \tilde{u}' \right)^2 + \tilde{u}^2 + \tilde{w}'^2 \right] \end{aligned} \tag{31}$$

Lemma 2. *Let $T \in [1,\infty]$, let $I = (0,T)$ and suppose that $f, g \in W^{1,2}_{\mathrm{loc}}(I)$ and*

$$e^{t/2}g + e^{-t/2}f' \in L^2(I), \quad f \in L^2(I), \quad g' \in L^2(I). \tag{32}$$

Then

$$g \in L^2(I), \quad e^{-t}f' \in L^2(I) \tag{33}$$

and

$$\|g\|^2_{L^2} + \|e^{-t}f'\|^2_{L^2} \leq C \left(\|e^{t/2}g + e^{-t/2}f'\|^2_{L^2} + \|f\|^2_{L^2} + \|g'\|^2_{L^2} \right). \tag{34}$$

Remark. For $T = \infty$ one can also prove a bound with the optimal exponential rate: the L^2 norms of $e^{t/2}g$ and $e^{-t/2}f'$ are bounded in terms by a constant times $\|e^{-t/2}f' + e^{t/2}g\|_{L^2} + \|f\|_{L^2} + \|g'\|_{L^2}$. To see this one uses the identity $(e^{-t/2}f')^2 + (e^{t/2}g)^2 = (e^{-t/2}f' + e^{t/2}g)^2 - 2f'g$ and integration by parts. The following example shows that one cannot estimate $e^{\beta t}g$ for any $\beta > \frac{1}{2}$. Let $\alpha > 0$ and $f = 2e^{-\alpha t}\sin e^{t/2}$, $g = -e^{-\alpha t}e^{-t/2}\cos e^{t/2}$. Then f, $e^t g + f'$ and g' are in $L^2((0,\infty))$ but $e^{\beta t}g \notin L^2((0,\infty))$ if $\beta \geq \frac{1}{2} + \alpha$.

Proof. Let

$$G(t) := -\int_t^T \mathrm{d}s\, e^{-s/2}(e^{-s/2}f'(s) + e^{s/2}g) + \int_t^T \mathrm{d}s\, e^{-s}f(s) - e^{-t}f(t). \tag{35}$$

Note that both integrals exist (even for $T = \infty$) since $e^{-s/2}f'(s) + e^{s/2}g \in L^2$, $f \in L^2$ and $e^{-s/2} \in L^2$. Moreover G is absolutely continuous and for a.e. t we have

$$G'(t) = e^{-t}f'(t) + g(t) - e^{-t}f(t) - (e^{-t}f(t))' = g(t). \tag{36}$$

Thus $G \in W^{2,2}_{loc}(I)$ and $G'' = g'$. Moreover by the Cauchy-Schwarz inequality

$$|G(t)| \leq e^{-t/2}\|e^{-s/2}f' + e^{s/2}g\|_{L^2} + e^{-t}\|f\|_{L^2} + e^{-t}f(t) \tag{37}$$

and this implies that

$$\|G\|_{L^2} \leq \|e^{-s/2}f'(s) + e^{s/2}\|_{L^2} + 2\|f\|_{L^2}. \tag{38}$$

For $a \in (0, T-1)$ we use the interpolation inequality

$$\|G'\|^2_{L^2((a,a+1))} \le C \left(\|G\|^2_{L^2((a,a+1))} + \|G''\|^2_{L^2((a,a+1))} \right) \tag{39}$$

For a proof see, e.g., [12] or derive a contradiction from the assumptions $\|G'_j\|_{L^2((0,1))} = 1$ and $\|G_j\|^2_{L^2((0,1))} + \|G''_j\|^2_{L^2((0,1))} \to 0$. Passage to the limit $a \downarrow 0$ and $a \uparrow T-1$ (if $T < \infty$) shows that the inequality also holds for $a = 0$ and $a = T-1$ (if $T < \infty$). If $T = \infty$ we sum the inequalities for $a \in \mathbb{N}$. If $T < \infty$ we denote by $[T]$ the integer part of T and sum the inequalities for $a = 0, \ldots, [T]-1$ and $a = T-1$. Since at most two of the intervals $(a, a+1)$ overlap we get

$$\|G'\|^2_{L^2} \le 2C \left(\|G\|^2_{L^2} + \|G''\|^2_{L^2} \right) \tag{40}$$

Since $G' = g$ and $G'' = g'$ the estimate for $\|g\|_{L^2}$ follows from (38). The estimate for $e^{-t} f'$ follows from the triangle inequality since $e^{-t} f' = e^{-t/2}(e^{-t/2} f' + e^{t/2} g) - g$. $\square$

We would like to apply the interpolation result with $f = \tilde{u}$ and $g = 2\tilde{w} + \tilde{w}^2$. We have $g' = 2(1+\tilde{w})\tilde{w}'$ and $E^{+,R}$ controls only the L^2 norm of $\tilde{w}'$ and not directly the L^2 norm of g'. We thus simulataneously prove an L^∞ bound for $\tilde{w}$ and an L^2 bound for g.

Lemma 3. *There exists a constant C with the following property. If $R > 1$ and $(u, w) \in W^{1,2}_{\text{loc}}([1, R))$ with $E(R) := E^+(u, w; (1, R)) < \infty$ then*

$$\sup_{[1,R]} |w| \le C(1 + E^{1/2}(R)), \tag{41}$$

$$\int_1^R \frac{dr}{r} \left[(2w + w^2)^2 + u'^2 \right] \le C(1 + E(R)^2), \tag{42}$$

$$R^{-1/2} |u(R)| \le C(1 + E(R)). \tag{43}$$

Proof. Let $R = e^T$, $\tilde{u}(t) = u(e^t)$, $\tilde{w}(t) = w(e^t)$ and $g = 2\tilde{w} + \tilde{w}^2$. To prove (41) we will assume in addition that $w \in L^\infty((1, R))$. This is no loss of generality since by the Sobolev embedding theorem $w \in L^\infty((1, R-\varepsilon))$ for all ε positive. If we have (41) with $R - \varepsilon$ instead of R for all $\varepsilon > 0$ we can then consider the limit $\varepsilon \downarrow 0$ to obtain the estimate for R.

Let

$$M := \sup_{[0,T]} |\tilde{w}| = \sup_{[1,R]} |w|. \tag{44}$$

If $M < 4$ there is nothing to show. We may thus assume $M \ge 4$. Then

$$\frac{1}{2} M^2 \le \sup g, \quad |g'| \le |2(1 + \tilde{w})\tilde{w}'| \le 4M|\tilde{w}'|. \tag{45}$$

By (31) we have

$$E(R) = \int_0^T \mathrm{d}t \left[\left(e^t g + \tilde{u}' \right)^2 + \tilde{u}^2 + \tilde{w}'^2 \right]. \tag{46}$$

Thus arguing as in Lemma 1 and using the interpolation estimate with $f = \tilde{u}$ we get

$$\frac{1}{4} M^4 \le \inf_{[0,T]} g^2 + (\sup_{[0,T]} g^2 - \inf_{[0,T]} g^2) \le \frac{1}{T} \int_0^T \mathrm{d}t\, g^2 + \int_0^T \mathrm{d}t\, g^2 + \int_0^T \mathrm{d}t\, g'^2 \tag{47}$$

$$\le 2C \int_0^T \mathrm{d}t \left[(e^{-t/2} \tilde{u}' + e^{t/2} g)^2 + \tilde{u}^2 \right] + (2C + 1) \int_0^T \mathrm{d}t\, g'^2 \tag{48}$$

$$\le CE(R) + 16M^2(2C + 1)E(R) \le \frac{1}{8} M^4 + C(1 + E(R)^2), \tag{49}$$

where we used Young's inequality $ab \le \frac{1}{8} a^2 + 2b^2$. This implies (41).

Now (42) follows directly from the interpolation estimate, (41) and (45). Indeed we have

$$\begin{aligned}
&\int_1^R \frac{\mathrm{d}r}{r}(2w+w^2)^2 = \int_0^T \mathrm{d}t\, g^2 \\
\leq& C\int_0^T \mathrm{d}t\, \left[(e^{-t/2}\tilde{u}' + e^{t/2}g)^2 + \tilde{u}^2 + g'^2\right] \\
\leq& C(1+E(R))\,E(R).
\end{aligned} \tag{50}$$

and the bound for u' follows by the triangle inequality since $\int_1^R r\mathrm{d}r(2w+w^2+u')^2 \leq E(R)$ and $r^{-1} \leq r$ on $[1,\infty)$.

Using again the interpolation inequality and the L^∞ bound for w we get

$$\begin{aligned}
&R^{-1}u^2(R) \leq \sup_{[0,T]} e^{-t}\tilde{u}^2 = \inf_{[0,T]} e^{-t}\tilde{u}^2 + (\sup_{[0,T]} e^{-t}\tilde{u}^2 - \inf_{[0,T]} e^{-t}\tilde{u}^2) \\
\leq& \frac{1}{T}\int_0^T \mathrm{d}t\, e^{-t}\tilde{u}^2 + 2\int_0^T \mathrm{d}t\, \left(e^{-t}\tilde{u}^2\right)' \\
\leq& 3\int_0^T \mathrm{d}t\, e^{-t}\tilde{u}^2 + \int_0^T \mathrm{d}t\, \tilde{u}^2 + \int_0^T \mathrm{d}t\, e^{-2t}\tilde{u}'^2 \leq C(1+E(R)^2).
\end{aligned} \tag{51}$$

Taking the square root we get (43). $\square$

Lemma 4. *There exists a constant C and $R_0 \geq 1$ such that for all $R \in [R_0,\infty)$ and all $(u,w) \in \mathcal{W}$ we have*

$$E^R(u,w) \geq \frac{1}{2}E^{+,R}(u,w) - C. \tag{52}$$

Moreover for all $(u,w) \in \mathcal{W}$ the limit

$$E(u,w) := \lim_{R\to\infty} E^R(u,w) \tag{53}$$

exists in $\mathbb{R}\cup\{\infty\}$ and

$$E(u,w) < \infty \quad \Longleftrightarrow \quad E^+(u,w) < \infty. \tag{54}$$

In addition, if $E(u,w) < \infty$ then

$$E(u,w) = E^+(u,w) + u(1) + \frac{1}{4} - \int_0^1 r\mathrm{d}r \frac{\psi^2}{r^2}. \tag{55}$$

Proof. The starting point is the relation (30)

$$E^R(u,w) = E^{+,R}(u,w) + u(1) - \frac{u(R)}{R} + \frac{1}{4}(1-R^{-2}) - \int_0^1 r\mathrm{d}r\, \frac{\psi^2}{r^2} \tag{56}$$

Note that with the notation of Lemma 3 we have

$$E^{+,R}(u,w) = E(R) + X_1, \quad \text{where } X_1 := E^+(u,w;(0,1)) \geq 0. \tag{57}$$

By (43)

$$\frac{|u(R)|}{R} \leq R^{-1/2}C(1+E(R)) \leq \frac{1}{4}E(R) + C \tag{58}$$

if $R \geq R_0 := 4C$.

Let $I = (0,1)$. By Lemma 1

$$\begin{aligned}
|\hat{u}(1)|^2 &\leq 2\|\hat{u}\|_{L^2(I;\mathrm{d}r/r)}\,\|\hat{u}'\|_{L^2(I;r\mathrm{d}r)} && (59)\\
&\leq 2\|\hat{u}\|_{L^2(I;\mathrm{d}r/r)}\left(\|\hat{u}'+\hat{w}^2-1\|_{L^2(I;r\mathrm{d}r)} + \|\hat{w}^2-1\|_{L^2(I;r\mathrm{d}r)}\right) && (60)\\
&\leq X_1 + 2X_1^{1/2}\|\hat{w}^2-1\|_{L^2(I;r\mathrm{d}r)}. && (61)
\end{aligned}$$

Again by Lemma 1 we have $\sup_{[0,1]} \hat{w}^2 \leq X_1$. Thus

$$\|\hat{w}^2 - 1\|_{L^2(I;r\mathrm{d}r)} \leq \sup |\hat{w}^2 - 1| \leq (1 + X_1) \tag{62}$$

and therefore $|\hat{u}(1)|^2 \leq (2X_1^{1/2} + X_1 + 2X_1^{3/2})$. Using Young's inequality $ab \leq \frac{1}{3}a^3 + \frac{2}{3}b^{3/2}$ first with $(a,b) = (X_1^{1/2}, 1)$ and then with $(a,b) = (1, X_1)$ we get $|\hat{u}(1)|^2 \leq 4(1 + X_1^{3/2})$. Finally we get

$$|u(1)| = |\hat{u}(1) - \frac{1}{2}| \leq 3 + 2X_1^{3/4} \leq 3 + \frac{1}{4}8^4 + \frac{1}{4}X_1, \tag{63}$$

where we used $ab \leq \frac{3}{4}a^{4/3} + \frac{1}{4}b^4$ with $a = \frac{1}{4}X^{3/4}$ and $b = 8$.

Combining this with (58) and (56) and using that $X_1 \leq E^{+,R}(u,w)$ and $E(R) \leq E^{+,R}(u,w)$ we obtain (52) (for $R \geq R_0$).

Now if $E^+(u,w) = \infty$ then it follows from (52) that $\lim_{R\to\infty} E^R(u,w) = \infty$. Assume now $E^+(u,w) < \infty$. Since $E^{+,R}(u,w) \leq E^+(u,w)$ it follows from Lemma 3 that $\lim_{R\to\infty} u(R)/R = 0$. In view of (56) we deduce that that $\lim_{R\to\infty} E^R(u,w)$ exists and

$$E(u,w) = E^+(u,w) + u(1) + \frac{1}{4} - \int_0^1 r\mathrm{d}r \frac{\psi^2}{r^2} < \infty. \tag{64}$$

$\square$

Corollary 1. *For the unrenormalized energy I_λ (cf. eq. (4)),*

$$|\log\lambda| - C \leq \lambda^{-2} \inf I_\lambda \leq |\log\lambda| + C \qquad \textit{for all } \lambda \in (0, R_0^{-1}).$$

Proof. By eq. (11),

$$\hat{E}^1_\lambda(\hat{u},\hat{w}) = \lambda^2 \hat{E}_1^{\lambda^{-1}}(\hat{u}_\lambda, \hat{w}_\lambda) \quad \text{where} \quad \hat{u}_\lambda = \lambda\hat{u}(\cdot/\lambda), \quad \hat{w}_\lambda = \hat{w}(\cdot/\lambda)\,. \tag{65}$$

Using the definition of $\hat{E}^R$, eq. (12), we get

$$\inf \hat{E}^1_\lambda \leq \lambda^2 \hat{E}^{\lambda^{-1}}(0,\psi) \leq C\lambda^2$$

which proves the upper bound since

$$I_\lambda = \hat{E}^1_\lambda + \lambda^2 \int_0^1 \psi(r/\lambda)^2 \mathrm{d}r/r = \hat{E}^1_\lambda + \lambda^2(C + |\log\lambda|)\,.$$

The lower bound follows from eq. (65) since by (52) we have

$$\hat{E}_1^{\lambda^{-1}}(\hat{u}_\lambda,\hat{w}_\lambda) = E^{\lambda^{-1}}(u_\lambda, w_\lambda) \geq -C \tag{66}$$

for $\lambda \leq 1/R_0$. $\square$

Now we are in a position to prove the existence of minimizers for the renormalized energy.

Theorem 2. *We have $\inf_{\mathcal{W}} E \in \mathbb{R}$ and the functional E attains its minimum in $\mathcal{W}$. Moreover there exists a minimizer (u,w) of E which satisfies*

$$w + \psi \geq 0 \quad \textit{a.e.} \tag{67}$$

Proof. By Lemma 4, E is bounded from below. Moreover $E(0,0) < \infty$. Thus $\inf E \in \mathbb{R}$. Let (u_j, w_j) be a minimizing sequence, i.e.,

$$E(u_j, w_j) \to \inf_{\mathcal{W}} E. \tag{68}$$

The energy is E does not change if we replace $\hat{w} = w + \psi$ by $|\hat{w}|$. We may thus assume in addition that

$$w_j + \psi \geq 0. \tag{69}$$

Since $\sup_j E(u_j, w_j)$ is bounded we deduce from (52) that

$$E^+(u_j, w_j) \le C \quad \forall j \in \mathbb{N}. \tag{70}$$

Let $0 < a < b < \infty$. Then it follows directly from the formula for E^+ that u_j and w_j' are bounded in $L^2((a,b))$. By Lemma 3 the sequence w_j is bounded in L^∞. Thus w_j and w_j^2 are bounded in $L^2((a,b))$. Since $u_j' + 2w_j + w_j^2$ is bounded in $L^2((a,b))$ it follows that u_j' is bounded in $L^2((a,b))$. Thus there exist a subsequence of (u_j, w_j) which converges weakly in $W^{1,2}((a,b))$. We can apply this argument with $a = 1/k$, $b = k$ for $k \in \mathbb{N}$, $k \ge 2$ and successively select subsequences. By a diagonalization argument there exists a single subsequence (still denoted by (u_j, w_j)) that converges weakly in $W^{1,2}_{loc}((0,\infty))$:

$$(u_j, w_j) \rightharpoonup (u, w) \quad \text{in } W^{1,2}_{loc}((0,\infty)). \tag{71}$$

By the compact Sobolev embedding this implies

$$(u_j, w_j) \to (u, w) \quad \text{locally uniformly in } (0,\infty) \tag{72}$$

In particular we have the weak convergences

$$2w_j + w_j^2 + u_j' \rightharpoonup 2w + w^2 + u \quad \text{in } L^2_{loc}((0,\infty)) \tag{73}$$

and

$$\hat{w}_j^2 - 1 + \hat{u}_j' \rightharpoonup \hat{w}^2 - 1 + \hat{u} \quad \text{in } L^2_{loc}((0,\infty)), \tag{74}$$

where $\hat{w}_j = w_j + \psi$, $\hat{u}_j = u_j + \psi/2r$, $\hat{w} = w + \psi$, $\hat{u} = u + \psi/2r$.

Weak lower semicontinuity of the L^2 norm implies that for $0 < a < 1 < b < \infty$.

$$\begin{aligned} &\int_a^1 r\mathrm{d}r \left[(\hat{w}^2 - 1 + \hat{u}'^2)^2 + \frac{\hat{u}^2}{r^2} + \frac{\hat{w}^2}{r^2} + \hat{w}'^2 \right] \\ \le &\liminf_{j\to\infty} \int_a^1 r\mathrm{d}r \left[(\hat{w}_j^2 - 1 + \hat{u}_j'^2)^2 + \frac{\hat{u}_j^2}{r^2} + \frac{\hat{w}_j^2}{r^2} + \hat{w}_j'^2 \right] \end{aligned} \tag{75}$$

and

$$\begin{aligned} &\int_1^b r\mathrm{d}r \left[(2w + w^2 + u'^2)^2 + \frac{u^2}{r^2} + w'^2 \right] \\ \le &\liminf_{j\to\infty} \int_a^1 r\mathrm{d}r \left[(2w_j + w_j^2 + u_j'^2)^2 + \frac{u_j^2}{r^2} + w_j'^2 \right]. \end{aligned} \tag{76}$$

Adding these two inequalities we get

$$E^+(u, w; [a,b]) \le \liminf_{j\to\infty} E^+(u_j, w_j), \tag{77}$$

where $E^+(u, w; [a,b])$ is defined as the sum of the terms on the left hand side of (75) and (76). Finally the monotone convergence theorem implies that we can take the limit $a \to 0$ and $b \to \infty$ in (77) and deduce

$$E^+(u, w) \le \liminf_{j\to\infty} E^+(u_j, w_j). \tag{78}$$

This in particular implies that $E^+(u, w) < \infty$ and thus $(u, w) \in \mathcal{W}$.

We now use the relation (55) between E^+ and E and the fact that $u_j(1) \to u(1)$ (see (72)) to deduce that

$$E(u, w) \le \liminf_{j\to\infty} E(u_j, w_j) = \inf_W E. \tag{79}$$

Thus (u, w) minimizes E in $\mathcal{W}$.

Finally the condition $w_j + \psi \ge 0$ implies that $w + \psi \ge 0$. $\square$

Remark 1 (Self-penetration of solutions)**.** The von Kármán model displays a pathology at the origin for the situation we want to model. Namely, the solutions we have found show interpenetration of matter. Consider again the Euler-Lagrange equation obtained by variation of $\hat{u}$,

$$\left(r\left(\hat{w}^2 - 1 + \hat{u}'\right)\right)' = \frac{\hat{u}}{r}\,.$$

Since $\hat{w} \to 0$ for $r \to 0$, the qualitative behaviour of solutions u near the origin is the same as the one of solutions of the linear equation

$$\left(r\left(\hat{u}' - 1\right)\right)' = \frac{\hat{u}}{r}\,.$$

The solutions of this latter equation are given by

$$\frac{1}{2} r \log r + C_1 r + C_2 r^{-1}.$$

The integration constant C_2 has to be set to zero to fulfill the boundary condition $\hat{u}(0) = 0$. Going back to eq. (9), we see that the value of U will be negative in some punctured neighbourhood of the origin and we have self-penetration of the solution (somewhere in the region $r \sim \exp\left(-\varepsilon^{-2}\right)$). We expect that this pathology could be cured by including nonlinear or higher order terms in u in our model. We refrain from doing so, since the main aspect of this work is the analysis of the solutions away from the origin.

4 Decay properties

We now turn our interest to the decay properties of minimizers. We first show that $\lim_{r\to\infty} w(r) = 0$ (if $w + \psi \geq 0$). This will level the field for an application of stable manifold theory, by which we prove that u and w decay like a stretched exponential $\exp(-c\sqrt{r})$.

Lemma 5. *Assume that $(u, w) \in \mathcal{W}$ with $E^+(u, w) < \infty$ and $w + \psi \geq 0$. Then*

$$\lim_{R\to\infty} w(R) = 0 \tag{80}$$

Proof. It follows from Lemma 3 and Lemma 1 that

$$\lim_{r\to\infty} 2w(r) + w^2(r) = 0. \tag{81}$$

Now $w(r) \geq -1$ and the function $F(s) = 2s + s^2 = (s + 1)^2 - 1$ has a continuous inverse on $[-1, \infty)$. Thus $\lim_{r\to\infty} w(r) = 0$. □

Now we show that minimizers actually have decay as $\exp(-c\sqrt{r})$ at infinity. We will use the following standard tool from stable manifold theory:

Theorem 3 ([8])**.** *Let $s_0 \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ a matrix with $k \leq n$ eigenvalues with negative real part and $n - k$ eigenvalues with positive real part , $F : \mathbb{R}^n \times [s_0, \infty) \to \mathbb{R}^n$ with the property that for every $\varepsilon > 0$, there exist $S \in [s_0, \infty)$ and $\delta_0 > 0$ such that*

$$|F(x, s) - F(\bar{x}, s)| \leq \varepsilon |x - \bar{x}|$$

whenever $|x - \bar{x}| \leq \delta_0$ and $s \geq S$. Then there exist $\delta > 0, \bar{s} > S$ such that for $|p| < \delta$ and $\bar{s} > S$, there exists a k-dimensional submanifold $\bar{M}(\bar{s})$ of $\mathbb{R}^n$ containing the origin such that the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}s} x(s) = Ax(s) + F(x, s)\,, \quad x(\bar{s}) = p \tag{82}$$

has a solution $x : [\bar{s}, \infty) \to \mathbb{R}^n$ *for any* $p \in \bar{M}(\bar{s})$*, and the property*

$$|x(s)| = o(\exp(-\sigma s)) \text{ as } s \to \infty$$

for any $\sigma > 0$ *such that the absolute values of the real parts of the eigenvalues of* A *are all bigger than* σ*. Furthermore there exists* $\eta > 0$ *independent of* $\bar{s}$ *such that, if* $p \notin \bar{M}(\bar{s})$*, then*

$$\|x\|_{L^\infty(s_0,\infty)} > \eta\,.$$

Proposition 2. *For any minimizer* (u, w) *of* E *with* $w \geq -\psi$*,*

$$|(u(r), u'(r), w(r), w'(r))| = o\left(\exp(-\sigma\sqrt{r})\right)$$

as $r \to \infty$ *for any* $\sigma < 2$*.*

Proof. Since E and E^+ only differ by a boundary term they lead to the same Euler-Lagrange equations. Thus for $r > 1$ the Euler Lagrange equations for (u, w) are the same as the Euler-Lagrange equations for the functional

$$\int_1^\infty r\mathrm{d}r \left[(2w(r) + w^2(r) + u'(r))^2 + \frac{u^2(r)}{r^2} + w'^2(r)\right]. \tag{83}$$

It turns out that these EL equations are not of the form required in Theorem 3 since the linear part is not autonomous (up to a contribution which decays as $r \to \infty$). We will make a change of variables to bring the EL equations in a suitable form. To motivate that change of variables it suffices to focus on the linearization, i.e., we may neglect the terms w^2 in the energy functional (as we already know $w \to 0$ at ∞). The linearized equations are

$$(r(2w + u'))' = \frac{u}{r} \tag{84}$$

$$2r(2w + u') = (rw')' = rw'' + w' \tag{85}$$

Differentiation of the second equation and use of the first yields $rw''' + 2w'' = 2u/r$. Thus $2u' = r^2w^{(4)} + 4rw^{(3)} + 2w''$ and inserting this into the second equation we get the linearized fourth order equation for w

$$\frac{1}{2}(r^2w^{(4)} + 4rw^{(3)} + 2w'') + 2w = \frac{1}{2}w'' + \frac{1}{2r}w' \tag{86}$$

Now we make the change of variables $w(r) = \underline{w}(r^\alpha)$. Then

$$w' = \alpha r^{\alpha-1}\underline{w}', \quad w^{(k)} = \alpha^k r^{k(\alpha-1)}\underline{w}^{(k)} + \text{lower order derivatives}. \tag{87}$$

This suggests to choose $\alpha = \frac{1}{2}$ so that the leading order term in the linear equation becomes $\frac{1}{32}\underline{w}^{(4)} + 2w = 0$. We will now derive the EL equations in the new variables in detail. It is most convenient to first transform the functional.

We make the change of variables

$$w(r) = \underline{w}(\sqrt{r}), \quad u(r) = \underline{u}(\sqrt{r}), \quad s = \sqrt{r}, \quad r = s^2. \tag{88}$$

Then

$$w'(r) = \frac{1}{2\sqrt{r}}\underline{w}'(\sqrt{r}) = \frac{1}{2s}\underline{w}'(s), \quad u'(r) = \frac{1}{2s}\underline{u}'(s). \tag{89}$$

Thus for $(u, w) \in \mathcal{W}$

$$\begin{aligned} E^+(u, w) \geq& \int_1^\infty r\mathrm{d}r \left[(2w(r) + w^2(r) + u'(r))^2 + \frac{u^2(r)}{r^2} + w'^2(r)\right] \\ =& \int_1^\infty 2s^3\mathrm{d}s \left[(2\underline{w}(s) + \underline{w}^2(s) + \frac{1}{2s}\underline{u}'(s))^2 + \frac{\underline{u}^2(s)}{s^4} + (\frac{1}{2s}\underline{w}'(s))^2\right] \\ =& 2\int_1^\infty \mathrm{d}s \left[s\left(s(2\underline{w} + \underline{w}^2) + \frac{1}{2}\underline{u}'\right)^2 + \frac{\underline{u}^2}{s} + \frac{1}{4}s\underline{w}'^2\right]. \end{aligned} \tag{90}$$

From (90) we easily obtain the Euler-Lagrange equations (for $s > 1$)

$$2s^2(1+\underline{w})\left(s(2\underline{w}+\underline{w}^2)+\frac{1}{2}\underline{u}'\right)=\frac{1}{4}(s\underline{w}')' \tag{91}$$

$$\frac{1}{2}\left[s\left(s(2\underline{w}+\underline{w}^2)+\frac{1}{2}\underline{u}'\right)\right]'=\frac{\underline{u}}{s}. \tag{92}$$

These equations first hold in the weak sense, but by standard elliptic regularity we get $\underline{w}\in W^{2,2}_{loc}$ and $\underline{u}\in W^{2,2}_{loc}$ and the equations hold a.e. By induction one easily sees that $(\underline{u},\underline{w})\in W^{k,2}_{loc}$ for all k and hence $(\underline{u},\underline{w})\in C^\infty$.

We choose $s_0 > 1$ large enough so that

$$\frac{1}{2}<1+\underline{w}(s)<\frac{3}{2}\ \text{for}\ s\geq s_0\,, \tag{93}$$

which is possible by (80). In this region we may divide eq. (91) by $2s(1+w)$ and get

$$s\left(s(2\underline{w}+\underline{w}^2)+\frac{1}{2}\underline{u}'\right)=\frac{1}{8s}\frac{1}{(1+\underline{w})}(s\underline{w}')'. \tag{94}$$

Then (92) becomes

$$\underline{u}(s)=\frac{1}{2}s\left[\frac{1}{8s(1+\underline{w})}(s\underline{w}')'\right]'. \tag{95}$$

Inserting this into (91) we get a fourth order equation for $\underline{w}$

$$(1+\underline{w})\left[s(2\underline{w}+\underline{w}^2)+\frac{1}{4}\left(s\left(\frac{(s\underline{w}')'}{8s(1+\underline{w})}\right)'\right)'\right]=\frac{1}{8s^2}(sw')'. \tag{96}$$

This can be rewritten as

$$\underline{w}^{(4)}(s)=-64\underline{w}(s)+g(x(s),s)+h(x(s),s) \tag{97}$$

where $x(s)=(\underline{w}^{(3)}(s),\underline{w}''(s),\underline{w}'(s),\underline{w}(s))$, and $g:\mathbb{R}^4\times\mathbb{R}^+\to\mathbb{R}$ contains the nonlinear terms in x, $h:\mathbb{R}^4\times\mathbb{R}^+\to\mathbb{R}$ the linear ones with coefficients $O(s^{-1})$. More precisely,

$$\begin{aligned} g(x,s)=&\frac{1}{1+\underline{w}}\left(2\underline{w}'\underline{w}^{(3)}+\frac{4}{s}\underline{w}''\underline{w}'+\underline{w}''^2-\frac{1}{s^2}\underline{w}'^2-\frac{1}{1+\underline{w}}\left(2\underline{w}'^2\underline{w}''+\frac{2}{s}\underline{w}'^3\right)\right)\\ &-96\underline{w}^2-32\underline{w}^3 \end{aligned} \tag{98}$$

$$h(x,s)=-\frac{2}{s}\underline{w}^{(3)}+\frac{5}{s^2}\underline{w}^{(2)}+\frac{3}{s^3}\underline{w}' \tag{99}$$

In particular, for $f:=g+h$, and given $\varepsilon>0$, there exist $S\geq s_0$, $\delta>0$ such that

$$|f(x,s)-f(\bar{x},s)|\leq\varepsilon|x-\bar{x}|$$

whenever $|x-\bar{x}|<\delta$ and $s>S$. Additionally, we have $f(0,s)=0$. Now we may rewrite eq. (97) as a system of first order equations,

$$\frac{\mathrm{d}}{\mathrm{d}s}x(s)^T=Ax(s)^T+F(x,s)$$

where $F:\mathbb{R}^4\times[s_0,\infty)\to\mathbb{R}^4$ is given by $F(x,s)=(f(x,s),0,0,0)^T$ and

$$A=\begin{pmatrix} 0 & -64\\ \mathrm{Id}_{3\times3} & 0\end{pmatrix}$$

The eigenvalues of A are $2(\pm1\pm i)$, i.e., A has two eigenvalues with positive real part and two with negative real part.

We already know that $\lim_{s\to\infty} \underline{w}(s) = \lim_{s\to\infty} w(s^2) = 0$. We will now show that

$$\lim_{s\to\infty} \underline{w}^{(3)}(s) = \lim_{s\to\infty} \underline{w}''(s) = \lim_{s\to\infty} \underline{w}'(s) = 0\,. \tag{100}$$

It follows that $\lim_{s\to\infty} x(s) = 0$. From Theorem 3, it follows that there exists $\bar{s}$ such that $x(\bar{s}) \in \bar{M}(\bar{s})$, and hence $|x(s)| = o(\exp(-\sigma s))$ for $\sigma < 2$. It remains to prove (100).

We first show $\underline{w}'' \in L^2((s_0,\infty); \mathrm{d}s/s)$. From (91) we get

$$|\underline{w}'' + s^{-1}\underline{w}'| \leq Cs \left| s(2\underline{w} + \underline{w}^2) + \frac{1}{2}\underline{u}' \right|. \tag{101}$$

Therefore

$$\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}\underline{w}''^2 \; \leq 2\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}\left[s^2 \left(s(2\underline{w} + \underline{w}^2) + \frac{1}{2}\underline{u}' \right)^2 + s^{-2}\underline{w}'^2 \right] < \infty$$

by (90). Again by (90) we have $\underline{w}' \in L^2((s_0,\infty); s\mathrm{d}s)$. Thus Lemma 1 yields

$$\lim_{s\to\infty} \underline{w}'(s) = 0. \tag{102}$$

Next we derive a weighted L^2 estimate for the third derivative $\underline{w}^{(3)}$. It follows from (95) that

$$\frac{\underline{u}}{s} = \frac{1}{16(1+\underline{w})}\left(\underline{w}^{(3)} + \frac{\underline{w}''}{s} - \frac{\underline{w}'}{s^2} - \frac{\underline{w}'\underline{w}''}{1+\underline{w}} - \frac{\underline{w}'^2}{s(1+\underline{w})} \right),$$

which implies (using the convergence of w and w')

$$\left|\underline{w}^{(3)}(s)\right|^2 \leq C\left(\left|\frac{\underline{u}(s)}{s}\right|^2 + \left|\frac{\underline{w}''}{s}\right|^2 + \left|\frac{\underline{w}'}{s^2}\right|^2 + \underline{w}''^2\underline{w}'^2 + \left|\frac{\underline{w}'^2}{s}\right|^2 \right). \tag{103}$$

With the possible exception of $\underline{w}''^2\underline{w}'^2$ all terms on the right hand side are integrable against $s\mathrm{d}s$. Thus we get for $s_1 > s_0$

$$\int_{s_0}^{s_1} s\mathrm{d}s \left|\underline{w}^{(3)}\right|^2 \leq C(1 + \sup_{[s_0,s_1]} |\underline{w}''|^2), \tag{104}$$

where C is controlled by $E^+(u,w)$ and in particular independent of s_1. Now we get as usual

$$\begin{aligned}
&\sup_{[s_0,S1]} |\underline{w}''|^2 - \inf_{[s_0,s_1]} |\underline{w}''|^2 \leq 2\int_{s_0}^{s_1} \mathrm{d}s|\underline{w}''\underline{w}^{(3)}| \\
\leq& 2\|\underline{w}''\|_{L^2((s_0,\infty);\mathrm{d}s/s)} C^{1/2}(1 + \sup_{[s_0,s_1]} |\underline{w}''|^2)^{1/2} \\
\leq& 4C\|\underline{w}''\|^2_{L^2((s_0,\infty);\mathrm{d}s/s)} + \frac{1}{4}(1 + \sup_{[s_0,s_1]} |\underline{w}''|^2)
\end{aligned} \tag{105}$$

Moreover

$$\inf_{[s_0,s_1]} \underline{w}''^2 \leq \frac{1}{\ln s_1/s_0}\int_{s_0}^{s_1} \frac{\mathrm{d}s}{s}\underline{w}''^2 \leq \frac{C}{\ln s_1/s_0}. \tag{106}$$

Thus absorbing the term $\frac{1}{4}(1 + \sup_{[s_0,s_1]} |\underline{w}''|^2)$ into the left hand side of the (105) and taking $s_1 \to \infty$ we get

$$\frac{3}{4}\sup_{[s_0,\infty]} \underline{w}''^2 \leq 4C\|\underline{w}''\|^2_{L^2((s_0,\infty);\mathrm{d}s/s)} < \infty \tag{107}$$

and by (104)

$$\int_{s_0}^{\infty} s\mathrm{d}s \left|\underline{w}^{(3)}\right|^2 < \infty. \tag{108}$$

Since $\underline{w}'' \in L^2((s_0, \infty); \mathrm{d}s/s)$ it follows that

$$\lim_{s\to\infty} \underline{w}''(s) = 0. \tag{109}$$

Finally we claim that $\underline{w}^{(4)} \in L^2((s_0, \infty); \mathrm{d}s/s)$. Indeed from the previous L^2 bounds we see immediately that $h \in L^2((s_0, \infty); \mathrm{d}s/s)$. Moreover the convergence of $\underline{w}'$ and $\underline{w}''$ imply that $g(x(s), s) \leq C(|w^{(3)}| + |\underline{w}'| + |\underline{w}''| + |\underline{w}|)$. By (42) we have

$$\int_{s_0}^{\infty} \frac{\mathrm{d}s}{s}(2\underline{w} + \underline{w}^2)^2 = \frac{1}{2}\int_{s_0^2}^{\infty} \frac{\mathrm{d}r}{r}(2w + w^2)^2 < \infty. \tag{110}$$

Since $\frac{5}{2} < 2 + \underline{w}(s) < \frac{7}{2}$, we get $\underline{w} \in L^2((s_0, \infty); \mathrm{d}s/s)$. Together with the weighted L^2 estimates for $\underline{w}^{(i)}$ for $i = 1, 2, 3$ we get $\underline{w}^{(4)} \in L^2((s_0, \infty); \mathrm{d}s/s)$. In combination with the estimate $\underline{w}^{(3)} \in L^2((s_0, \infty); s\mathrm{d}s)$ this implies that

$$\lim_{s\to\infty} \underline{w}^{(3)}(s) = 0. \tag{111}$$

Thus $\underline{w}^{(i)} = o(\exp(-\sigma s))$ as $s \to \infty$ for $i = 0, 1, 2, 3$ and all $\sigma < 2$. This implies $u(r), u'(r), w(r), w'(r) = o(\exp(-\sigma\sqrt{r}))$ as $r \to \infty$ for all $\sigma < 2$. □

Proof of Theorem 1. For $\lambda = 1$, this follows from Theorem 2 and Proposition 2. For $\lambda \neq 1$, we recall that by (11), we have $\hat{E}^R_\lambda(\hat{u}, \hat{w}) = \lambda^2 \hat{E}_1^{R/\lambda^2}(\hat{u}, \hat{w})$ and hence

$$\hat{E}_\lambda(\hat{u}, \hat{w}) = \lambda^2 \lim_{R\to\infty} \hat{E}_1^{R/\lambda}(\hat{u}_\lambda, \hat{w}_\lambda) = \lambda^2 E(u_\lambda, w_\lambda)$$

for all $(\hat{u}, \hat{w}) \in \mathcal{W}$, where we used the notation $\hat{u}_\lambda = \lambda^{-1}\hat{u}(\lambda\cdot)$, $\hat{w}_\lambda = \hat{w}(\lambda\cdot)$, $u_\lambda(r) = \hat{u}_\lambda(r) - \psi(r)/(2r)$, $w_\lambda = \hat{w}_\lambda - \psi$ and Lemma 4. Now all statements follow from the case $\lambda = 1$ already treated. □

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