ON THE UNITARY GLOBALIZATION OF COHOMOLOGICALLY INDUCED MODULES

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ABSTRACT. We describe the unitary globalization of cohomologically induced modules $A_{\mathfrak{q}}(\lambda)$. The purpose of the paper is to give a geometric realization of the unitarizable modules. Our results do not constitute a proof of unitarity.

1. INTRODUCTION

The orbit method suggests a close connection between irreducible representations of a Lie group $G_{\mathbf{R}}$ and co-adjoint orbits. In the case of nilpotent groups, unitary representations correspond to co-adjoint orbits. Kirillov used such correspondence to geometrically construct unitary representations of nilpotent groups. When $G_{\mathbf{R}}$ is a real reductive group, attached to elliptic co-adjoint orbits is a family of irreducible unitarizable $(\mathfrak{g}, K)$-modules, $A_{\mathfrak{q}}(\lambda)$. These are called cohomologically induced modules. The purpose of this paper is to give a geometric description of the corresponding $G_{\mathbf{R}}$ unitary representations.

The representation theory of $G_{\mathbf{R}}$ is more subtle than that of its Lie algebra. According to a theorem of Harish-Chandra, the space of K-finite vectors of an admissible irreducible $G_{\mathbf{R}}$-representation is an irreducible $(\mathfrak{g}, K)$-module. This relationship between $G_{\mathbf{R}}$ and $(\mathfrak{g}, K)$ modules is not bijective. The subtlety comes from the fact that different topologies yield different $G_{\mathbf{R}}$-modules. It is known, from the work of Casselman, Wallach [4] and Schmid [12], that each admissible irreducible $(\mathfrak{g}, K)$-module admits a maximal globalization (resp. minimal globalization) to a $G_{\mathbf{R}}$ module over a Frechet space, $\mathcal{F}$ (resp. to a $G_{\mathbf{R}}$ module over the topological dual of $\mathcal{F}$). When the $(\mathfrak{g}, K)$-module is an $A_{\mathfrak{q}}(\lambda)$, the maximal globalization (minimal globalization) corresponds to a Dolbeault cohomology representation, $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$, (corresponds to a compactly supported cohomology space). The unitary globalization should correspond to a topological space "in between" the minimal and the maximal globalization. This is the space we want to describe.

In [17], Vogan proposed to (a) explicitly describe the $G_{\mathbf{R}}$-invariant Hermitian form on the minimal globalization of $A_{\mathfrak{q}}(\lambda)$ (identified as a space of compactly supported cohomology) and (b) describe the unitary globalization as the completion of the compactly supported cohomology with respect to the Hermitian form given

in (a). This proposal amounts to finding an explicit $G_{\mathbf{R}}$-intertwining map from the Hermitian dual $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)^h$ to $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$. Moreover, such map is a "kernel type" transform. Indeed, if $(G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}$ is $G_{\mathbf{R}}/L_{\mathbf{R}}$ endowed with the opposite complex structure, then the kernel is determined by a diag $(G_{\mathbf{R}} \times G_{\mathbf{R}})$-invariant class $[w]$ in $H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$. The problem is to describe such a cohomology class.

Under some positivity assumptions on λ, $A_{\mathfrak{q}}(\lambda)$ can be also realized as the space of K-finite solutions of an elliptic differential operator $\mathcal{D}$, acting on sections of a bundle over $G_{\mathbf{R}}/K_{\mathbf{R}}$. This is indeed the content of ([[19], Corollary 38 and Section 7]). The geometric construction of Dolbeault cohomology is related to the solution space of $\mathcal{D}$ via the so-called Real Penrose transform. Moreover, the Real Penrose transform determines an isomorphism between the cohomology realization and $Ker\,\mathcal{D}$. See ([[19], Section 7]), ([[1], Section 10]) and [2]. In this paper we determine (a) the hermitian dual $(Ker\,\mathcal{D})^h$, (b) the space of continuous $G_{\mathbf{R}}$-intertwining maps from $(Ker\,\mathcal{D})^h$ to $Ker\,\mathcal{D}$. Such intertwining maps are also of "kernel type". We describe the kernels in terms of generalized spherical functions, F. In particular, when $A_{\mathfrak{q}}(\lambda)$ is the Harish-Chandra module of a representation in the discrete series, F is the function defined by Flensted-Jensen in [6]. The function F, given in Theorem 5.7, depends solely on the minimal K-type of $A_{\mathfrak{q}}(\lambda)$. This is consistent with the fact that $A_{\mathfrak{q}}(\lambda)$ is unitary if and only if the Hermitian form is definite on its bottom layer, see [17]. The cohomology class $[w]$ is completely determined by F.

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2. THE MAXIMAL GLOBALIZATION OF $A_{\mathfrak{q}}(\lambda)$

2.1. **Dolbeault Cohomology.** We recall results from [17] and [19] that will be relevant to our work. Our underlying group $G_{\mathbf{R}}$ is assumed to be real reductive with complexification G and Cartan involution Θ. We let $K_{\mathbf{R}}$ be the fixed point group of Θ, the maximal compact subgroup. We denote the Lie algebra of Lie groups by $\mathfrak{g}_{\mathbf{R}}$, $\mathfrak{k}_{\mathbf{R}}$ etc., and their complexifications by $\mathfrak{g}$, $\mathfrak{k}$, etc. Letting θ be the differential of Θ we write the decomposition of $\mathfrak{g}$ into ± 1 eigenspaces as $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$. We choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{k}$ and extend it to a Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ of $\mathfrak{g}$. Using the Killing form, $B(\cdot, \cdot)$, we consider $\mathfrak{t}^* \subset \mathfrak{h}^* \subset \mathfrak{g}^*$. Then an element $\lambda \in \mathfrak{t}^*$ is elliptic, and the orbit $G_{\mathbf{R}} \cdot \lambda \subset \mathfrak{g}^*$ is an elliptic co-adjoint orbit. We may identify this orbit with the homogeneous space $G_{\mathbf{R}}/L_{\mathbf{R}}$, where $L_{\mathbf{R}}$ is the centralizer in $G_{\mathbf{R}}$ of λ. On the other hand, λ defines a θ-stable parabolic subalgebra of $\mathfrak{g}$ as follows. Denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the roots of $\mathfrak{h}$ in $\mathfrak{g}$. Then the parabolic subalgebra associated to λ is $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where the Levi factor $\mathfrak{l}$ is spanned by $\mathfrak{h}$ and all root spaces $\mathfrak{g}^\alpha$ with $\langle \lambda, \alpha \rangle = 0$, and $\mathfrak{u}$ is spanned by all root spaces $\mathfrak{g}^\alpha$ with $\langle \lambda, \alpha \rangle > 0$. If $\mathcal{Q}$ is the normalizer of $\mathfrak{q}$ in G, one sees that $L_{\mathbf{R}} = \mathcal{Q} \cap G_{\mathbf{R}}$, so $G_{\mathbf{R}}/L_{\mathbf{R}}$ embeds

into the generalized flag variety $G/\mathcal{Q}$ as an open subset. In particular, $G_{\mathbf{R}}/L_{\mathbf{R}}$ has a $G_{\mathbf{R}}$-invariant complex structure; the antiholomorphic tangent space at the identity coset is naturally identified with $\mathfrak{g}/\mathfrak{q} \simeq \mathfrak{u}$. A similar construction makes $K_{\mathbf{R}}/(K_{\mathbf{R}} \cap L_{\mathbf{R}})$ into a complex compact submanifold of $G_{\mathbf{R}}/L_{\mathbf{R}}$.

Observe that each θ-stable parabolic subalgebra $\mathfrak{q}$ containing $\mathfrak{l}$ gives a complex structure on $G_{\mathbf{R}}/L_{\mathbf{R}}$; these are in fact all different. In the language of geometric quantization, these parabolic subalgebras are the invariant complex polarizations at λ. Typically, in geometric quantization, one chooses a particular polarization and this is what we will do here.

To attach a representation to $G_{\mathbf{R}} \cdot \lambda$, we assume that λ lifts to a character χ_λ of $L_{\mathbf{R}}$. Then there is a holomorphic homogeneous line bundle associated to χ_λ. If n is the complex dimension of $G_{\mathbf{R}}/L_{\mathbf{R}}$, it is natural to attach cohomology representations $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ to the orbit $G_{\mathbf{R}} \cdot \lambda$. The Dolbeault cohomology $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ can be computed, for example, by Leray covers, by C^∞-differential forms or by currents (differential forms with distribution coefficients). All these approaches yield the same cohomology groups, as vector spaces. Indeed, more is true. One can define strong topologies on $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ with respect to Leray covers, as well as with respect to C^∞-forms or currents, see [9]. (For example, the strong topology with respect to C^∞-forms is given by uniform convergence on compact sets for all derivatives of the coefficients when written in terms of coordinates in the local charts.)

Theorem 2.1 ([9], Theorem 2.1, Theorem 3.2). *The strong topologies on $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ with respect to Leray covers, C^∞-forms and currents all coincide.*

It is not *a priori* clear that the natural action of $G_{\mathbf{R}}$ (by left translation) on $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is continuous in the topology of Theorem 2.1. The difficulty is that a topology on $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is Hausdorff only if the operator $\overline{\partial}$ that defines the cohomology space has the closed range property in that given topology. This delicate issue was settled by Wong in [19] and [20]. Wong proved that (a) when $p = s = \dim(K_{\mathbf{R}}/(K_{\mathbf{R}} \cap L_{\mathbf{R}})$ $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is a non-zero continuous Fréchet representation (by showing that the image of $\overline{\partial}$ is closed in the C^∞-topology on forms), (b) $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)_{K-\text{finite}}$ is a cohomologically induced $(\mathfrak{g}, K)$-module, (c) $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is the maximal globalization of its underlying Harish-Chandra module in the sense of [12].

Remark 2.2. It is important to note that the results in [19] and [20] are more general than those stated above. On the one hand, [20] allows the inducing bundle to be infinite dimensional. On the other hand, the conditions on the line bundle in [19] are less restrictive that the ones used here. Indeed, for a fixed positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$ that contains $\Delta(\mathfrak{u})$ the main theorem in [19] holds for a one-dimensional representation χ_ν of L_R with weight ν, with $\langle \nu + \rho, \alpha \rangle > 0$ for roots α in $\mathfrak{u}$ (and ρ

equal to the half the sum of positive roots). In this paper we assume that

$$\langle \lambda, \alpha \rangle > 0 \text{ for all root } \alpha \in \Delta(\mathfrak{u}). \qquad (2.3)$$

We impose this, more restrictive, assumption on λ in order to have control on the K-type structure of $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)_{K-\text{finite}}$, see [16].

We keep the notation $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ and write $\mathfrak{u} = \mathfrak{u} \cap \mathfrak{p} \oplus \mathfrak{u} \cap \mathfrak{k}$. We let $\Delta(\mathfrak{u} \cap \mathfrak{p})$ stand for the set of weights in $\mathfrak{u} \cap \mathfrak{p}$ with respect to $\mathfrak{h}$ and we write $\rho(\mathfrak{u} \cap \mathfrak{p})$ for half the sum of the positive members of $(\mathfrak{u} \cap \mathfrak{p})$ with respect to $\mathfrak{h}$.

Theorem 2.4. *Suppose that $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \subset \mathfrak{l}$ be a Cartan subalgebra. Let $\lambda \in \mathfrak{t}^*$ be an integral weight and assume that*

$$\langle \lambda, \alpha \rangle > 0, \text{ for all root } \alpha \in \Delta(\mathfrak{u}).$$

Identify the elliptic co-adjoint orbit $G_{\mathbf{R}} \cdot \lambda$ with the homogeneous space $G_{\mathbf{R}}/L_{\mathbf{R}}$. Endow $G_{\mathbf{R}}/L_{\mathbf{R}}$ with the complex structure so that the antiholomorphic tangent space at the identity is identified with $\mathfrak{u}$. Let $s = dim_{\mathbf{C}}(K_{\mathbf{R}}/(L_{\mathbf{R}} \cap K_{\mathbf{R}}))$. Then,

(1) *The strong topology of Theorem 2.1 on $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is Hausdorff. In particular, the $\overline{\partial}$ Dolbeault operator has closed range.*
(2) $H^{n,p}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda) = 0$, *unless $p = s$.*
(3) *The continuous representation of $G_{\mathbf{R}}$ on $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)$ is irreducible and Hermitian. It is the maximal globalization of the underlying $(\mathfrak{g}, K)$-module.*
(4) $(\pi_\lambda, H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda))$ *contains with multiplicity one, the K-type with highest weight*

$$\mu = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

If μ' is the highest weight of a K-type occurring in $\pi_\lambda|_K$, then μ' is of the form

$$\mu' = \mu + \sum n_\alpha \, \alpha, \text{ with } n_\alpha \in \mathbb{N} \text{ and } \alpha \in \Delta(\mathfrak{u} \cap \mathfrak{p}).$$

Proof. Part (1) is proved in [19]. Parts (2) and (3) are results in [17], written here in the language of Dolbeault cohomology. Part (4) is Theorem 5.3 in [16]. $\square$

2.2. **The kernel of Schmid's $\mathcal{D}$-differential operator.** The maximal globalization of $A_\mathfrak{q}(\lambda)$ can be described as the solution space of an elliptic operator acting on the space of smooth sections of a bundle over $G_{\mathbf{R}}/K_{\mathbf{R}}$. Let (τ_μ, V_μ) be the minimal K-type occurring in $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)_{K-\text{finite}}$. The relevant vector bundle over $G_{\mathbf{R}}/K_{\mathbf{R}}$ is the one induced by V_μ. Indeed, such realization plays a key role in proving that the Dolbeault cohomology endowed with the topology of Theorem 2.1 is Hausdorff ([10] and [19]). This alternative realization of cohomologically induced modules will be important in our work. We start by recalling that $\mu = \lambda + 2\rho(\mathfrak{u} \cap \mathfrak{p})$ and that the highest weights of the irreducible K-modules occuring in $V_\mu \otimes \mathfrak{p}$ are of the form $\mu + \alpha$ with $\alpha \in \Delta(\mathfrak{p}, \mathfrak{t})$. Following [19] we introduce the following definition.

Definition 2.5. Let

$$V_\mu^- = \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} \tau_{\mu-\beta} \subset V_\mu \otimes \mathfrak{p}$$

and let $\mathbb{P} : V_\mu \otimes \mathfrak{p} \to V_\mu^-$ be the canonical projection. Choose $\{X_i\}$ an orthonormal basis of $\mathfrak{p}$ with respect to $(U, V) = -B(U, \overline{\theta V})$. For $F \in C^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu)$ define

$$\mathcal{D}F(g) = \sum_i \mathbb{P}\big[(X_i F)(g) \otimes \overline{X}_i\big]. \tag{2.6}$$

Proposition 2.7. *([19], Proposition 49) $\mathcal{D}$ is well defined (i.e. independent of the choice of basis). If λ is sufficiently positive, $\mathcal{D}$ is an elliptic operator.*

Observe that $C^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu)$, endowed with the topology of uniform convergence over compact subsets of functions and their derivatives, is a Fréchet space. The operator $\mathcal{D} : C^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu) \to C^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu^-)$ is continuous. Hence, the kernel space $Ker\,\mathcal{D}$ is closed in $C^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu)$ and it inherits the structure of Fréchet space. In order to emphasize the space on which $\mathcal{D}$ acts we write

$$Ker\,\mathcal{D} = C_\mathcal{D}^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu). \tag{2.8}$$

Theorem 2.9. *Keep the assumptions on Theorem 2.4. If $\langle \lambda - 2\rho(\mathfrak{l} \cap \mathfrak{k}), \alpha \rangle > 0$ for all root $\alpha \in \Delta(\mathfrak{u})$, then there exists a $G_\mathbf{R}$-equivariant map*

$$\mathcal{P} : H^{n,s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_\lambda) \to C_\mathcal{D}^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu).$$

The map $\mathcal{P}$ is an homeomorphism of topological spaces.

Proof. When $\mathrm{rank}(G_\mathbf{R}) = \mathrm{rank}(K_\mathbf{R})$ and $L_\mathbf{R} = T_\mathbf{R}$ is a maximal compact torus $\rho(\mathfrak{l} \cap \mathfrak{k}) = 0$ and the Theorem holds for all λ satisfying the positivity condition of Theorem 2.4. This result is proved in [11]. The general statement follows form ([19], Corollary 38 and the proof of Proposition 52). Let $\mathcal{N}$ be the normal bundle of the compact submanifold $K_\mathbf{R}/(L_\mathbf{R} \cap K_\mathbf{R})$ in $G_\mathbf{R}/L_\mathbf{R}$ and let (k) signify the k-th symmetric power. By ([19], Corollary 38) the Theorem holds if

$$H^i(K_\mathbf{R}/(L_\mathbf{R} \cap K_\mathbf{R}), \mathcal{L}_{\lambda+2\rho(\mathfrak{u})} \otimes (\mathcal{N}^*)^{(k)}) = 0 \text{ for all } i < s \text{ and all } k \geq 0. \tag{2.10}$$

The vanishing condition (2.10) follows from ([7], Theorem G). Our assumption on λ guarantees that the hypothesis of ([7], Theorem G) are satisfied.

$\square$

Remark 2.11. The map $\mathcal{P}$ is the Real Penrose transform in ([10], Lemma 7.1) ([19], Section 7), ([1], [2]) and ([21], Lecture 7) . The transform

$$\mathcal{P} : H^{n,s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_\lambda) \to C_\mathcal{D}^\infty(G_\mathbf{R}/K_\mathbf{R}, V_\mu)$$

is an integral transform; see [1]. If $\omega_c \in \wedge^s(\mathfrak{u} \cap \mathfrak{k})^*$ is a normalized top form, then

$$\mathcal{P}(w)(x) = \int_{K/(L \cap K)} \tau_\mu(k) v_\mu \, \langle w(xk), 1_\lambda \otimes \omega_c \rangle \, dk, \tag{2.12}$$

with v_μ a normalized highest weight vector in V_μ.

Under the positivity assumptions of Theorem 2.9 the transform $\mathcal{P}$ is injective onto $Ker\,\mathcal{D}$. Each $G \in Ker\,\mathcal{D}$ determines a unique cohomology class $[\eta_G]$. One way to determine a representative of $[\eta_G]$ is to follow the recursive procedure described in ([10], Lemma 7.1) while keeping track of the required $\mathfrak{l}$-equivariant property.

3. The minimal globalization of $A_{\mathfrak{q}}(\lambda)$

3.1. **Compactly supported cohomology.** Theorem 2.4 identifies the maximal globalization of cohomologically-induced modules as Dolbeault cohomology representations. In this section we summarize relevant information on the minimal globalization of $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)_{K\text{-finite}}$.

It is known that the minimal globalization $H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)_{K\text{-finite}}$ is the topological dual of its maximal globalization. Since we know that $\overline{\partial}$ has the closed range property, Serre duality implies that the minimal globalization occurs as compactly supported cohomology. As for Dolbeault cohomology, compactly supported cohomology can be calculated in different ways. In particular, such cohomology groups can be computed by using Leray covers, C^∞ compactly supported forms or compactly supported currents (forms with compactly supported distribution coefficients). All these approaches yield the same cohomology group as vector spaces. These cohomology spaces can be endowed with strong topologies as described in [9]. Theorem 2.1 holds for compactly supported cohomology.

Theorem 3.1. ([9], Theorem 2.1). *The strong topologies on $H_c^{0,q}(G_{\mathbf{R}}, L_{\mathbf{R}}, \mathcal{L}_\lambda)$ with respect to Leray covers, C_c^∞-forms and compactly supported currents coincide.*

Theorem 3.2. *In the setting of Theorem 2.4 and Theorem 3.1, write $\mathcal{L}_\lambda^*$ for the bundle dual to $\mathcal{L}_\lambda$.*

  (1) *There is a natural topological isomorphism*
  $$H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*)^* \simeq H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda).$$
  (2) *The topological space $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*)$ is Hausdorff.*

Proof. A more general version of this duality theorem is proved in ([9], Theorem 3.2). Also see [13] and the remarks in ([18], page 67).   □

Theorem 3.3 ([3], page 285; [18], Corollary 8.15)**.** *Keep the hypothesis of Theorem 2.4.*

  (1) $H_c^{0,q}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*) = 0$, *unless $q = n - s$.*
  (2) $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*)$ *is non-zero and it admits a continuous $G_{\mathbf{R}}$ action. The resulting representation is irreducible. It is the minimal globalization of the underlying $(\mathfrak{g}, K)$-module.*

(3) $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*)$ *admits an invariant Hermitian form.*

3.2. The topological dual of $Ker\,\mathcal{D}$.

Proposition 3.4. *Endow $C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$ with the strong topology relative to the smooth topology on $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$. Write*

$$\left(Ker\,\mathcal{D}\right)^\perp = \{\Lambda \in C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^* |\ \Lambda|_{Ker\,\mathcal{D}} = 0\}.$$

(1) *$C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp$, endowed with the quotient topology of the strong topology on $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*$, is homeomorphic to $C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*$ endowed with the strong topology.*

(2) *The topological spaces $C_c^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)/\left(Ker\,\mathcal{D}\right)^\perp \cap C_c^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$ and $C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*$ and are homeomorphic.*

Proof. The first statement of the Proposition follows from ([18], Prop.8.8 (2)). Indeed, the space of smooth sections of a finite-dimensional vector bundle, when endowed with the smooth topology, is a Fréchet nuclear space. Thus, it is reflexive. Also, $Ker\,\mathcal{D} \subset C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$ is a closed subspace, as $\mathcal{D}$ is a continuous operator. Hence, the hypothesis of ([18], Prop.8.8 (2)) is satisfied. We conclude that if $i : Ker\,\mathcal{D} \to C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$ is the natural inclusion, then the transpose map i^t induces the desired homeomorphism. This is,

$$i^t : C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp \to C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*,$$

is a homeomorphism of topological spaces.

In order to prove the second statement of the Proposition observe that $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp$ endowed with the quotient topology of the strong topology on $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*$ is Hausdorff. Indeed, the transpose $\mathcal{P}^t$ of the homeomorphism $\mathcal{P}$ in Theorem (2.9) is a homeomorphism from $\left(Ker\,\mathcal{D}\right)^*$ to the compactly supported cohomology $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda^*)$. As the cohomology space is Hausdorff (it is the minimal globalization of its underlying $(\mathfrak{g}, K)$-modules), $\left(Ker\,\mathcal{D}\right)^* \simeq C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp$ is Hausdorff. It follows, see ([14], Chapter 4), that the kernel of the continuous map $C_c^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu) \to C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp$ is closed. Hence, we have a continuous $G_{\mathbf{R}}$-equivariant map

$$T : C_c^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)/\left(Ker\,\mathcal{D}\right)^\perp \to C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp, \qquad (3.5)$$

where $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)^*/\left(Ker\,\mathcal{D}\right)^\perp$ is the minimal globalization of its underlying $(\mathfrak{g}, K)$-module, the Proposition follows. $\square$

4. HERMITIAN PAIRINGS

If E is a complete locally convex vector space, then its Hermitian dual E^h is given by

$E^h = \{\Lambda : E \to \mathbf{C} \text{ continuous} :$

$$\Lambda(av + bw) = \overline{a}\Lambda(v) + \overline{b}\Lambda(w) \text{ for } a, b \in \mathbf{C} \text{ and } v, w \in E\}.$$

The space E^h is conjugate linearly isomorphic to E^*. Using this identification E^h can be endowed with the strong topology; see ([18], Section 8.3). A Hermitian pairing between two complete locally convex vector spaces E and F is a separately continuous map

$$\langle \, , \, \rangle : E \times F \to \mathbf{C}$$

that is linear in the first variable and conjugate linear in the second variable. This space is in bijection with the space of continuous linear maps $L(E, F^h)$; see for example ([18], Section 9).

4.1. **Hermitian pairings on** $H_c^{0,n-s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_\lambda^*)$.

Theorem 4.1. [18] *Let* $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ *be a* θ-*stable parabolic subalgebra. Let* $\mathcal{Q}$ *be the analytic subgroup of* G *with Lie algebra* $\mathfrak{q}$. *Endow* $G_\mathbf{R}/L_\mathbf{R}$ *with the complex structure induced by the open embedding* $G_\mathbf{R}/L_\mathbf{R} \subset G/\mathcal{Q}$. *Let* $(G_\mathbf{R}/L_\mathbf{R})^{opp}$ *be the manifold* $G_\mathbf{R}/L_\mathbf{R}$ *endowed with the opposite complex structure.*

(1) *The Hermitian dual of* $H_c^{0,n-s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_{-\lambda})$ *is* $H^{n,s}((G_\mathbf{R}/L_\mathbf{R})^{opp}, \mathcal{L}_{-\lambda})$.
(2) *The space of separately continuous Hermitian pairings on* $H_c^{0,n-s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_{-\lambda})$ *is isomorphic to* $H^{(n,n)(s,s)}(G_\mathbf{R}/L_\mathbf{R} \times (G_\mathbf{R}/L_\mathbf{R})^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$.
(3) *The space of* $G_\mathbf{R}$-*invariant Hermitian forms on* $H_c^{0,n-s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_{-\lambda})$ *is the space of* $diag(G_\mathbf{R} \times G_\mathbf{R})$- *invariant real cohomology classes in* $H^{(n,n)(s,s)}(G_\mathbf{R}/L_\mathbf{R} \times (G_\mathbf{R}/L_\mathbf{R})^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$.

Remark 4.2.   (1) For a definition of *real cohomology class*, see ([18], page 339).
(2) *When* $H_c^{0,n-s}(G_\mathbf{R}/L_\mathbf{R}, \mathcal{L}_{-\lambda})$ *is computed by the complex of compactly supported smooth forms, the isomorphism in part (1) of Theorem 4.1 assigns to a compactly supported form* ϕ *the functional* Λ_ϕ *on* $H^{n,s}((G_\mathbf{R}/L_\mathbf{R})^{opp}, \mathcal{L}_{-\lambda})$ *given by*

$$\omega \in H^{n,s}((G_\mathbf{R}/L_\mathbf{R})^{opp}, \mathcal{L}_{-\lambda}) \to \Lambda_\phi(\omega) \in \mathbf{C}$$

$$\Lambda_\phi(\omega) = \int_{G_\mathbf{R}/L_\mathbf{R}} \phi \wedge \sigma(\omega).$$

Here σ *is the complex conjugation in cohomology induced by the map*

$$C^\infty((G_\mathbf{R}/L_\mathbf{R})^{opp}, \wedge^s\overline{\mathfrak{u}} \otimes \wedge^n\mathfrak{u} \otimes \mathbf{C}_{-\lambda}) \to C^\infty(G_\mathbf{R}/L_\mathbf{R}, \wedge^s\mathfrak{u} \otimes \wedge^n\overline{\mathfrak{u}} \otimes \mathbf{C}_\lambda). \quad (4.3)$$

(3) When $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda})$ is computed by the complex of compactly supported smooth forms, and $w(\cdot, \cdot)$ represents a smooth cohomology class in $H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$, the hermitian pairing in part (2) of Theorem 4.1 assigns to a compactly supported form ϕ the smooth Dolbeault cohomolgy class represented by $\eta(y) = \int_{G_{\mathbf{R}}/L_{\mathbf{R}}} \phi(x) \wedge w(x, y)$.

(4) When $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda})$ is identified with the space of conjugate linear continuous maps on $H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{opp}, \mathcal{L}_{-\lambda})$, the Hermitian pairing (2) in Theorem (4.1) assigns to a functional Λ, the Dolbeault cohomology class $\eta(y) = \Lambda\big(\sigma \otimes 1)(w(\cdot, y)\big)$. Here $\sigma \otimes 1$ is the conjugation in (4.3) applied to the "first variable".

(5) The space $H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$ is topologically isomorphic to the projective tensor
$H^{(n,s)}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda)\hat{\otimes}_\pi H^{(n,s)}((G_{\mathbf{R}}/L_{\mathbf{R}})^{opp}, \mathcal{L}_{-\lambda})$. See ([18], page 72) and ([14], Definition 43.2 and 43.5).

4.2. **Hermitian pairings in the $G_{\mathbf{R}}\backslash K_{\mathbf{R}}$-picture.** Write $(\tau_\mu^\vee, V_\mu^\vee)$ for the representation of $K_{\mathbf{R}}$ contragredient to (τ_μ, V_μ). Let T_σ denote the conjugate linear isomorphism from V_μ to $V_\mu^\vee$.

Definition 4.4. For F a smooth section of the vector bundle

$$(G_{\mathbf{R}} \times G_{\mathbf{R}}) \times_{K_{\mathbf{R}} \times K_{\mathbf{R}}} (V_\mu^\vee \otimes V_\mu),$$

and $g \in G_{\mathbf{R}}$ write $(R(1, g)F)(x, y) = F(x, yg)$ and use the same notation for the differential of the right action. Similarly define $R(g, 1)$. Choose $\{X_i\}$ an orthonormal basis of $\mathfrak{p}$ with respect to $(U, V) = -B(U, \overline{\theta V})$. Let $\mathbb{P}$ be the projection operator in Definition (2.6) and define the differential operator

$$1 \otimes \mathcal{D} : C^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu) \to C^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu^-)$$

by means of

$$[1 \otimes \mathcal{D}F](x, y) = \sum_i 1 \otimes \mathbb{P}\big[(R(1, X_i)F)(x, y) \otimes \overline{X_i}\big].$$

Similarly, define the operator $\underline{\mathcal{D}} \otimes 1$.

Definition 4.5. Let $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$ be the space of smooth sections of the vector bundle $G_{\mathbf{R}} \times G_{\mathbf{R}} \times_{K_{\mathbf{R}} \times K_{\mathbf{R}}} (V_\mu^\vee \otimes V_\mu)$ that are annihilated by both differential operators $1 \otimes \mathcal{D}$ and $\underline{\mathcal{D}} \otimes 1$. Endow the space $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$ with the strong topology relative to the smooth topology on the space of sections.

Proposition 4.6. $H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times G_{\mathbf{R}}/L_{\mathbf{R}}^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$ is homeomorphic to $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$.

Proof. We first prove that

$$C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee) \hat{\otimes}_\pi C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu) \text{ is homeomorphic to} \tag{4.7}$$
$$C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu).$$

It is not difficult to show, using ([14], Prop 44.1), that $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu) \simeq C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}) \hat{\otimes}_\pi V_\mu$. Using this observation and arguing as in ([14], Thm 51.6) we show that $C^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$ is canonically isomorphic to $C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee) \hat{\otimes}_\pi C^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu)$. $C^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$ has the structure of Souslin space, see ([14], page 556). As $\mathrm{Ker}(1 \otimes \mathcal{D})$ and $\mathrm{Ker}(\underline{\mathcal{D}} \otimes 1)$ are closed in $C^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$, we conclude that the space $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$ is Souslin. By ([14], Appendix Corollary 1), the surjective continuous map

$$C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu), \to \mathrm{Ker}(\underline{\mathcal{D}}) \hat{\otimes}_\pi \mathrm{Ker}(\mathcal{D}) \tag{4.8}$$

is open. This proves our claim.

Next, we recall (Remark (4.2), part 5) that the spaces

$$H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda}), \text{ and} \tag{4.9}$$
$$H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda) \hat{\otimes}_\pi H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_{-\lambda}).$$

are homeomorphic.

To complete the proof of the Proposition it is enough to argue that the tensor products in displays (4.9) and (4.7) are homeomorphic. In order to prove so, recall that under our assumptions on λ, the Penrose transforms maps of Theorem (2.9),

$$\mathcal{P} : H^{n,s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_\lambda) \to C_{\mathcal{D}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu) \text{ and}$$
$$\mathcal{P}_{\mathrm{opp}} : H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_{-\lambda}) \to C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)$$

are homeomorphism of topological spaces. Since the spaces under consideration are Fréchet, by ([14], Prop 43.9), $\mathcal{P} \hat{\otimes} \mathcal{P}_{\mathrm{opp}}$ implements the desired homeomorphism. (For a definition of $\mathcal{P} \hat{\otimes} \mathcal{P}_{\mathrm{opp}}$ see ([14], Definition 43.6).)

$\square$

Definition 4.10. Write $\mathcal{P} \otimes \mathcal{P}_{\mathrm{opp}}$ for the map that implements the homeomorphism from $H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$ to $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu^\vee \otimes V_\mu)$.

(1) Let $\mathcal{P}_{\mathrm{opp}}^h$ be the Hermitian transpose to the Penrose transform

$$\mathcal{P}_{opp} : H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_{-\lambda}) \to C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee).$$

That is,

$$\mathcal{P}^h_{\text{opp}} : \left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)\right)^h \to H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\text{opp}}, \mathcal{L}_{-\lambda})^h$$
$$f \to \mathcal{P}^h_{\text{opp}}(f) \text{ where}$$
$$\mathcal{P}^h_{\text{opp}}(f)(\omega) = f(\mathcal{P}_{\text{opp}}(\omega)) \text{ for each cohomology class } [\omega].$$

(2) Similarly, let $\left(\mathcal{P}^{-1}_{\text{opp}}\right)^h$ be the Hermitian transpose of $\mathcal{P}^{-1}_{\text{opp}}$. By definition, if $\eta \in H^{n,s}(G_{\mathbf{R}}/L^{\text{opp}}_{\mathbf{R}}, \mathcal{L}_{-\lambda})^h$ and $F \in C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)$, then $\left(\mathcal{P}^{-1}_{\text{opp}}\right)^h(\eta)(F) = \eta(\mathcal{P}^{-1}_{\text{opp}}F)$.

Lemma 4.11. *The composition*

$$\left(\mathcal{P}^{-1}_{opp}\right)^h \circ \mathcal{P}^h_{opp} : \left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)\right)^h \to \left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)\right)^h$$

is the identity map on $\left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)\right)^h$. *Similarly* $\mathcal{P}^h_{opp} \circ \left(\mathcal{P}^{-1}_{opp}\right)^h$ *is the identity map on* $H^{n,s}(\left(G_{\mathbf{R}}/L_{\mathbf{R}}\right)^{\overline{opp}}, \mathcal{L}_{-\lambda})^h$.

Proof. It is clear from the definitions. $\square$

Remark 4.12.

Using the explicit formula 2.12 one can show that

$$T_\sigma \mathcal{P}(w)(x) = \mathcal{P}_{\text{opp}}(\sigma w)(x)$$

where σ is the conjugation described in part (2) of Remark 4.2 and T_σ is the conjugate linear isomorphism from V_μ to $V^\vee_\mu$.

Theorem 4.13. *The space of separately continuous Hermitian pairings on* $\left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)\right)^h$ *is* $C^\infty_{\underline{\mathcal{D}}\times\mathcal{D}}(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu \otimes V^\vee_\mu)$.

Proof. An element $\phi \in C^\infty_{\underline{\mathcal{D}}\times\mathcal{D}}(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu \otimes V^\vee_\mu)$ defines a linear map

$$T_\phi : \left(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu))\right)^h \to C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V^\vee_\mu)$$
$$f \to f(T_\sigma \otimes 1 \; \phi).$$

We must show that the T_ϕ is continuous. Let ω be a representative of the cohomology class in $[\left(\mathcal{P} \otimes \mathcal{P}_{\text{opp}}\right)^{-1}(\phi)] \in H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{\text{opp}}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$. By ([18], Thm 8), ω defines a continuous linear map

$$T_\omega : H^{0,n-s}_c(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda}) \to H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\text{opp}}, \mathcal{L}_{-\lambda}).$$

We argue that the continuous composition $\mathcal{P}_{\mathrm{opp}} \circ T_\omega \circ \left(\mathcal{P}_{\mathrm{opp}}\right)^h$ is T_ϕ. Indeed, if $f \in \left(C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)\right)^h$, then

$$T_\omega \circ \left(\mathcal{P}_{\mathrm{opp}}\right)^h(f)(x) = \left(\mathcal{P}_{\mathrm{opp}}\right)^h(f)(\sigma \otimes 1 \, \omega(x, \cdot)) \qquad (4.14)$$
$$\text{by (2) in Remark 4.2}$$
$$= f\left(\sigma \otimes 1 \circ \mathcal{P}_{\mathrm{opp}} \, \omega(x, \cdot)\right)$$
$$\text{by definition of } \left(\mathcal{P}_{\mathrm{opp}}\right)^h.$$

Hence,

$$\mathcal{P}_{\mathrm{opp}} \circ T_\omega \circ \left(\mathcal{P}_{\mathrm{opp}}\right)^h(f) = f\left(\mathcal{P}_{\mathrm{opp}} \circ \sigma \otimes 1 \otimes \mathcal{P}_{\mathrm{opp}} \, \omega(\cdot, \cdot)\right)$$
$$= f\left(T_\sigma \otimes 1 \circ \mathcal{P} \otimes \mathcal{P}_{\mathrm{opp}} \, \omega(\cdot, \cdot)\right)$$
$$\text{by Remark 4.12}$$
$$= f\left(T_\sigma \otimes 1 \, \phi\right) = T_\phi(f).$$

This shows that T_ϕ is continuous.

To complete the proof we show that every continuous linear map $T : \left(C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)\right)^h \to C_{\underline{\mathcal{D}}}^\infty(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)$ is of the form T_ϕ for some section ϕ in $C_{\underline{\mathcal{D}} \times \mathcal{D}}^\infty(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times \overline{K}_{\mathbf{R}}), V_\mu \otimes V_\mu^\vee)$. Given such a map T, the composition $\mathcal{P}_{\mathrm{opp}}^{-1} \circ T \circ \left(\mathcal{P}_{\mathrm{opp}}^{-1}\right)^h$ is a continuous linear map from $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda})$ to $H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_{-\lambda})$. By ([18], Thm 8), there exists a cohomology class $[\omega] \in H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$ so that $\mathcal{P}_{\mathrm{opp}}^{-1} \circ T \circ \left(\mathcal{P}_{\mathrm{opp}}^{-1}\right)^h = T_\omega$. Hence, $T = \mathcal{P}_{\mathrm{opp}} \circ T_\omega \circ \left(\mathcal{P}_{\mathrm{opp}}\right)^h$. Now, the computation in (4.14) shows that $T = T_\phi$ for $\phi = \left(\mathcal{P} \otimes \mathcal{P}_{\mathrm{opp}}\right)(\omega)$.   □

4.3. Hermitian forms on $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda})$ in terms of Hermitian forms on $\left(Ker \, \underline{\mathcal{D}}\right)^h$.

Lemma 4.15. *Let* $[\omega] \in H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}} \times (G_{\mathbf{R}}/L_{\mathbf{R}})^{opp}, \mathcal{L}_\lambda \otimes \mathcal{L}_{-\lambda})$ *and let* $\phi = \mathcal{P} \otimes \mathcal{P}_{opp}(\omega)$. *Then,*

    (1) $\langle \, , \, \rangle_\omega$ *is* $diag(G_{\mathbf{R}} \times G_{\mathbf{R}})$-*invariant if and only if* $\langle \, , \, \rangle_\phi$ *is* $diag(G_{\mathbf{R}} \times G_{\mathbf{R}})$-*invariant.*

    (2) $\langle \, , \, \rangle_\omega$ *is positive definite if and only if* $\langle \, , \, \rangle_\phi$ *is positive definite.*

Proof. If $f_1, f_2 \in \big(\mathrm{Ker}(\underline{\mathcal{D}})\big)^h = \big(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)\big)^h$, then

$$\langle f_1, f_2 \rangle_\phi = f_1[T_\phi(f_2)] = f_1[\mathcal{P}_{\mathrm{opp}} \circ T_\omega \circ \big(\mathcal{P}_{\mathrm{opp}}\big)^h(f_2)]$$

by the argument in the proof of Theorem (4.13)

$$= (\mathcal{P}^h_{\mathrm{opp}} f_1)[T_\omega(\big(\mathcal{P}_{\mathrm{opp}}\big)^h(f_2))]$$

by the definition of $\mathcal{P}^h_{\mathrm{opp}}$

$$= \langle \mathcal{P}^h_{\mathrm{opp}}(f_1), \mathcal{P}^h_{\mathrm{opp}}(f_2) \rangle_\omega.$$

$\square$

5. A DESCRIPTION OF THE UNITARY GLOBALIZATION OF $A_{\mathfrak{q}}(\lambda)$

The aim of this section is to describe the unitary globalization of $A_{\mathfrak{q}}(\lambda)$. Our assumption on λ (2.3), guarantee that $A_{\mathfrak{q}}(\lambda)$ is irreducible and unitarizable [17]. It follows that the space of $\mathrm{diag}(G_{\mathbf{R}} \times G_{\mathbf{R}})$-invariant Hermitian forms on the cohomology space $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}}, \mathcal{L}_{-\lambda}) = H^{n,s}((G_{\mathbf{R}}/L_{\mathbf{R}})^{\mathrm{opp}}, \mathcal{L}_{-\lambda})^h$ is one-dimensional. By the results in sections 4.2 and 4.3, in order to identify the unitary globalization of $A_{\mathfrak{q}}(\lambda)$ it is enough to identify a $\mathrm{diag}\,(G_{\mathbf{R}} \times G_{\mathbf{R}})$-invariant section F in $C^\infty_{\underline{\mathcal{D}} \times \mathcal{D}}(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu \otimes V_\mu^\vee)$.

It is well known, see [8], that each admissible $(\mathfrak{g}, K)$-module may be realized as the space of K-finite vectors of some Hilbert space globalization. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ be a Hilbert space globalization of the admissible $(\mathfrak{g}, K)$-module $Ker(\mathcal{D})_{K-\mathrm{finite}}$. Write $(\pi_\lambda^\vee, \mathcal{H}_\lambda^\vee)$ with $\pi_\lambda^\vee(g) = \pi_\lambda^t(g^{-1})$, the contragredient representation. Let $\mathcal{H}_\lambda(\mu)$ be the K-isotypic subspace of $\mathcal{H}_\lambda$ for the minimal K-type τ_μ. Write E_μ for the orthogonal K-equivariant projection of $\mathcal{H}_\lambda$ onto $\mathcal{H}_\lambda(\mu)$. By Theorem 2.4 the multiplicity of $\mathcal{H}_\lambda(\mu)$ in $\mathcal{H}_\lambda$ is one. Choose a basis $\{v_i\}$ of $\mathcal{H}(\mu)$, orthonormal with respect to the Hilbert space inner product $\langle \ , \ \rangle$. Let $\{v_i^*\}$ be the dual basis. We show that the generalized spherical function

$$(x, y) \mapsto = \big(\frac{1}{\dim(\mathcal{H}(\mu))}\big) \sum_i v_i^*\big[\pi_\lambda(x^{-1}y) \cdot v_i\big] \tag{5.1}$$

$$= \big(\frac{1}{\dim(\mathcal{H}(\mu))}\big) \sum_i \pi_\lambda^\vee(x) v_i^*\big[\pi_\lambda(y) \cdot v_i\big]$$

when suitably interpreted, determines the $\mathrm{diag}(G_{\mathbf{R}} \times G_{\mathbf{R}})$-invariant Hermitian forms on $\big(C^\infty_{\underline{\mathcal{D}}}(G_{\mathbf{R}}/K_{\mathbf{R}}, V_\mu^\vee)\big)^h$.

5.1. A diag $(G_{\mathbf{R}} \times G_{\mathbf{R}})$-invariant section of $(G_{\mathbf{R}} \times G_{\mathbf{R}}/(K_{\mathbf{R}} \times K_{\mathbf{R}}), V_\mu \otimes V_\mu^\vee)$. We interpret the function in (5.1) as a smooth section of the bundle $(G_{\mathbf{R}} \times G_{\mathbf{R}}) \times_{(K_{\mathbf{R}} \times K_{\mathbf{R}})} (V_\mu \otimes V_\mu^\vee)$. To accomplish this, we (a) identify V_μ with $\mathcal{H}(\mu)$ and (b) realize $\mathcal{H}(\mu) \otimes \mathcal{H}^\vee(\mu)$, via Peter-Weyl Theorem, as a submodule of

$\mathrm{span}_{\mathbf{C}}\{K_{\mathbf{R}} \times K_{\mathbf{R}}$ matrix coefficients of $\mathcal{H}(\mu) \otimes \mathcal{H}^\vee(\mu)\}$.   Indeed, the $K_{\mathbf{R}} \times K_{\mathbf{R}}$-module $\mathcal{H}(\mu) \otimes \mathcal{H}^\vee(\mu)$ is equivalent to the $K_{\mathbf{R}} \times K_{\mathbf{R}}$ representation acting on

$$\mathrm{span}_{\mathbf{C}}\{(k_1, k_2) \to \langle (k_1 \times k_2) \cdot \sum_i (v_i \otimes v_i^*), v_j \otimes v_k^* \rangle |\ j, k \in \{1, \dots, \dim(\mathcal{H}(\mu))\}\}.$$

Observe that for $(x, y) \in G_{\mathbf{R}} \times G_{\mathbf{R}}$ fixed and $(k_1, k_2) \in K_{\mathbf{R}} \times K_{\mathbf{R}}$,

$$\mathrm{Trace}\big(E_\mu \circ \pi_\lambda(k_1^{-1} x^{-1} y k_2) \circ E_\mu\big) =$$

$$= \sum_i (k_1 \cdot v_i^*)\big[\pi_\lambda(x^{-1}y)k_2 \cdot v_i\big]$$

$$= \sum_i \sum_{j,k} \langle k_2 \cdot v_i, v_j \rangle \langle k_1 v_i^*, v_k^* \rangle\, v_k^*\big[\pi_\lambda(x^{-1}y)v_j\big]$$

$$= \sum_{j,k} \langle k_1 \times k_2 \cdot \sum_i (v_i \otimes v_i^*), v_j, \otimes v_k^* \rangle\, v_k^*\big(\big[\pi_\lambda(x^{-1}y)v_j\big]$$

$$\subset \mathrm{span}_{\mathbf{C}}\{\langle k_1 \times k_2 \cdot \sum_i (v_i \otimes v_i^*), v_j \otimes v_k^* \rangle$$

$$j, k \in \{1, \dots, \dim(\mathcal{H}(\mu))\}\}.$$

We summarize the above observation in the following Lemma.

Lemma 5.2. *The function*

$$F : G_{\mathbf{R}} \times G_{\mathbf{R}} \to V_\mu \otimes V_\mu^\vee \subset \mathrm{span}_{\mathbf{C}}\{\text{matrix coefficients of } V_\mu \otimes V_\mu^\vee\}$$

given by

$$F(x, y)(k_1, k_2) = \frac{1}{\dim(\mathcal{H}(\mu)}\, \mathrm{Trace}\big(E_\mu \circ \pi_\lambda(k_1^{-1} x^{-1} y k_2) \circ E_\mu\big),$$

defines a smooth section of the vector bundle

$$(G_{\mathbf{R}} \times G_{\mathbf{R}}) \times_{(K_{\mathbf{R}} \times K_{\mathbf{R}})} (V_\mu \otimes V_\mu^\vee).$$

Theorem 5.3. *The section F of the vector bundle $(G_{\mathbf{R}} \times G_{\mathbf{R}}) \times_{K_{\mathbf{R}} \times K_{\mathbf{R}}} (V_\mu \otimes V_\mu^\vee)$ given in Lemma (5.2) is annihilated by the differential operators $\underline{\mathcal{D}} \otimes 1$ and $1 \otimes \mathcal{D}$.*

Proof. We show that $(1 \otimes \mathcal{D})(F) = 0$. The proof of the identity $(\underline{\mathcal{D}} \otimes 1)(F) = 0$ is similar. According to Definition (4.4), given $\{X_\beta\}$ an orthonormal basis of $\mathfrak{p}$ consisting of root vectors, we must show that

$$(1 \otimes \mathbb{P})\big[\sum_{\beta \in \Delta(\mathfrak{p})} (R(1 \otimes X_\beta)F)(x, y) \otimes X_{-\beta}\big] = 0$$

where $\mathbb{P}$ is the canonical projection $\mathbb{P} : V_\mu \otimes \mathfrak{p} \to V_\mu^-$. Thus, it is enough to show that for each $\delta \in \Delta(\mathfrak{u} \cap \mathfrak{p})$,

$$\int_{K_{\mathbf{R}}} \tau_\mu^\vee \otimes \tau_\mu(1 \times k) \otimes \mathrm{Ad}(k) \big\{ \sum_{\beta \in \Delta(\mathfrak{p})} \big(R(1 \otimes X_\beta)F\big)(x, y) \otimes X_{-\beta}\big\} \overline{\chi_{\mu-\delta}(k)} dk = 0,$$

where $\chi_{\mu-\delta}$ is the character of the irreducible $K_{\mathbf{R}}$ module with highest weight $\mu-\delta$. We observe, as the definition of $\mathcal{D}$ is independent of the basis, that

$$\int_K \sum_{\beta\in\Delta(\mathfrak{p})} \left\{\left(R\big(1\otimes\mathrm{Ad}(k)X_\beta\big)F\right)(x,yk)\otimes\mathrm{Ad}(k)X_{-\beta}\right\}\overline{\chi_{\mu-\delta}(k)}dk \qquad (5.4)$$

$$= \int_K \sum_{\beta\in\Delta(\mathfrak{p})} \left\{\left(R\big(1\otimes X_\beta\big)F\right)(x,yk)\otimes X_{-\beta}\right\}\overline{\chi_{\mu-\delta}(k)}dk.$$

On the other hand,

$$R(1\otimes X_\beta)\,F(x,yk) = \sum_i v_i^*\left[\pi_\lambda(x^{-1}yk)\,X_\beta\cdot v_i\right].$$

As the vectors v_i are K-finite and $X_\beta\in\mathfrak{g}$, the vector $X_\beta\cdot v_i$ is K-finite. Thus, there exists a finite set $S\subset\hat{K}$ so that

$$X_\beta\cdot v_i = \sum_{\tau\in S}\mathbb{P}_\tau[X_\beta\cdot v_i]. \qquad (5.5)$$

Replacing identity (5.5) in the displayed formula (5.4) we get

$$\int_K \sum_{\beta\in\Delta(\mathfrak{p})} \left\{\left(R(1\otimes X_\beta)F\right)(x,yk)\otimes X_{-\beta}\right\}\overline{\chi_{\mu-\delta}(k)}dk = \qquad (5.6)$$

$$= \sum_{\beta\in\Delta(\mathfrak{p})}\sum_{i,\tau\in S}\int_K v_i^*\left[\pi_\lambda(x^{-1}y)\pi_\lambda(k)\,\mathbb{P}_\tau[X_\beta v_i]\right]\otimes X_{-\beta}\,\overline{\chi_{\mu-\delta}(k)}dk.$$

By Theorem 2.4 the K-type $\tau_{\mu-\delta}$ does not occur in $\pi_\lambda|_K$. Hence, the right hand side in equation (5.6) is zero. $\qquad\square$

Theorem 5.7. *The generalized spherical function*

$$F(x,y) = \frac{1}{dim(\mathcal{H}(\mu))}\ Trace\big(E_\mu\circ\pi_\lambda(x^{-1}y)\circ E_\mu\big)$$

determines the unique (up to a scalar) invariant Hermitian form on $\mathrm{Ker}\,(\mathcal{D})^h$. The cohomology class $[\omega]\in H^{(n,n)(s,s)}(G_{\mathbf{R}}/L_{\mathbf{R}}\times(G_{\mathbf{R}}/L_{\mathbf{R}})^{opp},\mathcal{L}_\lambda\otimes\mathcal{L}_{-\lambda})$ that corresponds to F via the Penrose transform determines the unique (up to scalar) Hermitian form on $H_c^{0,n-s}(G_{\mathbf{R}}/L_{\mathbf{R}},\mathcal{L}_{-\lambda})$.

Proof. The Theorem follows from combining Theorem 4.1, Theorem 4.13, Theorem 5.3 and Proposition 4.6. $\qquad\square$

When π_λ is a representation in the discrete series, $F(x,y)=\psi_\lambda(x^{-1}y)$, the function introduced by Flensted-Jensen in ([6], (7.11)). The image of the intertwining map T_{ψ_λ} is the space of square-integrable sections in $\mathrm{Ker}\,\underline{\mathcal{D}}$.

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