

Special Lagrangian Curvature

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Abstract: We define the notion of special Lagrangian curvature, showing how it may be interpreted as an alternative higher dimensional generalisation of two dimensional Gaussian curvature. We obtain first a local rigidity result for this curvature when the ambient manifold has negative sectional curvature. We then show how this curvature relates to the canonical special Legendrian structure of spherical subbundles of the tangent bundle of the ambient manifold. This allows us to establish a strong compactness result. In the case where the special Lagrangian angle equals $(n - 1)\pi/2$, we obtain compactness modulo a unique mode of degeneration, where a sequence of hypersurfaces wraps ever tighter round a geodesic.

Key Words: Gaussian curvature, Weingarten problems, special Lagrangian, special Legendrian, immersed submanifolds, compactness.

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1 - Introduction.

1.1 Curvature and Weingarten Problems.

Gaussian curvature is a fundamental object of study in differential geometry. Despite being simple to express, it leads to very hard problems involving highly non-linear PDEs. In [13] and [14], by showing how in two dimensions surfaces of constant Gaussian curvature may be studied in terms of pseudo-holomorphic curves in the unitary bundle of the ambient manifold, Labourie used the powerful machinery first developed by Gromov in [6] to obtain a number of elegant results concerning the existence and structure of constant Gaussian curvature surfaces in negatively curved spaces. Amongst other things, he applies these results in [12] to the study of geometrically finite, three dimensional, hyperbolic manifolds. These ideas were also used by the author in [21] to obtain a satisfying result relating geometrically finite surfaces of constant Gaussian curvature in hyperbolic space to holomorphic ramified coverings of the Riemann sphere.

In higher dimensions, the difficulty is aggravated by the loss of uniform ellipticity of the PDE defining Gaussian curvature. Nonetheless, by leading a frontal attack on this PDE, Rosenberg and Spruck were able to obtain in [17] strong existence results for complete hypersurfaces of constant Gaussian curvature immersed in hyperbolic space and satisfying certain boundary conditions. What the compactness properties of families of such hypersurfaces may be remains, however, an open question.

This paper arose from the attempt to generalise the techniques developed by Labourie to higher dimensions. We found that an alternative higher dimensional generalisation of Gaussian curvature yields a much more tractable problem. We thus obtain what we have chosen to call special Lagrangian curvature. This curvature is intimately related to the canonical special Legendrian structure of the unitary bundle (see the following section), and reduces in special cases to elementary combinations of more classical curvatures (see below).

This curvature may be defined in various ways. In this introduction, we present it in a form which is only defined for convex hypersurfaces. Although this definition is slightly technical, it exhibits the most clearly its geometric significance. Let A be a positive definite, symmetric matrix. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A . For $r > 0$, we define $SL_r(A)$ by:

$$SL_r(A) = \text{Arg}(\text{Det}(\text{Id} + irA)) = \sum_{i=1}^n \arctan(r\lambda_i).$$

SL_r is a strictly increasing function of r . Moreover, $SL_0 = 0$ and $SL_\infty = n\pi/2$. Thus, for all $\theta \in]0, n\pi/2[$, there exists a unique $r > 0$ such that:

$$SL_r(A) = \theta.$$

We now define $R_\theta(A) = r$. R_θ is invariant under the action of $O(n)$ on the space of positive definite, symmetric matrices, and it may thus be used to define curvature.

Special Lagrangian Curvature

Let (M, g) be a Riemannian manifold of dimension $(n+1)$, and let $\Sigma = (S, i)$ be an oriented immersed hypersurface in M (ie. an immersed submanifold of codimension 1). Let A be the shape (Weingarten) operator of Σ . We say that Σ is convex if and only if A is positive definite. When Σ is convex, for $\theta \in]0, n\pi/2[$, we define $R_\theta(\Sigma)$, the θ -special Lagrangian curvature of Σ by:

$$R_\theta(\Sigma) = R_\theta(A).$$

Of most interest is the case when θ is in the half-open interval $[(n-1)\pi/2, n\pi/2[$. Here, *convexity is naturally related to the curvature* (for lower values of θ , a sequence of hypersurfaces of constant special Lagrangian curvature can degenerate by ceasing to be convex). In particular, the case $\theta = (n-1)\pi/2$ yields the most interesting geometry. Here the special Lagrangian curvature has the simplest form, and here the compactness result yields a fascinating form of degeneration where hypersurfaces wrap ever more tightly round geodesics, thus generalising to higher dimensions the so-called ‘‘curtain surfaces’’, studied by Labourie in [13]. Examining the lower dimensional cases illustrates the natural geometric significance of this curvature:

- (i) when $n = 2$, $R_{\pi/2}^{-2} = K$, where K is the Gaussian curvature of Σ ;
- (ii) when $n = 3$, $R_{\pi}^{-2} = K/H$, where H is the mean curvature of Σ ; and
- (iii) in general $R_{(n-1)\pi/2}$ is the unique value of r such that:

$$\chi_n - r^2\chi_{n-2} + r^4\chi_{n-4} - \dots = 0,$$

where, for all i , we define the i 'th higher principal curvature χ_i of Σ such that, for all $t \in \mathbb{R}$:

$$\text{Det}(I + tA) = \sum_{i=0}^n \chi_i(A)t^i.$$

We thus see that the study of special Lagrangian curvature neatly forms a special case of the study of Weingarten hypersurfaces.

Although the form R_θ of the special Lagrangian curvature is more transparent to geometry, the form SL_r is much more tractable to analysis. Trivially, SL_r is constant and equal to θ if and only if R_θ is constant and equal to r . Therefore, in the sequel, we work with the r -special Lagrangian curvature, defined to be equal to $SL_r(A)$, and we will study hypersurfaces of constant r -special Lagrangian curvature.

The main results of this paper provide two key tools for the study of these hypersurfaces. The first is *local rigidity* (Theorem 1.3) in the case where the ambient manifold has negative sectional curvature, and the second, and by far the more significant, is *precompactness* (Theorem 1.4).

As an illustration of the precompactness result, in the very special case where $n = 3$ and $\theta = \pi$, a direct application of Theorem 1.4 yields:

Theorem 1.1

Let M be a Riemannian manifold of dimension 4. Let $\kappa > 0$ be a positive real number. Let $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ be a family of pointed, immersed hypersurfaces in M and let K_n and H_n be the Gaussian and mean curvatures respectively of Σ_n . Suppose that, for all n , $K_n/H_n = \kappa$, then, either:

- (i) $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ subconverges smoothly to a pointed, immersed hypersurface (Σ_0, p_0) in M ; or
- (ii) $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ contains a subsequence which degenerates by converging to a complete geodesic.

Remark: Large families of examples of such hypersurfaces are described in section 3 as well as the forthcoming paper [23].

Remark: The precise modes of convergence in both cases are described explicitly in Theorem 1.4.

Remark: This result suggests that K/H be considered as an alternative natural generalisation of two dimensional Gaussian curvature in the sense that it exhibits identical geometric behaviour to that described for Gaussian curvature by Labourie in [13].

1.2 Positive Special Legendrian Structures.

The results of this paper also apply in a very different setting: that of positive special Legendrian structures. Special Legendrian structures are the contact equivalent of special Lagrangian structures, which were first introduced in the landmark paper [8] of Harvey and Lawson and have since been of considerable interest to mathematicians and theoretical physicists (one excellent survey may be found in [11] and a thoroughly enjoyable read in [9]). We believe that, in this setting, our results will have wider applications beyond the scope of this paper.

We define a positive special Legendrian structure as follows: let (M, g) be a Riemannian manifold of dimension $(2n + 1)$. Let \mathcal{P} be a principal $\mathrm{SO}(n)$ bundle over M . We define the action φ of $\mathrm{SO}(n)$ over $\mathbb{R}^n \oplus \mathbb{R}^n$ by:

$$\varphi(A)(u, v) = (Au, Av).$$

Let ω_n be the canonical symplectic structure over $\mathbb{R}^n \oplus \mathbb{R}^n$. Let F be the associated bundle over M defined by the representation φ :

$$F = \mathcal{P} \otimes_{\varphi(\mathrm{SO}(n))} \mathbb{R}^n \oplus \mathbb{R}^n.$$

Since it is preserved by the action of $\mathrm{SO}(n)$, ω_n defines a form $\hat{\omega}_n$ over F . We define a positive special Legendrian structure over M to be a triple (\mathcal{P}, W, Φ) where:

- (i) \mathcal{P} is an $\mathrm{SO}(n)$ principal bundle over M ,
- (ii) W is a contact structure over M , and

(iii) $\Phi : F \rightarrow W$ is an isomorphism of vector bundles,

such that if we denote by ω the canonical symplectic form over W then $\omega = \Phi_*\hat{\omega}_n$. In fact, it suffices to assume that $\Phi_*\hat{\omega}_n$ and ω are merely colinear, but we impose the stricter condition of equality for ease of presentation.

We identify $\mathbb{R}^n \oplus \mathbb{R}^n$ with \mathbb{C}^n . Let $\Omega_n = dz^1 \wedge \dots \wedge dz^n$ be the canonical special Lagrangian form over $\mathbb{R}^n \oplus \mathbb{R}^n$. Similarly, we define the canonical Minkowski metric m_n over $\mathbb{R}^n \oplus \mathbb{R}^n$ by:

$$m_n((X, Y), (X, Y)) = \langle X, Y \rangle.$$

Both Ω_n and m_n are preserved by $\varphi(\text{SO}(n))$. They consequently induce forms $\hat{\Omega}_n$ and \hat{m}_n over F . Likewise, the canonical metric g_n on \mathbb{C}^n induces a metric \hat{g}_n over F . Thus, by pushing forward through the isomorphism Φ , we obtain forms over W that we may denote by Ω , m and g respectively. The positive special Legendrian structure thus defines a quadruplet of forms (Ω, m, ω, g) over the contact bundle W . Moreover, the metric over W may trivially be extended to a metric over M . Alternatively, given such a quadruplet, satisfying certain algebraic relations, we may deduce a positive special Legendrian structure over M , and we will adopt this latter point of view in the sequel.

The motivation for the introduction of a Minkowski metric arises from the following result of Jost & Xin and Yuan ([10] and [25]):

Theorem 4.1 [Jost, Xin, 2002], [Yuan, 2002]

Let Σ be a complete immersed special Lagrangian submanifold of $\mathbb{R}^n \oplus \mathbb{R}^n$ of type $C^{1,\alpha}$. If Σ is spacelike with respect to the canonical Minkowski metric m over $\mathbb{R}^n \oplus \mathbb{R}^n$ (i.e. if the restriction of m to $T\Sigma$ is non-negative), then $\hat{\Sigma}$ is an affine subspace.

Remark: In fact, the use of the positivity condition in [10] was itself inspired by the work [24] of Smoczyk concerning the Lagrangian mean curvature flow. Smoczyk showed that this condition ensures that the tangent space of the special Lagrangian submanifold remains within a convex subset of the Grassmannian of Lagrangian subspaces, and it is precisely this property that is used in [10] to prove the theorem.

It is this result, of Bernstein type, which forms the core of our main compactness result, (Theorem 1.2), which we reduce to the former by means of a blow-up type argument.

1.3 Principal Results - Positive Special Legendrian Submanifolds.

Let (M, g) be a Riemannian manifold of dimension $(2n + 1)$ and let (\mathcal{P}, W, Φ) be a special Legendrian structure over M . We define $\mathcal{SL}_\theta^+(M)$ to be the family of all complete, pointed, positive special Legendrian submanifolds immersed in M (see section 2.2). That is $(\Sigma, p) = (S, i, p)$ lies in $\mathcal{SL}_\theta^+(M)$ if and only if it is complete and:

$$\omega|_{T\Sigma}, \quad \text{Im}(e^{-i\theta}\Omega)|_{T\Sigma} = 0, \quad m|_{T\Sigma} \geq 0.$$

We define the canonical projection $\pi : \mathcal{SL}_\theta^+(M) \rightarrow M$ by:

$$\pi(\Sigma, p) = p.$$

If we provide $\mathcal{SL}_\theta^+(M)$ with the Cheeger/Gromov topology (see section 2.1) then the projection π is automatically continuous. We obtain the following result:

Theorem 1.2 Precompactness

The canonical projection $\pi : \mathcal{SL}_\theta^+ \rightarrow M$ is a proper mapping.

Remark: The same techniques may be used to yield an analogous result for positive special Lagrangian submanifolds.

1.4 Principal Results - Constant Special Lagrangian Curvature.

We now return our attention to the problem of hypersurfaces of constant r -special Lagrangian curvature.

Theorem 1.3 Local Rigidity

Let M be a Riemannian manifold of dimension $(n + 1)$ and of negative sectional curvature bounded above by -1 . Let $\Sigma = (S, i)$ and $\Sigma' = (S, i')$ be two compact, convex, immersed hypersurfaces of constant r -special Lagrangian curvature equal to $\theta \leq \arctan(r)$. If i' is sufficiently close to i in the C^0 topology, then $i' = i$.

This result may also be used in the construction of hypersurfaces of constant r -special Lagrangian curvature by smoothly deforming the metric of the ambient manifold. Indeed, rigidity ensures through the Fredholm alternative that, for small deformations, a smooth family of immersions may be constructed that appropriately follows these deformations. This method of construction will be used, in particular, in the forthcoming paper [23] to construct immersed hypersurfaces inside hyperbolic ends.

In order to state the precompactness result for hypersurfaces of constant special Lagrangian curvature, we require some notation. Let (M, g) be a Riemannian manifold of dimension $(n + 1)$. For $r > 0$, we define $S_r M$, the r -sphere bundle over M to be the bundle of spheres of radius r in TM . Let $\Sigma = (S, i)$ be an oriented immersed hypersurface in M . Let \mathbf{N} be the normal vector field over Σ . We define $\hat{\Sigma}_r = (S, \hat{i}_r)$, the r -Gauss lifting of Σ in $S_r M$ by:

$$\hat{i}_r = r\mathbf{N}.$$

We define $\mathcal{F}_{r,\theta}(M)$ to be the family of all pointed submanifolds (Σ, p) immersed in M , of constant r -special Lagrangian curvature equal to θ whose r -Gauss lifting on $S_r M$ is complete. We define $\hat{\mathcal{F}}_{r,\theta}(M)$ to be the family of all r -Gauss liftings of pointed hypersurfaces in $\mathcal{F}_{r,\theta}(M)$. We provide $\hat{\mathcal{F}}_{r,\theta}(M)$ with the pointed Cheeger/Gromov topology.

We show that $S_r M$ may be furnished with a canonical positive special Legendrian structure. We then show that $\hat{\mathcal{F}}_{\rho,\theta}(M)$ is a subset of $\mathcal{SL}_\theta^+(S_\rho M)$. Consequently, if we define the canonical projection π over $\hat{\mathcal{F}}_{\rho,\theta}(M)$ as before, we may use Theorem 1.2 to show that for every compact subset K of M , the subset $\pi^{-1}(K)$ of $\hat{\mathcal{F}}_{\rho,\theta}(M)$ is relatively compact in the space of complete, immersed submanifolds in $S_\rho M$. We obtain the following result concerning the topological boundary of $\hat{\mathcal{F}}_{\rho,\theta}(M)$ in $\mathcal{SL}_\theta^+(S_\rho M)$:

Theorem 1.4 Precompactness

Let M be a Riemannian manifold of dimension $(n + 1)$. Let $\rho \in \mathbb{R}^+$ be a positive real number.

- (i) If $\theta \in](n - 1)\pi/2, n\pi/2[$, then $\partial\hat{\mathcal{F}}_{\rho,\theta}(M)$ is empty.
- (ii) If $\theta = (n - 1)\pi/2$ and if (Σ, p) is a pointed immersed submanifold in $\partial\hat{\mathcal{F}}_{\rho,(n-1)\pi/2}(M)$, then Σ is a normal ρ -sphere bundle over a complete geodesic in M .

Remark: Heuristically, (ii) says that if $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ is a sequence of immersed hypersurfaces of constant r -special Lagrangian curvature equal to $(n - 1)\pi/2$, then, either this sequence subconverges to another such hypersurface, or it contains a subsequence that wraps ever more tightly around a complete geodesic, and this is the only mode of degeneration that may take place. This theorem thus generalises the result [13] of Labourie to higher dimensions.

Remark: The first case, where $\theta \in](n - 1)\pi/2, n\pi/2[$, follows trivially from the fact that if $SL_r(A)$ is not an integer multiple of $\pi/2$, then we automatically obtain a-priori upper and lower bounds on A . Thus, the shape operator of a hypersurface of constant r -special Lagrangian curvature equal to θ is bounded, the slope of the Gauss lifting is therefore also bounded, and this ensures that the Gauss lifting can never become vertical in the limit.

Remark: By using the results [16] and [3] of Pogorelov and Calabi instead of those of Jost & Xin and Yuan, the techniques used to prove Theorems 1.2 and 1.4 may be trivially adapted to yield an analogous, though slightly weaker, compactness result concerning hypersurfaces of constant Gaussian curvature (see Proposition 5.14).

1.5 Structure of This Paper.

This paper is structured as follows:

- (i) In part 2, we define the concepts used in the sequel, including special Lagrangian curvature and positive special Legendrian structures. We also prove a useful unique continuation principle.
- (ii) In part 3, we prove a weaker version of local rigidity, and we describe how this may be used to construct numerous examples of hypersurfaces of constant special Lagrangian curvature.
- (iii) In part 4, we use rescaling techniques to prove the compactness result for special Legendrian submanifolds (Theorem 1.2).
- (iv) In part 5, we prove the second compactness theorem (Theorem 1.4). This is done by first observing that the result can be expressed in terms of the vanishing of a certain positive function, and then showing that this function is superharmonic, and thus satisfies the maximum principle. This requires rather involved calculations of laplacians of functions defined over these hypersurfaces. We then use this result to complete the proof of local rigidity (Theorem 1.3).

(v) In the appendix, we briefly prove a version of the maximum principle required in the proof of Theorem 1.4.

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2 - Definitions and Unique Continuation.

2.1 Immersed Submanifolds and the Cheeger/Gromov Topology.

Let M be a smooth Riemannian manifold. An **immersed submanifold** is a pair $\Sigma = (S, i)$ where S is a smooth manifold and $i : S \rightarrow M$ is a smooth immersion. A **pointed immersed submanifold** in M is a pair (Σ, p) where $\Sigma = (S, i)$ is an immersed submanifold in M and p is a point in S . An **immersed hypersurface** is an immersed submanifold of codimension 1. We give S the unique Riemannian metric i^*g which makes i into an isometry. We say that Σ is **complete** if and only if the Riemannian manifold (S, i^*g) is.

A **pointed Riemannian manifold** is a pair (M, p) where M is a Riemannian manifold and p is a point in M . Let $(M_n, p_n)_{n \in \mathbb{N}}$ be a sequence of complete pointed Riemannian manifolds. For all n , we denote by g_n the Riemannian metric over M_n . We say that the sequence $(M_n, p_n)_{n \in \mathbb{N}}$ **converges** to the complete pointed manifold (M_0, p_0) in the **Cheeger/Gromov topology** if and only if for all n , there exists a mapping $\varphi_n : (M_0, p_0) \rightarrow (M_n, p_n)$, such that, for every compact subset K of M_0 , there exists $N \in \mathbb{N}$ such that for all $n \geq N$:

- (i) the restriction of φ_n to K is a C^∞ diffeomorphism onto its image, and
- (ii) if we denote by g_0 the Riemannian metric over M_0 , then the sequence of metrics $(\varphi_n^*g_n)_{n \geq N}$ converges to g_0 in the C^∞ topology over K .

We refer to the sequence $(\varphi_n)_{n \in \mathbb{N}}$ as a sequence of **convergence mappings** of the sequence $(M_n, p_n)_{n \in \mathbb{N}}$ with respect to the limit (M_0, p_0) . The convergence mappings are trivially not unique.

Let $(\Sigma_n, p_n)_{n \in \mathbb{N}} = (S_n, p_n, i_n)_{n \in \mathbb{N}}$ be a sequence of complete pointed immersed submanifolds in M . We say that $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ **converges** to $(\Sigma_0, p_0) = (S_0, p_0, i_0)$ in the **Cheeger/Gromov topology** if and only if $(S_n, p_n)_{n \in \mathbb{N}}$ converges to (S_0, p_0) in the Cheeger/Gromov topology, and, for every sequence $(\varphi_n)_{n \in \mathbb{N}}$ of convergence mappings of $(S_n, p_n)_{n \in \mathbb{N}}$ with respect to this limit, and for every compact subset K of S_0 , the sequence of functions $(i_n \circ \varphi_n)_{n \geq N}$ converges to the function $(i_0 \circ \varphi_0)$ in the C^∞ topology over K .

In an analogous manner, for all $k \geq 1$ and for all α , we may also define the $C^{k, \alpha}$ Cheeger/Gromov topology for manifolds and immersed submanifolds. In this case, the convergence mappings are of type $C^{k, \alpha}$ and the metrics converge in the $C^{k-1, \alpha}$ topology over each compact set.

2.2 Special Lagrangian and Legendrian Structures.

We identify \mathbb{R}^{2n} and \mathbb{C}^n . Using the complex coordinate functions, we define various geometric structures. We define the canonical **symplectic form** by:

$$\omega = \frac{i}{2} \sum_{k=1}^n dz^k \wedge d\bar{z}^k.$$

We define the canonical **Minkowski metric** by:

$$m = \operatorname{Re} \left(\frac{i}{2} \sum_{k=1}^n dz^k \otimes dz^k \right).$$

For all $\theta \in \mathbb{R}$, we define the θ -**special Lagrangian form** Ω_θ over $\mathbb{R}^n \oplus \mathbb{R}^n$ by:

$$\Omega_\theta = e^{-i\theta} dz^1 \wedge \dots \wedge dz^n.$$

The triple $(\omega, m, \Omega_\theta)$ defines a **positive special Lagrangian structure** over $\mathbb{R}^n \oplus \mathbb{R}^n$. The stabiliser group of this triple in $\operatorname{End}(\mathbb{R}^{2n})$ is the set of all matrices M of the form:

$$M = \begin{pmatrix} B & \\ & B \end{pmatrix},$$

where $B \in \operatorname{SO}(n)$. Let P be an n dimensional subspace of $\mathbb{R}^n \oplus \mathbb{R}^n$. P is said to be **θ -special Lagrangian** if and only if the restrictions of ω and Ω_θ to P vanish. P is said to be **positive** (or **spacelike**) if and only if the restriction of m to P is non-negative definite. In the sequel we will write SL_θ for θ -special Lagrangian and SL_θ^+ for positive θ -special Lagrangian.

Let M be a $(2n+1)$ dimensional manifold bearing a positive special Lagrangian structure as defined in the introduction. Let $\Sigma = (S, i)$ be an immersed submanifold in M . We say that Σ is **positive θ -special Legendrian** if and only if, for all $p \in S$:

- (i) $T_p \Sigma \subseteq W_{i(p)}$, and
- (ii) $T_p \Sigma$ is positive θ -special Lagrangian in $W_{i(p)}$.

We define the family $\mathcal{SL}_\theta^+(M)$ to be the family of all complete SL_θ^+ pointed submanifolds of M . We give this family the Cheeger/Gromov topology. Trivially, \mathcal{SL}_θ^+ forms a closed subset of the set of all complete pointed immersed submanifolds in M with respect to the Cheeger/Gromov topology. We define the canonical projection $\pi : \mathcal{SL}_\theta^+ \rightarrow M$ by:

$$\pi(\Sigma, p) = \pi((S, i), p) = i(p).$$

This mapping is trivially continuous.

2.3 The Positive Special Legendrian Structure Over $S_\rho M$.

Let M be a complete, oriented Riemannian manifold of dimension $(n+1)$. Let $\pi : TM \rightarrow M$ be the canonical projection. Let VTM be the vertical subbundle of TTM , and let HTM be the horizontal subbundle of the Levi-Civita connection of M . Thus:

$$TTM = HTM \oplus VTM \cong \pi^*TM \oplus \pi^*TM.$$

Using this identification of TTM with $\pi^*TM \oplus \pi^*TM$, for $X, Y \in \Gamma(M, TM)$, we define $\{X, Y\} \in \Gamma(TM, TTM)$ by:

$$i_H \oplus i_V \{X, Y\} = (\pi^*X, \pi^*Y).$$

Every vector field over TTM may be expressed locally in terms of a linear combination of such vector fields. For $X, Y, v \in T_pM$, we define $\{X, Y\}_v \in T_vTTM$ in an analogous manner.

For ρ a positive real number, let $S_\rho M$ be the set of vectors of length ρ in TM . We call $S_\rho M$ the ρ -sphere bundle over M . We define the tautological vector fields q^H and q^V over TM , by:

$$q^H(v) = \{v, 0\}_v, \quad q^V(v) = \{0, v\}_v.$$

Let $HS_\rho M$ and $VS_\rho M$ be the respective restrictions of the bundles HTM and VTM to $S_\rho M$. We define $\langle q^H \rangle^\perp$ (resp. $\langle q^V \rangle^\perp$) to be the subspace orthogonal to q^H (resp. q^V) in $HS_\rho M$ (resp. $VS_\rho M$). Trivially:

$$i_{\rho*} : TS_\rho M \cong HS_\rho M \oplus \langle q^V \rangle^\perp.$$

We define the subbundle $WS_\rho M$ of $TS_\rho M$ by:

$$WS_\rho M = \langle q^H \rangle^\perp \oplus \langle q^V \rangle^\perp.$$

$WS_\rho M$ defines a contact structure over $S_\rho M$. Moreover, if we denote by ω the canonical symplectic form over $WS_\rho M$, then:

$$\omega(\{X_1, X_2\}, \{Y_1, Y_2\}) = \langle X_2, Y_1 \rangle - \langle X_1, Y_2 \rangle.$$

In fact, using the metric of M , we identify TM and T^*M . The canonical Liouville form of T^*M then restricts to a contact form over $S_\rho M$ whose kernel is W . Moreover, the restriction to W of the exterior derivative of this form is colinear with ω . We thus obtain a second construction of ω .

The distributions $\langle q^H \rangle^\perp, \langle q^V \rangle^\perp \subseteq TTM$ are canonically isomorphic and inherit canonical metrics and orientations from TM . This defines a positive special Legendrian structure over W . We explicitly construct the forms ω, m, Ω_θ and g . We define J over $\langle q^H \rangle^\perp$ to be the canonical isometry sending this space into $\langle q^V \rangle^\perp$. J then uniquely extends to a complex structure over W .

Special Lagrangian Curvature

Let v_p be a point in $S_\rho M$. Let $E_1, \dots, E_n \in T_p M$ be such that $(E_1, \dots, E_n, v_p/\rho)$ forms an oriented, orthonormal basis. We define $X_1, \dots, X_n \in \langle q^H \rangle_v^\perp$, and $Y_1, \dots, Y_n \in \langle q^V \rangle_v^\perp$ by:

$$X_i = \{E_i, 0\}_v, \quad Y_i = \{0, E_i\}_v = JX_i \quad \forall i.$$

$X_1, \dots, X_n, Y_1, \dots, Y_n$ is an orthonormal basis of W_v . Let $(X^1, \dots, X^n, Y^1, \dots, Y^n)$ be the dual basis of W_v^* . We define:

$$\begin{aligned} \omega &= \sum_{i=1}^n X^i \wedge Y^i, \\ m &= \frac{1}{2} \sum_{i=1}^n (X^i \otimes Y^i + Y^i \otimes X^i), \\ \Omega_\theta &= e^{-i\theta} (X^1 + iY^1) \wedge \dots \wedge (X^n + iY^n), \\ g &= \sum_{i=1}^n X^i \otimes X^i + Y^i \otimes Y^i. \end{aligned}$$

The triplet $(\omega, m, \Omega_\theta)$ does not depend on $(E_i)_{1 \leq i \leq n}$. Since the form ω coincides with the canonical symplectic form over W , it follows that $(W, \omega, m, \Omega_\theta)$ does indeed define a positive special Legendrian structure over $S_\rho M$.

2.4 Normal Vector Fields, Second Fundamental Form, Convexity.

Let M be an oriented Riemannian manifold. Let $\Sigma = (S, i)$ be an oriented hypersurface immersed in M . Let $\mathbf{N}_\Sigma \in \Gamma(S, i^* TM)$ be the exterior unit normal of Σ in M . The sign of \mathbf{N}_Σ depends on the orientations of S and M . We define the **Weingarten operator** $A_\Sigma : TS \rightarrow TS$ and the **second fundamental form** II_Σ of Σ by:

$$\begin{aligned} A_\Sigma(X) &= \nabla_X \mathbf{N}_\Sigma, \\ II_\Sigma(X, Y) &= \langle A_\Sigma X, Y \rangle. \end{aligned}$$

Σ is said to be **convex** at $p \in S$ if and only if the symmetric bilinear form II_Σ is either positive or negative (but not mixed) at p . Through a slight abuse of language, we say that Σ is **convex** if it is convex at every point. In this case, by reversing the orientation of S if necessary, we may assume in the sequel that II_Σ is positive.

For $\rho \in (0, \infty)$, we define the **ρ -Gauss lifting** $\hat{\Sigma}_\rho = (S, \hat{i}_\rho)$ of Σ , which is an immersed submanifold of $S_\rho M$, by:

$$\hat{\Sigma}_\rho = (S, \hat{i}_\rho) = (S, \rho \mathbf{N}_\Sigma).$$

The relationship between hypersurfaces and immersed submanifolds is made clear by the following elementary lemma:

Lemma 2.1

Let Σ be an oriented immersed hypersurface in M . $\hat{\Sigma}_\rho$, the ρ -Gauss lifting of Σ , is a Legendrian submanifold of $S_\rho M$. Moreover, if Σ is convex, then $\hat{\Sigma}_\rho$ is positive (spacelike) with respect to the Minkowski metric m .

Conversely, if $\hat{\Sigma}$ is an immersed Legendrian submanifold in $S_\rho M$ such that $T\hat{\Sigma} \cap VS_\rho M = 0$, then there exists a unique oriented immersed hypersurface Σ in M such that $\hat{\Sigma}$ is the ρ -Gauss lifting of Σ . Moreover, if $\hat{\Sigma}$ is positive (spacelike) with respect to the Minkowski metric m , then Σ is convex.

2.5 Curvature.

Let us denote by $\text{Symm}(\mathbb{R}^n)$ the space of symmetric matrices over \mathbb{R}^n . We define $\Phi : \text{Symm}(\mathbb{R}^n) \rightarrow \mathbb{C}^*$ by:

$$\Phi(A) = \text{Det}(I + iA).$$

Since Φ never vanishes and $\text{Symm}(\mathbb{R}^n)$ is simply connected, there exists a unique analytic function $\tilde{\Phi} : \text{Symm}(\mathbb{R}^n) \rightarrow \mathbb{C}$ such that:

$$\tilde{\Phi}(I) = 0, \quad e^{\tilde{\Phi}(A)} = \Phi(A) \quad \forall A \in \text{Symm}(\mathbb{R}^n).$$

We define the function $\arctan : \text{Symm}(\mathbb{R}^n) \rightarrow (-n\pi/2, n\pi/2)$ by:

$$\arctan(A) = \text{Im}(\tilde{\Phi}(A)).$$

This function is trivially invariant under the action of $O(\mathbb{R}^n)$. If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A , then:

$$\arctan(A) = \sum_{i=1}^n \arctan(\lambda_i).$$

Let M be an oriented Riemannian manifold of dimension $n + 1$. Let $\Sigma = (S, i)$ be an oriented immersed hypersurface in M . For $\rho \in (0, \infty)$, we define $\text{SL}_\rho(\Sigma)$, the ρ -**special Lagrangian curvature** of Σ by:

$$\text{SL}_\rho(\Sigma) = \arctan(\rho A_\Sigma).$$

For $\theta \in [(n-1)\pi/2, n\pi/2[$ and $\rho \in (0, \infty)$, we define the family $\mathcal{F}_{\rho, \theta}(M)$ by:

$$\mathcal{F}_{\rho, \theta}(M) = \left\{ \begin{array}{l} (\Sigma, p) \text{ is an immersed, pointed hypersurface in } M, \\ (\Sigma, p) \text{ s.t. } \bullet \text{SL}_\rho(\Sigma) = \theta, \text{ and} \\ \bullet \hat{\Sigma}_\rho \text{ is a complete immersed submanifold of } S_\rho M. \end{array} \right\}.$$

We observe that hypersurfaces in $\mathcal{F}_{\rho, \theta}(M)$ are always convex. We define $\hat{\mathcal{F}}_{\rho, \theta}(M)$ to be the family of ρ -Gauss liftings of pointed hypersurfaces in $\mathcal{F}_{\rho, \theta}(M)$. Trivially, there is a canonical bijections between $\mathcal{F}_{\rho, \theta}(M)$ and $\hat{\mathcal{F}}_{\rho, \theta}(M)$. We furnish $\hat{\mathcal{F}}_{\rho, \theta}(M)$ with the Cheeger/Gromov topology. Let $\pi_\rho : S_\rho M \rightarrow M$ be the canonical projection. We define the canonical projection $\pi : \hat{\mathcal{F}}_{\rho, \theta}(M) \rightarrow M$ by:

$$\pi(\hat{\Sigma}, p) = \pi(S_\rho, \hat{i}_\rho, p) = (\pi_\rho \circ \hat{i}_\rho)(p) = i(p).$$

By definition of the Cheeger/Gromov topology, this projection is continuous.

Bearing in mind Lemma 2.1, we obtain the following result:

Lemma 2.2

For all $\rho \in (0, \infty)$ and for all $\theta \in [(n-1)\pi/2, n\pi/2[$:

$$\hat{\mathcal{F}}_{\rho, \theta} \subseteq \mathcal{SL}_\theta^+(S_\rho M).$$

Conversely, if $(\hat{\Sigma}, p) \in \mathcal{SL}_\theta^+(S_\rho M)$ and if $T\hat{\Sigma} \cap VS_\rho M = 0$, then $(\hat{\Sigma}, p)$ is the ρ -Gauss lifting of a convex immersed hypersurface in M and thus:

$$(\hat{\Sigma}, p) \in \hat{\mathcal{F}}_{\rho, \theta}.$$

Special Lagrangian Curvature

The study of convex immersed hypersurfaces of constant ρ -special Lagrangian curvature is a refinement of a special case of the more general theory of Weingarten hypersurfaces. Indeed, the family of hypersurfaces Σ satisfying $\text{SL}_\rho(\Sigma) \in \theta + \pi\mathbb{Z}$ coincides with the family of hypersurfaces Σ satisfying:

$$\sin(\theta) \sum_{k \geq 0} (-1)^k \rho^{2k} \chi_{2k}(\Sigma) - \cos(\theta) \sum_{k \geq 0} (-1)^k \rho^{2k+1} \chi_{2k+1}(\Sigma) = 0,$$

where, for all i , $\chi_i(\Sigma) = \chi_i(A)$ is the i 'th higher principal curvature of Σ as defined in the introduction. The special Lagrangian curvature deserves particular attention since, by Lemma 2.2, it reflects a uniformly elliptic problem arising from calibrated geometries. This uniform ellipticity permits us to obtain compactness results which are not necessarily valid for Weingarten hypersurfaces in general, even in such apparently simple cases as hypersurfaces of constant Gaussian curvature.

2.6 Unique Continuation.

The special Lagrangian equation linearises to a Laplacian. We thus obtain the following unique continuation principle:

Lemma 2.3

Let $(\Sigma, p) = (S, i, p)$ and $(\Sigma', p') = (S', i', p')$ be two special Legendrian submanifolds immersed in M . If $i(p) = i'(p') = q$ and if Σ and Σ' have the same ∞ -jet at q , then (Σ, p) is locally equivalent to (Σ', p') at q . In other words, there exist neighbourhoods U and U' of p and p' respectively, and a (locally unique) diffeomorphism $\varphi : U \rightarrow U'$ such that $\varphi(p) = p'$ and:

$$i' \circ \varphi = i.$$

Proof: Let $q = i(p) = i'(p')$. Let ω_0 be the canonical symplectic form over $\mathbb{R}^n \times \mathbb{R}^n$. Let β be a primitive of ω . Let us define the contact form α over $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ by:

$$\alpha = dt - \beta.$$

By Darboux's theorem (see, for example, [19]), for an appropriate choice of β , there exists:

- (i) a positive real number ϵ ,
- (ii) a neighbourhood U of q in M , and
- (iii) a diffeomorphism $\varphi : (B_\epsilon(0), 0) \rightarrow (U, p)$,

such that:

- (i) $\varphi_*\alpha$ is colinear with the contact structure over U , and
- (ii) $T_0\varphi \cdot (\{0\} \times \{0\} \times \mathbb{R}^n) = T_p\Sigma = T_{p'}\Sigma'$.

Since Σ and Σ' are Legendrian, there exists an open set Ω of 0 in \mathbb{R}^n and functions $f, f' : \Omega \rightarrow \mathbb{R}$ such that $\varphi^*(\Sigma, p)$ and $\varphi^*(\Sigma', p')$ are locally the graphs over Ω of (f, df) and (f', df') respectively.

Special Lagrangian Curvature

Let u be a function over Ω . There exists a function $M : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow GL(\mathbb{R}^n)$ such that the graph of (u, du) is θ -special Legendrian if and only if:

$$\text{Im}(e^{-i\theta} \text{Det}(I + iM^{-1}(x, u, du)D^2uM(x, u, du))) = 0.$$

Differentiating, we obtain, for all k , smooth functions a_k^{ij} , b_k^i and c_k over $\mathbb{R}^{(n+1)^2}$ such that:

$$\begin{aligned} a_k^{ij}(x, u, Du, D^2u)\partial_i\partial_j(\partial_k u) + b_k^i(x, u, Du, D^2u)\partial_i(\partial_k u) \\ + c_k(x, u, Du, D^2u)\partial_k u = 0. \end{aligned}$$

Moreover, a_k^{ij} is invertible close to the origin. Since f and f' both satisfy this relation, we have, for all k :

$$\begin{aligned} a_k^{ij}(x, f, Df, D^2f)\partial_i\partial_j\partial_k(f - f') \\ + (a_k^{ij}(x, f', Df', D^2f') - a_k^{ij}(x, f, Df, D^2f))\partial_i\partial_j\partial_k f' \\ + b_k^i(x, f, Df, D^2f)\partial_i\partial_k(f - f') \\ + (b_k^i(x, f', Df', D^2f') - b_k^i(x, f, Df, D^2f))\partial_i\partial_k f' \\ + c_k(x, f, Df, D^2f)\partial_k(f - f') \\ + (c_k(x, f', Df', D^2f') - c_k(x, f, Df, D^2f))\partial_k f' = 0. \end{aligned}$$

Since a_k^{ij} , b_k^i and c_k are smooth, since all the derivatives of f and f' are bounded near 0, and since a_k^{ij} is uniformly bounded below near zero, using Taylor's theorem and the preceding relation, we find that there exists $K \in \mathbb{R}^+$ such that, for all k :

$$|\Delta(\partial_k(f - f'))| \leq K(\|D^2(f - f')\| + \|D(f - f')\| + \|f - f'\|).$$

Consider the function w defined by:

$$w = (f - f', D(f - f')).$$

Using the preceding relation, we find that there exists $L \in \mathbb{R}^+$ such that:

$$|\Delta w| \leq L(\|Dw\| + \|w\|).$$

Since w vanishes to infinite order at the origin, it follows by Aronszajn's unique continuation principle (see [1] or [18]) that w vanishes identically in a neighbourhood of zero. Consequently f coincides with f' in a neighbourhood of zero and the result follows. \square

3 - Local Rigidity and Examples.

3.1 Infinitesimal Deformation of Hypersurfaces.

Let M be a Riemannian manifold. Let $\text{Exp} : TM \rightarrow M$ be the exponential mapping of M . Let $\Sigma = (S, i)$ be an immersed hypersurface of constant ρ -special Lagrangian curvature in M . Let \mathbf{N} be the normal vector field over Σ in M . For $f \in C^\infty(\Sigma)$, we define the family of immersed hypersurfaces $(\Sigma_{f,t})_{t \in (-\epsilon, \epsilon)} = (S, i_{f,t})_{t \in (-\epsilon, \epsilon)}$ by:

$$i_{f,t}(p) = \text{Exp}_{i(p)}(tf(p)\mathbf{N}(p)).$$

Special Lagrangian Curvature

For all t , let $N_{f,t}$ be the normal vector field over $\Sigma_{f,t}$. For all t , let $A_{f,t}$ be the Weingarten operator of $\Sigma_{f,t}$. In other words:

$$A_{f,t}u = \nabla_u N_{f,t}.$$

We define the operator of variation of ρ -special Lagrangian curvature \mathcal{L}_θ^ρ , such that:

$$\mathcal{L}_\theta^\rho f = \partial_t \text{Im}(e^{-i\theta} \text{Det}(I + i\rho A_{f,t}))|_{t=0}.$$

We aim to calculate explicitly the operator \mathcal{L}_θ^ρ . Let R be the Riemann curvature tensor of ∇ , let W to be the endomorphism of TS defined by:

$$i_* W(u) = R_{N_\Sigma i_* u} N_\Sigma,$$

and let $\text{Hess}(f)$ be the Hessian of the function f :

Lemma 3.1

If A is the Weingarten operator of Σ , then:

$$\begin{aligned} \mathcal{L}_\theta^\rho f &= \sqrt{I + \rho^2 A^2} (-\text{Tr}((I + \rho^2 A^2)^{-1} \rho \text{Hess}(f)) \\ &\quad + f \text{Tr}((I + \rho^2 A^2)^{-1} \rho W) - f \text{Tr}((I + \rho^2 A^2)^{-1} \rho A^2)). \end{aligned}$$

Proof: By Proposition 3.1.1 of [14]:

$$\partial_t A_{f,t}|_{t=0} = fW - \text{Hess}(f) - fA_{f,0}^2.$$

Bearing in mind that:

$$\text{Im}(e^{-i\theta} \text{Det}(I + i\rho A_{f,0})) = 0,$$

the result now follows by elementary calculation. \square

It follows that the operator \mathcal{L}_θ^ρ defined over a submanifold of constant ρ -special Lagrangian curvature equal to θ is elliptic. Since it acts over the space of real valued functions, it is of index zero, and we would thus like know under which conditions it is injective (and thus invertible). In [23] we prove:

Lemma 3.2

Suppose that the sectional curvature of M is less than $-\kappa < 0$ and suppose that the ρ -special Lagrangian curvature of Σ is less than or equal to $\arctan(\sqrt{\kappa}\rho)$, then the zeroth order term of the formula given in the preceding lemma is non-negative:

$$J = \text{Tr}((I + \rho^2 A^2)^{-1} \rho W) - \text{Tr}((I + \rho^2 A^2)^{-1} \rho A^2) \geq 0$$

Proof: This is an elementary, but long and technical exercise involving Lagrange multipliers. See [23] for details. \square

This yields the following partial version of Theorem 1.3

Lemma 3.3

Let M be a Riemannian manifold of dimension $(n + 1)$ and of negative sectional curvature bounded above by -1 . Let $\Sigma = (S, i)$ and $\Sigma' = (S, i')$ be two compact, convex immersed hypersurfaces of constant ρ -special Lagrangian curvature equal to $\theta \leq n \arctan(\rho)$. If i is sufficiently close to i' in the $C^{2,\alpha}$ topology, then (up to reparametrisation) $i' = i$.

Remark: A complete proof of Theorem 1.3 will follow immediately from Theorem 1.4.

Proof: This follows from the preceding two Lemmata and the implicit function theorem for smooth functions on Banach manifolds. \square

3.2 Construction of Examples.

Non trivial examples of hypersurfaces of constant special Lagrangian curvature may be constructed in various ways. Non complete examples may be trivially constructed by consider hypersurfaces of revolution and thus transforming the PDE into an ODE.

Another method of construction is the following: let M be a hyperbolic manifold of dimension $(n + 1)$. Let $N \subseteq M$ be a totally geodesic hypersurface. For all R , let N_R be the equidistant hypersurface at distance R from N . Elementary hyperbolic geometry allows us to calculate:

Lemma 3.4

$$SL_\rho(N_R) = n \arctan(\rho \tanh(R)).$$

Thus N_R satisfies the hypothesis of Lemma 3.2. Thus, using Lemmata 3.1 and 3.2 along with the implicit function theorem for smooth functions on Banach manifolds, we may show that for small, smooth deformations of the metric on M , there exists deformations of N_R whose special Lagrangian curvature remains constant.

In [23], we use an analogous technique to construct hypersurfaces of constant special Lagrangian curvature in hyperbolic ends.

4 - Compactness.

4.1 Preliminary Definitions and Results.

We collect results required to prove Theorem 1.2. First, in [10] and [25], Jost & Xin and Yuan prove the following result:

Theorem 4.1 [Jost, Xin, 2002], [Yuan, 2002]

If $\hat{\Sigma}$ is a complete immersed SL_θ^+ submanifold of $\mathbb{R}^n \oplus \mathbb{R}^n$ of type $C^{1,\alpha}$, then $\hat{\Sigma}$ is an affine subspace.

Remark: [10] and [25] obtain the same result simultaneously using very different techniques. Jost and Xin use the properties of harmonic maps into convex subsets of Grassmannians. Yuan uses geometric measure theory. [10] is a slightly stronger result than that of [25], although both are satisfactory for our purposes.

In [4], Corlette obtains a finiteness result for compact immersed submanifolds of a given Riemannian manifold, recalling the finiteness result of Cheeger. As indicated by Corlette, this finiteness result suggests an underlying compactness result. We introduce the following definition:

Definition 4.2

Let (M, g) be a Riemannian manifold. Let $X = (Y, i)$ be an immersed submanifold in M . Let ∇^i be the Levi-Civita covariant derivative generated over Y by the immersion i into (M, g) . Let $A(X)$ be the Weingarten operator of X . For all $k \geq 2$, we define $A_k(X)$ using the following recurrence relation:

$$\begin{aligned} A_2(X) &= A(X), \\ A_k(X) &= \nabla^i A_{k-1}(X) \quad \forall k \geq 3. \end{aligned}$$

If we denote by $\|\cdot\|_\Omega$ the C^0 norm over the set Ω , then the result of Corlette may be stated as follows:

Theorem 4.3 [Arzela, Ascoli, Corlette]

Let $(M_n, g_n, p_n)_{n \in \mathbb{N}}$ be a sequence of complete pointed manifolds which converges towards the pointed manifold (M_0, g_0, p_0) in the Cheeger/Gromov topology. For all $n \in \mathbb{N}$, let $\Sigma_n = (S_n, i_n)$ be a complete immersed submanifold in M_n , and suppose that there exists $q_n \in S_n$ such that $i_n(q_n) = p_n$. Let $K \in \mathbb{N}$ be a positive integer. Suppose that, for all $R \in \mathbb{R}^+$ there exists $B \in (0, \infty)$ and $N \in \mathbb{N}$ such that:

$$n \geq N \Rightarrow \|A_k(\Sigma)\|_{B_R(q_n)} \leq B \quad \forall k \leq K + 1.$$

Then, there exists a complete pointed immersed submanifold $(\Sigma_0, q_0) = (S_0, i_0, q_0)$ in M_0 of type $C^{K, \alpha}$ for all $\alpha \in (0, 1)$ such that:

- (i) $i_0(q_0) = p_0$, and
- (ii) after extraction of a subsequence, $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ converges to (Σ_0, p_0) in the $C^{K, \alpha}$ Cheeger/Gromov topology for all $\alpha \in (0, 1)$.

Proof: A proof may be found in [22]. \square

We also require the λ -maximum lemma. Let X be a topological space. Let $f : X \rightarrow \mathbb{R}$ be a real valued function. We say that f is **locally bounded** if and only if for all $P \in X$ there exists $B \in \mathbb{R}^+$ and a neighbourhood Ω of P in X such that:

$$Q \in \Omega \quad \Rightarrow \quad |f(Q)| \leq B.$$

Lemma 4.4 λ -maximum lemma

Let X be a metric space and let P be a point in X . Suppose that there exists $\delta > 0$ such that the closed δ -ball about P in X is complete. There exists $\lambda \in \mathbb{R}^+$, which only depends on δ such that, for every locally bounded function $f : X \rightarrow \mathbb{R}^+$ such that $f(P) \geq 1$, there exists $Q \in B(P, \delta)$ such that $f(Q) \geq f(P)$ and:

$$Q' \in B(Q, \epsilon) \Rightarrow f(Q') \leq \lambda f(Q),$$

where ϵ is given by the relation:

$$\epsilon = \lambda^{-1} f(Q)^{-1/2}.$$

Proof: It suffices to assume the contrary and thus obtain a point in X at which f is arbitrarily large, which is absurd. A detailed proof may be found, for example, in [20]. \square

4.2 The Compactness Theorem.

Bearing in mind Theorem 4.3, Theorem 1.2 is equivalent to the following result:

Lemma 4.5

Let M be a complete Riemannian manifold of dimension $2n + 1$. Let $\theta \in (-n\pi/2, n\pi/2)$ be an angle. Let $(W, m, \omega, \Omega_\theta)$ be a positive special Legendrian structure over M . Let $\pi : \mathcal{SL}_\theta^+(M) \rightarrow M$ be the canonical projection.

For every compact subset K of M , for every $R \in \mathbb{R}^+$ and for every $N \in \mathbb{N}$, there exists $B \in \mathbb{R}^+$ such that if $(\hat{\Sigma}, p) \in \pi^{-1}(K)$ then:

$$\|A_k(\hat{\Sigma})\|_{B_R(p)} \leq B \quad \forall k \leq N.$$

Proof: We assume the contrary and obtain a contradiction.

First step : We begin by constructing a sequence of pointed submanifolds.

We may assume that there exists a compact subset K of M , a sequence $(\Sigma_m, q_m)_{m \in \mathbb{N}} = (S_m, i_m, q_m)_{m \in \mathbb{N}}$ of pointed immersed submanifolds in $\mathcal{SL}_\theta^+(M)$, and $k \geq 2$ such that $q_m \in K$ for all m , and:

$$(\|A_k(\Sigma_m)(q_m)\|)_{m \in \mathbb{N}} \rightarrow \infty.$$

For all m , we define $\mathcal{A}_{k,m} : S_m \rightarrow \mathbb{R}$ by:

$$\mathcal{A}_{k,m} = \sum_{i=2}^k \|A_i(\hat{\Sigma}_m)\|^{1/i}.$$

For all m , for all ρ and for all $p \in S_m$, let $B(p, \rho)$ be the ball of radius ρ about p in S_m with respect to the metric i^*g . Let $\lambda > 1$ be a real number greater than 1. For all m , by

replacing q_m by a point p_m obtained using the λ -maximum lemma (Lemma 4.4), and by denoting $B_m = \mathcal{A}_{k,m}(p_m)$, we may suppose that:

$$q \in B \left(p_m, \frac{1}{\lambda B_m^{1/2}} \right) \Rightarrow \mathcal{A}_{k,m}(q) \leq \lambda B_m.$$

Second step : We simplify the problem by transforming the contact structure of M onto the canonical contact structure of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$.

Since K is compact, we may suppose that there exists p_0 to which $(p_m)_{m \in \mathbb{N}}$ converges. Let ω be the canonical symplectic form over $\mathbb{R}^n \times \mathbb{R}^n$. Let β be a primitive of ω . We define the contact form α over $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ by:

$$\alpha = dt - \beta.$$

By Darboux's theorem for families (see, for example, appendix C of [20]) we may assume that there exists $\epsilon > 0$ and, for all m , a neighbourhood U_m of p_m in M and a diffeomorphism $\varphi_m : (B_\epsilon(0), 0) \rightarrow (U_m, p_m)$ such that $(\varphi_m)_* \alpha$ is colinear with the contact structure over U_m . Moreover, we may assume that $(\varphi_m)_{m \in \mathbb{N}}$ converges to φ_0 in the C_{loc}^∞ topology.

We denote by \mathbb{R}_0^n the zero section over \mathbb{R}^n in the bundle $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. Since we may freely choose β , we may suppose that, for all m , the application $T_0 \varphi_m$ sends \mathbb{R}_0^n onto $T_{p_m} \Sigma_m$.

Third step : We rescale by dilating the metric over M by a constant factor which tends to infinity as m tends to infinity. The sequence of pointed manifolds thus obtained converges to a real vector space.

For all m , we define the metric g_m over M by:

$$g_m = B_m^2 g.$$

For all m , we define Δ_m , an endomorphism of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, by:

$$\Delta_m(t, q, p) = (B_m t, B_m q, B_m p).$$

For all m , we define $\phi_m : (B_{B_m \epsilon}(0), 0) \rightarrow (M, p_m)$ by:

$$\phi_m = \varphi_m \circ \Delta_m^{-1}.$$

Let R be a positive real number. Since $(B_m)_{m \in \mathbb{N}}$ tends to infinity, there exists $M \in \mathbb{N}$ such that, for $m \geq M$:

$$B_R(0) \subseteq B_{B_m \epsilon}(0).$$

Consequently, for $m \geq M$, ϕ_m is defined over $B_R(0)$ and is a diffeomorphism onto its image. Next:

$$\begin{aligned} \phi_m^* g_m(t, q, p) &= ((\Delta_m^{-1})^* \varphi_m^* g_m)(t, q, p) \\ &= (\varphi_m^* g)(t/B_m, q/B_m, p/B_m). \end{aligned}$$

Since $(\varphi_m^* g)_{m \in \mathbb{N}}$ converges in the C_{loc}^∞ topology towards $\varphi_0^* g$, the sequence of metrics $(\phi_m^* g_m)_{m \in \mathbb{N}}$ converges to a constant metric g_0 over $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. For the same reason, there exists a constant distribution W_0 in $T\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, a constant complex structure J_0 over W_0 , a constant symplectic form ω_0 over W_0 , a constant Minkowski metric m_0 over W_0 , and a constant special Lagrangian form Ω_0 over W_0 such that $(\phi_m^* W)_{m \in \mathbb{N}}$, $(\phi_m^* J)_{m \in \mathbb{N}}$, $(B_m^2 \phi_m^* \omega)_{m \in \mathbb{N}}$, $(B_m^2 \phi_m^* m)_{m \in \mathbb{N}}$ and $(B_m^2 \phi_m^* \Omega)_{m \in \mathbb{N}}$ converge towards W_0 , J_0 , ω_0 , m_0 and Ω_0 respectively in the C_{loc}^∞ topology.

Fourth step : We transform the pointed submanifolds living in M into submanifolds of the dilated ambient manifolds, and we show that the sequence of pointed submanifolds thus obtained converges towards an affine subspace of a real vector space in the Cheeger/Gromov topology.

For all m , and for all $\rho \in (0, \infty)$ we denote by $B(p_m, \rho)$ the ball of radius ρ about p_m in the manifold S_m with respect to the metric i^*g generated by the immersion i into the Riemannian manifold (M, g) . For sufficiently large m , we have:

$$B\left(p_m, \frac{1}{\lambda B_m^{1/2}}\right) \subseteq U_m.$$

Consequently, for sufficiently large m , we define the pointed immersed submanifold $\tilde{\Sigma}_m = (\tilde{S}_m, \tilde{i}_m)$ which is contained in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ by:

$$(\tilde{S}_m, \tilde{i}_m) = \left(B\left(p_m, \frac{1}{\lambda B_m^{1/2}}\right), \Delta_m \circ \varphi_m^{-1} \circ i_m \right).$$

For all m , we view $\tilde{\Sigma}_m$ as an immersed submanifold in $(B_{B_m \epsilon}(0), 0, \varphi_m^* g_m)$. Thus, if we define $\tilde{\mathcal{A}}_{k,m}$ for $\tilde{\Sigma}_m$ in the same way as $\mathcal{A}_{k,m}$ for Σ_m , we obtain:

$$\tilde{\mathcal{A}}_{k,m}(p_m) = 1,$$

and, for all $q \in S_m$:

$$\tilde{\mathcal{A}}_{k,m}(q) \leq \lambda.$$

By the Arzela-Ascoli-Corlette Theorem (Theorem 4.3), we may assume that there exists a complete pointed immersed submanifold $\tilde{\Sigma}_0 = (\tilde{S}_0, \tilde{i}_0)$ in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of type $C^{k-1, \alpha}$ for all α which passes by the origin such that $(\tilde{\Sigma}_m, p_m)_{m \in \mathbb{N}}$ tends towards $(\tilde{\Sigma}_0, p_0)$ in the $C^{k-1, \alpha}$ -Cheeger/Gromov topology for all α .

By using elliptic regularity along Theorem 4.1, we obtain the following result:

Lemma 4.6

$\tilde{\Sigma}_0$ is a linear subspace of $\{0\} \times \mathbb{R}^n \times \mathbb{R}^n$ and $(\tilde{\Sigma}_m)_{m \in \mathbb{N}}$ converges towards $\tilde{\Sigma}_0$ in the C^∞ Cheeger/Gromov topology.

Proof: $T\tilde{\Sigma}_0$ is contained in W_0 . Since W_0 is a constant distribution over $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, and since $\tilde{\Sigma}_0$ passes by the origin, it follows that $\tilde{\Sigma}_0$ is contained in a linear subspace X of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of dimension $2n$ which is parallel to W_0 . In fact, by construction of W_0 , we obtain:

$$X = \{0\} \times \mathbb{R}^n \times \mathbb{R}^n.$$

The restriction of $(\omega_0, m_0, \Omega_0)$ defines a positive special Lagrangian structure over X . Since $\tilde{\Sigma}_m$ is a positive special Lagrangian submanifold for all m , it follows that $\tilde{\Sigma}_0$ is also a positive special Lagrangian submanifold. Since $\tilde{\Sigma}_0$ is complete and of type $C^{1, \alpha}$, it follows by Jost and Xin's theorem (Theorem 4.1) that $\tilde{\Sigma}_0$ is an affine subspace of X .

Special Lagrangian Curvature

We denote by \mathbb{R}_0^n the zero section in the vector bundle $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ over \mathbb{R}^n . By construction $T_0 \tilde{\Sigma}_0$ coincides with \mathbb{R}_0^n and it follows that $\tilde{\Sigma}_0$ itself coincides with \mathbb{R}_0^n . Since the sequence $(\tilde{\Sigma}_m)_{m \in \mathbb{N}}$ converges towards $\tilde{\Sigma}$ in the $C^{k-1, \alpha}$ Cheeger/Gromov topology, we may assume that there exists $(\rho_m)_{m \in \mathbb{N}} \in]0, \infty[$ such that $(\rho_m)_{m \in \mathbb{N}} \rightarrow \infty$ and, for all m , a function $(t_m, q_m) : B_{\rho_m}(0) \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n$ and a neighbourhood Ω_m about p_m in S_m such that the image of Ω_m by \tilde{i}_m coincides with the graph of (t_m, q_m) over $B_{\rho_m}(0)$. The sequence of functions $(t_m, q_m)_{m \in \mathbb{N}}$ converges to zero in the $C_{\text{loc}}^{k-1, \alpha}$ topology. It suffices for our needs that this sequence converges in the $C_{\text{loc}}^{1, \alpha}$ topology.

For all $n \in \mathbb{N}$, we define the function $\mathcal{O}_m(p, t, q, A)$ for $p, q \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $A \in \mathbb{R}^{n^2}$ with $\|p\|, \|q\|, |t| < \epsilon B_m/3$ by:

$$\begin{aligned} \mathcal{O}_m(p, t, q, A) &= (\Omega_m)_{\left(\frac{t}{B_m}, \frac{p}{B_m}, \frac{q}{B_m}\right)} \left[(\partial_1, -\beta_{(p,q)}(\partial_1, A \cdot \partial_1), A \cdot \partial_1), \dots, \right. \\ &\quad \left. (\partial_m, -\beta_{(p,q)}(\partial_m, A \cdot \partial_m), A \cdot \partial_m) \right]. \end{aligned}$$

Since $\tilde{\Sigma}_m$ is Legendrian with respect to the form $\alpha = dt - \beta$, we obtain, for all m :

$$(dt_m)_p(\partial_i) = \beta_{(p, q_m(p))}(\partial_i, Dq_m \cdot \partial_i).$$

Since $\tilde{\Sigma}_m$ is special Legendrian with respect to the form Ω_m , we obtain, for all m :

$$\mathcal{O}_m(p, t_m, q_m, Dq_m) = 0.$$

For all $\theta \in \mathbb{R}$, we define $\mathcal{O}_{0, \theta}$ over $\mathbb{R}^{(n+1)^2} = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n^2}$ by:

$$\mathcal{O}_{0, \theta}(p, t, q, A) = \text{Im}(e^{i\theta} \text{Det}(I + iA)).$$

There exists $\theta \in \mathbb{R}$ such that the sequence of functions $(\mathcal{O}_m)_{m \in \mathbb{N}}$ converges towards $\mathcal{O}_{0, \theta}$ in the C_{loc}^∞ topology over $\mathbb{R}^{(m+1)^2}$. Without loss of generality, we may assume that $\theta = 0$ and we define $\mathcal{O}_0 = \mathcal{O}_{0, 0}$. For all m , by taking the derivative of \mathcal{O}_m , we obtain for all i a relation of the following form for $q_{m, i}$:

$$a_{m, i}^{jk} \partial_j \partial_k q_{m, i} + b_{m, i}^j \partial_j q_{m, i} + c_{m, i} q_{m, i} = d_{m, i},$$

where $a_{m, i}^{jk}$, $b_{m, i}^j$, $c_{m, i}$ and $d_{m, i}$ are smooth functions of (p, t_m, q_m, Dq_m) . For all i , we define $a_{0, i}$ by:

$$a_{0, i}^{pq}(p, t, q, A) = ((I + A^2)^{-1})^{pq}.$$

Since $(\mathcal{O}_m)_{m \in \mathbb{N}}$ tends towards \mathcal{O}_0 in the C_{loc}^∞ topology, the sequences $(b_{m, i}^j)_{m \in \mathbb{N}}$, $(c_{m, i})_{m \in \mathbb{N}}$ and $(d_{m, i})_{m \in \mathbb{N}}$ tend towards zero in the C_{loc}^∞ topology, and $(a_{m, i}^{jk})_{m \in \mathbb{N}}$ tends towards $a_{0, i}^{jk}$ in the C_{loc}^∞ topology. Since $(t_m, q_m)_{m \in \mathbb{N}}$ converges towards zero in the $C_{\text{loc}}^{1, \alpha}$ topology, it follows that the functions a, b, c et d are locally uniformly bounded in the $C^{0, \alpha}$ norm. Moreover, a_0 is locally bounded from below, and these relations are consequently uniformly elliptic. The Schauder estimates (see, for example, [5]) permit us to conclude that, for all

$i, (D^2 q_{m,i})_{n \in \mathbb{N}}$ is locally uniformly bounded in the $C^{0,\alpha}$ norm. It follows that $(q_m)_{m \in \mathbb{N}}$ is locally uniformly bounded in the $C^{2,\alpha}$ norm. Since, for all m , t_m may be obtained from q_m by integrating the form $\beta(p, q_m)$, it follows that $(t_m)_{m \in \mathbb{N}}$ is also locally uniformly bounded in the $C^{2,\alpha}$ norm. Working by induction, we may thus show that $(t_m, q_m)_{m \in \mathbb{N}}$ is locally uniformly bounded in the $C^{k,\alpha}$ norm for all $k \in \mathbb{N}$.

The classical Arzela-Ascoli theorem now permits us to show that every subsequence of $(t_m, q_m)_{m \in \mathbb{N}}$ contains a subsubsequence which converges in the C_{loc}^k topology for all k . The limit is necessarily zero. It thus follows that $(t_m, q_m)_{m \in \mathbb{N}}$ converges towards zero in the C_{loc}^∞ topology as m tends to infinity, and the result now follows. \square

Final step : We use continuity in order to obtain a contradiction.

We have shown that $(\tilde{\Sigma}_m, p_m)_{m \in \mathbb{N}}$ converges towards $(\tilde{\Sigma}_0, 0)$ in the Cheeger/Gromov topology. Moreover $(\tilde{\Sigma}_0, 0)$ is an affine plane. However, if we define $\tilde{\mathcal{A}}_{k,0}$ over $\tilde{\Sigma}_0$ as before, since $(\tilde{\Sigma}_m, p_m)$ converges towards $(\tilde{\Sigma}_0, 0)$ in particular in the C^k -Cheeger-Gromov topology, we obtain:

$$\tilde{\mathcal{A}}_{k,0}(0) = 1.$$

This is absurd. We thus obtain the desired contradiction, and the result now follows. \square

We now obtain Theorem 1.2 as an immediate corollary:

Proof of Theorem 1.2: It follows by the preceding lemma and the Arzela-Ascoli-Corlette Theorem (Theorem 4.3) that there exists a subsequence of $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ which converges in the C^k -Cheeger/Gromov topology for all k . In the case of immersed submanifolds, this is equivalent to convergence in the C^∞ -Cheeger-Gromov topology, and the result now follows. \square

5 - Degenerate Limits.

5.1 The Derivatives of the Weingarten Operator.

Let M be a Riemannian manifold. Let g denote the metric over M . Let $\rho \in (0, \infty)$ be a positive real number. Let $S_\rho M$ be the ρ -sphere bundle over M . Let $\Sigma = (S, i)$ be a convex, oriented, immersed hypersurface in M . Let $\hat{\Sigma}_\rho = (S, \hat{i}_\rho)$ be the ρ -Gauss lifting of Σ in $S_\rho M$. Let A be the Weingarten operator of Σ . We define \mathbb{A} by $\mathbb{A} = \rho A$.

Let p be a point in S . We work locally in a chart about p in S chosen such that the basis $\partial_1, \dots, \partial_n$ is orthonormal at the origin, and the matrix \mathbb{A} is diagonal at the origin.

The matrix \mathbb{A}^i_j is a $(1, 1)$ tensor. In the sequel, for T_β^α an arbitrary tensor over S , we denote by $T_{\beta;i}^\alpha$ the covariant derivative of T_β^α with respect to the Levi-Civita connection of i^*g in the direction ∂_i . Raising and lowering of indices will be carried out with respect to the metric $g_{ij} = i^*g$ (i.e. the metric inherited from M through the immersion i). Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of the matrix \mathbb{A} at the origin. Using reasoning inspired by Calabi [3], we establish various relations concerning the symmetry of the derivatives of \mathbb{A} .

Special Lagrangian Curvature

Breaking slightly with tradition, for an arbitrary curvature tensor R , we denote:

$$R_{ijkl} = \langle R_{\partial_i \partial_j} \partial_k, \partial_l \rangle.$$

Lemma 5.1

Let R^M and R^Σ be the Riemann curvature tensors of M and Σ respectively. Let η be the exterior normal vector to Σ . Then:

$$\begin{aligned} \mathbb{A}_{ik;j} - \mathbb{A}_{jk;i} &= -\rho R_{ij\eta k}^M, \\ \mathbb{A}_{ij;kl} - \mathbb{A}_{ij;lk} &= R_{ipkl}^\Sigma \mathbb{A}^p_j - R_{pjkl}^\Sigma \mathbb{A}^p_i. \end{aligned}$$

Proof: This follows trivially from the definition of curvature (see, for example, [20]). \square

Lemma 5.2

If Σ is of constant ρ -special Lagrangian curvature, then at the origin:

$$\begin{aligned} \sum_{p=1}^n \frac{1}{1+\lambda_p^2} \mathbb{A}_{pp;k} &= 0, \\ \sum_{p=1}^n \frac{1}{1+\lambda_p^2} \mathbb{A}_{pp;kl} &= \sum_{p,q=1}^n \frac{\lambda_p + \lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pq;k} \mathbb{A}_{pq;l}. \end{aligned}$$

Proof: This follows by differentiating the relation $\text{Im}(e^{-i\theta} \text{Det}(I + i\mathbb{A})) = 0$. \square

5.2 Covariant Derivatives and Laplacians.

Let ∇ denote the Levi-Civita covariant derivative of (S, i^*g) . Let $\hat{\nabla}$ denote the Levi-Civita covariant derivative of (S, \hat{i}_ρ^*g) . Let Γ^k_{ij} denote the Christoffel symbol of this covariant derivative with respect to ∇ :

$$\Gamma^k_{ij} \partial_k = (\hat{\nabla} - \nabla)_{\partial_i} \partial_j.$$

We observe that:

$$\hat{i}_\rho^*g(\cdot, \cdot) = i^*g((I + \mathbb{A}^2)\cdot, \cdot).$$

We denote by \mathbb{B}^{ij} the dual metric to \hat{i}_ρ^*g . Thus:

$$\mathbb{B}^{ij} (I + \mathbb{A}^2)_{jk} = \delta^i_k.$$

Lemma 5.3

Γ satisfies the following equation:

$$\Gamma^k_{ij} = \frac{\rho}{2} \mathbb{A}^p_j \mathbb{B}^{kq} R_{qip\eta}^M + \frac{\rho}{2} \mathbb{A}^p_i \mathbb{B}^{kq} R_{qjp\eta}^M + \frac{1}{2} \mathbb{B}^{kl} \mathbb{A}^p_l (\mathbb{A}_{pi;j} + \mathbb{A}_{pj;i}).$$

Proof: Analogously to the Koszul formula, we have:

$$2\langle(\hat{\nabla} - \nabla)_{\partial_i} \partial_j, (I + \mathbb{A}^2) \partial_k\rangle = \langle\partial_j, [\nabla_{\partial_i} (I + \mathbb{A}^2)] \partial_k\rangle - \langle\partial_i, [\nabla_{\partial_k} (I + \mathbb{A}^2)] \partial_j\rangle + \langle\partial_k, [\nabla_{\partial_j} (I + \mathbb{A}^2)] \partial_i\rangle.$$

Thus:

$$2(I + \mathbb{A}^2)_{kp} \Gamma^p_{ij} = \mathbb{A}^p_j (\mathbb{A}_{pk;i} - \mathbb{A}_{pi;k}) + \mathbb{A}^p_i (\mathbb{A}_{pk;j} - \mathbb{A}_{pj;k}) + \mathbb{A}^p_k (\mathbb{A}_{pi;j} + \mathbb{A}_{pj;i}).$$

The result now follows by Lemma 5.1. \square

Let v be a point in $S_\rho M$. For $X, Y, Z \in T_v S_\rho M$, we define $\mathcal{R}^a(X, Y, Z)$ and $\mathcal{R}^b(X, Y, Z)$ by:

$$\begin{aligned} \mathcal{R}^a(X, Y, Z) &= R^M(v, \pi_H(X), \pi_V(Y), \pi_H(Z)), \\ \mathcal{R}^b(X, Y, Z) &= R^M(v, \pi_V(X), \pi_H(Y), \pi_H(Z)). \end{aligned}$$

Let P be an n dimensional subspace of $W_v \subseteq TS_\rho M$. Let B^{ij} be the metric on P^* dual to that inherited by P from W_v through the canonical immersion. Let e_1, \dots, e_n be a basis for P . We define the vector $\xi(P) = \xi(P)^k e_k \in P$ by:

$$\xi(P)^k = -B^{ij} B^{kq} \mathcal{R}^a_{iqj} - B^{ij} B^{kq} \mathcal{R}^b_{iqj}.$$

$\xi(P)$ does not depend on the basis of P chosen. We thus define the vector field $\tilde{\xi}$ over $TS \cong T\hat{\Sigma}$ by:

$$\tilde{\xi}(p) = -\hat{i}_\rho^* \xi(T_p \hat{\Sigma}_p).$$

In the sequel, we will denote $\tilde{\xi}$ by ξ . In terms of the basis $\partial_1, \dots, \partial_n$ of TS :

$$\xi^q = -\rho \mathbb{B}^{ij} \mathbb{B}^{kq} \mathbb{A}^p_j R^M_{\eta ipk} - \rho \mathbb{B}^{ij} \mathbb{B}^{kq} \mathbb{A}^p_i R^M_{\eta pj k}.$$

Since ξ is defined in terms of \hat{i}_ρ , it remains meaningful as long as \hat{i}_ρ converges, even if i doesn't.

Lemma 5.4

For every function f over S :

$$\hat{\Delta} f + df(\xi) = \mathbb{B}^{ij} f_{;ij}.$$

Proof: First:

$$\hat{\Delta} f = \mathbb{B}^{ij} f_{;ij} - \mathbb{B}^{ij} \Gamma^k_{ij;k} f_{;k}.$$

By Lemma 5.3 and the symmetry of \mathbb{B} :

$$\mathbb{B}^{ij} \Gamma^k_{ij} = -\rho \mathbb{B}^{jq} \mathbb{B}^{kp} \mathbb{A}^i_q R^M_{\eta ipj} + \frac{1}{2} \mathbb{B}^{ij} \mathbb{B}^{kq} \mathbb{A}^p_q (\mathbb{A}_{pi;j} + \mathbb{A}_{pj;i}).$$

By Lemma 5.1:

$$\mathbb{A}_{pi;j} + \mathbb{A}_{pj;i} = \mathbb{A}_{ji;p} + \mathbb{A}_{ij;p} + \rho R^M_{\eta jip} + \rho R^M_{\eta jip}.$$

Using Lemma 5.2 and the symmetry of \mathbb{B} yields:

$$\mathbb{B}^{ij} (\mathbb{A}_{pi;j} + \mathbb{A}_{pj;i}) = 2\rho \mathbb{B}^{ij} R^M_{\eta jip}.$$

Thus:

$$\mathbb{B}^{ij} \Gamma^k_{ij} = -\rho \mathbb{B}^{ij} \mathbb{B}^{kq} \mathbb{A}^p_q R^M_{\eta ipj} - \rho \mathbb{B}^{jq} \mathbb{B}^{kp} \mathbb{A}^i_q R^M_{\eta ipj} = \xi^k.$$

The result now follows. \square

5.3 A Useful Function.

Let $HS_\rho M$ be the horizontal bundle of $S_\rho M$ associated to the Levi-Civita connection over M . Let π_H be the projection of W_ρ onto $HS_\rho M$. For all $\tau \in (0, \infty)$, we define f^τ by:

$$f^\tau = \text{Det}(\pi_H)^{\tau/2} = \text{Det}(I + \mathbb{A}^2)^{-\tau}.$$

For all τ , the function f^τ is defined even when $\hat{\Sigma}$ is vertical. We aim to calculate the Laplacian of f^τ . We first require:

Lemma 5.5

Let η be the exterior normal vector to Σ . Let $\nabla^M R^M$ be the covariant derivative of R^M with respect to the Levi-Civita connection over M . Then:

$$\mathbb{A}_{ij;kl} - \mathbb{A}_{kj;il} = -\rho(\nabla^M R^M)_{ik\eta j l} - R_{\eta k \eta j}^M \mathbb{A}_{li} - R_{i \eta \eta j}^M \mathbb{A}_{lk} - \mathbb{A}^p_l R_{ikpj}^M.$$

Proof: This follows by differentiating the first relation in Lemma 5.1. \square

For all i , we define $\hat{\partial}_{i,\rho}$ by:

$$\hat{\partial}_{i,\rho} = (\hat{i}_\rho)_* \partial_i = \{\partial_i, \rho A \partial_i\}.$$

Let $\pi : TS_\rho M \rightarrow W_\rho$ be the orthogonal projection onto W_ρ . We denote by $\nabla^{W_\rho} = \pi \circ \nabla^{S_\rho M}$ the covariant derivative of the distribution W_ρ inherited from the Levi-Civita connection of $S_\rho M$. We define \hat{A}^ρ_{ijk} over S by:

$$\begin{aligned} \hat{A}^\rho_{ijk} &= \omega(\nabla_{\hat{\partial}_{i,\rho}}^{W_\rho} \hat{\partial}_{j,\rho}, \hat{\partial}_{k,\rho}) + \frac{1}{2} R^M(\pi_H(\hat{\partial}_{i,\rho}), \pi_H(\hat{\partial}_{j,\rho}), v, \pi_H(\hat{\partial}_{k,\rho})) \\ &\quad + \frac{1}{2} R^M(v, \pi_V(\hat{\partial}_{j,\rho}), \pi_H(\hat{\partial}_{i,\rho}), \pi_V(\hat{\partial}_{k,\rho})) + \frac{1}{2} R^M(v, \pi_V(\hat{\partial}_{i,\rho}), \pi_H(\hat{\partial}_{j,\rho}), \pi_V(\hat{\partial}_{k,\rho})). \end{aligned}$$

In the sequel, we will refer to \hat{A}^ρ as the **adjusted Weingarten operator** of $\hat{\Sigma}_\rho$. We observe that $\|\hat{A}^\rho\|$ is controlled by the norm of R^M and the norm of the second fundamental form of $\hat{\Sigma}_\rho$.

Lemma 5.6

$$\hat{A}^\rho_{ijk} = \mathbb{A}_{jk;i}.$$

Proof: This is a trivial calculation (see, for example, [20]). \square

Lemma 5.7 Let \hat{A}^ρ be the adjusted Weingarten operator of $\hat{\Sigma}_\rho$. If $h = \text{Log}(\text{Det}(I + \mathbb{A}^2))$, then:

$$\begin{aligned} \mathbb{B}^{ij} h_{;ij} &= \sum_{pq,r} \frac{2(1+\lambda_p \lambda_q)}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} \\ &\quad - \sum_{p,q} \frac{(\lambda_p - \lambda_q)^2}{(1+\lambda_p^2)(1+\lambda_q^2)} R_{pqpq}^\Sigma \\ &\quad - C(T_p \hat{\Sigma}, M, \hat{A}^\rho), \end{aligned}$$

where C is a continuous function of:

- (i) the affine subspace $T_p \hat{\Sigma} \subseteq W_p$,
- (ii) the tensor \hat{A}^ρ defined over $T_p \hat{\Sigma}$, and
- (iii) the curvature of M and its first derivative.

Special Lagrangian Curvature

Proof: Differentiating h yields:

$$\begin{aligned} h_{;i} &= 2\mathrm{Tr}(\mathbb{A}(I + \mathbb{A}^2)^{-1}\mathbb{A}_{;i}), \\ h_{;ij} &= 2\mathrm{Tr}(\mathbb{A}_{;j}(I + \mathbb{A}^2)^{-1}\mathbb{A}_{;i}) \\ &\quad - 2\mathrm{Tr}(\mathbb{A}(I + \mathbb{A}^2)^{-1}[\mathbb{A}\mathbb{A}_{;j} + \mathbb{A}_{;j}\mathbb{A}](I + \mathbb{A}^2)^{-1}\mathbb{A}_{;i}) \\ &\quad + 2\mathrm{Tr}(\mathbb{A}(I + \mathbb{A}^2)^{-1}\mathbb{A}_{;ij}). \end{aligned}$$

Thus, at the origin:

$$\begin{aligned} h_{;ij} &= \sum_{p,q} \frac{2}{1+\lambda_p^2} \mathbb{A}_{pq;i} \mathbb{A}_{pq;j} - \sum_{p,q} \frac{2\lambda_p^2}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pq;i} \mathbb{A}_{pq;j} \\ &\quad - \sum_{p,q} \frac{2\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pq;i} \mathbb{A}_{pq;j} + \sum_p \frac{2\lambda_p}{1+\lambda_p^2} \mathbb{A}_{pp;ij}. \end{aligned}$$

Using the symmetry of \mathbb{A} with respect to p and q , we obtain:

$$h_{;ij} = \sum_{p,q} \frac{2 - 2\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pq;i} \mathbb{A}_{pq;j} + \sum_p \frac{2\lambda_p}{1+\lambda_p^2} \mathbb{A}_{pp;ij}.$$

We aim to eliminate the second derivatives in this expression.

$$\begin{aligned} \sum_{p,q} \frac{\lambda_p}{1+\lambda_p^2} \frac{1}{1+\lambda_q^2} \mathbb{A}_{pp;qq} &= \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{qq;pp} \\ &\quad + \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} [\mathbb{A}_{qp;qp} - \mathbb{A}_{qq;pp}] \\ &\quad + \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} [\mathbb{A}_{qp;pq} - \mathbb{A}_{qp;qp}] \\ &\quad + \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} [\mathbb{A}_{pp;qq} - \mathbb{A}_{qp;pq}]. \end{aligned}$$

By Lemmata 5.1 and 5.5:

$$\begin{aligned} \sum_{p,q} \frac{\lambda_p}{1+\lambda_p^2} \frac{1}{1+\lambda_q^2} \mathbb{A}_{pp;qq} &= \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{qq;pp} \\ &\quad + \sum_{p,q} \frac{\lambda_p(\lambda_q - \lambda_p)}{(1+\lambda_p^2)(1+\lambda_q^2)} R_{pqpp}^\Sigma + C_1(T_p \hat{\Sigma}, M), \end{aligned}$$

where C_1 depends continuously on $T_p \hat{\Sigma}$, the curvature of M and its first derivative. The last relation in Lemma 5.2 now yields:

$$\begin{aligned} \sum_{p,q} \frac{\lambda_p}{1+\lambda_p^2} \frac{1}{1+\lambda_q^2} \mathbb{A}_{pp;qq} &= \sum_{p,q,r} \frac{(\lambda_p + \lambda_q)\lambda_r}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} \\ &\quad + \sum_{p,q} \frac{\lambda_p(\lambda_q - \lambda_p)}{(1+\lambda_p^2)(1+\lambda_q^2)} R_{pqpp}^\Sigma + C_1(T_p \hat{\Sigma}, M). \end{aligned}$$

Thus:

$$\begin{aligned} \mathbb{B}^{ij} h_{;ij} &= \sum_p \frac{1}{1+\lambda_p^2} h_{;pp} \\ &= \sum_{p,q,r} \frac{2-2\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} + 2 \sum_{p,q} \frac{\lambda_p}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pp;qq} \\ &= \sum_{p,q,r} \frac{2-2\lambda_p\lambda_q+2\lambda_p\lambda_r+2\lambda_q\lambda_r}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} \\ &\quad + 2 \sum_{p,q} \frac{\lambda_p(\lambda_q - \lambda_p)}{(1+\lambda_p^2)(1+\lambda_q^2)} R_{pqpp}^\Sigma + C_1(T_p \hat{\Sigma}, M). \end{aligned}$$

Special Lagrangian Curvature

By the symmetry of \mathbb{A}_{ij} and Lemma 5.1:

$$\begin{aligned} \sum_{p,q,r} \frac{2\lambda_p\lambda_r+2\lambda_q\lambda_r}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} &= \sum_{p,q,r} \frac{4\lambda_q\lambda_r}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r}^2 \\ &= \sum_{p,q,r} \frac{4\lambda_q\lambda_r}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} (\mathbb{A}_{rq;p} - \rho R_{pr\eta q}^M)^2. \end{aligned}$$

Since:

$$\lambda_q\lambda_r R_{pr\eta q}^M = R^M(\pi_H(\hat{\partial}_{i,\rho}), \pi_V(\hat{\partial}_{r,\rho}), v, \pi_V(\hat{\partial}_{q,\rho})),$$

we obtain:

$$\begin{aligned} \mathbb{B}^{ij} h_{,ij} &= \sum_{p,q,r} \frac{2+2\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)(1+\lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r} \\ &\quad + 2 \sum_{p,q} \frac{\lambda_p(\lambda_q - \lambda_p)}{(1+\lambda_p^2)(1+\lambda_q^2)} R_{pqpp}^\Sigma + C(T_p \hat{\Sigma}, M, \hat{A}^\rho). \end{aligned}$$

The result now follows by the antisymmetry of R^Σ . \square

We now obtain the following result concerning f^τ and \hat{A}^ρ :

Lemma 5.8

With the same notation as in Lemma 5.7, if $\tau \leq 1/2n$ then:

$$\frac{1}{\tau f^\tau} \mathbb{B}^{ij} f_{,ij}^\tau \leq \sum_{p,q} \frac{(\lambda_p - \lambda_q)^2}{(1 + \lambda_p^2)(1 + \lambda_q^2)} R_{pqpp}^\Sigma + C(T_p \hat{\Sigma}, M, \hat{A}^\rho).$$

Proof: Trivially:

$$\frac{1}{\tau f^\tau} \mathbb{B}^{ij} f_{,ij} = \frac{1}{\tau (f^\tau)^2} \mathbb{B}^{ij} f_{,i}^\tau f_{,j}^\tau - \mathbb{B}^{ij} h_{,ij}.$$

The derivative of f^τ satisfies:

$$f_{,i}^\tau = -\tau f^\tau \sum_{j=1}^n \frac{\lambda_j}{(1 + \lambda_j^2)} \mathbb{A}_{jj;i}.$$

Thus:

$$\frac{1}{\tau (f^\tau)^2} \mathbb{B}^{ij} f_{,i}^\tau f_{,j}^\tau = \tau \sum_{p,q,r} \frac{1}{(1 + \lambda_r^2)} \frac{4\lambda_p\lambda_q}{(1 + \lambda_p^2)(1 + \lambda_q^2)} \mathbb{A}_{pp;r} \mathbb{A}_{qq;r}.$$

However, for all r , using the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} \sum_{p,q} \frac{4\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pp;r} \mathbb{A}_{qq;r} &= \left(\sum_p \frac{2\lambda_p}{(1+\lambda_p^2)} \mathbb{A}_{pp;r} \right)^2 \\ &\leq 4n \sum_p \frac{\lambda_p^2}{(1+\lambda_p^2)^2} \mathbb{A}_{pp;r}^2 \\ &\leq 4n \sum_{p,q} \frac{\lambda_p\lambda_q}{(1+\lambda_p^2)(1+\lambda_q^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r}. \end{aligned}$$

Consequently:

$$\frac{1}{\tau (f^\tau)^2} \mathbb{B}^{ij} f_{,i}^\tau f_{,j}^\tau \leq 2n\tau \sum_{p,q,r} \frac{2(1 + \lambda_p\lambda_q)}{(1 + \lambda_p^2)(1 + \lambda_q^2)(1 + \lambda_r^2)} \mathbb{A}_{pq;r} \mathbb{A}_{pq;r}.$$

Thus, if $2n\tau \leq 1$, the desired result follows from Lemmata 5.6 and 5.7. \square

To summarise, we obtain:

Lemma 5.9

If $\tau \leq 1/2n$, then:

$$\hat{\Delta}f^\tau + df^\tau(\xi) - Cf^\tau \leq 0.$$

where $C > 0$ is a constant depending continuously on the local geometries of $\hat{\Sigma}$ and M .

Proof: By definition of \mathbb{A} :

$$R_{pqpq}^\Sigma = R_{pqpq}^M - \rho^{-2}\lambda_p\lambda_q(1 - \delta_{pq}).$$

Since $\lambda_i \geq 0$ for all i :

$$\sum_{p,q} \frac{(\lambda_p - \lambda_q)^2}{(1 + \lambda_p^2)(1 + \lambda_q^2)} R_{pqpq}^\Sigma \leq \sum_{p,q} \frac{(\lambda_p - \lambda_q)^2}{(1 + \lambda_p^2)(1 + \lambda_q^2)} R_{pqpq}^M.$$

As before, the term on the right hand side depends continuously on the curvature M and may thus be incorporated into C . Thus, by Lemmata 5.4 and 5.8:

$$\hat{\Delta}f^\tau + df^\tau(\xi) - C(T_p\hat{\Sigma}, M, \tilde{A}_p)f^\tau \leq 0.$$

Since f^τ is positive, the result now follows by taking a constant upper bound for C . \square

5.4 Degeneration of Submanifolds.

We are now in a position to describe submanifolds in $\partial\mathcal{F}_{\rho,\theta}^+(M)(M)$. We require the following definition:

Definition 5.10

Let M be a manifold of dimension $(n + 1)$. Let $S_\rho M$ be the ρ -sphere bundle over M . Let $\pi : S_\rho M \rightarrow M$ be the canonical projection. Let $\hat{\Sigma} = (S, \hat{i})$ be an n dimensional immersed submanifold of $S_\rho M$. Let p be an arbitrary point of S . We say that $\hat{\Sigma}$ is **vertical** at p if and only if the kernel of $T_{i(p)}\pi$ has non-trivial intersection with $T_p\hat{\Sigma} = T_p\hat{i} \cdot T_pS$. For $k \in \mathbb{N}$, we say that $\hat{\Sigma}$ is **vertical of order k** at p if and only if:

$$\text{Dim}(\text{Ker}(T_{i(p)}\pi) \cap T_p\hat{\Sigma}) = k.$$

We say that Σ is **vertical** if and only if it is vertical at every point p in S .

Lemma 5.11

If (Σ, p) is a pointed immersed submanifold in $\partial\mathcal{F}_{\rho,\theta}^+(M)$, then Σ is vertical.

Proof: Let $(\hat{\Sigma}_0, p_0) = (S_0, \hat{i}_0, p_0)$ be a pointed, immersed submanifold in $\partial\hat{\mathcal{F}}_{\rho, \theta}(M)$. Let $(\Sigma_m, p_m)_{m \in \mathbb{N}} = (S_m, i_m, p_m)_{m \in \mathbb{N}}$ be a sequence of convex, pointed, immersed submanifolds in $\mathcal{F}_{\rho, \theta}(M)$ such that if, for all n , $(\hat{\Sigma}_m, p_m)$ is the ρ -Gauss lifting of (Σ_m, p_m) , then $(\hat{\Sigma}_m, p_m)_{m \in \mathbb{N}}$ converges to (Σ_0, p_0) in the Cheeger/Gromov topology.

For all m , let A_m be the Weingarten operator of Σ_m , and let $\lambda_{m,1}, \dots, \lambda_{m,n} \geq 0$ be the eigenvalues of $\rho A_m(p_m)$. We define the function $f_m : S_m \rightarrow \mathbb{R}$ by:

$$f_m = \text{Det}(I + \rho^2 A_m^2)^{-1/2n}.$$

We define f_0 over S_0 by $f_0 = f^{1/2n}(\Sigma_0)$. By Lemma 5.9, for all m :

$$\hat{\Delta} f_m + df_m(\xi_m) - C f_m \leq 0.$$

Since $(\hat{\Sigma}_m)_{m \in \mathbb{N}}$ converges, C may be chosen the same for all m . By the maximum principle (see Lemma A.1), if f_0 vanishes at a single point, then it vanishes everywhere. However, Σ_0 is vertical if and only if $f_0 = 0$ and the result follows. \square

5.5 The Geometry of Curtain Submanifolds.

We now prove Theorem 1.4. We recall the following result concerning vertical submanifolds:

Lemma 5.12

Let M be a Riemannian manifold. Let ρ be a positive real number. Let $\hat{\Sigma} = (S, \hat{i})$ be an immersed submanifold in $S_\rho M$. Let $\pi : S_\rho M \rightarrow M$ be the canonical projection. Let p be an arbitrary point in S . Suppose that there exists a neighbourhood Ω of p in S such that $\text{Dim}(T\hat{\Sigma} \cap VS_\rho M)$ is constant over Ω . Define k by $k = \text{Dim}(T\hat{\Sigma} \cap VS_\rho M)$. By restricting Ω if necessary, we may assume that there exists $\epsilon > 0$, a submanifold $\Sigma \subseteq M$ embedded into M and a diffeomorphism $\Phi : \Sigma \times (-\epsilon, \epsilon)^k \rightarrow \Omega$ such that, if we denote by $\pi_1 : \Sigma \times (-\epsilon, \epsilon)^k \rightarrow \Sigma$ the projection onto the first factor, then:

$$\pi \circ \Phi = \pi_1.$$

We now obtain the following result:

Lemma 5.13

Let M be a Riemannian manifold. Let ρ be a positive real number. Let $\theta \in [(n-1)\pi/2, n\pi/2[$ be an angle. Let $(\hat{\Sigma}, p) = (S, \hat{i}, p)$ be an immersed submanifold in $\partial\hat{\mathcal{F}}_{\rho, \theta}(M)$. If $\hat{\Sigma}$ is vertical, then $\hat{\Sigma}$ is a normal sphere bundle over a complete totally geodesic submanifold of M .

Proof: Let $(\Sigma_m, p_m)_{m \in \mathbb{N}} = (S_m, i_m, p_m)_{m \in \mathbb{N}}$ be a sequence of immersed submanifolds in $\hat{\mathcal{F}}_{\rho, \theta}^+(M)$ whose ρ -Gauss liftings converge to $(\hat{\Sigma}, p)$ in the Cheeger/Gromov topology.

Let q be a point in S . Suppose that $\text{Dim}(T\hat{\Sigma} \cap VS_\rho M)$ is constant in a neighbourhood of q . By Lemma 5.12, there exists a submanifold Σ of M , $k \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$, a neighbourhood Ω of q in S and a diffeomorphism $\Phi : \Sigma \times (-\epsilon, \epsilon)^k \rightarrow \Omega$ such that:

$$(\pi \circ \hat{i}) \circ \Phi = \pi_1.$$

Special Lagrangian Curvature

Let (x, y) be a point in $\Sigma \times (-\epsilon, \epsilon)^k$. Let X_1, \dots, X_n be a basis of $T_x \Sigma$. For all i , we have:

$$T\pi \cdot T(\hat{i} \circ \Phi) \cdot X_i = X_i.$$

Consequently, there exists $Y_1, \dots, Y_m \in T_x M$ such that, for all i :

$$T(\hat{i} \circ \Phi) \cdot X_i = \{X_i, Y_i\}.$$

Let $(q_m)_{m \in \mathbb{N}}$ be a sequence of points in $(S_m)_{m \in \mathbb{N}}$ which tends towards $\Phi(x, y)$. For all m , let $(E_{1,m}, \dots, E_{n,m})$ be vectors in $T_{q_m} S_m$ such that, for all i :

$$\begin{aligned} (T\hat{i}_m \cdot E_{i,m})_{m \in \mathbb{N}} &\rightarrow \{X_i, Y_i\} \\ \Rightarrow (T(\pi \circ \hat{i}_m) \cdot E_{i,m})_{m \in \mathbb{N}} &\rightarrow X_i. \end{aligned}$$

For all i and for all m , let $X_{i,m}, Y_{i,m} \in T_{(\pi \circ \hat{i}_m)(q_m)} M$ be such that:

$$T\hat{i}_m \cdot E_{i,m} = \{X_{i,m}, Y_{i,m}\}.$$

For all m , and for all i , since $X_{i,m}$ is a tangent vector of the hypersurface of which $\hat{\Sigma}_m$ is the ρ -Gauss lifting, we have:

$$\langle \hat{i}_m(q_m), X_{i,m} \rangle = 0.$$

Thus, by taking limits, we obtain:

$$\langle \hat{i}(\Phi(x, y)), X_i \rangle = 0 \quad \forall i.$$

Consequently, if we denote by $N\Sigma$ the normal ρ -sphere bundle over Σ , we find that, for all $q' \in \Omega$:

$$\hat{i}(q') \in N\Sigma.$$

Since $N\Sigma$ is of dimension n , it follows that \hat{i} sends Ω diffeomorphically onto an open subset of $N\Sigma$. Since $N\Sigma$ is θ -special Lagrangian, it follows by the unique continuation principle (Lemma 2.3) that Ω may be extended to an open set of S such that $\hat{i} : \Omega \rightarrow N\Sigma$ is a covering map.

For all i and for all m , since $\hat{\Sigma}_m$ is the ρ -Gauss lifting of a convex hypersurface, we have:

$$\langle X_{i,m}, Y_{i,m} \rangle \geq 0.$$

Consequently, taking limits, we obtain, for all i :

$$\langle X_i, Y_i \rangle \geq 0.$$

Thus, since \hat{i} is a diffeomorphism, for all $\{X, Y\} \in TN\Sigma$:

$$\langle X, Y \rangle \geq 0.$$

Special Lagrangian Curvature

Therefore, if N is a normal vector field of Σ , then, for all $X \in T_\Sigma$:

$$\langle X, \nabla_X N \rangle \geq 0.$$

However, $(-N)$ is also a normal vector field over Σ , and thus, for all $X \in T\Sigma$:

$$\begin{aligned} \langle X, \nabla_X N \rangle &= -\langle X, \nabla_X(-N) \rangle \leq 0 \\ \Rightarrow \langle X, \nabla_X N \rangle &= 0. \end{aligned}$$

Thus, by polarisation, for all $X, Y \in T\Sigma$ and for all N normal to Σ :

$$\langle X, \nabla_Y N \rangle = 0.$$

It thus follows that Σ is a totally geodesic submanifold. The result now follows by unique continuation. \square

Proof of Theorem 1.4: Let $(\hat{\Sigma}, p)$ be an immersed submanifold in $\partial\hat{\mathcal{F}}_{\rho,\theta}(M)$. By Lemma 5.11, $\hat{\Sigma}$ is vertical. By Lemma 5.13, there exists a complete totally geodesic immersed submanifold $\Sigma = (S, i)$ of codimension $k + 1$ in M such that if $N\Sigma$ is the k -sphere bundle over Σ in M , then \hat{i} defines a covering map from $\hat{\Sigma}$ onto $N\Sigma$. The special Lagrangian angle of $N\Sigma$ equals $k\pi/2 = \theta$. Thus, if $\theta \neq (n - 1)\pi/2$, we obtain a contradiction and it thus follows that $\partial\hat{\mathcal{F}}_{\rho,\theta} = \emptyset$. On the other hand, if $\theta = (n - 1)\pi/2$, then $k = (n - 1)$ and Σ is therefore a geodesic. The result now follows. \square

Finally we obtain Theorem 1.3 as a corollary:

Proof of Theorem 1.3: Let $\Sigma = (S, i)$ be a convex, immersed hypersurface in $\mathcal{F}_{\rho,\theta}(M)$. Let $(\Sigma_n)_{n \in \mathbb{N}} = (S_n, i_n)_{n \in \mathbb{N}}$ be a sequence of convex immersed submanifolds in $\mathcal{F}_{\rho,\theta}(M)$ such that $i_n \rightarrow i$ in the C^0 topology. For all n , let \hat{i}_n be the ρ -Gauss lifting of i_n . The convexity of Σ_n for all n and the convergence of $(i_n)_{n \in \mathbb{N}}$ to i_0 ensures that no subsequence of \hat{i}_n can converge to a sphere bundle over a geodesic. It thus follows by Theorem 1.4 that every subsequence of $(i_n)_{n \in \mathbb{N}}$ subconverges in the C^∞ Cheeger/Gromov topology (thus modulo reparametrisation) to an immersion i'_0 . Trivially, $i'_0 = i_0$. It thus follows that $(i_n)_{n \in \mathbb{N}}$ converges itself in C^∞ Cheeger/Gromov topology to i_0 . The result now follows by Lemma 3.3. \square

Finally, as remarked in the introduction, the techniques used to prove Theorems 1.2 and 1.4 may be trivially adapted to yield the following analogous result for Gaussian curvature:

Proposition 5.14

Let M be a Riemannian manifold of dimension $(n + 1)$. Let $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ be a sequence of complete, immersed submanifolds of constant Gaussian curvature. For all n , let A_n be the shape operator of Σ_n . If there exists $K > 0$ such that, for all n :

$$\|A_n\| \leq K,$$

then $(\Sigma_n, p_n)_{n \in \mathbb{N}}$ subconverges in the C^∞ Cheeger/Gromov sense to a complete, immersed submanifold of constant Gaussian curvature.

A - The Maximum Principle.

In this appendix, we briefly prove a version of the maximum principle required in the proof of Theorem 1.4.

Let Ω be an open subset of \mathbb{R}^k . Let $(g_n)_{n \in \mathbb{N}}$ be a sequence of smooth metrics over Ω converging smoothly over Ω to g_0 . For all $n \in \mathbb{N} \cup \{0\}$, let Δ_n be the Laplacian of g_n . Thus, if Γ_n is the Christoffel symbol of the Levi-Civita covariant derivative of g_n , then, for every $f : \Omega \rightarrow \mathbb{R}$:

$$\Delta_n f = (g_n)^{ij} \partial_i \partial_j f - (g_n)^{ij} (\Gamma_n)^k_{ij} \partial_k f.$$

Let $(b_n)_{n \in \mathbb{N}}$ be a sequence of smooth vector fields over Ω converging smoothly over Ω to b_0 .

Lemma A.1

Let c and α be strictly positive real numbers. For all $n \in \mathbb{N}$, let $f_n : \Omega \rightarrow]0, \infty[$ be a smooth, positive valued function such that:

$$\Delta_n f_n^\alpha + \langle b_n, df_n^\alpha \rangle - c f_n^\alpha \leq 0.$$

Suppose that $(f_n)_{n \in \mathbb{N}}$ converges C^∞ to $f_0 : \Omega \rightarrow [0, \infty[$.

If there exists $p \in \Omega$ such that $f_0(p) = 0$, then f_0 is identically zero.

Remark: Care is required since the function f_0^α may not be C^1 at 0.

Proof: Let B be a closed ball about p in Ω . For all n , we define the operator D_n by:

$$D_n f = \Delta_n f + \langle b_n, df \rangle - c f.$$

We consider this operator acting on the space of smooth functions over B which vanish on the boundary. Since $c < 0$, by the maximum principle (see [5]), this operator has trivial kernel on this space. Thus, by classical Fredholm theory, for any continuous function φ on ∂B , there exists a unique solution to the Dirichlet problem given by the operator D_n with boundary values equal to φ . In particular, for all $n \in \mathbb{N} \cup \{0\}$, since $c < 0$, there exists a unique function $u_n : B \rightarrow \mathbb{R}$ such that $\Delta_n u_n = 0$ and:

$$u_n|_{\partial B} = f_n^\alpha|_{\partial B}.$$

Since Δ_n converges C^∞ to Δ_0 and f_n^α converges C^0 to f_0^α , $(u_n)_{n \in \mathbb{N}}$ converges C^∞ to u_0 . For all $n \neq 0$:

$$\Delta_n (f_n^\alpha - u_n) \leq 0.$$

Since $f_n^\alpha - u_n$ is smooth, it follows by the maximum principle that $(f_n^\alpha - u_n)$ cannot have a non-positive minimum. Thus, for all $n \neq 0$:

$$f_n^\alpha \geq u_n.$$

Taking limits:

$$f_0^\alpha \geq u_0.$$

Thus $u_0(p) = 0$. Since $u_0 \geq 0$ along ∂B , it follows that u_0 has a non-positive minimum in the interior of B . Since u_0 is smooth, the strong maximum principle implies that u_0 vanishes identically. Since u_0 coincides with f_0^α along ∂B , and since B is arbitrary, it follows that f_0 vanishes identically in a neighbourhood of p . The result now follows by a standard open/closed argument. \square

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