WEIGHTS ON COHOMOLOGY, INVARIANTS OF SINGULARITIES, AND DUAL COMPLEXES

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Abstract. We study the weight filtration on the cohomology of a proper complex algebraic variety and obtain natural upper bounds on its size, when it is the exceptional divisor of a singularity. We also give bounds for the cohomology of links. The invariants of singularities introduced here gives rather strong information about the topology of rational and related singularities.

Given a divisor on a variety, the combinatorics governing the way the components intersect is encoded by the associated dual complex. This is the simplicial complex with p-simplices corresponding to $(p+1)$-fold intersections of components of the divisor. Kontsevich and Soibelman [KS, A.4] and Stepanov [Stp1] had independently observed that the homotopy type of the dual complex of a simple normal crossing exceptional divisor associated to a resolution of an isolated singularity is an invariant for the singularity. In fact, [KS], and later Thuillier [T] and Payne [P] have obtained homotopy invariance results for more general dual complexes, such as those arising from boundary divisors. In characteristic zero, all these results are consequences of the weak factorization theorem of Włodarczyk [Wlo], and Abramovich-Karu-Matsuki-Włodarczyk [AKMW]; Thuillier uses rather different methods based on Berkovich's non-Archimedean analytic geometry. As we show here, a slight refinement of factorization (theorems 7.6, 7.7) and of these techniques yields some generalizations this statement. This applies to divisors of resolutions of arbitrary not necessarily isolated singularities, and even in a more general context (discussed in the final section). We also allow dual complexes associated with nondivisorial varieties, such as fibres of resolutions of nonisolated singularites. Here is a slightly imprecise formulation of theorems 7.5 and 7.9.

Theorem 0.1. *Suppose that X is a smooth complete variety. Let $E \subset X$ be a divisor with simple normal crossings, or more generally a union of smooth subvarieties which is local analytically a union of intersections of coordinate hyperplanes. Then the homotopy type of the dual complex of E depends only the the complement $X - E$, and in fact only on its proper birational class.*

This theorem and related ones in the final section were inspired by the work of Payne [P, §5], Stepanov [Stp1], and Thuillier [T] mentioned above (and in hindsight also by [KS], although we were unaware of this paper at the time these results were completed).

To put the remaining results in context, we note that the present paper can be considered as the extended version of our earlier preprint [ABW], where we gave

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the bounds on the cohomology of the dual complex of singularities (not necessarily isolated). As we will explain below, these results are established in a more refined form in the present paper. To explain the motivation, we recall the following from the end of [Stp1]:

Question 0.2 (Stepanov)**.** *Is the dual complex associated to the exceptional divisor of a good resolution of an isolated rational singularity contractible?*

This seems to have been motivated by a result of Artin [A], that the exceptional divisor of a resolution of a rational surface singularity consists of a tree of rational curves. However, it appears to be somewhat overoptimistic. Payne [P, ex. 8.3] has found a counterexample, where the dual complex is homeomorphic to $\mathbb{RP}^2$. Nevertheless, a weaker form of this question have a positive answer. Namely that the higher Betti numbers of the dual complex associated to a resolution of an isolated rational vanish. As we have recently learned, this was first observed by Ishii [I, prop 3.2] two decades ago. Here we give a stronger statement and place it in a more general context. The key step is to study this problem from the much more general perspective of Deligne's weight filtration [D]. The point is that for a simple normal crossing divisor D, the weight zero part exactly coincides with the cohomology of the dual complex Σ_D:

Lemma 0.3 (Deligne)**.** $W_0(H^i(D,\mathbb{C})) = H^i(\Sigma_D,\mathbb{C})$

Because of this, W_0 was referred to as the "combinatorial part" in [ABW]. The inclusion in the lemma is induced by the map collapsing the components of D to the vertices of Σ_D (figure 1). This is explained in more detail in section 2.

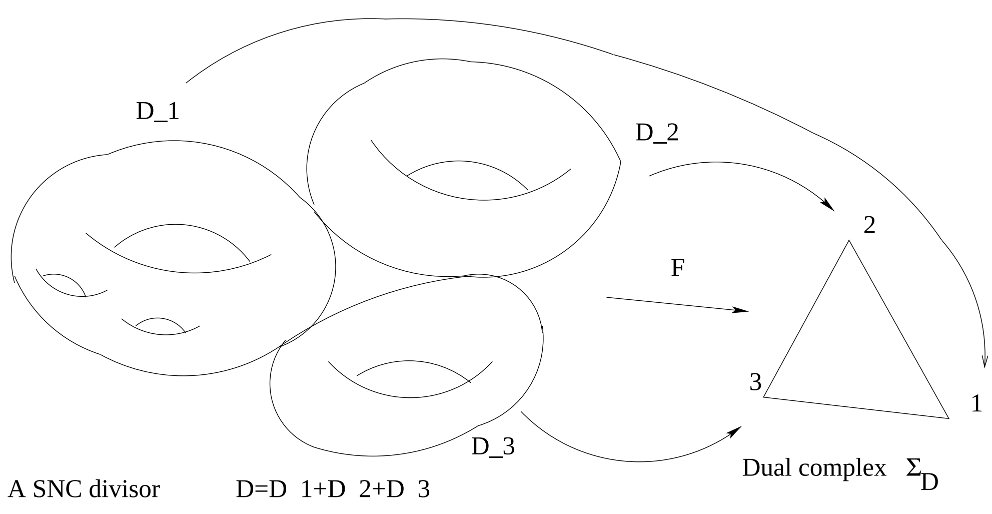

FIGURE 1.

The above result can be applied to the exceptional fibre of any resolution $f : X \to Y$ of any (not necessarily isolated) singularity, leading to the invariant

$$\dim(W_0(H^i(f^{-1}(y),\mathbb{C})) = \dim(H^i(\Sigma_{f^{-1}(y)},\mathbb{C}))$$

More generally, this suggests the study of $w_j^i(y) = \dim(W_j(H^i(f^{-1}(y)))$ and also the weight spaces of the intersection cohomology of the link. This is the principal goal of our paper. In precise terms, given a proper map of varieties $f : X \to Y$, we give a bound on $\dim W_j H^i(f^{-1}(y))$ in terms of the sum of $(i-p)$th cohomology of p-forms along the fibres for $p \leq j$. More concrete results can be extracted for appropriate classes of singularities. Specializing to $w_0^i(y)$, we find that it is bounded above by $\dim(R^i f_* \mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y$. This refines [I, prop 3.2]. The number $w_0^i(y)$ also vanishes for $0 < i < \dim Y - 1$ for a resolution of an isolated normal Cohen-Macaulay singularity (Y, y). This was initially observed in [ABW]. Payne [P] has given this an interesting reinterpretation as saying that the rational homotopy type of dual complex of a Cohen-Macaulay singularity is wedge of spheres. When (Y, y) is toroidal, we show that $w_j^i(y) = 0$ for $j < i/2$ and $i > 0$. We have similar bounds on $\dim W_j IH^i(L)$, where L is the link of y. These are reduced to the previous results by using the decomposition theorem.

The paper is organized as follows. In the first two sections we briefly introduce the elements of the theory of Deligne's weight filtration. In Sections 3 and 4, we discuss the bounds for W_0, and W_j. In Section 5 we introduce and study the invariants of singularities describing the topology of fibres of resolution. In Section 6 we study the invariants related to topology of the links of singularities. In Section 7 we prove theorems on boundary divisors. We note that in this paper the term "scheme", without further qualification, refers to a scheme of finite type over $\mathbb{C}$. A variety is a reduced scheme.

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1. Properties of Deligne's Weight filtration

Although the weight filtration is part of a much more elaborate story – the theory of mixed Hodge structures – it seems useful present the elementary features independently of this. A simple normal crossing divisor provides the key motivating example for the construction of the weight filtration. If X is a union of two components, the cohomology fits into the Mayer-Vietoris sequence. In general, it is computed by a Mayer-Vietoris (or Čech or descent) spectral sequence

$$E_1^{pq} = \bigoplus_{i_0 < \ldots < i_p} H^q(X^{i_0 \ldots i_p}) \Rightarrow H^{p+q}(X)$$

where $X^{i_0 i_1 \cdots}$ are intersections of components. The associated filtration on $H^*(X)$ is precisely the weight filtration $W_0 H^i(X) \subseteq W_1 H^i(X) \subseteq \ldots$. In explicit terms

$$W_j H^i(X) = \operatorname{im} H^i(X, \ldots 0 \to \bigoplus_{m_0 < m_1 \ldots} \pi_* \mathbb{Q}_{X^{m_0 \cdots m_{i-j}}} \to \ldots) \tag{1}$$

via the resolution

$$0 \to \mathbb{Q}_X \to \bigoplus_m \pi_* \mathbb{Q}_{X^m} \to \bigoplus_{m<n} \pi_* \mathbb{Q}_{X^{mn}} \to$$

where $\pi : X^{mn \cdots} \to X$ denote the inclusions.

More generally, we introduce the weight filtration in an axiomatic way. Given an complex algebraic variety or scheme X, Deligne [D] defined the weight filtration

$W_\bullet H^i_c(X)$ on the compactly supported rational cohomology. It is an increasing filtration possessing the following properties:

(W1) These subspaces are preserved by proper pullbacks.

(W2) If X is a divisor with simple normal crossings in a smooth complete variety, then $W_\bullet H^i_c(X) = W_\bullet H^i(X)$ is the filtration associated to the Mayer-Vietoris spectral sequence in (1). In particular, if X is smooth and complete, $W_j H^i(X) = 0$ for $j < i$ and $W_j H^i(X) = H^i(X)$ for $j \geq i$.

(W3) If $U \subseteq X$ is an open immersion and $Z = X - U$, then the standard exact sequence

$$\dots H^{i-1}_c(Z) \to H^i_c(U) \to H^i_c(X) \to \dots$$

restricts to an exact sequence

$$\dots W_j H^{i-1}_c(Z) \to W_j H^i_c(U) \to W_j H^i_c(X) \to \dots$$

(W4) Weights are multiplicative:

$$W_j H^i_c(X \times Y) = \bigoplus_{a+b=j, r+s=i} W_a H^r_c(X) \otimes W_b H^s_c(Y)$$

The goal of the remainder of this section is to prove that the axioms given above uniquely characterize the weight filtration on smooth or projective varieties. In fact, we will not need the last axiom for this. The proof of existence will be reviewed in the next section.

Lemma 1.1. *Suppose that $W_j H^i_c(-)$ are $W'_j H^i_c(-)$ are two families of filtrations satisfying (W1)-(W3). Then $W'_j H^i_c(X) = W_j H^i_c(X)$ for all smooth X.*

Proof. By resolution of singularities [Hir], we may choose a smooth compactification $\bar{X}$ of X so that $E = \bar{X} - X$ is divisor with simple normal crossings. Then from the exact sequence

$$\dots W_j H^{i-1}_c(\bar{X}) \to W_j H^{i-1}_c(E) \to W_j H^i_c(X) \to W_j H^i_c(\bar{X}) \to \dots$$

we deduce that

$$W_j H^i_c(X) = \begin{cases} H^i_c(X) & \text{if } j \geq i \\ \operatorname{coker}[H^{i-1}(\bar{X}) \to H^{i-1}(E)] & \text{if } j = i-1 \\ W_j H^{i-1}(E) & \text{if } j < i-1 \end{cases}$$

The filtration W' would have an identical description. $\square$

Lemma 1.2. *Assume that $W_j H^i_c(X)$ satisfies the axioms (W1)-(W3). Given a variety X with a closed set S and a desingularization $f : \tilde{X} \to X$ which is an isomorphism over $X - S$. Let $E = f^{-1}(S)$. Then there is an exact sequence*

$$\dots \to W_j H^{i-1}_c(E) \to W_j H^i_c(X) \to W_j H^i_c(\tilde{X}) \oplus W_j H^i_c(S) \to \dots$$

Proof. This follows from a diagram chase on

$$\begin{tikzcd} \dots W_j H^{i-1}_c(S) \arrow[r] \arrow[d] & W_j H^i_c(U) \arrow[r] \arrow[d, "="] & W_j H^i_c(X) \arrow[r] \arrow[d] & W_j H^i_c(S) \dots \arrow[d] \\ W_j H^{i-1}_c(E) \arrow[r] & W_j H^i_c(U) \arrow[r] & W_j H^i_c(\tilde{X}) \arrow[r] & W_j H^i(E) \end{tikzcd}$$

where $U = X - S$. $\square$

Proposition 1.3. *Suppose that $W_jH_c^i(-)$ are $W'_jH_c^i(-)$ are two families of filtrations satisfying (W1)-(W3). Then $W'_jH_c^i(X) = W_jH_c^i(X)$ for all projective X.*

Proof. The proof is inspired by the work of El Zein [E]. Choose an embedding $X \subset P = \mathbb{P}^n$. Let $\pi : \tilde{P} \to P$ be an embedded resolution of singularities such that $E = \pi^{-1}(X)$ is a divisor with simple normal crossings. By lemma 1.2, we have an exact sequence

$$\ldots \to W_jH^i(P) \to W_jH^i(\tilde{P}) \oplus W_jH_c^i(X) \to W_jH^i(E) \to W_jH^{i+1}(P) \ldots$$

This implies that

$$W_jH_c^i(X) = \begin{cases} H_c^i(X) & \text{if } j \geq i \\ W_jH^i(E) & \text{if } j < i \end{cases}$$

and this determines W_j uniquely. □

As an illustration of this method, we can recover an elementary description of the weight filtration on smooth or projective toric varieties due to Weber [W]. Fix an integer $p > 1$ (not necessarily prime). The map given by multiplication by p on tori extends to a "Frobenius-like" endomorphism $\phi_p : X \to X$ for any toric variety X [loc. cit.] When X is smooth and projective, ϕ_p acts on $H^i(X)$ by $p^{i/2}$. In general, we define $W_jH_c^i(X)$ to be the sum of eigenspaces of ϕ_p^* with eigenvalue p^k with $k \leq j/2$. This filtration can be checked to satisfy (W1)-(W3) restricted to the category of toric varieties.

Proposition 1.4 (Weber)**.** *If X is a toric variety, then W coincides with Deligne's weight filtration.*

When X is smooth or projective, this can be proved exactly as in lemma 1.1 and proposition 1.3 with the additional (realizable) constraint that the intermediate varieties and maps $X \subset \bar{X}$, $X \subset P$ and $\pi : \tilde{P} \to P$ are constructed within the category of toric varieties.

2. Simplicial resolutions

The general construction is based on simplicial resolutions. Perhaps, a motivating example is in order. Given a divisor X with simple normal crossings, the underlying combinatorics of how the components fit together is determined by the dual complex Σ_X. This is the simplicial complex having one vertex for each connected component; a simplex lies in Σ_X if and only the corresponding components meet. The cohomology of the dual complex is precisely $W_0H^i(X)$. In order to describe the rest of weight filtration in this fashion, we need the full simplicial resolution. Before describing it, we recall some standard material [D, GNPP, PS]. A simplicial object in a category is a diagram

$$\ldots X_2 \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} X_1 \begin{smallmatrix}\to\\\to\end{smallmatrix} X_0$$

with n face maps $\delta_i : X_n \to X_{n-1}$ satisfying the standard relation $\delta_i\delta_j = \delta_{j-1}\delta_i$ for $i < j$; this would be more accurately called a "strict simplicial" or "semisimplicial" object since we do not insist on degeneracy maps going backwards. The basic example of a simplicial set, i.e. simplicial object in the category of sets, is given by taking X_n to be the set of n-simplices of a simplicial complex on an ordered set of vertices. Let Δ^n be the standard n-simplex with faces $\delta'_i : \Delta^{n-1} \to \Delta^n$. Given a simplicial set or more generally a simplicial topological space, we can glue the $X_n \times \Delta^n$ together by identifying $(\delta_i x, y) \sim (x, \delta_i' y)$. This leads to a topological space $|X_\bullet|$ called the geometric realization, which generalizes the usual construction of the topological space associated a simplicial complex. Here is a basic example.

Example 2.1. *Suppose that X is a topological space given as union of open or closed subsets X^i. Let $X^{ij\cdots} = X^i \cap X^j \ldots$ denote the intersections. Then a simplicial space is given by taking X_n to be the disjoint union of the $(i+1)$-fold intersections $X^{i_0\cdots i_n}$. The face map δ_k is given by inclusions*

$$X^{i_0\cdots i_n} \subset X^{i_0\cdots \hat{i}_k \cdots i_n}\ (i_1 < \ldots < i_k)$$

When X is triangulable, and X^i are subcomplexes, then $|X_\bullet|$ and X are homotopy equivalent.

Example 2.2. *The above construction and comment applies to the case of an analytic space X with irreducible components X^i. Of particular interest is the case where X is a divisor with simple normal crossings; in this case $X_\bullet$ is referred to as the canonical simplicial resolution of X. Applying the connected component functor results in a simplicial complex $\pi_0(X_\bullet)$ which none other than the dual complex Σ_X.*

The notion of a simplicial resolution can be extended to arbitrary varieties as follows:

Theorem 2.3 (Deligne)**.** *Given any (possibly reducible) variety X, there exists a smooth simplicial variety $X_\bullet$, which we call a* simplicial resolution*, with a collection of proper morphisms $\pi_\bullet : X_\bullet \to X$ (commuting with face maps), called an augmentation, inducing a homotopy equivalence between $|X_\bullet|$ and X. Given a morphism $f : X \to Y$ there exists simplicial resolutions $X_\bullet, Y_\bullet$ and a morphism $f_\bullet : X_\bullet \to Y_\bullet$ compatible with f.*

The theorem is a consequence of resolution of singularities. Proofs can be found in [D, GNPP, PS]. Note that the original construction of Deligne results in a necessarily infinite diagram, whereas the method of Guillen, Navarro Aznar et. al yields a fairly economical resolution. The canonical simplicial resolution of normal crossing divisor is a simplicial resolution in this technical sense.

Example 2.4. *Following the method of* [GNPP] *we can construct a simplicial resolution of a variety X with isolated singularities as follows. Let $f : Y \to X$ be a resolution of singularities such that the exceptional divisor $E = \cup E^i$ is a divisor with simple normal crossings. Write $E^{ij} = E^i \cap E^j$ and $E_n = \coprod E^{i_0\cdots i_n}$. Let $S_0 \subset X$ be the set of singular points, $S_1 \subseteq S_0$ be the set of images of $\cup E^{ij}$ and so on. Then the simplicial resolution is given by*

$$\ldots E_1 \sqcup S_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} E_0 \sqcup S_1 \substack{\longrightarrow\\ \longrightarrow} Y \sqcup S_0$$

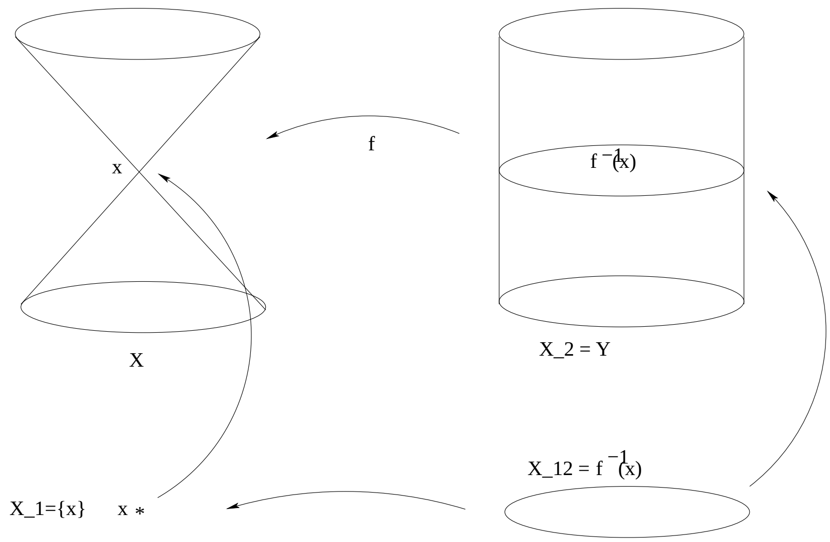
where the face maps are given by inclusions $S_i \to S_{i-1}$ on the second component. On the first component δ_k is given by

$$\begin{cases} E^{i_1\ldots i_n} \subset E^{i_i\ldots\hat{i}_k\ldots i_n} & \text{if } k \le n \\ f : E^{i_1\ldots i_n} \to S_{n-1} & \text{if } k = n+1 \end{cases}$$

Figure 2 depicts the simplest example of this, which is the simplicial resolution

$$X_1 = E \substack{\longrightarrow\\ \longrightarrow} X_0 = Y \sqcup x$$

with its augmentation to X, for an isolated singularity x resolved by a single blow up. Figure 3 depicts the geometric realization of this simplicial resolution, which is the cone over E in Y. This is homotopic to X.



Simplicial object {X_ij} associated to a single blow−up at x in X.

FIGURE 2.

Given a simplicial space, filtering $|X_\bullet|$ by skeleta $\bigcup_{n\leq N} X_n \times \Delta^n/\sim$ yields the spectral sequence

$$E_1^{pq} = H^q(X_p, A) \Rightarrow H^{p+q}(|X_\bullet|, A) \tag{2}$$

for any abelian group A. Part of the datum of this spectral sequence is the filtration on $H^*(|X_\bullet|, A)$ induced by skeleta. When applied to the canonical simplicial resolution of a divisor X with simple normal crossings, we recover the Mayer-Vietoris spectral sequence given earlier.

It is convenient to extend this. A simplicial sheaf on $X_\bullet$ is a collection of sheaves $\mathcal{F}_n$ on X_n with "coface" maps $\delta_i^{-1}\mathcal{F}_{n-1} \to \mathcal{F}_n$ satisfying the face relations.

Example 2.5. *The constant sheaves $\mathbb{Z}_{X_\bullet}$ with identities for coface maps forms a simplicial sheaf.*

Example 2.6. *Suppose that $\pi_\bullet : X_\bullet \to X$ is a morphism of spaces commuting with face maps. Then the pullback of any sheaf $\mathcal{F}_\bullet = \pi^{-1}\mathcal{F}$ is naturally a simplicial sheaf. The previous example is of this form, but not all are.*

Example 2.7. *If $X_\bullet$ is a simplicial object in the category of complex manifolds, then $\Omega^i_{X_\bullet}$ with the obvious maps, forms a simplicial sheaf. This example is not of the previous form.*

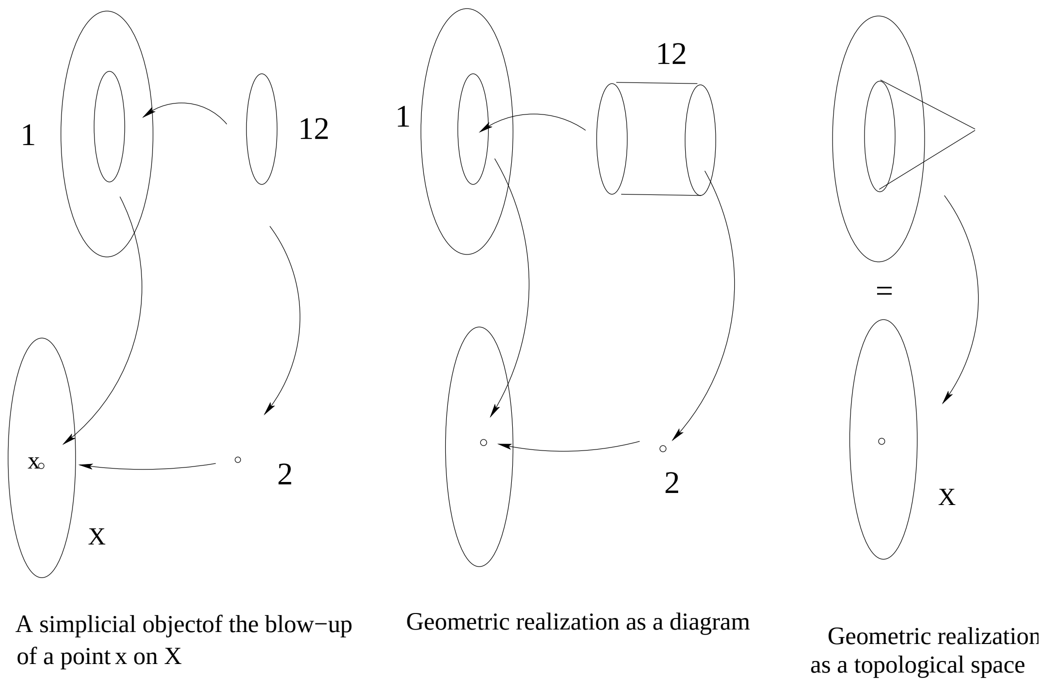


FIGURE 3.

We can define cohomology by setting

$$H^i(X_\bullet, \mathcal{F}_\bullet) = Ext^i(\mathbb{Z}_{X_\bullet}, \mathcal{F}_\bullet)$$

This generalizes sheaf cohomology in the usual sense, and it can be extended to the case where $\mathcal{F}_\bullet^\bullet$ is a bounded below complex of simplicial sheaves by using a hyper Ext. When $\mathcal{F} = A$ is constant, this coincides with $H^i(|X_\bullet|, A)$. But in general the meaning is more elusive. There is a spectral sequence

$$E_1^{pq}(\mathcal{F}_\bullet^\bullet) = H^q(X_p, \mathcal{F}_p^\bullet) \Rightarrow H^{p+q}(X_\bullet, \mathcal{F}_\bullet^\bullet) \tag{3}$$

generalizing (2). To be explicit, the differentials are given by the alternating sum of the compositions

$$H^q(X_p, \mathcal{F}_p^\bullet) \to H^q(X_{p+1}, \delta_i^{-1}\mathcal{F}_p^\bullet) \to H^q(X_{p+1}, \mathcal{F}_{p+1}^\bullet)$$

Filtering $\mathcal{F}^\bullet$ by the "stupid filtration" $\mathcal{F}_\bullet^{\geq n}$ yields a different spectral sequence

$$'E_1^{pq} = H^q(X_\bullet, \mathcal{F}_\bullet^p) \Rightarrow H^{p+q}(X_\bullet, \mathcal{F}_\bullet^\bullet) \tag{4}$$

Theorem 2.8 (Deligne)**.** *If $X_\bullet$ is a simplicial object in the category of compact Kähler manifolds and holomorphic maps. The spectral sequence* (2) *degenerates at E_2 when $A = \mathbb{Q}$.*

Remark 2.9. *The theorem follows from a more general result in* [D, 8.1.9]*. However the argument is very complicated. Fortunately, as pointed out in* [DGMS]*, this special case follows easily from the $\partial\bar{\partial}$-lemma. Here we give a more complete argument.*

Proof. It is enough to prove this after tensoring with $\mathbb{C}$. We can realize the spectral sequence as coming from the double complex $(E^\bullet(X_\bullet), d, \pm\delta)$, where $(E^\bullet, d)$ is the C^∞ de Rham complex, and δ is the combinatorial differential. (We are mostly going to ignore sign issues since they are not relevant here.) In fact this is a triple complex, since each $E^\bullet(-)$ is the total complex of the double complex $(E^{\bullet\bullet}(-), \partial, \bar\partial)$.

Given a class $[\alpha] \in H^i(X_j)$ lying in the kernel of δ, we have $\delta\alpha = d\beta$ for some $\beta \in E^{i-1}(X_{j+1})$ Then $d_2([\alpha])$ is represented by $\delta\beta \in E^{i-1}(X_{j+2})$. We will show this vanishes in cohomology. The ambiguity in the choice of β will turn out to be the key point.

By the Hodge decomposition, we can assume that α is pure of type (p, q). Therefore $\delta\alpha$ is also pure of this type. We can now apply the $\partial\bar\partial$-lemma [GH, p 149] to write $\alpha = \partial\bar\partial\gamma$ where $\gamma \in E^{p-1,q-1}(X_{j+1})$. This means we have two choices for β. Taking $\beta = \bar\partial\gamma$ shows that $d_2([\alpha])$ is represented by a form of pure type $(p-1, q)$. On the other hand, taking $\beta = -\partial\gamma$ shows that this class is of type $(p, q-1)$. Thus $d_2([\alpha]) \in H^{p-1,q} \cap H^{p,q-1} = 0$.

By what we just proved $\delta\alpha = d\beta, \delta\beta = d\eta$, and $\delta\eta$ represents $d_3([\alpha])$. It should be clear that one can kill this and higher differentials in the exact same way.

□

Corollary 2.10. *With the same assumptions as the theorem, the spectral sequence* (3) *degenerates at* E_2 *when* $\mathcal{F} = \mathcal{O}_{X_\bullet}$.

Proof. By the Hodge theorem, the spectral sequence for $\mathcal{F} = \mathcal{O}_{X_\bullet}$ is a direct summand of the spectral sequence for $\mathcal{F} = \mathbb{C}$. □

We are, at last, in a position to explain the *construction of the weight filtration* for a complete variety X. Choose a simplicial resolution $X_\bullet \to X$ as above. Then the spectral sequence (2) will then converge to $H^*(X, \mathbb{Q})$ when $A = \mathbb{Q}$. The weight filtration W is the induced increasing filtration on $H^*(X)$ indexed so that

$$W_q H^{p+q}(X)/W_{q-1} = E_\infty^{pq}$$

Although $X_\bullet$ is far from unique, Deligne [D] shows that W is well defined, and moreover that this part of the datum of the canonical mixed structure. By theorem 2.8, we obtain

Lemma 2.11. *We have* $W_{-1} = 0$ *and*

$$W_j H^i(X, \mathbb{Q})/W_{j-1} \cong H^{i-j}(\ldots \to H^j(X_p, \mathbb{Q}) \to H^j(X_{p+1}, \mathbb{Q}) \ldots)$$

Corollary 2.12. $W_j H^i(X) = H^i(X)$ *if* $j \geq i$ *and* $W_j H^i(X) = 0$ *if* $i - j > \dim X$.

Proof. The first part is an immediate consequence of the lemma. For the second, we observe that the work of Guillen, Navarro Aznar et. al. [GNPP],[PS, thm 5.26], shows that a simplicial resolution $X_\bullet$ can be chosen with length at most $\dim X$. So that $H^{i-j}(H^j(X_\bullet))$ is necessarily 0 for $i - j > \dim X$. □

Applying π_0 to $X_\bullet$ results in a simplicial set. We have a canonical map $X_\bullet \to \pi_0(X_\bullet)$ of simplicial spaces which induces a continuous map of

$$X \sim |X_\bullet| \to |\pi_0(X_\bullet)|$$

which is well defined up to homotopy.

Corollary 2.13. *The map on cohomology is injective and*

$$W_0 H^i(X, \mathbb{Q}) = \operatorname{im} H^i(|\pi_0(X_\bullet)|, \mathbb{Q})$$

Remark 2.14. *When X is a divisor with simple normal crossings, this says that $W_0H^i(X)$ is the cohomology of the dual complex. In this case the inclusion*

$$H^i(|\Sigma_X|, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$

can be constructed more directly. It is induced by a simple, but less canonical simplicial map $\phi : X \to |\Sigma_X|$ described in Stepanov [Stp2, lemma 3.2]. *Take the triangulation of X such that components X_i and their intersections*

$$X_{i_1,i_2,\ldots,i_k} := X_{i_1} \cap X_{i_2} \cap \ldots \cap X_{i_k}$$

are simpilicial subcomplexes. Denote by $\Delta_{i_1,i_2,\ldots,i_k}$ the simplex in the dual complex Σ_X which corresponds to $X_{i_1,i_2,\ldots,i_k}$. Then we make a barycentric subdivisions $\overline{\Sigma}$ of Σ and $\overline{\Sigma}_X$ of Σ_X. For any vertex v of $\overline{\Sigma}$ which lies in the minimal component $X_{i_1,i_2,\ldots,i_k}$ we put

$$\phi(v) := \text{the center of the simplex } \Delta_{i_1,i_2,\ldots,i_k}$$

See figure 4. This construction is homotopically equivalent to the construction in Corollary 2.13. In particular, it has connected fibres.

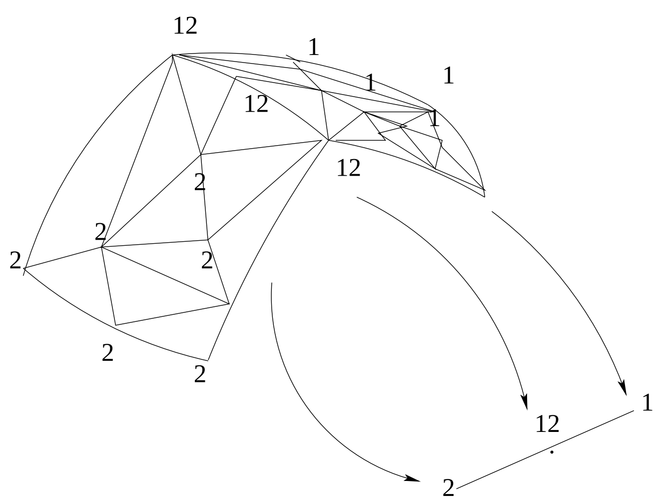


FIGURE 4.

We can now describe the construction of the weight filtration for an arbitrary variety U. Choose a compactification X. Denote the complement by $\iota : Z \subset X$. There exists simplicial resolutions $Z_\bullet \to Z$, $X_\bullet \to X$ and a morphism $\iota_\bullet : Z_\bullet \to X_\bullet$ covering ι. Then there is a new smooth simplicial variety *cone*$(\iota_\bullet)$ ([D, §6.3], [GNPP, IV §1.7]) whose geometric realization is homotopy equivalent to X/Z. We have a spectral sequence

$$E_1^{pq} = H_c^q(X_p - Z_p, \mathbb{Q}) \Rightarrow H_c^*(X - Z, \mathbb{Q}).$$

The weight filtration W is defined via this spectral sequence as above. Deligne [D] shows that conditions (W1), (W2), (W3) and (W4) are satisfied.

For any variety X, we can construct a simplicial variety $X_\bullet$ with $|X_\bullet|$ homotopic to X as in example 2.2. It is not a simplicial resolution in general, but it is dominated by one. If we apply π_0 to this simplicial variety, we get a simplicial set Σ_X canonically attached to X, that we will call the nerve or dual complex. There is a canonical map $H^i(|\Sigma_X|, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ coming from the spectral sequence (2) associated to this simplicial variety. From the above discussion, we can see that:

Lemma 2.15. *If X is complete, the image $H^i(|\Sigma_X|, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ lies in $W_0 H^i(X, \mathbb{Q})$. If X satisfies the assumptions of example 2.2, then these subspaces coincide.*

Lemma 2.16. *Let $\pi : \tilde{X} \to X$ be a resolution of a complete variety such that the exceptional divisor E has normal crossings. Let $S = \pi(E) \subset X$. Then $\dim W_0 H^i(X)$ is the $(i-1)$st Betti number b_{i-1} of the dual complex of E when $i > 2\dim(S)+1$. If S is nonsingular, then this holds for $i > 1$. When $i = 2\dim(S)+1$, $\dim W_0 H^i(X) = b_{i-1}$ minus the number of irreducible components of S of maximum dimension.*

Proof. This follows from lemma 1.2, and the above remarks. □

3. Bounds on W_0 of a fibre.

Suppose that X is a complete variety i.e., proper reduced scheme. Then in addition to the weight filtration, $H^i(X, \mathbb{C})$ carries a second filtration, called the Hodge filtration induced on the abutment $H^i(X, \Omega^\bullet_{X_\bullet}) \cong H^i(X, \mathbb{C})$ of the spectral sequence (4) for $\Omega^\bullet_{X_\bullet}$ for a simplicial resolution $\pi_\bullet : X_\bullet \to X$. By convention F is decreasing. We have $F^0 = H^i(X, \mathbb{C})$ and

$$F^0 H^i(X, \mathbb{C})/F^1 \cong H^i(X_\bullet, \mathcal{O}_{X_\bullet})$$

The filtration W induces the same filtration on $H^i(X_\bullet, \mathcal{O}_{X_\bullet})$ as the one coming from (3). In particular,

$$\begin{aligned} W_0 Gr_F^0 H^i(X, \mathbb{C}) &= H^i(\ldots \to H^0(X_p, \mathcal{O}) \to H^0(X_{p+1}, \mathcal{O}) \ldots) \\ &= H^i(\ldots \to H^0(X_p, \mathbb{C}) \to H^0(X_{p+1}, \mathbb{C}) \ldots) \\ &\cong W_0 H^i(X, \mathbb{C}) \end{aligned} \tag{5}$$

This means that Hodge filtrations becomes trivial on $W_0 H^i(X)$. So that this is a vector space and nothing more.

This leads to one of the main theorems of this paper.

Theorem 3.1.

(a) *Suppose that X is a proper (not necessarily reduced) scheme, then there is a canonical inclusion*

$$W_0 H^i(X, \mathbb{C}) \hookrightarrow H^i(X, \mathcal{O}_X),$$

which is a restriction of the natural map $\kappa : H^i(X, \mathbb{C}) \to H^i(X, \mathcal{O}_X)$ induced by the morphism of sheaves $\mathbb{C}_X \to \mathcal{O}_X$.

(b) *Suppose that $f : X \to Y$ a proper morphism of varieties. Then there is an inclusion $W_0 H^i(f^{-1}(y), \mathbb{C}) \hookrightarrow (R^i f_* \mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y$ for each $y \in Y$.*

Proof. By (5),

$$W_0 H^i(X, \mathbb{C}) = W_0 Gr_F^0 H^i(X, \mathbb{C})$$

thus $W_0H^i(X,\mathbb{C})$ injects into $Gr_F^0 H(X,\mathbb{C})$ under the canonical map $H^i(X,\mathbb{C}) \to Gr_F^0 H^i(X,\mathbb{C})$. Since there is a factorization

$$\begin{tikzcd} H^i(X,\mathbb{C}) \arrow[rr, "\kappa"] \arrow[dr] & & H^i(X,\mathcal{O}_X) \arrow[dl] \\ & Gr_F^0 H^i(X,\mathbb{C}) & \end{tikzcd}$$

the restriction of κ to $W_0H^i(X,\mathbb{C})$ is also necessarily injective. To be clear, we are factoring this as

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X) \to H^i(X_{red},\mathcal{O}_{X_{red}}) \to Gr_F^0 H^i(X,\mathbb{C})$$

For (b), let X_y be the reduced fibre over y, and $X_y^{(n)}$ the fibre with its nth infinitesimal structure. From (a), we have a natural inclusion $s : W_0H^i(X_y,\mathbb{C}) \hookrightarrow H^i(X_y,\mathcal{O}_{X_y})$. After choosing a simplicial resolution of the fibre $f_\bullet : \mathcal{X}_\bullet \to X_y$, s can be identified with the composition

$$E_2^{i0}(\mathbb{C}) \to E_2^{i0}(\mathcal{O}_{\mathcal{X}_\bullet}) \to H^i(X_y,\mathcal{O}_{X_y})$$

where the first map is induced by the natural map $\mathbb{C} \to \mathcal{O}$, and the last map is the edge homomorphism. Applying the same construction to the simplicial sheaf $f_\bullet^*\mathcal{O}_{X_y^{(n)}}$ yields a map s_n fitting into a commutative diagram

$$\begin{tikzcd} W_0H^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[dr, "s_n"] & & H^i(X_y,\mathcal{O}_{X_y}) \\ & H^i(X_y,\mathcal{O}_{X_y^{(n)}}) \arrow[ur] & \end{tikzcd}$$

Furthermore, these maps are compatible, thus they assemble into a map s_∞ to the limit. Together with the formal functions theorem [H, III 11.1], this yields a commutative diagram

$$\begin{tikzcd} W_0H^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[d, "s_\infty"] \arrow[drr, "s'"] & & H^i(X_y,\mathcal{O}_{X_y}) \\ \varprojlim H^i(X_y,\mathcal{O}_{X_y^{(n)}}) \arrow[r, "\sim"] & (R^if_*\mathcal{O}_X)_y\hat{} \arrow[r] & (R^if_*\mathcal{O}_X)_y \otimes \mathcal{O}_y/m_y \arrow[u] \end{tikzcd}$$

Since s is injective, the map labeled s' is injective as well. $\square$

Corollary 3.2. *Suppose that $f : X \to Y$ is a resolution of singularities.*

1. *If Y has rational singularities then $W_0H^i(f^{-1}(y),\mathbb{C}) = 0$ for $i > 0$.*
2. *If Y has isolated normal Cohen-Macaulay singularities, $W_0H^i(f^{-1}(y),\mathbb{C}) = 0$ for $0 < i < \dim Y - 1$*

Proof. There is no loss in assuming that the exceptional divisor has normal crossings. Then the first statement is an immediate consequence of the theorem. The second follows from the well known fact given below. We sketch the proof for lack of a suitable reference.

Proposition 3.3. *If $f : X \to Y$ is a resolution of a variety with isolated normal Cohen-Macaulay singularities, then $R^if_*\mathcal{O}_X = 0$ for $0 < i < \dim Y - 1$*

Sketch. We can assume that Y is projective. By the Kawamata-Viehweg vanishing theorem [Ka, V]

$$H^i(X, f^*L^{-1}) = 0, \quad i < \dim Y = n, \tag{6}$$

where L is ample. Replace L by L^N, with $N \gg 0$. Then by Serre vanishing and Serre duality (we use the CM hypothesis here)

$$H^i(Y, L^{-1}) = H^{n-i}(Y, \omega_Y \otimes L) = 0, \quad i < n. \tag{7}$$

The Leray spectral sequence together with (6) and (7) imply

$$H^0(R^i f_* \mathcal{O}_X \otimes L^{-1}) = 0, \quad i < n-1$$

Since the sheaves $R^i f_* \mathcal{O}_X$ have zero dimensional support, the proposition follows. □

□

Corollary 3.2 refines [I, prop 3.2]. Concerning the first conjecture, we have the following partial result in the rational homotopy category.

Corollary 3.4. *The rational homotopy type of the dual complex associated to a resolution of an isolated rational hypersurface singularity of dimension* ≥ 3 *is trivial.*

Proof. This follows from the corollary 3.2 and Stepanov's result that the dual complex associated to a resolution of an isolated hypersurface singularity of dimension ≥ 3 is simply connected [Stp2]. □

4. Bounds on higher weights of a fibre.

Theorem 3.1 can be refined to get bounds on W_j using ideas of du Bois [Du, PS] which we recall below. Given a simplicial resolution $\pi_\bullet : X_\bullet \to X$, we can construct the derived direct image

$$\tilde{\Omega}_X^p = \mathbb{R}\pi_{\bullet *}\Omega_{X_\bullet}^p$$

In more explicit terms, this can realized by the total complex of

$$\pi_{0*}G^\bullet(\Omega_{X_0}^p) \to \pi_{1*}G^\bullet(\Omega_{X_1}^p) \to \dots$$

where $G^\bullet$ is Godement's flasque resolution. These fit together into a bigger complex $\tilde{\Omega}_X^\bullet$ filtered by p

$$F^p \left\{ \begin{array}{ccccc} \pi_{0*}G^\bullet(\Omega_{X_0}^p) & \to & \pi_{1*}G^\bullet(\Omega_{X_0}^p) & \to \\ \downarrow & & \downarrow & \\ \pi_{0*}G^\bullet(\Omega_{X_0}^{p+1}) & \to & \pi_{1*}G^\bullet(\Omega_{X_0}^{p+1}) & \to \end{array} \right.$$

More precisely,

$$(\tilde{\Omega}_X^\bullet, F^p) = \mathbb{R}\pi_{\bullet *}(\Omega_{X_\bullet}^\bullet, \Omega_{X_\bullet}^{\geq p})$$

in the filtered derived category.

There is a natural map $\Omega_X^p \to \tilde{\Omega}_X^p$ from the pth exterior power of the sheaf of Kähler differentials. This is not a quasi-isomorphism in general. We summarize the basic properties:

1. As objects in the (filtered) derived category $(\tilde{\Omega}_X^\bullet, F)$ and $\tilde{\Omega}_X^p$ are independent of the simplicial resolution.
2. There is a map $(\Omega_X^\bullet, \Omega_X^{\geq \bullet}) \to (\tilde{\Omega}_X^\bullet, F)$, from the complex of Kähler differentials, such that composing with $\mathbb{C} \to \Omega_X^\bullet$ yields a quasi-isomorphism $\mathbb{C}_X \cong \tilde{\Omega}_X^\bullet$.

(3) The filtration F on $\tilde{\Omega}_X^\bullet$ induces the Hodge filtration on cohomology, and the associated spectral sequence degenerates at E_1 when X is proper.

From these statements, we extract

$$F^pH^i(X,\mathbb{C})/F^{p+1} \cong \mathbb{H}^i(X,\tilde{\Omega}_X^p)$$

when X is proper. If $X \subseteq Z$ is a closed immersion, then we get a map

$$\Omega_Z^\bullet|_X \to \tilde{\Omega}_X^\bullet$$

by composing the map in (2) with restriction to $\Omega_X^\bullet$.

Theorem 4.1.

(a) *Suppose that $X \subseteq Z$ is a closed immersion of proper scheme into another scheme, then there is a canonical inclusion*

$$H^i(X)/F^{j+1} \hookrightarrow \mathbb{H}^i(X,\Omega_Z^{\leq j}|_X)$$

(b) *Suppose that $X \subset Z$ is as in (a), then there is a canonical inclusion*

$$\dim W_jH^i(X,\mathbb{C}) \hookrightarrow \mathbb{H}^i(X,\Omega_Z^{\leq j}|_X)$$

(c) *Suppose that $f : X \to Y$ a proper morphism of varieties. Then there is an inclusion $W_jH^i(f^{-1}(y),\mathbb{C}) \hookrightarrow (\mathbb{R}^if_*\Omega_X^{\leq j})_y \otimes \mathcal{O}_y/m_y$ for each $y \in Y$*

Proof. By the remarks preceding the theorem, we have an isomorphism

$$H^i(X)/F^{j+1} \cong \mathbb{H}^i(X,\tilde{\Omega}_X^{\leq j})$$

For (a) it suffices to observe that this factors through $\mathbb{H}^i(X,\Omega_Z^{\leq j}|_X)$. So the corresponding map is injective.

By lemma 2.11 ,

$$Gr_j^WH^i(X,\mathbb{C}) = H^{i-j}(\ldots \to H^j(X_k,\mathbb{C}) \to H^j(X_{k+1},\mathbb{C})\ldots)$$

Therefore $F^{j+1} \cap W_jH^i(X) = 0$. So that the natural map

$$W_jH^i(X,\mathbb{C}) \to H^i(X)/F^{j+1}$$

is injective. Composing this with the map in (a) yields an injection

$$s : W_jH^i(X,\mathbb{C}) \to \mathbb{H}^i(X,\Omega_Z^{\leq j}|_X)$$

This proves (b).

The argument for (c) is basically a reprise of the proof of theorem 3.1 (b). Let X_y be the reduced fibre over y, and $X_y^{(n)}$ the fibre with its nth infinitesimal structure.

$$\begin{tikzcd}
W_jH^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[dr, "s_n"'] & & \mathbb{H}^i(X_y,\Omega_X^{\leq j}|_{X_y}) \\
& \mathbb{H}^i(X_y,\Omega_X^{\leq j})|_{X_y^{(n)}} \arrow[ur] &
\end{tikzcd}$$

Furthermore, these maps are compatible, thus they pass to map s_∞ to the limit. Then by the formal functions theorem, this yields a commutative diagram

$$\begin{tikzcd}
W_jH^i(X,\mathbb{C}) \arrow[rr, "s"] \arrow[d, "s_\infty"] \arrow[drr, "s'"] & & \mathbb{H}^i(X_y, \Omega^{\le j}_{X_y}) \\
\varprojlim \mathbb{H}^i(X_y, \Omega^{\le j}_{X_y^{(n)}}) \arrow[r, "\sim"] & (\mathbb{R}^i f_*\Omega^{\le j}_X)_y\hat{} \arrow[r] & (\mathbb{R}^i f_*\Omega^{\le j}_X)_y \otimes \mathcal{O}_y/m_y \arrow[u]
\end{tikzcd}$$

Since s is injective, the map labeled s' is injective as well. □

Corollary 4.2.

(1) *With the same assumptions as in (a), we have*

$$\dim W_jH^i(X) \le \sum_{p\le j} \dim H^{i-p}(X, \Omega^p_Z|_X)$$

(2) *With the same assumptions as in (c), we have*

$$\dim W_jH^i(f^{-1}(y)) \le \sum_{p\le j} \dim(R^{i-p}f_*\Omega^p_X)_y \otimes \mathcal{O}_y/m_y$$

In special cases, we can use this to get rather precise bounds. We need the following relative version of Danilov's theorem [Da, 7.6].

Proposition 4.3. *Suppose that $f : X \to Y$ is a projective toric between toric varieties, with X smooth. Then $R^qf_*\Omega^p_X = 0$ for $q > p$.*

Proof. The argument is similar to the proof of proposition 3.3. We can first assume that Y is affine and then replace it with a projective toric compactification. Let L be an ample line bundle on Y. After replacing L by L^N, with $N \gg 0$, Serre's vanishing implies that we can assume that $R^qf_*\Omega^p_X \otimes L$ is globally generated and that the Leray spectral sequence collapses to an isomorphism

$$H^q(X, \Omega^p_X \otimes f^*L) = H^0(R^qf_*\Omega^p_X \otimes L)$$

The left side vanishes by a theorem of Mavlyutov [M], and so the proposition follows. □

Proposition 4.4. *If $f : X \to Y$ is a projective toric morphism between toric varieties, then $W_jH^i(f^{-1}(y)) = 0$ when $j < i/2$.*

Proof. We have

$$\dim W_jH^i(f^{-1}(y)) \le \sum_{p\le j} \dim(R^{i-p}f_*\Omega^p_X)_y \otimes \mathcal{O}_y/m_y = 0$$

when $j < i/2$ by proposition 4.3. □

5. Invariants of singularities

The weight filtration can be applied to the study of the topology of the fibres of a resolution of singularities and their links.

Proposition 5.1. *Let Y be (the germ of) a variety with an isolated singularity $y \in Y$. Let $f : X \to Y$ be its resolution. We can assign to y the following invariants:*

$$w^i_j(y) := \dim W_j H^i(f^{-1}(y))$$

These are independent of the choice of resolution X, where $j < i$.

Proof. We can assume that Y is projective and that $U := Y \setminus \{y\}$ is nonsingular by resolving all points away from $y \in Y$, if necessary. This will not affect the exceptional fibre $f^{-1}(y)$ of the resolution. Since X is nonsingular we get $W_j(H^i(X)) = W_j(H^{i+1}(X)) = 0$ for $j < i$. It follows from the long exact sequence

$$\ldots W_j H^i_c(U) \to W_j H^i(X) \to W_j H^i(f^{-1}(y)) \to W_j H^{i+1}_c(U) \to W_j H^{i+i}(X) \to \ldots$$

that $W_j(H^i(f^{-1}(y)) \simeq W_j(H^{i+1}_c(U))$ is an isomorphism. The latter is independent of the fibre of the resolution $f : X \to Y$.

The proposition can also be deduced from [P, 7.1]. □

Proposition 5.2. *Let Y be a singular variety and $f : X \to Y$ be a resolution. We can assign to $y \in Y$ the following invariants,*

$$w^i_0(y) := \dim W_0 H^i(f^{-1}(y))$$
$$h^1(y) := \dim H^1(f^{-1}(y))$$

independently of the choice of resolution X.

Remark 5.3. *These invariants generalize the cohomology of the dual graph of the isolated singularities. If $X \to Y$ is a resolution of an isolated singularity $y \in Y$, such that $D = f^{-1}(y)$ is a SNC divisor then the homotopy type of Σ_D is independent of the resolution and, in particular, $w^i_0(y) = h^i(\Sigma_D)$ are also independent of the resolution in this situation. Thus propositions 5.1, and 5.2 generalize the observation for dual complexes to a more general situation.*

We first prove the following lemmas:

Lemma 5.4.

(1) *If X is projective nonsingular and irreducible, Y is an arbitrary variety and $\pi : X \times Y \to X$ is the projection then the natural homomorphism*
$$\pi^* : W_0(H^i(Y)) \to W_0(H^i(Y \times X))$$
is an isomorphism.

(2) *If X is projective nonsingular and irreducible with Betti number $b_1(X) = 0$, Y is an arbitrary variety and $\pi : X \times Y \to X$ is the projection then the natural homomorphism*
$$\pi^* : H^1(Y) \to H^1(Y \times X)$$
is an isomorphism.

Proof. (1) Note that, since X is projective and nonsingular we get $W_0(H^k(X)) = 0$, for $k > 0$. Since X is irreducible $W_0(H^0(X)) \simeq \mathbb{C}$. We have

$W_0(H^i(Y \times X)) = \bigoplus_{j+k=i} W_0(H^j(Y) \otimes W_0(H^k(X)) = W_0(H^i(Y)) \otimes W_0(H^0(X)) = W_0(H^i(Y)) \otimes \mathbb{C} \simeq W_0(H^i(Y))$

(2) $H^1(Y \times X)) = (H^0(Y) \otimes H^1(X)) \oplus (H^1(Y) \otimes H^0(X)) = H^1(Y) \otimes H^0(X)) = H^1(Y) \otimes \mathbb{C} = H^1(Y)$ □

Lemma 5.5. *If $E \to Y$ is a Zariski locally trivial bundle with fibre X then*

(1) *if X is nonsingular and projective,*

$$\pi^* : W_0(H^i(Y)) \to W_0(H^i(E))$$

is an isomorphism.

(2) *if $b_1(X) = 0$,*

$$\pi^* : H^1(Y) \to H^1(E)$$

is an isomorphism.

Proof. (1) We use induction on dimension of Y. Let $U \subset Y$ be an open set where the bundle $E_{|U} \simeq U \times X$ is trivial. By the previous Lemma

$$W_0(H^i(U)) \to W_0(H^i(E_{|U}) = W_0(H^i(U \times X))$$

is an isomorphism. Also, by the inductive assumption on dimension

$$W_0(H^i(Y \setminus U)) \to W_0(H^i(E_{|Y\setminus U}))$$

is an isomorphism. Apply the 5 lemma to the diagram

$$\begin{tikzcd}
\ldots W_0H_c^{i-1}(Y \setminus U) \arrow[r] \arrow[d, "\simeq"] & W_0H_c^i(U) \arrow[r] \arrow[d, "\simeq"] & W_0H_c^i(Y) \arrow[r] \arrow[d] & W_0H_c^i(Y \setminus U)\ldots \arrow[d, "\simeq"] \\
W_0H_c^{i-1}(E_{|Y\setminus U}) \arrow[r] & W_0H_c^i(E_{|U}) \arrow[r] & W_0H_c^i(E) \arrow[r] & W_0H_c^i(E_{|Y\setminus U})
\end{tikzcd}$$

(2) We argue as in (1). We may further assume that U contains no compact connected components. Thus $H_c^0(U) = 0$, and we get $H^2(E_{|U}) = H^2(U \times X) = (H_c^2(U) \otimes H_c^0(X)) \oplus (H_c^1(U) \otimes H_c^1(X)) \oplus (H_c^0(U) \otimes H_c^2(X)) = H_c^2(U) \otimes H_c^0(X) = H_c^2(U) \otimes \mathbb{C} \simeq H_c^2(U)$

We use the diagram.

$$\begin{tikzcd}
\ldots H_c^0(Y \setminus U) \arrow[r] \arrow[d, "\simeq"] & H_c^1(U) \arrow[r] \arrow[d, "\simeq"] & H_c^1(Y) \arrow[r] \arrow[d] & H_c^1(Y \setminus U) \arrow[r] \arrow[d, "\simeq"] & H_c^2(U)\ldots \arrow[d, "\simeq"] \\
H_c^0(E_{|Y\setminus U}) \arrow[r] & H_c^1(E_{|U}) \arrow[r] & H_c^1(E) \arrow[r] & H_c^1(E_{|Y\setminus U}) \arrow[r] & H_c^2(E_{|U})
\end{tikzcd}$$

□

Lemma 5.6. *If $f : X \to Y$ is a blow-up of a smooth centre on the nonsingular but not necessarily complete variety Y then $f^* : W_0(H^i(Y)) \to W_0(H^i(X))$ is an isomorphism.*

Proof. Let $C \subset Y$ be the smooth centre of the blow-up, and E be the exceptional divisor. Then $E \to C$ is a locally trivial bundle with the fibre isomorphic to $\mathbb{P}^k$, for some k. Set $U = Y \setminus C = X \setminus E$. Consider the following diagram and use the 5 lemma.

$$\begin{tikzcd}
\ldots W_0H_c^{i-1}(C) \arrow[r] \arrow[d, "\simeq"] & W_0H_c^i(U) \arrow[r] \arrow[d, "="] & W_0H_c^i(Y) \arrow[r] \arrow[d] & W_0H_c^i(C)\ldots \arrow[d, "\simeq"] \\
W_0H_c^{i-1}(E) \arrow[r] & W_0H_c^i(U) \arrow[r] & W_0H_c^i(X) \arrow[r] & W_0H_c^i(E)
\end{tikzcd}$$

□

Proof of Proposition 5.2. By the Weak Factorization theorem ([Wlo], [AKMW]), any two desingularizations can be connected by a sequence of blow-ups with smooth centres. Therefore it suffices to compare two resolutions $f : X \to Y$, and $\tilde{f} : \tilde{X} \to Y$, such that $\tilde{X} \to X$ is the blow up along a smooth centre C. Let $U = X \setminus f^{-1}(y)$, $\tilde{U} = \tilde{X} \setminus \tilde{f}^{-1}(y)$ and let E be the exceptional divisor of $\tilde{X} \to X$.

(1) Consider the diagram

$$\begin{tikzcd}
\ldots W_0H^{i-1}(f^{-1}(y)) \arrow[r] \arrow[d] & W_0H^i_c(U) \arrow[r] \arrow[d, "\simeq"] & W_0H^i(X) \arrow[r] \arrow[d, "\simeq"] & W_0H^i(f^{-1}(y)) \ldots \arrow[d] \\
W_0H^{i-1}(\tilde{f}^{-1}(y)) \arrow[r] & W_0H^i_c(\tilde{U}) \arrow[r] & W_0H^i_c(\tilde{X}) \arrow[r] & W_0H^i(\tilde{f}^{-1}(y))
\end{tikzcd}$$

It folows from lemma 5.5 that $W_0H^i_c(U) \to W_0H^i_c(\tilde{U})$ and $W_0H^i_c(X) \to W_0H^i_c(\tilde{X})$ are isomorphisms. By the diagram and 5 lemma we get that

$$W_0H^i(f^{-1}(y)) \to W_0H^i(\tilde{f}^{-1}(y))$$

is an isomorphism.

(2) Consider the diagram

$$\begin{tikzcd}
\ldots H^1(f^{-1}(y) \setminus C) \arrow[r] \arrow[d, "="] & H^1(f^{-1}(y)) \arrow[r] \arrow[d] & H^1_c(f^{-1}(y) \cap C) \arrow[r] \arrow[d, "\simeq"] & H^1(f^{-1}(y) \setminus C)) \ldots \arrow[d, "="] \\
H^0(\tilde{f}^{-1}(y) \setminus E) \arrow[r] & H^0(\tilde{f}^{-1}(y)) \arrow[r] & H^1_c(\tilde{f}^{-1}(y) \cap E) \arrow[r] & H^1(\tilde{f}^{-1}(y) \setminus E)
\end{tikzcd}$$

Note that $\tilde{f}^{-1}(y) \cap E \to f^{-1}(y) \cap C$ is a locally trivial $\mathbb{P}^k$-bundle. By the diagram and 5-Lemma we get that

$$H^1(f^{-1}(y)) \to H^1(\tilde{f}^{-1}(y))$$

is an isomorphism. $\square$

In general, if $j > 0$, $\dim W_jH^i(f^{-1}(y))$ may depend upon the resolution f. That is why we extend the above definition in two ways:

$$w^i_j(y) := \inf_{f:X\to Y} \dim W_jH^i(f^{-1}(y))$$

$$\bar{w}^i_j(y) := \sup_{f:X\to Y} \dim W_jH^i(f^{-1}(y))$$

$$h^i(y) := \inf_{f:X\to Y} \dim H^i(f^{-1}(y))$$

$$\bar{h}^i(y) := \sup_{f:X\to Y} (\dim H^i(f^{-1}(y))$$

where $f : X \to Y$ varies over all resolutions above. It follows immediately from the definition and the previous theorems that we have the following properties of the invariants $w^i_j(y)$, and $\bar{w}^i_j(y)$, $h^i(y)$, $\bar{h}^i(y)$.

Proposition 5.7.

(1) *Let $f : X \to Y$ be a resolution of Y. Then*

$$0 \le w^i_j(y) \le \dim W_jH^i(f^{-1}(y)) \le \bar{w}^i_j(y) \le \infty$$

$$0 \le h^i(y) \le \dim H^i(f^{-1}(y)) \le \bar{h}^i(y) \le \infty$$

(2)
$$0 \leq w_0^i(y) \leq w_1^i(y) \leq \ldots \leq w_i^i(y) = h^i(y) = w_{i+1}^i(y) = \ldots,$$
and
$$0 \leq \bar{w}_0^i(y) \leq \bar{w}_1^i(y) \leq \ldots \leq \bar{w}_i^i(y) = \bar{h}^i(y) = \bar{w}_{i+1}^i(y) = \ldots.$$

(3) *If* $g : Y_1 \to Y_2$ *is a smooth morphism and* $y \in Y_1$ *is a point such that* $w_j^i(y) = \bar{w}_j^i(y)$, *(respectively* $h^i(y) = \bar{h}^i(y)$*) then*
$$w_j^i(g(y)) = \bar{w}_j^i(g(y)) = w_j^i(y) = \bar{w}_j^i(y)$$
(respectively
$$h^i(g(y)) = \bar{h}^i(g(y)) = h^i(y) = \bar{h}^i(y)).$$

(4) *The equality* $w_j^i(y) = \bar{w}_j^i(y)$ *holds if*
  (a) y *is an isolated singularity and* $j < i$, *or* $j = 0$,
  (b) y *is arbitrary, and* $j = 0$, *or* $i \leq 1$.

(5) $w_j^i(y) = \bar{w}_j^i(y) = 0 \quad$ *if* $\quad i - j \geq \dim(Y) - 1$

(6) $h^i(y) = \bar{h}^i(y)$ *if* $i \leq 1$.

(7) $\bar{h}^i(y) = \infty$ *for* $2 \leq i \leq 2\dim_y Y - 2$

(8) $h^i(y) = \bar{h}^i(y) = 0$ *for* $i \geq 2\dim_y Y - 1$

Proof. (1) is obvious. (2) and (5) follows from corollary 2.12.

(3) Let $g : Y_1 \to Y_2$ be a smooth morphism. If $X_2 \to Y_2$ is a resolution then $X_1 := Y_{1 \times Y_2} X_2 \to Y_1$ is also a resolution of Y_1 with the same fibers, and there is a fiber square :

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 \end{array}$$

of resolutions with horizontal smooth maps and the result follows.

(4) and (6) follow from Propositions 5.1 and5.2.

(7) Blow-up a smooth centre C in the fibre $f^{-1}(y)$ of the resolution $f : X \to Y$ to obtain a new resolution $\tilde{f} : \tilde{X} \to Y$. Let E denote the exceptional fibre of $\tilde{f}$. Consider Mayer-Vietoris sequence

$$\ldots \to W_j H_c^{i-1}(E) \to W_j H_c^i(X) \to W_j H_c^i(\tilde{X}) \oplus W_j H_c^i(C) \to W_j H_c^i(E) \ldots$$

If $j = i$ then $W_j H_c^i() = H_c^i()$, and $W_j H^{i+1}(X) = 0$. Thus $H_c^i(\tilde{X}) \oplus H_c^i(C) \to H_c^i(E)$ is an epimorphism. Since $\tilde{f}^{-1}(y) \supseteq E$, the above morphism factors through

$$H_c^i(\tilde{X}) \oplus H_c^i(C) \to H_c^i(\tilde{f}^{-1}(y)) \oplus H_c^i(C) \to H_c^i(E).$$

Consequently the Mayer-Vietoris morphism

$$H_c^i(\tilde{f}^{-1}(y)) \oplus H_c^i(C) \to H_c^i(E)$$

is also an epimorphism for all i. Thus, in the Mayer-Vietoris sequence.

$$\ldots \to H_c^{i-1}(E) \to H_c^i(f^{-1}(y)) \xrightarrow{\psi} H_c^i(\tilde{f}^{-1}(y)) \oplus H_c^i(C) \xrightarrow{\phi} H_c^i(E) \ldots$$

we see ϕ is an epimorphism and ψ is a monomorphism. Therefore we get the short exact sequence.

$$0 \to H_c^i(f^{-1}(y)) \xrightarrow{\psi} H_c^i(\tilde{f}^{-1}(y)) \oplus H_c^i(C) \xrightarrow{\phi} H_c^i(E) \to 0$$

We are going to use blow-ups with centers which are either point or smooth elliptic curves. Then $\dim(C) \leq 1$, and $H^i_c(C) = 0$ for $i > 2$. We get

$$\dim(H^i_c(\tilde{f}^{-1}(y)) = \dim(H^i_c(f^{-1}(y)) + \dim(H^i_c(E)).$$

Since E is a locally trivial bundle over C with fibre $\mathbb{P}^l$, where $n = \dim(Y)$, $l = n-1$ if C is a point, and $l = n-2$ if C is an eliptic curve. As is well known, e.g. [BT, p 270],

$$H^i_c(E) \simeq \bigoplus_{j+k=i} H^j(C) \otimes H^k(\mathbb{P}^l).$$

In particular the Poincare polynomial P_E of E is equal to

$$P_E(t) = 1 + t^2 + \ldots + t^{2n-2}$$

for the blow-up at the point, and

$$P_E(t) = (1 + 2t + t^2)(1 + t^2 + \ldots + t^{2n-4})$$

for the blow-up at the elliptic curve. If we apply blow-ups at points and elliptic curves in in the fibre $f^{-1}(y)$ we can increase cohomology $h^i(f^{-1}(y))$, where $2 \leq i \leq 2n-2$.

(8) Follows from the inequality $\dim_{\mathbb{R}}(f^{-1}(y)) \leq 2\dim(Y) - 2$. □

Theorem 5.8.

(1) *If $y \in Y$ is a nonsingular point then*

$$h^i(y) = w^i_j(y) = 0, \text{ for } \quad i > 0.$$

$$\bar{w}^i_j(y) = w^i_j(y) = 0 \quad \text{for} \quad i > 0, \quad j < i.$$

(2) *If $y \in Y$ is toroidal (analytically equivalent to the germ of a toric variety) then*

$$w^i_j(y) = \bar{w}^i_j(y) = 0$$

for $i > 0$, $j < i/2$.

(3) *If $y \in Y$ is rational then*

$$w^i_0(y) = \bar{w}^i_0(y) = 0$$

for $i > 0$.

(4) *If $y \in Y$ is an isolated normal Cohen-Macaulay singularity*

$$w^i_0(y) = \bar{w}^i_0(y) = 0$$

for $i \neq 0$, $\dim Y - 1$

(5) *If $y \in Y$ is normal then $h^0(y) = 1$.*

Proof. (1) To see this, take the trivial resolution $Y \stackrel{\mathrm{id}}{\to} Y$, and apply (4) from the previous proposition.

(2) follows from Proposition 4.4.

(3) and (4) follow from Corollary 3.2.

(5) follows from Zariski's main Theorem. □

6. Weights of the link

The link L of a singularity (Y,y) is the boundary of a suitable small contractible neighbourhood of y. When (Y,y) has isolated singularities, $H^i(L)$ carries mixed Hodge structure by the identification

$$H^i(L) \cong H^{i+1}(Y, Y - \{y\})$$

when $i > 0$ (cf. [Stn]). More generally, work of Durfee and Saito [DS] shows that the intersection cohomology $IH^i(L)$ carries a mixed Hodge structure which is independent of any choices. (Among the various indexing conventions, we choose the one where $IH^i(L)$ coincides with ordinary cohomology $H^i(L)$ when L is a manifold, i.e. when (Y,y) is isolated. This means that $IH^i(L) = H^{i-\dim L}(k_{!*}(\mathbb{Q}_{L'}[\dim L]))$, where $k : L' \to L$ is the smooth locus.) The general case, which appeals to Saito's theory of mixed Hodge modules [Sa], is much more involved. By definition the mixed Hodge structure on the intersection cohomology of the link is given by right hand side of equation (8), below, computed in the category of mixed Hodge modules. We can apply our previous results to get bounds on the weights of these mixed Hodge structures.

These lead us to the following interesting invariants of singularities:

$$\ell^i_j(y) := \dim W_j IH^i(L),$$

where L is the link of singularity (Y,y).

Theorem 6.1. *If $i < \dim Y$, then*

$$\ell^i_j(y) \leq \dim W_j(H^i(f^{-1}(y))$$

for any desingularization $f : X \to Y$. In particular, for any singularity $y \in Y$,

$$\ell^i_j(y) \leq w^i_j(y).$$

Proof. Let $n = \dim Y$. By the decomposition theorem [BBD], $\mathbb{R}f_*\mathbb{Q}[n]$ decomposes into a sum of shifted perverse sheaves. This moreover holds in the derived category of mixed Hodge modules by the work of Saito [Sa]. By restricting to the smooth locus $k : Y' \to Y$, we can see that one of these summands is necessarily the intersection cohomology complex $IC_Y = k_{!*}\mathbb{Q}_{Y'}[n]$. Let $U = Y - y$ and denote the inclusions by $j : U \to Y$ and $\iota : y \to Y$. Then from [BBD, 2.1.11], we can conclude that $IC_Y = \tau_{\leq -1}\mathbb{R}j_* IC_U$, where $\tau_\bullet$ is the standard truncation operator [D, 1.4.6]. Since

$$IH^{i+n}(L) = H^i(\iota^*\mathbb{R}j_*IC_U) \tag{8}$$

$$= H^i(\iota^*(\tau_{\leq -1}\mathbb{R}j_*IC_U)) \tag{9}$$

for $i < 0$, it follows that $IH^{i+n}(L)$ is a summand of $H^i(\mathbb{R}f_*\mathbb{Q}[n]) = (H^{i+n}(f^{-1}(y))$ when $i < 0$. □

As a corollary we get

Theorem 6.2. *If $i < \dim Y$, then*

$$\ell^i_j(y) \leq w^i_j(y) \leq \sum_{p \leq j} \dim(R^{i-p}f_*\Omega^p_X)_y \otimes \mathcal{O}_y/m_y$$

for any desingularization $f : X \to Y$.

Proof. We use the previous theorem and apply the bounds from corollary 4.2. □

Remark 6.3. *For an isolated singularity, we can argue more directly, without mixed Hodge module theory, as in* [Stn, cor 1.12].

Example 6.4. *If $x \in X$ is a nonsingular point then $L \simeq S^{2n-1}$ and we get*
$\ell^i_j(x) = 0$ *for* $j < i$, *or* $j \geq i$ *and* $i \neq 0, 2n-1$,
$\ell^i_j(x) = 1$ *if* $j \geq i$ *and* $i = 0, 2n-1$

Corollary 6.5. *For a normal isolated Cohen-Macaulay (respectively rational) singularity,*

$$w^i_0(y) = \bar{w}^i_0(y) = \ell^i_0(y) = 0$$

for $0 < i < \dim Y - 1$ *(respectively* $i > 0$*). If* (Y, y) *is toroidal then*

$$w^i_j(y) = \bar{w}^i_j(y) = \ell^i_j(y) = 0$$

for $j < i/2$.

The natural conjecture which arises here is whether

Conjecture 6.6. *The invariants* $w^i_j(y), \bar{w}^i_j(y), \ell^i_j(y)$ *are upper semicontinuous.*

7. Boundary divisors and the Weak factorization theorem

Suppose we are given a smooth complete variety X, and a divisor with simple normal crossings $E \subset X$, or more generally a union of smooth subvarieties which is local analytically a union of intersections of coordinate hyperplanes. We will show that the homotopy type of the dual complex of E depends only the the complement $X - E$, and in fact only on its proper birational class. In fact, we will prove somewhat sharper results in theorems 7.5, 7.8 and 7.9 below. Stepanov in [Stp1] showed this result in the particular case, where E is the exceptional divisor of a resolution of an isolated singularity. Actually, his proof works for any boundary divisors which are complements a fixed open subset $U = X - E$. Thuillier in [T] proves a similar result in any characteristic. In our set up, the complement $X - E$ is not fixed. That is, we assume that for two different divisors E and E', there is a proper birational map $X - E \dashrightarrow X' - E'$. This implies the isomorphism of homotopy types of the dual complexes of E and E'. Payne, in his paper [P, §5], compares the homotopy types of the dual complexes of divisors of different log resolutions of a given singular variety. In our version, the situation is more general. In particular, we allow the maximal components of E to be of any dimension. This situation arises naturally when E is the fibre of a resolution of a nonisolated singularity.

We recall some constructions and notations from earlier sections, so that section can be read independently of the rest of the paper, Let $D = \cup D_i$ be a SNC (simple normal crossing) divisor on a nonsingular variety X. The dual complex is a CW-complex Σ_D whose cells are simplices $\Delta^j_{i_1,\ldots,i_k}$ corresponding to the irreducible components of $D^j_{i_1,\ldots,i_k}$ of the intersection $D_{i_1,\ldots,i_k} = D_{i_1}\cap, \ldots \cap D_{i_k}$. For any $\Delta = \Delta^j_{i_1,\ldots,i_k}$, and $\Delta' = \Delta^{j'}_{i'_1,\ldots,i'_s}$, where $\{i'_1, \ldots, i'_s\} \subset \{i_1, \ldots, i_k\}$ the simplex Δ' is a face of Δ if $D^{j'}_{i'_1,\ldots,i'_s} \supset D^j_{i_1,\ldots,i_k}$. If all the intersections $D_{i_1,\ldots,i_k} = D_{i_1}\cap, \ldots \cap D_{i_k}$ are irreducible then then Σ is a simplicial complex. In general the dual complex is a *quasicomplex*, i.e. it is a collection of simplices closed with respect to the face relation, and such that the intersection of two simplices is a union of some of their faces.

We recall some basic notions for quasicomplexes which refine those for simplicial complexes.

- By the $Star(\Delta, \Sigma)$, where $\Delta \in \Sigma$, we mean the set of all faces of S which contain Δ.
- For any set $S \subset \Sigma$ by $\overline{S} \subset \Sigma$ we mean the complex consisting of simplices of Σ and their faces.
- By the *link* we mean $L(\Delta, \Sigma_D) = \overline{Star(\Delta, \Sigma_D)} \setminus Star(\Delta, \Sigma_D)$.
- Let v_Δ be the barycentric centre of Δ. By the *stellar sudivision* of Σ at Δ (or at v_Δ) we mean

  $$\Sigma' := v_\Delta \cdot \Sigma := \Sigma \setminus Star(\Delta, \Sigma) \cup \overline{\{conv(v, \Delta) \mid \Delta \in L(\Delta, \Sigma)\}},$$

  where $conv(v_\Delta, \Delta)$ is the simplex spanned by v_Δ, and Δ.

Let Σ_D be a dual (quasi)-complex associated with D on a nonsingular X. Let Δ be a simplex in Σ_D and let $C = D(\Delta)$ be the corresponding intersection components of D. Then the blow-up $\sigma : X' \to X$ of C determines transformation of divisors $D \mapsto D' = \sigma^{-1}(D)$ which corresponds to the stellar subdivision $v_\Delta \cdot \Sigma_D$. Recall a well known lemma:

Lemma 7.1. *Let Σ_D be the dual (quasi)-complex associated with D on a nonsingular X. Let $X' \to X$ be a composition of blow-ups of all the intersection components starting from the components of the smallest dimension and ending at the blow-ups of the components of the highest dimension (divisors). Then the resulting quasi-complex $\Sigma_{D'}$ which is obtained from Σ_D by the successive stellar subdivisions is a simplicial complex.*

Proof. In the process we blow up (and thus eliminate) all the strict transforms of the intersections components of $D = \bigcup D_i$. Note that the center $C = D^j_{i_1,\ldots,i_k}$ is the lowest dimensional intersection component of the strict transforms of D_i so it intersects no other divisors D_i, except for $i = i_1, \ldots, i_k$.

By the induction centers of blow-ups and thus new exceptional divisors have irreducible intersections with the strict transforms of the intersections components of D and the already created exceptional divisors E. Finally we will have only irreducible intersections of the exceptional divisors.

$\square$

By this lemma, it always possible to reduce the situation to SNC divisor whose associated dual quasicomplex is a complex by applying additional blow-ups. We will not need to do this however.

Remark 7.2. *The blow-ups of the divisorial components coresponding to the stellar subdivisons at the vertices define identity transformations. They are introduced to simplify the considerations.*

Proposition 5.2 can be generalized in a few ways.

Proposition 7.3. *If $\phi : X \dashrightarrow X'$ is a proper birational map of two nonsingular projective varieties, such that its restriction $\phi_{|U} : U \dashrightarrow U'$ to open sets $U \subset X$ and $U' \subset X'$ is proper. Then there is a natural isomorphism*

$$W_0(H^i(X \setminus U)) \simeq W_0(H^i(X' \setminus U'))$$

Proof. This is identical to the proof of Proposition 5.2. $\square$

If the fibre $f^{-1}(y) = D_y$ of the resolution $f : X \to Y$ is a SNC divisor then the above Proposition 7.3 says that the cohomology $H^i(\Sigma_{D_y}) = W_0(H^i(D_y))$ of the dual complex Σ_{D_y} are independent of the resolution $f : X \to Y$. This suggests:

Theorem 7.4. *Let $f^{-1}(y) = D$ be the fibre of the resolution $f : Y \to X$ which is a SNC divisor. The homotopy type of the dual complex Σ_D corresponding to the fibre is independent of the resolution $f : Y \to X$.*

When the singularity is nonisolated, the fibres are usually not divisors. We consider a generalization later in theorem 7.9, which allows for general fibres. The theorem above is a generalization of the version of theorem for isolated singularities given in [Stp2, Stp1]. The following theorems 7.5, 7.8 generalize theorem 7.4, and the results of [P] for log-resolutions.

Theorem 7.5. *If $\phi : X \dashrightarrow X'$ is a birational map of two nonsingular projective varieties, such that its restriction $\phi_{|U} : U \dashrightarrow U'$ to open subsets $U \subset X$ and $U' \subset X'$ is proper. Assume that the boundary divisors $D = X \setminus U$ and $D' = X' \setminus U'$ have SNC. Then the dual complexes Σ_D and $\Sigma_{D'}$ are homotopically equivalent*

The method of the proof is identical with the one used in Stepanov [Stp1]. We just need the following simple strengthening of the Weak Factorization Theorem ([Wlo, AKMW]).

Theorem 7.6. *(*[Wlo, AKMW]*) Let $\phi : X \dashrightarrow X'$ be a birational map of smooth complete varieties such that its restriction $\phi_{|U} : U \to U'$ to open susbsets $U \subset X$ and $U' \subset X'$ is proper and which is an isomorphism over an open subset $V \subset U, U'$. Assume that the boundary divisors $D = X \setminus U$ and $D' = X' \setminus U'$ have SNC. Then there exists a weak factorization, that is a sequence of birational maps*

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} \ldots \overset{\phi_{n-1}}{\dashrightarrow} X_n = X',$$

such that $\phi_i : X = X_i \dashrightarrow X_{i+1}$ is either a blow-up or a blow-down along smooth centres. Moreover all the complements $D_i := X_i \setminus U_i$ are SNC divisors and all the centres have SNC with respect to D_i. Finally, the centres are disjoint from V.

We prove a more general version:

Theorem 7.7. *(*[Wlo, AKMW]*) Let $\phi : X \dashrightarrow X'$ be a birational map of smooth complete varieties. Let $D^1 \subset D^2 \subset \ldots \subset D^k = D$ be SNC divisors on X, and $D'^1 \subset D'^2 \subset \ldots \subset D'^k = D'$ be SNC divisors on X' such that for $U^i := X \setminus D^i$, and $U'^i := X' \setminus D'^i$ the restriction $f_{|U^i} : U^i \to U'^i$ is proper. Assume ϕ is an isomorphism over an open subset $V \subset U^k, U'^k$. Then there exists a weak factorization*

$$X = X_0 \overset{\phi_0}{\dashrightarrow} X_1 \overset{\phi_1}{\dashrightarrow} \ldots \overset{\phi_{n-1}}{\dashrightarrow} X_n = X',$$

as above, such that for the open subsets $U_i^j \subset X_i$ all the complements of $D_i^j := X_i \setminus U_i^j$ are SNC divisors and all the centres have SNC with D_i^j.

Proof. By a version of Hironaka principalization for an ideal sheaf $\mathcal{I}_{X\setminus V}$ (respectively $\mathcal{I}_{X'\setminus V}$) and SNC divisor D (respectively D') there exists a sequence of blow-ups $\pi : Y \to X$ (respectively $\pi' : Y' \to X'$) with smooth centres having SNC with the inverse images of D (respectively D'). Moreover, the total transform of $\mathcal{I}_{X\setminus V}$ (respectively of $\mathcal{I}_{X'\setminus V}$) is an ideal of an SNC divisor whose components have SNC with the inverse images of D (resp. D') (cf. [Kol],[Wlo2]).

There exists a weak factorization of $f : Y \to Y'$ such that all intermediate steps Y_i admit a morphism $f_i : Y_i \to Y$ or $f'_i : Y_i \to Y'$ ([Wlo], [AKMW]). Thus the full transform of $f_i^{-1}\pi^{-1}(\mathcal{I}_{D^j})$ (or $f_i'^{-1}\pi'^{-1}(\mathcal{I}_{D'^j})$) is principal and the complement $D_i^j := f_i^{-1}\pi^{-1}(D^j) = Y_i \setminus U_i^j$ is of codimension one for any $j = 1 \dots k$. Moreover all centres of the blow-ups have SNC with the complement divisor $E_i := Y_i \setminus V$ containing D_i^j. Thus D_i^j is a SNC divisor on Y_i and all the centres in the Weak Factorization have SNC with D_i^j. □

Proof of Theorem 7.5. The proof is an extension of the method mentioned in [Stp2]. By theorem 7.7, we can connect D and D' by blow-ups with smooth centres which have SNC with intermediate divisors which are complements of U.

Thus it is sufficient to consider the effect of a single blow-up. This is already done in [Stp1]. We describe this in order to keep the presentation self contained and to provide a model for what comes later. We also introduce some convenient notation here. Consider local coordinates $x_1, \dots, x_n$ on an open neighborhood $U_p \subset X$ of some point $p \in C \cap D$. We can assume that the coordinates $x_1, \dots, x_k$ describe the components of the divisor D, and that the centre C is described by $x_r, \dots, x_s$, where $r \leq s$. Consider three cases.

Case 1. The centre C is not contained in D (but has SNC with D).

This means that C is not contained in a component D_i. Thus no x_i, where $i \leq k$ vanishes on C, and consequently $k < r$. In this case the blow-up is defined by a coordinate transformation

$$x_1, \dots, x_k, \dots, x_{s-1}, x_s/x_r, \dots, x_{r-1}/x_r, x_r, x_{r+1} \dots, x_n$$

which do not change the configuration of components and thus the dual complex of the exceptional divisor.

$$\Sigma_{D'} = \Sigma_D.$$

Case 2. The centre $C = D_r \cap \ldots \cap D_s$ is the intersection of some divisorial components and corresponds to the face $\Delta_C := \Delta_{r,\dots,s}$. $(s < k.\ )$

The blow-up of C determines the stellar subdivision $\Sigma_{D'} = v_C \cdot \Sigma_D$ of Σ_D at the centre v_C of $\Delta_C = \Delta_{r,\dots,s}$. Thus topologically Σ_D and $\Sigma_{D'}$ are homeomorphic.

Case 3. The centre C is contained in the intersection $D_r \cap \ldots \cap D_k$ and is not contained in any smaller intersection. $(\ r < k \leq s)$

Extend locally the set of SNC divisorial components of D to the the set all divisors $D \cup E$ corresponding to the complete coordinate system $x_1, \dots, x_n$.

Consider the simplex Δ corresponding to the complete system of coordinates $x_1, \dots, x_n$. Its face $\Delta_{1,\dots,k}$ is also a face of Σ_D. Let $\Delta_{D,C} := \Delta_{s,\dots,k} \subset \Delta_{1,\dots,k}$ be the face of Σ_D corresponding to the minimal intersection component $D_r \cap \ldots \cap D_k$ which contain C. The blow-up of C corresponds to the stellar subdivision at the centre $v_C := v_{s,\dots,r}$ of the face $\Delta_C := \Delta_{s,\dots,r} \in \Sigma_E$. After the stellar subdivision the face $\Delta_{1,\dots,k}$ of Σ_D remains unchanged . We introduce the new vertex $v_C = v_{s,\dots,r}$ in the dual complex Σ_D corresponding to the exceptional divisor E. Denote by $D(\Delta)$, where $\Delta \in \Sigma_D$, the stratum which is the intersection of divisors corresponding to vertices of Δ.

Set

$$\Sigma_{D,C} := \{\Delta \in \Sigma_D \mid D(\Delta) \cap C \neq \emptyset\}$$

Note that by definition if $D(\Delta) \cap C \neq \emptyset$ then $D(\Delta) \cap D(\Delta_C) \neq \emptyset$. Thus

(*) the maximal simplexes of $\Sigma_{D,C}$ are in $Star(\Delta_{D,C}, \Sigma_D)$.

The new dual complex is given by

$$\Sigma_{D'} = \Sigma_D \cup \ v_C * \Sigma_{D,C},$$

where

$$v_C * \Sigma_C := \overline{\{conv(v_C, \Delta) \mid \Delta \in \Sigma_C\}}$$

is a cone over Σ_C with vertex v_C. In addition $conv(v_C, \Delta)$ is the simplex spanned by v_C, and Δ (a cone over Δ with vertex v_C).

Let $v_{D,C}$ denote the barycentre of $\Delta_{D,C}$. The complex $\Sigma_{D'}$ is homotopy equivalent to Σ_D. In one direction, the map $\beta : \Sigma_{D'} \to \Sigma_D$ is defined on the vertex $v_C \to v_{D,C}$ retracts $v_C * \Sigma_C$ to the Σ_C and is identical on Σ_D, and defines homotopy equivalence. The homotopy inverse map $\alpha : \Sigma_D \to \Sigma_{D'}$ is given by the inclusion. Then $\beta\alpha = id : \Sigma_D \to \Sigma_D$ and $\alpha\beta : \Sigma_{D'} \to \Sigma_{D'}$ is homotopic to the identity via

$$v_C \mapsto (1-t)v_{D,C} + tv_C.$$

Note that, by the condition (*), the interval $[v_{D,C}, v_C]$ is contained in Δ_C as well as in its subset $conv(\Delta_{D,C}, v_C) \in \Sigma_{D'}$. The point $v_{D,C}$ in the above construction can be replaced by any other point in $\Delta_{D,C}$. □

The above theorem can be easily extended to multiple divisors. This case was also considered in [P] in the context of log resolutions, and it is nearly the same as the previous one. Theorem 7.8 generalizes and implies the results in [P].

Theorem 7.8. *Let $\phi : X \dashrightarrow X'$ be a birational map of smooth complete varieties. Let $D^1 \subset D^2 \subset \ldots \subset D^k$ be SNC divisors on X, and $D'^1 \subset D'^2 \subset \ldots \subset D'^k$ be SNC divisors on X' such that for $U^i := X \setminus D^i$, and $U'^i := X' \setminus D'^i$ the restriction $\phi_{|U^i} : U^i \dashrightarrow U'^i$ is proper. Then the corresponding sequences of topological spaces $\Sigma^1 \subset \Sigma^2 \subset \ldots \subset \Sigma^k$ and $\Sigma'^1 \subset \Sigma'^2 \subset \ldots \subset \Sigma'^k$ are homotopically equivalent.*

Proof. As before, by the Weak factorization theorem it suffices to consider the effect of a single blow-up . We use the notation from the previous proof.

Assume $C = D(\Delta_C)$ is the intersection of some components of D^k. In the other cases the reasoning is the same. Let $r := \min\{i \mid C \subset D^i\}$ and $\ell := \min\{i \mid C = D(\Delta_{C,\Sigma^i})\}$ be the smallest index j such that the centre C is the intersection of the divisors in D^j.

Define for $i = r, \ldots, k$ the sequence of subcomplexes

$$\Sigma^i_C := \{\Delta \in \Sigma^i \mid D(\Delta) \cap C \neq \emptyset\}.$$

Then

$$\emptyset \subset \Sigma^r_C \subset \ldots \subset \Sigma^\ell_C = \overline{Star(\Delta_C, \Sigma_\ell)} \subset \ldots \subset \Sigma^k_C = \overline{Star(\Delta_C, \Sigma_k)}$$

The blow-up transforms $\Sigma^1 \subset \Sigma^2 \subset \ldots \subset \Sigma^k$ into $\Sigma'^1 \subset \Sigma'^2 \subset \ldots \subset \Sigma'^k$, where

(1) $\Sigma'^i = \Sigma^i$ for $1 \leq i \leq r$
(2) $\Sigma'^i = \Sigma^i \cup (v_C * \Sigma^i_C)$ for $r \leq i < \ell$
(3) $\Sigma'^i = v_C \cdot \Sigma^i$ for $\ell \leq i \leq k$.

Let $\Delta_{C,r} \in \Sigma_r$ be the face corresponding to the smallest intersection component of D^r containing C. Denote by $v_{C,r}$ the barycentre of $\Delta_{C,r} \in \Sigma_r$. There exists a map of topological spaces $\alpha : \Sigma^k \to \Sigma'^k$ which is an identity, and whose restrictions $\alpha_i : \Sigma^i \to \Sigma'^i$ are inclusions for $r \leq i < \ell$, and identities for $1 \leq i < r$, and $\ell \leq i \leq k$. There exists a map $\beta : \Sigma'^k \to \Sigma^k$ which is defined on the vertex

$$v_C \mapsto v_{C,r} \in \Sigma^r \subset \Sigma^{r+1} \subset \ldots \subset \Sigma^k$$

and identity on other vertices. Its restriction $\beta_i : \Sigma'^i \to \Sigma^i$ is a linear map homotopic to the identity for $\ell \le i \le k$, a retracts $v_C * \Sigma^i_C$ to the Σ^i_C and Σ'^i to Σ^i for $r \le i < \ell$ and is identity for $i \le r$.

The homotopy of the compositions $\alpha\beta : \Sigma^k \to \Sigma^k$ (and $\beta\alpha : \Sigma'^k \to \Sigma'^k$) with the identities are defined by

$$v_C \to (1-t)v_{C,r} + t \cdot v_C.$$

As before the interval $[v_C, v_{C,r}]$ is contained in Δ_C, as well as its subset

$$conv(v_C, \Delta_{C,r}) \subset conv(v_C, \Delta_{C,r+1}) \subset \ldots \subset conv(v_C, \Delta_{C,k})$$

(since $\Delta_{C,r}) \subset (v_C, \Delta_{C,r+1}) \subset \ldots \subset conv(\Delta_{C,k})$). The restriction of the homotopy equivalence $\Sigma_k \to \Sigma'_k$ defines homotopy equivalences between Σ_i and Σ'_i, for $i \le k$. □

Theorem 7.5 can be used, for instance, for studying fibers. In this situation the codimension one assumption is usually not satisfied. If the singular locus is not a finite set of points then the exceptional fibers are usually (generically) of lower dimension. That is why we generalize Theorems 7.4, and 7.5, 7.8 dropping the codimension one assumption.

Given a union $\bigcup E_i$ of nonsingular closed subvarieties E_i, we will say that it has simple normal crossings (SNC) if around every point E is locally analytically equivalent to a union of intersections of coordinate hyperplanes. We can define the dual complex Σ_E as above. We assume that E_i are maximal components and they are assigned vertices p_i. The simplices $\Delta_{p_{i_1},\ldots,p_{i_k}}$ correspond to the components of $E_{i_1}, \cap \ldots \cap E_{i_k}$ as before.

Theorem 7.9. *Suppose that $\phi : X \dashrightarrow X'$ is a birational map of two nonsingular complete varieties, such that its restriction $\phi|_U : U \dashrightarrow U'$ to open sets $U \subset X$ and $U' \subset X'$ is proper. Assume that the boundary sets $E = X \setminus U$ and $E' = X' \setminus U'$ are unions of the SNC components. Then the dual complexes Σ_E and $\Sigma_{E'}$ are homotopically equivalent.*

Proof. The theorem follows from the Lemma: □

Lemma 7.10. *Let $E = X \setminus U$ be an SNC set in a nonsingular variety X with components $E = \bigcup E_i$. There exists a sequence of blow-ups $X' \to X$ with centres which are intersections of the strict transforms of the maximal components of E such that $D = X' \setminus U$ is a SNC divisor, and there is a homotopy equivalence $\Sigma_D \to \Sigma_E$.*

Proof. Consider the smallest possible intersection component $E(\Delta_0)$ (i.e. intersection of some E_i) corresponding to the maximal face $\Delta_0 \in \Sigma_E$ of a certain dimension k. Consider the subcomplex Σ_{E,Δ_0} consisting of all faces Δ of Σ_E such that the corresponding intersection components $E(\Delta)$ are not contained in $E(\Delta_0)$. Note that Σ_{E,Δ_0} intersects with Δ_0 along a certain subcomplex Σ_{Δ_0}. The set $\Sigma^c_{E,\Delta_0} := \Sigma \setminus \Sigma_{E,\Delta_0}$ consists of simplices Δ for which $E(\Delta)$ are contained in $E(\Delta_0)$. By minimality condition for $E(\Delta_0)$, this means $E(\Delta) = E(\Delta_0)$. The simplices in Σ^c_{E,Δ_0} are the faces of Δ_0 with the property $E(\Delta) = E(\Delta_0)$.

After blow-up along the centre $C = E(\Delta_0)$, the set of maximal components will be enlarged by adding the exceptional divisor D_0 to the previous set $\{E_i\}$.

(If $E(\Delta_0)$ is not a maximal component the set of "old" maximal components E_i remains the same after the blow-up. If $E(\Delta_0) = E_i$ is a maximal component then the corresponding strict transform of E_i disappear.).

The complex Σ_{E,Δ_0} will remain the same after blow-up. The components defined by the afces in Σ^c_{E,Δ_0} will disappear. The simplices of $\Sigma_{\Delta_0} = \Sigma_{E,\Delta_0} \cap \Delta_0$ will be joined with a new vertex v_0 corresponding to the exceptional divisor D_0. The corresponding components $E(\Delta)$ intersect the centre but do not contain it. Thus their strict transform will intersect D_0. The simplices will form a complex $v_0 * \Sigma_{\Delta_0}$.

Then, the new complex is

$$\Sigma_{E,1} := \Sigma_{E,\Delta_0} \cup v_0 * \Sigma_{\Delta_0}.$$

There is a deformation retraction of $\Delta_0 \to v_0 * \Sigma_{\Delta_0}$ which is identity on $v_0 * \Sigma_{\Delta_0}$. Consider the stellar subdivision $v_0 \cdot \Delta_0$ of Δ_0 at its barycentre v_0.

Let $\{v_i\}_{i \in I_{k-1}}$ denotes the set of all the barycentres of all the $k-1$ dimensional faces in $(\Sigma^c_{E,\Delta_0}) \subset \partial(\Delta)$. Note that $\{v_i\}_{i \in I_{k-1}}$ lie in pairwise distinct simplices, and the star subdivisions at v_i commute. Take the stellar subdivisions

$$\{v_i\}_{i \in I_{k-1}} \cdot v_0 \cdot \Delta_0$$

of $v_0 \cdot \Delta_0$ at the centres $\{v_i\}_{i \in I_{k-1}}$. The linear map transform all $v_i \mapsto v_0$, and is identity on all other vertices of $\{v_i\}_{i \in I_{k-1}} \cdot v_0 \cdot \Delta_0$. It defines the homotopic retraction of

$$v_0 \cdot \Delta_0 = v_0 * \partial(\Delta_0) \simeq v_0 * (\{v_i\}_{i \in I_{k-1}} \cdot \partial(\Delta_0)) \to v_0 * ((\Sigma^c_{E,\Delta_0})^{k-2} \cup \Sigma_{\Delta_0}),$$

where $\partial(\Delta_0)$ consists of proper faces of Δ_0, and $(\Sigma^c_{E,\Delta_0})^{k-2}$ consists of all faces of Σ^c_{E,Δ_0} of dimension $\leq k-2$. Then take stellar subdivision of $v_0 * ((\Sigma^c_{E,\Delta_0})^{k-2})$ at the centres $\{v_i\}_{i \in I_{k-2}} \subset \partial(\Delta_0)$ of all the $k-2$ dimensional faces $(\Sigma^c_{E,\Delta_0}) \subset \Delta$. The linear map $v_i \to v_0$ defines the retraction

$$v_0 * ((\Sigma^c_{E,\Delta_0})^{k-2} \cup \Sigma_{\Delta_0}) \simeq v_0 * (\{v_i\}_{i \in I_{k-2}} \cdot (\Sigma^c_{E,\Delta_0})^{k-2} \cup \Sigma_{\Delta_0}) \to v_0 * ((\Sigma^c_{E,\Delta_0})^{k-3} \cup \Sigma_{\Delta_0}).$$

By continuing this process we get the retraction $\Delta_0 \to v_0 * \Sigma_{\Delta_0}$ which extends to the homotopic retraction

$$\Sigma_E = \Sigma_{E,\Delta_0} \cup \Delta_0 \to \Sigma_{E,1} = \Sigma_{E,\Delta_0} \cup v_0 * \Sigma_{\Delta_0}.$$

The complex $\overline{\Sigma}_{E,1} := \Sigma_{E,\Delta_0}$ is a subcomplex $\Sigma_{E,1}$ which corresponds to the strict transforms of the intersection components of E. The subcomplex $v_0 * \Sigma_{\Delta_0}$ in the complex $\Sigma_{E,1} = \overline{\Sigma}_{E,1} \cup v_0 * \Sigma_{\Delta_0}$, corresponds to the intersections of D with the components in $\Sigma_{\Delta_0} \subset \overline{\Sigma}_{E,1}$.

Suppose that the exceptional divisor D_0 contains a strict transform of an intersection component $E(\Delta)$. Then the center of blow-up $E(\Delta_0)$ on X contains $E(\Delta)$. By minimality $E(\Delta) = E(\Delta_0)$. But after the blow-up at $C = E(\Delta_0)$, the strict transforms of the components intersecting at $E(\Delta_0)$ and being normal crossings do not intersect anymore on X_1.

Thus D_0 does not contain any strict transforms of the intersection components of E. Since D_0 has SNC with these components it yields that D_0 and the strict transforms of the intersection components of E are described locally by different coordinates in a certain coordinate system. In particular, for any $\Delta, \Delta' \in \overline{\Sigma}_{E,1}$ we have the implication:

$$\emptyset \neq E(\Delta) \cap D \subset E(\Delta') \quad \Rightarrow \quad E(\Delta) \subset E(\Delta') \qquad (*)$$

Let $E(\Delta_1)$, where $\Delta_1 \in \overline{\Sigma}_{E_1}$ be the smallest possible intersection component (not contained in D) corresponding to the maximal face $\Delta_1 \in \overline{\Sigma}_{E_1}$ of a certain dimesion k_1. If $E(\Delta_1)$ intersects D the construction and reasoning are the same as before and we get (with the relevant notation):

$$\Sigma_{E,1} = \Sigma_{E,1,\Delta_1} \cup \Delta_1 \to \Sigma_{E,2} = \Sigma_{E,1,\Delta_1} \cup v_1 * \Sigma_{\Delta_1} = \overline{\Sigma}_{E_2} \cup v_0 * \Sigma_{\Delta_0} \cup v_1 * \Sigma_{\Delta_1}.$$

Assume that $E(\Delta_1)$ intersects D. Consider the subcomplex Σ_{E_1,Δ_1} consisting of all faces Δ of $\Sigma_{E,1}$ such that the corresponding intersection component $E(\Delta)$ are not contained in $E(\Delta_1)$.

Note that $\Sigma^c_{E,1,\Delta_1} := \Sigma_{E,1} \setminus \Sigma_{E,1,\Delta_1}$ consists of simplices Δ for which $E(\Delta)$ are contained in $E(\Delta_1)$. By minimality condition, this means $E(\Delta) = E(\Delta_1)$ or $E(\Delta) = E(\Delta_1) \cap D$. In particular, all the simplices of $\Sigma^c_{E,1,\Delta_1}$ are faces of $conv(\Delta_1, v_1)$. Also, by (*) a face Δ of Δ_1 is in $\Sigma^c_{E,1,\Delta_1}$ if $conv(\Delta, v_0)$ is in $\Sigma^c_{E,1,\Delta_1}$ (and vice versa).

Consider the intersection

$$\Sigma_{\Delta_1} := \Sigma_{E,1,\Delta_1} \cap \Delta_1.$$

By above $\Sigma_{E,1,\Delta_1}$ intersects with $conv(\Delta_1, v_0)$ along a subcomplex

$$\Sigma_{E,1,\Delta_1} \cap conv(\Delta_1, v_0) = v_0 * \Sigma_{\Delta_1}.$$

After blow-up at $C = E(\Delta_1)$ we construct the exceptional divisor D_1 corresponding to the new vertex v_1 which is a barycentre of Δ_1. The subcomplex Σ_{E_1,Δ_1} remains unchanged. Consider the stellar subdivision $v_1 \cdot \Delta_1$ of Δ_1 at its barycentre v_1.

There is a deformation retraction of

$$v_0 * \Delta_1 := conv(\Delta_1, v_0) \simeq v_1 \cdot (v_0 * \Delta_1) \to v_1 * v_0 * \Sigma_{\Delta_1}$$

which is identity on $v_1 * \Sigma_{\Delta_1}$. Take the stellar subdivisions

$$\{v_i\}_{i \in I_{k_1-1}} \cdot v_1 \cdot (v_0 * \Delta_1)$$

of $v_1 \cdot (v_0 * \Delta_1)$ at the barycentres $\{v_i\}_{i \in I_{k_1-1}} \subset \partial(\Delta_1)$ of all the $k_1 - 1$ dimensional faces $\Sigma^c_{E_1,\Delta_1} \cap \partial(\Delta_1) \subset \partial(\Delta_1)$.

Construct the linear map which transforms v_i to v_1, and which is identity on all other vertices of $\{v_i\}_{i \in I_{k_1-1}} \cdot v_1 \cdot conv(\Delta_1, v_0)$. It defines the homotopic retraction

$$v_0 * (v_1 \cdot \Delta_1) = v_1 * v_0 * \partial(\Delta_1) \to v_1 * v_0 * (((\Sigma^c_{E,\Delta_1})^{k_1-2} \cap \partial(\Delta_1)) \cup \Sigma_{\Delta_1}),$$

where $(\Sigma^c_{E,\Delta_1})^{k_1-2}$ consists of all faces of Σ^c_{E,Δ_1} of dimension $\leq k_1 - 2$. Then take stellar subdivision of at the centres $\{v_i\}_{i \in I_{k_1-2}} \cdot v_1 \cdot (v_0 * \Delta_1)$ at the barycentres of all the $k_1 - 2$ dimensional faces $(\Sigma^c_{E,\Delta_1}) \cap \partial(\Delta_1)$. The linear map $v_i \to v_1$ defines the retraction

$$v_1 * v_0 * (((\Sigma^c_{E,\Delta_1})^{k_1-2} \cap \partial(\Delta_1)) \cup \Sigma_{\Delta_1}) \to v_1 * v_0 * (((\Sigma^c_{E,\Delta_1})^{k_1-3} \cap \partial(\Delta_1)) \cup \Sigma_{\Delta_1}).$$

By continuing this process, we get the retraction $v_0 * \Delta_1 \to v_1 * v_0 * \Sigma_{\Delta_1}$ which extends to the deformation retraction

$$\Sigma_{E,1} = \Sigma_{E_1,\Delta_1} \cup v_0 * \Delta_1 = \Sigma_{E_1,\Delta_1} \cup v_0 * v_1 * \partial(\Delta_1) \to \Sigma_{E,2} = \Sigma_{E,1,\Delta_1} \cup v_0 * v_1 * \Sigma_{\Delta_1}.$$

Here $\overline{\Sigma}_{E,2} := \Sigma_{E,1,\Delta_1}$ is the subcomplex of $\Sigma_{E,1}$ and $\Sigma_{E,2}$ corresponding to the nonempty strict transforms of of the E-components.

We construct new divisors D_1 and D_2 intersecting transversally with the strict transforms of the E-components. The subcomplex $\overline{\Sigma}_{E,2}$ of $\Sigma_{E,2}$ defined by E-components is contained as a proper subset in $\overline{\Sigma}_{E,1} \subsetneq \Sigma_E$. By continuing this algorithm, we construct a sequence of blow-ups such that the subcomplex $\overline{\Sigma}_{E,k}$ of $\Sigma_{E,k}$ is empty. The resulting boundary set becomes divisorial. Note that we eliminate one by one all the simplices of Σ_E. The vertices of Σ_E will be eliminated upon the blowing up the corresponding maximal components at the very end of the process. $\square$

The lemma above can be extended to the multiple subvarieties case.

Corollary 7.11. *Let $E = X\setminus U$ be a union of the closed SNC components $E = \bigcup E_i$ on a nonsingular variety X. Let $E^1 \subset E^2 \subset \ldots \subset E^r = E$ be the filtration of unions of some components E_i. There exists a sequence of blow-ups $X' \to X$ with centres which are intersections of the strict transforms of maximal components of E^r such that the inverse images D^j of E^j is a SNC divisor, for $j = 1, \ldots, r$ and there is a homotopy equivalence of topological spaces $\Sigma_{D^1} \subset \Sigma_{D^2} \subset \ldots \subset \Sigma_{D^r}$ and $\Sigma_{E^1} \subset \Sigma_{E^2} \subset \ldots \subset \Sigma_{E^r}$.*

Proof. This is pretty much identical to the proof of the previous Lemma. Consider the smallest possible intersection component E_0 of $E = E^r$. Assume E_0 is a an intersection component of E^j but is not an intersection component of E^i for $i < j$. For any $i \geq j$ consider the maximal face $\Delta_{0,i} \in \Sigma_{E^i}$ corresponding to E_0. In particular $E(\Delta_{0,i}) = E_0$ for $i \geq j$, and

$$\Delta_{0,j} \subseteq \Delta_{0,j+1} \subseteq \ldots \subseteq \Delta_{0,k}$$

are face inclusions. Let $v_0 \in \Delta_{0,j}$ be its barycentre. Observe that for $i < j$ the blow-up of $C = E_0$ does not change Σ_{E^i}.

Let $i \geq j$. Consider the complex $\Sigma_{E^i,\Delta_{0,i}}$ consisting of all faces Δ of Σ_{E^i} such that the corresponding intersection components $E(\Delta)$ are not contained in $E(\Delta_0)$. As before the simplices in $\Sigma^c_{E^i,\Delta_{0,i}} := \Sigma_{E^i} \setminus \Sigma_{E^i,\Delta_{0,i}}$ are the faces of $\Delta_{0,i} \in \Sigma_{E^i}$ with the property $E(\Delta) = E(\Delta_{0,i})$. It follows that

$$\Sigma^c_{E^{i+1},\Delta_{0,i+1}} \setminus \Sigma^c_{E^i,\Delta_{0,i}} \subset \Delta_{0,i+1} \setminus \Delta_{0,i}.$$

The complexes $\Sigma_{E^i,\Delta_{0,i}}$ will remain the same after blow-up at E_0. The components defined by the faces in $\Sigma^c_{E^i,\Delta_{0,i}}$ will disappear. The simplices of $\Sigma_{\Delta_{0,i}} = \Sigma_{E^i,\Delta_{0,i}} \cap \Delta_{0,i}$ will be joined with a new vertex v_0 corresponding to the exceptional divisor D_0. They will form a complex $v_0 * \Sigma_{\Delta_{0,i}}$.

Thus for $i \geq j$ the new complex has the form $\Sigma_{E'^i} = \Sigma_{E^i,\Delta_{0,i}} \cup \quad v_0 * \Sigma_{\Delta_{0,i}}$. It is a subcomplex of $v_0 \cdot \Sigma_{E'^i}$ with deformation retraction defined on

$$v_0 * \Delta_{0,j} \subseteq v_0 \cdot \Delta_{0,j+1} \subseteq \ldots \subseteq v_0 \cdot \Delta_{0,r}$$

to

$$v_0 * \Sigma_{\Delta_{0,j}} \subseteq v_0 * \Sigma_{\Delta_{0,j+1}} \subseteq \ldots \subseteq v_0 * \Sigma_{\Delta_{0,r}},$$

The construction of the deformation retraction is almost identical as before. Note that $\Sigma_{\Delta_{0,i}} \cap \Delta_{0,i'} = \Sigma_{\Delta_{0,i'}}$ for $i \geq i' \geq j$. First we define the retraction

$$v_0 \cdot \Delta_{0,r} \to v_0 * (\Sigma_{\Delta_{0,r}} \cup \Delta_{0,r-1}),$$

eliminating all the faces in $\Sigma^c_{E^r,\Delta_{0,r}} \setminus \Delta_{0,r-1}$, and identical on Δ_{r-1}. (We use the same technique as in the proof of the previous Lemma.)

Then we retract

$$v_0 * \Delta_{0,r-1} \to v_0 * (\Sigma_{\Delta_{0,r-1}} \cup \Delta_{0,r-2}).$$

It defines retraction
$$v_0 * (\Sigma_{\Delta_{0,r}} \cup \Delta_{0,r-1}) \to v_0 * (\Sigma_{\Delta_{0,r}} \cup \Delta_{0,r-2}),$$
and
$$v_0 * \Delta_{0,r} \to v_0 * (\Sigma_{\Delta_{0,r}} \cup \Delta_{0,r-2}),$$

We continue this process down for $i = r, r-1, \ldots, j$ to get desired compatible retractions $v_0 * \Delta_{0,i} \to v_0 * (\Sigma_{\Delta_{0,i}})$.

Finally extending the above retractions by the identity yields the retraction of
$$\Sigma_{E^1} \subset \Sigma_{E^2} \subset \ldots \subset \Sigma_{E^r}$$
to
$$\Sigma_{E'^1} \subset \Sigma_{E'^2} \subset \ldots \subset \Sigma_{E'^r}.$$
The rest of the proof is the same. □

Corollary 7.12. *Let $E = X \setminus U$ (respectively $E' = X' \setminus U'$) be a union of the closed SNC components $E = \bigcup E_i$ (resp. $E' = \bigcup E_i'$) on a nonsingular variety X (resp. X'). Let $E^1 \subset E^2 \subset \ldots \subset E^r = E$ be the union of some components E_i, (resp. $E'^1 \subset E'^2 \subset \ldots \subset E'^r = E'$ be the union of some components E_i'.)*

Assume that there exists a proper birational map $\phi : X \dashrightarrow X'$ such that the restrictions $\phi_{X\setminus E^j} : X \setminus E^j \dashrightarrow X' \setminus E'^j$ are proper for $j = 1, \ldots, k$.

Then the corresponding sequences of topological spaces $\Sigma_{E^1} \subset \Sigma_{E^2} \subset \ldots \subset \Sigma_{E^r}$ and $\Sigma_{E'^1} \subset \Sigma_{E'^2} \subset \ldots \subset \Sigma_{E'^r}$ are homotopically equivalent.

Proof. Follows from the Corollary 7.11 and Theorem 7.8. □

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