October 14. 2012

COMMUTING INVOLUTIONS OF LIE ALGEBRAS, COMMUTING VARIETIES, AND SIMPLE JORDAN ALGEBRAS

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ABSTRACT. Let σ_1 and σ_2 be commuting involutions of a connected reductive algebraic group G with $\mathfrak g = \mathsf{Lie}(G)$. Let $\mathfrak g = \bigoplus_{i,j=0,1} \mathfrak g_{ij}$ be the corresponding $\mathbb Z_2 \times \mathbb Z_2$-grading. If $\{\alpha, \beta, \gamma\} = \{01, 10, 11\}$, then $[\ ,\ ] : \mathfrak g_\alpha \times \mathfrak g_\beta \to \mathfrak g_\gamma$, and the zero-fibre of this bracket is called a $\vec{\boldsymbol\sigma}$-commuting variety. The commuting variety of $\mathfrak g$ and commuting varieties related to one involution are particular cases of this construction. We develop a general theory of such varieties and point out some cases, when they have especially good properties. If G/G^{σ_1} is a Hermitian symmetric space of tube type, then one can find three conjugate pairwise commuting involutions σ_1, σ_2, and $\sigma_3 = \sigma_1\sigma_2$. In this case, any $\vec{\boldsymbol\sigma}$-commuting variety is isomorphic to the commuting variety of the simple Jordan algebra associated with σ_1. As an application, we show that if $\mathcal J$ is the Jordan algebra of symmetric matrices, then the product map $\mathcal J \times \mathcal J \to \mathcal J$ is equidimensional; while for all other simple Jordan algebras equidimensionality fails.

CONTENTS

2010 *Mathematics Subject Classification.* 14L30, 17B08, 17B40, 17C20, 22E46.

Key words and phrases. Semisimple Lie algebra, commuting variety, Cartan subspace, quaternionic decomposition, nilpotent orbit, Jordan algebra.

INTRODUCTION

The ground field $\Bbbk$ is algebraically closed and $\mathsf{char}\,\Bbbk = 0$. Let G be a connected reductive algebraic group with $\mathsf{Lie}(G) = \mathfrak g$. In 1979, Richardson proved that any pair of commuting elements of $\mathfrak g$ can be approximated by pairs of commuting semisimple elements [19]. More precisely, if $\mathfrak t \subset \mathfrak g$ is a Cartan subalgebra (CSA for short), then

$$\{(x, y) \in \mathfrak g \times \mathfrak g \mid [x, y] = 0\} = \overline{G{\cdot}(\mathfrak t \times \mathfrak t)}, \tag{0·1}$$

where 'bar' means the Zariski closure. The LHS is called the *commuting variety* of $\mathfrak g$, denoted $\mathfrak C(\mathfrak g)$. That is, $\mathfrak C(\mathfrak g)$ is the zero-fibre of the multiplication map $\mathfrak g \times \mathfrak g \xrightarrow{[\ ,\ ]} \mathfrak g$. It follows from (0·1) that $\mathfrak C(\mathfrak g)$ is irreducible and $\dim \mathfrak C(\mathfrak g) = \dim \mathfrak g + \mathsf{rk}\,\mathfrak g$. For arbitrary Lie algebras, e.g. for Borel subalgebras of $\mathfrak g$, the commuting variety can be reducible [24, p. 237].

There are several directions for generalising Richardson's work.

First, for given subvarieties $U, V \subset \mathfrak g$, one can consider the restriction of $[\ ,\ ]$ to $U \times V$ and study properties of $\mathfrak C(\mathfrak g) \cap (U \times V)$. For instance:

– Let σ be an involution of $\mathfrak g$ with the corresponding $\mathbb Z_2$-grading $\mathfrak g = \mathfrak g_0 \oplus \mathfrak g_1$. Taking $U = V = \mathfrak g_1$ yields the commuting variety $\mathfrak C(\mathfrak g_1) := \mathfrak C(\mathfrak g) \cap (\mathfrak g_1 \times \mathfrak g_1)$, which was considered first in [14]. Here the structure of $\mathfrak C(\mathfrak g_1)$ heavily depends on σ. If $\mathfrak g_1$ contains a CSA of $\mathfrak g$, then $\mathfrak C(\mathfrak g_1)$ is an irreducible normal complete intersection [14]. At the other extreme, if the symmetric space G/G_0 is of rank 1, then $\mathfrak C(\mathfrak g_1)$ is often reducible. In [17], the question of irreducibility of $\mathfrak C(\mathfrak g_1)$ is resolved for all but three involutions of simple Lie algebras, and the remaining cases are settled in [3]. It seems, however, that there is no simple rule to distinguish the involutions for which $\mathfrak C(\mathfrak g_1)$ is irreducible.

– Another natural possibility is to take $U = V = \mathcal N$, where $\mathcal N$ is the set of nilpotent elements of $\mathfrak g$. This leads to the *nilpotent commuting variety* of $\mathfrak g$, $\mathfrak C(\mathcal N)$, which is often reducible. However, $\mathfrak C(\mathcal N)$ is equidimensional, $\dim \mathfrak C(\mathcal N) = \dim \mathfrak g$, and the structure of irreducible components is well understood [18].

– An interesting situation with $U \ne V$ occurs if $\mathfrak g = \oplus_{i\in\mathbb Z}\mathfrak g(i)$ is $\mathbb Z$-graded, $U = \mathfrak g(i)$, and $V = \mathfrak g(-i)$, see [15, Sect. 3].

Second, one may look at commuting varieties related to other types of algebras. If $\mathcal A$ is any algebra, then $\mathfrak C(\mathcal A)$ is defined to be the zero fibre of the multiplication map $\mathcal A \times \mathcal A \to \mathcal A$. It is a natural task to study the commuting variety of a simple Jordan algebra. As far as I know, this problem has not been addressed before.

In this article, we elaborate on both directions outlined above. We study certain "commuting varieties" associated with $\mathbb Z_2 \times \mathbb Z_2$-gradings of $\mathfrak g$ (the first direction). It turns out that, for some gradings, these new commuting varieties are isomorphic to the commuting variety of simple Jordan algebras (the second direction). To describe our results more precisely, we need some notation. Let σ_1 and σ_2 be different commuting involutions of a connected reductive algebraic group G. This yields a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{i,j=0,1} \mathfrak{g}_{ij}, \text{ where } \mathfrak{g}_{ij} = \{x \in \mathfrak{g} \mid \sigma_1(x) = (-1)^i x \ \& \ \sigma_2(x) = (-1)^j x\}. \tag{0·2}$$

Then σ_1, σ_2, and $\sigma_3 = \sigma_1\sigma_2$ are pairwise commuting involutions, and following [25] we say that (0·2) is a *quaternionic decomposition* of $\mathfrak{g}$. For, if (α, β, γ) is any permutation of the set of indices $\{01, 10, 11\}$, then $[\mathfrak{g}_{00}, \mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_\gamma$. The conjugacy classes of pairs of commuting involutions are classified, see [10] and references therein. Therefore, it is not difficult to write down explicitly all the quaternionic decompositions of simple Lie algebras. This article is a continuation of [16], where we developed some theory on Cartan subspaces related to (0·2) and studied invariants of degenerations of isotropy representations involved.

Set $\vec{\boldsymbol{\sigma}} = (\sigma_1, \sigma_2, \sigma_3)$, and let G_{00} denote the connected subgroup of G with Lie algebra $\mathfrak{g}_{00}$. A *$\vec{\boldsymbol{\sigma}}$-commuting variety* is the zero-fibre of the bracket $[\ ,\ ] : \mathfrak{g}_\alpha \times \mathfrak{g}_\beta \longrightarrow \mathfrak{g}_\gamma$. Associated with (0·2), one has three essentially different such varieties that are parameterised by the choice of $\gamma \in \{01, 10, 11\}$. All these mappings are G_{00}-equivariant, and all $\vec{\boldsymbol{\sigma}}$-commuting varieties are G_{00}-varieties. The above-mentioned varieties $\mathfrak{E}(\mathfrak{g}_1)$ can be obtained as a special case of this construction, see Example 3.1. We usually stick to one particular choice of the commutator, $\varphi : \mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{01}$, and try to realise what assumptions on $\vec{\boldsymbol{\sigma}}$ imply good properties of $\mathfrak{E} := \varphi^{-1}(0)$ and other fibres of φ. Clearly, φ can be regarded as a quadratic map from $\mathfrak{g}_{1\star} := \mathfrak{g}_{10} \oplus \mathfrak{g}_{11}$ to $\mathfrak{g}_{01}$. Let $\mathfrak{c}_{1\star}$ be a Cartan subspace (=CSS) in $\mathfrak{g}_{1\star}$. Say that $\mathfrak{c}_{1\star}$ is *homogeneous* if it is σ_2-stable (or, equivalently, σ_3-stable), i.e., if $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$ with $\mathfrak{a}_{1j} \subset \mathfrak{g}_{1j}$. We prove that

• if $\mathfrak{c}_{1\star}$ is a homogeneous CSS, then the closure of $G_{00}{\cdot}\mathfrak{c}_{1\star}$ is an irreducible component of $\mathfrak{E}$ (Theorem 3.4). (Such irreducible components are said to be *standard*). However, there can be several standard component, of different dimension; and there can also exist some "non-standard" irreducible components.

• All homogeneous CSS in $\mathfrak{g}_{1\star}$ are G_{00}-conjugate (i.e., $\mathfrak{E}$ has only one standard component) if and only if $\dim \mathfrak{c}_{1\star} = \dim \mathfrak{c}_{10} + \dim \mathfrak{c}_{11}$, where $\mathfrak{c}_{1j}$ are CSS in $\mathfrak{g}_{1j}$ (Theorem 3.7).

• φ is dominant if and only if there exist $x \in \mathfrak{g}_{10}, y \in \mathfrak{g}_{11}$ such that $\mathfrak{z}_\mathfrak{g}(x)_{01} \cap \mathfrak{z}_\mathfrak{g}(y)_{01} = \{0\}$.

However, one cannot expect really good properties for φ and $\mathfrak{E}$ without extra assumptions. One natural assumption is that some of involutions in $\vec{\boldsymbol{\sigma}}$ are conjugate. Another possibility is that some of the σ_i's possess prescribed properties. Our more specific results are:

(1) If σ_1, σ_2 are conjugate, then φ is surjective and $\dim \varphi^{-1}(\xi) \geqslant \dim \mathfrak{g}_{11}$ for all $\xi \in \mathfrak{g}_{01}$ (Proposition 3.8). We also provide a method for detecting subvarieties of $\mathfrak{E}$ whose dimension is larger than $\dim \mathfrak{g}_{11}$. This exploits certain restricted root systems related to decomposition (0·2), see Section 5.

(2) If σ_1, σ_2 are involutions of maximal rank (hence they are conjugate), then φ is surjective and equidimensional, each irreducible component of $\mathfrak{E}$ is standard, and the scheme $\varphi^{-1}(0)$ is a reduced complete intersection (Theorem 4.1).

(3) Let $\mathfrak{g}$ be simple and σ a Hermitian involution (i.e., $\mathfrak{g}^\sigma$ is not semisimple). If the Hermitian symmetric space G/G^σ is of tube type, then there exists a commuting triple $\vec{\boldsymbol{\sigma}}$ such that each σ_i is conjugate to σ, and in this case $\mathfrak{E}$ is isomorphic to the commuting variety of the corresponding simple Jordan algebra, see Section 6.

(4) The relationship with $\vec{\boldsymbol{\sigma}}$-commuting varieties implies that the multiplication map $\mathcal{J} \times \mathcal{J} \xrightarrow{\circ} \mathcal{J}$ is equidimensional if and only if $\mathcal{J}$ is the Jordan algebra of symmetric matrices. The commuting variety of a simple Jordan algebra $\mathcal{J}$ is reducible, since $\mathcal{J} \times \{0\}$ and $\{0\} \times \mathcal{J}$ are always irreducible components; and there are certainly some other components.

(5) Results stated in (2) rely on an interesting property of $\mathbb{Z}_2$-gradings. For any $e \in \mathfrak{g}_0$, its centraliser in $\mathfrak{g}$ is also $\mathbb{Z}_2$-graded: $\mathfrak{g}^e = \mathfrak{g}_0^e \oplus \mathfrak{g}_1^e$. Then we prove that

$$\dim \mathfrak{g}_0^e + \mathsf{rk}\, \mathfrak{g} \geqslant \dim \mathfrak{g}_1^e$$

and the equality occurs only if $e = 0$ and σ is of maximal rank. However, the proof of this inequality (Theorem 4.4) is not quite uniform, and a better proof is welcome! The required case-by-case calculations are lengthy and tedious, so that not all of them are actually presented, and a part of them is placed in Appendix A. We hope that an *a priori* proof of this inequality might be related to a geometric property of centralisers of nilpotent elements in $\mathfrak{g}_0$, see Conjecture 4.6.

– Throughout, G is a connected reductive algebraic group and $\mathfrak{g} = \mathsf{Lie}(G)$. Then $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ is the centraliser of a subspace $\mathfrak{a} \subset \mathfrak{g}$, and the centraliser of $x \in \mathfrak{g}$ is denoted by $\mathfrak{z}_{\mathfrak{g}}(x)$ or $\mathfrak{g}^x$.

– $\mathsf{R}(\lambda)$ is a simple finite-dimensional G-module with highest weight λ.

– Algebraic groups are denoted by capital Roman letters and their Lie algebras are denoted by the corresponding lower-case gothic letters.

1. Preliminaries on involutions and commuting varieties

The set of all involutions of $\mathfrak{g}$ is denoted by $\mathsf{Inv}(\mathfrak{g})$. The group of inner automorphisms $\mathsf{Int}(G) \simeq G/Z(G)$ acts on $\mathsf{Inv}(\mathfrak{g})$ by conjugation. Two involutions are said to be *conjugate*, if they lie in the same $\mathsf{Int}(G)$-orbit. If $\sigma \in \mathsf{Inv}(\mathfrak{g})$, then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the corresponding $\mathbb{Z}_2$-grading of $\mathfrak{g}$, where $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid \sigma(x) = (-1)^i x\}$. We also say that $(\mathfrak{g}, \mathfrak{g}_0)$ is a *symmetric pair*. Whenever we wish to stress that $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are determined by σ, we write $\mathfrak{g}^\sigma$ and $\mathfrak{g}_1^{(\sigma)}$ for them. We assume that σ is induced by an involution of G, which is denoted by the same letter. The connected subgroup of G with Lie algebra $\mathfrak{g}_0$ is denoted by G_0. Hence G_0 is the identity component of $G^\sigma = \{g \in G \mid \sigma(g) = g\}$. The representation of G_0 in $\mathfrak{g}_1$ is the *isotropy representation* of the symmetric space G/G_0.

We freely use invariant-theoretic results on the G_0-action on $\mathfrak{g}_1$ obtained in [11]. A *Cartan subspace* (=CSS) is a maximal subspace of $\mathfrak{g}_1$ consisting of pairwise commuting semisimple elements. The Cartan subspaces are characterised by the following property:

(1·1) *Suppose that a subspace* $\mathfrak{a} \subset \mathfrak{g}_1$ *consists of pairwise commuting semisimple elements. Then* $\mathfrak{a}$ *is a* CSS *if and only if* $\mathfrak{z}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{g}_1 = \mathfrak{a}$ [11, Ch. I].

An element $x \in \mathfrak{g}_1$ is called G_0*-regular* if the orbit $G_0{\cdot}x$ is of maximal dimension. Let $\mathfrak{c}$ be a CSS of $\mathfrak{g}_1$. Below, we summarise some basic properties of the Cartan subspaces and isotropy representations:

– All CSS of $\mathfrak{g}_1$ are G_0-conjugate and $G_0{\cdot}\mathfrak{c}$ is dense in $\mathfrak{g}_1$;
– Every semisimple element of $\mathfrak{g}_1$ is G_0-conjugate to an element of $\mathfrak{c}$;
– A semisimple element $x \in \mathfrak{g}_1$ is G_0-regular $\Leftrightarrow$ $\mathfrak{z}_\mathfrak{g}(x) \cap \mathfrak{g}_1$ is a CSS;
– The orbit $G_0{\cdot}x$ is closed if and only if x is semisimple;
– The closure of $G_0{\cdot}x$ contains the origin if and only if x is nilpotent;
– The number of nilpotent G_0-orbits in $\mathfrak{g}_1$ is finite.

- We say that $\sigma \in \mathsf{Inv}(\mathfrak{g})$ is *of maximal rank* if $\mathfrak{g}_1$ contains a Cartan subalgebra of $\mathfrak{g}$.

As is well known, (1) $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_0 \leqslant \mathsf{rk}\,\mathfrak{g}$ for any σ, and the equality holds if and only if σ is of maximal rank; (2) all involutions of maximal rank are conjugate; (3) the involutions of maximal rank are inner *if and only if* all exponents of $\mathfrak{g}$ are odd.

Lemma 1.1 ([11, Prop. 5]). *For any* $x \in \mathfrak{g}_1$, *one has* $\dim \mathfrak{g}_0 - \dim \mathfrak{g}_0^x = \dim \mathfrak{g}_1 - \dim \mathfrak{g}_1^x$. *Equivalently,* $\dim G{\cdot}x = 2 \dim G_0{\cdot}x$ *for all* $x \in \mathfrak{g}_1$.

Consequently, if σ is of maximal rank, then

$$\dim \mathfrak{g}_1^x = \dim \mathfrak{g}_0^x + \mathsf{rk}\,\mathfrak{g}. \qquad (1{\cdot}2)$$

The property of having maximal rank is inheritable in the following sense.

Lemma 1.2. *Let* σ *be of maximal rank and* $x \in \mathfrak{g}_1$ *semisimple. Then the restriction of* σ *to* $\mathfrak{g}^x$ *and* $[\mathfrak{g}^x, \mathfrak{g}^x]$ *is also of maximal rank.*

The *commuting variety* associated with σ is

$$\mathfrak{C}(\mathfrak{g}_1) = \{(x, y) \in \mathfrak{g}_1 \times \mathfrak{g}_1 \mid [x, y] = 0\}. \qquad (1{\cdot}3)$$

That is, $\mathfrak{C}(\mathfrak{g}_1)$ is the zero-fibre of the commutator map $[\ ,\ ]_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. It is known that

- $\overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})}$ is always an irreducible component of $\mathfrak{C}(\mathfrak{g}_1)$ [14, Prop. 3.7];
- if σ is of maximal rank, then $\overline{G_0{\cdot}(\mathfrak{c} \times \mathfrak{c})} = \mathfrak{C}(\mathfrak{g}_1)$ and $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is equidimensional [14, Theorem 3.2]; moreover, all the fibres of $[\ ,\ ]_1$ are irreducible and normal [14, Cor. 4.4].
- $\mathfrak{C}(\mathfrak{g}_1)$ can be reducible [14, Example 3.5].

Example 1.3. Suppose that $\tilde{\mathfrak g} = \mathfrak g \oplus \mathfrak g$ and $\sigma(x, y) = (y, x)$. Then $\tilde{\mathfrak g}_0 = \Delta(\mathfrak g)$ and $\tilde{\mathfrak g}_1 = \{(x, -x) \mid x \in \mathfrak g\}$. Here the commutator $\tilde{\mathfrak g}_1 \times \tilde{\mathfrak g}_1 \to \tilde{\mathfrak g}_0$ coincides with the usual commutator $\mathfrak g \times \mathfrak g \to \mathfrak g$ and $\mathfrak E(\tilde{\mathfrak g}_1)$ is isomorphic to the usual commuting variety of a semisimple Lie algebra $\mathfrak g$. By a result of Richardson [19], $\mathfrak E(\mathfrak g)$ is irreducible and $\dim \mathfrak E(\mathfrak g) = \dim \mathfrak g + \mathsf{rk}\mathfrak g$.

A torus S of G is called *σ-anisotropic*, if $\sigma(s) = s^{-1}$ for all $s \in S$. All maximal σ-anisotropic tori are G_0-conjugate, and if $C \subset G$ is a maximal σ-anisotropic torus, then $\mathsf{Lie}(C)$ is a CSS in $\mathfrak g_1$. Recall that a *restricted root* of C is any non-trivial weight in the decomposition of $\mathfrak g$ into the sum of weight spaces of C. Write $\Psi^C(G/G_0)$ or just $\Psi(G/G_0)$ for the set of all restricted roots. Then

$$\mathfrak g = \mathfrak g^C \oplus \Big(\bigoplus_{\gamma \in \Psi(G/G_0)} \mathfrak g_\gamma \Big). \tag{1·4}$$

We use the additive notation for the operation in $\mathfrak X(C)$, the character group of C, and regard $\Psi(G/G_0)$ as a subset of the vector space $\mathfrak X(C) \otimes_{\mathbb Z} \mathbb R$. The set $\Psi(G/G_0)$ satisfies the usual axioms of finite root systems [6]. The notable difference from the structure theory of split semisimple Lie algebras is that the root system $\Psi(G/G_0)$ can be non-reduced and that multiplicities $m_\gamma = \dim \mathfrak g_\gamma$ ($\gamma \in \Psi(G/G_0)$) can be greater than 1.

For all involutions of simple Lie algebras, the restricted root systems and the respective multiplicities are known, see [6, Ch. X, Table VI].

2. Commuting involutions and quaternionic decompositions

Let σ_1 and σ_2 be different commuting involutions of $\mathfrak g$. The corresponding $\mathbb Z_2 \times \mathbb Z_2$-grading of $\mathfrak g$ is:

$$\mathfrak g = \bigoplus_{i,j=0,1} \mathfrak g_{ij}, \ \text{ where } \ \mathfrak g_{ij} = \{x \in \mathfrak g \mid \sigma_1(x) = (-1)^i x \ \ \& \ \ \sigma_2(x) = (-1)^j x\}. \tag{2·1}$$

We also say that it is a *quaternionic decomposition* of $\mathfrak g$ (determined by σ_1 and σ_2). Set $\sigma_3 := \sigma_1\sigma_2$ and $\vec{\boldsymbol\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. The pairwise commuting involutions σ_1, σ_2, and σ_3 are said to be *big*. The induced involutions on the fixed-point subalgebras $\mathfrak g^{\sigma_1}, \mathfrak g^{\sigma_2}, \mathfrak g^{\sigma_3}$ are said to be *little*. The same terminology applies to the corresponding $\mathbb Z_2$-gradings, isotropy representations, and CSS. Thus, associated with (2·1), one has three big and three little $\mathbb Z_2$-gradings. It is convenient for us to organise the summands of (2·1) in a 2×2 "matrix":

$$\mathfrak g = \begin{array}{c:cl} \mathfrak g_{00} & \mathfrak g_{01} & \\ \hdashline \mathfrak g_{10} & \mathfrak g_{11} & \sigma_1 \\ \sigma_2 & & \end{array} \tag{2·2}$$

Here the horizontal (resp. vertical) dotted line separates the eigenspaces of σ_1 (resp. σ_2), whereas two diagonals of this matrix represent the eigenspaces of σ_3. Hence the first row, first column, and the main diagonal represent the three little $\mathbb{Z}_2$-gradings (of $\mathfrak{g}^{\sigma_1}$, $\mathfrak{g}^{\sigma_2}$, and $\mathfrak{g}^{\sigma_3}$, respectively).

We repeatedly use the following notation for the eigenspaces of σ_1 and σ_2:

$$\mathfrak{g}^{\sigma_1} = \mathfrak{g}_{0\star} := \mathfrak{g}_{00} \oplus \mathfrak{g}_{01},\ \mathfrak{g}_{1\star} := \mathfrak{g}_{10} \oplus \mathfrak{g}_{11}, \qquad \mathfrak{g}^{\sigma_2} = \mathfrak{g}_{\star 0} := \mathfrak{g}_{00} \oplus \mathfrak{g}_{10},\ \mathfrak{g}_{\star 1} := \mathfrak{g}_{01} \oplus \mathfrak{g}_{11}.$$

Likewise, $G_{0\star}$ (resp. $G_{\star 0}$) is the connected subgroup of G corresponding to $\mathfrak{g}_{0\star}$ (resp. $\mathfrak{g}_{\star 0}$), G_{00} is the connected subgroup of G corresponding to $\mathfrak{g}_{00}$, etc. If $\mathfrak{q}$ is a $\vec{\boldsymbol{\sigma}}$-stable subalgebra of $\mathfrak{g}$, then $\mathfrak{q} = \bigoplus_{i,j} \mathfrak{q}_{ij}$ is the induced quaternionic decomposition of $\mathfrak{q}$, and Q, Q_{00} are the corresponding connected subgroups.

Following Vinberg [28, 0.3], we say that a triple $\{\sigma_1, \sigma_2, \sigma_3\} \subset \mathsf{Inv}(\mathfrak{g})$ is a *triad* if all three involutions are conjugate and $\sigma_1\sigma_2 = \sigma_3$. A complete classification of triads is obtained in [28, Sect. 3]. The triads lead to the "most symmetric" quaternionic decompositions. In [16], we considered less restrictive conditions on the σ_i's. We say that $\{\sigma_1, \sigma_2\} \subset \mathsf{Inv}(\mathfrak{g})$ is a *dyad* if σ_1, σ_2 are conjugate and $\sigma_1\sigma_2 = \sigma_2\sigma_1$ (no conditions on σ_3!).

The product of two conjugate involutions (not necessarily commuting) is always an inner automorphism of $\mathfrak{g}$. For, if $\sigma_2 = \mathrm{Int}(g){\cdot}\sigma_1{\cdot}\mathrm{Int}(g^{-1})$, then $\sigma_1\sigma_2 = \mathrm{Int}(\sigma_1(g)g^{-1})$. Therefore, any triad consists of inner involutions (but not any inner involution gives rise to a triad!). However, any involution can be a member of a dyad [16, Prop. 2.4]. But the third involution, σ_3, is then necessarily inner.

Proposition 2.1 (see [16, Prop. 2.2(1)]). *Suppose that $\mu \in \mathsf{Inv}(\mathfrak{g})$ is inner. Then there are commuting involutions of maximal rank, σ_1 and σ_2, such that $\mu = \sigma_1\sigma_2$. Moreover, σ_1 and σ_2 induce an involution of maximal rank of $\mathfrak{g}^\mu$.*

For $(ij) \ne (00)$, let $\mathfrak{c}_{ij}$ be a CSS of $\mathfrak{g}_{ij}$; that is, a little CSS related to the little $\mathbb{Z}_2$-grading $\mathfrak{g}_{00} \oplus \mathfrak{g}_{ij}$. There are also big CSS in the (-1)-eigenspaces of three big involutions:

$$\mathfrak{c}_{1\star} \subset \mathfrak{g}_{1\star},\ \mathfrak{c}_{\star 1} \subset \mathfrak{g}_{\star 1},\ \mathfrak{c}_{\star,1-\star} \subset \mathfrak{g}_{\star,1-\star} := \mathfrak{g}_{01} \oplus \mathfrak{g}_{10}.$$

Each little CSS can be included in two big CSS. E.g., because $\mathfrak{g}_{10} \subset \mathfrak{g}_{1\star}$ and $\mathfrak{g}_{10} \subset \mathfrak{g}_{\star,1-\star}$, one can choose Cartan subspaces $\mathfrak{c}_{1\star}$ and $\mathfrak{c}_{\star,1-\star}$ such that $\mathfrak{c}_{10} \subset \mathfrak{c}_{1\star}$ and $\mathfrak{c}_{10} \subset \mathfrak{c}_{\star,1-\star}$. If at least one equality occurs among all such inclusions, then this will be referred to as a *coincidence* of CSS (for a given quaternionic decomposition).

In [16], we obtained two sufficient conditions for a coincidence of CSS:

Theorem 2.2 (see [16, Thm. 3.3 & 3.7]).

(1) *Suppose that σ_1 is of maximal rank. Then*
   - *any little CSS $\mathfrak{c}_{11} \subset \mathfrak{g}_{11}$ is also a CSS in $\mathfrak{g}_{\star 1}$, i.e., for σ_2;*
   - *any little CSS $\mathfrak{c}_{10} \subset \mathfrak{g}_{10}$ is also a CSS in $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01}$, i.e., for σ_3.*

(2) *Suppose that $\{\sigma_1, \sigma_2\}$ is a dyad. Then any little CSS $\mathfrak{c}_{11} \subset \mathfrak{g}_{11}$ is also a CSS in $\mathfrak{g}_{1\star}$ or $\mathfrak{g}_{\star 1}$, i.e., for σ_1 or σ_2.*

The coincidences of CSS in Theorem 2.2(2) can formally be expressed as $\mathfrak{c}_{11} = \mathfrak{c}_{1\star}$ or $\mathfrak{c}_{11} = \mathfrak{c}_{\star 1}$, and likewise in all other possible cases. In view of (1·1), any coincidence of CSS can be restated as certain property of the little CSS in question. For instance, the first coincidence in Theorem 2.2(1) means that if $x \in \mathfrak{g}_{11}$ is a generic semisimple element (i.e., x belong to a unique little CSS), then $\mathfrak{z}_{\mathfrak{g}}(x)_{\star 1} = \mathfrak{z}_{\mathfrak{g}}(x)_{11} = \mathfrak{c}_{11}$, and hence $\mathfrak{z}_{\mathfrak{g}}(x)_{01} = 0$.

3. Commuting varieties and homogeneous Cartan subspaces

Consider a quaternionic decomposition (2·2). For any permutation (α, β, γ) of the set $\{01, 10, 11\}$, there is the commutator mapping $\varphi^{\gamma}_{\alpha,\beta} : \mathfrak{g}_\alpha \times \mathfrak{g}_\beta \to \mathfrak{g}_\gamma$. Clearly, $\varphi^{\gamma}_{\alpha,\beta}$ is G_{00}-equivariant. As our main interest is in fibres of this mapping, we do not distinguish $\varphi^{\gamma}_{\alpha,\beta}$ and $\varphi^{\gamma}_{\beta,\alpha}$. We concentrate on the following problems:

- When is $\varphi^{\gamma}_{\alpha,\beta}$ dominant?
- What is the dimension of $(\varphi^{\gamma}_{\alpha,\beta})^{-1}(0)$?
- How to describe the irreducible components of $(\varphi^{\gamma}_{\alpha,\beta})^{-1}(0)$?
- When is $\varphi^{\gamma}_{\alpha,\beta}$ equidimensional?

The variety $\mathfrak{C}^{\gamma}_{\alpha,\beta} = (\varphi^{\gamma}_{\alpha,\beta})^{-1}(0)$ is said to be a $\vec{\boldsymbol{\sigma}}$*-commuting variety*. For general quaternionic decompositions, one has three such varieties, and their properties can be rather different. We mainly restrict ourselves with considering the test case:

$$\varphi = \varphi^{01}_{10,11} : \mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{01}. \tag{3·1}$$

and also write $\mathfrak{C}$ in place of $\mathfrak{C}^{01}_{10,11}$. Note that we can regard φ as a quadratic map from $\mathfrak{g}_{1\star}$ to $\mathfrak{g}_{01}$, and $\mathfrak{C}$ as subvariety of $\mathfrak{g}_{1\star}$. The following example shows that the commuting variety in (1·3) is a particular case of this construction.

Example 3.1. Let $\mathfrak{g}$ be a reductive Lie algebra and σ an involution of $\mathfrak{g}$ with the corresponding $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Set $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}$ and define three involutions of $\tilde{\mathfrak{g}}$ as follows:

$$\sigma_1(x_1, x_2) = (\sigma(x_1), \sigma(x_2)), \quad \sigma_2(x_1, x_2) = (x_2, x_1), \quad \sigma_3 = \sigma_1\sigma_2.$$

Then $\tilde{\mathfrak{g}}^{\sigma_1} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$; $\tilde{\mathfrak{g}}^{\sigma_2} = \Delta(\mathfrak{g})$, the diagonal in $\mathfrak{g} \oplus \mathfrak{g}$; $\tilde{\mathfrak{g}}^{\sigma_3} = \{(x, \sigma(x)) \mid x \in \mathfrak{g}\}$.
Set $\Delta_-(M) := \{(m, -m) \mid m \in M\}$ for any subspace $M \subset \mathfrak{g}$. Then the corresponding quaternionic decomposition is:

$$\tilde{\mathfrak{g}} = \begin{array}{c|c} \Delta(\mathfrak{g}_0) & \Delta_-(\mathfrak{g}_0) \\ \hline \Delta(\mathfrak{g}_1) & \Delta_-(\mathfrak{g}_1) \end{array} \quad \sigma_1 \qquad \sigma_2$$

Upon the obvious identifications $\Delta(\mathfrak{g}_1) \simeq \Delta_-(\mathfrak{g}_1) \simeq \mathfrak{g}_1$, etc., our test commutator map $\tilde{\mathfrak{g}}_{10} \times \tilde{\mathfrak{g}}_{11} \to \tilde{\mathfrak{g}}_{01}$ becomes the commutator $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ associated with $\sigma \in \mathsf{Inv}(\mathfrak{g})$; whereas two other commutator maps are identified with the bracket $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$. Therefore, the concept of a $\vec{\boldsymbol{\sigma}}$-commuting variety provides a uniform setting for studying the fibres of both $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ and $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$.

Lemma 3.2. *Commutator map (3·1) is dominant if and only if there exist $x \in \mathfrak{g}_{10}$ and $y \in \mathfrak{g}_{11}$ such that $\mathfrak{z}_{\mathfrak{g}}(x)_{01} \cap \mathfrak{z}_{\mathfrak{g}}(y)_{01} = \{0\}$.*

Proof. A morphism of irreducible varieties is dominant if and only if its differential at some point is onto. As φ is bilinear, an easy computation shows that $d\varphi_{(x,y)}(\xi, \eta) = [x, \eta] + [\xi, y]$, $\xi \in \mathfrak{g}_{10}$, $\eta \in \mathfrak{g}_{11}$. Hence $\operatorname{Im} d\varphi_{(x,y)} = [\mathfrak{g}_{11}, x] + [\mathfrak{g}_{10}, y]$, and taking the orthogonal complement with respect to the restriction of the Killing form to $\mathfrak{g}_{01}$ yields $(\operatorname{Im} d\varphi_{(x,y)})^{\perp} = \mathfrak{z}_{\mathfrak{g}}(x)_{01} \cap \mathfrak{z}_{\mathfrak{g}}(y)_{01}$. □

As we see below, certain CSS in $\mathfrak{g}_{1\star}$ play an important role in describing irreducible components of $\mathfrak{E}$.

Definition 1. A big Cartan subspace $\mathfrak{c}_{1\star} \subset \mathfrak{g}_{1\star}$ is said to be *homogeneous* if it is σ_2-stable (or, equivalently, σ_3-stable). In other words, if one has $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$ with $\mathfrak{a}_{1j} \subset \mathfrak{g}_{1j}$.

Remark. A coincidence of CSS means that there is a homogeneous CSS of special form. For instance, if $\mathfrak{c}_{11} = \mathfrak{c}_{1\star}$, then $\mathfrak{c}_{11}$ is a homogeneous CSS in $\mathfrak{g}_{1\star}$, with trivial $\mathfrak{g}_{10}$-component.

Lemma 3.3. *(1) Homogeneous* CSS *always exist.*
(2) Moreover, if $x \in \mathfrak{g}_{10}, y \in \mathfrak{g}_{11}$ are commuting semisimple elements, then there exists a homogeneous CSS *in $\mathfrak{g}_{1\star}$ containing both of them.*

Proof. 1) Take a little CSS $\mathfrak{c}_{10}$ and consider the $\vec{\boldsymbol{\sigma}}$-stable reductive subalgebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{10})$. If $\tilde{\mathfrak{a}}_{11}$ is a little CSS in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{10})_{11}$, then $\mathfrak{c}_{10} \oplus \tilde{\mathfrak{a}}_{11}$ is a homogeneous CSS in $\mathfrak{g}_{1\star}$.
2) Consider the $\vec{\boldsymbol{\sigma}}$-stable reductive subalgebra $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(x) \cap \mathfrak{z}_{\mathfrak{g}}(y)$. By the previous part, there exists a homogeneous CSS in $\mathfrak{l}_{1\star}$, say $\tilde{\mathfrak{c}}_{1\star}$. Since x, y are central in $\mathfrak{l}$, we have $x, y \in \tilde{\mathfrak{c}}_{1\star}$. It is also clear that $\tilde{\mathfrak{c}}_{1\star}$ is a CSS in $\mathfrak{g}_{1\star}$. □

If $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$ is a homogeneous CSS, then $[\mathfrak{a}_{01}, \mathfrak{a}_{11}] = 0$ and hence $\overline{G_{00}{\cdot}\mathfrak{c}_{1\star}} \subset \mathfrak{E}$. However, a stronger result is true.

Theorem 3.4.

(i) *Let $\mathfrak{c}_{1\star}$ be a homogeneous* CSS *in $\mathfrak{g}_{1\star}$. Then $\overline{G_{00}{\cdot}\mathfrak{c}_{1\star}} \subset \mathfrak{E}$ is an irreducible component of $\mathfrak{E}$.*
(ii) *If two homogeneous* CSS *in $\mathfrak{g}_{1\star}$ are not G_{00}-conjugate, then the corresponding irreducible components are different.*

Proof. (i) The centraliser of $\mathfrak{c}_{1\star}$ is $\vec{\boldsymbol{\sigma}}$-stable. Hence $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{1\star}) = \bigoplus_{i,j=0,1} \mathfrak{a}_{ij}$, and here $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$. Recall that $\overline{G_{0\star}{\cdot}\mathfrak{c}_{1\star}} = \mathfrak{g}_{1\star}$. Therefore, $\dim \mathfrak{c}_{1\star} + \dim G_{0\star} - \dim \mathfrak{a}_{00} - \dim \mathfrak{a}_{01} = \dim \mathfrak{g}_{1\star}$. It follows that

$$\dim \overline{G_{00}{\cdot}\mathfrak{c}_{1\star}} = \dim \mathfrak{c}_{1\star} + \dim G_{00} - \dim \mathfrak{a}_{00} = \dim \mathfrak{g}_{1\star} - \dim \mathfrak{g}_{01} + \dim \mathfrak{a}_{01}. \tag{3·2}$$

On the other hand, let $x+y\in\mathfrak{c}_{1\star}$ ($x\in\mathfrak{g}_{10}, y\in\mathfrak{g}_{11}$). The proof of Lemma 3.2 shows that $\dim(\mathrm{Im}\, d\varphi_{(x,y)})=\dim\mathfrak{g}_{01}-\dim(\mathfrak{z}_{\mathfrak{g}}(x)_{01}\cap\mathfrak{z}_{\mathfrak{g}}(y)_{01})$. Now, if $x+y\in\mathfrak{c}_{1\star}$ is generic, then $\mathfrak{z}_{\mathfrak{g}}(x)\cap\mathfrak{z}_{\mathfrak{g}}(y)=\mathfrak{z}_{\mathfrak{g}}(x+y)=\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{1\star})$. Hence $\dim(\mathrm{Im}\, d\varphi_{(x,y)})=\dim\mathfrak{g}_{01}-\dim\mathfrak{a}_{01}$. This means that any irreducible component of $\mathfrak{E}$ containing (x,y) has dimension at most

$$\dim\mathfrak{g}_{1\star}-\dim(\mathrm{Im}\, d\varphi_{(x,y)})=\dim\mathfrak{g}_{1\star}-\dim\mathfrak{g}_{01}+\dim\mathfrak{a}_{01}.$$

Comparing with (3·2) shows that $\overline{G_{00}{\cdot}\mathfrak{c}_{1\star}}$ is an irreducible component of $\mathfrak{E}$ containing (x,y), and (x,y) is a smooth point of $\overline{G_{00}{\cdot}\mathfrak{c}_{1\star}}$.

(ii) As we have just shown, if $(x,y)\in\mathfrak{c}_{1\star}$ is generic, then it belongs to a unique irreducible component of $\mathfrak{E}$ (and to a unique CSS in $\mathfrak{g}_{1\star}$). □

Claim 3.5. *The number of G_{00}-orbits of homogeneous* CSS *in $\mathfrak{g}_{1\star}$ is finite.*

First proof. Since the number of irreducible components is finite, this readily follows from Theorem 3.4. However, this can also be proved in a different way. As the second proof has its own merits, we provide it below.

Second proof. Recall that $G_{00}\subset G_{0\star}$ are connected reductive groups and <u>all</u> big CSS in $\mathfrak{g}_{1\star}$ form a single $G_{0\star}$-orbit. Let $\mathfrak{c}_{1\star}$ be a homogeneous CSS. Set

$$N=\{g\in G_{0\star}\mid g{\cdot}\mathfrak{c}_{1\star}=\mathfrak{c}_{1\star}\},$$
$$\mathcal{M}=\{g\in G_{0\star}\mid g{\cdot}\mathfrak{c}_{1\star} \text{ is homogeneous }\}.$$

Note that N is reductive, but not connected, since N is mapped onto the (finite) little Weyl group associated with $\mathfrak{c}_{1\star}$. If $g\in\mathcal{M}$, $s\in G_{00}$, and $z\in N$, then $sgz\in\mathcal{M}$. Therefore, $\mathcal{M}$ is a union of (G_{00},N)-cosets, and our task is to prove that $G_{00}\backslash\mathcal{M}/N$ is finite.

If $g\in\mathcal{M}$, then $g{\cdot}\mathfrak{c}_{1\star}=\sigma_2(g)\mathfrak{c}_{1\star}$. Hence $g^{-1}\sigma_2(g)\in N$. Since $G_{00}\subset G^{\sigma_2}$, the map

$$\psi_{\mathcal{M}}: G_{00}\backslash\mathcal{M}\to N, \quad G_{00}g\mapsto g^{-1}\sigma_2(g)$$

is well-defined. Note that N is σ_2-stable and the range of $\psi_{\mathcal{M}}$ belongs to the closed subset

$$\mathcal{Q}=\mathcal{Q}(N)=\{g\in N\mid \sigma_2(g)=g^{-1}\}.$$

The twisted N-action on N is defined by $z\star x=zx\sigma_2(z)^{-1}$. Obviously, $\mathcal{Q}$ is stable under the twisted action of N. Moreover, $\psi_{\mathcal{M}}(gz)=z^{-1}\psi_{\mathcal{M}}(g)\sigma_2(z)$. Hence $\mathrm{Im}\,(\varphi_{\mathcal{M}})\subset\mathcal{Q}$ is the union of twisted N-orbits, and each twisted N-orbit gives rise to a G_{00}-orbit of homogeneous CSS. It follows from [21, Sect. 9] that $\mathcal{Q}$ is a finite union of twisted N-orbits, which is sufficient for our purpose. (See also remark below.) □

Remark 3.6. Richardson's results on twisted orbits [21, Sect. 9], specifically Proposition 9.1, are stated for a *connected* reductive group G, whereas we apply them to the reductive non-connected group N (in place of G). But his argument can easily be adjusted to cover the case ofvnon-connected reductive groups. That is, one can give a version of Richardson's Proposition 9.1 for non-connected groups G.

Definition 2. For a homogeneous CSS $\mathfrak{c}_{1\star} \subset \mathfrak{g}_{1\star}$, the irreducible component $\overline{G_{00}{\cdot}\mathfrak{c}_{1\star}} \subset \mathfrak{E}$ is said to be *standard*.

Since all big CSS in $\mathfrak{g}_{1\star}$ are $G_{0\star}$-conjugate, their centralisers in $\mathfrak{g}_{0\star}$ are essentially "the same". The centraliser in $\mathfrak{g}_{0\star}$ of a homogeneous CSS splits, and these splittings can be quite different. That is, $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{1\star})_{01}$ can be different for different homogeneous CSS, and this leads to a new phenomenon that standard irreducible components of $\mathfrak{E}$ may have different dimensions, cf. (3·2). Moreover, there can also be some "non-standard" irreducible components of $\mathfrak{E}$ that contain no semisimple elements at all.

By Theorem 3.4, a necessary condition for $\mathfrak{E}$ to be irreducible is that all homogeneous CSS in $\mathfrak{g}_{1\star}$ are G_{00}-conjugate, i.e., there is only one standard component. If $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$ is a homogeneous CSS with $\dim \mathfrak{a}_{1i} = d_i$, then (d_0, d_1) is called the *dimension vector*. Obviously, two homogeneous CSS with different dimension vectors are not G_{00}-conjugate.

Theorem 3.7. *1) If* $\mathfrak{c}_{1\star} = \mathfrak{a}_{10} \oplus \mathfrak{a}_{11}$ *is a homogeneous* CSS *with dimension vector* (d_0, d_1)*, then* $d_0 \leqslant \dim \mathfrak{c}_{10}$ *and* $d_1 \leqslant \dim \mathfrak{c}_{11}$*; hence* $\dim \mathfrak{c}_{1\star} \leqslant \dim \mathfrak{c}_{10} + \dim \mathfrak{c}_{11}$.

2) All homogeneous CSS *in* $\mathfrak{g}_{1\star}$ *are* G_{00}*-conjugate if and only if* $\dim \mathfrak{c}_{1\star} = \dim \mathfrak{c}_{10} + \dim \mathfrak{c}_{11}$.

Proof. 1) Being a toral subalgebra of $\mathfrak{g}_{1j}$, $\mathfrak{a}_{1j}$ is contained in a little CSS in $\mathfrak{g}_{1j}$.

2) "if" part: Let $\mathfrak{c}_{1\star}$ and $\tilde{\mathfrak{c}}_{1\star} = \tilde{\mathfrak{a}}_{10} \oplus \tilde{\mathfrak{a}}_{11}$ be two homogeneous CSS. By part 1), $\dim \mathfrak{a}_{01} = \dim \tilde{\mathfrak{a}}_{01} = \dim \mathfrak{c}_{10}$. Therefore, both $\mathfrak{a}_{01}$ and $\tilde{\mathfrak{a}}_{01}$ are little CSS, they are G_{00}-conjugate, and we may assume that $\mathfrak{a}_{01} = \tilde{\mathfrak{a}}_{01}$. Consider then the $\vec{\boldsymbol{\sigma}}$-stable reductive algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{10})$. As $\mathfrak{a}_{10}$ is a central toral subalgebra, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_{10}) = \mathfrak{a}_{10} \oplus \mathfrak{s}$, where $\mathfrak{s}$ is reductive and $\vec{\boldsymbol{\sigma}}$-stable. By construction, $\mathfrak{s}_{10} = \{0\}$ and $\mathfrak{a}_{11}, \tilde{\mathfrak{a}}_{11} \subset \mathfrak{s}_{11}$. Moreover, these are little CSS in $\mathfrak{s}_{11}$ (otherwise, $\mathfrak{c}_{1\star}$ or $\tilde{\mathfrak{c}}_{1\star}$ wouldn't be maximal). Therefore, $\mathfrak{a}_{01}$ and $\tilde{\mathfrak{a}}_{01}$ are S_{00}-conjugate, which implies that $\mathfrak{c}_{1\star}$ or $\tilde{\mathfrak{c}}_{1\star}$ are G_{00}-conjugate.

"only if" part: Assuming that $\dim \mathfrak{c}_{1\star} < \dim \mathfrak{c}_{10} + \dim \mathfrak{c}_{11}$, we construct two homogeneous CSS with different dimension vectors. First, let us take a little CSS $\mathfrak{c}_{10}$ and choose a little CSS in $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{10})_{11}$, say $\tilde{\mathfrak{a}}_{11}$. This yields a homogeneous CSS with dimension vector $(\dim \mathfrak{c}_{10}, \dim \mathfrak{c}_{1\star} - \dim \mathfrak{c}_{10})$. On the other hand, one can start with a little CSS $\mathfrak{c}_{11}$, etc., which yields a homogeneous CSS with dimension vector $(\dim \mathfrak{c}_{1\star} - \dim \mathfrak{c}_{11}, \dim \mathfrak{c}_{11})$. ☐

Note that $\dim \mathfrak{c}_{ij} > 0$ whenever $\mathfrak{g}_{ij} \neq \{0\}$. Therefore, a coincidence of CSS of the form $\mathfrak{c}_{11} = \mathfrak{c}_{1\star}$ or $\mathfrak{c}_{10} = \mathfrak{c}_{1\star}$ certainly excludes the possibility to have a unique standard component of $\mathfrak{E}$. For our test commutator (3·1), one may envisage several samples of good behaviour (not necessarily altogether):

(1) All irreducible components of $\mathfrak{E}$ are standard (possibly of different dimension);

(2) φ is surjective and equidimensional; hence, flat;

(3) $\mathfrak{E}$ has a unique standard component, but also may be some other components.

Property (3) always holds in the setting of Example 3.1, with any σ; and for σ of maximal rank, one gets a rare situation, where all three properties are satisfied. All quaternionic decompositions of simple Lie algebras can be written out explicitly, and then the presence of (3) amounts to a routine verification of the equality in Theorem 3.7(2).

Proposition 3.8. *Let $\{\sigma_1,\sigma_2\}$ be a dyad. Then $\dim\mathfrak g_{10}=\dim\mathfrak g_{01}$ and $\varphi:\mathfrak g_{10}\times\mathfrak g_{11}\to\mathfrak g_{01}$ is onto. (Therefore, $\dim\varphi^{-1}(\xi)\geqslant\dim\mathfrak g_{11}$ for all $\xi\in\mathfrak g_{01}$.) Moreover, $\{0\}\times\mathfrak g_{11}$ is a standard irreducible component of $\mathfrak E$ of minimal dimension.*

Proof. Since $\dim\mathfrak g^{\sigma_1}=\dim\mathfrak g^{\sigma_2}$, we have $\dim\mathfrak g_{10}=\dim\mathfrak g_{01}$. By Theorem 2.2(2), any little CSS $\mathfrak c_{11}\subset\mathfrak g_{11}$ is also a big CSS in $\mathfrak g_{1\star}$. Therefore, $\mathfrak c_{11}$ is a homogeneous CSS and $\overline{G_{00}{\cdot}\mathfrak c_{11}}=\mathfrak g_{11}$ is an irreducible component of $\mathfrak E$. Furthermore, if $x\in\mathfrak c_{11}$ is generic, then $\mathfrak z_{\mathfrak g}(x)\cap\mathfrak g_{1\star}=\mathfrak c_{11}$, i.e., $\mathfrak z_{\mathfrak g}(x)\cap\mathfrak g_{10}=\{0\}$. Therefore, $\dim[\mathfrak g_{10},x]=\dim\mathfrak g_{10}$, i.e., $[\mathfrak g_{10},x]=\mathfrak g_{01}$. □

4. Dyads of maximal rank and commuting varieties

Let $\{\sigma_1,\sigma_2\}$ be a dyad of maximal rank, i.e., both σ_1,σ_2 are of maximal rank. Recall that this implies that $\sigma_3=\sigma_1\sigma_2$ is inner, $\dim\mathfrak g_{01}=\dim\mathfrak g_{10}$, and, by Prop. 2.1, $\mathfrak g^{\sigma_3}=\mathfrak g_{00}\oplus\mathfrak g_{11}$ is a $\mathbb Z_2$-grading of maximal rank. In particular, $\mathfrak g_{11}$ contains a CSA of $\mathfrak g$ and any CSS in $\mathfrak g_{1\star}$ or $\mathfrak g_{\star 1}$ is a CSA. The main result of this section is

Theorem 4.1. *Let $\{\sigma_1,\sigma_2\}$ be a dyad of maximal rank. Then*

(i) *the commutator mapping $\varphi:\mathfrak g_{10}\times\mathfrak g_{11}\to\mathfrak g_{01}$ is surjective and equidimensional;*
(ii) *each irreducible component of $\mathfrak E=\varphi^{-1}(0)$ is standard, i.e., is the closure of the G_{00}-saturation of a homogeneous CSS in $\mathfrak g_{1\star}$;*
(iii) *the ideal of $\mathfrak E$ is generated by quadrics $\varphi^{\#}(\mathfrak g_{01}^*)$, where $\varphi^{\#}:\Bbbk[\mathfrak g_{01}]\to\Bbbk[\mathfrak g_{10}]\otimes\Bbbk[\mathfrak g_{11}]$ is the comorphism. (That is, the scheme $\varphi^{-1}(0)$ is a reduced complete intersection).*

Proof. If $\mathfrak q$ is a $\vec{\boldsymbol\sigma}$-stable reductive subalgebra of $\mathfrak g$, then $\mathfrak E_{\mathfrak q}$ stands for the zero-fibre of the commutator $\mathfrak q_{10}\times\mathfrak q_{11}\to\mathfrak q_{01}$. Clearly, $\mathfrak E_q\subset\mathfrak E=\mathfrak E_{\mathfrak g}$. Since σ_1 and σ_2 are of maximal rank, the centre of $\mathfrak g$, $\mathfrak z(\mathfrak g)$, is contained in $\mathfrak g_{11}$. Consequently, $\mathfrak E_{\mathfrak g}\simeq\mathfrak E_{[\mathfrak g,\mathfrak g]}\times\mathfrak z(\mathfrak g)$ and without loss of generality, we may assume that $\mathfrak g$ is semisimple.

By Proposition 3.8, φ is onto and $\dim\mathfrak E\geqslant\dim\mathfrak g_{11}$. In this situation, φ is equidimensional if and only if $\dim\mathfrak E=\dim\mathfrak g_{11}$. If $\mathfrak c_{1\star}$ is a homogeneous CSS, then it is necessarily a CSA of $\mathfrak g$. That is, $\mathfrak z_{\mathfrak g}(\mathfrak c_{1\star})_{01}=0$ for <u>all</u> homogeneous CSS and $\dim\overline{G_{00}{\cdot}\mathfrak c_{1\star}}=\dim\mathfrak g_{11}$. Hence all the standard components of $\mathfrak E$ have the same (expected) dimension, and for (i) and (ii) it suffices to prove that there is no other irreducible components.

To this end, we argue by induction on $\mathsf{rk}\,\mathfrak g=\dim\mathfrak c_{11}$.

– If $\dim\mathfrak c_{11}=1$, then $\mathfrak g=\mathfrak{sl}_2$ and the assertion is true.

– Suppose that $\mathsf{rk}\,\mathfrak g>1$ and the assertion holds for all dyads of maximal rank for semisimple algebras of rank smaller than $\mathsf{rk}\,\mathfrak g$.

1) Take $(x,y)\in\mathfrak{E}$, $y\in\mathfrak{g}_{11}$, and let $y=y_s+y_n$ be the Jordan decomposition. Then $[x,y_s]=0$. If $y_s\neq 0$, then $y_n\in\mathfrak{s}:=[\mathfrak{z}_{\mathfrak{g}}(y_s),\mathfrak{z}_{\mathfrak{g}}(y_s)]$ and $\mathsf{rk}\,\mathfrak{s}<\mathsf{rk}\,\mathfrak{g}$. By Lemma 1.2, $\sigma_i|_{\mathfrak{s}}$, $i=1,2$, are again involutions of maximal rank. Let $\mathfrak{z}$ denote the centre of $\mathfrak{z}_{\mathfrak{g}}(y_s)$, so that $\mathfrak{z}_{\mathfrak{g}}(y_s)=\mathfrak{z}\oplus\mathfrak{s}$ and $y_s\in\mathfrak{z}$. Since both σ_1 and σ_2 are of maximal rank, $\mathfrak{z}\subset\mathfrak{g}_{11}$ and hence $x\in\mathfrak{s}$. By the induction assumption, $(x,y_n)\in\mathfrak{s}_{10}\oplus\mathfrak{s}_{11}$ lies in a standard irreducible component of $\mathfrak{E}_{\mathfrak{s}}$. Obviously, adding a central summand does not affect this property, hence (x,y) lies in a standard component of $\mathfrak{E}_{\mathfrak{z}_{\mathfrak{g}}(y_s)}$. As $\mathsf{rk}\,\mathfrak{z}_{\mathfrak{g}}(y_s)=\mathsf{rk}\,\mathfrak{g}$, this also means that (x,y) lies in a standard component of $\mathfrak{E}$.

2) Hence it suffices to consider the case in which $y=y_n$. Write $\mathcal{N}_{11}$ for the closed set of all nilpotent elements in $\mathfrak{g}_{11}$. Let $\mathcal{K}$ be an irreducible component of $\mathfrak{E}$, hence $\dim\mathcal{K}\geqslant\dim\mathfrak{g}_{11}$. Then $\mathcal{K}_1:=\mathcal{K}\cap(\mathfrak{g}_{10}\times\mathcal{N}_{11})$ is a closed subvariety of $\mathcal{K}$. If $\mathcal{K}_1\neq\mathcal{K}$, then, by part 1), all the points in $\mathcal{K}\setminus\mathcal{K}_1$ belong to standard irreducible components. Consequently, $\mathcal{K}$ must be one of the standard components.

3) The next possibility is that $\mathcal{K}=\mathcal{K}_1$. Let $p:\mathfrak{g}_{10}\times\mathfrak{g}_{11}\to\mathfrak{g}_{11}$ be the projection. Then $p(\mathcal{K})\subset\mathcal{N}_{11}$, and therefore $\overline{p(\mathcal{K})}=\overline{G_{00}{\cdot}y}$ is the closure of a nilpotent G_{00}-orbit.

If $y=0$, then $\mathcal{K}=\mathfrak{g}_{10}\times\{0\}$. Let $\mathfrak{c}_{10}$ be a little CSS. The fact that $\overline{G_{00}{\cdot}(\mathfrak{c}_{10}\times\{0\})}=\mathfrak{g}_{10}\times\{0\}$ is an irreducible component of $\mathfrak{E}$ implies that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_{10})_{11}=\{0\}$, whence $\mathfrak{c}_{10}$ is also a CSS in $\mathfrak{g}_{1\star}$. That is, $\mathfrak{c}_{10}$ is a CSA of $\mathfrak{g}$. (Incidentally, this means that the (-1)-eigenspace of σ_3 contains a CSA, i.e., $\{\sigma_1,\sigma_2,\sigma_3\}$ is actually a triad.) Anyway, we see that if $y=0$, then such $\mathcal{K}$ appears to be a standard component.

4) Finally, we prove that the case in which $\mathcal{K}=\mathcal{K}_1$ and $y\neq 0$ is impossible. Assuming the contrary, we would have

$$\begin{aligned}\dim\mathfrak{g}_{11}&\leqslant\dim\mathcal{K}\leqslant\dim G_{00}{\cdot}y+\dim p^{-1}(y)\\ &=\dim\mathfrak{g}_{00}-\dim\mathfrak{z}_{\mathfrak{g}}(y)_{00}+\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}=\dim\mathfrak{g}_{11}-\dim\mathfrak{z}_{\mathfrak{g}}(y)_{11}+\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}.\end{aligned}$$

The last equality uses Lemma 1.1. Hence, the existence of such a component $\mathcal{K}$ would imply that $\dim\mathfrak{z}_{\mathfrak{g}}(y)_{11}\leqslant\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}$ for some nonzero $y\in\mathcal{N}_{11}\subset\mathfrak{g}_{11}$. One can rewrite the last condition so that it will only depend on the (inner) involution σ_3. Since $\{\sigma_1,\sigma_2\}$ is a dyad, we have $\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}=\dim\mathfrak{z}_{\mathfrak{g}}(y)_{01}$; and since σ_3 is inner and $\mathfrak{g}^{\sigma_3}=\mathfrak{g}_{00}\oplus\mathfrak{g}_{11}$ is a $\mathbb{Z}_2$-grading of maximal rank, we have $\dim\mathfrak{z}_{\mathfrak{g}}(y)_{11}=\dim\mathfrak{z}_{\mathfrak{g}}(y)_{00}+\mathsf{rk}\,\mathfrak{g}^{\sigma_3}=\dim\mathfrak{z}_{\mathfrak{g}}(y)_{00}+\mathsf{rk}\,\mathfrak{g}$, cf. (1·2). Then

$$\dim\mathfrak{z}_{\mathfrak{g}}(y)_{11}+\dim\mathfrak{z}_{\mathfrak{g}}(y)_{00}+\mathsf{rk}\,\mathfrak{g}=2\dim\mathfrak{z}_{\mathfrak{g}}(y)_{11}\leqslant 2\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}=\dim\mathfrak{z}_{\mathfrak{g}}(y)_{10}+\dim\mathfrak{z}_{\mathfrak{g}}(y)_{01}.$$

In other words, if the assumption were true, we would have

$$\dim(\mathfrak{z}_{\mathfrak{g}}(y)\cap\mathfrak{g}^{\sigma_3})+\mathsf{rk}\,\mathfrak{g}\leqslant\dim(\mathfrak{z}_{\mathfrak{g}}(y)\cap\mathfrak{g}_1^{(\sigma_3)}). \tag{4·1}$$

for some nonzero nilpotent $y\in\mathfrak{g}_{11}$. (Note that since $\mathfrak{g}^{\sigma_3}=\mathfrak{g}_{00}\oplus\mathfrak{g}_{11}$ is a $\mathbb{Z}_2$-grading of maximal rank, $\mathfrak{g}_{11}$ meets all nilpotent orbits in $\mathfrak{g}^{\sigma_3}$ [1]. Therefore, *a priori*, y can be any nonzero nilpotent element of $\mathfrak{g}^{\sigma_3}$.) However, Theorem 4.4 below shows that (4·1) is never satisfied if $y \neq 0$. This completes the proof of parts (i) and (ii).

For (iii), it suffices to prove that each irreducible component of $\mathfrak{E}$ contains a point (x, y) such that $d\varphi_{(x,y)}$ is onto, i.e., $\mathrm{Im}\, d\varphi_{(x,y)} = \mathfrak{g}_{01}$, cf. [20, Lemma 2.3]. Since each irreducible component of $\mathfrak{E}$ is the closure of the G_{00}-saturation of a homogeneous CSA , it contains a point (x, y) such that $\mathfrak{z}_{\mathfrak{g}}(x)_{01} \cap \mathfrak{z}_{\mathfrak{g}}(y)_{01} = \{0\}$ and then $d\varphi_{(x,y)}$ is onto, as shown in the proof of Lemma 3.2. □

Remark 4.2. 1) For any inner $\sigma \in \mathsf{Inv}(\mathfrak{g})$, there exist commuting involutions of maximal rank σ_1 and σ_2 such that $\sigma = \sigma_1\sigma_2$, see Prop. 2.1. Therefore, there are sufficiently many quaternionic decompositions, where Theorem 4.1 applies.

2) For an arbitrary dyad $\{\sigma_1, \sigma_2\}$, it can happen that all irreducible components of $\mathfrak{E}$ are standard, but they have different dimensions. That is, $\varphi : \mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{01}$ is not equidimensional, but still any pair of commuting elements in $\mathfrak{g}_{10} \times \mathfrak{g}_{11}$ can be approximated by a pair of commuting *semisimple* elements.

Example 4.3. Let σ_1 be an involution of $\mathfrak{g} = \mathfrak{so}_n$ such that $\mathfrak{g}^{\sigma_1} = \mathfrak{so}_{n-1}$. This can be included in a dyad $\{\sigma_1, \sigma_2\}$ such that $\mathfrak{g}^{\sigma_3} = \mathfrak{so}_{n-2} \times \mathfrak{so}_2$. The quaternionic decomposition is

$$\mathfrak{g} = \begin{array}{c|c} \mathfrak{so}_{n-2} & \mathsf{R}(\varpi_1) \\ \hline \mathsf{R}(\varpi_1) & \mathsf{R}(0) \end{array} \; \sigma_1 \, , \qquad \text{(vertical axis } \sigma_2\text{)}$$

where the trivial $\mathfrak{so}_{n-2}$-module $\mathsf{R}(0)$ is just the central torus $\mathfrak{so}_2$ in $\mathfrak{g}^{\sigma_3}$. Here $\dim \mathfrak{c}_{10} = \dim \mathfrak{c}_{11} = 1$ and the zero-fibre of multiplication $\mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{01}$ consists of two irreducible components, $\mathfrak{g}_{10} \times \{0\} \simeq \Bbbk^{n-2}$ and $\{0\} \times \mathfrak{g}_{11} \simeq \Bbbk$. Both components are standard.

The following auxiliary result does not refer to quaternionic decompositions; it concerns the case of a sole involution.

Theorem 4.4. *Let σ be an arbitrary involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the corresponding $\mathbb{Z}_2$-grading. For any nonzero $x \in \mathfrak{g}_0$, we have*

$$\dim \mathfrak{g}_0^x + \mathsf{rk}\, \mathfrak{g} - \dim \mathfrak{g}_1^x > 0. \tag{4·2}$$

More precisely, one always has $\dim \mathfrak{g}_0^x + \mathsf{rk}\, \mathfrak{g} - \dim \mathfrak{g}_1^x \geqslant 0$ and the equality only occurs if $x = 0$ and σ is of maximal rank.

Remark 4.5. For application to Theorem 4.1, we only need the case when x is nilpotent and σ is inner. But, surprisingly, the assertion appears to be absolutely general. Unfortunately, our proof is not conceptual, after all. Having successfully reduced the problem to non-even nilpotent elements of $\mathfrak{g}_0$, we then resort to case-by-case considerations. Certainly, there must be a better proof!

Proof. Note that $\dim G{\cdot}x$ is even and, therefore, the LHS in (4·2) is always even; hence the more accurate assertion is that $\dim\mathfrak g_0^x+\mathsf{rk}\,\mathfrak g-\dim\mathfrak g_1^x\geqslant 2$ for all nonzero $x\in\mathfrak g_0$.

1^o. If $x=0$, then we have $\dim\mathfrak g_0+\mathsf{rk}\,\mathfrak g-\dim\mathfrak g_1\geqslant 0$, and the equality holds if and only if σ is of maximal rank.

2^o. If x is nonzero semisimple, then $\mathfrak g^x$ is a σ-stable reductive subalgebra and x is a central element of $\mathfrak g^x$ that belongs to $\mathfrak g_0^x$. Write $\mathfrak g^x=\mathfrak z\oplus\mathfrak s$, where $\mathfrak s=[\mathfrak g^x,\mathfrak g^x]$ and $\mathfrak z$ is the centre. Then $\dim\mathfrak z_0>0$ and

$$\dim\mathfrak g_0^x+\mathsf{rk}\,\mathfrak g-\dim\mathfrak g_1^x=(\dim\mathfrak s_0+\mathsf{rk}\,\mathfrak s-\dim\mathfrak s_1)+2\dim\mathfrak z_0\geqslant 2.$$

3^o. If x is non-nilpotent, then using the Jordan decomposition $x=x_s+x_n$, we reduce the problem to the same property for the nilpotent element x_n in the σ-stable reductive subalgebra $\mathfrak z_{\mathfrak g}(x_s)$.

4^o. From now on, we assume that $x=e\in\mathfrak g_0$ is nonzero and nilpotent. Choose an $\mathfrak{sl}_2$-triple $\{e,h,f\}\subset\mathfrak g_0$. Suppose that e is even in $\mathfrak g$, i.e., the eigenvalues of $\mathrm{ad}\,h$ in $\mathfrak g$ are even. Then $\dim\mathfrak g^h=\dim\mathfrak g^e$ and $\dim\mathfrak g_0^h=\dim\mathfrak g_0^e$. Thus, the assertion is reduced to the same assertion for $h\in\mathfrak g_0$ and we are again in the setting of part 2^o.

5^o. Suppose that e is even in $\mathfrak g_0$, but not in $\mathfrak g$. That is, the eigenvalues of $\mathrm{ad}\,h$ in $\mathfrak g_0$ are even, but $\mathrm{ad}\,h$ has also some odd eigenvalues in $\mathfrak g_1$. Decomposing $\mathfrak g$ into the sum of σ-stable ideals, we may assume that either $\mathfrak g$ is simple or $\mathfrak g=\mathfrak s\oplus\mathfrak s$, where $\mathfrak s$ is simple and σ is the permutation involution. In the second case, if e is even in $\mathfrak g_0=\Delta(\mathfrak s)$, then e is also even in $\mathfrak g$. Therefore, without loss of generality, we may assume that $\mathfrak g$ is simple.

Let us decompose $\mathfrak g_1$ according to the parity of $\mathrm{ad}\,h$-eigenvalues: $\mathfrak g_1=\mathfrak g_1^{odd}\oplus\mathfrak g_1^{even}$. By assumption, $\mathfrak g_1^{odd}\neq 0$. Then $\tilde{\mathfrak g}:=[\mathfrak g_1^{odd},\mathfrak g_1^{odd}]\oplus\mathfrak g_1^{odd}$ is an ideal of $\mathfrak g$ that does not meet $\mathfrak g_1^{even}$. Therefore, $\tilde{\mathfrak g}=\mathfrak g$ and $\mathfrak g_1^{even}=0$. Hence $\mathfrak g_0^e=(\mathfrak g^e)^{even}$ and $\mathfrak g_1^e=(\mathfrak g^e)^{odd}$. Consider the $\mathbb Z$-grading of $\mathfrak g$ determined by the eigenvalues of h, $\mathfrak g=\bigoplus_{i\in\mathbb Z}\mathfrak g(i)$. The $\mathfrak{sl}_2$-theory shows that $\dim(\mathfrak g^e)^{even}=\dim\mathfrak g(0)$ and $\dim(\mathfrak g^e)^{odd}=\dim\mathfrak g(1)$. Hence $\dim\mathfrak g_0^e=\dim\mathfrak g(0)$ and $\dim\mathfrak g_1^e=\dim\mathfrak g(1)$. Finally, it follows from Vinberg's lemma [26, § 2.3] that the group $G(0)$ has finitely many orbits in $\mathfrak g(1)$, whence $\dim\mathfrak g(1)\leqslant\dim\mathfrak g(0)$. Thus, in this case the stronger inequality $\dim\mathfrak g_0^e\geqslant\dim\mathfrak g_1^e$ holds.

6^o. Thus, it remains to handle the case in which a nilpotent element $e\in\mathfrak g_0$ is not even. Here we do not know an a priori argument and resort to the case-by-case considerations.

7^o. If $\mathfrak g$ is a classical Lie algebra, then the nilpotent orbits in $\mathfrak g$ and $\mathfrak g_0$ are parameterised by partitions, and we use the explicit formulae for $\dim\mathfrak g^e$ and $\dim\mathfrak g_0^e$ in terms of partitions. Some of these calculations are presented in Appendix A.

8^o. If $\mathfrak g$ is an exceptional simple Lie algebra, then, for any non-even nilpotent element $e\in\mathfrak g_0$, we determine the corresponding nilpotent orbit in $\mathfrak g$ and then compare the dimensions of $\mathfrak g_0^e$ and $\dim\mathfrak g^e$. Being rather boring, the verification is, however, not very difficult.

For σ inner, we use the seminal work of Dynkin [4, Tables 16–20], where he computed, for all simple 3-dimensional subalgebras in exceptional Lie algebras, the "minimal including regular semisimple subalgebras" and the corresponding weighted Dynkin diagrams. See also comments on this article in [5, p. 309–312], where a few errors occurring in [4] are corrected.

To convey the idea, consider some examples related to an (inner) involution of $\mathfrak g = \mathbf{E}_8$ with $\mathfrak g_0 = \mathbf{D}_8 = \mathfrak{so}_{16}$. There are 33 non-even nilpotent orbits in $\mathfrak g_0$. (Recall that $e \in \mathfrak{so}_{16}$ is non-even if and only if the partition of e contains both odd and even parts.)

a) Let $e \in \mathfrak{so}_{16}$ be a nilpotent element corresponding to the partition $(11, 2, 2, 1)$. Using [7, Cor. 3.8(a)] or [12, Prop. 2.4], we obtain $\dim \mathfrak g_0^e = 16$. This partition also shows that a "minimal including regular semisimple subalgebra" of $\mathbf{D}_8$ containing e is of type $\mathbf{D}_6 + \mathbf{A}_1$. (Here $(11, 1)$ is the partition of the regular nilpotent element of $\mathbf{D}_6$ and any pair of equal parts (n, n) gives rise to the simple summand $\mathbf{A}_{n-1}$.) Then using [4, Table 20], we detect the simple 3-dimensional subalgebra in $\mathbf{E}_8$ with "minimal including regular semisimple subalgebra" of type $\mathbf{D}_6 + \mathbf{A}_1$. The corresponding nilpotent orbit has the modern label $\mathbf{E}_7(a_3)$ and here $\dim \mathfrak g^e = 28$. Hence $\dim \mathfrak g_1^e = 12$ and Eq. (4·2) holds.

b) Let $e \in \mathfrak{so}_{16}$ correspond to the partition $(7, 5, 2, 2)$. By [7, Cor. 3.8(a)], $\dim \mathfrak g_0^e = 22$. Here a "minimal including regular semisimple subalgebra" is of type $\mathbf{D}_6(a_2)+\mathbf{A}_1$, because the partition $(7, 5)$ determines the distinguished nilpotent orbit in $\mathbf{D}_6$, which is denoted by $\mathbf{D}_6(a_2)$. Using [4, Table 20], we detect the corresponding nilpotent orbit in $\mathfrak g$. This orbit is denoted nowadays by $\mathbf{E}_7(a_5)$ and here $\dim \mathfrak g^e = 42$.

c) Let $e \in \mathfrak{so}_{16}$ correspond to the partition $(7, 4, 4, 1)$. By [7, Cor. 3.8(a)], $\dim \mathfrak g_0^e = 22$. Here a "minimal including regular semisimple subalgebra" is of type $\mathbf{D}_4 + \mathbf{A}_3$. Using [4, Table 20], we detect the corresponding nilpotent orbit in $\mathfrak g$. This orbit is denoted nowadays by $\mathbf{D}_6(a_2)$ and here $\dim \mathfrak g^e = 44$.

If σ is outer, then $\mathfrak g$ is of type $\mathbf{E}_6$. In the respective two cases, we use the information on $e \in \mathfrak g_0$ for decomposing $\mathfrak g_1$ as $\langle e, h, f\rangle$-module, which allows to compute $\dim \mathfrak g_1^e$. $\square$

A case-free proof of Theorem 4.4 might be derived from the following conjectural invariant-theoretic property of centralisers. Recall that $\mathfrak g = \mathfrak g_0 \oplus \mathfrak g_1$ and $e \in \mathfrak g_0$. Let G_0^e be the connected subgroup of G_0 with Lie algebra $\mathfrak g_0^e$. Then G_0^e acts on $(\mathfrak g_1^e)^*$ and we write $\Bbbk((\mathfrak g_1^e)^*)^{G_0^e}$ for the field of G_0^e-invariant rational functions on $(\mathfrak g_1^e)^*$.

Conjecture 4.6. *For any $e \in \mathfrak g_0 \cap \mathcal N$, we have* $\operatorname{trdeg} \Bbbk((\mathfrak g_1^e)^*)^{G_0^e} \leqslant \mathsf{rk}\, \mathfrak g$.

By Rosenlicht's theorem [2, Ch. I.6], $\operatorname{trdeg} \Bbbk((\mathfrak g_1^e)^*)^{G_0^e} = \dim \mathfrak g_1^e - \max\limits_{\xi \in (\mathfrak g_1^e)^*} \dim G_0^e{\cdot}\xi$. If $e \neq 0$, then the one-dimensional unipotent group $\exp(te)$, $t \in \Bbbk$, acts trivially on $\mathfrak g_1^e$ and hence $\max_{\xi \in (\mathfrak g_1^e)^*} \dim G_0^e{\cdot}\xi \leqslant \dim \mathfrak g_0^e - 1$. Therefore, if the conjecture were true, we would obtain $\dim \mathfrak g_1^e - \dim \mathfrak g_0^e + 1 \leqslant \mathsf{rk}\, \mathfrak g$, as required. Perhaps, this can be related to the Elashvili conjecture, which asserts that $\operatorname{trdeg} \Bbbk((\mathfrak g^e)^*)^{G^e} = \mathsf{rk}\, \mathfrak g$ for all $e \in \mathcal N$.

Remark 4.7. Inequality (4·2) can be written as $\dim\mathfrak g_0^x > \dim\mathcal B_x$, where $\mathcal B_x$ is the variety of Borel subalgebras of $\mathfrak g$ containing x (the Springer fibre of x). [Recall that $\dim\mathcal B_x = (\dim\mathfrak g^x - \mathsf{rk}\,\mathfrak g)/2$.]

5. Commuting varieties and restricted root systems

Here we assume that $\{\sigma_1,\sigma_2\}$ is a dyad. As above, we consider the commutator map $\varphi:\mathfrak g_{10}\times\mathfrak g_{11}\to\mathfrak g_{01}$ and the $\vec{\boldsymbol\sigma}$-commuting variety $\mathfrak E=\varphi^{-1}(0)$. Then $\dim\mathfrak E\geqslant\dim\mathfrak g_{11}$ and $\mathfrak E$ have a standard irreducible component of expected dimension $\dim\mathfrak g_{11}$; namely, $\{0\}\times\mathfrak g_{11}$, see Proposition 3.8.

In this section, we describe a method for detecting subvarieties of $\mathfrak E$ of large dimension. This method is based on comparing restricted root systems for little and big symmetric spaces related to the quaternionic decomposition in question.

Take a little CSS $\mathfrak c_{11}\subset\mathfrak g_{11}$. Then, by Theorem 2.2(2), $\mathfrak c_{11}$ is also a CSS in $\mathfrak g_{1\star}$ and $\mathfrak g_{\star1}$, which is equivalent to that $\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{10}=\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{01}=\{0\}$ and $\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{11}=\mathfrak c_{11}$. Our idea is to replace $\mathfrak c_{11}$ with a proper subspace $\tilde{\mathfrak c}$ such that:

$$\tilde{\mathfrak c}\text{ still contains }G_{00}\text{-regular elements.}\tag{5·1}$$

Then we consider $\hat{\mathfrak c}:=\mathfrak z_{\mathfrak g}(\tilde{\mathfrak c})_{10}\times\tilde{\mathfrak c}\subset\mathfrak E$ and compute the dimension of $G_{00}{\cdot}\hat{\mathfrak c}$. Since $\overline{G_{00}{\cdot}\mathfrak c_{11}}=\mathfrak g_{11}$, we have

$$\dim G_{00}+\dim\mathfrak c_{11}-\dim\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{00}=\dim\mathfrak g_{11}.$$

Set $\mathfrak T_{00}(\hat{\mathfrak c})=\{g\in G_{00}\mid g{\cdot}y\in\hat{\mathfrak c}\ \text{ for generic }\ y\in\hat{\mathfrak c}\}$, and likewise for $\mathfrak c_{11}$. In view of (5·1), we have $\dim\mathfrak T_{00}(\hat{\mathfrak c})=\dim\mathfrak T_{00}(\mathfrak c_{11})=\dim\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{00}$. Then

$$\begin{aligned}\dim G_{00}{\cdot}\hat{\mathfrak c}&=\dim G_{00}+\dim\hat{\mathfrak c}-\dim\mathfrak T_{00}(\hat{\mathfrak c})\\&=\bigl(\dim G_{00}+\dim\mathfrak c_{11}-\dim\mathfrak z_{\mathfrak g}(\mathfrak c_{11})_{00}\bigr)+\bigl(\dim\mathfrak z_{\mathfrak g}(\tilde{\mathfrak c})_{10}-\dim\mathfrak c_{11}+\dim\tilde{\mathfrak c}\bigr)\\&=\dim\mathfrak g_{11}+\bigl(\dim\hat{\mathfrak c}-\dim\mathfrak c_{11}\bigr)\end{aligned}\tag{5·2}$$

Thus, we obtain a subvariety of larger dimension, if $\dim\mathfrak z_{\mathfrak g}(\tilde{\mathfrak c})_{10}+\dim\tilde{\mathfrak c}>\dim\mathfrak c_{11}$. Of course, it is not always possible to construct such a $\tilde{\mathfrak c}$. Our sufficient condition exploits restricted root systems. Set $\mathfrak h=\mathfrak g^{\sigma_3}$, and let H denote the corresponding connected (reductive) subgroup of G. Write $\bar\sigma$ for the restriction to H of σ_1 or σ_2.

Let $C_{11}=\exp(\mathfrak c_{11})\subset H\subset G$ be the corresponding torus. The coincidence of CSS means that C_{11} is a maximal σ_1-anisotropic torus in G and a maximal $\bar\sigma$-anisotropic torus in H. Accordingly, one obtains the inclusion of two restricted root systems relative to C_{11}:

$$\Psi(H/G_{00})\subset\Psi(G/G_{0\star}).$$

Identifying restricted roots and their differentials, one may consider restricted roots as linear forms on $\mathfrak c_{11}$. Then the set of G_{00}-regular elements of $\mathfrak c_{11}$ is $\{x \in \mathfrak c_{11} \mid \mu(x)\neq 0 \ \ \forall \mu \in \Psi(H/G_{00})\}$ and the set of $G_{0\star}$-regular elements is $\{x \in \mathfrak c_{11} \mid \mu(x)\neq 0 \ \ \forall \mu \in \Psi(G/G_{0\star})\}$.

Proposition 5.1. *Assume that* $\mu \in \Psi(G/G_{0\star})$ *and* $r\mu \notin \Psi(H/G_{00})$ *for any* $r \in \mathbb Q$. *If* $m_\mu > 1$, *then* $\dim \mathfrak E \geqslant \dim \mathfrak g_{11} + m_\mu - 1 > \dim \mathfrak g_{11}$.

Proof. Under this assumption, $\tilde{\mathfrak c} := \mathrm{Ker}\,(\mu) \subset \mathfrak c_{11}$ still contains G_{00}-regular elements, and $\dim \tilde{\mathfrak c} = \dim \mathfrak c_{11} - 1$. Furthermore, $\mathfrak z_{\mathfrak g}(\tilde{\mathfrak c})$ is $\vec{\boldsymbol\sigma}$-stable and $\mathfrak z_{\mathfrak g}(\tilde{\mathfrak c}) = \mathfrak z_{\mathfrak g}(\mathfrak c_{11}) \oplus \mathfrak g_\mu \oplus \mathfrak g_{-\mu}$. Recall that $\mathfrak z_{\mathfrak g}(\mathfrak c_{11})$ is contained in $\mathfrak g_{00} \oplus \mathfrak g_{11}$. Clearly, $\mathfrak g_\mu \oplus \mathfrak g_{-\mu}$ is also $\vec{\boldsymbol\sigma}$-stable and is contained in $\mathfrak g_{01} \oplus \mathfrak g_{10}$.

Since $\{\sigma_1, \sigma_2\}$ is a dyad, $\dim(\mathfrak g_\mu \oplus \mathfrak g_{-\mu}) \cap \mathfrak g_{10} = \dim(\mathfrak g_\mu \oplus \mathfrak g_{-\mu}) \cap \mathfrak g_{01} = m_\mu$. Hence $\dim \mathfrak z_{\mathfrak g}(\tilde{\mathfrak c})_{10} = m_\mu$, and the assertion follows from Eq. (5·2). □

Remark 5.2. 1) Such a construction gives nothing, if all root multiplicities in $\Psi(G/G_{0\star})$ are equal to 1. For instance, if σ_1 is of maximal rank.

2) The procedure described in the previous proof admits obvious modifications. Roughly speaking, if there are linearly independent roots $\mu_1, \mu_2, \dots$ in $\Psi(G/G_{0\star})$, with large multiplicities, such that $\mathbb Q$-span$\{\mu_1, \mu_2, \dots\} \cap \Psi(H/G_{00}) = \varnothing$, then one can take $\tilde{\mathfrak c} = \mathrm{Ker}\,(\mu_1, \mu_2, \dots)$, see Proposition 6.5 below.

Although it is convenient to stick to one specific $\vec{\boldsymbol\sigma}$-commuting variety in theoretical considerations, it may happen that in concrete examples different $\vec{\boldsymbol\sigma}$-commuting varieties exhibit different good (or bad) properties.

Example 5.3. Let σ_1 be an outer involution of $\mathfrak g = \mathfrak{sl}_{2n}$ with $\mathfrak g^{\sigma_1} = \mathfrak{sp}_{2n}$. In [16, Sect. 2], we gave a method for describing all the dyads including σ_1, which exploits the restricted root system $\Psi(G/G^{\sigma_1})$. This implies that one can find σ_2 conjugated σ_1 such that the inner involution $\sigma_3 = \sigma_1\sigma_2$ has the fixed-point subalgebra $\mathfrak h = \mathfrak{sl}_{2m} \oplus \mathfrak{sl}_{2n-2m} \oplus \mathfrak t_1$. The corresponding quaternionic decomposition is

$$\mathfrak{sl}_{2n} = \begin{array}{c|c} \mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-2m} & \mathsf R(\varpi_1)\mathsf R(\varpi_1') \\ \hline \mathsf R(\varpi_1)\mathsf R(\varpi_1') & \mathsf R(\varpi_2){+}\mathsf R(\varpi_2'){+}\mathsf R(0) \end{array} \ \ \sigma_1 \qquad (\text{vertical: } \sigma_2)$$

where ϖ_i (resp. ϖ_i') are fundamental weights of $\mathfrak{sp}_{2m}$ (resp. $\mathfrak{sp}_{2n-2m}$), and $\mathsf R(\lambda)$ is a simple module of the respective simple Lie algebra with highest weight λ.

• Here $G = SL_{2n}$, $G_{0\star} = Sp_{2n}$, $H = SL_{2m} \times SL_{2(n-m)} \times T_1$, and $G_{00} = Sp_{2m} \times Sp_{2(n-m)}$. According to Table VI in [6, Ch. X], we have $\Psi(G/G_{0\star}) = \mathbf A_{n-1}$, $\Psi(H/G_{00}) = \mathbf A_{m-1} + \mathbf A_{n-m-1}$, and all root multiplicities in $\Psi(G/G_{0\star})$ equals 4. Since $\Psi(H/G_{00})$ has fewer roots, Proposition 5.1 implies that $\mathfrak E$ has an irreducible component of dimension $> \dim \mathfrak g_{11} + (4-1)$ and our test map $\varphi : \mathfrak g_{10} \times \mathfrak g_{11} \to \mathfrak g_{01}$ is not equidimensional.

• Here $\dim \mathfrak{c}_{01} = \dim \mathfrak{c}_{10} = \min\{m, n-m\}$ and any big CSS in $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01}$ is of dimension $2\min\{m, n-m\}$. By Theorem 3.7(2), this means that all homogeneous CSS in $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01}$ are G_{00}-conjugate, and therefore, the $\vec{\boldsymbol{\sigma}}$-commuting variety related to the commutator $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01} \to \mathfrak{g}_{11}$ has a unique standard component.

Example 5.4. Let σ be an involution of $\mathfrak{g} = \mathbf{E}_7$ with $\mathfrak{g}^\sigma = \mathbf{D}_6 \times \mathbf{A}_1$. It can be included in two non-conjugate triads [10]. One of them has $\mathfrak{g}_{00} = \mathbf{D}_4 \times \mathbf{A}_1^3$, with quaternionic decomposition

$$\mathbf{E}_7 = \begin{array}{r:l} \mathbf{D}_4 \times \mathbf{A}_1^3 & \mathsf{R}(\varpi_4)\mathsf{R}(\varpi)\mathsf{R}(\varpi'') \\ \hdashline \mathsf{R}(\varpi_3)\mathsf{R}(\varpi)\mathsf{R}(\varpi') & \mathsf{R}(\varpi_1)\mathsf{R}(\varpi')\mathsf{R}(\varpi'') \\ \multicolumn{2}{c}{\sigma_2} \end{array} \quad \sigma_1$$

where $\varpi, \varpi', \varpi''$ are the fundamental weights of the simple factors of $\mathbf{A}_1^3$, and ϖ_i's are fundamental weights of $\mathbf{D}_4$. Here $\dim \mathfrak{g}_{ij} = 32$ for $(ij) \ne (00)$ and our test commutator map is

$$\varphi : \mathsf{R}(\varpi_3)\mathsf{R}(\varpi)\mathsf{R}(\varpi') \times \mathsf{R}(\varpi_1)\mathsf{R}(\varpi')\mathsf{R}(\varpi'') \to \mathsf{R}(\varpi_4)\mathsf{R}(\varpi)\mathsf{R}(\varpi'').$$

Using Table VI in [6, Ch. X], we find that $\mathsf{rk}(\mathbf{E}_7/\mathbf{D}_6 \times \mathbf{A}_1) = 4$ and the restricted root system $\Psi(\mathbf{E}_7/\mathbf{D}_6 \times \mathbf{A}_1)$ is of type $\mathbf{F}_4$; whereas $\mathsf{rk}(\mathbf{D}_6 \times \mathbf{A}_1/\mathbf{D}_4 \times \mathbf{A}_1^3) = \mathsf{rk}(\mathbf{D}_6/\mathbf{D}_4 \times \mathbf{A}_1^2) = 4$ and the corresponding root system is of type $\mathbf{B}_4$. The long (resp. short) roots of $\mathbf{B}_4$ are also long (resp. short) roots of $\mathbf{F}_4$, and the multiplicities are $m_{\text{long}} = 1$, $m_{\text{short}} = 4$. However, the root system $\mathbf{B}_4$ has fewer short roots than $\mathbf{F}_4$. Therefore, Proposition 5.1 applies here, and $\mathfrak{E}$ has an irreducible component of dimension at least $\mathfrak{m}_{\text{short}} - 1 + \dim \mathfrak{g}_{11} = 35$.

Example 5.5. Let σ be an involution of $\mathfrak{g} = \mathbf{F}_4$ with $\mathfrak{g}^\sigma = \mathbf{B}_4 = \mathfrak{so}_9$. Up to conjugacy, this involution can be included in a unique triad [10], with quaternionic decomposition

$$\mathbf{F}_4 = \begin{array}{r:l} \mathbf{D}_4 & \mathsf{R}(\varpi_4) \\ \hdashline \mathsf{R}(\varpi_3) & \mathsf{R}(\varpi_1) \\ \multicolumn{2}{c}{\sigma_2} \end{array} \quad \sigma_1 ,$$

where $\dim \mathsf{R}(\varpi_i) = 8$ and the main diagonal represent the little involution of $\mathfrak{g}^{\sigma_3} = \mathbf{B}_4 = \mathfrak{so}_9$. Our test commutator is the bilinear $\mathbf{D}_4$-equivariant mapping $\mathsf{R}(\varpi_3) \times \mathsf{R}(\varpi_1) \to \mathsf{R}(\varpi_4)$. Here $\mathsf{rk}(\mathbf{F}_4/\mathbf{B}_4) = 1$ and the restricted root system $\Psi(\mathbf{F}_4/\mathbf{B}_4)$ is of type $\mathbf{BC}_1$. The restricted root system $\Psi(\mathbf{B}_4/\mathbf{D}_4)$ is of type $\mathbf{C}_1$. Since all little and big CSS are one-dimensional, Proposition 5.1 does not help here. Actually, the only standard components of $\mathfrak{E}$ are $\mathfrak{g}_{10} \times \{0\}$ and $\{0\} \times \mathfrak{g}_{11}$, both of dimension 8. Below, we describe an "intermediate" non-standard irreducible component of dimension 11.

Let $x \in \mathfrak{g}_{11} \simeq \mathsf{R}(\varpi_1)$ be a nonzero nilpotent element. All such elements form a sole 7-dimensional SO_8-orbit. By Lemma 1.1, $\dim SO_9{\cdot}x = 2{\cdot}7 = 14$ and hence $\dim(\mathfrak{so}_9)^x = 22$. The only nilpotent SO_9-orbit of dimension 14 in $\mathfrak{so}_9$ is the orbit of short root vectors. The short roots of $\mathfrak{g}^{\sigma_3} = \mathbf{B}_4$ are also short roots of $\mathfrak{g} = \mathbf{F}_4$. Therefore, a "minimal including regular semisimple subalgebra" is $\tilde{\mathbf{A}}_1$ in Dynkin's notation. This implies that $\dim \mathfrak{z}_{\mathfrak{g}}(x) = 30$ and completely determines the dimension matrix of the spaces $\mathfrak{z}_{\mathfrak{g}}(x)_{ij}$: $\begin{array}{c|c} 21 & 4 \\ \hline 4 & 1 \end{array}$. Here the 1-dimensional space $\mathfrak{g}_{11}$ is just the line $\Bbbk x$. Then $\dim \overline{G_{00}{\cdot}(\mathfrak{z}_{\mathfrak{g}}(x)_{10} \oplus \Bbbk x)} = 4 + 7 = 11$. Using the projection $\mathfrak{E} \to \mathfrak{g}_{11}$, one can prove that $\overline{G_{00}{\cdot}(\mathfrak{z}_{\mathfrak{g}}(x)_{10} \oplus \Bbbk x)}$ is the only new irreducible component of $\mathfrak{E}$. It is contained in $\mathcal{N}_{10} \times \mathcal{N}_{11}$. Thus, $\mathfrak{E}$ has three irreducible components.

6. Triads of Hermitian involutions and simple Jordan algebras

In this section, $\mathfrak{g}$ is assumed to be simple. We say that $\sigma \in \mathsf{Inv}(\mathfrak{g})$ is *Hermitian* if $\mathfrak{g}_0$ is not semisimple. All these involutions are associated with $\mathbb{Z}$-gradings of $\mathfrak{g}$ with only three nonzero terms (*short gradings*), i.e., with parabolic subalgebras with abelian nilpotent radical. Let $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ be a short grading. Then $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ is a (maximal) parabolic subalgebra with abelian nilpotent radical, and one defines a Hermitian involution σ by letting $\mathfrak{g}^\sigma = \mathfrak{g}(0)$, $\mathfrak{g}_1^{(\sigma)} = \mathfrak{g}(-1) \oplus \mathfrak{g}(1)$.

Since $\mathfrak{g}$ is simple, the centre of $\mathfrak{g}(0)$ is one-dimensional and there is a *unique* $h \in \mathfrak{g}(0)$ such that $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = 2ix\}$. By [26, § 2.3], the reductive group $G(0)$ has finitely many orbits in $\mathfrak{g}(1)$. Let $\mathcal{O}$ be the dense $G(0)$-orbit in $\mathfrak{g}(1)$ and $e \in \mathcal{O}$. Set $\mathfrak{g}(i)^e = \mathfrak{g}(i) \cap \mathfrak{g}^e$.

For future reference, we provide a proof of the following well-known assertion.

Lemma 6.1. $h \in [\mathfrak{g}, e] \iff \mathfrak{g}(0)^e$ *is reductive.*

Proof. 1) If $h \in [\mathfrak{g}, e]$, then $h = [e, f]$ for some $f \in \mathfrak{g}(-1)$ and therefore, $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple. Then $\mathfrak{g}(0)^e = \mathfrak{z}_{\mathfrak{g}}(e, h, f)$, which is reductive.

2) For $e \in \mathcal{O}$, we have $\dim \mathfrak{g}(0)^e = \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1)$. Using the Kirillov–Kostant form associated with e, we see that $\dim \mathfrak{g}(-1) - \dim \mathfrak{g}(-1)^e = \dim \mathfrak{g}(0) - \dim \mathfrak{g}(0)^e$. Hence $\mathfrak{g}(-1)^e = 0$ and $\mathfrak{g}^e = \mathfrak{g}(0)^e \oplus \mathfrak{g}(1)$. Set $\mathfrak{k} = \mathfrak{g}(0)^e$, and let $(\ )^\perp$ denote the orthocomplement with respect the Killing form. Then $[\mathfrak{g}, e] = (\mathfrak{g}^e)^\perp = \mathfrak{g}(1) \oplus (\mathfrak{k}^\perp \cap \mathfrak{g}(0))$. Now, if $\mathfrak{k}$ is reductive, then the restriction of the Killing form to $\mathfrak{k}$ is non-degenerate and $\mathfrak{m} := \mathfrak{k}^\perp \cap \mathfrak{g}(0)$ is a $\mathfrak{k}$-stable complement to $\mathfrak{k}$ in $\mathfrak{g}(0)$. Since $\dim[\mathfrak{g}(-1), e] = \dim \mathfrak{g}(1) = \dim \mathfrak{g}(0) - \dim \mathfrak{k}$, we conclude that $\mathfrak{m} = [\mathfrak{g}(-1), e]$. Thus, e acts on $\mathfrak{g}$ as follows:

$$\left\{ \begin{array}{c} \mathfrak{g}(-1) \xrightarrow{\sim} \mathfrak{m} \xrightarrow{\sim} \mathfrak{g}(1) \to 0 \\ \mathfrak{k} \to 0 \end{array} \right. \tag{6·1}$$

Let $\{e, \tilde{h}, f\}$ be an $\mathfrak{sl}_2$-triple with $\tilde{h} \in \mathfrak{g}(0)$ and $f \in \mathfrak{g}(-1)$. Such a triple always exists, see [27, § 2]. Then Eq. (6·1) shows that $\mathfrak{g}$ is a sum of 3-dimensional and 1-dimensional $\mathfrak{sl}_2$-modules, and that $\mathfrak{g}^{\tilde{h}} = \mathfrak{k} \oplus \mathfrak{m}$. Since $\mathfrak{g}(0)$ has one-dimensional centre, one must have $\tilde{h} = h$. Thus, $h \in [\mathfrak{g}, e]$. ∎

Theorem 6.2. *Suppose that a Hermitian involution $\sigma = \sigma_1$ has the property that $\mathfrak{g}(0)^e$ is reductive. Then σ_1 can be included in a triad.*

Proof. Using the notation of the previous proof, we set $\mathfrak{k} = \mathfrak{g}(0)^e$ and take (the unique) $f \in \mathfrak{g}(-1)$ such that $h = [e, f]$. Then $\{e, h, f\}$ is an $\mathfrak{sl}_2$-triple, $[e, \mathfrak{g}(-1)] =: \mathfrak{m}$ is a complementary $\mathfrak{k}$-submodule to $\mathfrak{k}$ in $\mathfrak{g}(0)$, and $[e, [e, \mathfrak{g}(-1)]] = \mathfrak{g}(1)$. This also shows that $\mathfrak{g}(-1)$, $\mathfrak{m}$, and $\mathfrak{g}(1)$ are isomorphic $\mathfrak{k}$-modules.

In this case, $\mathfrak{k}$ is the fixed-point subalgebra of an involution of $\mathfrak{g}(0)$ and the (-1)-eigenspace of this involution is $\mathfrak{m}$ (see Proof of Prop. 3.3 in [13]). Let σ_2 denote this involution of $\mathfrak{g}(0)$. Then $\sigma_2(h) = -h$. We extend σ_2 to the whole of $\mathfrak{g}$ by letting $\sigma_2(e) = f$. Then $\sigma_2([x, e]) = [-x, f]$ for all $x \in \mathfrak{m}$, which defines σ_2 on $\mathfrak{g}(1)$ and shows that $\sigma_2(\mathfrak{g}(1)) \subset \mathfrak{g}(-1)$. Clearly, σ_1 and σ_2 commute. Furthermore, σ_1 and σ_2 are different involutions of the 3-dimensional simple subalgebra $\langle e, h, f\rangle$. This implies that σ_1, σ_2, and $\sigma_3 = \sigma_1\sigma_2$ are already conjugate with respect to $PSL_2 = \mathrm{Aut}\langle e, h, f\rangle$. In particular, $\{\sigma_1, \sigma_2, \sigma_3\}$ is a triad. $\square$

This theorem can be derived from the classification of triads, but our direct construction allows to visualise the resulting quaternionic decomposition rather explicitly. We have

$$\mathfrak{g} = \begin{array}{c:c} \mathfrak{k} & \mathfrak{m} \\ \hdashline [\mathfrak{m}, e-f] & [\mathfrak{m}, e+f] \\ \multicolumn{2}{c}{\sigma_2} \end{array} \;\; \sigma_1 \tag{6·2}$$

Here $h \in \mathfrak{m} = \mathfrak{g}_{01}$, $e + f \in [\mathfrak{m}, e - f] = \mathfrak{g}_{10}$, and $e - f \in [\mathfrak{m}, e + f] = \mathfrak{g}_{11}$. Note also that $\mathfrak{k} \oplus \mathfrak{m} = \mathfrak{g}(0)$ and $[\mathfrak{m}, e - f] \oplus [\mathfrak{m}, e + f] = \mathfrak{g}(1) \oplus \mathfrak{g}(-1)$.

Remark. If $\mathfrak{g}(0)^e$ is not reductive, then such a triad may not exist. For instance, if $\mathfrak{g} = \mathfrak{sl}_{2n}$ and $\mathfrak{g}_0 = \mathfrak{sl}_m \times \mathfrak{sl}_{2n-m} \times \mathfrak{t}_1$ with $n \neq m$ and m odd, then there is no respective triad, see [28, 3.2].

As is well known, if $\mathfrak{g}(0)^e$ is reductive, then $\mathfrak{g}(-1)$ has a structure of a simple Jordan algebra. Namely, for $x, y \in \mathfrak{g}(-1)$, we set

$$x \circ y = [x, [e, y]] \in \mathfrak{g}(-1).$$

Then $\{\mathfrak{g}(-1), \circ\}$ is a simple Jordan algebra [23, 9]. (See also [8, Sect. 4] for possible generalisations). Here $\mathfrak{k} = \mathfrak{g}_{00}$ is the Lie algebra of derivations of $\{\mathfrak{g}(-1), \circ\}$. The triad constructed in Theorem 6.2 is called a *Jordan triad*.

Definition 3. The *commuting variety* of a Jordan algebra $\{\mathcal{J}, \circ\}$ is

$$\mathfrak{C}(\mathcal{J}) = \{(x, y) \mid x \circ y = 0\} \subset \mathcal{J} \times \mathcal{J}.$$

The Jordan triad (6·2) provides a link between the commutator mapping $\varphi : \mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{01}$ and the commuting variety of the simple Jordan algebra $\mathfrak{g}(-1)$.

Theorem 6.3. *The commuting variety of the Jordan algebra* $\{\mathfrak{g}(-1), \circ\}$ *is isomorphic to the zero fibre of the commutator mapping* $\varphi : \mathfrak{g}_{10} \times \mathfrak{g}_{11} = [\mathfrak{m}, e-f] \times [\mathfrak{m}, e+f] \to \mathfrak{m} = \mathfrak{g}_{01}$.

Proof. Any element of $\mathfrak{m}$ can uniquely be written as $[x,e]$ with $x \in \mathfrak{g}(-1)$. So, if $[x,e],[y,e] \in \mathfrak{m}$ are arbitrary, then $[[x,e],e-f] \in \mathfrak{g}_{10}$ and $[[y,e],e+f] \in \mathfrak{g}_{11}$ are arbitrary and φ takes the corresponding pair to $\big[[[x,e],e-f],[[y,e],e+f]\big] \in \mathfrak{m} = \mathfrak{g}_{01}$. It is a good exercise in the Jacobi identity to check that

$$\big[[[x,e],e-f],[[y,e],e+f]\big] = 2\big[[[x,e],y],e\big].$$

(One should use the fact that $h=[e,f]$ is the defining element of the short grading. Hence $[[x,e],f]=2x$, etc.) Since $a=[[x,e],y] \in \mathfrak{g}(-1)$ and $\mathfrak{g}^e \cap \mathfrak{g}(-1)=0$, we have $[a,e]=0$ if and only if $a=0$. Therefore,

$$([[x,e],e-f],[[y,e],e+f]) \in \varphi^{-1}(0) \Leftrightarrow [[x,e],y]=0 \Leftrightarrow (x,y) \in \mathfrak{E}(\mathfrak{g}(-1)).$$ □

If $\mathcal{J}$ is a simple Jordan algebra, then the operator $L_x : \mathcal{J} \to \mathcal{J}$, $L_x(y) = x \circ y$, is invertible for almost all x. Therefore, $\mathcal{J} \times \{0\}$ and $\{0\} \times \mathcal{J}$ are two irreducible components of $\mathfrak{E}(\mathcal{J})$. Clearly, there are some other irreducible components. It is an interesting problem to determine all the components of $\mathfrak{E}(\mathcal{J})$ and their dimensions.

The list of Hermitian involutions leading to Jordan triads and simple Jordan algebras is given below. We point out the semisimple subalgebra $\mathfrak{s} = [\mathfrak{g}(0), \mathfrak{g}(0)]$ and the structure of $\mathfrak{g}(1)$ as $\mathfrak{s}$-module. Here the ϖ_i's are the fundamental weights of $\mathfrak{s}$.

	$\mathfrak{g}$	$\mathfrak{s}$	$\mathfrak{g}(1)$	$\mathfrak{k}$	$\mathcal{J}$	
1	$\mathfrak{sl}_{2n}$	$\mathfrak{sl}_n \oplus \mathfrak{sl}_n$	$\mathsf{R}(\varpi_1) \otimes \mathsf{R}(\varpi_1')$	$\mathfrak{sl}_n$	$n \times n$-matrices	$n \geqslant 1$
2	$\mathfrak{sp}_{2n}$	$\mathfrak{sl}_n$	$\mathsf{R}(2\varpi_1)$	$\mathfrak{so}_n$	symmetric $n \times n$-matrices	$n \geqslant 2$
3	$\mathfrak{so}_{4n}$	$\mathfrak{sl}_{2n}$	$\mathsf{R}(\varpi_2)$	$\mathfrak{sp}_{2n}$	skew-symm. $2n \times 2n$-matrices	$n \geqslant 2$
4	$\mathfrak{so}_{n+2}$	$\mathfrak{so}_n$	$\mathsf{R}(\varpi_1)$	$\mathfrak{so}_{n-1}$	spin-factor	$n \geqslant 4$
5	$\mathbf{E}_7$	$\mathbf{E}_6$	$\mathsf{R}(\varpi_1)$	$\mathbf{F}_4$	the Albert algebra	

Remark. The Jordan multiplication in the space Skew_{2n} of usual skew-symmetric matrices is defined as follows. If $A, B, J \in \mathsf{Skew}_{2n}$ and J is non-degenerate, then $A \circ B = \frac{1}{2}(AJB + BJA)$.

There are some coincidences for small n. Namely,

$$\text{Item 1 } (n=1) \ \simeq \ \text{Item 2 } (n=1)\,, \ \text{Item 1 } (n=2) \ \simeq \ \text{Item 4 } (n=3)\,.$$

Furthermore, if $n=1$ in Item 3, then $\mathfrak{g}$ is not simple. This explains the conditions on n given in the last column. For Item 2, the Hermitian involution (of $\mathfrak{sp}_{2n}$) is of maximal rank and the respective Jordan algebra is the algebra Sym_n of symmetric $n \times n$-matrices. Therefore, by Theorems 4.1 and 6.3, the multiplication morphism $\circ : \mathsf{Sym}_n \times \mathsf{Sym}_n \to \mathsf{Sym}_n$ is equidimensional, i.e., $\dim \mathfrak{E}(\mathsf{Sym}_n) = \dim \mathsf{Sym}_n = (n^2+n)/2$.

In all other cases, the multiplication morphism $\mathcal{J} \times \mathcal{J} \to \mathcal{J}$ is not equidimensional, see Proposition 6.5. Before checking this, we give an "elementary" explanation for the Jordan algebra of all matrices (Item 1).

Example 6.4. Let M be the associative (also Lie and Jordan) algebra of all $n \times n$-matrices. That is, we exploit the usual matrix product, the Lie bracket $[A, B] = AB - BA$, and the Jordan product $A \circ B = (AB + BA)/2$. Let $\chi(B) = \det(\lambda I - B) = \sum_i \chi_{n-i}(B)\lambda^i$ be the characteristic polynomial of a matrix B. Let $\mathfrak{z}^J(B)$ and $\mathfrak{z}^{Lie}(B)$ denote the Jordan and Lie centraliser of B, respectively. Consider the subvariety

$$\mathsf{M}^{\langle 2\rangle} = \{B \in \mathsf{M} \mid \chi_{2i+1}(B) = 0\ \forall i\}.$$

It is an irreducible complete intersection and $\operatorname{codim} \mathsf{M}^{\langle 2\rangle} = [n + 1/2]$ (cf. [22, Lemma 5.3]). We also need the dense open subset M^{reg} of regular elements (in the Lie algebra sense) and the subvariety

$$\mathsf{M}^{ev} = \{B \in \mathsf{M} \mid B \ \text{ is conjugate to } \ -B\}.$$

If $B \in \mathsf{M}^{ev}$ and $ABA^{-1} = -B$, then $A \in \mathfrak{z}^J(B)$ and the mapping $C \in \mathfrak{z}^{Lie}(B) \mapsto AC \in \mathfrak{z}^J(B)$ is a linear isomorphism. In particular, $\dim \mathfrak{z}^J(B) = \dim \mathfrak{z}^{Lie}(B)$. The following is clear:

- $\mathsf{M}^{\langle 2\rangle} \cap \mathsf{M}^{reg} \neq \varnothing$ (it contains a regular nilpotent element);
- $\mathsf{M}^{ev} \subset \mathsf{M}^{\langle 2\rangle}$ and $\mathsf{M}^{ev} \cap \mathsf{M}^{reg} \neq \varnothing$;

Claim. *We have* $\mathsf{M}^{\langle 2\rangle} \cap \mathsf{M}^{reg} \subset \mathsf{M}^{ev}$. *In particular,* $\dim \mathfrak{z}^J(B) = n$ *for almost all* $B \in \mathsf{M}^{\langle 2\rangle}$.

Proof. If $B \in \mathsf{M}^{\langle 2\rangle} \cap \mathsf{M}^{reg}$, then B and $-B$ are both regular and have the same Jordan blocks and the same eigenvalues. Hence B and $-B$ are conjugate. □

Let $\mathfrak{E}^J(\mathsf{M})$ denote the Jordan commuting variety and $p : \mathfrak{E}^J(\mathsf{M}) \to \mathsf{M}$ the projection to the first factor. The previous analysis implies that

$$\dim p^{-1}(\mathsf{M}^{\langle 2\rangle} \cap \mathsf{M}^{reg}) = \dim M^{\langle 2\rangle} + n = n^2 + [n/2].$$

Thus, $\dim \mathfrak{E}^J(\mathsf{M}) \geqslant n^2 + [n/2] > \dim \mathsf{M}$. One can prove that this yields an irreducible component of maximal dimension; that is, $\dim \mathfrak{E}^J(\mathsf{M}) = n^2 + [n/2]$.

The next table contains information on the restricted root systems associated with Jordan triads. For a Hermitian involution σ, we point out Lie algebras $\mathfrak{g}$, $\mathfrak{h} = \mathfrak{g}^\sigma$, $\mathfrak{g}_{00} = \mathfrak{k}$, the restricted root systems $\Psi(G/H)$ and $\Psi(H/G_{00})$, and the multiplicity of the short roots in $\Psi(G/H)$, denoted m_{short}. For all items in the table, the multiplicity of long roots in $\Psi(G/H)$ equals 1 and $\Psi(H/G_{00})$ is embedded in $\Psi(G/H)$ as a subset of *short* roots.

	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{g}_{00}$	$\Psi(G/H)$	m_{short}	$\Psi(H/G_{00})$
1	$\mathfrak{sl}_{2n}$	$\mathfrak{sl}_n \oplus \mathfrak{sl}_n \oplus \mathfrak{t}_1$	$\mathfrak{sl}_n$	$\mathbf{C}_n$	2	$\mathbf{A}_{n-1}$
2	$\mathfrak{sp}_{2n}$	$\mathfrak{gl}_n$	$\mathfrak{so}_n$	$\mathbf{C}_n$	1	$\mathbf{A}_{n-1}$
3	$\mathfrak{so}_{4n}$	$\mathfrak{gl}_{2n}$	$\mathfrak{sp}_{2n}$	$\mathbf{C}_n$	4	$\mathbf{A}_{n-1}$
4	$\mathfrak{so}_{n+2}$	$\mathfrak{so}_n \oplus \mathfrak{so}_2$	$\mathfrak{so}_{n-1}$	$\mathbf{C}_2$	$n-2$	$\mathbf{A}_1$
5	$\mathbf{E}_7$	$\mathbf{E}_6 \oplus \mathfrak{t}_1$	$\mathbf{F}_4$	$\mathbf{C}_3$	8	$\mathbf{A}_2$

The root system of type $\mathbf{C}_n$ has some short roots that are not roots of $\mathbf{A}_{n-1}$. Therefore, Proposition 5.1 guarantees the existence of a subvariety in $\mathfrak{E}(\mathcal{J})$ of dimension $\dim \mathcal{J} + m_{\text{short}} - 1$, which is larger than the dimension of a generic fibre if $m_{\text{short}} > 1$. However, a clever choice of $\tilde{\mathfrak{c}} \subset \mathfrak{c}_{11}$ (cf. Remark 5.2(2)) allows to get a better lower bound on $\dim \mathfrak{E}(\mathcal{J})$:

Proposition 6.5. *For all items in the table, we have* $\dim \mathfrak{E}(\mathcal{J}) \geqslant \dim \mathcal{J} + (m_{\text{short}} - 1)[r/2]$, *where* r *is the rank of* $\Psi(G/H)$.

Proof. Using Theorem 6.3, we identify $\mathfrak{E}(\mathcal{J})$ with the zero fibre of the quadratic covariant $\mathfrak{g}_{10} \times \mathfrak{g}_{11} \to \mathfrak{g}_{10}$ and work in the setting of Section 5. Let $\varepsilon_1, \dots, \varepsilon_r$ be the usual basis of $\mathfrak{X}(C_{11}) \otimes \mathbb{Q}$ such that the roots of $\Psi(G/H)$ are $\pm\varepsilon_i \pm \varepsilon_j$ ($i \neq j$) and $\pm 2\varepsilon_i$. The roots in $\Psi(H/G_{00})$ are $\pm(\varepsilon_i - \varepsilon_j)$. Therefore, $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01}$ is the sum of root spaces corresponding to $\pm(\varepsilon_i + \varepsilon_j)$ and $\pm 2\varepsilon_i$. Set

$$\tilde{\mathfrak{c}} = \{x \in \mathfrak{c}_{11} \mid (\varepsilon_i + \varepsilon_{r+1-i})(x) = 0, \ \ i = 1, 2, \dots, \left[\tfrac{r+1}{2}\right]\}.$$

Then $\dim \tilde{\mathfrak{c}} = [r/2]$, and we have $2[r/2]$ short roots of $\mathfrak{g}_{10} \oplus \mathfrak{g}_{01}$ vanishing on $\tilde{\mathfrak{c}}$. Moreover, if r is odd, then the long roots $\pm 2\varepsilon_{[r+1/2]}$ also vanish on $\tilde{\mathfrak{c}}$. Therefore,

$$\dim \mathfrak{z}_{\mathfrak{g}}(\tilde{\mathfrak{c}})_{10} = \frac{1}{2} \dim\big(\mathfrak{z}_{\mathfrak{g}}(\tilde{\mathfrak{c}}) \cap (\mathfrak{g}_{10} \oplus \mathfrak{g}_{01})\big) = \begin{cases} m_{\text{short}} \cdot \frac{r}{2} & \text{if } r \text{ is even,} \\ m_{\text{short}} \cdot \left[\frac{r}{2}\right] + 1 & \text{if } r \text{ is odd.} \end{cases}$$

In both cases, this yields $\dim G_{00} \cdot (\mathfrak{z}_{\mathfrak{g}}(\tilde{\mathfrak{c}})_{10} \oplus \tilde{\mathfrak{c}}) = \dim \mathfrak{g}_{11} + (m_{\text{short}} - 1)[r/2]$. $\square$

For the Jordan algebra of all matrices (related to a Hermitian involution of $\mathfrak{sl}_{2n}$), the above construction of $\tilde{\mathfrak{c}}$ gives exactly the subvariety of Example 6.4. It is plausible that the lower bound of Proposition 6.5 provides the exact value of $\dim(\mathfrak{E}(\mathcal{J}))$.

Remark 6.6. It is curious to observe that, for all Hermitian involutions leading to Jordan triads, the restricted root system is of type $\mathbf{C}_n$; whereas, for all other Hermitian involutions, the restricted root system Ψ is of type $\mathbf{BC}_n$. Namely, the symmetric pairs $\mathfrak{gl}_{n+m} \supset \mathfrak{gl}_n \times \mathfrak{gl}_m \times \mathfrak{t}_1$ ($n < m$) and $\mathfrak{so}_{4n+2} \supset \mathfrak{gl}_{2n+1}$ lead to $\Psi \simeq \mathbf{BC}_n$; and $\mathbf{E}_6 \supset \mathbf{D}_5 \times \mathfrak{t}_1$ leads to $\Psi \simeq \mathbf{BC}_2$.

Appendix A. Computations in classical Lie algebras

Here we provide some computations related to the proof of Theorem 4.4 for nilpotent elements in classical Lie algebras.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_s)$ be a partition and $e \in \mathfrak{gl}_n$ a nilpotent element corresponding to $\boldsymbol{\lambda}$, also denoted by $e \sim \boldsymbol{\lambda}$. Then $\sum \lambda_i = n$ and

$$\dim(\mathfrak{gl}_n)^e = n + 2\sum_{i<j} \min\{\lambda_i, \lambda_j\}, \quad \dim(\mathfrak{sl}_n)^e = \dim(\mathfrak{gl}_n)^e - 1. \tag{A·1}$$

If e is a nilpotent element in $\mathfrak{so}_n$ or $\mathfrak{sp}_{2n}$, with respective parity conditions on $\boldsymbol{\lambda}$, then

$$\dim(\mathfrak{sp}_{2n})^e = \frac{\dim(\mathfrak{gl}_{2n})^e + \#\{i \mid \lambda_i \text{ is odd}\}}{2}, \tag{A·2}$$

$$\dim(\mathfrak{so}_n)^e = \frac{\dim(\mathfrak{gl}_n)^e - \#\{i \mid \lambda_i \text{ is odd}\}}{2}. \tag{A·3}$$

See [7, (3.8)] and [12, 2.4]. Below, we consider several symmetric pairs with classical $\mathfrak{g}$ and check that (4·2) is satisfied for all nonzero nilpotent elements of $\mathfrak{g}_0$. There is no need in considering only non-even nilpotent element in $\mathfrak{g}_0$, since the computations go through without this assumption.

A.1. $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sl}_n, \mathfrak{so}_n)$. If $e \in \mathfrak{so}_n$ and $e \sim \boldsymbol{\lambda}$, then using (A·1) and (A·3) yields

$$\dim \mathfrak{g}_0^e = \frac{\dim(\mathfrak{gl}_n)^e - \#\{i \mid \lambda_i \text{ is odd}\}}{2}, \quad \dim \mathfrak{g}_1^e = \frac{\dim(\mathfrak{gl}_n)^e + \#\{i \mid \lambda_i \text{ is odd}\}}{2} - 1.$$

Therefore, $\dim \mathfrak{g}_0^e - \dim \mathfrak{g}_1^e + (n-1) = n - \#\{i \mid \lambda_i \text{ is odd}\}$. Here the parity condition means that each even part of $\boldsymbol{\lambda}$ occurs an even number of times. Since $e \neq 0$, i.e., $\boldsymbol{\lambda} \neq (1, \dots, 1)$, the minimal value is 2, and it is attained for $\boldsymbol{\lambda} = (3, 1^{n-3})$.

A.2. $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{sp}_{2n}, \mathfrak{gl}_n)$. If $e \in \mathfrak{gl}_n$ and $e \sim \boldsymbol{\lambda}$, then the partition of e as element of $\mathfrak{sp}_{2n}$ is obtained by doubling $\boldsymbol{\lambda}$, i.e., each part λ_i is replaced with (λ_i, λ_i). Then $\dim \mathfrak{g}_0^e = \dim(\mathfrak{gl}_n)^e$ is given by (A·1), and using (A·2) yields $\dim \mathfrak{g}_1^e = 2\left[\frac{\lambda_i+1}{2}\right] + 2\sum_{i<j} \min\{\lambda_i, \lambda_j\}$. Hence

$$\dim \mathfrak{g}_0^e - \dim \mathfrak{g}_1^e + n = 2n - 2\left[\frac{\lambda_i + 1}{2}\right] = n - \#\{i \mid \lambda_i \text{ is odd}\}.$$

For $e \neq 0$, the minimal value 2 is attained for $\boldsymbol{\lambda} = (2, 1^{n-2})$ or $(3, 1^{n-3})$.

A.3. $(\mathfrak{g}, \mathfrak{g}_0) = (\mathfrak{so}_{2n}, \mathfrak{gl}_n)$. If $e \in \mathfrak{gl}_n$ and $e \sim \boldsymbol{\lambda}$, then $\dim \mathfrak{g}_0^e = \dim(\mathfrak{gl}_n)^e$ is again given by (A·1), while using this time (A·3), we obtain $\dim \mathfrak{g}_1^e = 2\left[\frac{\lambda_i}{2}\right] + 2\sum_{i<j} \min\{\lambda_i, \lambda_j\}$. Hence the result is even better than in the previous case. Indeed, we have here $\dim \mathfrak{g}_0^e - \dim \mathfrak{g}_1^e \geqslant 0$.

A.4. $(\mathfrak{g},\mathfrak{g}_0)=(\mathfrak{sl}_{n+m},\mathfrak{sl}_n\times\mathfrak{sl}_m\times\mathfrak{t}_1)$. Here $n,m\geqslant 1$. A nilpotent element $e\in\mathfrak{g}_0$ is determined by two partitions, $e\sim(\boldsymbol{\lambda};\boldsymbol{\mu})=((\lambda_1,\dots,\lambda_k);(\mu_1,\dots,\mu_s))$. Using (A·1), we obtain

$$\dim\mathfrak{g}_0^e=n+m-1+2\sum_{i<j}\min\{\lambda_i,\lambda_j\}+2\sum_{i<j}\min\{\mu_i,\mu_j\},$$
$$\dim\mathfrak{g}_1^e=2\sum_{i,j}\min\{\lambda_i,\mu_j\}.$$

Therefore, $\dim\mathfrak{g}_0^e-\dim\mathfrak{g}_1^e+(n+m-1)=$
$$2\Bigl(n+m-1+\sum_{i<j}\min\{\lambda_i,\lambda_j\}+\sum_{i<j}\min\{\mu_i,\mu_j\}-\sum_{i,j}\min\{\lambda_i,\mu_j\}\Bigr).$$

Since $n=\sum_i\lambda_i$, $m=\sum_j\mu_j$, and $\sum_{i<j}\min\{\lambda_i,\lambda_j\}=\sum_{i\geqslant 2}(i-1)\lambda_i$, half of the RHS equals

$$\mathcal{F}(\boldsymbol{\lambda};\boldsymbol{\mu}):=\sum_{i=1}^k i\lambda_i+\sum_{j=1}^s j\mu_j-1-\sum_{i=1}^k\sum_{j=1}^s\min\{\lambda_i,\mu_j\}.$$

Arguing by induction, we prove that $\mathcal{F}(\boldsymbol{\lambda};\boldsymbol{\mu})\geqslant 0$ for all $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, and if $n+m\geqslant 3$, then $\mathcal{F}(\boldsymbol{\lambda};\boldsymbol{\mu})>0$.

1^o. First, $\mathcal{F}(1^n;1^m)=(n-m)^2/2+(n+m)/2-1$, which is positive if $(n,m)\neq(1,1)$.

2^o. The inequality is easily verified, if $\boldsymbol{\lambda}$ or $\boldsymbol{\mu}$ consists of only one part.

3^o. Suppose that $k\geqslant 2$ and $s\geqslant 2$. Write $\boldsymbol{\lambda}=(\lambda_1,\boldsymbol{\lambda}')$ and $\boldsymbol{\mu}=(\mu_1,\boldsymbol{\mu}')$. Then

$$\mathcal{F}(\boldsymbol{\lambda};\boldsymbol{\mu})=\mathcal{F}(\boldsymbol{\lambda}';\boldsymbol{\mu}')+\max\{\lambda_1,\mu_1\}+\sum_{i\geqslant 2}(\lambda_i-\min\{\lambda_i,\mu_1\})+\sum_{j\geqslant 2}(\mu_j-\min\{\lambda_1,\mu_j\})$$
$$\geqslant\mathcal{F}(\boldsymbol{\lambda}';\boldsymbol{\mu}')+\max\{\lambda_1,\mu_1\}\geqslant\max\{\lambda_1,\mu_1\}.$$

Here $\max\{\lambda_1,\mu_1\}$ arises as $\lambda_1+\mu_1-\min\{\lambda_1,\mu_1\}$.

We omit computations related to the remaining classical symmetric pairs $(\mathfrak{sl}_{2n},\mathfrak{sp}_{2n})$, $(\mathfrak{sp}_{2n+2m},\mathfrak{sp}_{2n}\times\mathfrak{sp}_{2m})$, and $(\mathfrak{so}_{n+m},\mathfrak{so}_n\times\mathfrak{so}_m)$.

Acknowledgements. Part of this work was done while I was visiting the Max-Planck Institut für Mathematik (Bonn).

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