MODULI OF MATHEMATICAL INSTANTON VECTOR BUNDLES WITH ODD c_2 ON PROJECTIVE SPACE

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Abstract. We study the problem of irreducibility of the moduli space I_n of rank-2 mathematical instanton vector bundles with second Chern class $n \geq 1$ on the projective space $\mathbb{P}^3$. The irreducibility of I_n was known for small values of n: for $n = 1$ it was proved by Barth (1977), for $n = 2$ by Hartshorne (1978), for $n = 3$ by Ellingsrud and Strømme (1981), for $n = 4$ by Barth (1981), for $n = 5$ by Coanda, Tikhomirov and Trautmann (2003). In this paper we prove the irreducibility of I_n for an arbitrary odd $n \geq 1$.

Bibliography: 22 items.

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1. Introduction

By a *mathematical n-instanton vector bundle* (shortly, a *n-instanton*) on 3-dimensional projective space $\mathbb{P}^3$ we understand a rank-2 algebraic vector bundle E on $\mathbb{P}^3$ with Chern classes

$$c_1(E) = 0, \quad c_2(E) = n, \quad n \geq 1, \tag{1}$$

satisfying the vanishing conditions

$$h^0(E) = h^1(E(-2)) = 0. \tag{2}$$

Denote by I_n the set of isomorphism classes of n-instantons. This space is nonempty for any $n \geq 1$ - see, e.g., [BT], [NT]. The condition $h^0(E) = 0$ for a n-instanton E implies that E is stable in the sense of Gieseker-Maruyama. Hence I_n is a subset of the moduli scheme $M_{\mathbb{P}^3}(2; 0, 2, 0)$ of semistable rank-2 torsion-free sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0,\ c_2 = n,\ c_3 = 0$. The condition $h^1(E(-2)) = 0$ for $[E] \in I_n$ (called the *instanton condition*) implies by semicontinuity that I_n is a Zariski open subset of $M_{\mathbb{P}^3}(2; 0, 2, 0)$, i.e. I_n is a quasiprojective scheme. It is called the *moduli scheme of mathematical n-instantons*.

In this paper we study the problem of the irreducibility of the scheme I_n. This problem has an affirmative solution for small values of n, up to $n = 5$. Namely, the cases $n = 1, 3, 3, 4$ and 5 were settled in papers [B1], [H], [ES], [B3] and [CTT], respectively. The aim of this paper is to prove the following result.

Theorem 1.1. *For each $n = 2m + 1,\quad m \geq 0$, the moduli scheme I_n of mathematical n-instantons is an integral scheme of dimension $8n - 3$.*

A guide to the paper is as follows. In section 3 we recall a well-known relation between mathematical n-instantons and nets of quadrics in a fixed n-dimensional vector space H_n over $\mathbf{k}$. The nets of quadrics are considered as vectors of the space $\mathbf{S}_n = S^2H_n^\vee \otimes \wedge^2V^\vee$, where $V = H^0(\mathcal{O}_{\mathbb{P}^3}(1))^\vee$, and those nets which correspond to n-instantons (we call them n-instanton nets) satisfy the so-called Barth's conditions - see definition (14). These nets constitute a locally closed subset $MI_n \subset$ of $\mathbf{S}_n$ which has a structure of a $GL(n)/\{\pm 1\}$-bundle over I_n. Thus the irreducibility of the moduli space I_n of n-instantons reduces to the irreducibility of the space MI_n of n-instanton nets of quadrics.

Section 4 is a study of some linear algebra related to a direct sum decomposition $\xi : H_{m+1} \oplus H_m \xrightarrow{\sim} H_{2m+1}$ giving the above embedding $H_{m+1} \hookrightarrow H_{2m+1}$. Using one result of section 11 we obtain here the relation (30) which is a key instrument for our further considerations. Also, the decomposition ξ enables us to relate $(2m+1)$-instantons E to rank-$(2m+2)$ symplectic vector bundles E_{2m+2} on $\mathbb{P}^3$ satisfying the vanishing conditions $h^0(E_{2m+2}) = h^2(E_{2m+2}(-2)) = 0$.

In section 6 we introduce a new set X_m as a locally closed subset of the vector space $\mathbf{S}_{m+1} \oplus \mathbf{\Sigma}_{m+1}$, where $\mathbf{\Sigma}_{m+1} = \mathrm{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$, defined by linear algebraic data somewhat similar to Barth's conditions. We prove that X_m, is isomorphic to a certain dense open subset $MI_{2m+1}(\xi)$ of MI_{2m+1} determined by the choice of the direct sum decomposition ξ above, where both X_m and $MI_{2m+1}(\xi)$ are understood as reduced schemes. This reduces the problem of the irreducibility of I_{2m+1} to that of X_m.

The last ingredient in the proof of Theorem 1.1 is a scheme Z_m introduced in section 7 as a locally closed subscheme of the affine space $\mathbf{S}_m^\vee \times \mathrm{Hom}(H_m, H_m^\vee \otimes \wedge^2 V^\vee)$ defined by explicit equations (see (76)). In section 7 we reduce the proof of Theorem 1.1 to the fact that Z_m is an integral locally complete intersection subscheme of the above mentioned affine space. This and other properties of Z_m are formulated in Theorem 7.2. The rest of the paper is devoted to the proof of Theorem 7.2.

In section 8 we start the proof of this Theorem by induction on m and prove a part of the induction step - see Proposition 8.1. The proof of it contains explicit computations in linear algebra. These computations seem to be somewhat cumbersome, and Remark 8.3 at the end of this section gives an explanation why these computations could not be essentially simplified.

Proposition 8.1 enables us then in section 9.1 to relate Z_m to the so-called t'Hooft instantons. As a result, in section 10 we finish the induction step in the proof of Theorem 7.2.

In Appendix (section 11) we prove two results of general position for nets of quadrics, which are used in the text.

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2. Notation and conventions

Our notations are mostly standard. The base field $\mathbf{k}$ is assumed to be algebraically closed of characteristic 0. We identify vector bundles with locally free sheaves. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules on an algebraic variety or scheme X, then $n\mathcal{F}$ denotes a direct sum of n copies of the sheaf $\mathcal{F}$, $H^i(\mathcal{F})$ denotes the i^{th} cohomology group of $\mathcal{F}$, $h^i(\mathcal{F}) := \dim H^i(\mathcal{F})$, and $\mathcal{F}^\vee$ denotes the dual to $\mathcal{F}$ sheaf, i.e. the sheaf $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If Z is a subscheme of X, by $\mathcal{I}_{Z,X}$ we denote the ideal sheaf corresponding to a subscheme Z. If $X = \mathbb{P}^r$ and t is an integer, then by $\mathcal{F}(t)$ we denote the sheaf $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(t)$. $[\mathcal{F}]$ will denote the isomorphism class of a sheaf $\mathcal{F}$. For any morphism of $\mathcal{O}_X$-sheaves $f : \mathcal{F} \to \mathcal{F}'$ and any $\mathbf{k}$-vector space U (respectively, for any homomorphism $f : U \to U'$ of $\mathbf{k}$-vector spaces) we will denote, for short, by the same letter f the induced morphism of sheaves $id \otimes f : U \otimes \mathcal{F} \to U \otimes \mathcal{F}'$ (respectively, the induced morphism $f \otimes id : U \otimes \mathcal{F} \to U' \otimes \mathcal{F}$).

Everywhere in the paper V will denote a fixed vector space of dimension 4 over $\mathbf{k}$ and we set $\mathbb{P}^3 := P(V)$. Also everywhere below we will reserve the letters u and v for denoting the two morphisms in the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{u} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{v} T_{\mathbb{P}^3}(-1) \to 0$. For any $\mathbf{k}$-vector spaces U and W and any vector $\phi \in \mathrm{Hom}(U, W \otimes \wedge^2 V^\vee) \subset \mathrm{Hom}(U \otimes V, W \otimes V^\vee)$ understood as a homomorphism $\phi : U \otimes V \to W \otimes V^\vee$ or, equivalently, as a homomorphism ${}^\sharp\phi : U \to W \otimes \wedge^2 V^\vee$, we will denote by $\widetilde{\phi}$ the composition $U \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{{}^\sharp\phi} W \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} W \otimes \Omega_{\mathbb{P}^3}(2)$, where ϵ is the induced morphism in the exact triple $0 \to \wedge^2 \Omega_{\mathbb{P}^3}(2) \xrightarrow{\wedge^2 v^\vee} \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} \Omega_{\mathbb{P}^3}(2) \to 0$ obtained by passing to the second wedge power in the dual Euler exact sequence. Also, shortening the notation, we will omit sometimes the subscript $\mathbb{P}^3$ in the notation of sheaves on $\mathbb{P}^3$, e.g., write $\mathcal{O}$, Ω etc., instead of $\mathcal{O}_{\mathbb{P}^3}$, $\Omega_{\mathbb{P}^3}$ etc., respectively.

Next, as above, for any integer $n \geq 1$ by H_n we understand a fixed n-dimensional vector space over $\mathbf{k}$. (E. g., one can take $\mathbf{k}^n$ for H_n.)

Everywhere in the paper for $m \geq 1$ we denote by $\mathbf{S}_m$ the vector space $S^2H_m^\vee \otimes \wedge^2 V^\vee$, respectively, by $\mathbf{\Sigma}_{m+1}$ the vector space $\mathrm{Hom}(H_m, H_{m+1}^\vee \otimes \wedge^2 V^\vee)$. For a given $\mathbf{k}$-vector space U (respectively, a direct sum $U \oplus U'$ of two $\mathbf{k}$-vector spaces) we will, abusing notations, denote by the same letter U (respectively, by $U \oplus U'$) the corresponding affine space $\mathbf{V}(U^\vee) = \mathrm{Spec}(Sym^* U^\vee)$ (respectively, the direct product of affine spaces $\mathbf{V}(U^\vee) \times \mathbf{V}(U'^\vee)$).

All the schemes considered in the paper are Noetherian. By an irreducible scheme we understand a scheme whose underlying topological space is irreducible. By an integral scheme we understand an irreducible reduced scheme. Also, by the dimension of a given scheme we understand below the maximum of dimensions of its irreducible components. By a general point of an irreducible (but not necessarily reduced) scheme $\mathcal{X}$ we mean any closed point belonging to some dense open subset of $\mathcal{X}$. An irreducible scheme is called generically reduced if it is reduced at a general point.

3. Some generalities on instantons. Set MI_n

In this Section we recall some well known facts about mathematical instanton bundles - see, e.g., [CTT].

For a given n-instanton E, the conditions (1), (2), Riemann-Roch and Serre duality imply

$$h^1(E(-1)) = h^2(E(-3)) = n, \quad h^1(E \otimes \Omega^1_{\mathbb{P}^3}) = h^2(E \otimes \Omega^2_{\mathbb{P}^3}) = 2n+2, \tag{3}$$

$$h^1(E) = h^2(E(-4)) = 2n-2.$$

$$h^i(E) = h^i(E(-1)) = h^{3-i}(E(-3)) = h^{3-i}(E(-4)) = 0, \quad i \neq 1, \quad h^i(E(-2)) = 0, \quad i \geq 0. \tag{4}$$

Furthermore, the condition $c_1(E) = 0$ yields an isomorphism $\wedge^2 E \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3}$, hence a symplectic isomorphism $j : E \xrightarrow{\simeq} E^\vee$ defined uniquely up to a scalar. Consider a triple (E, f, j) where E is an n-instanton, f is an isomorphism $H_n \xrightarrow{\simeq} H^2(E(-3))$ and $j : E \xrightarrow{\simeq} E^\vee$ is a symplectic structure on E. Note that, since E as a stable rank-2 bundle, it is a simple bundle, i. e. any automorphism φ of E has the form $\varphi = \lambda \mathrm{id}$ for some $\lambda \in \mathbf{k}^*$. Imposing the condition that φ is compatible with the symplectic structure j, i. e. $\varphi^\vee \circ j \circ \varphi = j$, we obtain $\lambda = \pm 1$. This leads to the following definition of equivalence of triples (E, f, j). We call two such triples (E, f, j) and $(E'f', j')$ equivalent if there is an isomorphism $g : E \xrightarrow{\simeq} E'$ such that $g_* \circ f = \lambda f'$ with $\lambda \in \{1, -1\}$ and $j = g^\vee \circ j' \circ g$, where $g_* : H^2(E(-3)) \xrightarrow{\simeq} H^2(E'(-3))$ is the induced isomorphism. We denote by $[E, f, j]$ the equivalence class of a triple (E, f, j). From this definition one easily deduces that the set $F_{[E]}$ of all equivalence classes $[E, f, j]$ with given $[E]$ is a homogeneous space of the group $GL(H_n)/\{\pm \mathrm{id}\}$.

Each class $[E, f, j]$ defines a point

$$A = A([E, f, j]) \in S^2 H_n^\vee \otimes \wedge^2 V^\vee \tag{5}$$

in the following way. Consider the exact sequences

$$0 \to \Omega^1_{\mathbb{P}^3} \xrightarrow{i_1} V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0, \tag{6}$$

$$0 \to \Omega^2_{\mathbb{P}^3} \to \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \Omega^1_{\mathbb{P}^3} \to 0, 0 \to \wedge^4 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-4) \to \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{i_2} \Omega^2_{\mathbb{P}^3} \to 0,$$

induced by the Koszul complex of $V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \stackrel{ev}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}$. Twisting these sequences by E and passing to cohomology in view of (2)-(4) gives the equalities $0 = h^0(E\otimes\Omega_{\mathbb{P}^3}) = h^3(E\otimes\Omega^2_{\mathbb{P}^3}) = h^2(E\otimes\Omega_{\mathbb{P}^3})$ and the diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & H^2(E(-4))\otimes\wedge^4V^\vee \arrow[r] & H^2(E(-3))\otimes\wedge^3V^\vee \arrow[r, "i_2"] \arrow[d, "A'"] & H^2(E\otimes\Omega^2_{\mathbb{P}^3}) \arrow[r] & 0 \\
0 & H^1(E)) \arrow[l] & H^1(E(-1))\otimes V^\vee \arrow[l] & H^1(E\otimes\Omega_{\mathbb{P}^3}) \arrow[l, "i_1"'] \arrow[u, "\cong"', "\partial"] & 0, \arrow[l]
\end{tikzcd}
\tag{7}
$$

where $A' := i_1\circ\partial^{-1}\circ i_2$. The Euler exact sequence (6) yields a canonical isomorphism $\omega_{\mathbb{P}^3} \stackrel{\simeq}{\to} \wedge^4V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(-4)$, and fixing an isomorphism $\tau:\mathbf{k}\stackrel{\simeq}{\to}\wedge^4V^\vee$ induces isomorphisms $\tilde\tau: V\stackrel{\simeq}{\to}\wedge^3V^\vee$ and $\hat\tau:\omega_{\mathbb{P}^3}\stackrel{\simeq}{\to}\mathcal{O}_{\mathbb{P}^3}(-4)$. Now the point A in (5) is defined as the composition

$$
\begin{aligned}
A: H_n\otimes V &\overset{\tilde\tau}{\underset{\simeq}{\to}} H_n\otimes\wedge^3V^\vee \overset{f}{\underset{\simeq}{\to}} H^2(E(-3))\otimes\wedge^3V^\vee \xrightarrow{A'} H^1(E(-1))\otimes V^\vee \overset{j}{\underset{\simeq}{\to}} \\
&\overset{j}{\underset{\simeq}{\to}} H^1(E^\vee(-1))\otimes V^\vee \overset{SD}{\underset{\simeq}{\to}} H^2(E(1)\otimes\omega_{\mathbb{P}^3})^\vee\otimes V^\vee \overset{\hat\tau}{\underset{\simeq}{\to}} H^2(E(-3))^\vee\otimes V^\vee \overset{f^\vee}{\underset{\simeq}{\to}} H_n^\vee\otimes V^\vee,
\end{aligned}
\tag{8}
$$

where SD is the Serre duality isomorphism. One checks that A is a skew symmetric map depending only on the class $[E,f,j]$ and not depending on the choice of τ, and that this point $A\in\wedge^2(H_n^\vee\otimes V^\vee)$ lies in the direct summand $\mathbf{S}_n = S^2H_n^\vee\otimes\wedge^2V^\vee$ of the canonical decomposition

$$\wedge^2\,(H_n^\vee\otimes V^\vee) = S^2H_n^\vee\otimes\wedge^2V^\vee\oplus\wedge^2H_n^\vee\otimes S^2V^\vee. \tag{9}$$

Here $\mathbf{S}_n$ is the space of nets of quadrics in H_n. Following [B3], [T1] and [T2] we call A the *n-instanton net of quadrics* corresponding to the data $[E,f,j]$.

Denote $W_A := H_n\otimes V/\ker A$. Using the above chain of isomorphisms we can rewrite the diagram (7) as

$$
\begin{tikzcd}
0 \arrow[r] & \ker A \arrow[r] & H_n\otimes V \arrow[r, "c_A"] \arrow[d, "A"] & W_A \arrow[r] \arrow[d, "\cong"', "q_A"] & 0 \\
0 & \ker A^\vee \arrow[l] & H_n^\vee\otimes V^\vee \arrow[l] & W_A^\vee \arrow[l, "c_A^\vee"'] & 0. \arrow[l]
\end{tikzcd}
\tag{10}
$$

Here in view of (3) $\dim W_A = 2n+2$ and $q_A: W_A\stackrel{\simeq}{\to}W_A^\vee$ is the induced skew-symmetric isomorphism. An important property of $A = A([E,f,j])$ is that the induced morphism of sheaves

$$a_A^\vee: W_A^\vee\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{c_A^\vee} H_n^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{ev} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) \tag{11}$$

is an epimorphism such that the composition $H_n\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{a_A} W_A\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{q_A} W_A^\vee\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{a_A^\vee} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)$ is zero, and $E = \ker(a_A^\vee\circ q_A)/\operatorname{Im} a_A$. Thus A defines a monad

$$\mathcal{M}_A:\quad 0\to H_n\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{a_A} W_A\otimes\mathcal{O}_{\mathbb{P}^3}\xrightarrow{a_A^\vee\circ q_A} H_n^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1)\to 0 \tag{12}$$

with the cohomology sheaf E,

$$E = E(A) := \ker(a_A^\vee\circ q_A)/\operatorname{Im} a_A. \tag{13}$$

Note that passing to cohomology in the monad $\mathcal{M}_A$ twisted by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and using (13) yields the isomorphism $f: H_n\stackrel{\simeq}{\to}H^2(E(-3))$. Furthermore, the simplecticity of the form q_A in the monad $\mathcal{M}_A$ implies that there is a canonical isomorphism of $\mathcal{M}_A$ with its dual monad, and this isomorphism induces the symplectic isomorphism $j: E\stackrel{\simeq}{\to}E^\vee$. Thus, the data $[E,f,j]$

are recovered from the net A. This leads to the following description of the moduli space I_n. Consider the *set of n-instanton nets of quadrics*

$$MI_n := \left\{ A \in \mathbf{S}_n \;\middle|\; \begin{array}{c} (i)\ \mathrm{rk}(A : H_n \otimes V \to H_n^\vee \otimes V^\vee) = 2n+2, \\ (ii)\ \text{the morphism } a_A^\vee : W_A^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \to H_n^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \\ \text{defined by } A \text{ in (11) is surjective,} \\ (iii)\ h^0(E_2(A)) = 0,\ \text{where } E_2(A) := \ker(a_A^\vee \circ q_A)/\operatorname{Im} a_A \\ \text{and } q_A : W_A \xrightarrow{\simeq} W_A^\vee \text{ is a symplectic isomorphism} \\ \text{defined by } A \text{ in (10)} \end{array} \right\} \tag{14}$$

The conditions (i)-(iii) here are called *Barth's coditions*. These conditions show that MI_n is naturally endowed with a structure of a locally closed subscheme of the vector space $\mathbf{S}_n$. Moreover, the above description shows that there is a morphism $\pi_n : MI_n \to I_n :\ A \mapsto [E(A)]$, and it is known that this morphism is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology - cf. [CTT]. Here by construction the fibre $\pi_n^{-1}([E])$ over an arbitrary point $[E] \in I_n$ coincides with the homogeneous space $F_{[E]}$ of the group $GL(H_n)/\{\pm\mathrm{id}\}$ described above. Hence the irreducibility of $(I_n)_{red}$ is equivalent to the irreducibility of the scheme $(MI_n)_{red}$.

The definition (14) yields the following.

Theorem 3.1. *For each $n \geq 1$, the space of n-instanton nets of quadrics MI_n is a locally closed subscheme of the vector space $\mathbf{S}_n$ given locally at any point $A \in MI_n$ by*

$$\binom{2n-2}{2} = 2n^2 - 5n + 3 \tag{15}$$

equations obtained as the rank condition (i) in (14).

Note that from (15) it follows that

$$\dim_{[A]} MI_n \geq \dim \mathbf{S}_n - (2n^2 - 5n + 3) = n^2 + 8n - 3 \tag{16}$$

at any point $A \in MI_n$. On the other hand, by deformation theory for any n-instanton E we have $\dim_{[E]} I_n \geq 8n-3$. This agrees with (16), since $MI_n \to I_n$ is a principal $GL(H_n)/\{\pm\mathrm{id}\}$-bundle in the étale topology.

Let $\mathcal{S}_n = \{[E] \in I_n |$ there exists a line $l \in \mathbb{P}^3$ of maximal jump for E, i.e. such a line l that $h^0(E(-n)|_l) \neq 0\}$. It is known [S] that $\mathcal{S}_n$ is a closed subset of I_n of dimension $6n+2$, and I_n is smooth along $\mathcal{S}_n$. Thus, since $\dim_{[E]} I_n \geq 8n-3$ at any $[E] \in I_n$, it follows that

$$I_n' := I_n \smallsetminus \mathcal{S}_n \tag{17}$$

is an open subset of I_n and $(I_n')_{red}$ is dense open in $(I_n)_{red}$; respectively,

$$MI_n' := \pi_n^{-1}(I_n') \tag{18}$$

is an open subset of MI_n and we have a dense open embedding

$$(MI_n')_{red} \xhookrightarrow{\text{dense open}} (MI_n)_{red}\ . \tag{19}$$

For technical reasons we will below restrict ourselves to MI_n' instead of MI_n.

Remark 3.2. There exist smooth points of I_n - see, e.g., [NT]. Hence, there exist smooth points in MI_n.

4. Decomposition $H_{2m+1} \simeq H_{m+1} \oplus H_m$ and related constructions

4.1. One result of general position for $(2m+1)$-instanton nets.
Fix a positive integer $m \geq 3$ and, for a given $(2m+1)$-instanton vector bundle $[E] \in I'_{2m+1}$, fix an isomorphism $f : H_{2m+1} \xrightarrow{\simeq} H^2(E(-3))$ and set

$$H_{4m} := H^2(E(-4)), \quad W_{4m+4} := H^1(E \otimes \Omega_{\mathbb{P}^3})^\vee. \tag{20}$$

(Here we keep in mind the equalities (3) for $n = 2m+1$.) In this notation, the lower exact triple in (7) can be rewritten as:

$$0 \to W^\vee_{4m+4} \to H^\vee_{2m+1} \otimes V^\vee \xrightarrow{mult} H^\vee_{4m} \to 0 \tag{21}$$

We formulate now the following result of general position for $(2m+1)$-instanton nets of quadrics which will be important for further study.

Theorem 4.1. *Let $m \geq 3$ and let E be a $(2m+1)$-instanton, $[E] \in I'_{2m+1}$, supplied with an isomorphism $f : H_{2m+1} \xrightarrow{\simeq} H^2(E(-3))$ and set $W_{4m+4} = H^1(E \otimes \Omega_{\mathbb{P}^3})^\vee$, so that there is the injection $W^\vee_{4m+4} \hookrightarrow H^\vee_{2m+1} \otimes V^\vee$ defined in (21). Then for a generic m-dimensional subspace V_m of $H^\vee_{2m+1}$ one has*

$$W^\vee_{4m+4} \cap V_m \otimes V^\vee = \{0\}.$$

The proof of this Theorem has rather technical character, and we leave it to the end of the paper - see Appendix (section 11).

4.2. Decomposition $H_{2m+1} \simeq H_{m+1} \oplus H_m$.
Fix an isomorphism

$$\xi : H_{m+1} \oplus H_m \xrightarrow{\simeq} H_{2m+1} \tag{22}$$

and let

$$H_{m+1} \overset{i_{m+1}}{\hookrightarrow} H_{m+1} \oplus H_m \overset{i_m}{\hookleftarrow} H_m \tag{23}$$

be the injections of direct summands. For a given $(2m+1)$-instanton vector bundle E, $[E] \in I'_{2m+1}$, fix an isomorphism $f : H_{2m+1} \xrightarrow{\simeq} H^2(E(-3))$ and a symplectic structure $j : E \xrightarrow{\simeq} E^\vee$. The data $[E, f, j]$ define a net of quadrics $A \in MI'_{2m+1}$ (see section 3), and the exact triple (21) is naturally identified with the dual to the triple $0 \to \ker A \to H_{2m+1} \otimes V \to W_A \to 0$ and fits in diagram (10) for $n = 2m+1$

$$\begin{tikzcd} 0 \arrow[r] & \ker A \arrow[r] & H_{2m+1} \otimes V \arrow[r, "c_A"] \arrow[d, "A"] & W_A \arrow[r] \arrow[d, "\cong"', "q_A"] & 0 \\ 0 & \ker A^\vee \arrow[l] & H^\vee_{2m+1} \otimes V^\vee \arrow[l] & W^\vee_A \arrow[l, "c^\vee_A"'] & 0. \arrow[l] \end{tikzcd} \tag{24}$$

Consider the composition

$$j_{\xi,A} : H_{m+1} \otimes V \overset{i_{m+1}}{\hookrightarrow} H_{m+1} \otimes V \oplus H_m \otimes V \overset{\xi}{\xrightarrow{\simeq}} H_{2m+1} \otimes V \xrightarrow{c_A} W_A. \tag{25}$$

Under these notations Theorem 4.1 can be reformulated in the following way:

(*) *Assume $m \geq 3$ and let A be an arbitrary $(2m+1)$-net from MI'_{2m+1}. Then for a generic isomorphism $\xi : H_{2m+1} \xrightarrow{\simeq} H_{m+1} \oplus H_m$ one has*

$$\ker A \cap (\xi \circ i_{m+1})(H_{m+1} \otimes V) = \{0\}. \tag{26}$$

Equivalently, $j_{\xi,A} : H_{m+1} \otimes V \to W_A$ is an isomorphism.

Consider the direct sum decomposition corresponding to the isomorphism (22)

$$\widetilde{\xi}:\mathbf{S}_{m+1}\ \oplus\ \mathbf{\Sigma}_{m+1}\ \oplus\ \mathbf{S}_m \xrightarrow{\sim} \mathbf{S}_{2m+1} \tag{27}$$

and let

$$\mathbf{S}_{2m+1}\twoheadrightarrow \mathbf{S}_{m+1}: A\mapsto A_1(\xi),\quad \mathbf{S}_{2m+1}\twoheadrightarrow \mathbf{\Sigma}_{m+1}: A\mapsto A_2(\xi),\quad \mathbf{S}_{2m+1}\twoheadrightarrow \mathbf{S}_{m}: A\mapsto A_3(\xi) \tag{28}$$

be the projections onto direct summands. By definition, $A_1(\xi)$ considered as a skew-symmetric homomorphism $H_{m+1}\otimes V\to H^\vee_{m+1}\otimes V^\vee$ coincides with the composition

$$A_1(\xi):\ H_{m+1}\otimes V \xrightarrow{j_{\xi,A}} W_A \underset{\simeq}{\xrightarrow{q_A}} W_A^\vee \xrightarrow{j^\vee_{\xi,A}} H^\vee_{m+1}\otimes V^\vee. \tag{29}$$

This means that assertion (*) can be reformulated as:

(**) *Assume $m\ge 3$ and let A be an arbitrary $(2m+1)$-net from MI'_{2m+1}. Then for a generic isomorphism ξ in (22) the skew-symmetric homomorphism $A_1(\xi):\ H_{m+1}\otimes V\to H^\vee_{m+1}\otimes V^\vee$ is invertible.*

Now, using the notation (28), we can represent the net $A\in\mathbf{S}_{2m+1}$ considered as a homomorphism $A: H_{m+1}\otimes V\oplus H_m\otimes V\to H^\vee_{m+1}\otimes V^\vee\oplus H^\vee_m\otimes V^\vee$ by the $(8m+4)\times(8m+4)$-matrix of homomorphisms

$$A=\begin{pmatrix} A_1(\xi) & A_2(\xi)\\ -A_2(\xi)^\vee & A_3(\xi)\end{pmatrix}.$$

This matrix is of rank $4m+4$ according to Barth's condition (i) in (14). On the other hand, by (**) we have $\mathrm{rk}A_1(\xi)=4m+4$, i.e. ranks of A and of its submatrix $A_1(\xi)$ coincide. This yields, after multiplying the matrix A by the invertible matrix of homomorphisms

$$\begin{pmatrix} A_1(\xi)^{-1} & \mathbf{0}\\ A_2(\xi)^\vee\circ A_1(\xi)^{-1} & \mathrm{id}_{H^\vee_m\otimes V^\vee}\end{pmatrix}$$

from the left, the following relation between the matrices $A_1(\xi),\ A_2(\xi)$ and $A_3(\xi)$:

$$A_3(\xi)=-A_2(\xi)^\vee\circ A_1(\xi)^{-1}\circ A_2(\xi), \tag{30}$$

Remark 4.2. This relation means that $A_3(\xi)$ is uniquely determined by $A_1(\xi)$ and $A_2(\xi)$. We will use this important observation systematically in the sequel.

For $m\ge 1$ let Isom_{2m+1} be the set of all isomorphisms ξ in (22) and set

$$MI_{2m+1}(\xi):=\{A\in MI'_{2m+1}\mid \text{the skew}-\text{symmetric homomorphism } A_1(\xi) \text{ in (29)} \\ \text{is invertible}\},\quad \xi\in \mathrm{Isom}_{2m+1}. \tag{31}$$

In these notations we have the following result.

Theorem 4.3. *For $m\ge 3$ the following statements hold.*

(i) There exists a dense subset Isom^0_{2m+1} of Isom_{2m+1} such that the sets $MI_{2m+1}(\xi),\ \xi\in \mathrm{Isom}^0_{2m+1}$, constitute an open cover of MI'_{2m+1} .

(ii) There exists a dense open subset $\mathrm{Isom}^{00}_{2m+1}$ of Isom_{2m+1} contained in Isom^0_{2m+1} such that the sets $MI_{2m+1}(\xi),\ \xi\in\mathrm{Isom}^{00}_{2m+1}$, are dense open subsets of MI'_{2m+1}.

(iii) For any $\xi\in\mathrm{Isom}^0_{2m+1}$ and any $A\in MI_{2m+1}(\xi)$ the relation (30) is true.

Proof. (i)-(ii) Let $MI'_{2m+1}=M_1\cup\ldots\cup M_s$ be a decomposition of MI'_{2m+1} into irreducible components. Consider the set $U:=\{(A,\xi)\in MI'_{2m+1}\times\mathrm{Isom_{2m+1}}\mid A_1(\xi):\ H_{m+1}\otimes V\to H^\vee_{m+1}\otimes V^\vee$ is invertible $\}$ with projections $MI'_{2m+1}\xleftarrow{p} U\xrightarrow{q}\mathrm{Isom_{2m+1}}$, and let $U_i:=U\cap M_i\times\mathrm{Isom_{2m+1}}$ with the induced projections $M_i\xleftarrow{p_i}U_i\xrightarrow{q_i}\mathrm{Isom_{2m+1}},\ \ i=1,\ldots,s$. By definition, U is open in $MI'_{2m+1}\times\mathrm{Isom_{2m+1}}$, hence each U_i is open in $M_i\times\mathrm{Isom_{2m+1}}$. Moreover, the property (**) implies that $p_i(U_i)=M_i$, so that U_i is nonempty, hence dense in $M_i\times\mathrm{Isom_{2m+1}}$

since both M_i and $\mathrm{Isom_{2m+1}}$ are irreducible. (Note that $\mathrm{Isom_{2m+1}}$ is irreducible as a principal homogeneous space of the group $GL(2m+1)$.) Hence $q_i(U_i)$ contains a dense open subset, say, W_i of $\mathrm{Isom_{2m+1}}$. Set $\mathrm{Isom}^0_{2m+1} := \underset{1\le i\le s}{\cup} q_i(U_i)$ and $\mathrm{Isom}^{00}_{2m+1} := \underset{1\le i\le s}{\cap} W_i$. By construction, the sets $MI_{2m+1}(\xi) \simeq q^{-1}(\xi),\ \xi \in \mathrm{Isom}^0_{2m+1}$, constitute an open cover of MI'_{2m+1}. Respectively, for any $\xi \in \mathrm{Isom}^{00}_{2m+1}$ and each i, $1 \le i \le s$, the set $q_i^{-1}(\xi)$ is nonempty open, hence dense subset in M_i. This yields that, for $\xi \in \mathrm{Isom}^{00}_{2m+1}$, the set $MI'_{2m+1}(\xi)) \simeq q^{-1}(\xi) = \underset{1\le i\le s}{\cup} q_i^{-1}(\xi)$ is dense open in MI'_{2m+1}.

(iii) This follows from (30) and (**). $\square$

We will need below the following lemma.

Lemma 4.4. *For $\xi \in \mathrm{Isom}^0_{2m+1}$ and $A \in MI_{2m+1}(\xi)$, set*

$$B := A_1(\xi), \quad C := A_2(\xi). \tag{32}$$

Then the following statements hold.

(i) Consider a subbundle morphism

$$\alpha_{\xi,A} := j^{-1}_{\xi,A} \circ a_A \circ \xi : (H_{m+1} \oplus H_m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \to H_{m+1} \otimes V \otimes \mathcal{O}_{\mathbb{P}^3}. \tag{33}$$

Then there exists an epimorphism

$$\lambda_{\xi,A} : \mathrm{coker}(B \circ \alpha_{\xi,A}) \twoheadrightarrow H^\vee_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(1). \tag{34}$$

making commutative the diagram

$$\begin{tikzcd} H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "can"] \arrow[dr, "u^\vee"'] & \mathrm{coker}(B \circ \alpha_{\xi,A}) \arrow[d, "\lambda_{\xi,A}"] \\ & H^\vee_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(1), \end{tikzcd} \tag{35}$$

where can is the canonical surjection.

(ii) Consider the commutative diagram

$$\begin{tikzcd} & H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) & & & \\ 0 \arrow[r] & (H_{m+1} \oplus H_m) \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[u, two heads] \arrow[r, "B\circ\alpha_{\xi,A}"] & H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "can"] & \mathrm{coker}(B \circ \alpha_{\xi,A}) \arrow[r] & 0 \\ 0 \arrow[r] & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[u, hook, "i_{m+1}"] \arrow[r, "B\circ u"] & H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[u, equal] \arrow[r, "v\circ B^{-1}"] & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \arrow[u, two heads, "\epsilon_{\xi,A}"] \arrow[r] & 0 \\ & & & H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \arrow[u, hook, "\tau_{\xi,A}"] & \end{tikzcd} \tag{36}$$

where $\tau_{\xi,A}$ and $\epsilon_{\xi,A}$ are the induced morphisms. Then the morphism $\tau_{\xi,A}$ is a subbundle morphism fitting in a commutative diagram

$$\begin{tikzcd} H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "v\circ B^{-1}"] & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \\ H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[u, "C\circ u"] \arrow[r, equal] & H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1). \arrow[u, hook, "\tau_{\xi,A}"] \end{tikzcd} \tag{37}$$

Proof. (i) Consider the commutative diagram

(38)

$$\begin{tikzcd}
H_{2m+1}\otimes\mathcal{O}(-1) \arrow[r, tail, "a_A"] & W_A\otimes\mathcal{O} \arrow[r, "q_A", "\simeq"'] & W_A^\vee\otimes\mathcal{O} \arrow[r, two heads, "a_A^\vee"] \arrow[d, "\simeq"', "j_{\xi,A}^\vee"] & H_{2m+1}^\vee\otimes\mathcal{O}(1) \arrow[d, "\simeq"', "\xi^\vee"] \\
(H_{m+1}\oplus H_m)\otimes\mathcal{O}(-1) \arrow[u, "\xi", "\simeq"'] \arrow[r, tail, "\alpha_{\xi,A}"] & H_{m+1}\otimes V\otimes\mathcal{O} \arrow[u, "j_{\xi,A}", "\simeq"'] \arrow[r, "B", "\simeq"'] & H_{m+1}^\vee\otimes V^\vee\otimes\mathcal{O} \arrow[r, two heads, "\alpha_{\xi,A}^\vee"] \arrow[dr, "u^\vee"'] & (H_{m+1}\oplus H_m)^\vee\otimes\mathcal{O}(1) \arrow[d, "i_{m+1}^\vee"] \\
H_{m+1}\otimes\mathcal{O}(-1) \arrow[u, hook, "i_{m+1}"] \arrow[ur, "u"'] & & & H_{m+1}^\vee\otimes\mathcal{O}(1)
\end{tikzcd}$$

Here the upper triple is the monad (12) for $n = 2m+1$. Whence the statement (i) follows.

(ii) Standard diagram chasing using (30), (32) and diagram (36). □

4.3. Remarks on t'Hooft instantons.

Consider the set

$$I^{tH}_{2m+1} := \{[E] \in I_{2m+1} \mid h^0(E(1)) \neq 0\},$$

of *t'Hooft instanton bundles* and the corresponding set of *t'Hooft instanton nets*

$$MI^{tH}_{2m+1} := \pi_n^{-1}(I^{tH}_{2m+1}).$$

We collect some well-known facts about I^{tH}_{2m+1} in the following Lemma - see [BT], [NT], [T2, Prop. 2.2].

Lemma 4.5. *Let $m \geq 1$. Then the following statements hold.*

(i) I^{tH}_{2m+1} is an irreducible $(10m+9)$-dimensional subvariety of I_{2m+1}. Respectively, MI^{tH}_{2m+1} is an irreducible $(4m^2+14m+10)$-dimensional subvariety of I_{2m+1}.

(ii) $I^{tH*}_{2m+1} := I^{tH}_{2m+1} \cap I'_{2m+1}$ is a smooth dense open subset of I^{tH}_{2m+1} and

$$h^0(E(1)) = 1, \qquad [E] \in I^{tH*}_{2m+1}. \tag{39}$$

(iii) MI^{tH*}_{2m+1} is a smooth dense open subset of the set

$$TH_{2m+1} := \{A \in \mathbf{S}_{2m+1} | A = \sum_{i=1}^{2m+2} h^2 \otimes w, \text{ where } h \in H^\vee_{2m+1}, w \in \wedge^2 V^\vee,\ w\wedge w = 0\}.$$

We are going to extend the statement of Theorem 4.3 to the cases $m = 1$ and 2. To this end, for $m = 1, 2$ and $\xi \in \mathrm{Isom}_{2m+1}$ consider the sets $MI_{2m+1}(\xi)$ defined in (31) and set

$$MI''_{2m+1} := \underset{\xi\in\mathrm{Isom}_{2m+1}}{\cup} MI_{2m+1}(\xi), \quad m = 1, 2. \tag{40}$$

For $m = 1, 2$, fix an isomorphism $\xi^0 \in \mathrm{Isom}_{2m+1}$, $\xi^0 : H_{m+1} \oplus H_m \xrightarrow{\sim} H_{2m+1}$ and fix a basis $\{h_1, ..., h_{2m+1}\}$ in $H^\vee_{2m+1}$ such that $\{h_1, ..., h_m\}$ in $H^\vee_{2m+1}$ and $\{h_{m+2}, ..., h_{2m+1}\}$ in $H^\vee_{2m+1}$; respectively, let $e_1, ..., e_4$ be some fixed basis in $V^\vee$. Consider the nets $A^{(m)} \in TH_{2m+1}$, $m = 1, 2$, defined as follows

$$A^{(1)} = h_1^2 \otimes (e_1\wedge e_2 + e_3\wedge e_4) + h_2^2 \otimes (e_1\wedge e_3 + e_4 \wedge e_2), \tag{41}$$

$$A^{(2)} = h_1^2 \otimes (e_1\wedge e_2 + e_3\wedge e_4) + h_2^2 \otimes (e_1\wedge e_3 + e_4\wedge e_2) + h_3^2 \otimes (e_1\wedge e_4 + e_2\wedge e_3).$$

It is an exercise to show that, in the notation of (28), the homomorphisms

$$A_1^{(m)}(\xi^0) : H_{m+1}\otimes V \to H^\vee_{m+1} \otimes V^\vee, \quad m = 1, 2,$$

are invertible. On the other hand, for a given $\xi \in \mathrm{Isom}_{2m+1}$, the condition that a homomorphism $A_1(\xi) : H_{m+1} \otimes V \to H^\vee_{m+1} \otimes V^\vee$ is invertible is an open condition on the net $A \in TH_{2m+1}$, respectively, on the net $A \in \mathbf{S}_{2m+1}$. Since the sets MI'_{2m+1}, $m = 1, 2$, are irreducible (see [CTT]), this together with Lemma 4.5 yields the following corollary.

Corollary 4.6. *(i) For $m=1,2$ the set MI''_{2m+1} is a dense open subset of MI'_{2m+1} and of MI_{2m+1}, and the statement of Theorem 4.3 extends to the cases $m=1$ and 2, if we substitute MI'_{2m+1} by MI''_{2m+1} and take for $\mathrm{Isom}^0_{2m+1}=\mathrm{Isom}^{00}_{2m+1}$ any nonempty open subset of Isom_{2m+1} contained in the set $\{\xi\in\mathrm{Isom}_{2m+1}\mid MI_{2m+1}(\xi)\neq\emptyset\}$.*

(ii) Let $m\ge 1$. The set

$$MI^{tH**}_{2m+1}:=\begin{cases}MI''_{2m+1}\cap MI^{tH*}_{2m+1}, & m=1,2,\\ MI^{tH*}_{2m+1}, & m\ge 3,\end{cases}$$

is a dense open subset of MI^{tH*}_{2m+1}, respectively, of MI^{tH}_{2m+1}.

(iii) For $m\ge 1$ let

$$MI^{tH}_{2m+1}(\xi):=MI^{tH**}_{2m+1}\cap MI_{2m+1}(\xi),\quad \xi\in\mathrm{Isom}_{2m+1}.$$

The set

$$\mathrm{Isom}^{tH}_{2m+1}:=\{\xi\in\mathrm{Isom}_{2m+1}\mid MI^{tH}_{2m+1}(\xi)\neq\emptyset\}\tag{42}$$

is a dense open subset of Isom_{2m+1} such that MI^{tH**}_{2m+1} is covered by dense open subsets $MI^{tH}_{2m+1}(\xi)$, $\xi\in\mathrm{Isom}^{tH}_{2m+1}$.

Remark 4.7. From the definition of the sets Isom^0_{2m+1}, $MI^{tH}_{2m+1}(\xi)$ and $\mathrm{Isom}^{tH}_{2m+1}$ it follows immediately that $\mathrm{Isom}^{tH}_{2m+1}\subset\mathrm{Isom}^0_{2m+1}$ and $MI^{tH}_{2m+1}(\xi)\subset MI_{2m+1}(\xi)$ for $\xi\in\mathrm{Isom}^{tH}_{2m+1}$.

Now (19), Theorem 4.3 and Corollary 4.6 yield

Corollary 4.8. *Let $m\ge 1$. Then for any $\xi\in\mathrm{Isom}^0_{2m+1}$ (respectively, for any $\xi\in\mathrm{Isom}^{00}_{2m+1}$) the scheme $(MI_{2m+1}(\xi))_{red}$ is open (respectively, dense open) in $(MI_{2m+1})_{red}$. In particular,*

$$\dim_A MI_{2m+1}(\xi)=\dim_A MI_{2m+1},\quad A\in MI_{2m+1}(\xi),\quad \xi\in\mathrm{Isom}^{00}_{2m+1}.\tag{43}$$

5. Invertible nets of quadrics from $\mathbf{S}_{m+1}$ and symplectic rank-$(2m+2)$ bundles

5.1. Construction of symplectic rank-$(2m+2)$ bundles from invertible nets of quadrics from $\mathbf{S}_{m+1}$.

In this subsection we show that each invertible net of quadrics $B\in\mathbf{S}_{m+1}$ naturally leads to a construction of a symplectic rank-$(2m+2)$ vector bundle $E_{2m+2}(B)$ on $\mathbb{P}^3$. Let us introduce more notation. Set

$$\mathbf{S}^0_{m+1}:=\{B\in\mathbf{S}_{m+1}\mid B:H_{m+1}\otimes V\to H^\vee_{m+1}\otimes V^\vee \text{ is an invertible homomorphism}\}.\tag{44}$$

The set $\mathbf{S}^0_{m+1}$ is a dense open subset of the vector space $\mathbf{S}_{m+1}$, and it is easy to see that for any $B\in\mathbf{S}^0_{m+1}$ the following conditions are satisfied.
(1) The morphism $\widetilde{B}:H_{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to H^\vee_{m+1}\otimes\Omega_{\mathbb{P}^3}(1)$ induced by the homomorphism $B:H_{m+1}\otimes V\to H^\vee_{m+1}\otimes V^\vee$ is a subbundle morphism, i.e.

$$E_{2m+2}(B):=\mathrm{coker}(\widetilde{B})\tag{45}$$

is a vector bundle of rank $2m+2$ on $\mathbb{P}^3$. This follows from the diagram
(46)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H_{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\widetilde{B}"] \arrow[d, "u"] & H_{m+1}^\vee\otimes\Omega_{\mathbb{P}^3}(1) \arrow[r, "e"] \arrow[d, "v^\vee"] & E_{2m+2}(B) \arrow[r] & 0 \\
 & H_{m+1}\otimes V\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "B", "\simeq"'] \arrow[d, "v"] & H_{m+1}^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[d, "u^\vee"] & & \\
0 \to E_{2m+2}(B)^\vee \arrow[r] & H_{m+1}\otimes T_{\mathbb{P}^3}(-1) \arrow[r, "\widetilde{B}^\vee"] \arrow[d] & H_{m+1}^\vee\otimes\mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

(2) The homomorphism ${}^\sharp B : H_{m+1} \to H_{m+1}^\vee \otimes \wedge^2 V^\vee$ induced by $B : H_{m+1}\otimes V \to H_{m+1}^\vee \otimes V^\vee$ is injective. This follows from the commutative diagram extending the upper horizontal triple in (46)

(47)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H_{m+1}^\vee\otimes T_{\mathbb{P}^3}(-2) \arrow[r, equal] \arrow[d] & H_{m+1}^\vee\otimes T_{\mathbb{P}^3}(-2) \arrow[d] & \\
0 \arrow[r] & H_{m+1}\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "{}^\sharp B"] \arrow[d, equal] & H_{m+1}^\vee\otimes\wedge^2V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "can"] \arrow[d, "w"] & H^0(E_{2m+2}(B)(1))\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d, "ev"] & 0 \\
0 \arrow[r] & H_{m+1}\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "\widetilde{B}"] & H_{m+1}^\vee\otimes\Omega_{\mathbb{P}^3}(2) \arrow[r, "e"] \arrow[d] & E_{2m+2}(B)(1) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0, &
\end{tikzcd}$$

where w is the morphism induced by the morphism v from the Euler exact sequence in (46). From this diagram we obtain an isomorphism

$$\operatorname{coker}({}^\sharp B) \simeq H^0(E_{2m+2}(B)(1)). \tag{48}$$

(3) Diagram (46) and the Five-Lemma yield an isomorphism

$$\theta : E_{2m+2}(B) \xrightarrow{\sim} E_{2m+2}(B)^\vee \tag{49}$$

which is in fact symplectic,

$$\theta^\vee = -\theta,$$

since the homomorphism $B : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee$ is skew-symmetric. The isomorphism θ together with the upper triple from (46) and its dual fits in the commutative diagram

(50)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "\widetilde{B}"] \arrow[d, equal] & H_{m+1}^\vee \otimes \Omega_{\mathbb{P}^3}(1) \arrow[r, "e"] \arrow[d, "v^\vee"] & E_{2m+2}(B) \arrow[r] \arrow[d, "e^\vee \circ \theta"] & 0 \\
0 \arrow[r] & H_{m+1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r, "B \circ u"] & H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "v \circ B^{-1}"] \arrow[d, "u^\vee"] & H_{m+1} \otimes T_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d, "\widetilde{B}^\vee"] & 0 \\
 & & H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r, equal] \arrow[d] & H_{m+1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Note that the upper horizontal triple in (46) immediately implies

$$h^0(E_{2m+2}(B)) = h^i(E_{2m+2}(B)(-2)) = 0, \quad i \geq 0. \tag{51}$$

5.2. Relation between instantons and rank-$(2m+2)$ symplectic bundles.

For $m \geq 1$ let $\xi \in \mathrm{Isom}^0_{2m+1}$ and $A \in MI_{2m+1}(\xi)$. In this subsection we relate an instanton vector bundle $E(A)$ to a symplectic rank-$(2m+2)$ vector bundle $E_{2m+2}(B)$ for $B = A_1(\xi)$. We will show that $E(A)$ is a cohomology sheaf of the monad (55) defined by the data (ξ, A) with $E_{2m+2}(B)$ in the middle - see Lemma 5.1.

In fact, since $\xi \in \mathrm{Isom}^0_{2m+1}$, the homomorphism $B : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee$ by definition lies in $\mathbf{S}^0_{m+1}$. Hence by Lemma 4.4 the diagram (37) holds. This diagram together with (50) implies $\widetilde{B}^\vee \circ \tau_{\xi,A} = 0$ (note that in (37) $\mathrm{im}(C \circ u) \subset H_{m+1}^\vee \otimes \Omega_{\mathbb{P}^3}(1)$ since $C \in \mathbf{\Sigma}_{m+1}$), so that there exists a morphism

$$\rho_{\xi,A} : H_m \otimes \mathcal{O}(-1) \to E_{2m+2}(B) \tag{52}$$

such that $\tau_{\xi,A} = e^\vee \circ \theta \circ \rho_{\xi,A}$. Since $\tau_{\xi,A}$ is a subbundle morphism, $\rho_{\xi,A}$ is also a subbundle morphism. Moreover, diagrams (37) and (50) yield a commutative diagram

(53)
$$\begin{tikzcd}
H_{m+1}^\vee \otimes \Omega_{\mathbb{P}^3}(1) \arrow[rr, "e"] \arrow[dd, "v^\vee"] & & E_{2m+2}(B) \arrow[dd, "e^\vee \circ \theta"] \\
 & H_m \otimes \mathcal{O}(-1) \arrow[ul, "\sharp C"'] \arrow[ur, "\rho_{\xi,A}"] \arrow[dl, "\widetilde{C}"] \arrow[dr, "\tau_{\xi,A}"'] & \\
H_{m+1}^\vee \otimes V^\vee \otimes \mathcal{O} \arrow[rr, "v \circ B^{-1}"] & & H_{m+1} \otimes T_{\mathbb{P}^3}(-1).
\end{tikzcd}$$

Diagrams (50) and (53) yield a commutative diagram

(54)
$$\begin{tikzcd}
H_m\otimes\mathcal{O}(-1) \arrow[rrr, "\widetilde{C}"] \arrow[dr, "\rho_{\xi,A}"'] \arrow[drr, "\sharp C"] \arrow[ddd, "D_C"'] & & & H^\vee_{m+1}\otimes V^\vee\otimes\mathcal{O} \arrow[ddd, "\simeq"', "B^{-1}"] \\
& E_{2m+2}(B) \arrow[d, "\simeq"', "\theta"] & H^\vee_{m+1}\otimes\Omega_{\mathbb{P}^3}(1) \arrow[l, two heads, "e"] \arrow[ur, "v^\vee"] \arrow[d, "e^\vee\circ\theta\circ e"] & \\
& E_{2m+2}(B)^\vee \arrow[r, tail, "e^\vee"] \arrow[dl, "\rho^\vee_{\xi,A}"'] & H_{m+1}\otimes T_{\mathbb{P}^3}(-1) \arrow[dll, "\sharp C^\vee"] & \\
H^\vee_m\otimes\mathcal{O}(1) & & & H_{m+1}\otimes V\otimes\mathcal{O}, \arrow[lll, "\widetilde{C}^\vee"] \arrow[ul, two heads, "v"']
\end{tikzcd}$$

where $D_C := -\widetilde{C}^\vee\circ B^{-1}\circ\widetilde{C} = -u^\vee\circ(C^\vee\circ B^{-1}\circ C)\circ u$ is the zero map. In fact, by (30) and (32) we have $D_C = p_2(A_3(\xi))$, where $p_2 : \wedge^2(H_n^\vee\otimes V^\vee)\to\wedge^2 H_n^\vee\otimes S^2V^\vee$ is the projection onto the second direct summand of the decomposition (9). Since by (28) $A_3(\xi)$ lies in the first direct summand of (9) it follows that $D_C = 0$. We thus obtain a monad

$$0\to H_m\otimes\mathcal{O}(-1)\xrightarrow{\rho_{\xi,A}} E_{2m+2}(B)\xrightarrow{\rho^\vee_{\xi,A}\circ\theta} H_m^\vee\otimes\mathcal{O}(1)\to 0 \tag{55}$$

with cohomology sheaf

$$E_2(\xi,A) := \ker(\rho^\vee_{\xi,A}\circ\theta)/\operatorname{Im}\rho_{\xi,A} \tag{56}$$

which is a vector bundle since $\rho_{\xi,A}$ is a subbundle morphism. Furthermore, by (51) it follows from the monad (55) that $E_2(\xi,A)$ is a $(2m+1)$-instanton,

$$[E_2(\xi,A)]\in I_{2m+1}. \tag{57}$$

Lemma 5.1. *$E_2(\xi,A)\simeq E(A)$, where the sheaf $E(A)$ is defined in (13).*

Proof. Diagram chasing using (30), (36)-(38), (46)-(47) and (50). □

6. Scheme X_m. An isomorphism between X_m and an open subset of the space $(MI_{2m+1})_{red}$

In this section we introduce a locally closed subset X_m of the vector space $\mathbf{S}_{m+1}\oplus\mathbf{\Sigma}_{m+1}$ and prove in Theorem 6.1 below that this subset, considered as a reduced scheme, is isomorphic to the reduced scheme $(MI_{2m+1}(\xi))_{red}$ for any $\xi\in\mathrm{Isom}^0_{2m+1}$. The set X_m is defined as follows:
(58)

$$X_m := \left\{ (B,C)\in\mathbf{S}^0_{m+1}\times\mathbf{\Sigma}_{m+1} \;\middle|\; \begin{array}{c} (i)\ (C^\vee\circ B^{-1}\circ C : H_m\otimes V\to H_m^\vee\otimes V^\vee)\in\mathbf{S}_m, \\[1ex] (ii)\ \text{the map } (H_{m+1}\oplus H_m)\otimes\mathcal{O}\xrightarrow{(B,C)\circ u} H^\vee_{m+1}\otimes V^\vee\otimes\mathcal{O}(1) \\ \text{is a subbundle morphism,} \\ (iii)\ \text{the composition } \hat{C} : H_m\overset{\sharp C}{\to} H^\vee_{m+1}\otimes\wedge^2V^\vee \overset{can}{\twoheadrightarrow} \\ H^\vee_{m+1}\otimes\wedge^2V^\vee/\operatorname{Im}(\sharp B)\simeq H^0(E_{2m+2}(B)(1)) \text{ yields} \\ \text{a subbundle morphism} \\ H_m\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\overset{\rho_{B,C}}{\to} E_{2m+2}(B), \\ \text{i.e. } \rho^\vee_{B,C} \text{ is surjective and } E_2(B,C) := \operatorname{Ker}({}^t\rho_{B,C})/\operatorname{Im}(\rho_{B,C}) \\ \text{is locally free} \end{array} \right\}.$$

By definition X_m is a locally closed subset of $\mathbf{S}^0_{m+1} \times \mathbf{\Sigma}_{m+1}$. Hence it is naturally endowed with the structure of a reduced scheme.

Note that in the condition (iii) of the definition of X_m we set ${}^t\rho_{B,C} := \rho^\vee_{B,C} \circ \theta$, where $\theta : E_{2m+2}(B) \overset{\simeq}{\to} E^\vee_{2m+2}(B)$ is the natural symplectic structure on $E_{2m+2}(B)$ defined in (49).

Theorem 6.1. *Let $m \geq 1$ and let $\xi \in \mathrm{Isom}^0_{2m+1}$.*

(i) There is an isomorphism of reduced schemes

(59) $$f_m : \ (MI_{2m+1}(\xi))_{red} \overset{\simeq}{\to} X_m : \ A \mapsto (A_1(\xi), A_2(\xi)).$$

(ii) The inverse isomorphism is given by the formula

(60) $$g_m : \ X_m \overset{\simeq}{\to} (MI_{2m+1}(\xi))_{red} : \ (B, C) \mapsto \ \widetilde{\xi}(B,\ C,\ -C^\vee \circ B^{-1} \circ C).$$ [1]

Proof. (i) We first show that the image of the map $f_m : \ (MI_{2m+1}(\xi))_{red} \to \mathbf{S}^0_{m+1} \times \Sigma^{in}_{m,m+1}$ lies in X_m, i.e. satisfies the conditions (i)-(iii) in the definition of X_m. Indeed, the condition (i) is automatically satisfied, since (28) and (30) give $-C^\vee \circ B^{-1} \circ C = -A_2(\xi)^\vee \circ A_1(\xi)^{-1} \circ A_2(\xi) = A_3(\xi) \in S^2H^\vee_m \otimes \wedge^2 V^\vee$. Next, the morphism $\rho_{B,C}$ defined in (58.iii) above coincides by its definition with the morphism $\rho_{\xi,A}$ defined in (52). In fact, the upper triangle of the diagram (53) twisted by $\mathcal{O}(1)$ and the lower part of the diagram (47) fit in the diagram

(61)
$$\begin{tikzcd}
0 \arrow[r] & H_{m+1} \otimes \mathcal{O} \arrow[r, "\sharp B"] \arrow[dd, equal] & H^\vee_{m+1} \otimes \wedge^2 V^\vee \otimes \mathcal{O} \arrow[rr, "can"] \arrow[dd, "w"'] & & H^0(E_{2m+2}(B)(1)) \otimes \mathcal{O} \arrow[r] \arrow[dd, "ev"] & 0 \\
 & & & H_m \otimes \mathcal{O} \arrow[ul, "\sharp C"'] \arrow[ur, "\widehat{C}"] \arrow[dl, "\widetilde{C}"'] \arrow[dr, "\rho_{\xi,A}"] & & \\
0 \arrow[r] & H_{m+1} \otimes \mathcal{O} \arrow[r, "\widetilde{B}"] & H^\vee_{m+1} \otimes \Omega(2) \arrow[rr, "e"] & & E_{2m+2}(B)(1) \arrow[r] & 0,
\end{tikzcd}$$

where the composition $\widehat{C} = can \circ C$ is defined in the condition (iii) of the definition of X_m. Whence

(62) $$\rho_{B,C} = \rho_{\xi,A}.$$

Since $\rho_{\xi,A}$ is a subbundle morphism, the condition (iii) is satisfied and, moreover, $\widehat{C}$ is a subbundle morphism as well. Thus, the lower part of the diagram (61) shows that the morphism $(\widetilde{B}, \widetilde{C}) : (H_{m+1} \oplus H_m) \otimes \mathcal{O} \to H^\vee_{m+1} \otimes \Omega(2)$ is a subbundle morphism. Hence its composition with the subbundle morphism $v^\vee : H^\vee_{m+1} \otimes \Omega(2) \hookrightarrow H^\vee_{m+1} \otimes V \otimes \mathcal{O}(1)$ is a subbundle morphism as well. By definition, this composition coincides with $(B, C) \circ u$. Hence the condition (ii) in the definition of X_m is satisfied.

This shows that $f_m((MI_{2m+1}(\xi))_{red})$ lies in X_m. Finally, the equality $g_m \circ f_m = id$ follows directly from (28) and (30).

(ii) We first prove that the image of the map

(63) $$g_m : X_m \to \mathbf{S}_{2m+1} : \ (B, C) \mapsto (B,\ C,\ C^\vee \circ B^{-1} \circ C)$$ [2]

lies in $(MI_{2m+1}(\xi))_{red}$. In fact, the subbundle morphism $\mathcal{A} := (B, C) \circ u : (H_{m+1} \oplus H_m) \otimes \mathcal{O} \to H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O}(1)$ and its dual extend to the right and left exact sequence

(64) $$0 \to (H_{m+1} \oplus H_m) \otimes \mathcal{O}(-1) \overset{\mathcal{A}}{\to} H^\vee_{m+1} \otimes V^\vee \otimes \mathcal{O} \overset{\mathcal{A}^\vee \circ B^{-1}}{\to} (H_{m+1} \oplus H_m)^\vee \otimes \mathcal{O}(1) \to 0.$$

[1]Here we use the decomposition (27) fixed by the choice of ξ.

[2] We identify here the triple $(B,\ C,\ C^\vee \circ B^{-1} \circ C)$ with a point in $S^2H^\vee_{2m+1} \otimes \wedge^2 V^\vee$ via the decomposition (27).

Furthermore, by definition $\mathcal{A}^\vee\circ B^{-1}\circ\mathcal{A} = u^\vee\circ A\circ u$, where A is the matrix $\begin{pmatrix} B & C\\ -C^\vee & -C^\vee\circ B^{-1}\circ C\end{pmatrix}$. Since the condition (i) of (58) is satisfied, under the direct sum decomposition (27) this matrix A can be treated as an element of $\mathbf{S}_{2m+1}$. Hence $u^\vee\circ A\circ u=0$, i.e. (64) is a monad. We will show that its cohomology bundle

$$E(B,C):=\ker(\mathcal{A}^\vee\circ B^{-1})/\operatorname{Im}\mathcal{A}$$

is an $(2m+1)$-instanton, and this will give the desired inclusion $g(X_m)\subset(MI_{2m+1}(\xi))_{red}$. For this, consider the diagram (36) in which we substitute $B\circ\alpha_{\xi,A}$ by $\mathcal{A}$, denote $\mathcal{G}:=\operatorname{coker}\mathcal{A}$, and change the notation for $\tau_{\xi,A}$ and $\epsilon_{\xi,A}$, respectively, to $\tau_{B,C}$ and $\epsilon_{B,C}$:

$$\begin{tikzcd}
 & H_m\otimes\mathcal{O}_{\mathbb{P}^3}(-1) & & & \\
0 \arrow[r] & (H_{m+1}\oplus H_m)\otimes\mathcal{O}_{\mathbb{P}^3}(-1) \arrow[u, two heads] \arrow[r, "\mathcal{A}"] & H^\vee_{m+1}\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "can"] & \mathcal{G} \arrow[r] & 0\\
0 \arrow[r] & H_{m+1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1) \arrow[u, hook, "i_{m+1}"] \arrow[r, "B\circ u"] & H^\vee_{m+1}\otimes V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[u, equal] \arrow[r, "v\circ B^{-1}"] & H_{m+1}\otimes T_{\mathbb{P}^3}(-1) \arrow[u, two heads, "\epsilon_{B,C}"] \arrow[r] & 0\\
 & & & H_m\otimes\mathcal{O}_{\mathbb{P}^3}(-1). \arrow[u, hook, "\tau_{B,C}"] &
\end{tikzcd} \tag{65}$$

In these notations the diagram (50) becomes the display of the antiselfdual monad

$$0\to H_{m+1}\otimes\mathcal{O}(-1)\xrightarrow{B\circ u} H^\vee_{m+1}\otimes V^\vee\otimes\mathcal{O}\xrightarrow{u^\vee} H^\vee_{m+1}\otimes\mathcal{O}(1)\to 0 \tag{66}$$

with the symplectic cohomology sheaf $E_{2m+2}(B)$:

$$E_{2m+2}(B)=\ker(u^\vee)/\operatorname{Im}(B\circ u). \tag{67}$$

Moreover, as in (52) and (53) we obtain a subbundle morphism

$$\rho_{B,C}:H_m\otimes\mathcal{O}(-1)\to E_{2m+2}(B) \tag{68}$$

such that

$$\tau_{B,C}=e^\vee\circ\theta\circ\rho_{B,C}, \tag{69}$$

where $\theta: E_{2m+2}(B)\xrightarrow{\simeq} E_{2m+2}(B)$ is a symplectic structure on $E_{2m+2}(B)$. In addition, as in (51) we have

$$h^0(E_{2m+2}(B))=h^i(E_{2m+2}(B)(-2))=0,\quad i\ge 0. \tag{70}$$

Furthermore, the antiselfdual monads (64) and (66) recover the antiselfdual monad (55) which in view of (62) becomes

$$0\to H_m\otimes\mathcal{O}(-1)\xrightarrow{\rho_{B,C}} E_{2m+2}(B)\xrightarrow{\rho^\vee_{B,C}\circ\theta} H^\vee_m\otimes\mathcal{O}(1)\to 0. \tag{71}$$

with the cohomology sheaf $E(B,C)$,

$$E(B,C)=\ker(\rho^\vee_{B,C}\circ\theta)/\operatorname{Im}(\rho_{B,C}). \tag{72}$$

Now (70) and (71) yield $h^0(E(B,C))=h^i(E(B,C)(-2))=0,\quad i\ge 0$, i.e. $E(B,C)$ is an $(2m+1)$-instanton.

Thus $\operatorname{Im} g_m\subset I_{2m+1}(\xi)$. The fact that $f_m\circ g_m=id$ follows directly from (59) and (60). $\square$

Remark 6.2. Note that, since the morphism $\widehat{C}$ in the diagram (61) is injective, it follows from this diagram that, for any $m \geq 1$, $\xi \in \mathrm{Isom}^0_{2m+1}$ and any $A \in MI_{2m+1}(\xi)$, the monomorphisms $H_{m+1} \overset{\sharp A_1(\xi)}{\hookrightarrow} H^\vee_{m+1} \otimes \wedge^2 V^\vee \overset{\sharp A_2(\xi)}{\hookleftarrow} H_m$ satisfy the condition $\mathrm{Im}(^\sharp A_1(\xi)) \cap \mathrm{Im}(^\sharp A_2(\xi)) = \{0\}$, i. e. $\dim \mathrm{Span}(\mathrm{Im}(^\sharp A_1(\xi)),\ \mathrm{Im}(^\sharp A_2(\xi))) = 2m+1$.

7. Scheme Z_m. Reduction of the irreducibility of X_m to the irreducibility of Z_m. Proof of main theorem

7.1. Scheme $\widehat{Z}_m$ and its open subset Z_m. In this subsection we introduce a new set Z_m as a locally closed subset of a certain vector space (see (77)) and endow it with a natural scheme structure. We then formulate Theorem 7.2 on the irreducibility of Z_m. This Theorem plays a key role in the proof of irreducibility of I_{2m+1} which we give in subsection 7.2. The proof of Theorem 7.2 will be given in the next section.

Set

$$\mathbf{\Lambda}_m := \wedge^2 H^\vee_m \otimes S^2 V^\vee, \quad \mathbf{\Phi}_m := \mathrm{Hom}(H_m, H^\vee_m \otimes \wedge^2 V^\vee), \tag{73}$$

and

$$(\mathbf{S}^\vee_m)^0 := \{D \in \mathbf{S}^\vee_m \mid D : H^\vee_m \otimes V^\vee \to H_m \otimes V \text{ is invertible}\}. \tag{74}$$

Note that $(\mathbf{S}^\vee_m)^0$ is a dense open subset of $\mathbf{S}^\vee_m$ and there is a canonical isomorphism

$$\mathbf{S}^0_m \xrightarrow{\simeq} (\mathbf{S}^\vee_m)^0 :\ A \mapsto A^{-1}. \tag{75}$$

Consider the sets

$$\widehat{Z}_m := \left\{ (D, \phi) \in \mathbf{S}^\vee_m \times \mathbf{\Phi}_m \;\middle|\; \begin{array}{c} \Theta(D,\phi) := \phi^\vee \circ D \circ \phi : H_m \otimes V \to \\ \to H^\vee_m \otimes V^\vee \text{ satisfies the condition} \\ \Theta(D,\phi) \in \mathbf{S}_m \end{array} \right\}. \tag{76}$$

and

$$Z_m := \widehat{Z}_m \cap (\mathbf{S}^\vee_m)^0 \times \mathbf{\Phi}_m \tag{77}$$

(here we understand a point $D \in \mathbf{S}^\vee_m$ as a homomorphism $H^\vee_m \otimes V^\vee \to H_m \otimes V$) and let $\overline{Z}_m$ be the closure of Z_m in $\mathbf{S}^\vee_m \times \mathbf{\Phi}_m$. By definition, Z_m is an open subset of $\widehat{Z}_m$, respectively, a dense open subset of $\overline{Z}_m$.

Note that there is a standard decomposition

$$\wedge^2(H^\vee_m \otimes V^\vee) = \mathbf{S}_m \oplus \mathbf{\Lambda}_m$$

with induced projection onto the second summand

$$q_m :\ \wedge^2(H^\vee_m \otimes V^\vee) \to \mathbf{\Lambda}_m \tag{78}$$

and the morphism

$$h : \mathbf{S}^\vee_m \oplus \mathbf{\Phi}_m \to \mathbf{\Lambda}_m : (D, \phi) \mapsto q_m(\Theta(D,\phi)).$$

By the definition of $\widehat{Z}_m$ we obtain

$$\widehat{Z}_m = h^{-1}(0). \tag{79}$$

Clearly, the point $(0,0)$ belongs to $\widehat{Z}_m$, i. e. $\widehat{Z}_m$ is nonempty.

Convention: We endow $\widehat{Z}_m$ with the structure of a scheme-theoretic fibre $h^{-1}(0)$ of the morphism h. Respectively, Z_m inherits the structure of an open subscheme of $\widehat{Z}_m$.

Remark 7.1. From (79) it follows that $\widehat{Z}_m$ may be considered as the zero-scheme $(h^*\mathbf{s}_{taut})_0$ of the section $h^*\mathbf{s}_{taut}$ of the trivial vector bundle $\mathbf{\Lambda}_m \otimes \mathcal{O}_{\mathbf{S}_m \oplus \mathbf{\Phi}_m}$, where $\mathbf{s}_{taut}$ is the tautological section of the trivial vector bundle $\mathbf{\Lambda}_m \otimes \mathcal{O}_{\mathbf{\Lambda}_m}$ of rank $\dim \mathbf{\Lambda}_m = 5m(m-1)$ over $\mathbf{\Lambda}_m$. We thus obtain the following estimate for the dimension of $\widehat{Z}_m$ at each point $z \in \widehat{Z}_m$,

$$\dim_z \widehat{Z}_m = \dim h^{-1}(0) \geq \dim(\mathbf{S}_m^\vee \times \mathbf{\Phi}_m) - \dim \mathbf{\Lambda}_m = 3m(m+1) + 6m^2 - 5m(m-1) = 4m(m+2). \tag{80}$$

In particular, if Z_m is nonempty, then

$$\dim_z Z_m \geq 4m(m+2), \qquad z \in Z_m. \tag{81}$$

In the next subsection we will use the following result about Z_m.

Theorem 7.2. *(i) Z_m is an integral locally complete intersection scheme of dimension $4m(m+2)$.*

(ii) The natural morphism $p_m : Z_m \to (\mathbf{S}_m^\vee)^0 : \ (D, \phi) \mapsto D$ is surjective.

We begin the proof of this theorem in section 8 and finish in section 10.

7.2. Proof of the main theorem.

In this subsection we give the proof of Theorem 1.1. Set

$$\widetilde{X}_m := \{(D, C) \in (\mathbf{S}_{m+1}^\vee)^0 \times \mathbf{\Sigma}_{m+1} \mid (C^\vee \circ D \circ C : H_m \otimes V \to H_m^\vee \otimes V^\vee) \in \mathbf{S}_m\}. \tag{82}$$

The set $\widetilde{X}_m$ has a natural structure of a closed subscheme of $(\mathbf{S}_{m+1}^\vee)^0 \times \mathbf{\Sigma}_{m+1}$ defined by the equations

$$C^\vee \circ D \circ C \in \mathbf{S}_m. \tag{83}$$

Since the conditions (ii) and (iii) in the definition (58) of X_m are open and X_m is nonempty (see Theorem 6.1), it follows immediately in view of (75) that X_m is a nonempty open subset of $(\widetilde{X}_m)_{red}$,

$$\emptyset \neq X_m \overset{\text{open}}{\hookrightarrow} (\widetilde{X}_m)_{red}. \tag{84}$$

Fix a direct sum decomposition

$$H_{m+1} \xrightarrow{\sim} H_m \oplus \mathbf{k}.$$

Under this isomorphism any homomorphism

$$C \in \mathbf{\Sigma}_{m+1} = \mathrm{Hom}(H_m, H_{m+1}^\vee) \otimes \wedge^2 V^\vee, \quad C : H_m \otimes V \to H_{m+1}^\vee \otimes V^\vee, \tag{85}$$

can be represented as a homomorphism

$$C : H_m \otimes V \to H_m^\vee \otimes V^\vee \ \oplus \ \mathbf{k}^\vee \otimes V^\vee, \tag{86}$$

i.e. as a matrix of homomorphisms

$$C = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{87}$$

where

$$\phi \in \mathrm{Hom}(H_m, H_m^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_m, \quad \psi \in \mathbf{\Psi}_m := \mathrm{Hom}(H_m, (\mathbf{k})^\vee) \otimes \wedge^2 V^\vee. \tag{88}$$

Respectively, any homomorphism $D \in (\mathbf{S}_{m+1}^\vee)^0 \subset \mathbf{S}_{m+1}^\vee = S^2 H_{m+1} \otimes \wedge^2 V \subset \mathrm{Hom}(H_{m+1}^\vee \otimes V^\vee, H_{m+1} \otimes V)$ can be represented as a matrix of homomorphisms

$$D = \begin{pmatrix} D_1 & \lambda \\ -\lambda^\vee & \mu \end{pmatrix}, \tag{89}$$

where

$$D_1 \in \mathbf{S}_m^\vee \subset \mathrm{Hom}(H_m^\vee \otimes V^\vee, H_m \otimes V), \tag{90}$$

$$\lambda \in \mathbf{L}_m := \mathrm{Hom}(\mathbf{k}^\vee, H_m) \otimes \wedge^2 V, \quad \mu \in \mathbf{M}_m := \mathrm{Hom}(\mathbf{k}^\vee, \mathbf{k}) \otimes \wedge^2 V.$$

From (87) and (89) it follows that the homomorphism

$$C^\vee \circ D \circ C : H_m \otimes V \to H_m^\vee \otimes V^\vee, \qquad C^\vee \circ D \circ C \in \wedge^2(H_m^\vee \otimes V^\vee),$$

can be represented as

$$C^\vee \circ D \circ C = \phi^\vee \circ D_1 \circ \phi + \phi^\vee \circ \lambda \circ \psi - \psi^\vee \circ \lambda^\vee \circ \phi + \psi^\vee \circ \mu \circ \psi. \tag{91}$$

By (87)-(90) we have

$$\mathbf{S}_{m+1}^\vee \times \mathbf{\Sigma}_{m+1} = \mathbf{S}_m^\vee \times \mathbf{\Phi}_m \times \mathbf{\Psi}_m \times \mathbf{L}_m \times \mathbf{M}_m,$$

and there are well-defined morphisms

$$\tilde{p}_m : \widetilde{X}_m \to \mathbf{L}_m \oplus \mathbf{M}_m : (D_1, \phi, \psi, \lambda, \mu) \mapsto (\lambda, \mu).$$

and

$$p_m := \tilde{p}_m | \overline{X}_m : \overline{X}_m \to \mathbf{L}_m \oplus \mathbf{M}_m,$$

where $\overline{X}_m$ is the closure of X_m in $(\mathbf{S}_{m+1}^\vee)^0 \times \mathbf{\Sigma}_{m+1}$. We now invoke the following proposition, the proof of which is postponed to Section 11.

Proposition 7.3. *Let $m \geq 1$. Then, for any point $D \in (\mathbf{S}_{m+1}^\vee)^0$ and a general choice of the decomposition $H_{m+1} \xrightarrow{\sim} H_m \oplus \mathbf{k}$, the induced homomorphism D_1 in the matrix of homomorphisms D in (89) is nondegenerate.*

According to this proposition, we fix such a decomposition $H_{m+1} \xrightarrow{\sim} H_m \oplus \mathbf{k}$ for which the homomorphism $D_1 : H_m^\vee \otimes V^\vee \to H_m \otimes V$ in (89) is nondegenerate, i.e. $D_1 \in (\mathbf{S}_m^\vee)^0$.

Let $\mathcal{X}$ be any irreducible component of X_m and let $\overline{\mathcal{X}}$ be its closure in $\overline{X}_m$. Fix a point $z = (D_1, \phi, \psi, \lambda, \mu) \in \mathcal{X}$ not lying in the components of X_m different from $\mathcal{X}$. Consider the morphism

$$f : \ \mathbb{A}^1 \to \overline{\mathcal{X}} : \ t \mapsto (D_1, t^2\phi, t\psi, t\lambda, t^2\mu), \quad f(1) = z. \tag{92}$$

(This morphism is well-defined by (91).) By definition, the point $f(0) = (D_1, 0, 0, 0, 0)$ lies in the fibre $p_m^{-1}(0,0)$. Hence, $p_m^{-1}(0,0) \cap \overline{\mathcal{X}} \neq \emptyset$. In other words,

$$\rho^{-1}(0,0) \neq \emptyset, \qquad \textit{where} \quad \rho := p_m | \overline{\mathcal{X}}. \tag{93}$$

Now from (91) and the definition of $\widetilde{X}_m$ it follows that

$$\tilde{p}_m^{-1}(0,0) = \{(D_1, \phi, \psi) \in (\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m \times \mathbf{\Psi}_m \mid \phi^\vee \circ D_1 \circ \phi \in \mathbf{S}_m\}. \tag{94}$$

Comparing this with the definition (77) of Z_m we see that, set-theoretically, $\tilde{p}_m^{-1}(0,0) = Z_m \times \mathbf{\Psi}_m$, so that

$$\rho^{-1}(0,0) \overset{\text{sets}}{\subset} p_m^{-1}(0,0) \overset{\text{sets}}{=} \tilde{p}_m^{-1}(0,0) \overset{\text{sets}}{=} Z_m \times \mathbf{\Psi}_m. \tag{95}$$

Respectively, scheme-theoretically we have embeddings of schemes

$$\rho^{-1}(0,0) \overset{\text{schemes}}{\subset} p_m^{-1}(0,0) \overset{\text{schemes}}{\subset} \tilde{p}_m^{-1}(0,0) \overset{\text{schemes}}{=} Z_m \times \mathbf{\Psi}_m. \tag{96}$$

From (95) and Theorem 7.2 it follows, in particular, that

$$\dim \rho^{-1}(0,0) \leq \dim p_m^{-1}(0,0) \leq \dim Z_m + \dim \mathbf{\Psi}_m = 4m(m+2) + 6m = 4m^2 + 14m. \tag{97}$$

Hence in view of (93)

$$\dim \overline{\mathcal{X}} \leq \dim \rho^{-1}(0,0) + \dim \mathbf{L}_m + \dim \mathbf{M}_m \leq 4m^2 + 14m + 6m + 6 = 4m^2 + 20m + 6. \tag{98}$$

On the other hand, formula (16) for $n = 2m+1$, equality (43) and Theorem 6.1(ii) show that, for any point $x \in \mathcal{X}$ such that $A := g_m(x) \in MI_{2m+1}(\xi)$,

$$4m^2 + 20m + 6 = (2m+1)^2 + 8(2m+1) - 3 \le \dim_A MI_{2m+1}(\xi) = \dim \overline{\mathcal{X}}. \tag{99}$$

Comparing (98) with (99) we see that all inequalities in (97)-(99) are equalities. In particular,

$$\dim \rho^{-1}(0,0) = \dim(Z_m \times \mathbf{\Psi}_m) = \dim \overline{\mathcal{X}} - \dim(\mathbf{L}_m \times \mathbf{M}_m). \tag{100}$$

Since by Theorem 7.2 the scheme Z_m is integral and so $Z_m \times \mathbf{\Psi}_m$ is integral as well, (96) and (100) yield isomorphisms of integral schemes

$$\rho^{-1}(0,0) \overset{\text{schemes}}{=} p_m^{-1}(0,0) \overset{\text{schemes}}{=} \tilde{p}_m^{-1}(0,0) \overset{\text{schemes}}{=} Z_m \times \mathbf{\Psi}_m. \tag{101}$$

Now we formulate the following Lemma, the proof of which we leave to the reader.

Lemma 7.4. *Let $f : X \to Y$ be a morphism of reduced schemes, where Y is a smooth integral scheme. Assume that there exists a closed point $y \in Y$ such that for any irreducible component X' of X the following conditions are satisfied:*
(a) $\dim f^{-1}(y) = \dim X' - \dim Y$,
(b) the scheme-theoretic embedding of fibres $(f|_{X'})^{-1}(y) \subset f^{-1}(y)$ is an isomorphism of integral schemes.
Then
(i) there exists an open subset U of Y containing the point y such that the morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is flat,
(ii) X is integral and
(iii) X is smooth at any smooth point of $f^{-1}(y)$.

Applying the assertions (i)-(ii) of this lemma to $X = X_m,\ X' = \mathcal{X},\ Y = \mathbf{L}_m \times \mathbf{M}_m,\ y = (0,0), f = p_m$, and using (100) and (101), we obtain that X_m is an integral scheme of dimension $4m^2 + 20m + 6$.

It follows now from Corollary 4.8 and Theorem 6.1 that $(MI_{2m+1})_{red}$ is irreducible of dimension $4m^2 + 20m + 6 = n^2 + 8n - 3$ for $n = 2m+1$, i.e. the inequality (16) becomes the strict equality. This together with Theorem 3.1 implies that MI_{2m+1} is a locally complete intersection subscheme of the vector space $\mathbf{S}_{2m+1}$. We use now the following easy lemma, the proof of which is left to the reader.

Lemma 7.5. *Let $\mathcal{X}$ be an irreducible locally complete intersection subscheme of a smooth integral scheme $\mathcal{Y}$ such that $\mathcal{X}$ is smooth at some point. Then $\mathcal{X}$ is integral.*

Applying this Lemma to $\mathcal{X} = MI_{2m+1},\ \mathcal{Y} = \mathbf{S}_{2m+1}$ and using Remark 3.2, we obtain that MI_{2m+1} is integral. Since $\pi_{2m+1} : MI_{2m+1} \to I_{2m+1} :\ A \mapsto [E(A)]$ is a principal $GL(H_{2m+1})/\{\pm id\}$-bundle in the étale topology (see section 3), it follows that I_{2m+1} is integral of dimension $16m + 5 = 8n - 3$ for $n = 2m+1$. This finishs the proof of Theorem 1.1.

Remark 7.6. Consider the natural projections $p_I : X_m \to \mathbf{L}_m \times \mathbf{M}_m \times \mathbf{\Psi}_m$, $p_{II} : X_m \to \mathbf{S}_m \times \mathbf{L}_m \times \mathbf{M}_m \times \mathbf{\Psi}_m \simeq \mathbf{S}_{m+1} \times \mathbf{\Psi}_m$ and $p : X_m \overset{p_{II}}{\to} \mathbf{S}_{m+1} \times \mathbf{\Psi}_m \overset{pr_1}{\to} \mathbf{S}_{m+1}$. From (101) it follows that $p_I^{-1}(0,0,0) \simeq Z_m$. On the other hand, Theorem 7.2 shows that the projection $p' : Z_m \overset{p_m}{\to} (\mathbf{S}_m^\vee)^0 \simeq \mathbf{S}_m^0 \overset{\text{open}}{\hookrightarrow} \mathbf{S}_m$ is dominant, hence, for a general point $D_1 \in \mathbf{S}_m$, the fibre $p'^{-1}(D_1)$ is an integral scheme of dimension $\dim Z_m - \dim \mathbf{S}_m = m(m+5)$. This fibre in view of the equality $p_I^{-1}(0,0,0) \simeq Z_m$ coincides with the fibre $p_{II}^{-1}(D_1,0,0,0)$, and we thus have $\dim p_{II}^{-1}(D_1,0,0,0) = 5m(m+1) = 4m^2 + 20m + 6 - (3(m+1)(m+2)/2 + 6m) = \dim X_m - \dim(\mathbf{S}_m \times \mathbf{\Psi_m})$. Thus, applying Lemma 7.4 to $X = X' = X_m,\ Y = \mathbf{L}_m \times \mathbf{M}_m,\ y = (D_1,0,0,0), f = p_{II}$, we obtain that p_{II} is a dominant morphism. A fortiori,

$$p : X_m \to \mathbf{S}_{m+1} :\ (D,\phi) \to D$$

is a dominant morphism.

8. Study of Z_m. Beginning of the proof of Theorem 7.2

In this section we begin proving Theorem 7.2 on the irreducibility of Z_m. In subsection 8.1 we first treat the case $m = 1$. Next, we obtain explicit equations of Z_m under a fixed decompomposition of H_m into a direct sum of H_{m-1} and $\mathbf{k}$. In subsection 8.2 we formulate the main result of this section - Proposition 8.1 - which is a part of the induction step in the proof of Theorem 7.2. (The rest of the proof of Theorem 7.2 will be given in the last subsection of Section 10.) In subsections 8.3-8.5 we study in detail the explicit equations of Z_m and as a result obtain the proof of Proposition 8.1.

8.1. Explicit equations of Z_m in $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m$. We proceed to the proof of the irreducibility of Z_m by increasing induction on m. For $m = 1$ clearly $\mathbf{\Lambda}_m = 0$, so that the equations $\{\Theta_1(D_1, \phi_1) \in \mathbf{S}_1\}$ of Z_1 in $(\wedge^2(\mathbf{k}^\vee \otimes V^\vee))^0$ are empty, i.e. scheme-theoretically we have

$$Z_1 = (\wedge^2(\mathbf{k}^\vee \otimes V^\vee))^0 \times \mathbf{\Phi}_1 \overset{open}{\hookrightarrow} \mathbb{A}^{12}.$$

Thus Z_1 is integral as a dense open subset of $\mathbb{A}^{12}$.

Now fix an isomorphism

$$H_{m-1} \oplus \mathbf{k} \xrightarrow{\sim} H_m : ((a_1, ..., a_{m-1}), a_m) \mapsto (a_1, ..., a_m). \tag{102}$$

Under this isomorphism any homomorphism

$$\phi : H_m \otimes V \to H_m^\vee \otimes V^\vee, \qquad \phi \in \mathbf{\Phi}_m = \mathrm{Hom}(H_m, H_m^\vee \otimes \wedge^2 V^\vee). \tag{103}$$

can be represented as a homomorphism

$$\phi : H_{m-1} \otimes V \oplus \mathbf{k} \otimes V \to H_{m-1}^\vee \otimes V^\vee \oplus \mathbf{k}^\vee \otimes V^\vee, \tag{104}$$

i.e. as a matrix of homomorphisms

$$\phi = \begin{pmatrix} \phi_{m-1} & \chi \\ \psi & \theta \end{pmatrix}, \tag{105}$$

where

$$\begin{gathered} \phi_{m-1} \in \mathbf{\Phi}_{m-1} = \mathrm{Hom}(H_{m-1}, H_{m-1}^\vee \otimes \wedge^2 V^\vee), \quad \psi \in \mathbf{\Psi}_{m-1} := \mathrm{Hom}(H_{m-1}, \mathbf{k}^\vee \otimes \wedge^2 V^\vee), \\ \chi \in \mathrm{Hom}(\mathbf{k}, H_{m-1}^\vee \otimes \wedge^2 V^\vee) = \mathbf{\Psi}_{m-1}, \qquad \theta \in \mathbf{B}_\theta := \mathrm{Hom}(\mathbf{k}, \mathbf{k}^\vee \otimes \wedge^2 V^\vee) = \mathbf{S}_1. \end{gathered} \tag{106}$$

Respectively, a homomorphism

$$D \in \mathbf{S}_m^\vee \subset \mathrm{Hom}(H_m^\vee \otimes V^\vee, H_m \otimes V) \tag{107}$$

can be represented as a matrix of homomorphisms

$$D = \begin{pmatrix} D_{m-1} & a \\ -a^\vee & \alpha \end{pmatrix}, \tag{108}$$

where

$$\begin{gathered} D_{m-1} \in \mathbf{S}_{m-1}^\vee \subset \mathrm{Hom}(H_{m-1}^\vee \otimes V^\vee,\ H_{m-1} \otimes V), \\ a \in \mathrm{Hom}(\mathbf{k}^\vee, H_{m-1} \otimes \wedge^2 V) = \mathbf{\Psi}_{m-1}^\vee, \qquad \alpha \in \mathbf{B}_\alpha := \mathrm{Hom}(\mathbf{k}^\vee, \mathbf{k} \otimes \wedge^2 V). \end{gathered} \tag{109}$$

Note that the data (106) and (109) yield isomorphisms

$$\mathbf{S}_m^\vee \xrightarrow{\simeq} \mathbf{B}_\alpha \times \mathbf{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee, \quad \mathbf{\Phi}_m \xrightarrow{\simeq} \mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{B}_\theta, \tag{110}$$

and hence an isomorphism

$$\mathbf{S}_m^\vee \times \mathbf{\Phi}_m \xrightarrow{\simeq} \mathbf{B}_\theta \times \mathbf{B}_\alpha \times \mathbf{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee \times \mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1} : \tag{111}$$

$$(D,\phi)\mapsto(\theta,\alpha,a,D_{m-1},\phi_{m-1},\psi,\chi).$$

From (105) and (108) it follows that the homomorphism

$$\Theta(D,\phi):=\phi^\vee\circ D\circ\phi: H_m\otimes V\to H_m^\vee\otimes V^\vee,\qquad \Theta(D,\phi)\in\wedge^2(H_m^\vee\otimes V^\vee),$$

can be represented as a matrix of homomorphisms

$$\Theta(D,\phi)=\begin{pmatrix}\Theta_1(D,\phi) & b(D,\phi)\\ -b(D,\phi)^\vee & \beta(D,\phi)\end{pmatrix}, \tag{112}$$

where

$$\begin{gathered}\Theta_1(D,\phi):=\phi_{m-1}^\vee\circ D_{m-1}\circ\phi_{m-1}+\phi_{m-1}^\vee\circ a\circ\psi-\psi^\vee\circ a^\vee\circ\phi_{m-1}+\psi^\vee\circ\alpha\circ\psi\in\\ \in\wedge^2(H_{m-1}^\vee\otimes V^\vee)\subset\mathrm{Hom}(H_{m-1}^\vee\otimes V^\vee,H_{m-1}\otimes V),\\ b(D,\phi):=\phi_{m-1}^\vee\circ D_{m-1}\circ\chi+\phi_{m-1}^\vee\circ a\circ\theta-\psi^\vee\circ a^\vee\circ\chi+\psi^\vee\circ\alpha\circ\theta\in\\ \in\mathrm{Hom}(H_{m-1}\otimes V,\mathbf{k}^\vee\otimes V^\vee),\\ \beta(D,\phi):=\chi^\vee\circ D_{m-1}\circ\chi+\chi^\vee\circ a\circ\theta-\theta^\vee\circ a^\vee\circ\chi+\theta^\vee\circ\alpha\circ\theta\in\mathbf{B}_\theta.\end{gathered}\tag{113}$$

In these notations Z_m can be described as

$$Z_m=\left\{(D,\phi)\in(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m\;\middle|\;\begin{array}{l}(i)\ \Theta_1(D,\phi)\in\mathbf{S}_{m-1},\\ (ii)\ b(D,\phi)\in\mathbf{\Psi}_{m-1}\end{array}\right\}. \tag{114}$$

(Note that the condition $\beta(D,\phi)\in\mathbf{S}_1$ here is empty.)

We thus have the following explicit equations of Z_m in the open subset $(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m$ of the variety $\mathbf{S}_m^\vee\times\mathbf{\Phi}_m$, where we consider $\mathbf{S}_m^\vee\times\mathbf{\Phi}_m$ as the direct product $\mathbf{B}_\theta\times\mathbf{B}_\alpha\times\mathbf{\Psi}_{m-1}^\vee\times\mathbf{S}_{m-1}^\vee\times\mathbf{\Phi}_{m-1}\times\mathbf{\Psi}_{m-1}\times\mathbf{\Psi}_{m-1}$ via (111):

$$\Theta_1(D,\phi):=\phi_{m-1}^\vee\circ D_{m-1}\circ\phi_{m-1}+\phi_{m-1}^\vee\circ a\circ\psi-\psi^\vee\circ a^\vee\circ\phi_{m-1}+\psi^\vee\circ\alpha\circ\psi\in\mathbf{S}_{m-1}, \tag{115}$$

$$b(D,\phi):=\phi_{m-1}^\vee\circ D_{m-1}\circ\chi+\phi_{m-1}^\vee\circ a\circ\theta-\psi^\vee\circ a^\vee\circ\chi+\psi^\vee\circ\alpha\circ\theta\in\mathbf{\Psi}_{m-1}. \tag{116}$$

These equations will be used systematically in the next subsections.

8.2. Part of induction step in the proof of Theorem 7.2.

We first introduce some more notation. Set

$$\begin{gathered}(\wedge^2V)^0:=\{a\in\wedge^2V\mid a:V^\vee\to V\text{ is an isomorphism}\},\\ (\wedge^2V^\vee)^0:=\{a\in\wedge^2V^\vee\mid a:V\to V^\vee\text{ is an isomorphism}\}.\end{gathered}$$

Consider the projective space $P(\wedge^2V^\vee)$ together with the Grassmannian $G=G(1,3)\subset P(\wedge^2V^\vee)$ embedded by Plücker. Take any two points $a\in(\wedge^2V)^0$ and $b\in(\wedge^2V^\vee)^0$ such that the corresponding points $<a^{-1}>$ and $<b>$ in $P(\wedge^2V^\vee)$ are distinct. The projective line $P^1(a,b):=\mathrm{Span}(<a^{-1}>,<b>)$ joining these points intersects the quadric G in two points, say, $\{y_1,y_2\}$, not necessarily distinct, and let $\mathbb{P}^1_{(i)}(a,b),\ i=1,2$, be the two disjoint lines in $\mathbb{P}^3$ corresponding to the points y_1,y_2. Set

$$L(a,b):=\mathbb{P}^1_{(1)}(a,b)\sqcup\mathbb{P}^1_{(2)}(a,b). \tag{117}$$

Next, note that there are natural isomorphisms $\mathbf{S}_1^\vee\simeq\wedge^2V$ and $\mathbf{\Phi}_1^\vee\simeq\wedge^2V^\vee$, and, for any $m\ge2$, the induced isomorphisms

$$U_{\mathbf{S}}:=\overset{m}{\underset{i=1}{\oplus}}(\mathbf{S}_1^\vee)_{(i)}\simeq\overset{m}{\underset{1}{\oplus}}\wedge^2V,\quad U_{\mathbf{\Phi}}:=\overset{m}{\underset{i=1}{\oplus}}(\mathbf{\Phi}_1)_{(i)}\simeq,\overset{m}{\underset{i=1}{\oplus}}\wedge^2V^\vee, \tag{118}$$

where $(\mathbf{S}_1^\vee)_{(i)}$ and $(\mathbf{\Phi}_1)_{(i)}$ are copies of $\mathbf{S}_1^\vee$ and $\mathbf{\Phi}_1$, respectively. Furthermore, any isomorphism

$$h:\underbrace{H_1\oplus\ldots\oplus H_1}_{m}\xrightarrow{\simeq}H_m \tag{119}$$

induces embeddings $U_{\mathbf{S}}\hookrightarrow\mathbf{S}_m^\vee$ and $U_{\mathbf{\Phi}}\hookrightarrow\mathbf{\Phi}_m$, hence an embedding

$$\tau_h:U_{\mathbf{S}}\times U_{\mathbf{\Phi}}\hookrightarrow\mathbf{S}_m^\vee\times\mathbf{\Phi}_m. \tag{120}$$

Note also that the set

$$(121)\quad W_{\mathbf{S\Phi}} := \{((D_{(1)},...,D_{(m)}),(\phi_{(1)},...,\phi_{(m)})) \in U_{\mathbf{S}} \times U_{\mathbf{\Phi}} \mid \text{the subsets } L(D_{(i)},\phi_{(i)}) \text{ of } \mathbb{P}^3,$$
$$1 \le i \le m, \text{ are well defined, pairwise disjoint and not lying on a quadric}\}$$

is clearly a dense open subset of $U_{\mathbf{S}} \times U_{\mathbf{\Phi}}$.

The aim of the rest of this section is to prove the following proposition which is a part of the induction step $m-1 \rightsquigarrow m$ in the proof of Theorem 7.2.

Proposition 8.1. *Let $m \ge 2$ and let Z_{m-1} satisfy the statement of Theorem 7.2. Then there exists an irreducible component Z of Z_m such that:*

(i) let $Z_m = Z \cup Y$ be the decomposition of Z_m into components; then $Z^0 := Z \smallsetminus (Z \cap Y)$ is an integral locally complete intersection subscheme of $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m$;

(ii) $\dim Z = 4m(m+2)$ and the natural projection $p_m|_Z : Z \to (\mathbf{S}_m^\vee)^0 : \ (D,\phi) \mapsto D$ is dominant;

(iii) there exists an isomorphism h in (119) such that, in the notations (120) and (121), $Z \cap \tau_h(W_{\mathbf{S\Phi}}) \neq \emptyset$.

Before proving this proposition we need some preliminary remarks.

First, consider the case $m=2$. In this case $D_{m-1} = D_1 \in \wedge^2 V$, $\phi_{m-1} = \phi_1 \in \wedge^2 V^\vee$ and $a, \alpha \in \wedge^2 V$, $\psi, \chi, \theta \in \wedge^2 V^\vee$ so that the equations (115) become empty, and the equations (116) become:

$$(122)\qquad (\phi_1 \circ D_1 - \psi_1 \circ a) \circ \chi - (\phi_1 \circ a - \psi_1 \circ \alpha) \circ \theta \in \wedge^2 V^\vee.$$

Now one can easily check that, for a general point $x = (D_1, \phi_1, \psi, a, \alpha) \in (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^{\times 4}$, the equations (122) as a linear system on the pair $(\chi, \theta)) \in (\wedge^2 V^\vee)^{\times 2}$ has maximal rank equal 10. Thus the space F_x of solutions of this system as a subspace of $(\wedge^2 V^\vee)^{\times 2}$ has dimension 2. This means that there exists a component Z of Z_2 with projection $p_Z : Z \to (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^{\times 4}$: $(D_1, \phi_1, \psi, a, \alpha, \chi, \theta) \mapsto (D_1, \phi_1, \psi, a, \alpha)$ with a smooth fibre $F_x = p_Z^{-1}(x)$ of dimension 2. Hence, in particular, Z is generically reduced and $\dim Z \le \dim((\wedge^2 V)^0 \times (\wedge^2 V^\vee)^{\times 4}) + 2 = 32$. On the other hand, since (122) is a system of 10 equations of Z_2 in $(\mathbf{S}_2^\vee)^0 \times \mathbf{\Phi}_2$, it follows that Z as irreducible component of Z_2 has dimension $\ge \dim((\mathbf{S}_2^\vee)^0 \times \mathbf{\Phi}_2) - 10 = 42 - 10 = 32$. Hence $\dim Z = 32$ and p_Z is dominant. As a corollary, the projection $p_2|Z : Z \to (\mathbf{S}_2^\vee)^0$: $(D_1, \phi_1, \psi, a, \alpha, \chi, \theta) \mapsto (D_1, a, \alpha)$ is also dominant. Moreover, since F_x is smooth and $p_Z(Z)$ is smooth as a dense open subset of $(\wedge^2 V)^0 \times (\wedge^2 V^\vee)^{\times 4}$, it follows that Z is generically reduced. Now we use the following remark.

Remark 8.2. Let $\tilde{\mathcal{X}}$ be a locally closed subscheme of an affine space $\mathbb{A}^M$ defined locally by N equations. Let $\mathcal{X}$ be an irreducible component of $\tilde{\mathcal{X}}$ and let $\mathcal{X}^0$ be a complement in $\mathcal{X}$ of its intersection with the union of other possible components of $\tilde{\mathcal{X}}$. Let $\mathcal{X}$ be generically reduced and let $\dim \mathcal{X} = M - N$. Then $\mathcal{X}^0$ is an integral locally complete intersection subscheme of $\mathbb{A}^M$.

Applying this remark to the case $\mathcal{X} = Z_2$, $\mathbb{A}^{42} = (\wedge^2 V)^0 \times (\wedge^2 V^\vee)^{\times 6}$, we obtain from the above that the statements (i)-(ii) of Proposition 8.1 are true for Z. Now an explicit computation shows that the statement (iii) of this Proposition is also true for Z. We thus have proved Proposition 8.1 for $m = 2$.

We proceed now to the proof of Proposition 8.1 for $m \ge 3$. For this, note that, by the assumption, Z_{m-1} is an integral subscheme of $(\mathbf{S}_{m-1}^\vee)^0 \times \mathbf{\Phi}_{m-1}$ such that

$$\dim Z_{m-1} = 4(m^2 - 1)$$

and the natural projection $p_{m-1} : Z_{m-1} \to (\mathbf{S}_{m-1}^\vee)^0 : \ (D_{m-1}, \phi_{m-1}) \mapsto D_{m-1}$ is surjective:

$$(123)\qquad p_{m-1}(Z_{m-1}) = (\mathbf{S}_{m-1}^\vee)^0.$$

Hence, since $\dim(\mathbf{S}_{m-1}^\vee)^0 = 3m(m-1)$ and so $\dim Z_{m-1} - \dim(\mathbf{S}_{m-1}^\vee)^0 = (m-1)(m+4)$, it follows that the set

(124)
$$(\mathbf{S}_{m-1}^\vee)^{int} := \{D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^0 \mid \text{the fibre } p_m^{-1}(D_{m-1}) \text{ is integral of dimension } (m-1)(m+4)\}$$

is a dense open subset of $(\mathbf{S}_{m-1}^\vee)^0$; respectively,

$$Z_{m-1}^{int} := p_{m-1}^{-1}((\mathbf{S}_{m-1}^\vee)^{int}) \tag{125}$$

is a dense open subset of Z_{m-1}.

Next, using (111) and the embedding $Z_m \hookrightarrow \mathbf{S}_m^\vee \times \mathbf{\Phi}_m$ consider the projections

$$\begin{aligned} &pr_m : \mathbf{S}_m^\vee \times \mathbf{\Phi}_m \to \mathbf{B}_\theta \times \mathbf{B}_\alpha \times \mathbf{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee : \quad (D, \phi) = (\theta, \alpha, a, D_{m-1}, \phi_{m-1}, \psi, \chi) \mapsto \\ &\mapsto (\theta, \alpha, a, D_{m-1}), \qquad \pi_m := pr_m|_{Z_m} : Z_m \to \mathbf{B}_\theta \times \mathbf{B}_\alpha \times \mathbf{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee. \end{aligned} \tag{126}$$

We are going now to study the fibre

$$\pi_m^{-1}(y^0)$$

of the projection π_m over the point

$$y^0 := (\theta^0, \alpha^0, 0, D_{m-1}) \in \mathbf{B}_\theta \times \mathbf{B}_\alpha \times \mathbf{\Psi}_{m-1}^\vee \times (\mathbf{S}_{m-1}^\vee)^0, \tag{127}$$

where

$$\alpha^0 = (p_{ij}) \in \wedge^2 V^\vee \simeq \mathbf{B}_\alpha, \quad \theta^0 = (q_{ij}) \in \wedge^2 V^\vee \simeq \mathbf{B}_\theta, \quad p_{ij}, q_{ij} \in \mathbf{k}. \tag{128}$$

[3]

Note that, by the definition of π_m, the fibre $\pi_m^{-1}(y^0)$ naturally lies in $\mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1}$:

$$\pi_m^{-1}(y^0) \subset \mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1}. \tag{129}$$

Thus, substituting (127) into (115) and (116), we obtain the equations of $\pi_m^{-1}(y^0)$ as a subscheme of $\mathbf{\Phi}_{m-1} \times \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1}$ as equations in the variables ϕ_{m-1}, χ and ψ:

$$\phi_{m-1}^\vee \circ D_{m-1} \circ \phi_{m-1} + \psi^\vee \circ \alpha^0 \circ \psi \in \mathbf{S}_{m-1}, \tag{130}$$

$$\phi_{m-1}^\vee \circ D_{m-1} \circ \chi + \psi^\vee \circ \alpha^0 \circ \theta^0 \in \mathbf{\Psi}_{m-1}. \tag{131}$$

For an arbitrary point y^0 in (127), where $D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^0$, consider the set

$$F(\theta^0, \alpha^0, D_{m-1}) := \pi_m^{-1}(y^0) \cap \{\chi = \psi = 0\}. \tag{132}$$

It follows from (130) that

$$F(\theta^0, \alpha^0, D_{m-1}) \simeq \{\phi_{m-1} \in \mathbf{\Phi}_{m-1} \mid \phi_{m-1}^\vee \circ D_{m-1} \circ \phi_{m-1} \in \mathbf{S}_{m-1}\}. \tag{133}$$

Hence, $\underset{D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^0}{\cup} F(\theta^0, \alpha^0, D_{m-1}) = \{(\theta^0, \alpha^0)\} \times Z_{m-1}$. Moreover, the definition (124) implies that for $D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^{int}$ the set $F(\theta^0, \alpha^0, D_{m-1})$ is irreducible of dimension $(m-1)(m+4)$ and, by (111), (125) and (132),

$$\underset{D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^{int}}{\cup} F(\theta^0, \alpha^0, D_{m-1}) = \{(\theta^0, \alpha^0, 0)\} \times Z_{m-1}^{int} \times \{(0,0)\}. \tag{134}$$

8.3. Proof of Proposition 8.1: case m odd, first computations. In this subsection we prove Proposition 8.1 for the case of odd m,

$$m = 2p + 1, \quad p \geq 1.$$

Fix decompositions

$$H_{m-1} \simeq \underbrace{H_2 \oplus \ldots \oplus H_2}_{p}, \quad H_2 \simeq H_1 \oplus H_1. \tag{135}$$

[3]Here and below we use a fixed basis $e_1, \ldots, e_4$ of V in order to understand points of $\wedge^2 V$ and $\wedge^2 V^\vee$ as skew 4×4- matrices.

Under these decompositions consider the points $D^{\Delta}_{m-1} \in (\mathbf{S}^{\vee}_{m-1})^0$ and $\phi^{\Delta}_{m-1} \in \mathbf{\Phi}_{m-1}$ given by the matrices [4]

$$D^{\Delta}_{m-1} := \underbrace{D_2 \oplus ... \oplus D_2}_{p}, \quad \phi^{\Delta}_{m-1} = \phi^{\Delta}_{m-1}(N,a,d,f,g) := \underbrace{\phi_2 \oplus ... \oplus \phi_2}_{p}, \tag{136}$$

where

$$D_2 = D' \oplus D'' \in \mathbf{S}^{\vee}_2, \quad D' = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \in \wedge^2 V, \quad D'' = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \in \wedge^2 V, \tag{137}$$

$$\phi_2 = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \in \mathbf{\Phi}_2, \quad \phi_{11} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & N \\ & & -N & \end{pmatrix}, \quad \phi_{22} = \begin{pmatrix} & & 1 & \\ & & & N \\ -1 & & & \\ & -N & & \end{pmatrix}, \quad N \in \mathbf{k}, \tag{138}$$

$$\phi_{12} = \begin{pmatrix} & & & f \\ & & g & \\ & -g & & \\ -f & & & \end{pmatrix}, \quad \phi_{21} = \begin{pmatrix} & & a & f \\ & & -g & d \\ -a & g & & \\ -f & -d & & \end{pmatrix} \in \wedge^2 V^{\vee}, \quad a,d,f,g \in \mathbf{k}.$$

One easily checks that

$$(\phi^{\Delta}_{m-1})^{\vee} \circ D^{\Delta}_{m-1} \circ \phi^{\Delta}_{m-1} \in \mathbf{S}_{m-1}, \tag{139}$$

hence the point $(D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1}) \in \mathbf{S}^{\vee}_{m-1} \times \mathbf{\Phi}_{m-1}$ lies in $\widehat{Z}_{m-1}$. Moreover, since $D^{\Delta}_{m-1} \in (\mathbf{S}^{\vee}_{m-1})^0$, it follows that

$$(D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1}) \in Z_{m-1}. \tag{140}$$

In addition, it follows from (139) that the equations (130) are automatically satisfied for any $\psi \in \Psi_{m-1}$. Now, substituting the data $(\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1})$ into (131), we obtain the equations on (χ, ψ):

$$(\phi^{\Delta}_{m-1})^{\vee} \circ D^{\Delta}_{m-1} \circ \chi + \psi^{\vee} \circ \alpha^0 \circ \theta^0 \in \mathbf{\Psi}_{m-1}. \tag{141}$$

Set

$$W(\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1}) := \{(\chi, \psi) \in \mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1} \mid (\chi, \psi) \text{ satisfies } (141)\}. \tag{142}$$

Note that, since the equations (141) on (χ, ψ) are linear, it follows that $W(D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1}, \alpha^0, \theta^0)$ is a linear subspace of the vector space $\mathbf{\Psi}_{m-1} \times \mathbf{\Psi}_{m-1} \simeq \mathbf{\Psi}^{\vee}_{m-1} \oplus \mathbf{\Psi}_{m-1}$.

Find the dimension of the vector space $W(\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1})$. For this, using the decompositions (135) we represent χ and ψ as p-ples

$$\chi = (\chi_1, ..., \chi_p), \quad \psi = (\psi_1, ..., \psi_p), \quad \psi_k, \chi_k \in \mathbf{\Psi}_2, \ k = 1, ..., p, \tag{143}$$

where

$$\chi_k = (X_k, Y_k), \quad \psi_k = (A_k, B_k), \quad X_k, Y_k, A_k, B_k \in \wedge^2 V^{\vee}, \tag{144}$$

and

$$X_k = (x^{(k)}_{ij}), \ Y_k = (y^{(k)}_{ij}), \ A_k = (a^{(k)}_{ij}), \ B_k = (b^{(k)}_{ij}) \tag{145}$$

[4] Here and everywhere below the empty entries of matrices mean zeroes. Besides, we use the standard notation $A = A_1 \oplus ... \oplus A_n$ for a direct sum A of matrices $A_1, ..., A_n$ which is a block matrix with diagonal blocks $A_1, ..., A_n$ and the zero rest blocks.

are skew-symmetric 4×4-matrices. Inserting D^{Δ}_{m-1} and ϕ^{Δ}_{m-1} from (136) into the system of equations (141) we rewrite this system as

$$\phi_2^{\vee} \circ D_2 \circ \chi_k + \psi_k^{\vee} \circ \alpha^0 \circ \theta^0 \in \mathbf{\Psi}_2, \quad k = 1, ..., p. \tag{146}$$

Substituting here D_2, ϕ_2 and θ^0 from (137), (138) and (128) and denoting $x_1^{(k)} = x_{12}^{(k)}, x_2^{(k)} = x_{34}^{(k)}, x_3^{(k)} = x_{13}^{(k)}, x_4^{(k)} = x_{14}^{(k)}, x_5^{(k)} = x_{23}^{(k)}, x_6^{(k)} = x_{24}^{(k)}, x_7^{(k)} = y_{12}^{(k)}, x_8^{(k)} = y_{34}^{(k)}, x_9^{(k)} = y_{13}^{(k)}, x_{10}^{(k)} = y_{14}^{(k)}, x_{11}^{(k)} = y_{23}^{(k)}, x_{12}^{(k)} = y_{24}^{(k)}, x_{13}^{(k)} = a_{12}^{(k)}, x_{14}^{(k)} = a_{34}^{(k)}, x_{15}^{(k)} = a_{13}^{(k)}, x_{16}^{(k)} = x_{14}^{(k)}, x_{17}^{(k)} = x_{23}^{(k)}, x_{18}^{(k)} = x_{24}^{(k)}, x_{19}^{(k)} = b_{12}^{(k)}, x_{20}^{(k)} = b_{34}^{(k)}, x_{21}^{(k)} = b_{13}^{(k)}, x_{22}^{(k)} = b_{14}^{(k)}, x_{23}^{(k)} = b_{23}^{(k)}, x_{24}^{(k)} = b_{24}^{(k)}$, we rewrite the system (146) as

$$\sum_{j=1}^{24} m_{ij} x_j^{(k)} = 0, \quad i = 1, ..., 20, \quad k = 1, ..., p, \tag{147}$$

where $M := (m_{ij})$ is the 20×24-matrix with entries depending on $N, a, d, f, g, p_{ij}, q_{ij}$.

Now a direct computation of the matrix $\mathbf{M} = (m_{ij})$ for

$$N = 101, \ a = 4, \ d = 6, \ f = 2, \ g = 5, \tag{148}$$

$$\begin{gathered} p_{12} = -9, \ p_{13} = -2, \ p_{14} = -4, \ p_{23} = 6, \ p_{24} = -3, \ p_{34} = -7, \\ q_{12} = -4, q_{13} = -4, q_{14} = -2, q_{23} = 3, q_{24} = -7, q_{34} = 8, \end{gathered} \tag{149}$$

shows that $\mathbf{M}$ is the upper left block submatrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{\psi} & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{0} & \mathbf{M}_{\psi} \end{pmatrix} \tag{150}$$

of the block matrix $\widetilde{\mathbf{M}}$ given below in (181)-(186). From (150) and (182)-(186) it follows by an explicit computation that

$$\mathrm{rk}\mathbf{M} = 20. \tag{151}$$

Hence, since the matrix of the system (147) is a direct sum of p copies of matrix $\mathbf{M}$, it follows that its rank equals

$$p \cdot \mathrm{rk}\mathbf{M} = 20p = 10(m-1). \tag{152}$$

Next, denote by

$$\boldsymbol{\phi}_{m-1}, \quad \text{resp.,} \quad \boldsymbol{\alpha}, \ \boldsymbol{\theta} \tag{153}$$

the matrices obtained by inserting the entries (148) into the matrix ϕ^{Δ}_{m-1} in (136), respectively, the entries (149) into the matrices α^0 and θ^0 in (128). In this notation, denoting by $R(\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1})$ the rank of the linear system (141) as a function of $\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1}$ we rewrite (152) as

$$R(\boldsymbol{\theta}, \boldsymbol{\alpha}, D^{\Delta}_{m-1}, \boldsymbol{\phi}_{m-1}) = 10(m-1). \tag{154}$$

Note that $(D^{\Delta}_{m-1}, \boldsymbol{\phi}_{m-1}) \in Z_{m-1}$ by (140), and by (125) Z^{int}_{m-1} is irreducible and dense open in Z_{m-1}. In addition, since the maximal value of $R(\theta^0, \alpha^0, D_{m-1}, \phi_{m-1})$ equals $10(m-1)$, the condition $R(\theta^0, \alpha^0, D_{m-1}, \phi_{m-1}) = 10(m-1)$ imposed on the point $(D_{m-1}, \phi_{m-1}) \in Z_{m-1}$ is open. Hence it follows from (154) that

a) the set $(Z^{int}_{m-1})^0 := \{(D_{m-1}, \phi_{m-1}) \in Z^{int}_{m-1} \mid R(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}, \phi_{m-1}) = 10(m-1)\}$ is dense open in Z^{int}_{m-1}, hence also in Z_{m-1}. By (123) this implies that

b) there exists a dense open subset $(\mathbf{S}^{\vee}_{m-1})^*$ of $(\mathbf{S}^{\vee}_{m-1})^{int}$ such that, for $D_{m-1} \in (\mathbf{S}^{\vee}_{m-1})^*$, the set

$$F(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1})^0 := F(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}) \cap (Z^{int}_{m-1})^0$$

where $F(\theta^0, \alpha^0, D_{m-1})$ is defined in (132), is an integral scheme of dimension $(m-1)(m+4)$ and it is a dense open subset of $F(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1})$.

Now for $D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^*$ set

$$\mathbf{F} := \pi_m^{-1}(\boldsymbol{\theta}, \boldsymbol{\alpha}, 0, D_{m-1}), \quad F = F(D_{m-1}) := F(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}) = \mathbf{F} \cap \{\chi = \psi = 0\}.$$

From a) and b) it follows similar to (134) that $\underset{D_{m-1}\in(\mathbf{S}_{m-1}^\vee)^*}{\cup} F(D_{m-1})$ is dense open in $\{(\boldsymbol{\theta}, \boldsymbol{\alpha})\} \times Z_{m-1}^{int} \times \{(0,0)\}$, hence

$$\overline{\underset{D_{m-1}\in(\mathbf{S}_{m-1}^\vee)^*}{\cup} F(D_{m-1})} = \{(\boldsymbol{\theta}, \boldsymbol{\alpha}, 0)\} \times \widehat{Z}_{m-1} \times \{(0,0)\}, \tag{155}$$

where the closure is taken in $\mathbf{S}_m^\vee \times \boldsymbol{\Phi}_m$ and we use the isomorphism (111).

Take an arbitrary point $D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^*$. By b) $F = F(D_{m-1})$ is integral of dimension $(m-1)(m+4)$ and contains a dense open subset F^0 such that, for any point $w = (\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}, \phi'_{m-1}) \in F^0$, one has $R(w) := R(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}, \phi'_{m-1}) = 10(m-1)$. Fix such a point w which is smooth on F. We are going now to compute the dimension of the tangent space $T_w\mathbf{F}$.

Note that by (129) we consider $\mathbf{F}$ as lying in $\boldsymbol{\Phi}_{m-1} \times \boldsymbol{\Psi}_{m-1} \times \boldsymbol{\Psi}_{m-1}$. Hence the equations of the tangent space

$$T_w\mathbf{F}$$

are given by differentiating at w the equations (130) and (131):

$$d\phi_{m-1}^\vee|_{\phi'_{m-1}} \circ D_{m-1} \circ \phi_{m-1} + {\phi'}_{m-1}^\vee \circ D_{m-1} \circ d\phi_{m-1}|_{\phi'_{m-1}} \in \mathbf{S}_{m-1}, \tag{156}$$

$${\phi'}_{m-1}^\vee \circ D_{m-1} \circ d\chi|_0 + d\psi|_0^\vee \circ \alpha^0 \circ \theta^0 \in \boldsymbol{\Psi}_{m-1}. \tag{157}$$

Here the equations (156) coincide with the equations obtained by differentiating at w the equations $\phi_{m-1}^\vee \circ D_{m-1} \circ \phi_{m-1} \in \mathbf{S}_{m-1}$ defining F as a subscheme of $\boldsymbol{\Phi}_{m-1}$. Since w is a smooth point of F^0, it follows that the equations (156) define the tangent space $T_wF^0 = T_wF$ as a subspace of $T_{\phi'_{m-1}}\boldsymbol{\Phi}_{m-1}$ and

$$\dim T_wF = \dim F = (m-1)(m-4). \tag{158}$$

On the other hand, the equations (157) just coincide with (131) via identifying $(\chi|_0, d\psi|_0)$ with (χ, ψ), i.e. they are the equations of the subspace $W(w) = W(\boldsymbol{\theta}, \boldsymbol{\alpha}, D_{m-1}, \phi'_{m-1})$ in $\Psi_{m-1} \oplus \Psi_{m-1}$. Hence $\dim W(w) = \dim(\Psi_{m-1} \oplus \Psi_{m-1}) - R(w) = 12(m-1) - 10(m-1) = 2(m-1)$. In view of (158) we have

$$\dim_w \mathbf{F} \le \dim T_w\mathbf{F} = \dim T_wF + \dim W(w) = (m-1)(m+4) + 2(m-1) = m^2 + 5m - 6. \tag{159}$$

Note that, since $D_{m-1} \in (\mathbf{S}_{m-1}^\vee)^0$ and $\boldsymbol{\alpha} \in \mathbf{S}_1^0$ (see (128)), it follows that $D = D_{m-1} \oplus \boldsymbol{\alpha} \in (\mathbf{S}_m^\vee)^0$, so that

$$w \in Z_m. \tag{160}$$

In addition, $\dim(\mathbf{B}_\theta \times \mathbf{B}_\alpha \times \boldsymbol{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee) = \dim(\mathbf{B}_\theta \times \mathbf{S}_m^\vee) = 6 + 3m(m+1) = 3m^2 + 3m + 6$. Counting the dimension of the fibres of $\pi_m : Z_m \to \mathbf{B}_\theta \times \mathbf{B}_\alpha \times \boldsymbol{\Psi}_{m-1}^\vee \times \mathbf{S}_{m-1}^\vee \simeq \mathbf{B}_\theta \times \mathbf{S}_m^\vee$ and using (159) we obtain

$$\dim_w Z_m \le \dim_w \mathbf{F} + \dim(\mathbf{B}_\theta \times \mathbf{S}_m^\vee) \le (m^2 + 5m - 6) + (3m^2 + 3m + 6) = 4m(m+2).$$

Comparing this with (81) we see that the above inequalities on dimensions are strict equalities. In particular, $\dim_w Z_m = 4m(m+2)$ and $\dim_w \mathbf{F} = \dim T_w\mathbf{F} = m^2 + 5m - 6$ and $\dim \pi_m(Z_m) = (3m^2 + 3m + 6) = \dim(\mathbf{B}_\theta \times \mathbf{S}_m^\vee)$. This together with the assertion (iii) of Lemma 7.4 implies that there exists a unique irreducible component, say, Z of Z_m passing through w such that:

(i) $\dim Z = 4m(m+2)$ and Z_m, respectively, Z is smooth at w; hence, in notations of Proposition 8.1(i), Z^0 is an integral locally complete intersection subscheme of $(\mathbf{S}_m^\vee)^0 \times \boldsymbol{\Phi}_m$ (we use here Remark 8.2);

(ii) $\pi_m(Z)$ is dense in $\mathbf{B}_\theta \times \mathbf{S}_m^\vee$; respectively, $p_m(Z) = pr_{\mathbf{S}}(\pi_m(Z))$ is dense in $\mathbf{S}_m^\vee$, where $pr_{\mathbf{S}} : \mathbf{B}_\theta \times \mathbf{S}_m^\vee \to \mathbf{S}_m^\vee$ is the projection. This gives proof of the statements (i) and (ii) of Proposition 8.1.

Moreover, by a) and b) above, $F = F(D_{m-1}) \subset Z$ for $D_{m-1} \in (\mathbf{S}^\vee_{m-1})^*$, so that (155) implies the existence of an embedding

$$\{(\boldsymbol{\theta}, \boldsymbol{\alpha}, 0)\} \times \widehat{Z}_{m-1} \times \{(0,0)\} \subset \overline{Z}, \tag{161}$$

where $\overline{Z}$ is the closure of Z in $\mathbf{S}^\vee_m \times \mathbf{\Phi}_m$. In particular, similar to (160) we have in view of (140):

$$w^0 := (\boldsymbol{\theta}, \boldsymbol{\alpha}, 0, D^\Delta_{m-1}, \phi^\Delta_{m-1}, 0, 0) \in Z. \tag{162}$$

8.4. Proof of Proposition 8.1: case m odd, last computations. In this subsection we prove the last statement (iii) of Proposition 8.1 in case of odd m. For this, consider the following modification of the data (136)-(138):

$$\begin{aligned} D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}) &:= D_2(c, f_1, g_1) \oplus ... \oplus D_2(c, f_p, g_p), \\ \phi^\Delta_{m-1}(\varepsilon, \mathbf{f}, \mathbf{g}) &:= \phi_2(\varepsilon, f_1, g_1) \oplus ... \oplus \phi_2(\varepsilon, f_p, g_p), \end{aligned} \tag{163}$$

where

$$D_2(c, f_i, g_i) = \begin{pmatrix} D'(c, f_i, g_i) & \\ & D'' \end{pmatrix} \in \mathbf{S}^\vee_2, \tag{164}$$

$$D'(c, f_i, g_i) = \begin{pmatrix} & -1 & & cg_i \\ 1 & & cf_i & \\ & -cf_i & & 1 \\ -cg_i & & -1 & \end{pmatrix}, \ i = 1, ..., p, \quad D'' = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix} \in \wedge^2 V,$$

$$\phi_2(\varepsilon, f_i, g_i) = \begin{pmatrix} \phi_{11} & \phi_{12}(\varepsilon, f_i, g_i) \\ \phi_{21}(\varepsilon, f_i, g_i) & \phi_{22} \end{pmatrix} \in \mathbf{\Phi}_2, \quad \phi_{11} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & N \\ & & -N & \end{pmatrix}, \tag{165}$$

$$\phi_{22} = \begin{pmatrix} & & 1 & \\ & & & N \\ -1 & & & \\ & -N & & \end{pmatrix}, \quad \phi_{12}(\varepsilon, f_i, g_i) = \begin{pmatrix} & & & \varepsilon f_i \\ & & \varepsilon g_i & \\ & -\varepsilon g_i & & \\ -\varepsilon f_i & & & \end{pmatrix},$$

$$\phi_{21}(\varepsilon, f_i, g_i) = \begin{pmatrix} & & \varepsilon a & \varepsilon f_i \\ & & -\varepsilon g_i & \varepsilon d \\ -\varepsilon a & \varepsilon g_i & & \\ -\varepsilon f_i & -\varepsilon d & & \end{pmatrix} \in \wedge^2 V^\vee, \quad c, \epsilon, N, a, d, f_i, g_i \in \mathbf{k}, \ i = 1, ..., p,$$

and where $\mathbf{f} = (f_1, ..., f_p), \mathbf{g} = (g_1, ..., g_p) \in \mathbf{k}^p$. One easily checks that $(\phi^\Delta_{m-1}(\varepsilon))^\vee \circ D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}) \circ \phi^\Delta_{m-1}(\varepsilon) \in \mathbf{S}_{m-1}$, hence the point

$$(D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}), \phi^\Delta_{m-1}(\varepsilon, \mathbf{f}, \mathbf{g})) \in \mathbf{S}^\vee_{m-1} \times \mathbf{\Phi}_{m-1}$$

lies in $\widetilde{Z}_{m-1}$. Moreover, since $(D^\Delta_{m-1}(0, \mathbf{f}, \mathbf{g}) = D^\Delta_{m-1} \in (\mathbf{S}^\vee_{m-1})^0$ and $(\mathbf{S}^\vee_{m-1})^0$ is open in $\mathbf{S}^\vee_{m-1}$, it follows that, for any $\mathbf{f}, \mathbf{g} \in \mathbf{k}^p$ there exists some dense open subset $\mathcal{U}(\mathbf{f}, \mathbf{g})$ of $\mathbf{k}$ such that $D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}) \in (\mathbf{S}^\vee_{m-1})^0, \quad c \in \mathcal{U}(\mathbf{f}, \mathbf{g})$. Hence, $(D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}), \phi^\Delta_{m-1}(\varepsilon, \mathbf{f}, \mathbf{g})) \in Z_{m-1}$ for $c \in \mathcal{U}(\mathbf{f}, \mathbf{g})$, so that, since Z_{m-1} is closed in $\mathbf{S}^\vee_{m-1} \times \mathbf{\Phi}_{m-1}$,

$$(D^\Delta_{m-1}(c, \mathbf{f}, \mathbf{g}), \phi^\Delta_{m-1}(\varepsilon, \mathbf{f}, \mathbf{g})) \in Z_{m-1}, \quad c, \varepsilon \in \mathbf{k}, \quad \mathbf{f}, \mathbf{g} \in \mathbf{k}^p. \tag{166}$$

In particular, take $c = 1$ and $\varepsilon = 0$ in (163)-(165). It follows immediately that the point

$$w(\mathbf{f}, \mathbf{g}, \theta^0, \alpha^0) := (D^\Delta_{m-1}(1, \mathbf{f}, \mathbf{g}) \oplus \alpha^0, \phi^\Delta_{m-1}(0, \mathbf{f}, \mathbf{g}) \oplus \theta^0), \quad (\theta^0, \alpha^0) \in \wedge^2 V^\vee \times \wedge^2 V,$$

is the image of the point

$$((D'(1,f_1,g_1),...,D'(1,f_p,g_p),\underbrace{D'',...,D''}_{p},\alpha^0),(\underbrace{\phi_{11},...,\phi_{11}}_{p},\underbrace{\phi_{22},...,\phi_{22}}_{p},\theta^0))\in U_{\mathbf{S}}\times U_{\mathbf{\Phi}}$$

under the embedding $\tau_h: U_{\mathbf{S}}\times U_{\mathbf{\Phi}}\hookrightarrow \mathbf{S}_m^\vee\times\mathbf{\Phi}_m$ defined (up to a permutation of direct summands) as in (119)-(120) via the isomorphism

$$h:\underbrace{H_1\oplus...\oplus H_1}_{m}\xrightarrow{\simeq} H_m,\ m=2p+1, \tag{167}$$

determined by the decompositions (135).

On the other hand, by (111) and (166) we have $w(\mathbf{f},\mathbf{g},\boldsymbol{\theta},\boldsymbol{\alpha})\in\{(\boldsymbol{\theta},\boldsymbol{\alpha},0)\}\times\widehat{Z}_{m-1}\times\{(0,0)\}$, so that, in view of (161), $w(\mathbf{f},\mathbf{g},\boldsymbol{\theta},\boldsymbol{\alpha})\in\overline{Z}$. Thus,

$$w(\mathbf{f},\mathbf{g},\boldsymbol{\theta},\boldsymbol{\alpha})\in Z\cap\tau_h(U_{\mathbf{S}}\times U_{\mathbf{\Phi}}),\quad \mathbf{f},\mathbf{g}\in\mathbf{k}^p. \tag{168}$$

Note that $D^\Delta_{m-1}(1,\mathbf{0},\mathbf{0})=D^\Delta_{m-1}$, hence it follows from the definition of $w(\mathbf{f},\mathbf{g},\theta^0,\alpha^0)$ that the point $w(\mathbf{0},\mathbf{0},\boldsymbol{\theta},\boldsymbol{\alpha})$ lies in Z_m (cf. (162)). Since the condition $w(\mathbf{f},\mathbf{g},\theta^0,\alpha^0)\in Z_m$ on the point $(\mathbf{f},\mathbf{g},\theta^0,\alpha^0)\in\mathbf{k}^{2p}\times\wedge^2V^\vee\times\wedge^2V$ is open, we obtain from (168) that there exists a dense open subset $\mathcal{U}\in\mathbf{k}^{2p}\times\wedge^2V^\vee\times\wedge^2V$ such that

$$w(\mathbf{f},\mathbf{g},\theta^0,\alpha^0)\in Z_m\cap\tau_h(U_{\mathbf{S}}\times U_{\mathbf{\Phi}}),\quad (\mathbf{f},\mathbf{g},\theta^0,\alpha^0)\in\mathcal{U}. \tag{169}$$

Next, one easily sees that, for general $f_i,g_i\neq 0$ the points $D'(0,f_i,g_i),D'(1,f_i,g_i)$ lie in $(\wedge^2V)^0$ and, moreover, the projective plane Span$(<D'(0,f_i,g_i)^{-1}>,<D'(1,f_i,g_i)^{-1}>,<\phi_{11}>)$ in $P(\wedge^2V^\vee)$ intersects the Grassmannian $G=G(1,3)$ in a smooth conic. This immediately implies that, in the notation of (117), for a general choice of $f_1,g_1,f_2,g_2\in\mathbf{k}$, the sets $L(D'(1,f_1,g_1)^{-1},\phi_{11})$ and $L(D'(1,f_2,g_2)^{-1},\phi_{11})$ are well defined and disjont. In other words, using the notation of (118) and considering the projection onto the direct summand

$$pr_{ij}:U_{\mathbf{S}}\times U_{\mathbf{\Phi}}\to((\mathbf{S}_1^\vee)_{(i)}\oplus(\mathbf{S}_1^\vee)_{(j)})\times((\mathbf{\Phi}_1)_{(i)}\oplus(\mathbf{\Phi}_1)_{(j)})\simeq(\mathbf{S}_1^\vee\oplus\mathbf{S}_1^\vee)\times(\mathbf{\Phi_1}\oplus\mathbf{\Phi_1})$$

for any $1\le i<j\le m$ and taking the dense open subset W_{ij} of $U_{\mathbf{S}}\times U_{\mathbf{\Phi}}$ defined as

$$W_{ij}:=pr_{ij}^{-1}(\{((D_1,D_2),(\phi_1,\phi_2))\in(\mathbf{S}_1^\vee\oplus\mathbf{S}_1^\vee)\times(\mathbf{\Phi}_1\oplus\mathbf{\Phi}_1)\mid \text{the subsets } L(D_1^{-1},\phi_1) \text{ and } L(D_2^{-1},\phi_2) \text{ of } \mathbb{P}^3 \text{are well defined and disjoint}\})$$

are well defined, pairwise disjoint we obtain in view of (169) that

$$Z\cap\tau_h(W_{12})\neq\emptyset. \tag{170}$$

Now since the set Isom_m of all isomorphisms h in (167) is a principal homogeneous space of the group $GL(H_m)$ which is connected, it follows from (170) that $Z_m\cap\tau_h(W_{ij})\neq\emptyset$ for a general $h\in\mathrm{Isom}_m$ and any pair $(i,j),\ 1\le i<j\le m$. Hence, since $W_{\mathbf{S\Phi}}=\bigcap\limits_{1\le i<j\le m}W_{ij}$ by the definition (121) of $W_{\mathbf{S\Phi}}$, we deduce that $Z\cap\tau_h(W_{\mathbf{S\Phi}})\neq\emptyset$. This finishes the proof of Proposition 8.1 for m odd.

8.5. Proof of Proposition 8.1: case m even.

The proof of Proposition 8.1 for the case of even m,

$$m=2p+4,\quad p\ge 0.^5$$

is completely parallel to that given above for the case of odd m. Namely, similar to (135) fix the decompositions

$$H_{m-1}\simeq H_3\oplus\underbrace{H_2\oplus...\oplus H_2}_{p},\quad H_2\simeq H_1\oplus H_1,\quad H_3\simeq H_1\oplus H_1\oplus H_1. \tag{171}$$

[5] Note that we start with $m=4$ since the case $m=2$ has been already treated in subsection 8.2.

Under these decompositions, similar to (136) consider the points $D^{\Delta}_{m-1} \in (\mathbf{S}^{\vee}_{m-1})^0$ and $\phi^{\Delta}_{m-1} \in \mathbf{\Phi}_{m-1}$ given by the matrices with diagonal blocks

$$D^{\Delta}_{m-1} := D_3 \oplus \underbrace{D_2 \oplus ... \oplus D_2}_{p}, \quad \phi^{\Delta}_{m-1} = \phi^{\Delta}_{m-1}(N,a,d,f,g,\lambda) := \phi_3 \oplus \underbrace{\phi_2 \oplus ... \oplus \phi_2}_{p}, \tag{172}$$

$$D_3 = D_2 \oplus D', \qquad \phi_3 = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \lambda\phi_{21} \\ \phi_{31} & \lambda\phi_{12} & \phi_{11} \end{pmatrix} \in \mathbf{\Phi}_3, \qquad \lambda \in \mathbf{k}, \tag{173}$$

where $D_2,\ D'$ and $\phi_2,\ \phi_{i,j},\ i,j = 1,2$, are given by (137)-(138) and

$$\phi_{13} = (r_{ij}) \in \wedge^2 V^{\vee}, \quad \phi_{31} = (s_{ij}) \in \wedge^2 V^{\vee}, \tag{174}$$

where $r_{ij}, s_{ij} \in \mathbf{k}$ satisfy the additional relations

$$r_{i3} + r_{i4} = s_{i3} + s_{i4}, \quad i = 1,2. \tag{175}$$

We now proceed along the same lines as before. In particular, it follows from (138) and (172)-(175) that the relations (139) and (140) are satisfied for the point $(D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1})$. Hence, as before, the equations (130) are automatically satisfied for any $\psi \in \Psi_{m-1}$. Now, substituting the data $(\theta^0, \alpha^0, D^{\Delta}_{m-1}, \phi^{\Delta}_{m-1})$ from (128) and (172)-(174) into (131), we obtain the equations on (χ, ψ):

$$(\phi^{\Delta}_{m-1})^{\vee} \circ D^{\Delta}_{m-1} \circ \chi + \psi^{\vee} \circ \alpha^0 \circ \theta^0 \in \mathbf{\Psi}_{m-1}. \tag{176}$$

Next, using the decompositions (171) we represent χ and ψ as $(p+1)$-ples (cf. (143))

$$\chi = (\chi_0, ..., \chi_p), \quad \psi = (\psi_0, ..., \psi_p), \quad \psi_0, \chi_0 \in \mathbf{\Psi}_3, \quad \psi_k, \chi_k \in \mathbf{\Psi}_2,\ k = 1, ..., p, \tag{177}$$

where $\chi_k = (X_k, Y_k),\ \psi_k = (A_k, B_k),\ k = 1, ..., p$, are the same matrices of variables as in (144), and $\chi_0 = (X_0, Y_0, Z_0), \quad \psi_0 = (A_0, B_0, C_0), \quad X_0, Y_0, Z_0, A_0, B_0, C_0 \in \wedge^2 V^{\vee}$, i.e.

$$X_0 = (x^{(0)}_{ij}),\ Y_0 = (y^{(0)}_{ij}),\ Z_0 = (z^{(0)}_{ij}),\ A_0 = (a^{(0)}_{ij}),\ B_0 = (b^{(0)}_{ij}),\ C_0 = (c^{(0)}_{ij}). \tag{178}$$

are skew-symmetric 4×4-matrices of variables. Using the same notation for variables $x^{(k)}_1, ..., x^{(k)}_{24}, k = 1, ..., p$, as in (147) and introducing new variables $x^{(0)}_1, ..., x^{(0)}_{36}$ as follows: $x^{(0)}_1 = x^{(0)}_{12}, x^{(0)}_2 = x^{(0)}_{34}, x^{(0)}_3 = x^{(0)}_{13}, x^{(0)}_4 = x^{(0)}_{14}, x^{(0)}_5 = x^{(0)}_{23}, x^{(0)}_6 = x^{(0)}_{24}, x^{(0)}_7 = y^{(0)}_{12}, x^{(0)}_8 = y^{(0)}_{34}, x^{(0)}_9 = y^{(0)}_{13}, x^{(0)}_{10} = y^{(0)}_{14}, x^{(0)}_{11} = y^{(0)}_{23}, x^{(0)}_{12} = y^{(0)}_{24}, x^{(0)}_{13} = z^{(0)}_{12}, x^{(0)}_{14} = z^{(0)}_{34}, x^{(0)}_{15} = z^{(0)}_{13}, x^{(0)}_{16} = z^{(0)}_{14}, x^{(0)}_{17} = z^{(0)}_{23}, x^{(0)}_{18} = z^{(0)}_{24}, x^{(0)}_{19} = a^{(0)}_{12}, x^{(0)}_{20} = a^{(0)}_{34}, x^{(0)}_{21} = a^{(0)}_{13}, x^{(0)}_{22} = a^{(0)}_{14}, x^{(0)}_{23} = a^{(0)}_{23}, x^{(0)}_{24} = a^{(0)}_{24}, x^{(0)}_{25} = b^{(0)}_{12}, x^{(0)}_{26} = b^{(0)}_{34}, x^{(0)}_{27} = b^{(0)}_{13}, x^{(0)}_{28} = b^{(0)}_{14}, x^{(0)}_{29} = b^{(0)}_{23}, x^{(0)}_{30} = b^{(0)}_{24}, x^{(0)}_{31} = c^{(0)}_{12}, x^{(0)}_{32} = c^{(0)}_{34}, x^{(0)}_{33} = c^{(0)}_{13}, x^{(0)}_{34} = c^{(0)}_{14}, x^{(0)}_{35} = c^{(0)}_{23}, x^{(0)}_{36} = c^{(0)}_{24}$, we rewrite the system (176) similar to (147) as

$$\sum_{j=1}^{36} \tilde{m}_{ij} x^{(0)}_j = 0, \quad \sum_{j=1}^{24} m_{ij} x^{(k)}_j = 0, \quad i = 1, ..., 20, \quad k = 1, ..., p. \tag{179}$$

A direct computation of the matrices $\mathbf{M} = (m_{ij})$ and $\widetilde{\mathbf{M}} = (\tilde{m}_{ij})$ for the above chosen values (148),(149) of $N, a, d, f, g, p_{ij}, q_{ij}$ in (138) and (172) and, respectively, for the following values values of $\lambda,\ r_{ij},\ s_{ij}$ in (173) and (174) satisfying (175):

$$\lambda = -2,\ r_{12} = 3,\ r_{13} = 7,\ r_{14} = -2,\ r_{23} = 4,\ r_{24} = -6,\ r_{34} = -8, \tag{180}$$

$$s_{12} = -8, s_{13} = -3, s_{14} = 8, s_{23} = -2, s_{24} = 0, s_{34} = -5,$$

show that $\mathbf{M}$ is the block matrix (150) and $\widetilde{\mathbf{M}}$ is the block matrix

$$\widetilde{\mathbf{M}} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{\psi} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{0} & \mathbf{M}_{\psi} & \mathbf{0} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{\psi} \end{pmatrix} \tag{181}$$

with blocks

$$\mathbf{M}_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{M}_{12} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 & -2 \\ 0 & 0 & 5 & 0 & -2 & -5 \\ 0 & 0 & 0 & 5 & 2 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{182}$$

$$\mathbf{M}_{21} = \begin{pmatrix} 0 & 0 & 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & -5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 & 0 & 0 \\ -5 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{M}_{22} = \begin{pmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{183}$$

$$\mathbf{M}_{13} = \begin{pmatrix} 0 & 0 & -6 & -4 & -2 & -7 \\ 0 & 0 & 6 & -4 & -2 & 7 \\ -7 & -7 & -5 & 0 & 0 & 0 \\ 2 & 2 & 0 & -5 & 0 & 0 \\ -4 & -4 & 0 & 0 & -5 & 0 \\ 6 & 6 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & -12 & -8 \\ 0 & 0 & -8 & 0 & 14 & 0 \\ 0 & 0 & -4 & -14 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 & -4 \end{pmatrix}, \quad \mathbf{M}_{23} = \begin{pmatrix} 0 & 0 & 0 & 10 & -4 & 0 \\ 0 & 0 & 0 & 10 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 10 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \tag{184}$$

$$\mathbf{M}_{31} = \begin{pmatrix} 0 & 0 & 0 & 2 & 8 & 3 \\ 0 & 0 & 0 & 2 & 8 & -3 \\ 3 & 3 & -13 & 0 & 0 & 0 \\ -8 & -8 & 0 & -13 & 0 & 0 \\ 2 & 2 & 0 & 0 & -13 & 0 \\ 0 & 0 & 0 & 0 & 0 & -13 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 & -6 & 0 \\ 0 & 0 & 16 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}, \quad \mathbf{M}_{32} = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 4 & 0 \\ 0 & 0 & -4 & -4 & 0 & 4 \\ 0 & 0 & -10 & 0 & 4 & 10 \\ 0 & 0 & 0 & -10 & -4 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{185}$$

(186)

$$\mathbf{M}_{33}=\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&100&0&0&0\\ 0&0&0&100&0&0\\ 0&0&0&0&100&0\\ 0&0&0&0&0&100\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},\quad \mathbf{M}_{\psi}=\begin{pmatrix} -20&0&20&66&-5&-47\\ 0&3&40&-38&-37&-57\\ 57&-47&-82&-38&-22&0\\ 37&5&7&-79&0&-22\\ -38&66&-28&0&-62&-38\\ 40&-20&0&-28&7&-59\\ -56&0&0&0&40&132\\ 0&-76&-76&0&-114&0\\ 44&0&-10&-94&0&0\\ 0&-14&0&80&0&-74 \end{pmatrix}.$$

Now as in (151) we have $\mathrm{rk}\mathbf{M}=20$. Respectively, from (181)-(186) we obtain by an explicit computation that $\mathrm{rk}\widetilde{\mathbf{M}}=30$. Hence, since the matrix of the system (179) is a direct sum of matrix $\widetilde{\mathbf{M}}$ and p copies of matrix $\mathbf{M}$, it follows that its rank equals

$$\mathrm{rk}\widetilde{\mathbf{M}}+p\cdot\mathrm{rk}\mathbf{M}=30+20p=10(m-1). \tag{187}$$

Denote now by $R(\theta^0,\alpha^0,D^{\Delta}_{m-1},\phi^{\Delta}_{m-1})$ the rank of the linear system (176), equivalent to (179, as a function of $\theta^0,\alpha^0,D^{\Delta}_{m-1},\phi^{\Delta}_{m-1}$. It follows from (187) that, similar to (153), there exist values $\boldsymbol{\phi}_{m-1},\boldsymbol{\alpha},\boldsymbol{\theta}$ of $\phi^{\Delta}_{m-1},\alpha^0,\theta^0$, respectively, such that, as in (154),

$$R(\boldsymbol{\theta},\boldsymbol{\alpha},D^{\Delta}_{m-1},\boldsymbol{\phi}_{m-1})=10(m-1). \tag{188}$$

Repeating now the arguments from subsection 8.3 and using (188), we obtain the inclusions (161) and (162) for the above chosen data $\boldsymbol{\theta},\boldsymbol{\alpha},D^{\Delta}_{m-1},\phi^{\Delta}_{m-1}$.

Finally, using (172)-(174), we modify appropriately the matrices (163)-(165), so that, arguing as in subsection 8.4 and using the inclusions (161) and (162), we deduce that $Z\cap\tau_h(W_{\mathbf{S\Phi}})\neq\emptyset$. This finishes the proof of Proposition 8.1 for m even.

Remark 8.3. In perfoming the above computations of the rank of the linear system (131) one might try to simplify the shape of the matrices ϕ_2 in (138). E.g., in order to do computations simultaneously for odd and even values of m, one might set $\phi_{12}=\phi_{21}=0$. However, under these constraints the experiments with computations for arbitrary values of parameters N,p_{ij},q_{ij} give at best the value $9(m-1)$ for the rank of the system (131), which is insufficient for further arguments. Respectively, in case of m even one might also try to simplify the shape of the matrix ϕ_3 in (173). E.g., one might set $\phi_{13}=\phi_{31}=0$, and this would satisfy the equations (130). However, experiments with computations in this case for arbitrary values of the parameters $N,p_{ij},q_{ij},a,d,f,g,\lambda$ give at best the value 29 for the rank of the matrix $\widetilde{\mathbf{M}}$ which is also insufficient.

9. Geometric meaning of Z_m. Its relation to t'Hooft instantons

9.1. One property of the component Z of the scheme Z_m. In this subsection we prove one openness property of the component Z of $Z_m,\ m\geq 3$, introduced in Proposition 8.1 - see Lemma 9.2 below.

Take an arbitrary point

$$D\in(\mathbf{S}^{\vee}_m)^0.$$

Then in the notation of (67) we obtain a symplectic rank-$2m$ vector bundle

$$E_{2m}(D^{-1})$$

(see (45) and (49) where we take $2m$ instead of $2m+2$ and put $B = D^{-1}$) and a natural epimorphism

$$c_D : H_m^\vee \otimes \wedge^2 V^\vee \twoheadrightarrow W_{5m} := H_m^\vee \otimes \wedge^2 V^\vee / \mathrm{im}(^\sharp(D^{-1})) \simeq H^0(E_{2m}(D^{-1})(1)), \quad \dim W_{5m} = 5m.$$

Now take an arbitrary point

$$z = (D, \phi) \in Z_m.$$

Here the morphism ϕ understood as a homomorphism $^\sharp\phi : H_m \to H_m^\vee \otimes \wedge^2 V^\vee$ defines the diagram

$$\begin{tikzcd} & & H_m \arrow[d, "^\sharp\phi"'] \arrow[dr, "s(z)"] & & \\ 0 \arrow[r] & H_m \arrow[r, "^\sharp(D^{-1})"] & H_m^\vee \otimes \wedge^2 V^\vee \arrow[r, "c_D"] & W_{5m} \arrow[r] & 0. \end{tikzcd} \tag{189}$$

The lower horizontal triple in (189) yields the diagram

$$\begin{tikzcd} 0 \arrow[r] & H_m \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "^\sharp(D^{-1})"] \arrow[d, equal] & H_m^\vee \otimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "c_D"] \arrow[d, two heads, "ev"] & W_{5m} \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d, two heads, "ev"] & 0 \\ 0 \arrow[r] & H_m \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\widetilde{D^{-1}}"] & H_m^\vee \otimes \Omega_{\mathbb{P}^3}(2) \arrow[r, "c_D"] & E_{2m}(D^{-1})(1) \arrow[r] & 0. \end{tikzcd} \tag{190}$$

Moreover, the diagrams (189) and (190) define the composition

$$s_z : \ H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s(z)} W_{5m} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{ev}{\twoheadrightarrow} E_{2m}(D^{-1}). \tag{191}$$

Note that the relation $\phi^\vee \circ D \circ \phi \in \mathbf{S}_m$ following from the definition of Z can be easily rewritten as

$$^t s_z \circ s_z = 0, \tag{192}$$

where $^t s_z := s_z^\vee \circ \theta$ and $\theta : E_{2m}(D^{-1}) \xrightarrow{\sim} E_{2m}(D^{-1})^\vee$ is the symplectic structure on $E_{2m}(D^{-1})$ defined as in (49). We have an antiselfdual complex

$$0 \to H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s_z} E_{2m}(D^{-1}) \xrightarrow{^t s_z} H_m^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0. \tag{193}$$

Now, according to statement (iii) of Proposition 8.1, take a point

$$z = (D, \phi) \in Z \cap \tau_h(W_{\mathbf{S\Phi}}), \tag{194}$$

where h is a fixed decomposition (119), and consider the induced decompositions

$$D = D_1 \oplus ... \oplus D_m, \quad \phi := \phi_1 \oplus ... \oplus \phi_m, \quad (D_i, \phi_i) \in (\wedge^2 V^\vee)^0 \times (\wedge^2 V)^0, \tag{195}$$

such that

$$\mathbf{L} := \overset{m}{\underset{i=1}{\cup}} L(D_i, \phi_i) = \overset{m}{\underset{i=1}{\sqcup}} L(D_i, \phi_i). \tag{196}$$

is a disjoint union of $2m$ lines in $\mathbb{P}^3$. Moreover, for this point z we have

$$E_{2m}(D^{-1}) = \overset{m}{\underset{i=1}{\oplus}} E_2(D_i^{-1}), \tag{197}$$

where $E_2(D_i^{-1}),\ i = 1, ..., m$, are rank-2 null-correlation bundles.

Under the decomposition (119) the diagrams (189) and (190) decompose into the direct sums of m diagrams

$$\begin{tikzcd} & & \mathbf{k} \arrow[d, "^\sharp\phi_i"'] \arrow[dr, "s_i(z)"] & & \\ 0 \arrow[r] & \mathbf{k} \arrow[r, "^\sharp(D_i^{-1})"] & \wedge^2 V^\vee \arrow[r, "c_{D_i}"] & W_{5(i)} \arrow[r] & 0, \end{tikzcd} \tag{198}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\sharp(D_i^{-1})"] \arrow[d, equal] & \wedge^2V^\vee\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r, "c_{D_i}"] \arrow[d, "ev"] & W_{5(i)}\otimes\mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d, "ev"] & 0\ , \quad i=1,...,m, \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r, "\widetilde{D_i^{-1}}"] & \Omega_{\mathbb{P}^3}(2) \arrow[r, "c_{D_i}"] & E_2(D_i^{-1})(1) \arrow[r] & 0 \end{tikzcd} \tag{199}$$

in which we substitute $\mathbf{k}$ for H_1 and set $W_{5(i)} := \wedge^2V^\vee/\mathrm{im}(^\sharp(D_i^{-1}:\mathbf{k}\to\wedge^2V^\vee)),\ \dim W_{5(i)} = 5,\ i=1,...,m$.

Note that the decomposition (119) induces a decomposition of the complex (193) into a direct sum of m comlexes

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{s_i}E_2(D_i^{-1})\xrightarrow{^ts_i}\mathcal{O}_{\mathbb{P}^3}(1)\to 0,\quad i=1,...,m. \tag{200}$$

Here the sections $0\ne s_i\in H^0(E_2(D_i^{-1})(1))\simeq W_{5(i)}$ understood as homomorphisms $\mathbf{k}\to W_{5(i)}$ coincide by construction with homomorphisms $s_i(z)$ in the diagram (198). Hence the homomorphism $s(z)$ in the diagram (189) is also injective as the direct sum of $s_i(z)$'s. This means that $\mathrm{im}(^\sharp\phi)\cap\mathrm{im}(^\sharp(D^{-1}))=\{0\}$ i.e.

$$z\in((\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m)^* := \{(D,\phi)\in(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m \mid \text{the homomorphism } ^\sharp\phi: H_m\to H_m^\vee\otimes\wedge^2V^\vee \tag{201}$$
$$\text{is injective and } \mathrm{im}(^\sharp\phi)\cap\mathrm{im}(^\sharp(D^{-1}))=\{0\}\ \}.$$

Next, from the definition of $\mathbf{L}$ and the construction of the morphisms $s_z, s_i, i=1,...,m$, (see (189)–(199), (191) and (200)) it follows that these complexes are exact except in their righthand terms and

$$\mathrm{coker}(^ts_z)=\mathcal{O}_{\mathbf{L}}(1),\quad \mathrm{coker}(^ts_i)=\mathcal{O}_{L(D_i,\phi_i)}(1),\quad (s_i)_0=L(D_i,\phi_i),\quad i=1,...,m, \tag{202}$$

Remark 9.1. An arbitrary point $D\in(\mathbf{S}_m^\vee)^0$ defines a point

For an arbitrary embedding

$$j: H_{m-1}\hookrightarrow H_m$$

and an arbitrary point $z\in(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m$ there is defined an induced morphism of sheaves

$$s_z(j): H_{m-1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{j}H_m\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\xrightarrow{s_z}E_{2m}(D^{-1}). \tag{203}$$

Let $e_1,...,e_m$ be the basis of H_m related to the decomposition (119) and set

$$H_{m-1}:=\mathrm{Span}(e_1,...,e_{m-1}).$$

Consider the monomorphism

$$j_0: H_{m-1}\hookrightarrow H_m:\ e_i\mapsto e_i+e_{i+1},\quad i=1,...,m-1. \tag{204}$$

Since $\mathbf{L}$ is a disjoint union of pairs of lines $L(D_i,\phi_i),\ i=1,...,m$, it follows from (202) and (204) that $s_z(j_0)$ is a subbbundle morphism, i.e.

$$\mathrm{coker}(^ts_z(j_0))=0. \tag{205}$$

Now for a given monomorphism $j: H_{m-1}\hookrightarrow H_m$ consider the following conditions on a point $z=(D,\phi)\in Z$:

(**I**) the composition $s_z(j)=s_z\circ j: H_{m-1}\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to E_{2m}(D^{-1})$ is a subbbundle morphism;

(**II**) $s_z: H_m\otimes\mathcal{O}_{\mathbb{P}^3}(-1)\to E_{2m}(D^{-1})$ is an injective morphism of sheaves (but not a subbundle morphism).

Note that the conditions (**I**) and (**II**) are open conditions on the point $z\in Z_m$. The condition (**I**) is satisfied for the point z from (194) and the embedding j_0 by (205). The condition (**II**) is satisfied for this point z in view of (202). Thus, since the set $((\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m)^*$ defined in (201) is dense open in $(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m$, we obtain the following result.

Lemma 9.2. *(i) There exists a monomorphism $j : H_{m-1} \hookrightarrow H_m$ such that the sets*

$$Z(j) := \{z = (D, \phi) \in Z \cap ((\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m)^* \mid z \text{ satisfies the conditions (i) and } (\mathbf{II}) \text{ above}\},$$

$$Z(j, \mathbf{I}) := \{z = (D, \phi) \in Z \cap ((\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m)^* \mid z \text{ satisfies the condition } (\mathbf{I}) \text{ above}\}$$

are dense open subsets of Z, and we have open embeddings $Z(j) \hookrightarrow Z(j, \mathbf{I}) \hookrightarrow Z$. The same is true for a generic monomorphism $j : H_{m-1} \hookrightarrow H_m$.

(ii) Fix a monomorphism $j : H_{m-1} \hookrightarrow H_m$. Then the sets

$$Z_m(j) := \{z = (D, \phi) \in Z_m \cap ((\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m)^* \mid z \text{ satisfies the conditions } (\mathbf{I}) \text{ and } (\mathbf{II}) \text{ above}\}$$

$$Z_m(j, \mathbf{I}) := \{z = (D, \phi) \in Z_m \cap ((\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m)^* \mid z \text{ satisfies the conditions } (\mathbf{I}) \text{ and } (\mathbf{II}) \text{ above}\}$$

are open subsets of Z_m. Respectively, let $\widetilde{Z}$ be an arbitrary irreducible component of Z_m. Then the sets

$$\widetilde{Z}(j) := \widetilde{Z} \cap Z_m(j), \quad \widetilde{Z}(j, \mathbf{I}) := \widetilde{Z} \cap Z_m(j, \mathbf{I}) \tag{206}$$

are open subsets of $\widetilde{Z}$.

9.2. Relation between Z and t'Hooft instantons. Morphism $\lambda_{(j)} : Z_m \to \mathbf{S}_{2m-1}$.

In this subsection we relate the open subset $\widetilde{Z}(j)$ of Z_m introduced in Lemma 9.2(ii) to t'Hooft instantons - see Lemma 9.3.

In the notation of Lemma 9.2, assume that $\widetilde{Z}(j) \neq \emptyset$ and take an arbitrary point $z = (D, \phi) \in \widetilde{Z}(j)$, so that the symplectic vector bundle $E_{2m}(D^{-1})$ satisfies the diagrams (189)-(190). Respectively, the morphism of sheaves s_z defined in (191) is injective - see the definition of condition (ii) above. In addition, s_z satisfies the relation (192) which clearly implies the relation

$$^t s_z(j) \circ s_z(j) = 0 \tag{207}$$

for the subbundle morphism $s_z(j)$, i.e. we obtain a monad

$$0 \to H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{s_z(j)} E_{2m}(D^{-1}) \xrightarrow{^t s_z(j)} H_{m-1}^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \tag{208}$$

From the diagram (190) we deduce the equalities $h^i(E_{2m}(D^{-1})(-2)) = 0, \quad i \geq 0$, hence the cohomology sheaf of the monad (208) is an instanton bundle

$$E_2(z, j) := \mathrm{Ker}(^t s_z(j)) / \mathrm{Im}(s_z(j)), \quad [E_2(z, j)] \in I_{2m-1}. \tag{209}$$

Now consider the subvariety $I_{2m-1}^{tH} \subset I_{2m-1}$ of t'Hooft instanton bundles (see subsection 4.3),

$$I_{2m-1}^{tH} = \{[E] \in I_{2m-1} \mid h^0(E(1)) \neq 0\}.$$

Lemma 9.3. *(i) In notations of Lemma 9.2(i) let $Z(j) \neq \emptyset$ and let $z = (D, \phi)$ be an arbitrary point of $Z(j)$. Then the bundle $E_2(z, j)$ is a t'Hooft instanton bundle, i.e. $[E_2(z, j)] \in I_{2m-1}^{tH}$;*

(ii) In notations of Lemma 9.2(iii) let $\widetilde{Z}(j) \neq \emptyset$. Take an arbitrary point $z \in Z'(j)$. Then the monad (208) is well defined and its cohomology bundle $E_2(z, j)$ is a t'Hooft bundle;

(iii) Fix an isomorphism

$$\xi : H_m \oplus H_{m-1} \xrightarrow{\simeq} H_{2m-1}, \quad \xi \in \mathrm{Isom}_{2m-1}. \tag{210}$$

Then there is a well defined morphism

$$\lambda_{(j)} : Z_m \to \mathbf{S}_{2m-1} : \ z = (D, \phi) \mapsto A = \widetilde{\xi}(D^{-1}, \phi \circ j, -(\phi \circ j)^\vee \circ D \circ (\phi \circ j)). \tag{211}$$

such that

$$\lambda_{(j)}(Z_m(j)) \subset MI_{2m-1}^{tH}(\xi). \tag{212}$$

Proof. (i) Consider the complexes (193) and (208) and set

$$\mathcal{H}_{m-1} := H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad \mathcal{H}_m := H_m \otimes \mathcal{O}_{\mathbb{P}^3}(-1), \quad \mathcal{K}_{m+1} := \operatorname{coker} s_z(j), \quad \mathcal{K}_m := \operatorname{coker} s_z.$$

The complexes (193) and (208) are antiselfdual, hence they extend to a commutative diagram (213)

$$\begin{tikzcd}
 & & & & & E_2(z,j) \arrow[d, tail, "\tau"'] & \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[l, dotted, "u_z"'] \arrow[dl, "\alpha"] \\
 & & \mathcal{H}_{m-1} \arrow[r, tail, "s_z(j)"] \arrow[dl, "j"'] & E_{2m}(D^{-1}) \arrow[rr, two heads] \arrow[dl, equal] \arrow[ddd, "{}^t s_z(j)"'] & & \mathcal{K}_{m+1} \arrow[dl, "\beta"'] \arrow[ddd, "\delta"'] & \\
 & \mathcal{H}_m \arrow[r, tail, "s_z"] \arrow[dl] & E_{2m}(D^{-1}) \arrow[rr, two heads] \arrow[ddd, "{}^t s_z"'] & & \mathcal{K}_m \arrow[ddd, "\gamma"'] & & \\
\mathcal{O}_{\mathbb{P}^3}(-1) & & & & & & \\
 & & & \mathcal{H}^\vee_{m-1} \arrow[rr, equal] & & \mathcal{H}^\vee_{m-1} & \\
 & & \mathcal{H}^\vee_m \arrow[ur, "j^\vee"] \arrow[rr, equal] & & \mathcal{H}^\vee_m \arrow[ur, "j^\vee"] & & \\
 & \mathcal{O}_{\mathbb{P}^3}(1) \arrow[ur, tail] \arrow[rr, equal] & & \mathcal{O}_{\mathbb{P}^3}(1), \arrow[ur, tail] & & &
\end{tikzcd}$$

in which $\alpha, \beta, \gamma, \delta$ and τ are the induced morphisms. In this diagram we have $\beta \circ \alpha = 0$ and $j^\vee \circ \gamma \circ \beta = \delta$. Hence $\delta \circ \alpha = 0$. This implies that α factors through the morphism τ, i.e. there exists an injection $u_z : \mathcal{O}_{\mathbb{P}^3}(-1) \to E_2(z,j)$ such that $\alpha = \tau \circ u_z$. This injection u_z is a nonzero section $u_z \in H^0(E_2(z,j)(1))$. Hence $E_2(z,j)$ is a t'Hooft bundle.

(ii) Repeat the above argument.

(iii) This immediately follows from Lemma 5.1 since (208)-(209) coincides with (55)-(56) after sustituting $m-1$ for m and putting $B = D^{-1}$. □

Remark 9.4. From the diagram (9.3) it follows that the point $z \in Z(j)$ (respectively, the point $z \in \widetilde{Z}(j)$) defines not only a t'Hooft bundle $[E_2(z,j)]$, but also a proportionality class $< u_z >$ of a section $0 \neq u_z \in H^0(E_2(z,j))$. Moreover, the pointwise constructions (over $z \in \widetilde{Z}(j)$) of Lemma 9.3 clearly globalize to $\mathbb{P}^3 \times \widetilde{Z}(j)$. In particular, the morphism $\lambda_{(j)} : \widetilde{Z}(j) \to \mathbf{S}_{2m-1}$ defines a subbundle morphism of sheaves

$$\widetilde{\mathbf{A}}_Z : \mathcal{O}_{\widetilde{Z}(j)} \to \mathbf{S}_{2m-1} \otimes \mathcal{O}_{\widetilde{Z}(j)}, \tag{214}$$

i.e., equivalently, a family of instanton nets of quadrics

$$\mathbf{A}_Z : H_{2m-1} \otimes V \otimes \mathcal{O}_{\widetilde{Z}(j)} \to H^\vee_{2m-1} \otimes V^\vee \otimes \mathcal{O}_{\widetilde{Z}(j)}. \tag{215}$$

Let $\pi_Z : \mathbb{P}^3 \times \widetilde{Z}(j) \to \widetilde{Z}(j)$ be the projection. By construction we have a rank $4m$ bundle $\mathbf{W}_Z := \mathrm{im}\mathbf{A}_Z$ on $\widetilde{Z}(j)$ and the correspondig monad $0 \to H_{2m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{\widetilde{Z}(j)} \to \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_Z \to H^\vee_{2m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathcal{O}_{\widetilde{Z}(j)} \to 0$ with the cohomology rank 2 bundle $\mathbf{E}_Z$ such that $\mathbf{E}_Z|_{\mathbb{P}^3 \times \{z\}} = E_2(z,j), \ z \in \widetilde{Z}(j)$. This monad, together with relative Serre duality for the projection π_Z, defines in a standard way an isomorphism of locally free $\mathcal{O}_{\widetilde{Z}(j)}$-sheaves

$$f_Z : \ H_{2m-1} \otimes \mathcal{O}_{\widetilde{Z}(j)} \xrightarrow{\simeq} \mathbf{G}_Z := (\mathrm{Ext}^1_{\pi_Z}(\mathbf{E}_Z(-3), \omega_{\pi_Z}))^\vee \tag{216}$$

relativizing the pointwise isomorphisms $f : H_{2m-1} \xrightarrow{\simeq} H^2(E_2(z,j)(-3))$ (cf. Section 3) and Serre duality $H^2(E_2(z,j)(-3)) \xrightarrow{\simeq} (\mathrm{Ext}^1(E_2(z,j)(-3), \omega_{\mathbb{P}^3}))^\vee$. (Here we set $\mathbf{E}_Z(k) := \mathbf{E}_Z \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\widetilde{Z}(j)},\ k \in \mathbb{Z}$.) In addition, the sections $u_z \in H^0(E_2(z,j)),\ z \in \widetilde{Z}(j)$, glue up to a section

$$u : \ \mathcal{O}_{\mathbb{P}^3 \times \widetilde{Z}(j)} \to \mathbf{E}_Z(1). \tag{217}$$

9.3. Description of the fibers of the morphism $\lambda_{(j)} : Z_m(j) \to \mathbf{S}_{2m-1}$.

In this subsection we will give a description of the fibres of the morphism $\lambda_{(j)} : Z_m(j) \to \mathbf{S}_{2m-1}$ and of its restriction onto Z, $\lambda_j := \lambda_{(j)}|_Z : Z \to \mathbf{S}_{2m-1}$. The precise statement is given in Lemma 9.5 below.

To formulate the result on the fibres, note that the point $z = (D, \phi) \in Z_m(j)$ defines the monad (208) with the cohomology bundle $E_2(z,j)$ with $[E_2(z,j)] \in I^{tH}_{2m-1}$ (see Lemma 9.3). The display of this monad twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ is

$$\begin{tikzcd}
 & & E_2(z,j)(1) \arrow[d] \\
H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, tail, "s_z(j)"] & E_{2m}(D^{-1})(1) \arrow[r, two heads, "\epsilon"] \arrow[dr, "{}^t s_z(j)"'] & \mathrm{coker}(s_z(j)) \arrow[d] \\
 & & H^\vee_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(2).
\end{tikzcd} \tag{218}$$

Note that from (39) and the definition of MI^{tH}_{2m-1} it follows that $h^0(E_2(z,j)(1)) \le 2$. Hence, passing to sections in the diagram (218) we obtain a well defined epimorphism

$$b(z,j) := h^0({}^t s_z(j)) : H^0(E_{2m}(D^{-1})(1)) \xrightarrow{h^0(\epsilon)} H^0(\mathrm{coker}(s_z(j))) \xrightarrow{can} \tag{219}$$

$$\twoheadrightarrow H^0(\mathrm{coker}(s_z(j)))/H^0(E_2(z,j)(1)) \simeq \left\{ \begin{array}{ll} \mathbf{k}^{4m}, & \text{if } h^0(E_2(z,j)(1)) = 1 \\ \mathbf{k}^{4m-1}, & \text{if } h^0(E_2(z,j)(1)) = 2 \end{array} \right\} \hookrightarrow H^\vee_{m-1} \otimes S^2 V^\vee.$$

(Note that $h^0(E_2(1)) \le 2$ for any $[E_2] \in I^{tH}_{2m-1}$.) In addition, as in Remark 6.2, where we take $m-1$ instead of m, it follows that

$$\mathrm{im}({}^\sharp D^{-1}) \cap \mathrm{im}({}^\sharp \phi \circ j) = \{0\}, \quad \dim \mathrm{Span}(\mathrm{im}({}^\sharp D^{-1}),\ \mathrm{im}({}^\sharp \phi \circ j)) = 2m-1. \tag{220}$$

Consider the epimorphism $c_D : H^\vee_m \otimes \wedge^2 V^\vee \twoheadrightarrow H^0(E_{2m}(D^{-1})(1))$ in this triple (see the diagram (190)) and set

$$V(z,j) := c_D^{-1}(\ker b(z,j)). \tag{221}$$

From (219) it follows immediately that

$$V(z,j) \simeq \left\{ \begin{array}{ll} \mathbf{k}^{2m}, & \text{if } h^0(E_2(z,j)(1)) = 1, \\ \mathbf{k}^{2m+1}, & \text{if } h^0(E_2(z,j)(1)) = 2. \end{array} \right. \tag{222}$$

Now observe that the complex (208) is well defined for any $z \in Z_m$ and any $j : H_{m-1} \hookrightarrow H_m$ since the condition (207) is a closed condition satisfied for any $z \in Z_m$ (this complex now might be apriori not left- and right-exact). Hence the homomorphisms $b(z,j) = h^0({}^t s_z(j)) : H^0(E_{2m}(D^{-1})(1)) \to H^\vee_{m-1} \otimes S^2 V^\vee$ and $c_D : H^\vee_m \otimes \wedge^2 V^\vee \twoheadrightarrow H^0(E_{2m}(D^{-1})(1))$ are well defined, and we define the set $V(z,j)$ by the same formula (221). Since Z is irreducible, from (221) it follows by semicontinuity that

$$\dim V(z,j) \ge 2m, \quad z \in Z. \tag{223}$$

Lemma 9.5. *Let j be as in Lemma 9.2.*

(i) For any point $z \in Z_m$ the fibre of the morphism $\lambda_{(j)} : Z_m \to \mathbf{S}_{2m-1}$ through the point z is a reduced scheme naturally identified with $V(z,j)$:

$$\lambda_{(j)}^{-1}(\lambda_{(j)}(z)) \xrightarrow{\simeq} V(z,j), \tag{224}$$

where $V(z,j)$ is defined in (221). Hence, in particular, for any $z \in Z$, $\dim \lambda_{(j)}^{-1}(\lambda_{(j)}(z)) \geq 2m$.

(ii) Let Z_1 be the union of all possible irreducible components of Z_m distinct from Z and let $Z_0(j) := Z(j) \smallsetminus Z_1$. Consider the morphism $\lambda_j := \lambda_{(j)}|_Z : Z \to \mathbf{S}_{2m-1}$. Then for any $z \in Z_0(j)$ one has a natural isomorphism

$$\lambda_j^{-1}(\lambda_j(z)) \xrightarrow{\simeq} V(z,j), \tag{225}$$

where the dimension of $V(z,j)$ is given by (222), and, for an arbitrary $z \in Z$,

$$\lambda_j^{-1}(\lambda_j(z)) \subset \lambda_{(j)}^{-1}(\lambda_j(z)) = V(z,j), \quad \dim \lambda_j^{-1}(\lambda_j(z)) \geq 2m. \tag{226}$$

If $z \in Z(j,\mathbf{I})$, then the dimension of $V(z,j)$ in (226) is given by (222).

(iii) Let $\widetilde{Z}$ be an arbitrary irreducible component of Z_m, let Z_1 be the union of all possible irreducible components of Z_m distinct from $\widetilde{Z}$ and let $\widetilde{Z}_0(j) := \widetilde{Z}(j) \smallsetminus Z_1$. Consider the morphism $\tilde{\lambda}_j := \lambda_{(j)}|_{\widetilde{Z}} : \widetilde{Z} \to \mathbf{S}_{2m-1}$. Then for any $z \in \widetilde{Z}_0(j)$ one has the natural isomorphism

$$\tilde{\lambda}_j^{-1}(\tilde{\lambda}_j(z)) \xrightarrow{\simeq} V(z,j), \tag{227}$$

where the dimension of $V(z,j)$ is given by (222), and, for an arbitrary $z \in \widetilde{Z}$,

$$\tilde{\lambda}_j^{-1}(\tilde{\lambda}_j(z)) \subset \lambda_{(j)}^{-1}(\tilde{\lambda}_j(z)) = V(z,j), \quad \dim \tilde{\lambda}_j^{-1}(\tilde{\lambda}_j(z)) \geq 2m. \tag{228}$$

Proof. (i) Consider the spaces $\mathbf{\Lambda}_m = \wedge^2 H_m^\vee \otimes S^2V^\vee$ and $\mathbf{\Lambda}_{m-1} = \wedge^2 H_{m-1}^\vee \otimes S^2V^\vee$ together with projections $q_m : \wedge^2(H_m^\vee \otimes V^\vee) \to \mathbf{\Lambda}_m$ and $q_{m-1} : \wedge^2(H_{m-1}^\vee \otimes V^\vee) \to \mathbf{\Lambda}_{m-1}$, respectively (cf. (73) and (78)). Fix a monomorphism $j_{\mathbf{k}} : \mathbf{k} \hookrightarrow H_m$ such that $j(H_{m-1}) \cap \mathbf{k} = \{0\}$, i. e. we have a direct sum decomposition of H_m together with embeddings of summands

$$H_m = H_{m-1} \oplus \mathbf{k}, \quad H_{m-1} \overset{j}{\hookrightarrow} H_m \overset{j_{\mathbf{k}}}{\hookleftarrow} \mathbf{k}. \tag{229}$$

This decomposition induces a direct sum decomposition of $\mathbf{\Lambda}$ together with projections

$$\mathbf{\Lambda}_m = \mathbf{\Lambda}_{m-1} \oplus \mathrm{Hom}(\mathbf{k}, H_{m-1}^\vee \otimes S^2V^\vee), \quad \mathbf{\Lambda}_{m-1} \overset{pr'}{\leftarrow} \mathbf{\Lambda}_m \overset{pr''}{\to} \mathrm{Hom}(\mathbf{k}, H_{m-1}^\vee \otimes S^2V^\vee). \tag{230}$$

Now the equations of Z_m in $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m$ are

$$\mathcal{A} := q_m(\phi^\vee \circ D \circ \phi) = 0. \tag{231}$$

Next, consider the diagram (54) twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$, in which we substitute $m-1$ for m, set $B = D^{-1}$ and put $s_z(j)$ instead of $\rho_{\xi,A}$ and $\phi \circ j$ instead of $\widetilde{C}$, respectively. Proceeding to sections in this diagram and, respectively, to sections in the diagram (218) we see that the condition

$$0 = pr'(\mathcal{A}) := q_{m-1}((\phi \circ j)^\vee \circ D \circ (\phi \circ j)) = b(z,j) \circ e(z) \tag{232}$$

is automatically satisfied, where $e(z)$ is a homomorphism $e(z) = h^0(s_z(j)) : H_{m-1} \to H^0(E_{2m}(B)(1))$. (Clearly, the vanishing of $pr'(\mathcal{A})$ can be equivalently rewritten as the condition that ${}^\sharp\phi \circ j$ embeds H_{m-1} in $V(z,j)$.) Hence the equations (231) are equivalent to the equations

$$pr''(\mathcal{A}) = b(z,j) \circ c(z) \circ {}^\sharp\phi \circ j_{\mathbf{k}} = 0, \tag{233}$$

which in view of the definition (221) mean that

$${}^\sharp\phi|_{\mathbf{k}} \subset V(z,j)$$

Thus, since the point $\lambda_{(j)}(z)$ is given, so that the points D and $\phi \circ j$ are determined by $\lambda_{(j)}(z)$ (see (211)), it follows that the point $(D, \phi) \in \lambda_{(j)}(z)^{-1}(\lambda_{(j)}(z))$ is determined by the data $^\sharp\phi|_{\mathbf{k}}$. Hence, the above inclusion implies that $\lambda_{(j)}(z)^{-1}(\lambda_{(j)}(z)) \simeq V(z, j)$.

(ii)-(iii) follow from (i).

Note that the above argument can be illustrateded by the diagram

(234)
$$
\begin{tikzcd}
 & & H_m \arrow[rr, equal] \arrow[dd] & & H_m \arrow[dd, "^\sharp B"] & \\
H_{m-1} \arrow[r, hook, "j"] \arrow[drr, hook, "^\sharp\phi\circ j"'] \arrow[dd, equal] & H_m \arrow[dr, "^\sharp\phi"] & & \mathbf{k} \arrow[ll, hook', "j_{\mathbf{k}}"'] \arrow[dl, "^\sharp\phi|_{\mathbf{k}}"] & & \\
 & 0 \arrow[r] & V(z,j) \arrow[rr] \arrow[dd] & & H_m^\vee \otimes \wedge^2 V^\vee \arrow[r] \arrow[dd, "c_D"] & H_{m-1}^\vee \otimes S^2 V^\vee \arrow[dd, equal] \\
H_{m-1} \arrow[drr, hook] \arrow[drrrr, hook, "e(z)"] & & & & & \\
 & 0 \arrow[r] & \ker b(z,j) \arrow[rr] & & H^0(E_{2m}(D^{-1})(1)) \arrow[r, "b(z,j)"] & H_{m-1}^\vee \otimes S^2 V^\vee.
\end{tikzcd}
$$

□

Remark 9.6. Note here that, as it follows from the proof of this Lemma, for $z = (D, \phi) \in Z$ the fiber $V(z, j) = \lambda_{(j)}^{-1}(\lambda_{(j)}(z)) \subset H_m^\vee \otimes \wedge^2 V^\vee$ of the morphism $\lambda_{(j)}$ naturally lies in $\{D\} \times \mathbf{\Phi}_m$ via the embedding $j_{\mathbf{k}}^* : H_m^\vee \otimes \wedge^2 V^\vee \hookrightarrow \mathrm{Hom}(H_m, H_m^\vee) \otimes \wedge^2 V^\vee = \mathbf{\Phi}_m = \{D\} \times \mathbf{\Phi}_m$ induced by the embedding $j_{\mathbf{k}} : \mathbf{k} \hookrightarrow H_m$.

Lemma 9.7. *Consider the set* $R_Z = \{z = (D, \phi) \in Z \mid \mathrm{rank}\ s(z) \le m - 2\}$ *where the homomorphism* $s(z) = c_D \circ {}^\sharp\phi : H_m \to W_{5m}$ *is defined for* $z = (D, \phi)$ *in (189). Then*

$$\mathrm{codim}_Z R_Z \ge 2.$$

Proof. Fix a monomorphism $j : H_{m-1} \hookrightarrow H_m$ satisfying the conditions of Lemma 9.2, so that $Z(j)$ is nonempty, hence dense in Z. and take any point $z = (D, \phi) \in Z$. From the definition of the set $V(z, j)$ (see (221)) it follows that, for $z \in R_Z$, one has a natural inclusion $c_D^{-1}(\mathrm{im}\ s(z)) \subset \lambda_j^{-1}(\lambda_j(z)) \subset V(z, j)$ (cf. the diagram (234), so that the diagram (189) and the definition of R_Z imply $\dim c_D^{-1}(\mathrm{im}\ s(z)) \le \mathrm{rank}\ s(z) + m \le 2m - 2$. Hence by Lemma 9.5(ii) $\mathrm{codim}_{\lambda_j^{-1}(\lambda_j(z))} c_D^{-1}(\mathrm{im}\ s(z)) \ge 2$. Thus we have an inclusion $R_Z \simeq \bigcup_{z \in Z} c_D^{-1}(\mathrm{im}\ s(z)) \subset \bigcup_{z \in Z} \lambda_j^{-1}(\lambda_j(z)) = Z$, which together with the last inequality yields the Lemma. □

10. Complete family of t'Hooft sheaves with $c_2 = 2m - 1$. End of the proof of Theorem 7.2

In this section we construct a complete $(10m - 1)$-dimensional family T of t'Hooft $(2m - 1)$-instsantons and their degenerations (we call these degenerations *t'Hooft sheaves*). The family T will be used to prove that the variety Z studied in the previous two sections coincides with Z_m. This finishes the proof of Theorem 7.2.

10.1. Construction of a complete family $\mathbf{E} \to \mathbf{T}$ of $(2m - 1)$-t'Hooft sheaves.

Consider the subvariety $I_{2m-1}^{tH} \subset I_{2m-1}$ of t'Hooft $(2m-1)$-instantons. We first recall the following two properties of an arbitrary t'Hooft instanton $[E] \in I_{2m-1}^{tH},\ m \geq 1$, - see [BT] and [NT]:

(i) $h^0(E(1)) \leq 2$;

(ii) for any section $0 \neq s \in H^0(E(1))$ the zero scheme $Z_s = (s)_0$ is locally contained in a smooth surface;

(iii)$(Z_s)_{red}$ is a disjoint union of lines $l_1, ..., l_r,\ 1 \leq r \leq 2m$, and $\mathcal{O}_{Z_s} = \overset{r}{\underset{i=1}{\oplus}} \mathcal{O}_{Z_i}$, where for each $i,\ 1 \leq i \leq r$, the scheme Z_i has a filtration by subschemes $l_i = Z_{1i} \subset Z_{1i} \subset ... \subset Z_{m_i,i} = Z_i$ for some $m_i \geq 1$, with $\mathrm{Supp}(Z_{ji}) = l_i$ such that, if $m_i \geq 2$, then

$$\mathcal{O}_{Z_{j-1,i}} = \mathcal{O}_{Z_{ji}}/\mathcal{O}_{l_i},\ 2 \geq j \geq m_i; \tag{235}$$

For a given integer $d \geq 1$ consider the Hilbert scheme $\mathcal{H}_d := \mathrm{Hilb}^d G$ of 0-dimensional subschemes of length d of the Grassmannian $G = G(1,3)$ of lines in $\mathbb{P}^3$, and let $\Gamma_{\mathcal{H}_d} \subset G \times \mathcal{H}_d$ be the universal family with projections $G \overset{p_d}{\leftarrow} \Gamma_{\mathcal{H}_d} \overset{q_d}{\rightarrow} \mathcal{H}_d$. For a given point $x \in \mathcal{H}_d$ we denote by Y_x the corresponding 0-dimensional subscheme $p_d(q_d^{-1}(x))$ of G. We call a point $x \in \mathcal{H}_d$ *curvilinear* if there exists an integer $b \geq 1$, a partition $d = d_1 + ... + d_b,\ d_i \geq 1$, and points $x_i \in \mathcal{H}_{d_i},\ 1 \leq i \leq b$, such that

(a) for each $i,\ 1 \leq i \leq b$, the subscheme $Y_{x_i} \subset G$ is isomorphic to $\mathrm{Spec}(\mathbf{k}[t]/(t^{d_i+1}))$, and

(b) Y_x is a disjoint union $Y_x = Y_{x_1} \sqcup ... \sqcup Y_{x_b}$.

Set $\mathcal{H}_d^{curv} := \{x \in \mathcal{H}_d \mid x \text{ is curvilinear}\}$. It is well known (and easily seen) that $\mathcal{H}_d^{curv}$ is an open smooth $4d$-dimensional subscheme of $\mathcal{H}_d$. Next, let $\Gamma \subset \mathbb{P}^3 \times G$ be the graph of incidence, together with projections $\mathbb{P}^3 \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} G$. From the above properties (i)-(iii) we deduce now the following lemma.

Lemma 10.1. *For each $[E] \in I_{2m-1}^{tH}$ and $0 \neq s \in H^0(E(1))$, there exists a curvilinear point $x = x([E], s) \in \mathcal{H}_{2m}^{curv}$ such that $Z_s \overset{sets}{=} p(q^{-1}(Y_x))$ and the scheme structure of Z_s coincides with that given by formula*

$$\mathcal{O}_{Z_s} = p_* q^* \mathcal{O}_{Y_x}. \tag{236}$$

Proof. Since by (ii) the support of Z_s is a disjoint union of lines; hence from the definition of curviliear schemes we deduce that it is enough to consider the case when Z_s is a single line, say, l with a nonreduced structure, i.e. there is a filtration of Z_s by subschemes

$$l = Z_1 \subset Z_2 \subset ... \subset Z_{2m} = Z_s,\ m \geq 2, \tag{237}$$

such that the following triples are exact (see (235)):

$$0 \to \mathcal{O}_l \to \mathcal{O}_{Z_2} \to \mathcal{O}_l \to 0\ ,\ \ .\ .\ .\ ,\ \ 0 \to \mathcal{O}_l \to \mathcal{O}_{Z_{2m}} \to \mathcal{O}_{Z_{2m-1}} \to 0. \tag{238}$$

From the first triple in (238), (ii) and the Ferrand construction [BF, §1] it follows that $\mathcal{O}_l$ is a factor-sheaf of the conormal sheaf $N_{l/\mathbb{P}^3} \simeq 2\mathcal{O}_{\mathbb{P}^3}$ and that the surjection $N_{l/\mathbb{P}^3} \twoheadrightarrow \mathcal{O}_l$ gives a double structure on l coinciding with the scheme structure of Z_2. This surjection implies that Z_2 lies as a scheme on a smooth quadric, say, Q passing through l. Choose homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ on $\mathbb{P}^3$ such that

(1) $l = \{x_2 = x_3 = 0\},\ Q = \{x_0x_2 - x_1x_3 = 0\}$, and

(2) let $\mathbb{P}^3 = U_0 \cup U_1$ be the open cover of $\mathbb{P}^3$ by the sets $U_i = \{x_i \neq 0\},\ i = 0, 1$; then the ideal of $Z_2 \cap U_i$ in $\mathbf{k}[U_i]$ is generated by x_2/x_0 and $(x_3/x_0)^2$ for $i = 0$ and, respectively, by x_3/x_1 and $(x_2/x_1)^2$ for $i = 1$.

Let $S_1, .., S_c$ be quasiprojective smooth surfaces in $\mathbb{P}^3$ such that the sets $Z_{(k)} := Z_s \cap S_k,\ k = 1, ..., c$, constitute an open cover of Z_s. (Such surfaces exist because of (ii).) Set $Z_{(ik)} := Z_{(k)} \cap U_i,\ i = 0, 1,\ k = 1, ..., c$. From (1)-(iii) and (1)-(2) follows the property

(3) for $k = 1, ..., c$ the ideal $I_{Z_{(ik)}}$ of $Z_{(ik)}$ in $\mathcal{O}[U_i \cap S_k]$ is generated by $(x_3/x_0)^{2m+1}$ for $i = 0$ and, respectively, by $(x_2/x_1)^{2m+1}$ for $i = 1$.

Since by (1) the elements $x_3/x_0 \in \mathcal{O}[Z_{(0k)}]$ and $x_2/x_1 \in \mathcal{O}[Z_{(1k)}]$ coincide in $\mathcal{O}[Z_{(0k)} \cap Z_{(1k)}]$, $k = 1, ..., c$, it follows that there are well defined homomorphisms $\mathbf{k}[t]/(t^{2m+1}) \to \mathcal{O}[Z_{(ik)}]$: $1\mathrm{mod}(t^{2m+1}) \mapsto 1\mathrm{mod}I_{Z_{(ik)}}$ and $t\mathrm{mod}(t^{2m+1}) \mapsto (x_3/x_0)\mathrm{mod}I_{Z_{(0k)}}$ for $i = 0$, respectively, $t\mathrm{mod}(t^{2m+1}) \mapsto (x_2/x_1)\mathrm{mod}I_{Z_{(1k)}}$ for $i = 1$, which are compatible on $Z_{(0k)} \cap Z_{(1k)}$. This defines a morphism $\pi_Z : Z_s \to \mathrm{Spec}(\mathbf{k}[t]/(t^{2m+1}))$. Set $\tau_i := \mathrm{Spec}(\mathbf{k}[t]/(t^{i+1}))$, $i = 0, ..., 2m$. From the definition of the morphism π_Z and exact triples (238) it follows that, for $i = 2, .., 2m$, the (nilpotent) ideal sheaf $\mathcal{I}_i := \mathcal{I}_{\tau_{i-1},\tau_i} \subset \mathcal{O}_{\tau_i}$ satisfies the isomorphism $mult : \mathcal{I}_i \otimes_{\mathcal{O}_{\tau_i}} \mathcal{O}_{Z_i} \xrightarrow{\simeq} \mathcal{I}_{Z_i}$: $a \otimes \bar{1} \mapsto \pi_Z^*(a)$. Hence, by [HL, Lemma 2.13] the morphism π_Z is a flat family of lines over τ_{2m}, so that it defines an embedding $\tau_{2m} = \mathrm{Spec}(\mathbf{k}[t]/(t^{2m+1})) \hookrightarrow G$, i.e. a curvilinear point $x \in \mathcal{H}_{2m}$ such that $p :\ q^{-1}(Y_x) \xrightarrow{\simeq} Z_s$ is an isomorphism. Lemma is proved. □

Remark 10.2. One easily sees that $\mathcal{H}_{2m}^{tH-curv} := \{x \in \mathcal{H}_{2m}^{curv} \mid x = x([E], s)$ for some $[E] \in I_{2m-1}^{tH}$ and $0 \neq s \in H^0(E(1))\}$ is a dense open subset of $\mathcal{H}_{2m}^{curv}$. We thus consider its closure $\overline{\mathcal{H}_{2m}^{tH-curv}} = \overline{\mathcal{H}_{2m}^{curv}}$ in $\mathrm{Hilb}^{2m}G$. Fix a desingularization $\mathcal{H}$ of $\overline{\mathcal{H}_{2m}^{tH-curv}}$. $\mathcal{H}$ is a smooth integral scheme, and there is thea graph of incidence $\Gamma_{\mathcal{H}} \subset G \times \mathcal{H}$ with projections $G \xleftarrow{p_{\mathcal{H}}} \Gamma_{\mathcal{H}} \xrightarrow{q_{\mathcal{H}}} \mathcal{H}$.

Consider the subcheme $\widetilde{\mathbf{L}}_{\mathcal{H}} = \Gamma_{\mathcal{H}} \times_{G \times \mathcal{H}} \Gamma \times \mathcal{H}$ of $\Gamma \times \mathcal{H}$ and set

$$\mathbf{L}_{\mathcal{H}} := pr_1(\widetilde{\mathbf{L}}_{\mathcal{H}}),$$

where $pr_1 : \Gamma \times \mathcal{H} \to \mathbb{P}^3 \times \mathcal{H}$ is the projection. We endow $\mathbf{L}_{\mathcal{H}}$ with the structure of a subscheme of $\mathbb{P}^3 \times \mathcal{H}$ via setting

$$\mathcal{O}_{\mathbf{L}_{\mathcal{H}}} := pr_{1*}(\mathcal{O}_{\widetilde{\mathbf{L}}_{\mathcal{H}}}).$$

Since the sheaf $pr_{1*}(\mathcal{O}_{\widetilde{\mathbf{L}}_{\mathcal{H}}})$ is clearly flat over $\mathcal{H}$, in order to prove that the above definition is consistent, one has to check it fibrewise with respect to the projection $p_L : \mathbf{L}_{\mathcal{H}} \to \mathcal{H}$. Thus, taking any point $y \in \mathcal{H}$ and the corresponding 0-dimensional scheme $Z = Z_y$ of G, respectively, the subscheme $\widetilde{L}_y = q^{-1}(Z_y)$ of $P^3 \times G$, we have to check that the sheaf $p_*\mathcal{O}_{\Gamma_y}$ is the structure sheaf of a certain subscheme L_y of $\mathbb{P}^3$ supported at $p(\widetilde{L}_y)$. Take any closed point $z \in Z_y$ and set $\tilde{l} = q^{-1}(z)$, respectively, $l = p(\tilde{l})$. Also, take an arbitrary point $\tilde{x} \in \tilde{l}$, respectively, $x = p(\tilde{x}) \in l$. Applying the functor p_* to the composition of surjections $\mathcal{O}_\Gamma \twoheadrightarrow \mathcal{O}_{L_y} \twoheadrightarrow \mathcal{O}_{\tilde{l}} \twoheadrightarrow \mathbf{k}_{\tilde{x}}$ we obtain a surjection $\mathcal{O}_{\mathbb{P}^3} = p_*\mathcal{O}_\Gamma \twoheadrightarrow p_*\mathbf{k}_{\tilde{x}} = \mathbf{k}_x$ as the composition $\mathcal{O}_{\mathbb{P}^3} \xrightarrow{\epsilon} p_*\mathcal{O}_{\tilde{L}_y} \twoheadrightarrow \mathbf{k}_x$. Hence, by Nakayama's lemma ϵ is an epimorphism, as stated. Note that, by construction, the scheme L_y has a filtration by subschemes as in (237)-(238):

$$\emptyset = L_0 = L_1 \subset L_2 \subset ... \subset L_{2m} = L_y, \quad \mathcal{O}_{L_{i-1}} = \mathcal{O}_{L_i}/\mathcal{O}_{l_i}, \quad 1 \leq i \leq 2m, \tag{239}$$

where $l_1, ..., l_{2m}$ are lines in $\mathbb{P}^3$, not necessarily distinct, corresponding to closed points of the scheme Z_y.

Remark 10.3. Consider the set $\mathcal{H}_s := \{x \in \mathcal{H}_{2m}^{tH-curv} \mid x = x([E], s)$ for some $[E] \in I_{2m-1}^{tH}$ with $h^0(E(1)) \geq 2\}$. $\mathcal{H}_s$ is a closed subset of $\mathcal{H}_{2m}^{tH-curv}$ and it is well known (see, e.g., [BT]) that the condition $x([E], s) \in \mathcal{H}_s$ is equivalent to the condition that the scheme $Z_s = (s)_0$ lies on a smooth quadric in $\mathbb{P}^3$. This is, in turn, equivalent to saying that the 0-dimensional subscheme Y_x of G lies one a projective plane $\mathbb{P}^2$ in $\mathbb{P}^5 = \mathrm{Span}(G)$ intersecting G in a smooth conic (i.e. a general plane in $\mathbb{P}^5$. Whence it follows that $\dim \mathcal{H}_s = \mathrm{length}(Y_x) + \dim G(2, \mathbb{P}^5) = 2m + 9$. Respectively,

$$\mathrm{codim}_{\mathcal{H}} \mathcal{H}_s = 8m - (2m + 9) = 6m - 9 > 2, \qquad m \geq 2. \tag{240}$$

Now let $pr_2 : \mathbb{P}^3 \times \mathcal{H}$ be the projection and consider the flat over $\mathcal{H}$ sheaf $\mathcal{I}_{\mathbf{L}}(1) := \mathcal{I}_{\mathbf{L},\mathbb{P}^3 \times \mathcal{H}} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_{\mathcal{H}}$ and the relative Ext-sheaf

$$\mathbf{F} = \mathrm{Ext}^1_{pr_2}(\mathcal{I}_{\mathbf{L}}(1), \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathcal{H}}).$$

A standard computation using (239) shows that the sheaf $\mathbf{F}$ satisfies the base change isomorphism

$$b_y : \mathbf{F} \otimes \mathbf{k}_y \xrightarrow{\simeq} \mathrm{Ext}^1(\mathcal{I}_{L_y,P^3}(1), \mathcal{O}_{\mathbb{P}^3}(-1)) \simeq \mathbf{k}^{2m}, \quad y \in \mathcal{H}. \tag{241}$$

Hence $\mathbf{F}$ is a locally free $\mathcal{O}_{\mathcal{H}}$-sheaf of rank $2m$. We thus have a smooth integral $(10m-1)$-dimensional scheme $\mathbf{T} = \mathbf{Proj}(\mathbf{F}^\vee)$ with structure morphism $p_{\mathbf{T}} : \mathbf{T} \to \mathcal{H}$ and the Grothendieck sheaf $\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)$. In particular, $\mathbf{T}$ is a smooth variety of dimension

$$\dim \mathbf{T} = \dim \mathcal{H} + \mathrm{rk}\mathbf{F}^\vee - 1 = 8m + 2m - 1 = 10m - 1. \tag{242}$$

Moreover, let $\mathbf{p_T} = id_{\mathbb{P}^3} \times p_{\mathbf{T}} : \mathbb{P}^3 \times \mathbf{T} \to \mathbb{P}^3 \times \mathcal{H}$ be the projection and set $\mathbf{L_T} := \mathbf{p_T}^{-1}(\mathbf{L})$. On $\mathbb{P}^3 \times \mathbf{T}$ there is a universal family of (classes of) extensions of sheaves - see, e.g., [L, Cor. 4.5]:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(1) \to \mathbf{E} \to \mathcal{I}_{\mathbf{L_T}}(1) \to 0, \tag{243}$$

where $\mathcal{I}_{\mathbf{L_T}} := \mathcal{I}_{\mathbf{L_T},\mathbb{P}^3\times\mathbf{T}}$. By construction, for any closed point $t \in \mathbf{T}$ the sheaf $E_t = E|_{\mathbb{P}^3\times\{t\}}$ is a nontrivial extension of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to E_t \to \mathcal{I}_{L_y}(1) \to 0, \;\; y = p_T(t), \tag{244}$$

hence

(i) E_t is a stable rank-2 sheaf (i.e. $[E_t] \in M_{\mathbb{P}^3}(2;0,2,0)$), which satisfies the condition $h^0(E_y(1)) > 0$; furthermore, from (244) and (239) it follows easily that

(ii) $h^0(E_t(-2)) = 0$;

(iii) there exists a dense open subset $\mathbf{T}'$ of $p_{\mathbf{T}}^{-1}(\mathcal{H}_{2m}^{tH-curv})$, hence also of $\mathbf{T}$ such that, for $t \in \mathbf{T}'$, E_t is locally free, i.e. E_t is a t'Hooft bundle;

(iv) there exists a dense open subset $\mathbf{T}''$ of $\mathbf{T}'$ such that, for $t \in \mathbf{T}''$, $h^0(E_t(1)) = 1$; furthermore, for any two distinct points $t, t' \in \mathbf{T}''$ one has $E_t \not\simeq E_{t'}$.

The properties (i)-(iv) mean that there is a well defined modular morphism $\mathbf{f} : \mathbf{T} \to M_{\mathbb{P}^3}(2;0,2,0) : t \mapsto [E_t]$ such that

$$\mathbf{f}(\mathbf{T}) = \overline{I_{2m-1}^{tH}} \tag{245}$$

is the closure of I_{2m-1}^{tH} in $M_{\mathbb{P}^3}(2;0,2,0)$. Moreover, $f|_{\mathbf{T}^0}$ is injective. We thus call the family $\mathbf{E} \to \mathbf{T}$ *the complete* $(10m-1)$*-dimensional family of t'Hooft sheaves.*

Note also that the property (iii) above implies that

$$\mathrm{Supp}\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathbf{E}, \mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}) \subset \mathbb{P}^3 \times \partial\mathbf{T}, \quad \partial\mathbf{T} := \mathbf{T} \smallsetminus \mathbf{T}'. \tag{246}$$

Remark 10.4. Assume that we are given a vector bundle $\mathbf{E}_B$ on $\mathbb{P}^3 \times B$ such that, (i) for each $b \in B$, $E_b = \mathbf{E}_B|_{\mathbb{P}^3\times\{b\}}$ is a t'Hooft bundle, (ii) there is given a morphism $u_B : \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{N}_B \to \mathbf{E}_B$ nonvanishing for any $b \in B$, where $\mathcal{N}_B$ is some invertible sheaf on B. Then $\operatorname{coker} u_B = \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_B \otimes \mathcal{I}_{\mathbf{L}_B,\mathbb{P}^3\times B}$ where $\mathbf{L}_B = \underset{b\in B}{\cup} Z_b$ is a union of subschemes Z_b of $\mathbb{P}^3$ described in Lemma 10.1. We thus have an extension $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_B \xrightarrow{u_B} \mathbf{E}_B \to \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathcal{O}_B \otimes \mathcal{I}_{\mathbf{L}_B,\mathbb{P}^3\times B} \to 0$. It follows in a standard way from [L] that there exists a morphism $r : B \to \mathbf{T}'$ such that the last extension is obtained via applying the functor $(id_{\mathbb{P}^3} \times r)^*$ to the triple (243). In particular, applying this remark to the bundle $\mathbf{E}_Z$ on $\mathbb{P}^3 \times Z(j)$ and the morphism u in (217), i.e. taking $B = Z(j)$ and $u_B = u$, we obtain the morphism $r = r_{\mathbf{T}} : Z(j) \to \mathbf{T}'$ such that

$$(id_{\mathbb{P}^3} \times r_{\mathbf{T}})^*\mathbf{E} = \mathbf{E}_Z, \qquad r_{\mathbf{T}}^*\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1) = \mathcal{O}_{Z(j)}. \tag{247}$$

10.2. A family of nets of quadrics A associated to the family $\mathbf{E} \to \mathbf{T}$.

In this subsection we construct associated to $\mathbf{E} \to \mathbf{T}$ a family of nets of quadrics which will be used below. For this we first note that, by (239) and (244), we obtain the following equalities for a sheaf E_t in (244):

$\dim \mathrm{Ext}^1(E_t(-4),\omega_{\mathbb{P}^3}) = \dim \mathrm{Ext}^2(E_t,\omega_{\mathbb{P}^3}) = 4m-4$, $\dim \mathrm{Ext}^1(E_t(-3),\omega_{\mathbb{P}^3}) = \dim \mathrm{Ext}^2(E_t(-1),\omega_{\mathbb{P}^3}) = 2m-1$, $\mathrm{Ext}^i(E_t,\omega_{\mathbb{P}^3}) = \mathrm{Ext}^i(E_t(-1),\omega_{\mathbb{P}^3}) = \mathrm{Ext}^{3-i}(E_t(-3),\omega_{\mathbb{P}^3}) = \mathrm{Ext}^{3-i}(E_t(-4),\omega_{\mathbb{P}^3}) = 0,\ i\neq 2$, and $\mathrm{Ext}^i(E_t(-2),\omega_{\mathbb{P}^3}) = 0,\ i\geq 0$,
where $t\in\mathbf{T}$ is an arbitrary point and $\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4)$. Therefore, applying the functor $\mathrm{Ext}^i_\pi(-,\omega_\pi)$ to the sheaves $\mathbf{E}(-j) := \mathbf{E}\otimes\mathcal{O}_{\mathbb{P}^3}(-j)\boxtimes\mathcal{O}_{\mathbf{T}},\ 0\leq j\leq 4$, where $\pi:\mathbb{P}^3\times\mathbf{T}\to\mathbf{T}$ is the projection, the sheaf $\mathbf{E}$ is defined in (243) and $\omega_\pi = \omega_{\mathbb{P}^3}\boxtimes\mathcal{O}_{\mathbf{T}}$, and using base change for relative Ext-sheaves we obtain that the sheaves

$$\mathbb{F}_i := \mathrm{Ext}^2_\pi(\mathbf{E}(-i),\omega_\pi),\quad \mathbb{G}_i := \mathrm{Ext}^1_\pi(\mathbf{E}(i-4),\omega_\pi),\quad i=0,1, \tag{248}$$

are locally free $\mathcal{O}_{\mathbf{T}}$-sheaves of ranks, respectively,

$$\mathrm{rk}\mathbb{F}_0 = \mathrm{rk}\mathbb{G}_0 = 4m-4,\quad \mathrm{rk}\mathbb{F}_1 = \mathrm{rk}\mathbb{G}_1 = 2m-1, \tag{249}$$

and

$$\mathrm{Ext}^i_\pi(\mathbf{E},\omega_\pi) = \mathrm{Ext}^i_\pi(\mathbf{E}(-1),\omega_\pi) = \mathrm{Ext}^{3-i}_\pi(\mathbf{E}(-3),\omega_\pi) = \mathrm{Ext}^{3-i}_\pi(\mathbf{E}(-4),\omega_\pi) = 0,\ i\neq 2, \tag{250}$$
$$\mathrm{Ext}^i_\pi(\mathbf{E}(-2),\omega_\pi) = 0,\ i\geq 0,$$

Similarly, we obtain that $\mathbb{H} := R^1\pi_*(\mathbf{E}(-1))$ is a locally free $\mathcal{O}_{\mathbf{T}}$-sheaf of rank

$$\mathrm{rk}\mathbb{H} = 2m-1. \tag{251}$$

Using (244) we also see that the sheaf $\mathbb{H}$ duality commutes with the base change. Hence, there is a relative Serre-Grothendieck duality isomorphism (see, e.g., [K])

$$SD:\mathbb{F}_1 \xrightarrow{\simeq} \mathbb{H}^\vee. \tag{252}$$

Next, the local-to-relative spectral sequence $E_2^{p,q} = R^p\pi_*\mathcal{E}xt^q_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathbf{E}(-3),\omega_\pi) \Rightarrow \mathrm{Ext}^{p+q}_\pi(\mathbf{E}(-3),\omega_\pi)$ gives an exact sequence $0\to R^1\pi_*(\mathbf{E}^\vee(-1))\to\mathbb{G}_1\to \pi_*\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathbf{E}(-3),\omega_\pi)$, where by (246) $\mathrm{Supp}\pi_*\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathbf{E}(-3),\omega_\pi)\subset\partial\mathbf{T}$. Since $\mathrm{codim}_{\mathbf{T}}\partial\mathbf{T}\geq 1$, dualizing this sequence we obtain an injective morphism of $\mathcal{O}_{\mathbf{T}}$-sheaves

$$0\to\mathbb{G}_1^\vee \xrightarrow{\alpha} (R^1\pi_*(\mathbf{E}^\vee(-1)))^\vee \tag{253}$$

Next, dualizing the triple (243) and using the fact that $\mathrm{codim}_{\mathbb{P}^3\times\mathbf{T}}\mathbf{L}_{\mathbf{T}} = 2$ we obtain an exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)\boxtimes\mathcal{O}_{\mathbf{T}}\to\mathbf{E}^\vee\to\mathcal{O}_{\mathbb{P}^3}(1)\boxtimes\mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1)\to\mathcal{E}xt^2_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathcal{O}_{\mathbf{L}_{\mathbf{T}}}(1),\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}})\to \tag{254}$$
$$\to\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}}}(\mathbf{E},\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}})\to 0,$$

so that $\det\mathbf{E}^\vee = \pi^*\mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1)$. Hence, as $\mathbf{T}$ is a smooth integral scheme, it follows by [H1, Prop. 1.10] that

$$\mathbf{E}^{\vee\vee}\simeq\mathbf{E}^\vee\otimes(\det\mathbf{E}^\vee)^{-1} = \mathbf{E}^\vee\otimes\pi^*\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1).$$

Dualizing (243) twice we see that the canonical morphism $can:\mathbf{E}\to\mathbf{E}^{\vee\vee}\simeq\mathbf{E}^\vee\otimes\pi^*\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)$ is injective, and we obtain an exact sequence $0\to\mathbf{E}(-1)\xrightarrow{can}\mathbf{E}(-1)^\vee\otimes\pi^*\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)\to\mathrm{coker}(can)\to 0$, where $\mathrm{Supp}\,\mathrm{coker}(can)\subset\mathbb{P}^3\times\partial\mathbf{T}$. Applying to this triple the functor $R^i\pi_*$ and using the fact that $\mathbb{H}$ is locally free on $\mathbf{T}$, we thus obtain an exact sequence $0\to\mathbb{H}\xrightarrow{g} R^1\pi_*(\mathbf{E}^\vee(-1))\otimes\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)\to\mathrm{coker}(g)\to 0$, where $\mathrm{Supp}\,\mathrm{coker}(g)\subset\partial\mathbf{T}$. Dualizing this sequence we obtain an injective morphism of $\mathcal{O}_{\mathbf{T}}$-sheaves $\beta:(R^1\pi_*(\mathbf{E}^\vee(-1)))^\vee\to\mathbb{H}^\vee\otimes\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)$. Composing it with the morphism α from (253) and the inverse of the relative duality isomorphism SD from (252) we obtain an injective morphism of locally free $\mathcal{O}_{\mathbf{T}}$-sheaves

$$\gamma = SD^{-1}\circ\beta\circ\alpha:\mathbb{G}_1^\vee\to\mathbb{F}_1\otimes\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1). \tag{255}$$

In view of the property (iii) above (245) one easily sees that γ is an isomorphism when restricted onto $\mathbf{T}'$:

$$\gamma|_{\mathbf{T}'}:\mathbb{G}_1^\vee|_{\mathbf{T}'}\xrightarrow{\simeq}\mathbb{F}_1\otimes\mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)|_{\mathbf{T}'}.$$

(In fact, the restriction of γ onto an arbitrary point $t \in \mathbf{T}'$ is just the Serre duality isomorphism $H^2(E_t(-3)) \xrightarrow{\simeq} H^1(E_t(-1))^\vee$ for a t'Hooft instanton E_t.)

Next, the resolution of the diagonal Δ on $\mathbb{P}^3 \times \mathbb{P}^3$ extends to a diagram of sheaves

(256)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-4) \boxtimes \mathcal{O}_{\mathbb{P}^3} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-4) \boxtimes \mathcal{O}_{\mathbb{P}^3} \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-3) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-3) \boxtimes \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-3) \boxtimes T_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes T_{\mathbb{P}^3}(-2) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \wedge^2 V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-2) \boxtimes \Omega_{\mathbb{P}^3}(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \Omega_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes V^\vee \otimes \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^3}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{\Delta, \mathbb{P}^3 \times \mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_\Delta \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $\boldsymbol{\rho} : \mathbb{P}^3 \times \mathbf{T} \times \mathbb{P}^3 \to \mathbb{P}^3 \times \mathbb{P}^3$ and $\boldsymbol{\pi} = \pi \times id_{\mathbb{P}^3} : \mathbb{P}^3 \times \mathbf{T} \times \mathbb{P}^3 \to \mathbf{T} \times \mathbb{P}^3$ be the projections and denote $\omega_{\boldsymbol{\pi}} = \omega_\pi \boxtimes \mathcal{O}_{\mathbb{P}^3}$. Applying the functor $\mathrm{Ext}^i_{\boldsymbol{\pi}}(-, \omega_{\boldsymbol{\pi}})$ to the diagram $\boldsymbol{\rho}^*(256) \otimes \mathbf{E} \boxtimes \mathcal{O}_{\mathbb{P}^3}$ and using (248), (250) and base change we obtain the commutative diagram of sheaves on $\mathbb{P}^3 \times \mathbf{T} \simeq \mathbf{T} \times \mathbb{P}^3$:

(257)

$$\begin{tikzcd}
 & & & 0 & 0 & & \\
 & & 0 & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{G}_0 \arrow[l] \arrow[u] & \mathbb{M} \arrow[l] \arrow[u] & \mathbf{E}^\vee \arrow[l] & 0 \arrow[l] \\
 & 0 & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbb{G}_1 \arrow[l] \arrow[u] & \mathcal{O}_{\mathbb{P}^3} \boxtimes V^\vee \otimes \mathbb{G}_1 \arrow[l, "\mathbf{e}"'] \arrow[u] & \Omega_{\mathbb{P}^3}(1) \boxtimes \mathbb{G}_1 \arrow[l] \arrow[u] & 0 \arrow[l] & \\
 & 0 & T_{\mathbb{P}^3}(-1) \boxtimes \mathbb{F}_1 \arrow[l] \arrow[u] & \mathcal{O}_{\mathbb{P}^3} \boxtimes V \otimes \mathbb{F}_1 \arrow[l] \arrow[u, "id_{\mathcal{O}} \boxtimes \mathbb{A}'"'] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{F}_1 \arrow[l, "\mathbf{i}"'] \arrow[u, "\mathbf{a}"'] & 0 \arrow[l] & \\
0 & \mathbf{E}^\vee \arrow[l] & \mathbb{K} \arrow[l] \arrow[u] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{F}_0 \arrow[l] \arrow[u] & 0 \arrow[l] \arrow[u] & & \\
 & & 0 \arrow[u] & 0 \arrow[u] & & &
\end{tikzcd}$$

where we denote $\mathbb{K} = \mathrm{Ext}^2_{\boldsymbol{\pi}}(\boldsymbol{\rho}^* \mathcal{I}_{\Delta, \mathbb{P}^3 \times \mathbb{P}^3} \otimes \mathbf{E} \boxtimes \mathcal{O}_{\mathbb{P}^3}, \omega_{\boldsymbol{\pi}})$, $\mathbb{M} = \operatorname{coker} \mathbf{a}$ and where $\mathbb{A}'$ is a morphism $V \otimes \mathbb{F}_1 \to V^\vee \otimes \mathbb{G}_1$ given by this diagram.

Now set $\mathbb{W} := \mathrm{im} \mathbb{A}'$ and let $\epsilon_{\mathbb{A}'} : V \otimes \mathbb{F}_1 \twoheadrightarrow \mathbb{W}$, $i_{\mathbb{A}'} : \mathbb{W} \rightarrowtail V^\vee \otimes \mathbb{G}_1$, $\mathbf{g} : \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W} \xrightarrow{id_{\mathcal{O}} \boxtimes i_{\mathbb{A}'}} \mathcal{O}_{\mathbb{P}^3} \boxtimes V^\vee \otimes \mathbb{G}_1 \xrightarrow{\mathbf{e}} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbb{G}_1$ and $\mathbf{f} : \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{F}_1 \xrightarrow{\mathbf{i}} \mathcal{O}_{\mathbb{P}^3} \boxtimes V \otimes \mathbb{F}_1 \xrightarrow{id_{\mathcal{O}} \boxtimes \epsilon_{\mathbb{A}'}} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W}$ be the induced morphisms. From (249) and the middle vertical sequence in (257) it follows that

$\mathbb{W}$ is a locally free $\mathcal{O}_{\mathbf{T}}$-sheaf of rank $4m$:

$$\mathrm{rk}\mathbb{W} = 4m. \tag{258}$$

Moreover, the diagram (257) gives the monad with the cohomology sheaf $\mathbf{E}^\vee$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{F}_1 \xrightarrow{\mathbf{f}} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbb{G}_1 \to 0, \quad \mathbf{E}^\vee = \ker \mathbf{g}/\mathrm{im}\mathbf{f}. \tag{259}$$

Remark 10.5. One can, of course, obtain the monad (259) from the Beilinson spectral sequence with E_1-term $E_1^{p,q} = \mathrm{Ext}_\pi^{3-q}(\mathbf{E} \otimes \Omega_{\mathbb{P}^3}^{-p}(-p) \boxtimes \mathcal{O}_{\mathbf{T}}, \omega_\pi)$ (cf. [OSS, Ch. II, 3.1.4]). However, we use here the diagram (257) because it will be also used below in producing the monad (266) and Lemma 10.6.

Next, from the definition of the morphisms $\mathbf{f}, \mathbf{g}$ and γ follows the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{G}_1^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \arrow[r, "\mathbf{e}^\vee"] \arrow[d, "\gamma"'] & \mathcal{O}_{\mathbb{P}^3} \boxtimes V \otimes \mathbb{G}_1^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \arrow[d, "\gamma"'] \\ \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{F}_1 \arrow[r, "\mathbf{i}"] & \mathcal{O}_{\mathbb{P}^3} \boxtimes V \otimes \mathbb{F}_1 \end{tikzcd} \tag{260}$$

is commutative. Thus, the composition $\mathbb{A} : V \otimes \mathbb{G}_1^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \xrightarrow{\gamma} V \otimes \mathbb{F}_1 \xrightarrow{\mathbb{A}'} V^\vee \otimes \mathbb{G}_1$ fits in the (left- and right-exact) complex of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbb{G}_1^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \xrightarrow{\mathbf{e}^\vee} \mathcal{O}_{\mathbb{P}^3} \boxtimes V \otimes \mathbb{G}_1^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \xrightarrow{id_{\mathcal{O}} \boxtimes \mathbb{A}} \tag{261}$$
$$\to \mathcal{O}_{\mathbb{P}^3} \boxtimes V^\vee \otimes \mathbb{G}_1 \xrightarrow{\mathbf{e}} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbb{G}_1 \to 0$$

and $\mathrm{im}\mathbb{A} \subset \mathbb{W}$. In addition, by construction for any $t \in \mathbf{T}'$ the homomorphism $\mathbb{A} \otimes \mathbf{k}_t$ in view of Serre duality $H := H^2(E_t(-3)) \xrightarrow{\simeq} H^1(E_t(-1))$ coincides with the skew-symmetric middle vertical homomorphism $A : V \otimes H \to V^\vee \otimes H^\vee$ in (10) for $E = E_t$ and $n = 2m-1$. Hence, $\mathbb{A}$ is skew-symmetric, $\mathbb{A} \in H^0(\wedge^2(V^\vee \otimes \mathbb{G}_1) \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(1))$. We thus obtain the induced skew-symmetric morphism $\mathbf{q} : \mathbb{W}^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1)) \to \mathbb{W}$ which yields a decomposition of $\mathbb{A}$ as $\mathbb{A} = i_{\mathbb{A}'} \circ \mathbf{q} \circ i_{\mathbb{A}'}^\vee$. This decomposition, being restricted onto an arbitrary point $t \in \mathbf{T}'$, gives the rightmost square in (10). In particular, it follows that

$$\mathbb{A} \in H^0(\wedge^2 V^\vee \otimes S^2\mathbb{G}_1) \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(1)), \tag{262}$$

and that $\mathbf{q}|_{\mathbf{T}'}$ is an isomorphism. We thus consider the dense open subset $\mathbf{T}_0$ of $\mathbf{T}$ containing $\mathbf{T}'$ which is defined as

$$\mathbf{T}_0 := \{t \in \mathbf{T} \mid \mathbf{q}|_{\mathbb{P}^3 \times \{t\}} : \mathbb{W}^\vee \otimes \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1) \otimes \mathbf{k}_t \to \mathbb{W} \otimes \mathbf{k}_t \text{ is an isomorphism}\}, \tag{263}$$
$$\mathbf{T}_0 \supset \mathbf{T}'.$$

Denote

$$\mathbf{W} := \mathbb{W}^\vee, \quad \mathbf{W}_0 := \mathbf{W}|_{\mathbf{T}_0}, \quad \mathbf{q}_0 := \mathbf{q}|_{\mathbf{T}_0}, \quad \boldsymbol{\mathcal{L}} := \mathcal{O}_{\mathbf{T}/\mathcal{H}}(-1), \quad \boldsymbol{\mathcal{L}}_0 := \boldsymbol{\mathcal{L}}|_{\mathbf{T}_0}, \quad \mathbf{E}_0 = \mathbf{E}|_{\mathbf{T}_0}, \tag{264}$$
$$\mathbf{g}_0 := \mathbf{g}^\vee|_{\mathbf{T}_0}, \quad \mathbf{G} := \mathbb{G}_1^\vee, \quad \mathbf{G}_0 = \mathbf{G}|_{\mathbf{T}_0}.$$

In this notation the complex (261) induces the following right- and left-exact complex

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G} \otimes \boldsymbol{\mathcal{L}} \xrightarrow{\mathbf{g}} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W} \otimes \boldsymbol{\mathcal{L}} \xrightarrow{\mathbf{q}} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}^\vee \xrightarrow{\mathbf{g}^\vee} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbf{G}^\vee \to 0, \tag{265}$$

Standard diagram chasing with (257)-(261) shows that the restriction of the monad (259) onto $\mathbb{P}^3 \times \mathbf{T}_0$ coincides with the restriction onto $\mathbb{P}^3 \times \mathbf{T}_0$ of the complex (265) and is isomorphic to a (antiselfdual) monad

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G}_0 \otimes \boldsymbol{\mathcal{L}}_0 \xrightarrow{\mathbf{g}_0} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_0 \otimes \boldsymbol{\mathcal{L}}_0 \underset{\simeq}{\xrightarrow{\mathbf{q}_0}} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_0^\vee \xrightarrow{\mathbf{g}_0^\vee} \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbf{G}_0^\vee \to 0, \tag{266}$$
$$\mathbf{E}_0^\vee = \ker \mathbf{g}_0^\vee/\mathrm{im}(\mathbf{q}_0 \circ \mathbf{g}_0).$$

From this monad and (263) immediately follows

Lemma 10.6. $\mathbf{E}_0$ *is a locally free* $\mathcal{O}_{\mathbb{P}^3\times\mathbf{T}_0}$*-sheaf, i.e.* $\mathbf{T}_0=\mathbf{T}'$.

Consider the variety $\mathbf{Y}:=\mathbf{Proj}(\mathcal{H}om(\mathbf{G},H_{2m-1}\otimes\mathcal{O}_{\mathbf{T}}))$ with the projection $p_{\mathbf{Y}}:\mathbf{Y}\to\mathbf{T}$ and set $\mathbf{G}_{\mathbf{Y}}:=p_{\mathbf{Y}}^*\mathbf{G},\quad \boldsymbol{\mathcal{L}}_{\mathbf{Y}}:=p_{\mathbf{Y}}^*\boldsymbol{\mathcal{L}}\otimes\mathcal{O}_{\mathbf{Y}/\mathbf{T}}(-1)$. The universal morphism

$$\tau:\ H_{2m-1}\otimes\mathcal{O}_{\mathbf{Y}}\otimes\mathcal{O}_{\mathbf{Y}/\mathbf{T}}(-1)\to\mathbf{G}_{\mathbf{Y}} \tag{267}$$

on $\mathbf{Y}$ together with the family $p_{\mathbf{Y}}^*\mathbb{A}:\mathbf{G}_{\mathbf{Y}}\otimes V\otimes\boldsymbol{\mathcal{L}}_{\mathbf{Y}}\to\mathbf{G}_{\mathbf{Y}}^\vee\otimes V^\vee$ yields a family of nets of quadrics $\mathbf{A}:H_{2m-1}\otimes V\otimes\boldsymbol{\mathcal{L}}_{\mathbf{Y}}\to H_{2m-1}^\vee\otimes V^\vee\otimes\mathcal{O}_{\mathbf{Y}}$, i.e., equivalently, the morphism

$$\mathbf{A}:\boldsymbol{\mathcal{L}}_{\mathbf{Y}}\to S^2H_{2m-1}^\vee\otimes\wedge^2V^\vee\otimes\mathcal{O}_{\mathbf{Y}}=\mathbf{S}_{2m-1}\otimes\mathcal{O}_{\mathbf{Y}}. \tag{268}$$

We call $\mathbf{A}$ *the family of nets of quadrics associated to the family* $\mathbf{E}\to\mathbf{T}$.

Now consider the principal $PGL(H_{2m-1})$-bundle $p_{\mathbf{Y}_0}:\mathbf{Y}_0:=P(\mathcal{I}som(H_{2m-1}\otimes\mathcal{O}_{\mathbf{T}_0},\mathbf{G}_0))\to\mathbf{T}_0$ together with the natural open embedding $\mathbf{Y}_0\overset{i_0}{\hookrightarrow}\mathbf{Y}$ such that $p_{\mathbf{Y}_0}=p_{\mathbf{Y}}\circ i_0$ and set $\mathbf{A}_0:=\mathbf{A}|_{\mathbf{Y}_0},\ \boldsymbol{\mathcal{L}}_{\mathbf{Y}_0}:=\boldsymbol{\mathcal{L}}_{\mathbf{Y}}|_{\mathbf{Y}_0},\ \mathbf{W}_{\mathbf{Y}_0}:=p_{\mathbf{Y}_0}^*\mathbf{W}_0$. The monad $p_{\mathbf{Y}_0}^*(266)$:

$$0\to\mathcal{O}_{\mathbb{P}^3}(-1)\boxtimes H_{2m-1}\otimes\boldsymbol{\mathcal{L}}_{\mathbf{Y}_0}\to\mathcal{O}_{\mathbb{P}^3}\boxtimes\mathbf{W}_{\mathbf{Y}_0}\otimes\boldsymbol{\mathcal{L}}_{\mathbf{Y}_0}\overset{\simeq}{\to}\mathcal{O}_{\mathbb{P}^3}\boxtimes\mathbf{W}_{\mathbf{Y}_0}^\vee\to\mathcal{O}_{\mathbb{P}^3}(1)\boxtimes H_{2m-1}^\vee\otimes\mathcal{O}_{\mathbf{Y}_0}\to 0, \tag{269}$$

Now pick a monomorphism $j:H_{m-1}\hookrightarrow H_m$ and let $\widetilde{Z}$ be any irreducible component of Z_m. Assume that $\widetilde{Z}(j)$ is nonempty, hence dense in $\widetilde{Z}$ according to Lemma 9.2 (in particular, such j exists for $\widetilde{Z}=Z$ by the same Lemma). Consider the morphism $r_{\mathbf{T}}:\widetilde{Z}(j)\to\mathbf{T}'$ defined in (247). Note that from the definition (248) of the locally free $\mathcal{O}_{\mathbf{T}}$-sheaf $\mathbb{G}_1=\mathrm{Ext}^1_\pi(\mathbf{E}(-3),\omega_\pi)$ it follows that the formation of $\mathbb{G}_1^\vee$ commutes with the base change. In particular, the definition (247) of the morphism r_Z and the definition (216) of the sheaf $\mathbf{G}_Z$ imply that $\mathbf{G}_Z=r_{\mathbf{T}}^*\mathbb{G}_1^\vee$. Hence the isomorphism (216) gives a subbundle morphism

$$i_Z:\mathcal{O}_{\widetilde{Z}(j)}\to\mathcal{H}om(H_{2m-1}\otimes\mathcal{O}_{\widetilde{Z}^*(j)},\mathbf{G}_Z)=r_{\mathbf{T}}^*\mathcal{H}om(H_{2m-1}\otimes\mathcal{O}_{\mathbf{T}},\mathbb{G}_1^\vee), \tag{270}$$
$$\mathrm{im}i_Z\subset\mathcal{I}som(H_{2m-1}\otimes\mathcal{O}_{\mathbf{T}_0},\mathbf{G}_0).$$

Now the well known universal property of $\mathbf{Proj}$ (see [H, Ch. III, Prop. 7.12]) and the last inclusion in (270) show that the morphism $r_{\mathbf{T}}:\widetilde{Z}(j)\to\mathbf{T}'=\mathbf{T}_0$ (here we use Lemma 10.6) lifts to the morphism $r_{\mathbf{Y}}:\widetilde{Z}(j)\to\mathbf{Y}_0$ giving the factorization of $r_{\mathbf{T}}$:

$$r_{\mathbf{T}}:\widetilde{Z}(j)\overset{r_{\mathbf{Y}}}{\to}\mathbf{Y}_0\overset{p_{\mathbf{Y}_0}}{\to}\mathbf{T}_0 \tag{271}$$

such that

$$\widetilde{\mathbf{A}}_Z=r_{\mathbf{Y}}^*\mathbf{A}, \tag{272}$$

where $\widetilde{\mathbf{A}}_Z:\mathcal{O}_{\widetilde{Z}(j)}\to\mathbf{S}_{2m-1}\otimes\mathcal{O}_{\widetilde{Z}(j)}$ is the family of nets of quadrics (214) and $\mathbf{A}$ is the net (268). Moreover, consider the total space $\mathbf{V}=\mathrm{Spec}(S^\cdot_{\mathcal{O}_{\mathbf{Y}}}\boldsymbol{\mathcal{L}}_{\mathbf{Y}}^{-1})$ of the vector bundle $\boldsymbol{\mathcal{L}}_{\mathbf{Y}}$ and let $\mathbf{V}_0=V\smallsetminus\{0\text{-section}\}$ be the complement of the 0-section in $\mathbf{V}$, with the projection $\rho:\mathbf{V}_0\to\mathbf{Y}$. The morphism $r_{\mathbf{Y}}:\widetilde{Z}(j)\to\mathbf{Y}_0$ naturally lifts to a morphism $r_{\mathbf{V}}:\widetilde{Z}(j)\to\mathbf{V}_0$, i.e. $r_{\mathbf{Y}}$ factorizes as $r_{\mathbf{Y}}=\rho\circ r_{\mathbf{V}}$:

$$\begin{tikzcd} \widetilde{Z}(j) \arrow[r, "r_{\mathbf{V}}"] \arrow[d, "r_{\mathbf{Y}}"'] & \mathbf{V}_0 \arrow[d, "\rho"] \\ \mathbf{Y}_0 \arrow[r, hook] & \mathbf{Y}. \end{tikzcd} \tag{273}$$

so that, by (272),

$$\widetilde{\mathbf{A}}_Z=r_{\mathbf{V}}^*\rho^*\mathbf{A}. \tag{274}$$

Next, there is a well defined morphism $\mu:\mathbf{V}_0\to\mathbf{S}_{2m-1}:v\mapsto(\rho^*\mathbf{A})(\mathbf{s}(v))$ where $\mathbf{s}$ is the canonical section of $\rho^*\boldsymbol{\mathcal{L}}_{\mathbf{Y}}\simeq\mathcal{O}_{\mathbf{V}_0}$. Now (274) means that $\tilde{\lambda}_j=\mu\circ r_{\mathbf{V}}$:

$$\tilde{\lambda}_j:\widetilde{Z}(j)\overset{r_{\mathbf{Y}}}{\to}\mathbf{V}_0\overset{\mu}{\to}\mathbf{S}_{2m-1} \tag{275}$$

where the morphism $\tilde{\lambda}_j : \widetilde{Z} \to \mathbf{S}_{2m-1}$ is defined in Lemma 9.5(ii).

Remark 10.7. By definition, the morphism $r_{\mathbf{V}}$ considered in the diagram above is well defined as the morphism $r_{\mathbf{V}} : Z_m(j) \to \mathbf{V}$.

10.3. Irreducibility of Z_m.

Take an arbitrary point $z_0 = (D_0, \phi_0) \in Z$ with $\phi_0 \neq 0$. According to Lemma 9.2(i) there exists a monomorphism $j : H_{m-1} \hookrightarrow H_m$ such that $Z(j)$ is a dense open subset of Z. Hence there exists a smooth affine curve C with a marked point $0 \in C$ and a morphism $g : C \to Z$ such that $g(0) = z_0$ and $g(C^*) \subset Z(j)$ where $C^* := C \smallsetminus \{0\}$. For any $x \in C$ set $(D_x, \phi_x) := g(x)$. Here, for all $x \in C$, by definition $A_1(x) := D_x^{-1}$ is an isomorphism $H_m \otimes V \xrightarrow{\simeq} H_m^\vee \otimes V^\vee$ and also $A_2(x) := \phi_x \circ j$ is a homomorphism $H_{m-1} \otimes V \xrightarrow{\simeq} H_m^\vee \otimes V^\vee$. Hence, picking an isomorphism $\xi : H_m \oplus H_{m-1} \xrightarrow{\simeq} H_{2m-1}$, we may consider the matrix $A(x) = \begin{pmatrix} A_1(x) & A_2(x) \\ -A_2(x)^\vee & A_3(x) \end{pmatrix}$ with $A_3(x) = -A_2(x)^\vee \circ A_1(x)^{-1} \circ A_2(x)$ as a homomorhism (net of quadrics) $A(x) : H_{2m-1} \otimes V \to H_{2m-1}^\vee \otimes V^\vee$ of rank

$$\mathrm{rk} A(x) = \mathrm{rk} A_1(x) = 4m, \quad x \in C. \tag{276}$$

We thus have a family of nets of quadrics $\mathbf{A}_C = \{A(x)\}_{x \in C}$ and its restriction $\mathbf{A}_{C^*} = \{A(x)\}_{x \in C} | C^*$.[6]

Consider the composition $r_{\mathbf{Y}} \circ g : C^* \to \mathbf{Y}_0 \hookrightarrow \mathbf{Y}$. Since Y is projective, this morphism extends to the morphism $\psi_{\mathbf{Y}} : C \to \mathbf{Y}$ such that $\mathbf{A}_C = \psi_{\mathbf{Y}}^* \mathbf{A}$. As $A(0) \neq 0$, it follows that $\psi_{\mathbf{Y}}$ lifts to the morphism $\psi_{\mathbf{V}} : C \to \mathbf{V}_0$ such that $\psi_{\mathbf{Y}} = \rho \circ \psi_{\mathbf{V}}$. We also have the composition $\psi_{\mathbf{T}} = p_{\mathbf{Y}} \circ \psi_{\mathbf{Y}} : C \to \mathbf{T}$ and the commutative diagram

$$\begin{tikzcd} H_{2m-1} \otimes V \otimes \mathcal{O}_C \arrow[r, "\mathbf{A}_C"] \arrow[d, "\tau_C"'] & H_{2m-1}^\vee \otimes V^\vee \otimes \mathcal{O}_C \\ \mathbf{G}_C \otimes V \arrow[r, "\psi_{\mathbf{Y}}^* \mathbb{A}"] & \mathbf{G}_C^\vee \otimes V^\vee \arrow[u, "\tau_C^\vee"'] \end{tikzcd} \tag{277}$$

where $\mathbf{G}_C := \psi_{\mathbf{Y}}^* \mathbf{G}_{\mathbf{Y}}$, $\tau_C := \psi_{\mathbf{Y}}^* \tau$ and τ is the universal morphism (267). Consider the $\mathcal{O}_C$-sheaves $\mathbf{W}_C = H_{2m-1} \otimes V \otimes \mathcal{O}_C / \ker \mathbf{A}_C$ and $\mathbb{W}_C = \mathbf{G}_C \otimes V / \ker \mathbf{A}_C$ and the morphisms $\mathbf{e}_C : H_{2m-1} \otimes V \otimes \mathcal{O}_C \twoheadrightarrow \mathbf{W}_C$, $e_C : \mathbf{G}_C \otimes V \twoheadrightarrow \mathbb{W}_C$, $\mathbf{q}_C : \mathbf{W}_C \to \mathbf{W}_C^\vee$, $q_C : \mathbb{W}_C \to \mathbb{W}_C^\vee$ and $\epsilon : \mathbf{W}_C \to \mathbb{W}_C$ induced by the diagram (277), so that

$$\mathbf{q}_C = \epsilon^\vee \circ q_C \circ \epsilon, \tag{278}$$

$$\epsilon \circ \mathbf{e}_C = e_C \circ \tau_C. \tag{279}$$

The condition (276) means that $\mathbf{W}_C$ is a locally free rank-$4m$ $\mathcal{O}_C$-sheaf and $\mathbf{q}_C$ is an isomorphism. Hence (278) implies that $\mathbb{W}_C$ is a locally free rank-$4m$ $\mathcal{O}_C$-sheaf and q_C is an isomorphism. This together with Lemma 10.6 precisely means that

$$\psi_{\mathbf{Y}}(C) \subset \mathbf{Y}_0, \quad \text{resp.,} \quad \psi_{\mathbf{T}}(C) \subset \mathbf{T}_0. \tag{280}$$

Consider the compositions $\mathbf{a}_C : \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes H_{2m-1} \otimes \mathcal{O}_C \to V \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes H_{2m-1} \otimes \mathcal{O}_C \xrightarrow{\mathbf{e}_C} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_C$ and $a_C : \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G}_C \to V \otimes \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{G}_C \xrightarrow{e_C} \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W}_C$ and the diagram of induced complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes H_{2m-1} \otimes \mathcal{O}_C \arrow[r, "\mathbf{a}_C"] \arrow[d, "\tau_C"'] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_C \arrow[r, "{}^t\mathbf{a}_C"] \arrow[d, "\epsilon", "\simeq"'] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes H_{2m-1}^\vee \otimes \mathcal{O}_C \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G}_C \arrow[r, "a_C"] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W}_C \arrow[r, "{}^t a_C"] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbf{G}_C^\vee \arrow[r] \arrow[u, "\tau_C^\vee"'] & 0. \end{tikzcd} \tag{281}$$

[6]Equivalently, using Lemma 9.3(iii), one can define $A(x)$ as $\lambda_j(g(x))$, $x \in C$.

From (280) it follows now that the lower complex in this diagram is a genuine monad which is by construction obtained by applying the functor $(id_{\mathbb{P}^3} \times \psi_{\mathbf{T}})^*$ to the monad (266). In particular, its cohomology sheaf $\mathbb{E}_C$ is a rank-2 bundle. Also, by construction, these two complexes are isomorphic over C^*. However, the upper complex is apriori not right- and left-exact when restricted to $\mathbb{P}^3 \times \{0\}$. We are going to show that, in fact, it is isomorphic to the lower monad, hence it is left- and right-exact, i.e. it is a monad.

For this, consider the monomorphism $\mathbf{i}_m : H_m \rightarrowtail H_{2m-1}$ given by the isomorphism ξ above, let $\boldsymbol{\alpha} : H_m \otimes V \otimes \mathcal{O}_C \rightarrowtail H_{2m-1} \otimes V \otimes \mathcal{O}_C \twoheadrightarrow \mathbf{W}_C$, $\beta : H_m \otimes \mathcal{O}_C \rightarrowtail H_{2m-1} \otimes \mathcal{O}_C \overset{\tau_C}{\twoheadrightarrow} \mathbf{G}_C$ be the induced morphisms and set $\mathbf{G}_m := \mathrm{im}\beta$, $\mathbf{G}_{m-1} := \mathbf{G}_C/\mathbf{G}_m$. From (276) it follows that α is an isomorphism and, respectively, the induced morphism $\alpha : \mathbf{G}_m \otimes V \to \mathbb{W}_C$ is an isomorphism. Hence by (277)-(279) β is injective, $\mathbf{G}_m$ is a locally free rank-m $\mathcal{O}_C$-sheaf, the morphism $\mathbf{G}_m \rightarrowtail \mathbf{G}_C$ is a subbundle morphism, hence $\mathbf{G}_{m-1}$ is a locally free rank-$(m-1)$ $\mathcal{O}_C$-sheaf. We now have the induced diagram of isomorphic monads obtained similar to (281):

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes H_m \otimes \mathcal{O}_C \arrow[r, "\boldsymbol{\alpha}_C"] \arrow[d, "\simeq"', "\beta_C"] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbf{W}_C \arrow[r, "{}^t\boldsymbol{\alpha}_C"] \arrow[d, "\simeq"', "\epsilon"] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes H^\vee_{2m-1} \otimes \mathcal{O}_C \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G}_m \arrow[r, "\alpha_C"] & \mathcal{O}_{\mathbb{P}^3} \boxtimes \mathbb{W}_C \arrow[r, "{}^t\alpha_C"] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbf{G}^\vee_m \arrow[r] \arrow[u, "\simeq", "\beta^\vee_C"'] & 0.
\end{tikzcd} \tag{282}$$

with the isomorphism $\delta : \mathbf{E}_{2m} \xrightarrow{\sim} \mathbb{E}_{2m}$ of the rank-$2m$ cohomology sheaves of these monads. (Note that, by construction, $\mathbf{E}_{2m} = \underset{x\in C}{\cup} E_{2m}(D_x^{-1})$.) In addition, the diagram of natural morphisms

$$\begin{tikzcd}
0 \arrow[r] & H_m \otimes V \otimes \mathcal{O}_C \arrow[r, "\mathbf{i}_m"] \arrow[d, "\simeq"', "\beta"] & H_{2m-1} \otimes V \otimes \mathcal{O}_C \arrow[r] \arrow[d, "\tau_C"] & H_{m-1} \otimes V \otimes \mathcal{O}_C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & \mathbf{G}_m \otimes V \arrow[r, "i_m"] & \mathbf{G}_{2m-1} \otimes V \arrow[r] & \mathbf{G}_{m-1} \otimes V \arrow[r] & 0.
\end{tikzcd}$$

satisfying the relations $\boldsymbol{\alpha} = \mathbf{e}_C \circ \mathbf{i}_m$, $\alpha = e_C \circ i_m$, $\alpha \circ \beta = \epsilon \circ \boldsymbol{\alpha}$, together with the diagrams (281)-(282), yields a diagram of factor-complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes H_{m-1} \otimes \mathcal{O}_C \arrow[r, "\bar{\boldsymbol{\alpha}}_C"] \arrow[d, "\gamma_C"'] & \mathbf{E}_{2m} \arrow[r] \arrow[d, "\simeq"', "\delta"] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes H^\vee_{m-1} \otimes \mathcal{O}_C \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathbf{G}_{m-1} \arrow[r] & \mathbb{E}_{2m} \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(1) \boxtimes \mathbf{G}^\vee_{m-1} \arrow[r] \arrow[u, "\gamma^\vee_C"'] & 0
\end{tikzcd} \tag{283}$$

where $\bar{\boldsymbol{\alpha}}_C$ is the induced morphism. By the above, this diagram becomes an isomorphism of monads when restricted onto $\mathbb{P}^3 \times C^*$. To show that it is an isomorphism everywhere, it is enough to show that $\gamma_C \otimes \mathbf{k}(0) : H_{m-1} \to \mathbf{G}_{m-1} \otimes \mathbf{k}(0)$ is an isomorphism. Passing to sections in the left square of the diagram (283)$\otimes \mathcal{O}_{\mathbb{P}^3}(-1) \boxtimes \mathcal{O}_C$, we see that this condition is equivalent to the injectivity of homomorphism of sections $h^0(\bar{\boldsymbol{\alpha}}_C \otimes \mathbf{k}(0)) : H_{m-1} \to H^0(E_{2m}(D_0^{-1})(1))$. But this homomorphism exactly coincides with the composition

$$s_{z_0}(j) : H_{m-1} \overset{j}{\hookrightarrow} H_m \xrightarrow{s(z_0)=\sharp\phi_0} H^0(E_{2m}(D_0^{-1})(1)).$$

Now from the definition of the subset R_Z of Z defined in Lemma 9.7 it follows that the injectivity of the map $s_{z_0}(j)$ is true for any point $z_0 \in Z \smallsetminus R_Z$ and a generic monomorphism $j : H_{m-1} \hookrightarrow H_m$. Hence, for such point $z_0 = (D_0, \phi_0)$ the restriction of the upper complex in (283) onto $\mathbb{P}^3 \times \{0\}$ is a monad: $0 \to H_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \overset{s_{z_0}(j)}{\to} E_{2m}(D_0^{-1}) \overset{{}^ts_{z_0}(j)}{\to} H^\vee_{m-1} \otimes \mathcal{O}_{\mathbb{P}^3}(1) \to 0$, which by definition coincides with the monad (208) for $z = z_0$. (As a corollary we obtain that the diagrams (281) and (283) are the diagrams of isomorphisms of monads for this z_0.) In other words, $z_0 \in \widehat{Z}(j)$ where the set $\widehat{Z}(j)$ was defined in Lemma 9.2(i).

We thus have proved the following statement.

Proposition 10.8. *For any point $z \in Z \setminus R_Z$ there exists a monomorphism $j : H_{m-1} \hookrightarrow H_m$ such that $z \in Z(j, \mathbf{I})$.*

Consider the morphism $r_{\mathbf{V}} : \widetilde{Z}(j) \to \mathbf{V}_0$ defined in diagram (273). By (275) we have

$$\tilde{\lambda}_j|_{\widetilde{Z}(j)} = \mu \circ r_{\mathbf{V}}. \tag{284}$$

We now prove the following proposition.

Proposition 10.9. *Take any irreducible component $\widetilde{Z}$ of Z_m and any monomorphism $j : H_{m-1} \hookrightarrow H_m$ such that $\widetilde{Z}(j)$ is nonempty. Then the morphism $r_{\mathbf{V}} : \widetilde{Z}(j, \mathbf{I}) \to \mathbf{V}_0$* [7] *is dominating and, for a general point $z \in \widetilde{Z}(j, \mathbf{I})$, the fibre $r_{\mathbf{V}}^{-1}(r_{\mathbf{V}}(z))$ coincides with $V(z, j)$ where $V(z, j)$ is defined in (221). Moreover, $\dim \widetilde{Z}(j, \mathbf{I}) = 4m(m+2)$, and there exists a dense open subset Z' of $\widetilde{Z}(j, \mathbf{I})$ such that*

$$\dim V(z, j) = 2m, \quad z \in Z', \tag{285}$$

$$r_{\mathbf{V}}^{-1}(r_{\mathbf{V}}(z)) = \tilde{\lambda}_j^{-1}(\tilde{\lambda}_j(z)) = \lambda_{(j)}^{-1}(\lambda_j(z)) = V(z, j), \quad z \in Z', \tag{286}$$

$$\operatorname{codim}_{\widetilde{Z}(j,\mathbf{I})}(\widetilde{Z}(j, \mathbf{I}) \setminus Z') \geq 2. \tag{287}$$

Proof. First, since by definition $\widetilde{Z}(j, \mathbf{I})$ is an open subset of Z_m, we have by (81) $\dim \widetilde{Z} = \dim \widetilde{Z}(j, \mathbf{I}) \geq 4m(m+2)$.

Next, set $\mathbf{V}_{00} := \rho^{-1}(\mathbf{Y}_0)$. According to the diagram (273) we have $r_{\mathbf{V}}(\widetilde{Z}(j, \mathbf{I})) \subset \mathbf{V}_{00}$. Consider the composition of projections

$$\mathbf{p} : \mathbf{V}_{00} \xrightarrow{\rho} \mathbf{Y}_0 \xrightarrow{p_{\mathbf{Y}_0}} \mathbf{T}_0 \xrightarrow{p_{\mathbf{T}}} \mathcal{H}^0 := \mathcal{H}_{2m}^{tH-curv},$$

$$\mathbf{p}_j : \widetilde{Z}(j, \mathbf{I}) \xrightarrow{r_{\mathbf{V}}} \mathbf{V}_{00} \xrightarrow{\mathbf{p}} \mathcal{H}^0.$$

Since the projections ρ, $p_{\mathbf{Y}_0}$ and $p_{\mathbf{T}}$ are smooth fibrations with fibers of dimensions, respectively, 1, $(2m-1)^2 - 1$ and $(2m-1)$, and $\dim \mathcal{H}^0 = \dim \mathcal{H} = 8m$ (cf. (242)), it follows that

$$\dim \mathbf{V}_{00} = \dim(\text{fibre of } \mathbf{p}) + \dim \mathcal{H}^0 = 2m(2m-1) + 8m = 4m^2 + 6m. \tag{288}$$

Whence,

$$\begin{aligned} \dim\{\text{generic fibre of } r_{\mathbf{V}} : \widetilde{Z}(j, \mathbf{I}) \to \mathbf{V}_{00}\} &\geq \dim \widetilde{Z}(j, \mathbf{I}) - \dim \mathbf{V}_{00} \geq \\ &\geq 4m(m+2) - (4m^2 + 6m) = 2m. \end{aligned} \tag{289}$$

Now take an arbitrary point $z \in \widetilde{Z}(j, \mathbf{I})$ and set $v := r_{\mathbf{V}}(z)$, $A := \tilde{\lambda}_j(z)$. From (284) it follows that $A = \mu(v)$ and so by Lemma 9.5(ii)

$$r_{\mathbf{V}}^{-1}(v) \subset \tilde{\lambda}_j^{-1}(A) = V(z, j), \tag{290}$$

where $V(z, j)$ is described in (221). Using Remark 10.3, we rewrite (221) as:

$$\dim V(z, j) = \begin{cases} 2m, & \text{if } \mathbf{p}(z) \in \mathcal{H}^*, \\ 2m+1, & \text{if } \mathbf{p}(z) \in \mathcal{H}_s, \end{cases} \tag{291}$$

where we set

$$\mathcal{H}^* := \mathcal{H}^0 \setminus \mathcal{H}_s.$$

As $\mathbf{p} : \mathbf{V}_{00} \to \mathcal{H}^0$ is a smooth fibration with fibres of dimension $2m(2m-1)$ (see (288)), formulas (240), (288), (290) and (291) yield

$$\dim \mathbf{p}_j^{-1}(\mathcal{H}_s) \leq 4m^2 + 6m - 3 + (2m+1) = 4m(m+2) - 2 < \dim \widetilde{Z}(j, \mathbf{I}). \tag{292}$$

[7] See the definition of the sets $\widetilde{Z}(j, \mathbf{I})$ in (206).

Thus, $\mathbf{p}_j(\widetilde{Z}(j,\mathbf{I}))\not\subset\mathcal{H}_s$, i.e. there is a dense open subset Z' of $\widetilde{Z}(j,\mathbf{I})$ such that $\mathbf{p}_j(Z')\subset\mathcal{H}^*$. In particular, (290) and (291) imply

$$\dim r_{\mathbf{V}}^{-1}(r_{\mathbf{V}}(z))\le\dim V(z,j)=2m,\quad z\in Z'. \tag{293}$$

On the other hand, since Z' is dense in $\widetilde{Z}(j,\mathbf{I})$, (289) yields $\dim r_{\mathbf{V}}^{-1}(r_{\mathbf{V}}(z))\ge 2m,\ z\in Z'$. Comparing this with (293), (288) and the inequality $\dim Z'\ge 4m(m+2)$, we obtain that

$$\dim r_{\mathbf{V}}(Z')=\dim\mathbf{V}_{00}=4m^2+6m, \tag{294}$$

$$r_{\mathbf{V}}^{-1}(r_{\mathbf{V}}(z))=V(z,j),\quad \dim V(z,j)=2m,\quad z\in Z'. \tag{295}$$

Moreover, (287) follows from (292). Now since the minimal possible dimension of $V(z,j)$ is $2m$, the equality (286) follows from Lemma 9.5(ii-iii) (see (226) and (228)) by the semicontinuity of dimension of fibres of a morphism of irreducible varieties. This together with (294) and (295) yields Proposition. □

Now we are ready to finish the proof of Theorem 7.2.

End of the proof of Theorem 7.2.

(i) We prove the irreducibility of Z_m, and the surjectivity of the projection $p_m: Z\to(\mathbf{S}_m^\vee)^0:(D,\phi)\mapsto D$ will be a by-product of this proof. First, Z_m contains an irreducible component Z introduced in Proposition 8.1. Assume that there exists another irreducible component Z' of Z_m. Let $b:\mathbf{\Phi}_m\smallsetminus\{0\}\to P(\mathbf{\Phi}_m)$ be the canonical projection and $\mathbf{b}:=id\times b:(\mathbf{S}_m^\vee)^0\times(\mathbf{\Phi}_m\smallsetminus\{0\})\to(\mathbf{S}_m^\vee)^0\times P(\mathbf{\Phi}_m)$ be the induced projection. The equations of Z_m in $(\mathbf{S}_m^\vee)^0\times\mathbf{\Phi}_m$ (see (76)-(77)) are homogeneous with respect to affine coordinates in $\mathbf{\Phi}_m$, hence there exist irreducible closed subsets $\underline{Z}$ and $\underline{Z}'$ and the closed subset $\underline{Z}_m$ in $(\mathbf{S}_m^\vee)^0\times P(\mathbf{\Phi}_m)$ such that $Z=\mathbf{b}^{-1}(\underline{Z})\cup\{0\}$, respectively, $Z'=\mathbf{b}^{-1}(\underline{Z}')\cup\{0\}$, respectively, $Z_m=\mathbf{b}^{-1}(\underline{Z}_m)\cup\{0\}$. Moreover, by construction $\underline{Z}$ and $\underline{Z}'$ are irreducible components of $\underline{Z}_m$.

Take any point

$$y=(D_0,<\phi>)\in\underline{Z}'\smallsetminus\underline{Z}'\cap\underline{Z} \tag{296}$$

and consider the projective space $\mathbb{P}=\{D_0\}\times P(\mathbf{\Phi}_m)$, $\dim\mathbb{P}=6m^2-1$. By definition, the sets $\underline{Z}_m(D_0)=\underline{Z}'\cap P_D$ and $\underline{Z}'(D_0)=\underline{Z}'\cap P_D$ are closed subsets of $\mathbb{P}$ such that

$$y\in\underline{Z}'(D_0)\subset\underline{Z}_m(D_0) \tag{297}$$

and by Remark7.1 we have $\mathrm{codim}_{\mathbb{P}}\underline{Z}'(D_0)\le 5m(m-1)$

$$\dim_{\mathbb{P}}\underline{Z}_m(D_0)\ge m^2+5m-1\ge 1,\quad m\ge 1. \tag{298}$$

By definition, $\underline{Z}_m(D_0)$ is given in $\mathbb{P}$ by $5m(m-1)$ global equations of the form $\phi^\vee\circ D_0\circ\phi\in\mathbf{S}_m$. Hence, in view of (298) $\underline{Z}_m(D_0)$ is connected.

Next, by Proposition 8.1(ii) the morphism $pr_1: Z\to(\mathbf{S}_m^\vee)^0:(D,\phi)\mapsto D$ is dominant, so that the induced projective morphism $\underline{Z}\to(\mathbf{S}_m^\vee)^0:(D,<\phi>)\mapsto D$ is also dominant, hence surjective,[8] since $\underline{Z}$ is closed in $(\mathbf{S}_m^\vee)^0\times P(\mathbf{\Phi}_m)$. In particular, the set $\underline{Z}(D_0)=\underline{Z}\cap\mathbb{P}$ is a nonempty closed subset of $\underline{Z}_m(D_0)$. In addition, by (296) $y\in\underline{Z}_m(D_0)\smallsetminus\underline{Z}(D_0)$. Hence, since $\underline{Z}_m(D_0)$ is connected, it contains an irreducible component, say, $\underline{Z}''(D_0)$ distinct from $\underline{Z}(D_0)$ and intersecting $\underline{Z}(D_0)$. Let $\underline{Z}''$ be an irreducible component of $\underline{Z}_m$ containing $\underline{Z}''(D_0)$, hence distinct from $\underline{Z}(D_0)$. We thus have

$$\underline{Z}\cap\underline{Z}''\ne\emptyset. \tag{299}$$

Let $Z''=\mathbf{b}^{-1}(\underline{Z}'')\cup\{0\}$. By construction Z'' is an irreducible component of Z_m such that, in view of (299), there exists a point

$$z=(D,\phi)\in Z\cap Z'',\qquad \phi\ne 0. \tag{300}$$

[8]This clearly implies the surjectivity of projection $p_m=pr_1: Z\to(\mathbf{S}_m^\vee)^0$.

Since Z_m is given in $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m$ by $5m(m-1)$ equations (see (79)) and Z has dimension $4m(m+2)$ (Proposition 8.1). Hence, outside of its intersection with other irreducible components of Z_m, Z is a locally complete intersection of codimension $5m(m-1)$ in $(\mathbf{S}_m^\vee)^0 \times \mathbf{\Phi}_m$. Now it follows easily from the connectedness in codimension 1 of locally complete intersections (see [H2]) that through any point of intersection of Z with other components of Z_m (e.g., through the point z in (300)) there passes a component, say, $\widetilde{Z}$ of Z_m, distinct from Z, such that $\mathrm{codim}_Z Z \cap \widetilde{Z} = 1$.

Take any irreducible component F of $Z \cap \widetilde{Z}$ having codimension 1 in Z. From Lemma 9.7 it follows now that the set $F' := F \smallsetminus (R_Z \cap \{$union of all possible components of $Z \cap \widetilde{Z}$ distinct from $F\})$ is dense open in F. Take any point $z \in F'$. By Proposition 10.8 there exists a monomorphism $j : H_{m-1} \hookrightarrow H_m$ such that $z \in Z(j, \mathbf{I})$. Then by Proposition 10.9, in which we take Z for $\widetilde{Z}$, it follows that:

1) there exists a dense open subset Z' of $Z(j, \mathbf{I})$ such that $F^* := F' \cap Z'$ is dense open in F (see (287)),

2) for any point $z \in F^*$, $\lambda_j^{-1}(\lambda_j(z)) = \tilde{\lambda}_j^{-1}(\tilde{\lambda}_j(z)) = V(z, j) \simeq \mathbf{k}^{2m}$. (In fact, apply formula (286) to $Z(j, \mathbf{I})$ and to $\widetilde{Z}(j, \mathbf{I})$, respectively). The last equality means that

$$z = (D, \phi) \in V(z, j) \subset Z \cap \widetilde{Z}, \quad \dim V(z, j) = 2m. \tag{301}$$

Now we obtain from (301) and diagram (234) that there exists a monomorphism $j'_{\mathbf{k}} : \mathbf{k} \hookrightarrow V(z, j)$ for which the induced homomorphism $^\sharp\phi' := (^\sharp\phi \circ j, j'_{\mathbf{k}}) : H_m = H_{m-1} \oplus \mathbf{k} \to V(z, j) \hookrightarrow H_m^\vee \otimes \wedge^2 V^\vee$ is such that, in notations of (189), the point $z' = (D, \phi') \in V(z, j)$ satisfies the condition:

$$\textit{the composition } s(z') : H_m \to H_m^\vee \otimes \wedge^2 V^\vee \overset{c_D}{\twoheadrightarrow} H^0(E_{2m}(D^{-1})(1)) \textit{ is injective.}$$

Az $z \in Z(j, \mathbf{I})$, the composition $s(z') \circ j : H_{m-1} \to H^0(E_{2m}(D^{-1})(1))$ is also injective. This together with the above condition and exactly means that the point $z' \in V(z, j) \subset$ satisfies both conditions ($\mathbf{I}$) and ($\mathbf{II}$) in the definition of $\widetilde{Z}(j)$ in Lemma 9.2. It follows now from (301) that $\widetilde{Z}(j)$ is nonempty.

We are now in conditions of Proposition 10.9 which we apply to the irreducible sets $Z(j)$ and $\widetilde{Z}(j)$. Consider the morphism $r_{\mathbf{V}} : Z_m(j) \to \mathbf{V}^0$ and its restrictions $r := r_{\mathbf{V}}|_{Z(j)}$ and $\tilde{r} := r_{\mathbf{V}}|_{\widetilde{Z}(j)}$. Then according to Proposition 10.9 there exist dense open subsets Z' of $Z(j)$ and, respectively, $\widetilde{Z}'$ of $\widetilde{Z}(j)$, such that $\mathbf{V}' := r(Z') = \tilde{r}(\widetilde{Z}')$. Now, for a general point $v \in \mathbf{V}'$ and an arbitrary point $z \in r^{-1}(v) \cap Z'$, one has by (286):

$$r^{-1}(v) = V(z, j) = \lambda_{(j)}^{-1}(v) = \tilde{r}^{-1}(v).$$

This is clearly a contradiction, since, by assumption, $Z(j)$ and $\widetilde{Z}(j)$ are distinct varieties. Hence Z_m is irreducible.

The surjectivity of the morphism $p_m : Z_m \to (\mathbf{S}_m^\vee)^0$ was already mentioned in the footnote 7 above. Theorem 7.2 is proved.

11. Appendix: two results of general position

In this Appendix we prove Theorem 4.1 and Proposition 7.3.

11.1. Proof of Theorem 4.1.

We first need to recall some definitions and standard facts from theory of determinantal varieties.

Definition 11.1. Let U and U' be two vector spaces of dimensions respectively m and n, where $m \geq n$. Consider the projective space $P(U \otimes U')$. We say that *a point $x \in P(U \otimes U')$ has rank r* (and denote this as $\mathrm{rk}(x) = r$), if

(i) there exist unique subspaces $U_r(x) \subset U$ and $U'_r(x) \subset U'$ of dimensions $\dim U_r(x) = \dim U'_r(x) = r$ such that $x \in P(U_r(x) \otimes U'_r(x))$, and

(ii) there do not exist subspaces $\tilde{U} \subset U$ and $\tilde{U}' \subset U'$ of dimension $\dim \tilde{U} = \dim \tilde{U}' < r$ such that $x \in P(\tilde{U} \otimes \tilde{U}')$.

The following Lemma is a well known fact from the theory of determinantal varieties (see, e. g., [R]).

Lemma 11.2. *Each point $x \in P(U \otimes U')$ has a uniquely defined rank $\mathrm{rk}(x)$, $1 \leq \mathrm{rk}(x) \leq n$. Moreover, for a given point $x \in P(U \otimes U')$ of rank $\mathrm{rk}(x) = r$ such that $x \in W \otimes W'$ for some subspaces $W \subset U$ and $W' \subset U'$, the subspaces $U_r(x) \subset U$ and $U'_r(x) \subset U'$ of dimensions $\dim U_k(x) = \dim U'_k(x) = r$ defined in (i) above are such that $U_r(x) \subset W$ and $U'_r(x) \subset W'$.*

Proof. According to Definition 11.1 in which we put $U = H^\vee_{2m+1}$, $U' = V^\vee$, each point $x \in P(H^\vee_{2m+1} \otimes V^\vee)$ has rank $1 \leq \mathrm{rk}(x) \leq \dim V^\vee = 4$ [9] . Thus

$$P(W^\vee_{4m+4}) = \overset{4}{\underset{r=1}{\cup}} Z_r, \tag{302}$$

where

$$Z_r := \{x \in P(W^\vee_{4m+4}) \mid rk(x) = r\}, \quad 1 \leq r \leq 4,$$

are locally closed subsets of $P(W^\vee_{4m+4})$. Consider the Grassmannian

$$G := G(m, H^\vee_{2m+1})$$

and its locally closed subsets

$$\Sigma_r := \{V_m \in G \mid V_m \supset U_r(x) \text{ for some point } x \in Z_r\}, \quad 1 \leq r \leq 4. \tag{303}$$

In view of Lemma 11.2 the condition $x \in Z_r \cap P(V_m \otimes V^\vee)$ means that $x \in Z_r \cap P(U_r \otimes V^\vee)$ for some r-dimensional subspace $U_r = U_r(x) \subset V_m$. This together with (302) and (303) shows that

$$\{V_m \in G \mid P(V_m \otimes V^\vee) \cap P(W^\vee_{4m+4}) \neq \emptyset\} = \overset{4}{\underset{r=1}{\cup}} \Sigma_r.$$

Now the theorem says that $\overset{4}{\underset{r=1}{\cup}} \Sigma_r \underset{\neq}{\subset} G$. Thus, to prove the theorem, it is enough to show that

$$\dim \Sigma_r < \dim G, \quad 1 \leq r \leq 4. \tag{304}$$

We are starting now the proof of (304) for $r = 4, 3, 2, 1$.

(i) **Case $\mathbf{r = 4}$.** Set $\Gamma_4 := \{(x, U) \in P(W^\vee_{4m+4}) \times G(4, H^\vee_{2m+1}) \mid \mathrm{rk}(x) = 4 \text{ and } U = U_4(x)\}$ and let $P(W^\vee_{4m+4}) \overset{p_4}{\leftarrow} \Gamma_4 \overset{q_4}{\rightarrow} G(4, H^\vee_{2m+1})$ be the projections. By construction, $p_4(\Gamma_4) = Z_4$, and by the definition 11.1(i) the projection $p_4 : \Gamma_4 \to Z_4$ is a bijection. Hence

$$\dim q_4(\Gamma_4) \leq \dim \Gamma_4 = \dim Z_4 \leq \dim P(W^\vee_{4m+4}) = 4m + 3.$$

By construction we have the graph of incidence

$$\Pi_4 = \{(U, V_m) \in q_4(\Gamma_4) \times \Sigma_4 \mid U \subset V_m\}$$

with surjective projections $q_4(\Gamma_4) \overset{pr_1}{\leftarrow} \Pi_4 \overset{pr_2}{\rightarrow} \Sigma_4$ and a fibre

$$pr_1^{-1}(U) \simeq G(m-4, H^\vee_{2m+1}/U) \tag{305}$$

[9]Everywhere in this proof by the rank of a point x of a given subspace of $P(H^\vee_{2m+1} \otimes V^\vee)$ we understand its rank as of a point in $P(H^\vee_{2m+1} \otimes V^\vee)$.

over an arbitrary point $U \in q_4(\Gamma_4)$. (In fact, the condition $U \subset V_m \subset H^\vee_{2m+1}$ means that $V_m/U \in G(m-4, H^\vee_{2m+1}/U)$.) Hence

$$\dim \Sigma_4 \leq \dim \Pi_4 = \dim q_4(\Gamma_4) + \dim G(m-4, H^\vee_{2m+1}/U) \leq 4m+3+(m-4)(m+1) = m(m+1)-1 =$$

$$= \dim G - 1 < \dim G,$$ i.e. (304) is true for $r = 4$.

(ii) **Case $\mathbf{r = 3}$**. Consider the projection $f_3 : Z_3 \to P(V^\vee)^\vee = \mathbb{P}^3 : x \mapsto V_3(x)$, where the pair of 3-dimensional spaces $(U_3(x), V_3(x))$, $U_3(x) \subset H^\vee_{2m+1}$ and $V_3(x) \subset V^\vee$, is determined uniquely by the point x via the condition $x \in P(U_3(x) \otimes V_3(x))$, since $\mathrm{rk}(x) = 3$ (see Definition 11.1 and Lemma 11.2). Now for a given 3-dimensional subspace $V_3 \subset V^\vee$ set

$$\Sigma_3(V_3) = \{V_m \in G \mid V_m \supset U_3(x) \text{ for some point } x \in f_3^{-1}(V_3)\}. \tag{306}$$

Comparing this with (303) for $r = 3$ we obtain

$$\Sigma_3 = \bigcup_{V_3 \subset V^\vee} \Sigma_3(V_3). \tag{307}$$

Note that a priori f_3 is not necessarily surjective. Hence,

$$\dim \Sigma_3 \leq \dim \Sigma_3(V_3) + 3. \tag{308}$$

We are going to obtain an estimate for the dimension of $\Sigma_3(V_3)$ for an arbitrary 3-dimensional subspace V_3 of $V^\vee$. This subspace defines a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & \Omega_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{I}_z(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_z \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathbf{k}_z \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd} \tag{309}$$

where $z = P(\ker : V \twoheadrightarrow V_3^\vee)$ is a point in $\mathbb{P}^3$ and the sheaf F has an $\mathcal{O}_{\mathbb{P}^3}$-resolution $0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to 3\mathcal{O}_{\mathbb{P}^3}(-2) \to F \to 0$. Twisting this resolution by the vector bundle E and passing to cohomology we obtain the equalities $H^1(F \otimes E) \simeq H^2(E(-3)) = H_{2m+1}$, $H^2(F \otimes E) = 0$. Respectively, passing to cohomology in diagram (309) twisted by E and using the above equalities and evident relations $H^0(E \otimes \mathbf{k}_z) \simeq \mathbf{k}^2$, $H^1(E \otimes \mathbf{k}_z) = 0$ implies the diagram

(310)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \mathbf{k}^2 \arrow[d, tail] & \\
0 \arrow[r] & H_{2m+1} \arrow[r] \arrow[d] & W^{\vee}_{4m+4} \arrow[r] \arrow[d] & H^1(E \otimes \mathcal{I}_z(-1)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{\vee}_{2m+1} \otimes V_3 \arrow[r, "\lambda"] \arrow[d] & H^{\vee}_{2m+1} \otimes V^{\vee} \arrow[r] \arrow[d, "mult"] & H^{\vee}_{2m+1} \arrow[r] \arrow[d] & 0 \\
\mathbf{k}^2 \arrow[r, tail] & H^1(E \otimes \mathcal{I}_z) \arrow[r] \arrow[d] & H^{\vee}_{4m} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0. & &
\end{tikzcd}$$

In this diagram the composition $\epsilon := mult \circ \lambda$ is surjective. Hence, setting $W_{2m+3}(V_3) := \ker \epsilon$, where $\dim W_{2m+3}(V_3) = 2m+3$, we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_{2m+3}(V_3) \arrow[r] \arrow[d] & W^{\vee}_{4m+4} \arrow[r] \arrow[d] & H^{\vee}_{2m+1} \arrow[d, equal] \\
0 \arrow[r] & H^{\vee}_{2m+1} \otimes V_3 \arrow[r, "\lambda"] \arrow[d, "\epsilon"] & H^{\vee}_{2m+1} \otimes V^{\vee} \arrow[r] \arrow[d, "mult"] & H^{\vee}_{2m+1} \\
 & H^{\vee}_{4m} \arrow[r, equal] \arrow[d] & H^{\vee}_{4m} \arrow[d] & \\
 & 0 & 0 &
\end{tikzcd}$$

which yields the relation

$$W_{2m+3}(V_3) = H^{\vee}_{2m+1} \otimes V_3 \cap W^{\vee}_{4m+4}, \tag{311}$$

where the intersection is taken in $H^{\vee}_{2m+1} \otimes V^{\vee}$. Set

$$Z_3(V_3) := \{x \in P(W_{2m+3}(V_3)) \mid \mathrm{rk}(x) = 3\}.$$

The relation (311) and Lemma 11.2 imply the bijection

$$Z_3(V_3) \overset{\simeq}{\to} f_3^{-1}(V_3). \tag{312}$$

Consider the graph of incidence $\Gamma_3(V_3) := \{(x, U) \in Z_3(V_3) \times G(3, H^{\vee}_{2m+1}) \mid U = U_3(x)\}$ with projections $Z_3(V_3) \overset{p_3}{\leftarrow} \Gamma_3(V_3) \overset{q_3}{\to} G(3, H^{\vee}_{2m+1})$. By Lemma 11.2, $p_3(\Gamma_3(V_3)) = Z_3(V_3)$ and the projection $p_3 : \Gamma_3(V_3) \to Z_3(V_3)$ is a bijection. Hence

$$\dim q_3(\Gamma_3(V_3)) \le \dim \Gamma_3(V_3) = \dim Z_3(V_3) \le \dim P(W_{2m+3}(V_3)) = 2m+2. \tag{313}$$

Consider the graph of incidence

$$\Pi_3(V_3) = \{(U, V_m) \in q_3(\Gamma_3(V_3)) \times \Sigma_3(V_3) \mid U \subset V_m\}$$

with projections $q_3(\Gamma_3(V_3)) \overset{pr_1}{\leftarrow} \Pi_3(V_3) \overset{pr_2}{\to} \Sigma_3(V_3)$ and a fibre

$$pr_1^{-1}(U) \simeq G(m-3, H^{\vee}_{2m+1}/U) \tag{314}$$

over an arbitrary point $U \in q_3(\Gamma_3(V_3))$ (cf. (305)). The projection $\Pi_3(V_3) \overset{pr_2}{\to} \Sigma_3(V_3)$ is surjective in view of (312). Hence, using (313), we obtain

$\dim \Sigma_3(V_3) \le \dim \Pi_3(V_3) = \dim q_3(\Gamma_3(V_3)) + \dim G(m-3, H^\vee_{2m+1}/U) \le 2m+2+(m-3)(m+1) = = m^2 - 1$. This together with (308) and the assumption $m \ge 3$ yields $\dim \Sigma_3 \le m^2 + 2 = \dim G + 2 - m < \dim G$, i.e. (304) holds for $r = 3$.

Before proceeding to the case $r = 2$ we need to make a small digression on jumping lines of E. Introduce some more notation. For a given line $l \subset \mathbb{P}^3$ we have $E|l \simeq \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d)$ for a well-defined nonnegative integer d called the *jump of* $E|l$ and denoted also by $d_E(l)$; respectively, the line l is called a *jumping line of jump* d *of* E. Set $G_{2,4} := G(2, V^\vee)$ and $J_k(E) := \{l \in G_{2,4} \mid d_E(l) \ge k\}$, $J^*_k(E) := J_k(E) \setminus J_{k+1}(E),\ 0 \le k$. From the semicontinuity of $E|l,\ l \in G_{2,4}$, it follows that $J_k(E)$ (resp., $J^*_k(E)$) is a closed (resp., locally closed) subset of $G_{2,4},\ k \ge 0$. Moreover, by a well-known theorem of Grauert-Mülich, $J^*_0(E)$ is a dense open subset of $G_{2,4}$. Next, since $E \in I'_{2m+1}$, it follows that

$$J_{2m+1}(E) = \emptyset, \tag{315}$$

so that

$$J_{2m-1}(E) = J^*_{2m-1}(E) \sqcup J^*_{2m}(E). \tag{316}$$

We will use below the following lemma.

Lemma 11.3. *Let* $E \in I'_{2m+1}$. *Then*
1. $\dim J_{2m-1}(E) \le 1$.
2. $\dim J^*_k(E) \le 3$ *for* $1 \le k \le 2m-2$.

Proof. (1) Suppose the contrary, i.e. $\dim J_{2m-1}(E) \ge 2$. Take any irreducible surface $S \subset J_{2m-1}(E)$ and let D be the degree of S with respect to the sheaf $\mathcal{O}_{G_{2,4}}(1)$. Fix an integer $r \ge 5$ and take any irreducible curve C belonging to the linear series $\left|\mathcal{O}_{G_{2,4}}(r)|_S\right|$. Then the degree $\deg C$ w.r.t. $\mathcal{O}_{G_{2,4}}(1)$ equals to Dr, hence $\deg C \ge 5$. Hence by [C, Lemma 6] there exist two distinct lines, say, $l_1, l_2 \in C$, which intersect in $\mathbb{P}^3$. Let the plane $\mathbb{P}^2$ be the span of l_1 and l_2 in $\mathbb{P}^3$. Now the exact triple $0 \to E(-2)|_{\mathbb{P}^2} \to E|_{\mathbb{P}^2} \to E|_{l_1 \cup l_2} \to 0$ implies

$$H^0(E|_{\mathbb{P}^2}) \to H^0(E|_{l_1\cup l_2}) \to H^1(E(-2)|_{\mathbb{P}^2}). \tag{317}$$

Next, as $[E] \in I_{2m+1}$, we have $h^0(E(-1)) = h^1(E(-2)) = 0$, hence the exact triple $0 \to E(-2) \to E(-1) \to E(-1)|_{\mathbb{P}^2} \to 0$ implies

$$H^0(E(-1)|_{\mathbb{P}^2}) = 0. \tag{318}$$

Now assume $h^0(E|_{\mathbb{P}^2}) > 0$. Then a section $0 \ne s \in H^0(E|_{\mathbb{P}^2})$ defines an injection $\mathcal{O}_{\mathbb{P}^2} \overset{s}{\hookrightarrow} E|_{\mathbb{P}^2}$. This injection and (318) show that the zero-set Z of the section s is 0-dimensional and the injection s extends to a triple $0 \to \mathcal{O}_{\mathbb{P}^2} \overset{s}{\to} E|_{\mathbb{P}^2} \to \mathcal{I}_{Z,\mathbb{P}^2} \to 0$. Whence

$$h^0(E|_{\mathbb{P}^2}) \le 1. \tag{319}$$

Furthermore, equality (318) together with Riemann-Roch and Serre duality for the vector bundle $E(-1)|_{\mathbb{P}^2}$ shows that $h^1(E(-2)|_{\mathbb{P}^2}) = 2m+1$. Whence in view of (317) and (318) we obtain

$$h^0(E|_{l_1\cup l_2}) \le 2m+2. \tag{320}$$

On the other hand, let $x := l_1 \cap l_2$. Since by construction $l_1, l_2 \in J_{2m-1}(E)$, it follows from (316) that either $E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1-2m)$, or $E|_{l_i} \simeq \mathcal{O}_{\mathbb{P}^2}(2m) \oplus \mathcal{O}_{\mathbb{P}^2}(-2m)$, hence $h^0(E \otimes \mathcal{I}_{x,l_i}) \ge 2m-1,\ i = 1,2$. This clearly implies $h^0(E|_{l_1\cup l_2}) \ge h^0(E \otimes \mathcal{I}_{x,l_1\cup l_2}) \ge h^0(E \otimes \mathcal{I}_{x,l_1}) + h^0(E \otimes \mathcal{I}_{x,l_2}) = 4m-2$. Comparing this with (320) we obtain the inequality $2m+2 \ge 4m-2$, i.e. $m \le 2$. This contradicts to the assumption $m \ge 3$. Hence, the assertion (1) follows.

(2) This is an immediate corollary of the theorem of Grauert-Mülich. The lemma is proved. □

(iii) **Case r = 2**. Here our notation and argument are completely parallel to those in the case $r = 3$ above. Consider a morphism $f_2 : Z_2 \to G_{2,4} : x \mapsto V_2(x)$, where the pair of 2-dimensional spaces $(U_2(x), V_2(x))$, $\quad U_2(x) \subset H^\vee_{2m+1}$ and $V_2(x) \subset V^\vee$, is determined uniquely by the point x via the condition $x \in P(U_2(x) \otimes V_2(x))$, since $\mathrm{rk}(x) = 2$ (see Lemma 11.2).

According to (315) we may assume that $l \in J^*_k(E)$ for some $0 \le k \le 2m$, i.e.

$$h^0(E|l) = 2, \quad h^1(E|l) = 0, \quad \text{if} \quad l \in J^*_0(E),$$

respectively,

$$h^0(E|l) = k+1, \quad h^1(E|l) = k-1, \quad \text{if} \quad l \in J^*_k(E), \quad 1 \le k \le 2m. \tag{321}$$

Now, for $1 \le k \le 2m$ and a given subspace $V_2 \in J^*_k$, set

$$\Sigma_{2,k}(V_2) = \{V_m \in G \mid V_m \supset U_2(x) \text{ for some point } x \in f_2^{-1}(V_2)\}. \tag{322}$$

Then similarly to (307) we have

$$\Sigma_2 = \bigcup_{k=0}^{2m} \bigcup_{V_2 \in J^*_k} \Sigma_{2,k}(V_2).$$

Hence, in view of Lemma 11.3

$$\dim \Sigma_2 \le \max_{\substack{V_2 \in J^*_k \\ 0 \le k \le 2m}} (\dim \Sigma_{2,k}(V_2) + \dim J^*_k). \tag{323}$$

We are going to obtain an estimate for the dimension of $\Sigma_{2,k}(V_2)$ for an arbitrary 2-dimensional subspace V_2 in J^*_k, $\;0 \le k \le 2m$. This subspace defines a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-2) \arrow[r, "s"] \arrow[d] & \Omega_{\mathbb{P}^3} \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_2 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & V_2' \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_l \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_l \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd} \tag{324}$$

where $V_2' := V^\vee/V_2$, $l = P((V_2')^\vee)$ is a line in $\mathbb{P}^3$, and $F := \mathrm{coker}\, s$. Passing to cohomology in the diagram (324) twisted by E, we obtain the diagram

(325)
$$
\begin{tikzcd}
 & & 0 \arrow[d] & H^0(E|l) \arrow[d, tail] & \\
 & & W^\vee_{4m+4} \arrow[r, equal] \arrow[d] & H^1(E\otimes F) \arrow[d] & \\
0 \arrow[r] & H^\vee_{2m+1}\otimes V_2 \arrow[r] \arrow[d, equal] & H^\vee_{2m+1}\otimes V^\vee \arrow[r] \arrow[d, "mult"] & H^\vee_{2m+1}\otimes V_2' \arrow[r] \arrow[d, "\epsilon_2"] & 0 \\
H^0(E|l) \arrow[r, tail] & H^1(E\otimes\mathcal{I}_l) \arrow[r] & H^\vee_{4m} \arrow[r, "\epsilon_1"] \arrow[d] & H^1(E|l) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}
$$

Assume first that $1 \le k \le 2m$. (The case $k=0$ is treated below.) In this case (321) and the diagram (325) lead to the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_{k+1}(V_2) \arrow[r] \arrow[d] & W^\vee_{4m+4} \arrow[r] \arrow[d] & \ker\epsilon_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^\vee_{2m+1}\otimes V_2 \arrow[r] \arrow[d] & H^\vee_{2m+1}\otimes V^\vee \arrow[r] \arrow[d, "mult"] & H^\vee_{2m+1}\otimes V_2' \arrow[r] \arrow[d, "\epsilon_2"] & 0 \\
0 \arrow[r] & \ker\epsilon_1 \arrow[r] \arrow[d] & H^\vee_{4m} \arrow[r, "\epsilon_1"] \arrow[d] & H^1(E|l) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}
$$

where we set $W_{k+1}(V_2) := H^0(E|l)$. Here according to (321) we have $\dim W_{k+1}(V_2) = k+1,\quad \dim\ker\epsilon_1 = 4m-k+1,\quad \dim\ker\epsilon_2 = 4m-k+3$. This diagram yields the relation (cf. (311))

(326)
$$W_{k+1}(V_2) = H^\vee_{2m+1}\otimes V_2 \cap W^\vee_{4m+4},$$

where the intersection is taken in $H^\vee_{2m+1}\otimes V^\vee$. Set

$$Z_{2,k}(V_2) := \{x \in P(W_{k+1}(V_2)) \mid \mathrm{rk}(x) = 2\}.$$

The relation (326) and Lemma 11.2 imply the bijection

(327)
$$Z_{2,k}(V_2) \overset{\simeq}{\to} f_2^{-1}(V_2).$$

Consider the graph of incidence $\Gamma_{2,k}(V_2) := \{(x,U) \in Z_{2,k}(V_2)\times G(2,H^\vee_{2m+1}) \mid U = U_2(x)\}$ with projections $Z_{2,k}(V_2) \overset{p_2}{\leftarrow} \Gamma_{2,k}(V_2) \overset{q_2}{\to} G(2,H^\vee_{2m+1})$. By construction, $p_2(\Gamma_{2,k}(V_2)) = Z_{2,k}(V_2)$ and the projection $p_2 : \Gamma_{2,k}(V_2) \to Z_{2,k}(V_2)$ is a bijection. Hence

(328)
$$\dim q_2(\Gamma_{2,k}(V_2)) \le \dim\Gamma_{2,k}(V_2) = \dim Z_{2,k}(V_2) \le \dim P(W_{k+1}(V_2)) = k.$$

Consider the graph of incidence

$$\Pi_{2,k}(V_2) = \{(U,V_m) \in q_2(\Gamma_{2,k}(V_2))\times\Sigma_{2,k}(V_2) \mid U \subset V_m\}$$

with projections $q_2(\Gamma_{2,k}(V_2)) \overset{pr_1}{\leftarrow} \Pi_{2,k}(V_2) \overset{pr_2}{\to} \Sigma_{2,k}(V_2)$ and a fibre

$$pr_1^{-1}(U) \simeq G(m-2, H^\vee_{2m+1}/U)$$

over an arbitrary point $U \in q_2(\Gamma_{2,k}(V_2))$ (cf. (305) and (314)). The projection $\Pi_{2,k}(V_2) \overset{pr_2}{\to} \Sigma_{2,k}(V_2)$ is surjective in view of (327). Hence using (328) we obtain

(329) $$\dim \Sigma_{2,k}(V_2) \le \dim \Pi_{2,k}(V_2) = \dim q_2(\Gamma_{2,k}(V_2)) + \dim G(m-2, H^\vee_{2m+1}/U) \le$$
$$\le k + (m-2)(m+1) = m^2 - m - 2 + k = \dim G - (2m - k + 2), \quad 1 \le k \le 2m.$$

Now consider the case $k = 0$. In this case one has $h^0(E|l) = 2$ and, respectively, $\dim q_2(\Gamma_{2,0}(V_2)) \le \dim \Gamma_{2,0}(V_2) = \dim Z_{2,0}(V_2) \le \dim P(W_1(V_2)) = 1$, instead of (328). Hence, similar to the above we obtain for $k = 0$:

$$\dim \Sigma_{2,0}(V_2) \le 1 + (m-2)(m+1) = m^2 - m - 1 = \dim G - (2m+1).$$

The last inequality together with (329), (323), Lemma 11.3 and the assumption $m \ge 3$ yields $\dim \Sigma_2 < \dim G$, i.e. (304) is true for $r = 2$.

(iv) **Case $\mathbf{r = 1}$**. Again the notation and argument goes along the same lines as in cases $r = 4, 3$ and 2 above. Consider the projection $f_1 : Z_1 \to P(V^\vee) = (\mathbb{P}^3)^\vee : x \mapsto V_1(x)$, where the pair of 1-dimensional spaces $(U_1(x), V_1(x))$, $U_1(x) \subset H^\vee_{2m+1}$ and $V_1(x) \subset V^\vee$, is determined uniquely by the point x via the condition $x \in P(U_1(x) \otimes V_1(x))$, since $\mathrm{rk}(x) = 1$ (see Lemma 11.2). Now for a given subspace $V_1 \in (\mathbb{P}^3)^\vee$ set

$$\Sigma_1(V_1) := \{V_m \in G \mid V_m \supset U_1(x) \text{ for some point } x \in f_1^{-1}(V_1)\}.$$

Then similar to (307) we have

(330) $$\Sigma_1 = \underset{V_1 \in (\mathbb{P}^3)^\vee}{\cup} \Sigma_1(V_1).$$

Hence,

(331) $$\dim \Sigma_1 \le \dim \Sigma_1(V_1) + 3.$$

We are going to obtain an estimate for the dimension of $\Sigma_1(V_1)$ for an arbitrary 1-dimensional subspace V_1 of $V^\vee$. This subspace V_1 defines a commutative diagram

(332)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega_{\mathbb{P}^3} \arrow[r, equal] \arrow[d] & \Omega_{\mathbb{P}^3} \arrow[d] & \\
0 \arrow[r] & V_1 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d, equal] & V^\vee \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & V_3 \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^3}(-1) \arrow[r] & \mathcal{O}_{\mathbb{P}^3} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Note that to the point $V_1 \in (\mathbb{P}^3)^\vee$ there corresponds a projective plane $P(V_1)$ in $\mathbb{P}^3$ and set $B(E) := \{V_1 \in (\mathbb{P}^3)^\vee \mid h^0(E|_{P(V_1)}) \ne 0\}$. It is known that, for $m \ge 1$, $\dim B(E) \le 2$ (see [B1]). Moreover, in view of (319),

(333) $$h^0(E|_{P(V_1)}) = 1, \quad V_1 \in B(E).$$

Passing to cohomology in diagram (332) twisted by E and using the equality $h^0(E) = 0$ for $[E] \in I_{2m+1}$ we obtain the diagram

(334)
$$
\begin{tikzcd}
 & & 0 \arrow[d] & H^0(E|_{P(V_1)}) \arrow[d] & \\
 & & W^\vee_{4m+4} \arrow[r, equal] \arrow[d] & W^\vee_{4m+4} \arrow[d] & \\
0 \arrow[r] & H^\vee_{2m+1}\otimes V_1 \arrow[r, "\lambda"] \arrow[d, equal] & H^\vee_{2m+1}\otimes V^\vee \arrow[r] \arrow[d, "mult"] & H^\vee_{2m+1}\otimes V_3 \arrow[r] \arrow[d] & 0 \\
H^0(E|_{P(V_1)}) \arrow[r, tail] & H^\vee_{2m+1} \arrow[r] & H^\vee_{4m} \arrow[r] \arrow[d] & H^1(E|_{P(V_1)}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}
$$

Let $V_1 \in B(E)$. Setting $W_1(V_1) := \ker(mult\circ\lambda) = H^0(E|_{P(V_1)})$, where by (333) $\dim W_1(V_1) = 1$, we obtain from (334) a commutative diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & W_1(V_1) \arrow[r] \arrow[d] & W^\vee_{4m+4} \arrow[r] \arrow[d] & W^\vee_{4m+4}/W_1(V_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^\vee_{2m+1}\otimes V_1 \arrow[r, "\lambda"] \arrow[d, "\epsilon"] & H^\vee_{2m+1}\otimes V^\vee \arrow[r] \arrow[d, "mult"] & H^\vee_{2m+1}\otimes V_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^\vee_{2m+1}/W_1(V_1) \arrow[r] \arrow[d] & H^\vee_{4m} \arrow[r] \arrow[d] & H^1(E|_{\mathbb{P}^2(V_1)}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0, &
\end{tikzcd}
$$

hence a relation

(335)
$$W_1(V_1) = H^\vee_{2m+1}\otimes V_1 \cap W^\vee_{4m+4},$$

where the intersection is taken in $H^\vee_{2m+1}\otimes V^\vee$. Set

$$Z_1(V_1) := \emptyset \ \text{ if } \ V_1 \neq B(E), \quad \text{respectively,} \ \ Z_1(V_1) := P(W_1(V_1)) = \{pt\} \ \text{ if } \ V_1 \in B(E).$$

The relation (335) and Lemma 11.2 imply the bijection

(336)
$$Z_1(V_1) \overset{\simeq}{\to} f_1^{-1}(V_1), \quad V_1 \in (\mathbb{P}^3)^\vee,$$

Consider the graph of incidence $\Gamma_1(V_1) := \{(x,U) \in Z_1(V_1) \times P(H^\vee_{2m+1}) \mid U = U_1(x)\}$ with projections $Z_1(V_1) \overset{p_1}{\leftarrow} \Gamma_1(V_1) \overset{q_1}{\to} P(H^\vee_{2m+1})$. By construction, $p_1(\Gamma_1(V_1)) = Z_1(V_1)$ and the projection $p_4 : \Gamma_1(V_1) \to Z_1(V_1)$ is a bijection. Hence

(337)
$$\dim q_1(\Gamma_1(V_1)) \leq \dim \Gamma_1(V_1) = \dim Z_1(V_1) \leq 0.$$

Consider the graph of incidence

$$\Pi_1(V_1) = \{(U, V_m) \in q_1(\Gamma_1(V_1)) \times \Sigma_1(V_1) \mid U \subset V_m\}$$

with projections $q_1(\Gamma_1(V_1)) \overset{pr_1}{\leftarrow} \Pi_1(V_1) \overset{pr_2}{\to} \Sigma_1(V_1)$ and a fibre

$$pr_1^{-1}(U) \simeq G(m-1, H^\vee_{2m+1}/U)$$

over an arbitrary point $U \in q_1(\Gamma_1(V_1))$. The projection $\Pi_1(V_1) \overset{pr_2}{\to} \Sigma_1(V_1)$ is surjective in view of (336). Hence using (337) we have

$\dim \Sigma_1(V_1) \leq \dim \Pi_1(V_1) = \dim q_1(\Gamma_1(V_1)) + \dim G(m-1, H^\vee_{2m+1}/U) \leq 0 + (m-1)(m+1) =$
$= m^2 - 1$. This together with (331) and the assumption $m \geq 3$ yields $\dim \Sigma_1 \leq m^2 + 2 = \dim G + 2 - m < \dim G$, i.e. (304) holds for $r = 1$. Theorem is proved. □

11.2. Proof of Proposition 7.3.

Before giving the proof of this Proposition, we need some preliminary arguments. For any point $B \in \mathbf{S}_{m+1}$ let $\hat{B} : S^2 H_{m+1} \to \wedge^2 V^\vee$ denote the induced homomorphism. We have a morphism of affine varieties

$$\mathbf{b} : \ H_{m+1} \times \mathbf{S}_{m+1} \to \wedge^2 V^\vee : \ (h, B) \mapsto \hat{B}(h \otimes h). \tag{338}$$

Fix a basis e_1, e_2, e_3, e_4 in V. Then the point $B \in \mathbf{S}_{m+1}$ considered as a homomorphism $B : H_{m+1} \otimes V \to H^\vee_{m+1} \otimes V^\vee$ can be represented by a skew-symmetric block matrix

$$B = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & -A_{34} & 0 \end{pmatrix} \tag{339}$$

where $A_{ij} \in S^2 H^\vee_{m+1}$, $1 \leq i < j \leq 4$. Here we consider A_{ij} as the quadratic forms

$$H_{m+1} \to \mathbf{k} : x \mapsto A_{ij}(x), \ 1 \leq i < j \leq 4, \tag{340}$$

on H_{m+1}. Respectively, in the projective space $P(H_{m+1}) \simeq \mathbb{P}^m$ there are defined quadrics

$$Q_{ij}(B) := \{< x > \in P(H_{m+1}) \mid A_{ij}(x) = 0\}, \qquad 1 \leq i < j \leq 4. \tag{341}$$

Let $K \subset \wedge^2 V^\vee$ be the cone of decomposable vectors, $K = \{w \in \wedge^2 V^\vee | \operatorname{rk}(w : V \to V^\vee) \leq 2\}$, and, for $m \geq 1$, set

$$M_{m+1} := \{B \in \mathbf{S}_{m+1} | \ \mathbf{b}(H_{m+1} \times \{B\}) \subset K\}. \tag{342}$$

By construction, M_{m+1} is a closed subset of $\mathbf{S}_{m+1}$, and we consider it as a reduced subscheme of $\mathbf{S}_{m+1}$.

Consider first the cases $m = 0, 1$ and 2. An explicit computation shows that

(i) M_1, M_2 and M_3 are irreducible and, moreover,

$$M_1 = K, \quad M_{m+1} \subset \mathbf{S}_{m+1} \smallsetminus (\mathbf{S}_{m+1})^0, \quad \operatorname{codim}_{\mathbf{S}_{m+1}} M_{m+1} = 2, \quad m = 1, 2; \tag{343}$$

(ii) $M_3^* := \{B \in M_3 | \ Y_3(B) := Q_{13}(B) \cap Q_{23}(B)$ is a 4-ple of distinct points in the projective plane $P(H_3)\}$ is a dense open subset of M_3.

Now proceed to the case $m \geq 3$. In this case, set

$$\mathbf{S}^*_{m+1} := \{B \in \mathbf{S}_{m+1} | \ Y_{m+1}(B) := Q_{13}(B) \cap Q_{23}(B) \text{ is an integral codimension 2 subscheme of the projective space } P(H_{m+1})\}. \tag{344}$$

Since $m \geq 3$, $\mathbf{S}^*_{m+1}$ is a dense open subset of $\mathbf{S}_{m+1}$.

Lemma 11.4. *For $m \geq 3$ let $B \in \mathbf{S}^*_{m+1} \cap M_{m+1}$. Then $B \notin \mathbf{S}^0_{m+1}$.*

Proof. We represent a given point $B \in \mathbf{S}^*_{m+1} \cap M_{m+1}$ by matrix (339). Then, under the notation (340), for $x \in H_{m+1}$, we obtain a skew-symmetric (4×4)-matrix with entries in $\mathbf{k}$

$$B(x) = \begin{pmatrix} 0 & A_{12}(x) & A_{13}(x) & A_{14}(x) \\ -A_{12}(x) & 0 & A_{23}(x) & A_{24}(x) \\ -A_{13}(x) & -A_{23}(x) & 0 & A_{34}(x) \\ -A_{14}(x) & -A_{24}(x) & -A_{34}(x) & 0 \end{pmatrix}. \tag{345}$$

The condition $B \in M_{m+1}$ by definition means that the matrix $B(x)$ is degenerate, i.e. its Pfaffian vanishes identically as a polynomial function on H_{m+1}:

$$A_{12}(x)A_{34}(x) - A_{13}(x)A_{24}(x) + A_{14}(x)A_{23}(x) \equiv 0, \quad x \in H_{m+1}. \tag{346}$$

Since $B \in \mathbf{S}^*_{m+1}$, from (341) and (344) it follows that the quadrics $Q_{13}(B)$ and $Q_{23}(B)$ are integral and their intersection $Y := Y_{m+1}(B)$ is integral of codimension 2 in $P(H_{m+1})$. In this case (346) implies that either $Q_{12}(B) \supset Y$, or $Q_{34}(B) \supset Y$. Let, say, $Q_{34}(B) \supset Y_{m+1}(B)$. This means that $A_{34}(x) \in H^0(\mathcal{I}_{Y,\mathbb{P}^m}(2))$. Now, passing to sections of the exact triple $0 \to \mathcal{O}_{\mathbb{P}^m}(-2) \to 2\mathcal{O}_{\mathbb{P}^m} \xrightarrow{A_{13}(x),A_{23}(x)} \mathcal{I}_{Y,\mathbb{P}^m}(2) \to 0$, we obtain that $A_{34}(x) = \alpha A_{13}(x) + \beta A_{23}(x)$ for some $\alpha, \beta \in \mathbf{k}$. Substituting this relation into (346) we obtain a relation $A_{13}(x)(\alpha A_{12}(x) - A_{24}(x)) + A_{23}(x)(\beta A_{12}(x) + A_{14}(x)) \equiv 0$. Since Q_{13} and Q_{23} are integral, the last relation implies that either

(i) $A_{23} = \lambda A_{13},\ A_{24} - \alpha A_{12} = \lambda(\beta A_{12} + A_{14})$ for some $\lambda \in \mathbf{k}$, or
(ii) $\beta A_{12} + A_{14} = \mu A_{13},\ A_{24} - \alpha A_{12} = \mu A_{23}$ for some $\mu \in \mathbf{k}$.

Substituting the relations (i) into (339) and denoting $\gamma = \alpha + \lambda\beta$, we obtain

$$B = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & \lambda A_{13} & \gamma A_{12} + \lambda A_{14} \\ -A_{13} & -\lambda A_{23} & 0 & \gamma A_{13} \\ -A_{14} & -\gamma A_{12} - \lambda A_{14} & -\gamma A_{13} & 0 \end{pmatrix}. \tag{347}$$

Adding the multiplied by λ first block column of this matrix to its fourth block column, and then performing a similar operation with block rows, we obtain the matrix

$$B' = \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & \lambda A_{13} & \lambda A_{14} \\ -A_{13} & -\lambda A_{13} & 0 & 0 \\ -A_{14} & -\lambda A_{14} & 0 & 0 \end{pmatrix} \tag{348}$$

which is degenerate. Hence B is also degenerate. A similar computation with relations (ii) also gives the degenerateness of B. Lemma is proved. □

From Lemma 11.4 it follows that, for any irreducible component M'_{m+1} of M_{m+1},

$$1 \le \mathrm{codim}_{\mathbf{S}_{m+1}} M'_{m+1} \le 2, \quad m \ge 3. \tag{349}$$

Indeed, from this Lemma we obtain that $\mathbf{S}^*_{m+1} \cap \mathbf{S}^0_{m+1} \cap M_{m+1} = \emptyset$. Since $\mathbf{S}^*_{m+1} \cap \mathbf{S}^0_{m+1}$ is a dense open subset of $\mathbf{S}_{m+1}$, it follows that $M_{m+1} \ne \mathbf{S}_{m+1}$, i.e. $1 \le \mathrm{codim}_{\mathbf{S}_{m+1}} M_{m+1}$. On the other hand, K is a nonempty divisor in $\wedge^2 V^\vee$, hence $\mathbf{b}^{-1}_{m+1}(K)$ is a nonempty divisor of $H_{m+1} \times \mathbf{S}_{m+1}$. Since M_{m+1} is nonempty (in fact, $\{0\} \in M_{m+1}$), counting of dimensions of the fibres of the natural projection $\mathbf{b}^{-1}_{m+1}(K) \to \mathbf{S}_{m+1}$ shows that, for any irreducible component M'_{m+1} of M_{m+1}, $\mathrm{codim}_{\mathbf{S}_{m+1}} M'_{m+1} \le 2$, and (349) follows.

Lemma 11.5. *For $m \ge 3$ let M'_{m+1} be any irreducible component of M_{m+1}. Then $\mathbf{S}^*_{m+1} \cap M'_{m+1} \ne \emptyset$. Hence $\mathbf{S}^*_{m+1} \cap M'_{m+1}$ is a dense open subset of M'_{m+1}.*

Proof. 1) Consider first the case $m = 3$. Choose coordinates $x_1, ..., x_4$ in H_4 and let H_1 and H_3 be the subspaces of H_4 given by the equations $x_1 = x_2 = x_3 = 0$ and $x_4 = 0$, respectively. The direct sum decomposition $H_4 = H_1 \oplus H_3$ induces the iclusion of a direct summand $\mathbf{S}_1 \oplus \mathbf{S}_3 \hookrightarrow \mathbf{S}_4$. Considering this inclusion as an embedding of an affine subspace $\mathbf{S}_1 \times \mathbf{S}_3 \hookrightarrow \mathbf{S}_4$, we obtain from (343) and from the definition (342) that

$$M_3 = (\{0\} \times \mathbf{S}_3) \cap M_4, \quad K = (\mathbf{S}_1 \times \{0\}) \cap M_4. \tag{350}$$

This together with (349) and the irreducibility of M_3 (see property (i) above) implies that, for an arbitrary irreducible component M'_4 of M_4,

$$M_3 = (\{0\} \times \mathbf{S}_3) \cap M'_4, \quad K = (\mathbf{S}_1 \times \{0\}) \cap M'_4. \tag{351}$$

Note that (343) and (349) imply that

$$\operatorname{codim}_{\mathbf{S}_4} M'_4 = 2. \tag{352}$$

Take any point $B' \in M^*_3$, and let $A_{i3}(B')(x_1, x_2, x_3)$ be the quadratic forms on H_3 corresponding to the entries $A_{i3}(B')$, $i = 1, 2$, of the matrix B'. Then the set $Y_3(B')$ is given in the projective space $P(H_3)$ by the equations $\{A_{i3}(B')(x_1, x_2, x_3) = 0,\ i = 1, 2\}$. Now take an arbitrary point $B'' \in \mathbf{S}_1 \simeq \wedge^2 V^\vee$ and, according to (339), consider B'' as a skew-symmetric matrix $(a_{ij}(B''))$. Then the point $B := (B', B'') \in \mathbf{S}_1 \times \mathbf{S}_3$ determines the scheme $Y_4(B)$ (see (344)) which is given in the projective space $P(H_4)$ by the equations

$$A_{i3}(B')(x_1, x_2, x_3) - a_{i3}(B'')x_4^2 = 0, \quad i = 1, 2. \tag{353}$$

Consider the sets $U' = \{(B', B'') \in \mathbf{S}_1 \times \mathbf{S}_3 | Y_3(B') = Q_{13}(B') \cap Q_{23}(B')$ is a 4-ple of distinct points in the plane $P(H_3)\}$ and $U'' = \{(B', B'') \in \mathbf{S}_1 \times \mathbf{S}_3 | a_{i3}(B'') \neq 0,\ i = 1, 2\}$. These sets dense open subsets of $\mathbf{S}_1 \times \mathbf{S}_3$, and from (351) and the property (ii) above it follows that $M''_4 := M'_4 \cap U' \cap U''$ is a dense open subset of M'_4. Now for any point $B = (B', B'') \in M''_4$ the equations (353) can be rewritten as follows

$$\begin{gathered} A(x_1, x_2, x_3) := A_{13}(B')(x_1, x_2, x_3)a_{23}(B'') - A_{23}(B')(x_1, x_2, x_3)a_{13}(B'') = 0, \\ A_{13}(B')(x_1, x_2, x_3) - a_{13}(B'')x_4^2 = 0. \end{gathered} \tag{354}$$

Consider the conic $C(B) = \{A(x_1, x_2, x_3) = 0\}$ in $\mathbb{P}^2$. Then $M'''_4 = \{B \in M''_4 \mid C(B) \text{ is integral}\}$ is a dense open subset of M''_4. By construction, the set $\{A_{13}(B')(x_1, x_2, x_3) = 0\} \cap C(B)$ coincides with the set $Y_3(B')$ which by definition is a 4-ple of distinct points in $\mathbb{P}^2$. Therefore the equations (354) defining $Y_4(B)$ show that $Y_4(B)$ is a double cover of $C(B)$ ramified in $Y_3(B')$, hence it is an integral elliptic quartic curve in $\mathbb{P}^3$. In other words, $B \in \mathbf{S}^*_4 \cap M'_4$. This means that $M'''_4 \subset \mathbf{S}^*_4 \cap M'_4$, so that $\mathbf{S}^*_4 \cap M'_4$ is dense open in M'_4.

2) The argument in the case $m \geq 4$ is similar to the above. Choose coordinates $x_1, ..., x_{m+1}$ in H_{m+1} and let H_{m-3} and H_4 be the subspaces of H_{m+1} given by the equations $x_1 = ... = x_4 = 0$ and $x_5 = ... = x_{m+1} = 0$, respectively. The direct sum decomposition $H_{m+1} = H_{m-3} \oplus H_4$ induces the iclusion of a direct summand $\mathbf{S}_4 \hookrightarrow \mathbf{S}_{m+1}$. Considering this inclusion as an embedding of an affine subspace $\mathbf{S}_4 \hookrightarrow \mathbf{S}_{m+1}$, we obtain from the definition (342) that, similar to (351),

$$M_4 = \mathbf{S}_4 \cap M_{m+1}. \tag{355}$$

Now let M'_{m+1} be any irreducible component of M_{m+1}. From (349), (352) and (355) it follows that, for any irreducible component M'_4 of $\mathbf{S}_4 \cap M'_{m+1}$, the set $M'^*_4 = \mathbf{S}^*_4 \cap M'_4$ is a dense open subset of M'_4. By definition, an arbitrary point $B \in M'^*_4$ is such that $Y_4(B)$ is an integral quartic curve in $\mathbb{P}^3$. From the construction of the embedding $\mathbf{S}_4 \hookrightarrow \mathbf{S}_{m+1}$ it follows now that, for this point B considered as a point in M'_{m+1}, the scheme $Y_{m+1}(B)$ is a cone in $P(H_{m+1})$ over $Y_4(B)$. Hence $Y_{m+1}(B)$ is an integral codimension 2 subscheme of $P(H_{m+1})$, i.e. $B \in \mathbf{S}^*_{m+1}$. This means that $\mathbf{S}^*_{m+1} \cap M'_{m+1}$ is a dense open subset of M'_{m+1}. □

Corollary 11.6. *For any $m \geq 0$, $M_{m+1} \subset \mathbf{S}_{m+1} \smallsetminus \mathbf{S}^0_{m+1}$.*

Proof. For $m \leq 2$ this statement follows from (343). Let $m \geq 3$ and let M'_{m+1} be any irreducible component of M_{m+1}. By Lemma 11.4 $\mathbf{S}^*_{m+1} \cap M'_{m+1} \subset \mathbf{S}_{m+1} \smallsetminus \mathbf{S}^0_{m+1}$. Since $\mathbf{S}_{m+1} \smallsetminus \mathbf{S}^0_{m+1}$ is a closed subset of $\mathbf{S}_{m+1}$ and by Lemma 11.5 the set $\mathbf{S}^*_{m+1} \cap M'_{m+1}$ is a dense open subset of an irreducible set M'_{m+1}, it follows that $M'_{m+1} \subset \mathbf{S}_{m+1} \smallsetminus \mathbf{S}^0_{m+1}$. □

We are now ready to prove Proposition 7.3.

Proof of Proposition 7.3. Let $D \in (\mathbf{S}^\vee_{m+1})^0$, i.e. D is a nondegenerate homomorphism $D : H^\vee_{m+1} \otimes V^\vee \to H_{m+1} \otimes V$. Assume that, for any monomorphism $j : H^\vee_m \hookrightarrow H^\vee_{m+1}$, the composition $j_D := j^\vee \circ D \circ j : H^\vee_m \otimes V^\vee \to H_m \otimes V$ is degenerate. We will show that this leads to a contradiction. For this, represent j dually as a monomorphism $j_{\mathbf{k}} : \mathbf{k} \hookrightarrow H_{m+1}$. Consider the nondegenerate homomorphism $B := D^{-1} : H_{m+1} \otimes V \to H_{m+1}^\vee \otimes V^\vee$ and the induced skew-symmetric homomorphism $j_B := j_{\mathbf{k}}^\vee \circ B \circ j_{\mathbf{k}} : V \simeq \mathbf{k} \otimes V \to \mathbf{k}^\vee \otimes V^\vee \simeq V^\vee$. Then the degenerateness of j_D is equivalent to the degenerateness of j_B. As above, the homomorphism B can be represented by a skew-symmetric matrix (339). In this notation, the degenerateness of the homomorphism j_B for any $j_{\mathbf{k}} : \mathbf{k} \hookrightarrow H_{m+1}$ just means that, for any vector $x \in H_{m+1}$, the skew-symmetric (4×4)-matrix $B(x)$ in (345) is degenerate, i.e., by definition, $B \in M_{m+1}$. Then by Corollary 11.6 B is degenerate. This contradiction proves Proposition.

References

[BF] **Banica C., Forster O.** *Multiplicity structures on space curves*, The Lefschetz centennial conference, part I, Contemp. Math., vol. **58**, Amer. Math. Soc., Providence, RI, 1986, pp. 47-64.

[B1] **Barth W.** *Some properties of stable rank-2 vector bundles on* $\mathbf{P}_n$, Math. Ann. **226** (1977), 125-150.

[B2] **Barth W.** *Moduli of vector bundles on the projective plane*, Inventiones Math. **42** (1977), 63-91.

[B3] **Barth W.** *Irreducibility of the Space of Mathematical Instanton Bundles with Rank 2 and* $c_2 = 4$, Math. Ann.**258** (1981), 81-106.

[BT] **Böhmer W., Trautmann G.** *Special instanton bundles and Poncelet curves*, Lecture Notes in Math., **1273** (1987), 325-336.

[C] **Coandă I.** *On Barth's restriction theorem*, Journ. reine u. angew. Mathematik **428**, 97-110, 1992.

[CTT] **Coandă I., Tikhomirov A., Trautmann G.** *Irreducibility and Smoothness of the moduli space of mathematical 5–instantons over* P_3, Intern. J. Math., **14**, No.1 (2003), 1-45.

[ES] **Ellingsrud G., Strømme S.A.** *Stable rank-2 bundles on* P_3 *with* $c_1 = 0$ *and* $c_2 = 3$, Math. Ann. **255** (1981), 453-463.

[H0] **Hartshorne R.** *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[H] **Hartshorne R.** *Stable vector bundles of rank 2 on* P_3, Math. Ann., **238** (1978), 229-280.

[H1] **Hartshorne R.** *Stable reflexive sheaves*, Math. Ann., **254** (1980), 121-176.

[H2] **Hartshorne R.** *Complete intersections and connectedness*, Amer. J. of Math., **84** (1962), No. 3, 497-508.

[HL] **Huybrechts D., Lehn M.** *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[K] **Kleiman S.** *Relative duality for quasi-coherent sheaves*, Compositio math., **41** , Fasc. 1 (1980), 39-60.

[L] **Lange H.** *Universal Families of Extensions,* J. of Algebra **83** (1983), 101-112.

[LeP] **LePotier J.** *Sur l'espace de modules des fibrés de Yang et Mills*, In: Mathématique et Physique, Seminaire de l'Ecole Normale Supérieure 1979-1982, Birkhäuser 1983, pp. 65-137.

[NT] **Nüssler Th., Trautmann G.** *Multiple Koszul structures on lines and instanton bundles,* Int. J. Math. 5, No.3 (1994), 373-388.

[OSS] **Okonek C., Schneider M., Spindler H.**, *Vector Bundles on Complex Projective Spaces*, Progress in Math., Bd. 3, Birkhäuser, 1980.

[R] **Room T.G.** *The geometry of determinantal loci,* Cambridge, 1938.

[S] **Skiti M.** *Sur une famille de fibrés instantons,* Math. Z. **225** (1997), 373-294.

[T1] **Tjurin A.N.**, *The instanton equations for* $(n+1)$*-superpositions of marked* $\mathrm{ad}\, T\mathbf{P}^3 \oplus \mathrm{ad}\, T\mathbf{P}^3$, J. Reine Angew. Math., 341 (1983), 131–146.

[T2] **Tjurin A.N.**,*On the superpositions of mathematical instantons*, in Arithmetic and geometry, Vol. II, vol. 36 of Progr. Math., Birkhäuser, Boston, MA, 1983, pp. 433–450.

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