

ON GAMMA QUOTIENTS AND INFINITE PRODUCTS

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ABSTRACT. Convergent infinite products, indexed by all natural numbers, in which each factor is a rational function of the index, can always be evaluated in terms of finite products of gamma functions. This goes back to Euler. A purpose of this note is to demonstrate the usefulness of this fact through a number of diverse applications involving multiplicative partitions, entries in Ramanujan's notebooks, the Chowla–Selberg formula, and the Thue–Morse sequence. In addition, we propose a numerical method for efficiently evaluating more general infinite series such as the slowly convergent Kepler–Bouwkamp constant.

1. INTRODUCTION

Recall that an infinite product $\prod_{k=1}^{\infty} a(k)$ is said to converge if the sequence of its partial products converges to a nonzero limit. In this note we are especially interested in the case when $a(k)$ is a rational function of k . Assuming that the infinite product converges, $a(k)$ is then necessarily of the form

$$(1) \quad a(k) = \frac{(k + \alpha_1) \cdots (k + \alpha_n)}{(k + \beta_1) \cdots (k + \beta_n)}$$

with $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n complex numbers, none of which are negative integers, such that $\alpha_1 + \dots + \alpha_n = \beta_1 + \dots + \beta_n$. To see that this is the case, note first that clearly $a(k) \rightarrow 1$ as $k \rightarrow \infty$ so that $a(k)$ can be factored into linear terms as on the right-hand side of (1). On the other hand, if $a(k) = 1 + ck^{-1} + O(k^{-2})$ as $k \rightarrow \infty$ then convergence of the infinite product (and divergence of the harmonic series) forces $c = \alpha_1 + \dots + \alpha_n - \beta_1 - \dots - \beta_n = 0$.

These infinite products always have a finite-term evaluation in terms of Euler's gamma function [30, Sec. 12.13].

Theorem 1.1. *Let $n \geq 1$ be an integer, and let $\alpha_1, \dots, \alpha_n$ and β_1, \dots, β_n be nonzero complex numbers, none of which are negative integers. If $\alpha_1 + \dots + \alpha_n = \beta_1 + \dots + \beta_n$, then*

$$(2) \quad \prod_{k \geq 0} \frac{(k + \alpha_1) \cdots (k + \alpha_n)}{(k + \beta_1) \cdots (k + \beta_n)} = \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_n)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}.$$

Otherwise, the infinite product in (2) diverges.

This result is a simple consequence of Euler's infinite product definition (3) of the gamma function, see the beginning of Section 2. It is, however, scarcely stated explicitly in the literature. For instance, while the table [14] contains several pages

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of special cases of (2), some of which are rather generic in nature, it does not list (2) or an equivalent version thereof. An incidental objective of this note is therefore to advertise (2) and to illustrate its usefulness during the course of the applications given herein.

This note was motivated by a result, discussed in Section 3, which recently appeared in [8] as part of a study of multiplicative partitions. In Section 3 we also apply Theorem 1.1 to two entries in Ramanujan's (lost) notebook [3].

The first novel contribution of this note may be found in Section 4, where we propose an approach to the numerical evaluation of certain general, not necessarily rational, infinite products, which is based upon Theorem 1.1 and Padé approximation. We illustrate this approach by applying it to the Kepler–Bouwkamp constant, defined as the infinite product $\prod_{k=3}^{\infty} \cos(\pi/k)$. Due to its infamously slow convergence, various procedures for its numerical evaluation have been discussed in the literature [7], [13], [28]. The present approach has the advantage that it does not rely on developing alternative, more rapidly convergent, expressions for the Kepler–Bouwkamp constant.

In Section 5 we discuss properties of short gamma quotients at rational arguments. In particular, we offer an alternative proof of a result established in [24] and [17]. In the light of our proof, this result may be interpreted as a (much simpler) version of the Chowla–Selberg formula [25] in the case of principal characters.

Finally, in Section 6, consideration of an infinite product defined in terms of the Thue–Morse sequence naturally leads us to a curious open problem posed by Shallit.

2. PROOF AND BASIC EXAMPLES

We commence with supplying a proof of Theorem 1.1 and giving a number of basic examples.

Proof of Theorem 1.1. Euler's definition gives the gamma function as

$$(3) \quad \Gamma(z) = \lim_{m \rightarrow \infty} \frac{m^z m!}{z(z+1) \cdots (z+m)},$$

which is valid for all $z \in \mathbb{C}$ except for negative integers z . Thus,

$$\begin{aligned} \prod_{j=1}^n \frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)} &= \lim_{m \rightarrow \infty} \prod_{j=1}^n m^{\beta_j - \alpha_j} \prod_{k=0}^m \frac{\alpha_j + k}{\beta_j + k} \\ &= \lim_{m \rightarrow \infty} \prod_{k=0}^m \prod_{j=1}^n \frac{\alpha_j + k}{\beta_j + k}, \end{aligned}$$

where, for the second equality, we make use of the fact that the sum of the α_j is the same as the sum of the β_j . \square

In a similarly straight-forward manner, see [30, Sec. 12.13], an alternative proof of Theorem 1.1 follows from the Weierstrassian infinite product

$$\frac{1}{\Gamma(1+z)} = e^{\gamma z} \prod_{k \geq 1} \left(1 + \frac{z}{k}\right) e^{-z/k}.$$

Example 2.1. In the case $\alpha_1 = z$, $\alpha_2 = -z$, $\beta_1 = \beta_2 = 0$, Theorem 1.1 yields the famous

$$(4) \quad \prod_{k \geq 1} \left(1 - \frac{z^2}{k^2}\right) = \frac{1}{\Gamma(1-z)\Gamma(1+z)} = \frac{\sin(\pi z)}{\pi z}.$$

Similarly, one finds, for non-integral z , the slightly less well-known

$$\prod_{k \geq 1} \frac{(k-z)(k+z-1)}{(k-1/2)^2} = \frac{\Gamma(1/2)^2}{\Gamma(1-z)\Gamma(z)} = \sin(\pi z).$$

Both representations clearly reflect the reflection formula,

$$(5) \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

for the gamma function.

Example 2.2. Theorem 1.1 also gives an immediate proof of Wallis' product

$$\frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{6 \cdot 6}{5 \cdot 7} \cdots = \prod_{k \geq 0} \frac{(2k+2)(2k+2)}{(2k+1)(2k+3)} = \frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(1)\Gamma(1)} = \frac{\pi}{2}.$$

Alternatively, this evaluation can be seen as a corollary to (4). Several generalizations of Wallis' product, similarly based on Theorem 1.1, are discussed in the recent [27] and [4].

Example 2.3. As another illustration of Theorem 1.1, we evaluate

$$\prod_{k \geq 1} \left(1 - \frac{(-1)^k}{(2k+1)^3}\right).$$

This is Entry 89.6.12 in [14] where it is incorrectly listed with value $\pi^3/32$, but is corrected in an erratum. Let $\rho = (3 + i\sqrt{3})/8$. The correct, though more involved, value is

$$\begin{aligned} \prod_{k \geq 1} \left(1 - \frac{(-1)^k}{(2k+1)^3}\right) &= \prod_{k \geq 1} \left(1 - \frac{1}{(4k+1)^3}\right) \prod_{k \geq 1} \left(1 + \frac{1}{(4k-1)^3}\right) \\ &= \prod_{k \geq 1} \frac{k(k+\rho)(k+\bar{\rho})}{(k+1/4)^3} \prod_{k \geq 1} \frac{k(k-\rho)(k-\bar{\rho})}{(k-1/4)^3} \\ &= \frac{\Gamma(1+1/4)^3}{\Gamma(1+\rho)\Gamma(1+\bar{\rho})} \frac{\Gamma(1-1/4)^3}{\Gamma(1-\rho)\Gamma(1-\bar{\rho})} \\ &= \left(\frac{\pi/4}{\sin(\pi/4)}\right)^3 \frac{\sin(\pi\rho)}{\pi\rho} \frac{\sin(\pi\bar{\rho})}{\pi\bar{\rho}} \\ &= \frac{\pi}{12} \left[1 + \sqrt{2} \cosh\left(\frac{\sqrt{3}}{4}\pi\right)\right] \end{aligned}$$

For the last equality we used that $|\sin(x+iy)|^2 = (\cosh(2y) - \cos(2x))/2$.

Example 2.4. This example briefly indicates that Theorem 1.1 also applies to infinite products with individual terms removed, for instance, for convergence. Let

$\xi_n = e^{2\pi i/n}$ and $z \notin \{0, 1, 2, \dots\}$. The evaluation

$$\prod_{k \geq 0} \frac{k^n - z^n}{k^n + z^n} = \prod_{j=1}^{2n} \Gamma(z \xi_{2n}^j)^{(-1)^{j+1}}$$

follows from Theorem 1.1 as well. Let m be a nonnegative integer. Since the residue of the gamma function $\Gamma(z)$ at $z = -m$ is $(-1)^m/m!$, we have, as $z \rightarrow m$,

$$(m^n - z^n)\Gamma(-z) \rightarrow \frac{(-1)^m}{m!} n m^{n-1},$$

and hence

$$\prod_{k \geq 0, k \neq m} \frac{k^n - m^n}{k^n + m^n} = (-1)^m m! \frac{2m}{n} \prod_{j=1}^{2n-1} \Gamma(-m \xi_{2n}^j)^{(-1)^{j+1}}.$$

This example is discussed in much more detail in [5, Section 1.2], where it is also noted that the gamma functions can be replaced by trigonometric functions when n is even.

3. FURTHER APPLICATIONS

3.1. Multiplicative partitions. The original motivation for this note was the following result, which recently appeared in [8] as part of a study of multiplicative partitions.

Theorem 3.1. ([8, Theorem 4.2]) *For integers $n \geq 2$,*

$$\prod_{k \geq 2} \frac{1}{1 - k^{-n}} = \prod_{j=1}^{n-1} \Gamma(2 - \xi_n^j) = n \prod_{j=1}^{n-1} \Gamma(1 - \xi_n^j)$$

where $\xi_n = e^{2\pi i/n}$.

Proof. Apply Theorem 1.1 to the rational function $\frac{(k+2)^n}{(k+2)^{n-1}}$ to obtain the first equality. For the second part, recall that $\Gamma(x+1) = x\Gamma(x)$. \square

To put Theorem 3.1 into context, let a_n denote the number of multiplicative partitions of the natural number n . For instance, $a_{18} = 4$ because $18 = 2 \cdot 9 = 2 \cdot 3 \cdot 3 = 3 \cdot 6$. In analogy with Euler's infinite product formula for the zeta function, the Dirichlet generating series for the a_n is the product

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{k \geq 2} (1 + k^{-s} + k^{-2s} + \dots) = \prod_{k \geq 2} \frac{1}{1 - k^{-s}}.$$

The values in Theorem 3.1 are values of this Dirichlet series at positive integers and, as such, analogous to the zeta values $\zeta(n)$.

3.2. A product considered by Ramanujan. In [22], see also [3, Chapter 16], Ramanujan considers the product

$$(6) \quad \phi(\alpha, \beta) = \prod_{n=1}^{\infty} \left\{ 1 + \left(\frac{\alpha + \beta}{n + \alpha} \right)^3 \right\},$$

and shows that $\phi(\alpha, \beta)$ can be expressed in "finite terms" when the difference $\alpha - \beta$ is an integer. If one includes values of the gamma function in the notion of finite

term, then the product (6) can always be evaluated in finite terms. Indeed, using the factorization

$$1 + \left(\frac{\alpha + \beta}{n + \alpha}\right)^3 = \frac{(n + 2\alpha + \beta) \left(n + \frac{\alpha - \beta + (\alpha + \beta)\sqrt{-3}}{2}\right) \left(n + \frac{\alpha - \beta - (\alpha + \beta)\sqrt{-3}}{2}\right)}{(n + \alpha)^3},$$

we have, by Theorem 1.1,

$$(7) \quad \phi(\alpha, \beta) = \frac{\Gamma(1 + \alpha)^3}{\Gamma(1 + 2\alpha + \beta) \Gamma\left(1 + \frac{\alpha - \beta + i(\alpha + \beta)\sqrt{3}}{2}\right) \Gamma\left(1 + \frac{\alpha - \beta - i(\alpha + \beta)\sqrt{3}}{2}\right)}.$$

We observe that, if $\alpha - \beta$ is an integer, then the product

$$\Gamma\left(1 + \frac{\alpha - \beta + i(\alpha + \beta)\sqrt{3}}{2}\right) \Gamma\left(1 + \frac{\alpha - \beta - i(\alpha + \beta)\sqrt{3}}{2}\right)$$

can be simplified using the reflection formula (5). As noted by Ramanujan, one arrives at an evaluation of $\phi(\alpha, \beta)$ in terms of hyperbolic functions only.

Example 3.2. In the case $\alpha = \beta$, we have from (7)

$$(8) \quad \begin{aligned} \phi(\alpha, \alpha) &= \frac{\Gamma(1 + \alpha)^3}{\Gamma(1 + 3\alpha) \Gamma(1 + i\alpha\sqrt{3}) \Gamma(1 - i\alpha\sqrt{3})} \\ &= \frac{\Gamma(1 + \alpha)^3 \sinh(\pi\alpha\sqrt{3})}{\Gamma(1 + 3\alpha) \pi\alpha\sqrt{3}}, \end{aligned}$$

as in [22, equation (6)] and [3, equation (16.3.1)].

3.3. Another product considered by Ramanujan. In the next example we employ Theorem 1.1 to rewrite an integral, considered by Ramanujan in his lost notebook [3]. By doing so, we obtain a Mellin–Barnes integral which provides further context for the integral evaluation and leads to natural generalizations.

In [3, Entry 4.9.1] the following integral due to Ramanujan is recorded:

$$(9) \quad \begin{aligned} &\int_0^\infty \left(\frac{1 + x^2/b^2}{1 + x^2/a^2}\right) \left(\frac{1 + x^2/(b+1)^2}{1 + x^2/(a+1)^2}\right) \left(\frac{1 + x^2/(b+2)^2}{1 + x^2/(a+2)^2}\right) \cdots dx \\ &= \frac{\sqrt{\pi}}{2} \frac{\Gamma(a + 1/2) \Gamma(b) \Gamma(b - a - 1/2)}{\Gamma(a) \Gamma(b - 1/2) \Gamma(b - a)}, \end{aligned}$$

where $0 < a < b - \frac{1}{2}$. Using the factorization

$$\frac{1 + x^2/(b+k)^2}{1 + x^2/(a+k)^2} = \frac{(k+a)^2(k+b+ix)(k+b-ix)}{(k+b)^2(k+a+ix)(k+a-ix)},$$

as well as Theorem 1.1, the integrand can be rewritten as

$$\prod_{k=0}^{\infty} \frac{1 + x^2/(b+k)^2}{1 + x^2/(a+k)^2} = \frac{\Gamma(b)^2 \Gamma(a+ix) \Gamma(a-ix)}{\Gamma(a)^2 \Gamma(b+ix) \Gamma(b-ix)}.$$

Equation 9 is therefore equivalent to the Mellin–Barnes integral

$$(10) \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s) \Gamma(a-s)}{\Gamma(b+s) \Gamma(b-s)} ds = \frac{1}{2\sqrt{\pi}} \frac{\Gamma(a)}{\Gamma(b)} \frac{\Gamma(a+1/2) \Gamma(b-a-1/2)}{\Gamma(b-1/2) \Gamma(b-a)}.$$

We note that, using the duplication formula

$$(11) \quad \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right),$$

the right-hand side of (10) can be simplified to yield

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(a-s)}{\Gamma(b+s)\Gamma(b-s)} ds = \frac{\Gamma(2a)\Gamma(2b-2a-1)}{\Gamma(b-a)^2\Gamma(2b-1)}.$$

In this final, somewhat more canonical, form it is straight-forward to find natural generalizations in the literature, such as

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(c-s)}{\Gamma(b+s)\Gamma(d-s)} ds = \frac{\Gamma(a+c)\Gamma(b+d-a-c-1)}{\Gamma(b-a)\Gamma(d-c)\Gamma(b+d-1)},$$

which is proved in [21] using Parseval's formula.

4. NUMERICAL APPLICATIONS

4.1. Numerical evaluations of products. In this section, we consider general infinite products $\prod_{k=1}^{\infty} a(k)$, where $a(k)$ is not necessarily a rational function. The goal is to present a simple yet efficient way to obtain accurate numerical evaluations of such infinite products for certain $a(k)$, even when the original product converges very slowly. The approach is based on approximating $a(k)$ by a rational function and using Theorem 1.1 to express the result as a finite product of gamma functions.

We illustrate this approach for the Kepler–Bouwkamp constant [11, Sec. 6.3]

$$(12) \quad \prod_{k=3}^{\infty} \cos\left(\frac{\pi}{k}\right) = 0.1149420448532962\dots$$

This constant is motivated by a geometric construction. Start with a circle of unit radius and inscribe an equilateral triangle, inscribe the triangle with another circle which is then inscribed with a square, inscribe the square with yet another circle which is inscribed with a regular pentagon, and so on as in Figure 1. The radii of the inscribing circles then approach a limit which is the Kepler–Bouwkamp constant (12). For further references and the history of this constant, including various approaches to its numerical computation, we refer to [7], [11, Sec. 6.3], [13] and [28].

The product (12), however, converges rather slowly. For instance, truncating the product after 10^4 terms only results in four correct digits. The usual approach to computing the Kepler–Bouwkamp constant — taken, for instance, in [7] and [28] — is to first develop more rapidly converging expressions for (12). On the other hand, we will demonstrate how one can use (12) to evaluate the Kepler–Bouwkamp constant in a completely automated way to, say, 100 digits in a matter of a few seconds.

A natural choice for approximating a function $f(x)$, such as $\cos(\pi x)$, by a rational function is to use a Padé approximant [10, Chapter 4]. The Padé approximant of order $[m, n]$ is the rational function of numerator degree m and denominator degree n , whose Maclaurin series agrees with the one of $f(x)$ to order $m+n$ (the highest possible order). For our purposes we are interested only in Padé approximants of order $[n, n]$. For instance, the $[2, 2]$ Padé approximant of $\cos(x)$ is

$$r_2(x) = \frac{12 - 5x^2}{12 + x^2} = \cos(x) + O(x^6).$$

We can now approximate the product (12) by the corresponding product of rational functions, which, using Theorem 1.1, evaluates to a finite product of gamma

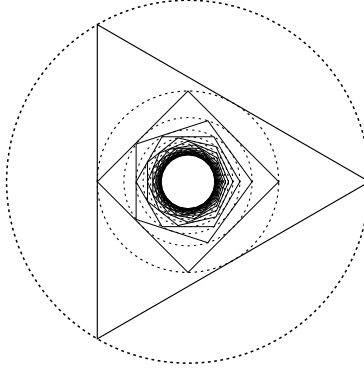


FIGURE 1. Kepler–Bouwkamp constant as ratio of inner circle and outer circle

functions:

$$\prod_{k=3}^{\infty} r_2\left(\frac{\pi}{k}\right) = \prod_{k=3}^{\infty} \frac{12k^2 - 5\pi^2}{12k^2 + \pi^2} = \frac{\Gamma\left(3 - \frac{i}{6}\sqrt{3}\pi\right) \Gamma\left(3 + \frac{i}{6}\sqrt{3}\pi\right)}{\Gamma\left(3 - \frac{1}{6}\sqrt{15}\pi\right) \Gamma\left(3 + \frac{1}{6}\sqrt{15}\pi\right)},$$

This approximation agrees with (12) to three decimal digits. More accurate approximations can be obtained by using the Padé approximant only for $k \geq N$, thus approximating (12) with

$$\left[\prod_{k=3}^{N-1} \cos\left(\frac{\pi}{k}\right) \right] \left[\prod_{k=N}^{\infty} r_2\left(\frac{\pi}{k}\right) \right] = \left[\prod_{k=3}^{N-1} \cos\left(\frac{\pi}{k}\right) \right] \frac{\Gamma\left(N - \frac{i}{6}\sqrt{3}\pi\right) \Gamma\left(N + \frac{i}{6}\sqrt{3}\pi\right)}{\Gamma\left(N - \frac{1}{6}\sqrt{15}\pi\right) \Gamma\left(N + \frac{1}{6}\sqrt{15}\pi\right)}.$$

For $N = 10$ this results in 6 correct digits, and 11 correct digits for $N = 100$. Of course, more accurate approximations are obtained if the order n of the Padé approximant is increased. Table 1 shows the number of correct decimal digits that one obtains for various modest choices of n and N (instead of truncating the entries of Table 1 to integers, we adopt the convention that two numbers A and B agree to $-\log_{10}|A - B|$ decimal digits). None of the computations took more than 5 seconds on a usual laptop using Mathematica 7 and without optimizing the computation.

N	3	4	5	10	100	1000
$n = 2$	3.19	4.00	4.57	6.22	11.3	16.3
$n = 4$	6.87	8.22	9.21	12.1	21.3	30.3
$n = 6$	11.2	13.1	14.5	18.7	31.9	45.0
$n = 8$	16.1	18.5	20.3	25.7	43.0	60.1
$n = 10$	21.4	24.3	26.5	33.1	54.5	75.5
$n = 12$	27.0	30.4	33.0	40.8	66.2	91.3
$n = 14$	32.9	36.8	39.7	48.8	78.3	107.
$n = 16$	39.0	43.4	46.7	57.0	90.5	124.

TABLE 1. Number of correct digits for various approximations to (12)

We note that this method for approximating an infinite product $\prod_{k=1}^{\infty} a(k)$ only relies on computing a Padé approximant for the factor $a(k)$, numerically finding

the zeros and poles of this rational function in order to apply Theorem 1.1, and numerically evaluating the resulting finite product of gamma functions. All these operations are efficiently and easily available in any computer algebra system. This makes the present approach a rather versatile tool in numerically evaluating a number of slowly converging infinite products.

Remark 4.1. Because of their special relevance in number theory it is natural to ask if the above numerical procedure can be applied to products indexed by primes, such as Artin's constant

$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p(p-1)}\right) \approx 0.373956.$$

However, no immediate transfer appears available. Fortunately, a very efficient method for numerically evaluating such products has been developed in [18]. We cannot resist to remark that, for Artin's constant, the corresponding product over all integers is given by

$$\prod_{n \geq 2} \left(1 - \frac{1}{n(n-1)}\right) = -\frac{1}{\pi} \cos\left(\frac{\sqrt{5}}{2}\pi\right) \approx 0.296675.$$

4.2. Numerical evaluations of sums. Note that series $\sum_{k=0}^{\infty} a_k$ are related to products via the obvious

$$(13) \quad \sum_{k=0}^{\infty} a_k = \log \left(\prod_{k=0}^{\infty} \exp(a_k) \right).$$

The approach for numerically evaluating infinite products may thus also be applied to certain series.

In case of the exponential function, explicit formulas exist [10, Example 4.2.2] for the Padé approximants of any order. In particular, letting

$$f_n(x) = \sum_{j=0}^n \frac{(2n-j)!n!}{(2n)!j!(n-j)!} x^j = \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{2n}{j}} \frac{x^j}{j!},$$

the $[n, n]$ Padé approximation of $\exp(x)$ is given by

$$\frac{f_n(x)}{f_n(-x)}.$$

In fact, as explained in the lovely article [9], these approximations to the exponential function are already implicit in Hermite's 1873 paper on the transcendence of e , and thus predate the systematic study of Padé approximations by Hermite's eponymous student Padé in his 1892 thesis.

For instance, in the case of the Riemann zeta function we obtain the approximations

$$(14) \quad \zeta(m) = \sum_{k=1}^{\infty} \frac{1}{k^m} \approx \log \left(\prod_{k=1}^{\infty} \frac{f_n(k^{-m})}{f_n(-k^{-m})} \right) =: \zeta_n(m)$$

by replacing the exponential function by its $[n, n]$ Padé approximation in (13). Table 2 gives some indication on the quality of this approximation (again, we say that two numbers A and B agree to $-\log_{10} |A - B|$ decimal digits). For instance, $\zeta_{10}(3)$ agrees with $\zeta(3)$ to 25 digits. Indeed, a few computations quickly suggest that, for

any integer $m \geq 2$, $\zeta_{10}(m)$ agrees with $\zeta(m)$ to 25 decimal digits. Analogously, the data of Table 2 (to the precision given) applies for any integer $m \geq 2$. This observation is, to some degree, explained next.

n	2	3	4	5	6	7	8	9	10
$m = 3$	2.83	4.99	7.39	9.99	12.8	15.6	18.7	21.8	25.0

TABLE 2. Number of digits of $\zeta_n(3)$ that agree with $\zeta(3)$

Clearly, we have $\zeta(m) \rightarrow 1$ as $m \rightarrow \infty$. On the other hand, we see from (14) that

$$\lim_{m \rightarrow \infty} \zeta_n(m) = \log \left(\frac{f_n(1)}{f_n(-1)} \right).$$

For instance,

$$(15) \quad \lim_{m \rightarrow \infty} \zeta_3(m) = \log \left(\frac{193}{71} \right) \approx 1.000010312,$$

which elucidates the entry $4.99 \approx -\log_{10} |1 - \log(193/71)|$ for $n = 3$ in Table 2. The fact that the rational numbers $f_n(1)/f_n(-1)$, including $19/7$ for $n = 2$ and $193/71$ for $n = 3$, are convergents to the continued fraction of Euler's number e is further discussed in [9].

5. SHORT GAMMA QUOTIENTS

In this section we discuss a few properties of gamma quotients whose arguments are rational numbers. While no closed forms are known for $\Gamma(1/n)$ when $n > 2$, it turns out that surprisingly short products of gamma functions at rational arguments have simple evaluations [33], [24], [6], [29], [17], [20]. For instance, as proposed in [12] and shown in [20],

$$(16) \quad \Gamma \left(\frac{1}{14} \right) \Gamma \left(\frac{9}{14} \right) \Gamma \left(\frac{11}{14} \right) = 4\pi^{3/2}.$$

Let $\Phi(n)$ denote the set of integers between 1 and n which are coprime to n . For example, $\Phi(12) = \{1, 5, 7, 11\}$. Let $\phi(n)$ represent the totient function, that is, the size of $\Phi(n)$. The general result we present in Theorem 5.1 below has been established in [24] and, independently, in [17] (a nice generalization is developed in [20], see Remark 5.3). In both cases, the proof is based on Möbius inversion. We offer an alternative proof which rests upon Lerch's identity [2, Theorem 1.3.4]

$$(17) \quad \log \Gamma(x) = \frac{\partial}{\partial s} \zeta(s, x) \Big|_{s=0} + \frac{1}{2} \log(2\pi),$$

where $\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}$ is the Hurwitz zeta function.

Theorem 5.1. *If n is not a prime power, then*

$$(18) \quad \prod_{k \in \Phi(n)} \Gamma \left(\frac{k}{n} \right) = (2\pi)^{\phi(n)/2}.$$

If $n = p^m$ for a prime p , then (18) holds with the right-hand side divided by \sqrt{p} .

Proof. Let χ be the principal character modulo n (that is, $\chi(k) = 1$ if k is coprime to n , and $\chi(k) = 0$ otherwise), and let p_1, \dots, p_r be the distinct prime factors of n . Then the Dirichlet L -function associated to χ differs from the Riemann zeta function

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

only in that the factors corresponding to the primes p_1, \dots, p_r are missing. In other words,

$$(19) \quad L(\chi, s) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s} = \zeta(s) \prod_{j=1}^r (1 - p_j^{-s}).$$

On the other hand,

$$n^s L(\chi, s) = \sum_{k=1}^n \chi(k) \zeta\left(s, \frac{k}{n}\right).$$

Taking the derivative with respect to s and applying Lerch's identity (17), we have

$$\sum_{k=1}^n \chi(k) \log \Gamma\left(\frac{k}{n}\right) = \log(n) L(\chi, 0) + L'(\chi, 0) + \frac{1}{2} \phi(n) \log(2\pi).$$

It follows from (19) that $L(\chi, 0) = 0$. Similarly, it follows that $L'(\chi, 0) = 0$ provided that $r > 1$. This proves equation (18) unless n is a prime power. Lastly, if $n = p^m$ for some prime p , then

$$L'(\chi, 0) = \log(p) \zeta(0) = -\frac{1}{2} \log(p),$$

which proves (18) in the remaining case. \square

Example 5.2. In the case $n = 14$, we find

$$\Gamma\left(\frac{1}{14}\right) \Gamma\left(\frac{3}{14}\right) \Gamma\left(\frac{5}{14}\right) \Gamma\left(\frac{9}{14}\right) \Gamma\left(\frac{11}{14}\right) \Gamma\left(\frac{13}{14}\right) = (2\pi)^3.$$

In light of (16), it follows that

$$\Gamma\left(\frac{3}{14}\right) \Gamma\left(\frac{5}{14}\right) \Gamma\left(\frac{13}{14}\right) = 2\pi^{3/2},$$

which, as in the case of (16), also follows from the results of [20], which generalize Theorem 5.1 and are outlined in Remark 5.3. Combining both products and using Theorem 1.1, one therefore obtains

$$\prod_{k \geq 0} \frac{(k + 3/14)(k + 5/14)(k + 13/14)}{(k + 1/14)(k + 9/14)(k + 11/14)} = 2.$$

Remark 5.3. Note that $\Phi(n)$, the set of integers between 1 and n which are coprime to n , is a group with respect to multiplication modulo n . Assume that $n > 1$ is odd, and let A be (a coset of) the cyclic subgroup of $\Phi(2n)$ generated by $n + 2$. It is shown in [20] that

$$\prod_{k \in A} \Gamma\left(\frac{k}{2n}\right) = 2^{b(A)} \pi^{|A|/2},$$

with $|A|$ denoting the cardinality of A and $b(A)$ the number of elements in A which exceed n . In particular, this lets us construct lots of identities as in Example 5.2. To wit, for any A as above,

$$\prod_{k \in A} \frac{\Gamma\left(\frac{k}{2n}\right)}{\Gamma\left(1 - \frac{k}{2n}\right)} = 2^{2b(A) - |A|}.$$

For instance, with $n = 31$, one choice is $A = \{1, 33, 35, 39, 47\}$ and we find

$$\prod_{k \geq 0} \frac{(k + 15/62)(k + 23/62)(k + 27/62)(k + 29/62)(k + 61/62)}{(k + 1/62)(k + 33/62)(k + 35/62)(k + 39/62)(k + 47/62)} = 8.$$

Remark 5.4. Equation (18) may be used to produce the identity

$$\begin{aligned} \frac{\phi(n)}{2} \log(2\pi) &= \sum_{x \in \Phi(n)} \log \Gamma(x/n) \\ &= \sum_{x \in \Phi(n)} \log \Gamma(1 - x/n) \\ &= \sum_{x \in \Phi(n)} \left[\gamma \frac{x}{n} + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} (x/n)^k \right] \\ &= \gamma \frac{\phi(n)}{2} + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \sum_{x \in \Phi(n)} (x/n)^k, \end{aligned}$$

where

$$\gamma = \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k}$$

is Euler's constant. This may be re-arranged to obtain

$$(20) \quad \frac{\log(2\pi) - \gamma}{2} = \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \frac{1}{\phi(n)} \sum_{x \in \Phi(n)} (x/n)^k,$$

again valid for n which are not a prime power. Note that the expression on the left-hand side of (20) has the interesting property that it is independent of n . The sums

$$\Psi_k(n) := \sum_{x \in \Phi(n)} x^k$$

have been studied in [26], where, among other results, it is shown that

$$\Psi_k(n) = \frac{n^{k+1}}{k+1} \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k+1}{2m} \frac{B_{2m}}{n^{2m}} \prod_{p|n} (1 - p^{2m-1}).$$

Remark 5.5. We note that the proof of Theorem 5.1 naturally extends to certain more general gamma quotients. For instance, consider the real character $\chi(n) = \left(\frac{-d}{n}\right)$ where $-d < 0$ is a negative fundamental discriminant (fundamental discriminants are squarefree integers congruent to 1 modulo 4, or multiples by -4 or -8 of such numbers) with associated class number $h = h(-d)$. Then

$$(21) \quad w\zeta(s)L(\chi, s) = \sum_{j=1}^h Z_{Q_j}(s),$$

where w is the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, Q_1, \dots, Q_h are non-equivalent reduced binary quadratic forms with discriminant $-d$, and Z_Q denotes the Epstein zeta function

$$Z_Q(s) = \sum_{n,m} \frac{1}{Q(m,n)^s},$$

with the sum extending over all integers n, m such that $(n, m) \neq (0, 0)$. The *Kronecker limit formula* — see [19] for a nice proof — shows that

$$Z_Q(s) = \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \dots,$$

with $a_{-1} = 2\pi/\sqrt{d}$ and

$$a_0 = \frac{4\pi}{\sqrt{d}}(\gamma - \log(d^{1/4}\sqrt{2y}|\eta(z)|^2)),$$

where $z = x + iy$ is the solution to $Q(z, 1) = 0$ in the upper half-plane, and $\eta(\tau)$ is the Dedekind eta function. We now expand both sides of (21) around $s = 1$ and equate the constant terms. For the right-hand side we employ the Kronecker limit formula, while for the left-hand side we proceed as in the proof of Theorem 5.1. Using $L(\chi, 0) = \frac{2h}{w}$, this eventually yields

$$(22) \quad \prod_{m=1}^d \Gamma\left(\frac{m}{d}\right)^{\binom{-d}{m}} = \left[\prod_{j=1}^h 4\pi\sqrt{dy_j}|\eta(z_j)|^4 \right]^{2/w}.$$

This is the well-known Chowla–Selberg formula [25]. We note that $w = 2$ unless $d = -3$, in which case $w = 6$, or $d = -4$, in which case $w = 4$. Generalizations of the Chowla–Selberg formula exist, for instance, to arbitrary negative discriminants and to genera of binary quadratic forms; we refer to [15] and the references therein. Finally, we remark that, for fixed d , the individual eta values occurring in (22) differ only by an algebraic factor.

Remark 5.6. Following [16], a *period* is a number which is the value of an integral of an algebraic function over an algebraic domain. For example, as noted in the introduction to [16], the numbers $\Gamma(p/q)^q$ are periods because they may be represented as beta integrals. More generally, let $\alpha_1, \dots, \alpha_n$ be rational numbers. Then $\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)$ is a period whenever $\Gamma(\alpha_1 + \dots + \alpha_n)$ is a period. To see this, note that

$$\Gamma(\alpha_1) \cdots \Gamma(\alpha_n) = B(\alpha_1, \alpha_2)B(\alpha_1 + \alpha_2, \alpha_3) \cdots B(\alpha_1 + \dots + \alpha_{n-1}, \alpha_n)\Gamma(\alpha_1 + \dots + \alpha_n),$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt$$

is the beta function. Due to the integral representation its values for rational α, β are periods. It now suffices to observe that periods form a ring.

A similar argument shows that gamma quotients arising from infinite products of rational functions with rational roots are always a quotient of two periods.

Finally, we remark that, while the ring of periods is countable, it is an open problem, [16, Problem 3], to exhibit at least one number which is provably not a period. For instance, it would be a surprise to many if $1/\pi$ were a period.

6. PRODUCTS INVOLVING THE THUE–MORSE SEQUENCE

The Thue–Morse sequence t_j is defined by $t_j = 1$ if the number of ones in the binary representation of j is odd and $t_j = 0$ otherwise. For further information on this sequence and its occurrences in various contexts a beautiful reference is [1]. Let $p(j) = (-1)^{t_j}$. Then Theorem 1.1 implies that, for $m > 1$,

$$(23) \quad \prod_{k \geq 1} \prod_{j=0}^{2^m-1} (k+j)^{p(j)} = \prod_{j=0}^{2^m-1} (j!)^{-p(j)}.$$

As we will see below, the right-hand side of (23) further simplifies.

Remark 6.1. Note that $p(j) = 1$ for exactly half of the 2^m many numbers $j = 0, 1, \dots, 2^m - 1$, so that the left-hand side of (23) indeed converges when $m > 1$. Denote with S_1 the set of integers among $0, 1, \dots, 2^m - 1$ with $p(j) = 1$. Similarly, S_{-1} consists of those with $p(j) = -1$. Not only are the two sets equinumerous, but also $\sum_{j \in S_1} j = \sum_{j \in S_{-1}} j$ and, in fact, as discovered by Prouhet in 1851, see [32] or [1],

$$\sum_{j \in S_1} j^n = \sum_{j \in S_{-1}} j^n$$

for all $n = 0, 1, \dots, m - 1$.

Example 6.2. For instance, when $m = 3$ then $0^n + 3^n + 5^n + 6^n = 1^n + 2^n + 4^n + 7^n$ for $n = 0, 1, 2$ as well as

$$\prod_{k \geq 1} \frac{k(k+3)(k+5)(k+6)}{(k+1)(k+2)(k+4)(k+7)} = \frac{2!4!7!}{3!5!6!} = \frac{7}{3 \cdot 5}.$$

In the next lemma, we observe that the right-hand side of (23) always simplifies as it did in Example 6.2.

Lemma 6.3. For integers $m > 1$,

$$(24) \quad \prod_{k \geq 1} \prod_{j=0}^{2^m-1} (k+j)^{p(j)} = \prod_{j=0}^{2^{m-1}-1} (2j+1)^{p(j)}.$$

Proof. Note that $p(2j)$ and $p(2j+1)$ are always of opposite sign. Thus,

$$\prod_{j=0}^{2^m-1} (j!)^{-p(j)} = \prod_{j=0}^{2^{m-1}-1} \left[\frac{(2j)!}{(2j+1)!} \right]^{p(2j+1)}.$$

It only remains to use that $p(2j+1) = -p(j)$, which follows from the definition of the Thue–Morse sequence. \square

Example 6.4. Combining Example 6.2 with the corresponding product for $m = 4$, one finds

$$\prod_{k \geq 1} \frac{(k+9)(k+10)(k+12)(k+15)}{(k+8)(k+11)(k+13)(k+14)} = \frac{11 \cdot 13}{9 \cdot 15}.$$

From computing a few more instances, we are led to observe that both sides of (24) appear to approach $1/2$ as $m \rightarrow \infty$. In other words,

$$(25) \quad \lim_{m \rightarrow \infty} \prod_{j=0}^{2^m-1} (2j+1)^{p(j)} = \frac{1}{2}.$$

To see that this is indeed true, it is of advantage to also consider

$$(26) \quad f_m(x) = \prod_{j=0}^{2^m-1} (x+j)^{p(j)}.$$

This product arises in

$$\prod_{k \geq 0} \prod_{j=0}^{2^m-1} (x+k+j)^{p(j)} = \prod_{j=0}^{2^{m-1}-1} (x+2j)^{p(j)} = f_m\left(\frac{1}{2}\right),$$

which is a natural extension of (24) and can be proved in the same way.

Now the truth of (25) can be seen from the solution [23] to the problem [31] proposed by Woods. Indeed, denoting with $f(x)$ the limit of $f_m(x)$ as $m \rightarrow \infty$, we need to show that $f(1/2) = 1/2$. Note that

$$f_m\left(x + \frac{1}{2}\right) = \prod_{j=0}^{2^m-1} \left(x + \frac{1}{2} + j\right)^{p(j)} = \prod_{j=0}^{2^m-1} (2x + 2j + 1)^{-p(2j+1)}.$$

Proceeding as in [23], we thus find that

$$(27) \quad f_{m+1}(2x)f_m\left(x + \frac{1}{2}\right) = f_m(x).$$

With appropriate care one may now apply L'Hôpital's rule to obtain

$$f\left(\frac{1}{2}\right) = \lim_{x \rightarrow 0} \frac{f(x)}{f(2x)} = \frac{f'(0)}{2f'(0)} = \frac{1}{2},$$

as desired.

Remark 6.5. Combining terms as in the proof of Lemma 6.3, we note that

$$(28) \quad f(x) = \lim_{m \rightarrow \infty} \prod_{j=0}^{2^m-1} (x+j)^{p(j)} = \prod_{j=0}^{\infty} \left(\frac{2j+x}{2j+x+1}\right)^{p(j)},$$

because the infinite product converges. It further follows from (27) that $f(1)^2 = f(1/2)$ and hence $f(1) = 1/\sqrt{2}$. In light of (28) this result is equivalently expressed as

$$(29) \quad P = \prod_{j=0}^{\infty} \left(\frac{2j+1}{2j+2}\right)^{p(j)} = \frac{1}{\sqrt{2}},$$

which is the evaluation asked for in problem [31]. A beautiful alternative solution, avoiding analytic tools such as L'Hôpital's rule, is given in [1]. The clever alternative proof considers, besides P , the product

$$Q = \prod_{j=1}^{\infty} \left(\frac{2j}{2j+1}\right)^{p(j)},$$

and shows that

$$\begin{aligned}
 PQ &= \frac{1}{2} \prod_{j=1}^{\infty} \left(\frac{j}{j+1} \right)^{p(j)} \\
 &= \frac{1}{2} \prod_{j=1}^{\infty} \left(\frac{2j+1}{2j+2} \right)^{p(2j+1)} \prod_{j=1}^{\infty} \left(\frac{2j}{2j+1} \right)^{p(2j)} \\
 &= \frac{1}{2} P^{-1} Q.
 \end{aligned}$$

Cancelling Q , one has again derived $P = 1/\sqrt{2}$. On the other hand, the quantity Q is much more mysterious. It is not even known whether Q is irrational, let alone transcendental. Jeffrey Shallit has offered \$25 for an answer to this question.

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