

# Syntomic cohomology and Beilinson's Tate conjecture for $K_2$

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# 1 Introduction

The Tate conjecture for Chow groups is one of the most important question in arithmetic geometry. It appears in various areas explicitly or implicitly and often plays a key role. Although there have been significant progresses on this conjecture (e.g., for abelian varieties), very few things are known about its analogue for higher  $K$ -theory raised by Beilinson (cf. [Ja] 5.19). In this paper we are mainly concerned with the Tate conjecture for  $K_2$ , which asserts that for a nonsingular variety  $U$  over a number field  $F$  (or more generally a finitely generated field over a prime field) the étale chern class map

$$c_{\text{ét}} : K_2(U) \otimes \mathbb{Q}_p \longrightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F}$$

is surjective. Here  $G_F := \text{Gal}(\overline{F}/F)$  is the absolute Galois group,  $\overline{U}$  denotes  $U \otimes_F \overline{F}$  and the superscript  $G_F$  denotes the fixed part by  $G_F$ .

In this paper, we focus on an elliptic surface  $\pi : X \rightarrow C$  over a  $p$ -adic local field  $K$  which is absolutely unramified. Let  $D = \sum D_i$  be the sum of the split multiplicative fibers of  $\pi$ , and put  $U = X - D$ . Assume that  $X$  and  $C$  have projective smooth models  $\mathcal{X}$  and  $\mathcal{C}$  over the integer ring  $R$  of  $K$ , respectively, and that the closure  $\mathcal{D} \subset \mathcal{X}$  of  $D$  has normal crossings. Put  $\mathcal{U} := \mathcal{X} - \mathcal{D}$ . Then we introduce a space of *formal Eisenstein series*

$$\text{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \subset \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D}))$$

(see §5.2 for details), where the right hand side is the space of global 2-forms with log poles along  $\mathcal{D}$ . One of our main results asserts that

$$\text{Im}(H_{\text{syn}}^2(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \rightarrow \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D}))) \subset \text{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}, \quad (1.0.1)$$

where  $H_{\text{syn}}^*(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2))$  denotes the syntomic cohomology of  $\mathcal{X}$  with log poles along  $\mathcal{D}$  due to Kato and Tsuji. From this result, we will further deduce inequalities

$$\dim_{\mathbb{Q}_p} c_{\text{ét}}(K_2(U) \otimes \mathbb{Q}_p) \leq \dim_{\mathbb{Q}_p} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F} \leq \text{rank}_{\mathbb{Z}_p} \text{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \quad (1.0.2)$$

using the fact that the Fontaine-Messing map  $H_{\text{syn}}^2(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \rightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K}$  is surjective (§5.3). The inclusion (1.0.1) is an extension of Beilinson's theorem on Eisenstein symbols in the following sense. When  $\pi : X \rightarrow X(\Gamma)$  is the universal family of elliptic curves over a modular curve  $X(\Gamma)$ , the space  $E(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$  consists of the (usual) Eisenstein series of weight 3 ([A2] §8.3). Beilinson proved that it is spanned by the dlog image of the Eisenstein symbols in  $K_2(U)$  ([Be]), so that

$$\text{Im}(K_2(U) \otimes \mathbb{Q}_p \rightarrow \Gamma(X, \Omega_{X/K}^2(D))) = E(\mathcal{X}, \mathcal{D})_{\mathbb{Q}_p} = \mathbb{Q}_p^{\# \text{ of cusps}}$$

under our notation. We can thus regard (1.0.1) as a partial extension of Beilinson's theorem to arbitrary elliptic surfaces. A distinguished feature is that it gives a new upper bound of the rank of  $H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F}$  in view of the fact that the rank of  $E(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$  is strictly less than the number of split multiplicative fibers in some cases (§6, see also [ASat] §5). The proof of (1.0.1) is quite different from that of the theorem of Beilinson. A key ingredient is the  $p$ -adic Hodge theory, in particular a detailed computation on the syntomic cohomology of Tate curves over 2-dimensional complete local rings.

The inequality (1.0.2) is a key tool in our application to Beilinson's Tate conjecture for  $K_2$ . In fact, if one can construct enough elements in  $K_2(U)$  (e.g. by symbols) so that the dimension of  $c_{\text{ét}}(K_2(U))$  is equal to the upper bound, then the equalities hold in (1.0.2) and one can conclude that  $c_{\text{ét}}$  is surjective. In §6.1 we will give a number of such examples. It is remarkable that as a consequence one has non-trivial elements in the *Selmer group*  $H_f^1(G_K, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2))/\text{NS}(\overline{X}))$  of Bloch-Kato [BK2] with large rank (§6.2).

In §7, we present another application of (1.0.1) to the finiteness of torsion 0-cycles. It is a folklore conjecture that  $\text{CH}^m(X)$  is finitely generated  $\mathbb{Z}$ -module for a projective smooth variety  $X$  over a number field, which is a widely open problem unless  $m = 1$ . The finiteness of torsion part  $\text{CH}^m(X)_{\text{tor}}$  supports the question. When the base field is a  $p$ -adic local field, the Chow group is no longer finitely generated. Although most people had believed that the finiteness of torsion cycles remains true even for  $p$ -adic local fields, counter-examples were found recently ([RSr], [ASai]). On the other hand all such examples are *not* defined over number fields, i.e. the minimal fields of definition are not number fields. Therefore we are naturally lead to the following modified question:

*Let  $X_0$  be a projective nonsingular variety over a number field  $F$ . Let  $K$  be a completion of  $F$  at a finite place  $v$  and put  $X = X_0 \otimes_F K$ . Then is  $\text{CH}^m(X)_{\text{tors}}$  finite?*

It is in fact a crucial question whether the  $p$ -primary torsion part  $\text{CH}^m(X)\{p\}$  is finite. When  $m = 2$ , the finiteness of  $\text{CH}^2(X)\{p\}$  is reduced to the study of the  $p$ -adic regulator on  $K_1$  by a recent work of Saito and the second author [SS1], that is,  $\text{CH}^2(X)\{p\}$  is finite if the  $p$ -adic regulator map

$$\varrho : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H_g^1(G_K, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2)))$$

is surjective onto the  $g$ -part of Bloch-Kato [BK2] (3.7). When  $H^2(X, \mathcal{O}_X) = 0$ , this map is well-known to be surjective even when the minimal field of definition is not a number field (cf. [CTR1], [CTR2], [S] 3.6). However when  $H^2(X, \mathcal{O}_X) \neq 0$ , the question becomes more

difficult, and nobody has found an affirmative or negative example so far. In fact, the Selmer group  $H_f^1(G_K, H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2))/\text{NS}(\overline{X}))$  of Bloch-Kato is no longer zero (cf. Lemma 7.2.2), so one needs to construct *integral indecomposable* elements of  $K_1$  generating the Selmer group over  $\mathbb{Q}_p$ , which is a crucial difficulty there. We achieve it in the same way as in §6.2 and give the first example of a surface  $X$  over a  $p$ -adic local field with  $H^2(X, \mathcal{O}_X) \neq 0$  such that the  $p$ -adic regulator map  $\varrho$  is surjective and hence the torsion of  $\text{CH}^2(X)$  is finite (Theorem 7.0.3, Corollary 7.0.5, §7.3).

This paper is organized as follows. In §2, we review and fix the notation for (log) syntomic cohomology and Tate curves. In §3, we state the main result on Tate curves (Theorem 3.2.3) and prove it admitting a key commutative key diagram. §4 is devoted to the proof of the key diagram. In §5 we prove the main results on elliptic surfaces over  $p$ -adic fields. In §6, we apply them to the study of Beilinson's Tate conjecture for  $K_2$ . In §7 we give an example of elliptic  $K3$  surface over  $\mathbb{Q}_p$  with finitely many torsion 0-cycles.

## 2 Preliminaries

For a scheme  $X$  over a ring  $A$  and an  $A$ -algebra  $B$  we write  $X_B := X \times_A B$ . For an integer  $n$  which is invertible on  $X$ ,  $\mathbb{Z}/n(1)$  denotes the étale sheaf  $\mu_n$  of  $n$ -th roots of unity. We often write  $\mathbb{Z}/n(m)$  ( $m \in \mathbb{N}$ ) for the étale sheaf  $\mu_n^{\otimes m}$ . For a function  $f \in \Gamma(X, \mathcal{O}_X)$  which is not a zero divisor, we put

$$X[f^{-1}] := \text{Spec}(\mathcal{O}_X[T]/(fT - 1)),$$

which is the maximal open subset of  $X$  where  $f$  is invertible.

### 2.1 Syntomic cohomology

Let  $p$  be a prime number. For a scheme  $T$ , we put

$$T_n := T \otimes \mathbb{Z}/p^n.$$

**Definition 2.1.1** *Let  $T$  be a scheme.*

- (1) *A morphism  $\varphi : T \rightarrow T$  over  $\mathbb{F}_p$  is called the absolute Frobenius endomorphism, if the underlying morphism of topological spaces is the identity map and the homomorphism  $\varphi^* : \mathcal{O}_T \rightarrow \varphi_* \mathcal{O}_T = \mathcal{O}_T$  sends  $x \mapsto x^p$ . Here the equality  $\varphi_* \mathcal{O}_T = \mathcal{O}_T$  means the natural identification.*
- (2) *A morphism  $\varphi : T \rightarrow T$  over  $\mathbb{Z}_p$  is called a Frobenius endomorphism, if  $\varphi \otimes \mathbb{Z}/p : T_1 \rightarrow T_1$  is the absolute Frobenius endomorphism.*

Let  $X$  be a scheme which is flat over  $\mathbb{Z}_p$ . Assume the following condition:

**Condition 2.1.2** *There exists a closed immersion  $X \hookrightarrow Z$  satisfying the following conditions (cf. [Ka4] 2.4):*

- (1)  *$Z_n$  has  $p$ -bases over  $\mathbb{Z}/p^n$  locally for any  $n \geq 1$  (loc. cit. Definition 1.3).*
- (2)  *$Z$  has a Frobenius endomorphism, or more weakly,  $Z_n$  has a Frobenius endomorphism  $\varphi_n$  for each  $n \geq 1$  and the morphism  $\varphi_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$  agrees with  $\varphi_n$  for any  $n \geq 1$ .*
- (3) *Let  $D_n$  be the PD-envelope of  $X_n$  in  $Z_n$  compatible with the canonical PD-structure on the ideal  $(p) \subset \mathbb{Z}/p^n$ . For  $i \geq 1$ , let  $J_{D_n}^{[i]} \subset \mathcal{O}_{D_n}$  be the  $i$ -th divided power of the ideal  $J_{D_n} := \text{Ker}(\mathcal{O}_{D_n} \rightarrow \mathcal{O}_{X_n})$ . For  $i \leq 0$ , we put  $J_{D_n}^{[i]} := \mathcal{O}_{D_n}$ . Then the following sequence is exact for any  $m, n \geq 1$  and any  $i \geq 0$ :*

$$J_{D_{m+n}}^{[i]} \xrightarrow{\times p^m} J_{D_{m+n}}^{[i]} \xrightarrow{\times p^n} J_{D_{m+n}}^{[i]} \longrightarrow J_{D_n}^{[i]} \longrightarrow 0. \quad (2.1.3)$$

We give some examples of  $Z$  which satisfy the condition (1):

**Example 2.1.4** Let  $W = W(k)$  be the Witt ring of a perfect field  $k$ , and let  $W[[t]]$  be the formal power series ring over  $W$  with  $t$  an indeterminate. Then  $W_n[[t]] := W[[t]]/(p^n)$  has a  $p$ -basis over  $\mathbb{Z}/p^n$  for any  $n \geq 1$ . Indeed,  $W_n[[t]]$  is flat over  $\mathbb{Z}/p^n$  and  $t \in k[[t]]$  is a  $p$ -basis of  $k[[t]]$  over  $\mathbb{F}_p$ . Therefore  $t \in W_n[[t]]$  is a  $p$ -basis over  $\mathbb{Z}/p^n$  by [Ka3] Proposition 1.4. More generally,  $Z_n$  has  $p$ -bases locally over  $\mathbb{Z}/p^n$  in the following cases:

- (i)  $Z$  is smooth of finite type over  $W[[t]]$ .
- (ii)  $Z$  is flat of finite type over  $W[[t]]$  and  $Z_1$  is a regular semistable family over  $k[[t]]$ .

The following fact provides a sufficient condition for  $X \hookrightarrow Z$  to satisfy the condition (3):

**Proposition 2.1.5 (Fontaine-Messing / Kato [Ka4] Lemma 2.1)** Let  $X \hookrightarrow Z$  be a closed immersion satisfying the conditions (1) and (2) in Condition 2.1.2. Assume that  $Z$  is locally noetherian, and  $\text{Ker}(\mathcal{O}_{Z,x} \rightarrow \mathcal{O}_{X,x})$  is generated by an  $\mathcal{O}_{Z,x}$ -regular sequence for any  $x \in X_1$ . Then  $X \hookrightarrow Z$  satisfies the condition (3) in Condition 2.1.2.

For  $0 \leq r \leq p-1$ , we define the syntomic complex  $\mathcal{S}_n(r)_{X,Z}$  (with respect to the embedding  $X \hookrightarrow Z$ ) as follows. Let  $\mathbb{J}_{n,X,Z}^{[r]}$  be the complex of sheaves on  $(X_1)_{\text{ét}}$

$$J_{D_n}^{[r]} \xrightarrow{d} J_{D_n}^{[r-1]} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^1 \xrightarrow{d} \cdots \xrightarrow{d} J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^q \xrightarrow{d} \cdots,$$

where  $J_{D_n}^{[r]}$  is placed in degree 0. Put  $\mathbb{E}_{n,X,Z} := \mathbb{J}_{n,X,Z}^{[0]}$ . By the assumption that  $0 \leq r \leq p-1$ , the Frobenius endomorphism on  $Z_{n+r}$  induces a homomorphism of complexes

$$f_r := \overline{p^{-r} \cdot \varphi_{n+r}^*} : \mathbb{J}_{n,X,Z}^{[r]} \longrightarrow \mathbb{E}_{n,X,Z}$$

(see [Ka4] p. 411 for details).

**Definition 2.1.6** For  $0 \leq r \leq p-1$ , we define the complex  $\mathcal{S}_n(r)_{X,Z}$  on  $(X_1)_{\text{ét}}$  as the mapping fiber of

$$1 - f_r : \mathbb{J}_{n,X,Z}^{[r]} \longrightarrow \mathbb{E}_{n,X,Z}.$$

More precisely, the degree  $q$ -part of  $\mathcal{S}_n(r)_{X,Z}$  is

$$(J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^q) \oplus (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n}^{q-1})$$

and the differential operator is given by

$$(x, y) \longmapsto (dx, x - f_r(x) - dy).$$

We define the syntomic cohomology of  $X$  with coefficients in  $\mathcal{S}_n(r)$  as the hypercohomology groups of this complex:

$$H_{\text{syn}}^*(X, \mathcal{S}_n(r)) := \mathbb{H}_{\text{ét}}^*(X, \mathcal{S}_n(r)_{X,Z}).$$

This notation is well-defined, because the image of the complex  $\mathcal{S}_n(r)_{X,Z}$  in the derived category is independent of  $X \hookrightarrow Z$  as in Condition 2.1.2 ([Ka4] p. 412). See also Remark 2.2.8 below.

**Proposition 2.1.7** For  $m, n \geq 1$ , there is an exact sequence of complexes on  $(X_1)_{\text{ét}}$

$$\mathcal{S}_{m+n}(r)_{X,Z} \xrightarrow{\times p^m} \mathcal{S}_{m+n}(r)_{X,Z} \xrightarrow{\times p^n} \mathcal{S}_{m+n}(r)_{X,Z} \longrightarrow \mathcal{S}_n(r)_{X,Z} \longrightarrow 0.$$

Consequently, there is a short exact sequence

$$0 \longrightarrow \mathcal{S}_m(r)_{X,Z} \xrightarrow{p^n} \mathcal{S}_{m+n}(r)_{X,Z} \longrightarrow \mathcal{S}_n(r)_{X,Z} \longrightarrow 0.$$

*Proof.* The assertion follows from the exactness of (2.1.3) and the fact that  $\Omega_{Z_n}^q$  is free over  $\mathbb{Z}/p^n$  ([Ka4] Lemma 1.8).  $\square$

**Remark 2.1.8** Assume that the identity map  $X \rightarrow X$  satisfies the conditions (1)–(3) in Condition 2.1.2. Then we have  $D_n = X_n$ , i.e.,

$$\mathbb{J}_{n,X,X}^{[r]} = \Omega_{X_n}^{\bullet \geq r} \quad \text{and} \quad \mathbb{E}_{n,X,X} = \Omega_{X_n}^{\bullet},$$

and there is a short exact sequence of complexes

$$0 \longrightarrow \mathcal{S}_n(r)_{X,X} \longrightarrow \Omega_{X_n}^{\bullet \geq r} \xrightarrow{1-f_r} \Omega_{X_n}^{\bullet} \longrightarrow 0. \quad (2.1.9)$$

**Remark 2.1.10** Syntomic complexes can be defined in a more general situation by a gluing argument in the derived category (see [Ka1] Remark 1.8 and [Ka4] Lemma 2.2).

## 2.2 Syntomic cohomology with log poles

The notation remain as in §2.1. The aim of this subsection is to define of the syntomic cohomology of a regular scheme  $X$  with log poles along a simple normal crossing divisor  $\mathcal{D}$  following Tsuji ([Ts2] §2), which is necessary to formulate Theorem 3.2.3 below. The case that  $\mathcal{D}$  is empty corresponds to the syntomic cohomology defined in §2.1. To give precise arguments, we will use the terminology in log geometry [Ka3].

**Definition 2.2.1** Let  $(T, M_T)$  be a log scheme.

- (1) ([Ka3] Definition 4.7) A morphism  $\varphi : (T, M_T) \rightarrow (T, M_T)$  over  $\mathbb{F}_p$  is called the *absolute Frobenius endomorphism*, if the underlying morphism  $T \rightarrow T$  is the absolute Frobenius endomorphism in the sense of Definition 2.1.1 (1) and the homomorphism  $\varphi^* : M_T \rightarrow \varphi_* M_T = M_T$  is the multiplication by  $p$ . Here the equality  $\varphi_* M_T = M_T$  means the natural identification obtained from the fact that the underlying morphism of  $\varphi$  of topological spaces is the identity map.
- (2) A morphism  $\varphi : (T, M_T) \rightarrow (T, M_T)$  over  $\mathbb{Z}_p$  is called a *Frobenius endomorphism*, if  $\varphi \otimes \mathbb{Z}/p : (T_1, M_{T_1}) \rightarrow (T_1, M_{T_1})$  is the absolute Frobenius endomorphism. Here  $M_{T_1}$  denotes the inverse image log structure of  $M_T$  onto  $T_1$  (loc. cit. (1.4)).

Let  $(X, M)$  be a fine log scheme such that  $X$  is flat over  $\mathbb{Z}_p$  (as a usual scheme). The main example we are concerned with is the following case:

**Example 2.2.2** *Let  $X$  be a regular scheme which is flat over  $\mathbb{Z}_p$ . Let  $\mathcal{D}$  be a simple normal crossing divisor on  $X$ , which may be empty. Put  $U := X - \mathcal{D}$  and let  $j : U \hookrightarrow X$  be the natural open immersion. We define the sheaf  $M$  of monoids on  $X_{\text{ét}}$  as*

$$M := \mathcal{O}_X \cap j_* \mathcal{O}_U^\times.$$

The canonical map  $M \rightarrow \mathcal{O}_X$  is a fine log structure on  $X$ , which we call the log structure associated with  $\mathcal{D}$ .

For  $n \geq 1$ , we write  $M_n$  for the inverse image log structure of  $M$  onto  $X_n$ . We assume the following condition, which is a logarithmic variant of Condition 2.1.2:

**Condition 2.2.3** *There exist exact closed immersions  $i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$  of log schemes for  $n \geq 1$  which satisfy the following conditions for all  $n \geq 1$  (cf. [Ts2] §2):*

(0)  $(Z_n, M_{Z_n})$  is fine. We have

$$(Z_n, M_{Z_n}) \xrightarrow{\sim} (Z_{n+1}, M_{Z_{n+1}}) \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$$

as log schemes, and the morphism  $i_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$  agrees with  $i_n$ .

- (1)  $(Z_n, M_{Z_n})$  has  $p$ -bases locally over  $\mathbb{Z}/p^n$  with the trivial log structure (loc. cit. Definition 1.4).
- (2)  $(Z_n, M_{Z_n})$  has a Frobenius endomorphism  $\varphi_n$  and the morphism  $\varphi_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n$  agrees with  $\varphi_n$ .
- (3) Let  $(D_n, M_{D_n})$  be the PD-envelope of  $(X_n, M_n)$  in  $(Z_n, M_{Z_n})$  which is compatible with the canonical PD-structure on the ideal  $(p) \subset \mathbb{Z}/p^n$  ([Ka3] Definition (5.4)). For  $i \geq 1$ , let  $J_{D_n}^{[i]} \subset \mathcal{O}_{D_n}$  be the  $i$ -th divided power of the ideal  $J_{D_n} = \text{Ker}(\mathcal{O}_{D_n} \rightarrow \mathcal{O}_{X_n})$ . For  $i \leq 0$ , we put  $J_{D_n}^{[i]} := \mathcal{O}_{D_n}$ . Then the following sequence is exact for any  $m, n \geq 1$  and any  $i \geq 0$ :

$$J_{D_{m+n}}^{[i]} \xrightarrow{\times p^m} J_{D_{m+n}}^{[i]} \xrightarrow{\times p^n} J_{D_{m+n}}^{[i]} \longrightarrow J_{D_n}^{[i]} \longrightarrow 0. \quad (2.2.4)$$

We give some examples of  $(Z_n, M_{Z_n})$  which satisfy the condition (1):

**Example 2.2.5** *Let  $W[[t]]$  be as in Example 2.1.4. We endow  $\text{Spec}(W[[t]])$  with a pre-log structure  $\mathbb{N} \rightarrow W[[t]]$  by sending 1 to  $t$ , and write  $N$  for the associated log structure on  $\text{Spec}(W[[t]])$ . Then  $(W_n[[t]], N_n)$  has a  $p$ -basis over  $\mathbb{Z}/p^n$  for any  $n \geq 1$ , which one can check in a similar way as in Example 2.1.4. More generally, a fine log scheme  $(Z_n, M_{Z_n})$  which is log smooth over  $(W_n[[t]], N_n)$  has  $p$ -bases locally over  $\mathbb{Z}/p^n$ . Indeed,  $(Z_n, M_{Z_n})$  has  $p$ -bases locally over  $(W_n[[t]], N_n)$  by [Ts2] Lemma 1.5. Hence  $(Z_n, M_{Z_n})$  has  $p$ -bases locally over  $\mathbb{Z}/p^n$  by loc. cit. Proposition 1.6 (2).*

The following fact is a logarithmic variant of Propostion 2.1.5:

**Proposition 2.2.6 (Tsuji [Ts2] Corollary 1.9)** *Let  $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$  be a system of exact closed immersions satisfying the conditions (0)–(2) in Condition 2.2.3. Assume that  $Z_n$  is locally noetherian, and that  $\text{Ker}(\mathcal{O}_{Z_n, x} \rightarrow \mathcal{O}_{X_n, x})$  is generated by an  $\mathcal{O}_{Z_n, x}$ -regular sequence for any  $x \in X_1$ . Then  $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$  satisfies the condition (3) in Condition 2.2.3.*

For  $0 \leq r \leq p - 1$ , we define a log syntomic complex  $\mathcal{S}_n(r)_{(X, M), (Z_*, M_{Z_*})}$  as follows. Let  $\mathbb{J}_{n, (X, M), (Z_*, M_{Z_*})}^{[r]}$  be the complex of sheaves on  $(X_1)_{\text{ét}}$

$$J_{D_n}^{[r]} \xrightarrow{d} J_{D_n}^{[r-1]} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^1 \xrightarrow{d} \cdots \xrightarrow{d} J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^q \xrightarrow{d} \cdots,$$

where  $J_{D_n}^{[r]}$  is placed in degree 0 and  $\omega_{(Z_n, M_{Z_n})}^*$  denotes the differential module of  $(Z_n, M_{Z_n})$  [Ka3] (1.7). The arrows  $d$  denote the derivations defined in [Ts2] Corollary 1.10. Put

$$\mathbb{E}_{n, (X, M), (Z_*, M_{Z_*})} := \mathbb{J}_{n, (X, M), (Z_*, M_{Z_*})}^{[0]}.$$

By the assumption that  $0 \leq r \leq p - 1$ , the Frobenius endomorphism on  $(Z_{n+r}, M_{Z_{n+r}})$  induces a homomorphism of complexes

$$f_r := \overline{p^{-r} \cdot \varphi_{n+r}^*} : \mathbb{J}_{n, (X, M), (Z_*, M_{Z_*})}^{[r]} \longrightarrow \mathbb{E}_{n, (X, M), (Z_*, M_{Z_*})}$$

(see [Ts2] p. 540 for details).

**Definition 2.2.7** *For  $0 \leq r \leq p - 1$ , we define the complex  $\mathcal{S}_n(r)_{(X, M), (Z_*, M_{Z_*})}$  on  $(X_1)_{\text{ét}}$  as the mapping fiber of*

$$1 - f_r : \mathbb{J}_{n, (X, M), (Z_*, M_{Z_*})}^{[r]} \longrightarrow \mathbb{E}_{n, (X, M), (Z_*, M_{Z_*})}$$

(cf. Definition 2.1.6). We define the syntomic cohomology of  $(X, M)$  with coefficients in  $\mathcal{S}_n(r)$  as the hypercohomology groups of this complex:

$$H_{\text{syn}}^*((X, M), \mathcal{S}_n(r)) := \mathbb{H}_{\text{ét}}^*(X, \mathcal{S}_n(r)_{(X, M), (Z_*, M_{Z_*})}).$$

This notation is well-defined, because the image of the complex  $\mathcal{S}_n(r)_{(X, M), (Z_*, M_{Z_*})}$  in the derived category is independent of embedding systems as in Condition 2.2.3 by Remark 2.2.8 below. In the case of Example 2.2.2, we put

$$H_{\text{syn}}^*(X(\mathcal{D}), \mathcal{S}_n(r)) := H_{\text{syn}}^*((X, M), \mathcal{S}_n(r))$$

for simplicity.

**Remark 2.2.8** *Take another embedding system  $\{i'_n : (X_n, M_n) \hookrightarrow (Z'_n, M_{Z'_n})\}_{n \geq 1}$  as in Condition 2.2.3, and consider the embedding system*

$$\{i_n \times i'_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n}) \times_{\mathbb{Z}/p^n} (Z'_n, M_{Z'_n}) =: (Z''_n, M_{Z''_n})\}_{n \geq 1}.$$

We define a Frobenius endomorphism on  $(Z_n'', M_{Z_n}'')$  ( $n \geq 1$ ) as the fiber product of those of  $(Z_n, M_{Z_n})$  and  $(Z_n', M_{Z_n}')$ . Then this embedding system satisfies the conditions (0)–(3) in Condition 2.2.3 as well (see [Ka4] Proof of Lemma 2.2, and [Ts2] Proposition 1.8), and there are natural quasi-isomorphisms of complexes on  $(X_1)_{\text{ét}}$

$$\mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \xrightarrow{\text{qis}} \mathcal{S}_n(r)_{(X,M),(Z_*'',M_{Z_*}'')} \xleftarrow{\text{qis}} \mathcal{S}_n(r)_{(X,M),(Z_*',M_{Z_*}')}$$

by [Ts2] Corollary 1.11. Hence the image of the complex  $\mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}$  in the derived category is independent of embedding systems. Moreover, this fact verifies that log syntomic cohomology groups are contravariantly functorial for morphisms  $(X, M) \rightarrow (X', M')$  of log schemes which satisfy Condition 2.2.3 (see also [Ka1] p. 212).

**Definition 2.2.9** For  $r, r' \geq 0$  with  $r + r' \leq p - 1$ , we define a product structure

$$\mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \otimes \mathcal{S}_n(r')_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathcal{S}_n(r + r')_{(X,M),(Z_*,M_{Z_*})}$$

by

$$(x, y) \otimes (x', y') \longmapsto (xx', (-1)^q xy' + f_{r'}(x')y),$$

where

$$\begin{aligned} (x, y) \in \mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})}^q &= (J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^q) \oplus (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^{q-1}), \\ (x', y') \in \mathcal{S}_n(r')_{(X,M),(Z_*,M_{Z_*})}^{q'} &= (J_{D_n}^{[r'-q']} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^{q'}) \oplus (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^{q'-1}). \end{aligned}$$

The following proposition follows from the same arguments as for Proposition 2.1.7 (use [Ts2] Corollary 1.9 instead of [Ka4] Lemma 1.8):

**Proposition 2.2.10** For  $\{i_n : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$  as before and integers  $m, n \geq 1$ , there is a short exact sequence of complexes on  $(X_1)_{\text{ét}}$

$$0 \longrightarrow \mathcal{S}_m(r)_{(X,M),(Z_*,M_{Z_*})} \xrightarrow{p^n} \mathcal{S}_{m+n}(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow 0.$$

The following standard fact relates the syntomic cohomology with de Rham cohomology.

**Proposition 2.2.11** For  $0 \leq r \leq p - 1$  and  $i \geq 0$ , there is a canonical map

$$\epsilon^i : H_{\text{syn}}^i((X, M), \mathcal{S}_n(r)) \longrightarrow \mathbb{H}_{\text{ét}}^i(X_n, \omega_{(X_n, M_n)}^{\bullet \geq r}),$$

which is compatible with product structures and contravariantly functorial in  $(X, M)$ .

*Proof.* Fix a system of embeddings  $(X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})$  ( $n \geq 1$ ) as before. There are natural maps

$$\mathcal{S}_n(r)_{(X,M),(Z_*,M_{Z_*})} \longrightarrow \mathbb{J}_{n,(X,M),(Z_*,M_{Z_*})}^{[r]} \longrightarrow \omega_{(X_n, M_n)}^{\bullet \geq r},$$

where the first arrow arises from the definition of the syntomic complex, and the second arrow is induced by the natural maps

$$\mathcal{O}_{X_n} \otimes_{\mathcal{O}_{D_n}} (J_{D_n}^{[r-q]} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^q) \longrightarrow \begin{cases} 0 & (q \leq r-1) \\ \omega_{(X_n, M_n)}^q & (q \geq r). \end{cases}$$

Let  $\epsilon_{n, (X, M), (Z_*, M_{Z_*})}$  be the composite of the above natural maps of complexes, and define the desired map  $\epsilon^i$  as that induced by  $\epsilon_{n, (X, M), (Z_*, M_{Z_*})}$ . One can easily check that the image of  $\epsilon_{n, (X, M), (Z_*, M_{Z_*})}$  in the derived category is independent of the choice of an embedding system, by repeating the arguments in Remark 2.2.8. Thus we obtain the proposition.  $\square$

**Remark 2.2.12** *If the identity maps  $(X_n, M_n) \rightarrow (X_n, M_n)$  ( $n \geq 1$ ) satisfy the conditions (0)–(3) in Condition 2.2.3, then we have  $(D_n, M_{D_n}) = (X_n, M_n)$ , i.e.,*

$$\mathbb{J}_{n, (X, M), (X_*, M_*)}^{[r]} = \omega_{(X_n, M_n)}^{\bullet \geq r} \quad \text{and} \quad \mathbb{E}_{n, (X, M), (X_*, M_*)} = \omega_{(X_n, M_n)}^\bullet,$$

and there is a short exact sequence of complexes

$$0 \longrightarrow \mathcal{S}_n(r)_{(X, M), (X_*, M_*)} \xrightarrow{\epsilon_n} \omega_{(X_n, M_n)}^{\bullet \geq r} \xrightarrow{1-f_r} \omega_{(X_n, M_n)}^\bullet \longrightarrow 0. \quad (2.2.13)$$

for  $0 \leq r \leq p-1$ .

We provide a spectral sequence computing syntomic cohomology with log poles, which will be used in §4.4 below.

**Proposition 2.2.14** *Let  $X, \mathcal{D}$  and  $M$  be as in Example 2.2.2, assume that  $\mathcal{D}$  is flat over  $\mathbb{Z}_p$ . Let  $\{\mathcal{D}_i\}_{i \in I}$  be the irreducible components of  $\mathcal{D}$ . Put  $X^{(0)} := X$  and*

$$X^{(m)} := \coprod_{\{i_1, i_2, \dots, i_m\} \subset I} \mathcal{D}_{i_1} \times_X \cdots \times_X \mathcal{D}_{i_m}$$

for  $m \geq 1$ , where for each subset  $\{i_1, i_2, \dots, i_m\} \subset I$ , the indices are pairwise distinct. Assume the following conditions:

- (i) *The identity maps  $(X_n, M_n) \rightarrow (X_n, M_n)$  satisfy the conditions (0)–(3) in Condition 2.2.3 for all  $n \geq 0$ .*
- (ii) *The identity maps  $X^{(m)} \rightarrow X^{(m)}$  satisfy the conditions (1)–(3) in Condition 2.1.2 for all  $m \geq 0$ .*
- (iii) *The given Frobenius endomorphisms on  $X_{n+r}$  and  $(X^{(0)})_{n+r}$  are compatible under the canonical finite morphism  $(X^{(0)})_{n+r} \rightarrow X_{n+r}$ .*
- (iv) *For any  $m \geq 0$ , the given Frobenius endomorphisms on  $(X^{(m)})_{n+r}$  and  $(X^{(m+1)})_{n+r}$  are compatible under the canonical finite morphism  $(X^{(m+1)})_{n+r} \rightarrow (X^{(m)})_{n+r}$ .*

Fix an ordering on the set  $I$ . Then for  $0 \leq r \leq p-1$ , there is a spectral sequence of syntomic cohomology groups

$$E_1^{a,b} = H_{\text{syn}}^{2a+b}(X^{(-a)}, \mathcal{S}_n(a+r)) \implies H_{\text{syn}}^{a+b}(X(\mathcal{D}), \mathcal{S}_n(r)).$$

*Proof.* By (i) and (ii), the syntomic complexes  $\mathcal{S}_n(r)_{(X,M),(X^*,M^*)}$  and  $\mathcal{S}_n(r)_{X^{(m)},X^{(m)}}$  are defined for  $0 \leq r \leq p-1$  and  $m \geq 0$ . These complexes are computed as in Remarks 2.1.8 and 2.2.12. By (iii) and (iv), there is a natural ‘filtration’ on  $\mathcal{S}_n(r)_{(X,M),(X^*,M^*)}$  as follows:

$$\begin{aligned} 0 &\longrightarrow \mathcal{S}_n(r)_{X,X} \xrightarrow{\alpha_0} \mathcal{S}_n(r)_{(X,M),(X^*,M^*)} \longrightarrow C_{1,n}^\bullet \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{S}_n(r)_{X^{(m)},X^{(m)}}[-m] \xrightarrow{\alpha_m} C_{m,n}^\bullet \longrightarrow C_{m+1,n}^\bullet \longrightarrow 0 \quad (m \geq 1) \\ &C_{m,n}^\bullet = 0 \quad (\text{if } X^{(m)} = \emptyset), \end{aligned}$$

where we have omitted the indication of the direct image of sheaves under the canonical finite morphisms  $(X^{(m)})_1 \rightarrow X_1$ . The arrow  $\alpha_0$  denotes the natural inclusion of complexes, and the arrow  $\alpha_m$  for  $m \geq 1$  is given by the alternate sum of the inverse of Poincaré residue mappings whose signs are determined by the fixed ordering on  $I$ . The spectral sequence in question is obtained from this filtration.  $\square$

The following fact relates the syntomic cohomology with étale cohomology and plays an important role in this paper.

**Theorem 2.2.15 (Tsuij [Ts1] §3.1)** *Let  $X, \mathcal{D}$  and  $M$  be as in Example 2.2.2 ( $\mathcal{D}$  may be empty). Assume that  $(X, M)$  satisfies Condition 2.2.3 and that there exists a henselian local ring  $\mathcal{R}$  which is faithfully flat over  $\mathbb{Z}_p$  and such that  $X$  is proper over  $\mathcal{R}$ . Put  $U := X - \mathcal{D}$ . Then for  $0 \leq r \leq p-2$  and  $i \geq 0$ , there is a canonical homomorphism*

$$c^i : H_{\text{syn}}^i(X(\mathcal{D}), \mathcal{S}_n(r)) \longrightarrow H_{\text{ét}}^i(U[p^{-1}], \mathbb{Z}/p^n(r)),$$

which is compatible with product structures and contravariantly functorial in  $(X, M)$ .

*Proof.* Since  $X$  is regular and flat over  $\mathbb{Z}_p$ , we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_p t^{\{r\}} \longrightarrow \text{Fil}_p^r A_{\text{crys}}(\overline{A^h}) \xrightarrow{1 - \frac{\varphi}{p^r}} A_{\text{crys}}(\overline{A^h}) \longrightarrow 0$$

for an affine open subset  $\text{Spec}(A) \subset X$  (see [Ts1] p. 245). We obtain a morphism

$$c : \mathcal{S}_n(r)_{(X,M)} \longrightarrow \iota^* Rj_* \mathbb{Z}/p^n(r) \quad (X_1 \xrightarrow{\iota} X \xleftarrow{j} U[p^{-1}])$$

in the derived category of étale sheaves on  $X_1$  by repeating the arguments in loc. cit. pp. 316–322, and obtain the desired map  $c^i$  by the proper base-change theorem:

$$c^i : H_{\text{syn}}^i(X(\mathcal{D}), \mathcal{S}_n(r)) \xrightarrow{c} H_{\text{ét}}^i(X_1, \iota^* Rj_* \mathbb{Z}/p^n(r)) \xleftarrow{\sim} H_{\text{ét}}^i(U[p^{-1}], \mathbb{Z}/p^n(r)).$$

See [Ts2] pp. 544–545 for the functoriality.  $\square$

## 2.3 Symbol maps

For a scheme  $Z$ , we write  $\mathcal{O}(Z)$  for  $\Gamma(Z, \mathcal{O}_Z)$ , for simplicity. We first review étale symbol maps. Let  $X$  be a scheme and let  $n$  be a positive integer which is invertible on  $X$ . Then we have a short exact sequence on  $X_{\text{ét}}$ , called *the Kummer sequence*

$$0 \longrightarrow \mathbb{Z}/n(1) \longrightarrow \mathcal{O}_X^\times \xrightarrow{\times n} \mathcal{O}_X^\times \longrightarrow 0.$$

See the beginning of §2 for the definition of the étale sheaf  $\mathbb{Z}/n(1)$ . Taking étale cohomology groups, we get a connecting map

$$\mathcal{O}(X)^\times/n \hookrightarrow H_{\text{ét}}^1(X, \mathbb{Z}/n(1)). \quad (2.3.1)$$

We write  $\{x\}^{\text{ét}}$  for the image of  $x \in \mathcal{O}(X)^\times$  under this map. Taking cup products, we obtain a map

$$(\mathcal{O}(X)^\times)^{\otimes r}/n \longrightarrow H_{\text{ét}}^r(X, \mathbb{Z}/n(r)), \quad (2.3.2)$$

which sends  $x_1 \otimes x_2 \otimes \cdots \otimes x_r$  (each  $x_i \in \mathcal{O}(X)^\times$ ) to  $\{x_1\}^{\text{ét}} \cup \{x_2\}^{\text{ét}} \cup \cdots \cup \{x_r\}^{\text{ét}}$ . By an argument of Tate [T] Proposition 2.1, this map annihilates Steinberg relations in  $(\mathcal{O}(X)^\times)^{\otimes r}$ , i.e., the elements of the form

$$x_1 \otimes x_2 \otimes \cdots \otimes x_r \text{ with } x_i + x_j = 0 \text{ or } 1 \text{ for some } i \neq j$$

map to 0 under the map (2.3.2). Consequently, we get a map

$$K_r^M(\mathcal{O}(X))/n \longrightarrow H_{\text{ét}}^r(X, \mathbb{Z}/n(r)), \quad (2.3.3)$$

which we call *the étale symbol map*. When  $X$  is the spectrum of a field, we often call this map *the Galois symbol map*.

**Remark 2.3.4** (1) *Since we have  $H_{\text{ét}}^1(X, \mathcal{O}_X^\times) \simeq \text{Pic}(X)$  by Hilbert's theorem 90, the map (2.3.1) is bijective if  $X$  is the spectrum of a UFD (e.g., a field).*

(2) *If  $r = 2$  and  $X = \text{Spec}(F)$  with  $F$  a field, then the map (2.3.3) is bijective by the Merkur'ev-Suslin theorem [MS].*

We next review syntomic symbol maps ([FM] p. 205, [Ka1] Chapter I §3, [Ts1] §2.2, [Ts2] p. 542). Let  $(X, M)$  be a log scheme which is flat over  $\mathbb{Z}_p$  and satisfies Condition 2.2.3. Fix an embedding system  $\{i : (X_n, M_n) \hookrightarrow (Z_n, M_{Z_n})\}_{n \geq 1}$  as in Condition 2.2.3. We define the complex  $C_n$  as

$$C_n := (1 + J_{D_n} \longrightarrow M_{D_n}^{\text{gp}}) \quad (1 + J_{D_n} \text{ is placed in degree } 0),$$

where for a sheaf of  $\mathcal{M}$  of commutative monoids,  $\mathcal{M}^{\text{gp}}$  denotes the associated sheaf of abelian groups. We define the map of complexes

$$s : C_{n+1} \longrightarrow \mathcal{S}_n(1)_{(X, M), (Z_*, M_{Z_*})}$$

as the map

$$s^0 : 1 + J_{D_{n+1}} \longrightarrow J_{D_n}, \quad a \longmapsto \log(a)$$

in degree 0, and the map

$$\begin{aligned} s^1 : M_{D_{n+1}}^{\text{gp}} &\longrightarrow (\mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \omega_{(Z_n, M_{Z_n})}^1) \oplus \mathcal{O}_{D_n} \\ b &\longmapsto (d\log(b), p^{-1} \log(b^p \varphi_{n+1}(b)^{-1})) \end{aligned}$$

in degree 1. Here  $\varphi_{n+1}(b)b^{-p}$  belongs to  $1 + p\mathcal{O}_{D_{n+1}}$  and the logarithm  $\log(b^p \varphi_{n+1}(b)^{-1})$  belongs to  $p\mathcal{O}_{D_{n+1}}$ . The notation ‘ $p^{-1}$ ’ means the inverse image under the isomorphism

$$p : \mathcal{O}_{D_n} \xrightarrow{\sim} p\mathcal{O}_{D_{n+1}} \quad (\text{cf. (2.2.4)}).$$

One can easily check that the maps  $s^0$  and  $s^1$  yield a map of complexes. Since there is a natural quasi-isomorphism  $C_{n+1} \xrightarrow{\sim} M_{n+1}^{\text{gp}}[1]$ , the map  $s$  induces a morphism

$$M_{n+1}^{\text{gp}}[1] \longrightarrow \mathcal{S}_n(1)_{(X, M), (Z_*, M_{Z_*})}$$

in the derived category, which is independent of the choice of embedding systems.

Now suppose further that  $X$ ,  $\mathcal{D}$  and  $M$  be as in Example 2.2.2, and let  $j : U := X - \mathcal{D} \hookrightarrow X$  be the natural open immersion. Then we have  $M^{\text{gp}} = j_* \mathcal{O}_U^\times$ , and obtain a map

$$\mathcal{O}(U_{n+1})^\times \longrightarrow H_{\text{syn}}^1(X(\mathcal{D}), \mathcal{S}_n(1)). \quad (2.3.5)$$

We often write  $\{x\}^{\text{syn}} \in H_{\text{syn}}^1(X(\mathcal{D}), \mathcal{S}_n(1))$  for the image of  $x \in \mathcal{O}(U_{n+1})^\times$ . This map and the product structure of syntomic cohomology (Definition 2.2.9) give rise to a map

$$K_r^M(\mathcal{O}(U_{n+1})) \longrightarrow H_{\text{syn}}^r(X(\mathcal{D}), \mathcal{S}_n(r)) \quad (2.3.6)$$

for  $0 \leq r \leq p-1$  (cf. [Ka1] Proposition 3.2), which we call *the syntomic symbol map*.

**Remark 2.3.7** When  $M$  is the trivial log structure  $\mathcal{O}_X^\times$ , we obtain a symbol map

$$K_r^M(\mathcal{O}(X_{n+1})) \longrightarrow H_{\text{syn}}^r(X, \mathcal{S}_n(r))$$

for  $0 \leq r \leq p-1$ . If the identity map  $X \rightarrow X$  satisfies the conditions (1)–(3) in Condition 2.1.2, then the connecting homomorphism induced by (2.1.9)

$$H_{\text{dR}}^0(X_n) \longrightarrow H_{\text{syn}}^1(X, \mathcal{S}_n(1))$$

sends  $a \in H_{\text{dR}}^0(X_n)$  to  $\{1 + pa\}^{\text{syn}}$ , where  $1 + pa$  is well-defined in  $\mathcal{O}(X_{n+1})^\times$ . One can easily check this fact directly from the definition of the symbol map.

**Theorem 2.3.8 (Tsuji [Ts1] Proposition 3.2.4)** Let  $X$ ,  $\mathcal{D}$  and  $M$  be as in Example 2.2.2, and assume that  $(X, M)$  satisfies the assumptions in Theorem 2.2.15. Put  $U := X - \mathcal{D}$ . Then there is a commutative diagram

$$\begin{array}{ccccc} K_r^M(\mathcal{O}(U)) & \longrightarrow & K_r^M(\mathcal{O}(U_{n+1})) & \xrightarrow{(2.3.6)} & H_{\text{syn}}^r(X(\mathcal{D}), \mathcal{S}_n(r)) \\ & \searrow & & & \downarrow c^r \\ & & K_r^M(\mathcal{O}(U)[p^{-1}]) & \xrightarrow{(2.3.3)} & H_{\text{ét}}^r(U[p^{-1}], \mathbb{Z}/p^n(r)) \end{array}$$

for  $0 \leq r \leq p-2$ , where  $c^r$  denotes the canonical map in Theorem 2.2.15.

*Proof.* The case  $r = 1$  follows from the same arguments as in [Ts1] Proposition 3.2.4. The general case follows from the previous case by the compatibility of these arrows with product structures (Theorem 2.2.15).  $\square$

We state a syntomic analogue of the facts in Remark 2.3.4, which will be used in §4.4 below.

**Theorem 2.3.9** *Let  $R$  be a henselian discrete valuation ring whose fraction field  $L$  has characteristic zero and whose residue field  $F$  has characteristic  $p$ . Then the syntomic symbol map*

$$K_r^M(R)/p^n \longrightarrow H_{\text{syn}}^r(R, \mathcal{S}_n(r))$$

*is surjective for  $r = 2$  (and  $p \geq 5$ ), and bijective for  $r = 1$  (and  $p \geq 3$ ).*

*Proof.* We prove only the case  $r = 2$ . The case  $r = 1$  follows from a similar (and simpler) arguments as below and the details are left to the reader. We first show that the following sequence of Milnor  $K$ -groups is exact:

$$K_2^M(R) \longrightarrow K_2^M(L) \xrightarrow{\partial} K_1^M(F) \longrightarrow 0, \quad (2.3.10)$$

where the first arrow is the natural pull-back of symbols. The arrow  $\partial$  is the boundary map of Milnor  $K$ -groups, which is obviously surjective. We show the exactness at  $K_2^M(L)$ . Indeed, we have a localization sequence of algebraic  $K$ -groups

$$K_2(R) \longrightarrow K_2(L) \xrightarrow{d} K_1(F)$$

and natural isomorphisms  $K_2(L) \simeq K_2^M(L)$  and  $K_1(F) \simeq F^\times$ . The arrow  $d$  agrees with  $\partial$  (up to a sign) under these isomorphisms. Moreover the natural map  $K_2^M(R) \rightarrow K_2(R)$  is surjective ([Sr] p. 17 Remark). Hence the sequence (2.3.10) is exact at  $K_2^M(L)$ .

Put  $\eta := \text{Spec}(F)$ , and consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} K_2^M(R)/p^n & \longrightarrow & K_2^M(L)/p^n & \xrightarrow{\partial} & F^\times/p^n & \longrightarrow & 0 \\ \downarrow & & \downarrow \wr (2.3.3) & & \downarrow d\log & & \\ 0 \longrightarrow & H_{\text{syn}}^2(R, \mathcal{S}_n(2)) & \xrightarrow{c^2} & H_{\text{ét}}^2(L, \mathbb{Z}/p^n(2)) & \xrightarrow{\partial'} & H_{\text{ét}}^0(\eta, W_n \Omega_{\eta, \log}^1) & \longrightarrow 0 \end{array}$$

where the upper row is the exact sequence (2.3.10) modulo  $p^r$ . The arrow  $\partial'$  is the boundary map of Galois cohomology ([Ka2] §1) and the lower row is exact by Kurihara [Ku]. The central vertical arrow is bijective by Remark 2.3.4 (2). The right vertical arrow is bijective by the short exact sequence

$$0 \longrightarrow \mathcal{O}_\eta^\times \xrightarrow{\times p^n} \mathcal{O}_\eta^\times \xrightarrow{d\log} W_n \Omega_{\eta, \log}^1 \longrightarrow 0$$

on  $\eta_{\text{ét}}$  and Remark 2.3.4 (1). Thus we obtain the lemma.  $\square$

We end this section with the following standard fact on Milnor  $K$ -groups, which will be used in the proof of Proposition 3.4.4 below:

**Proposition 2.3.11** *Let  $R$  be a henselian discrete valuation ring whose residue field  $F$  has characteristic  $p$ . Let  $L$  be the fraction field of  $R$ . Assume that  $F$  is infinite and that  $\Omega_{F,\log}^2 = 0$ . Then the natural map  $K_2^M(R)/p^n \rightarrow K_2^M(L)/p^n$  is injective for any  $n \geq 1$ .*

*Proof.* Since  $F$  is infinite, we have  $K_2^M(R) = K_2(R)$  by a theorem of van der Kallen ([Sr] p. 17 Remark). There is a localization exact sequence of algebraic  $K$ -groups

$$K_2(F) \longrightarrow K_2(R) \longrightarrow K_2(L) \longrightarrow K_1(F)(= F^\times).$$

The last arrow is surjective, as  $R$  is a principal ideal domain. We decompose this sequence into two exact sequences

$$\begin{aligned} K_2(F) &\longrightarrow K_2(R) \longrightarrow M \longrightarrow 0, \\ 0 &\longrightarrow M \longrightarrow K_2(L) \longrightarrow F^\times \longrightarrow 0. \end{aligned}$$

Since  $F^\times$  is  $p$ -torsion-free, we have  $M/p^n \hookrightarrow K_2(L)/p^n$ . On the other hand, we have

$$K_2(F)/p^n = K_2^M(F)/p^n = 0$$

by the assumption that  $\Omega_{F,\log}^2 = 0$  ([BK1] Theorem 2.1) and  $K_2(R)/p^n \simeq M/p^n$ . Hence

$$K_2^M(R)/p^n = K_2(R)/p^n \simeq M/p^n \hookrightarrow K_2(L)/p^n = K_2^M(L)/p^n$$

as required. □

By Theorem 2.3.9 and Proposition 2.3.11, we obtain the following consequence, which will not be used in the rest of this paper:

**Corollary 2.3.12** *Let  $R$  be as in Theorem 2.3.9. Assume  $p \geq 5$  and that the residue field  $F$  satisfies the assumptions in Proposition 2.3.11. Then the symbol map  $K_2^M(R)/p^n \rightarrow H_{\text{syn}}^2(R, \mathcal{S}_n(2))$  is bijective.*

## 2.4 Tate curve

Let  $B$  be a noetherian complete local ring with  $6^{-1} \in B$ . Let  $q \in B$  be an element which is contained in the maximal ideal of  $B$  and not nilpotent. Put

$$A := B[q^{-1}].$$

The Tate curve  $E = E_q$  over  $A$  with period  $q$  is the projective completion in  $\mathbb{P}_A^2$  of the affine curve

$$y^2 + xy = x^3 + a_4(q)x + a_6(q) \quad \text{on } \text{Spec}(A[x, y]),$$

where  $a_4(q)$  and  $a_6(q) \in B$  are defined as follows:

$$a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad a_6(q) = - \sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5) q^n}{12(1 - q^n)}.$$

We review the Tate parameterization of  $E$ . Let  $O \in E(A)$  be the infinite point. The series

$$\begin{aligned} x(\alpha) &= \sum_{n \in \mathbb{Z}} \frac{q^n \alpha}{(1 - q^n \alpha)^2} - 2 \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \\ y(\alpha) &= \sum_{n \in \mathbb{Z}} \frac{(q^n \alpha)^2}{(1 - q^n \alpha)^3} + \sum_{n \geq 1} \frac{q^n}{(1 - q^n)^2} \end{aligned}$$

converge for all  $\alpha \in A^\times - q^{\mathbb{Z}}$ . They induce an injective homomorphism

$$A^\times / q^{\mathbb{Z}} \longrightarrow E(A), \quad \alpha \longmapsto \begin{cases} (x(\alpha) : y(\alpha) : 1) & (\alpha \notin q^{\mathbb{Z}}) \\ O & (\alpha \in q^{\mathbb{Z}}), \end{cases}$$

which is bijective if  $B$  is a complete discrete valuation ring ([Si] V Theorem 3.1). This assignment is not algebraic, but we have the following algebraic byproduct. Let  $u$  be an indeterminate, and put

$$\mathcal{B}_0 := \varprojlim_n B[u, u^{-1}]/(q^n) \quad \text{and} \quad \mathcal{A}_0 := \mathcal{B}_0[q^{-1}].$$

**Proposition 2.4.1** *There is an injective homomorphism of  $B$ -algebras*

$$B[x, y]/(y^2 + xy - x^3 - a_4(q)x - a_6(q)) \longrightarrow \mathcal{B}_0[(1 - u)^{-1}]$$

given by the ‘ $q$ -adic presentation’ of  $(x(u), y(u))$ :

$$\begin{aligned} x &\longmapsto \frac{u}{(1 - u)^2} + \sum_{d \geq 0} \sum_{m|d} m(u^m + u^{-m} - 2)q^d \\ y &\longmapsto \frac{u^2}{(1 - u)^3} + \sum_{d \geq 0} \sum_{m|d} m \left( \frac{m-1}{2} u^m - \frac{m+1}{2} u^{-m} + 1 \right) q^d. \end{aligned}$$

Moreover, this ring homomorphism induces a dominant morphism of schemes

$$\beta_0 : \text{Spec}(\mathcal{A}_0) \rightarrow E.$$

*Proof.* The assertion follows from the same arguments as in [Si] p. 425 Proof of Theorem 3.1 (c).  $\square$

**Definition 2.4.2** (1) We define the canonical invariant 1-form  $\omega_E \in \Gamma(E, \Omega_E^1)$  as

$$\omega_E := \frac{dx}{2y + x},$$

which maps to  $u^{-1}du$  under the pull-back map

$$\beta_0^* : \Gamma(E, \Omega_E^1) \hookrightarrow \Omega_{\mathcal{B}_0}^1.$$

(2) We define the theta function  $\theta(u, q) \in \mathcal{B}_0$  as

$$\theta(u, q) := (1 - u) \prod_{n=1}^{\infty} (1 - q^n u)(1 - q^n u^{-1}), \quad (2.4.3)$$

which satisfies

$$\theta(qu, q) = \theta(u^{-1}, q) = -u^{-1}\theta(u, q).$$

The following proposition follows from standard facts on theta functions and theta divisors, whose details are left to the reader.

**Proposition 2.4.4** Put  $K := \text{Frac}(B)$ , and let  $K(E)$  be the function field of  $E$ .

(1) A function  $f(u) \in \text{Frac}(\mathcal{B}_0)$  given by a finite product

$$f(u) = c \prod_i \frac{\theta(\alpha_i u, q)}{\theta(\beta_i u, q)} \quad (c, \alpha_i, \beta_i \in A^\times) \quad (2.4.5)$$

is  $q$ -periodic, if  $\prod_i \alpha_i / \beta_i = 1$ .

(2) The ring homomorphism in Proposition 2.4.1 induces a natural inclusion

$$\left\{ c \prod_i \frac{\theta(\alpha_i u, q)}{\theta(\beta_i u, q)} \mid c, \alpha_i, \beta_i \in A^\times \text{ with } \prod_i \alpha_i / \beta_i = 1 \right\} \hookrightarrow K(E)^\times.$$

**Proposition 2.4.6** Let  $f$  be the structural morphism  $E[p^{-1}] \rightarrow \text{Spec}(A[p^{-1}])$ . Then there is an exact sequence of étale sheaves on  $\text{Spec}(A[p^{-1}])$

$$0 \longrightarrow \mathbb{Z}/p^n(1) \longrightarrow R^1 f_* \mathbb{Z}/p^n(1) \longrightarrow \mathbb{Z}/p^n \longrightarrow 0. \quad (2.4.7)$$

*Proof.* See [DR] VII.1.13. □

## 2.5 Frobenius endomorphism on Tate curves

Let  $p$  be a prime number at least 5, and let  $B, A$  and  $q$  be as in the beginning of §2.4. We assume here that  $B$  is a  $\mathbb{Z}_p$ -algebra, and that  $B$  has a Frobenius endomorphism  $\phi$ . Here a Frobenius endomorphism means a ring endomorphism compatible with the absolute Frobenius endomorphism on  $B/(p)$ . Let  $E_q = E_{q,A}$  be the Tate curve over  $A$  with period  $q$ . There is a canonical morphism

$$\text{can} : E_{\phi(q)} \longrightarrow E_q$$

induced by the ring homomorphism

$$\begin{aligned} A[x, y]/(y^2 + xy - x^3 - a_4(q)x - a_6(q)) \\ \longrightarrow A[x, y]/(y^2 + xy - x^3 - a_4(\phi(q))x - a_6(\phi(q))) \end{aligned}$$

sending

$$x \mapsto x, \quad y \mapsto y, \quad a \mapsto \phi(a) \quad (a \in A).$$

We define  $W : E_{\phi(q)} \rightarrow E_q$  as the composite

$$W : E_{\phi(q)} \xrightarrow{\text{can}} E_q \xrightarrow{[p]} E_q,$$

where  $[p]$  denotes the multiplication by  $p$  (with respect to the group structure on  $E_q$ ).

**Remark 2.5.1** *Under the map in Proposition 2.4.1, the map  $W$  corresponds to the endomorphism of  $\text{Frac}(\mathcal{B}_0)$  sending*

$$u \mapsto u^p, \quad b \mapsto \phi(b) \quad (b \in B).$$

In what follows, we assume that

$$\phi(q) = q^p \quad \text{and} \quad \phi(a) = a \quad (a \in \mathbb{Z}_p).$$

There is an isogeny over  $\text{Spec}(A)$

$$\iota : E_{q^p} \longrightarrow E_q$$

corresponding to the identity map of  $\text{Frac}(\mathcal{B}_0)$ , which is an analogue of the natural projection  $\mathbb{C}^\times/q^{p\mathbb{Z}} \rightarrow \mathbb{C}^\times/q^{\mathbb{Z}}$ . The morphism  $W$  factors through  $\iota$ . Indeed,  $\iota$  is surjective and we have

$$\text{Ker}(\iota) \subset \text{Ker}([p] : E_{q^p} \rightarrow E_{q^p})$$

as finite étale group schemes over  $\text{Spec}(A)$ . Thus  $W$  gives rise to an endomorphism

$$\varphi : E_q \longrightarrow E_q. \tag{2.5.2}$$

We show that  $\varphi$  is a Frobenius endomorphism:

**Lemma 2.5.3** *Put  $(E_q)_1 := E_q \otimes_A A/(p) = E_q \otimes \mathbb{F}_p$ . Then the morphism  $\varphi_1 : (E_q)_1 \rightarrow (E_q)_1$  induced by  $\varphi$  is the absolute Frobenius endomorphism of  $(E_q)_1$ .*

*Proof.* Define the Frobenius endomorphism  $\phi$  on  $\mathbb{Z}_p((q)) := \mathbb{Z}_p[[q]][q^{-1}]$  as  $\phi(q) = q^p$  and  $\phi(a) = a$  for  $a \in \mathbb{Z}_p$ . The natural ring homomorphism  $\mathbb{Z}_p((q)) \rightarrow A$  is compatible with the Frobenius endomorphisms. Note that  $E_q = E_{q,A}$  is obtained from  $E' = E_{q,\mathbb{Z}_p((q))}$  by scalar extension, and that we have

$$\varphi = \varphi' \otimes \phi$$

under the identification  $E_q = E' \otimes_{\mathbb{Z}_p((q))} A$ . Here  $\varphi'$  mean the map (2.5.2) defined for  $E'$ . Therefore it is enough to consider the case that  $A = \mathbb{Z}_p((q))$ . Then  $\varphi_1$  agrees with  $\varphi''$ , the map (2.5.2) defined for  $E'' = E_{q,\mathbb{F}_p((q))}$ . Under the morphism  $\beta_0$  in Proposition 2.4.1 with  $B = \mathbb{F}_p[[q]]$ ,  $\varphi''$  corresponds to the absolute Frobenius endomorphism of the field

$$\mathcal{A}_0 = \mathcal{B}_0[q^{-1}] = \left( \varprojlim_n \mathbb{F}_p[[q]][u, u^{-1}]/(q^n) \right) [q^{-1}]$$

(cf. Remark 2.5.1), which shows that  $\varphi''$  is the absolute Frobenius morphism. □

### 3 De Rham regulator of Tate curves

In this section, we assume  $p \geq 5$ . The main result of this section will be stated in Theorem 3.2.3 below.

#### 3.1 Setting

Let  $R$  be a  $p$ -adic integer ring which is unramified over  $\mathbb{Z}_p$ , and let  $k$  be the residue field of  $R$ . Let  $q_0$  be an indeterminate. We define the rings  $A$  and  $B$  as

$$B := R[[q_0]], \quad A := B[q_0^{-1}].$$

$B$  is a 2-dimensional regular complete local domain, and  $A$  is a Dedekind domain. Let  $\widehat{A}$  be the  $p$ -adic completion of  $A$ :

$$\widehat{A} := \varprojlim_{n \geq 1} A/p^n,$$

which is a complete discrete valuation ring whose maximal ideal is generated by  $p$ . Put

$$L := \widehat{A}[p^{-1}].$$

Let  $r$  be a positive integer prime to  $p$ , and put  $q := q_0^r$ . Let  $E = E_q$  be the Tate curve over  $\text{Spec}(A)$  with period  $q$ . Let  $\mathcal{E}'$  be the projective curve over  $\text{Spec}(B)$  defined by the following homogeneous equation in  $\mathbb{P}_B^2 = \text{Proj}(B[x, y, z])$ :

$$\mathcal{E}' : y^2z + xyz = x^3 + a_4(q)xz^2 + a_6(q)z^3$$

(see §2.1 for  $a_4(q)$  and  $a_6(q)$ ), which is a projective flat model of  $E$  over  $B$ . By blowing-up  $\mathcal{E}'$  along the locus  $\{x = y = q_0 = 0\}$  up to  $(r - 1)$ -times, we get a regular scheme  $\mathcal{E}$ , which is a generalized elliptic curve in the sense of [DR] II.1.12. The divisor  $\mathcal{D} := \{q_0 = 0\} \subset \mathcal{E}$  is the standard Néron  $r$ -gon over  $\text{Spec}(R)$ , and the structural morphism  $\mathcal{E} \rightarrow \text{Spec}(B)$  is smooth outside of the intersection loci of two distinct irreducible components of  $\mathcal{D}$ . There is a cartesian diagram

$$\begin{array}{ccc} E & \hookrightarrow & \mathcal{E} \\ \pi_E \downarrow & \square & \downarrow \pi_{\mathcal{E}} \\ \text{Spec}(A) & \hookrightarrow & \text{Spec}(B), \end{array}$$

where  $\pi_E$  is projective smooth and  $\pi_{\mathcal{E}}$  is projective flat. The horizontal arrows are open embeddings.

**Lemma 3.1.1** *Let  $M$  be the log structure on  $\mathcal{E}$  associated with  $\mathcal{D}$ , as in Example 2.2.2. Then the identity maps  $(\mathcal{E}_n, M_n) \rightarrow (\mathcal{E}_n, M_n)$  ( $n \geq 1$ ) satisfy the conditions (0)–(3) in Condition 2.2.3. Consequently, the log syntomic cohomology  $H_{\text{syn}}^*(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(r))$  is defined for  $0 \leq r \leq p - 1$  (Definition 2.2.7).*

*Proof.* The condition (0) is obvious, and (3) follows from Proposition 2.2.6. The condition (1) follows from Example 2.2.5, and (2) follows from Lemma 4.1.3 below.  $\square$

**Remark 3.1.2** Similarly, the identity map  $\mathcal{E} \rightarrow \mathcal{E}$  satisfies the conditions (1)–(3) in Condition 2.1.2, by Example 2.1.4 and Lemma 4.1.3 below. Consequently, the syntomic cohomology  $H_{\text{syn}}^*(\mathcal{E}, \mathcal{S}_n(r))$  ( $0 \leq r \leq p-1$ ) is computed by the distinguished triangle

$$0 \longrightarrow \mathcal{S}_n(r)_{\mathcal{E}, \mathcal{E}} \longrightarrow \Omega_{\mathcal{E}_n}^{\bullet \geq r} \xrightarrow{1-f_r} \Omega_{\mathcal{E}_n}^{\bullet} \longrightarrow 0$$

by Remark 2.1.8.

We put

$$H_{\text{syn}}^*(-, \mathcal{S}_{\mathbb{Z}_p}(r)) := \varprojlim_{n \geq 1} H_{\text{syn}}^*(-, \mathcal{S}_n(r)), \quad H_{\text{ét}}^*(-, \mathbb{Z}_p(r)) := \varprojlim_{n \geq 1} H_{\text{ét}}^*(-, \mathbb{Z}/p^n(r)).$$

We define the map of Kähler differential forms

$$\tau_{\infty}^{\text{dR}} : \Gamma(E, \Omega_E^2) \longrightarrow \Omega_A^1$$

as the composite of natural isomorphisms

$$\Gamma(E, \Omega_E^2) \simeq \Gamma(E, \Omega_A^1 \otimes_A \Omega_{E/A}^1) \simeq \Omega_A^1 \otimes_A \Gamma(E, \Omega_{E/A}^1)$$

and the map

$$\Omega_A^1 \otimes_A \Gamma(E, \Omega_{E/A}^1) \longrightarrow \Omega_A^1, \quad \eta \otimes \omega_E \mapsto \eta \quad (\eta \in \Omega_A^1).$$

Here  $\omega_E$  denotes the canonical invariant 1-form on  $E$  defined in Definition 2.4.2 (1), and we have used the fact that  $\Gamma(E, \Omega_{E/A}^1)$  is a free  $A$ -module generated by  $\omega_E$ .

### 3.2 Main result on Tate curves

Let  $\zeta_1, \dots, \zeta_d \in R$  be roots of unity which form a basis of  $R$  over  $\mathbb{Z}_p$ , whose existence is verified by the assumption that  $R$  is absolutely unramified. An arbitrary formal Laurant power series  $f(q_0) \in R[[q_0, q_0^{-1}]]$  is expanded into a power series of the form

$$f(q_0) = \sum_{j \leq 0} b_j q_0^j + \sum_{j \geq 1} \left( \frac{a_{1j} \zeta_1 q_0^j}{1 - \zeta_1 q_0^j} + \dots + \frac{a_{dj} \zeta_d q_0^j}{1 - \zeta_d q_0^j} \right) \quad (a_{ij} \in \mathbb{Z}_p, b_j \in R) \quad (3.2.1)$$

and the coefficients  $a_{ij}$  and  $b_j$  are uniquely determined by  $f(q_0)$ . We say that  $f(q_0)$  is a *formal power series of Eisenstein type* (in  $R[[q_0, q_0^{-1}]]$ ) if

$$(E1) \quad c_k = 0 \text{ for } k < 0 \text{ and } c_0 \in \mathbb{Z}_p,$$

$$(E2) \quad a_k^{(j)} \in k^2 \mathbb{Z}_p \text{ for all } j \text{ and } k \geq 1.$$

The condition (E2) does not depend on the choice of  $\zeta_i$  ([A2] Lemma 3.4). Moreover,  $f(q_0) \in R[[q_0, q_0^{-1}]]$  is of Eisenstein type if and only if so is it in  $R'[[q_0, q_0^{-1}]]$  for an  $p$ -adic integer ring  $R'$  which is unramified over  $R$ .

**Remark 3.2.1** *The reason we call ‘Eisenstein’ is the following fact. Suppose that  $f(q_0)$  is the  $q_0$ -expansion of a modular form of weight 3 at a cusp. Then  $f(q_0)$  is of Eisenstein type if and only if it is a linear combination of the usual Eisenstein series of weight 3 ([A2] §8.3).*

The main result of this section deals with the image of a ‘de Rham regulator map’ from syntomic cohomology

$$\mathrm{reg}_{\mathrm{dR}} : H_{\mathrm{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\epsilon^2} \varprojlim_{n \geq 1} \Gamma(E_n, \Omega_{E_n}^2) \xrightarrow{\tau_{\infty}^{\mathrm{dR}}} \Omega_{\widehat{A}}^1 \xrightarrow{f \cdot \frac{dq_0}{q_0} \mapsto f} \widehat{A},$$

where  $\tau_{\infty}^{\mathrm{dR}}$  denote the map defined in §3.1. See Proposition 2.2.11 for  $\epsilon^2$ .

**Theorem 3.2.3** *Assume that  $f(q_0) \in \widehat{A}$  is contained in the image of  $\mathrm{reg}_{\mathrm{dR}}$ . Then it is a formal power series of Eisenstein type.*

### 3.3 A commutative diagram

In the rest of this section, we prove Theorem 3.2.3 assuming a commutative diagram (3.3.1) below. For  $n \geq 1$ , let  $\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}$  be the composite of canonical maps of étale cohomology groups

$$\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}} : H_{\acute{\mathrm{e}}\mathrm{t}}^2(E_L, \mathbb{Z}/p^n(2)) \longrightarrow H_{\acute{\mathrm{e}}\mathrm{t}}^1(L, H_{\acute{\mathrm{e}}\mathrm{t}}^1(E_{\overline{L}}, \mathbb{Z}/p^n(2))) \longrightarrow H_{\acute{\mathrm{e}}\mathrm{t}}^1(L, \mathbb{Z}/p^n(1)).$$

Here the second arrow is induced by the map  $H_{\acute{\mathrm{e}}\mathrm{t}}^1(E_{\overline{L}}, \mathbb{Z}/p^n(1)) \rightarrow \mathbb{Z}/p^n$  in (2.4.7). The first arrow is obtained from the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H_{\acute{\mathrm{e}}\mathrm{t}}^a(L, H_{\acute{\mathrm{e}}\mathrm{t}}^b(E_{\overline{L}}, \mathbb{Z}/p^n(2))) \implies H_{\acute{\mathrm{e}}\mathrm{t}}^{a+b}(E_L, \mathbb{Z}/p^n(2))$$

and the fact that

$$E_2^{0,2} \simeq H_{\acute{\mathrm{e}}\mathrm{t}}^0(L, \mathbb{Z}/p^n(1)) = 0,$$

where we have used the assumption that  $R$  is unramified over  $\mathbb{Z}_p$ .

A key ingredient of the proof of Theorem 3.2.3 is the following commutative diagram:

$$\begin{array}{ccc} H_{\acute{\mathrm{e}}\mathrm{t}}^2(E_L, \mathbb{Z}_p(2)) & \xrightarrow{\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}} & H_{\acute{\mathrm{e}}\mathrm{t}}^1(L, \mathbb{Z}_p(1)) & (3.3.1) \\ \uparrow c^2 & & \uparrow c^1 & \\ H_{\mathrm{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{\tau_{\infty}^{\mathrm{syn}}} & H_{\mathrm{syn}}^1(A, \mathcal{S}_{\mathbb{Z}_p}(1)) & \\ \downarrow \epsilon^2 & & \downarrow d\log & \\ \varprojlim_{n \geq 1} \Gamma(E_n, \Omega_{E_n}^2) & \xrightarrow{\tau_{\infty}^{\mathrm{dR}}} & \Omega_{\widehat{A}}^1 & \end{array}$$

Here  $c^1$  and  $c^2$  are canonical maps in Theorem 2.2.15, and  $d\log$  is given by logarithmic differentials (see Theorem 2.3.9 for the isomorphism  $s$ ):

$$H_{\mathrm{syn}}^1(A, \mathcal{S}_{\mathbb{Z}_p}(1)) \rightarrow H_{\mathrm{syn}}^1(\widehat{A}, \mathcal{S}_{\mathbb{Z}_p}(1)) \xrightarrow{s} \varprojlim_{n \geq 1} \widehat{A}^\times / p^n \xrightarrow{f \mapsto \frac{df}{f}} \varprojlim_{n \geq 1} \Omega_{\widehat{A}/p^n}^1 = \Omega_{\widehat{A}}^1.$$

We will construct  $\tau_\infty^{\text{syn}}$  in §4.2 and prove the commutativity of the squares in §4.4 below.

As a preliminary of the proof of Theorem 3.2.3, we prove Lemma 3.3.2 below. For  $n \geq 1$ , let  $\nu$  be the isomorphism

$$\nu : H_{\text{ét}}^1(L, \mathbb{Z}/p^n(1)) \simeq L^\times/p^n.$$

in Remark 2.3.4(1).

**Lemma 3.3.2** *For any  $n \geq 0$ , the composite map*

$$H_{\text{ét}}^2(E_L, \mathbb{Z}/p^n(2)) \xrightarrow{\nu \circ \tau_\infty^{\text{ét}}} L^\times/p^n \xrightarrow{a \mapsto \{a, q_0\}} K_2^M(L)/p^n$$

is zero.

*Proof.* Since  $q = q_0^r$  and  $(r, p) = 1$  by definition, we may replace the second arrow with the assignment  $a \mapsto \{a, q\}$ . We consider the following commutative diagram, whose top row is an exact sequence arising from (2.4.7):

$$\begin{array}{ccccc} H_{\text{ét}}^1(L, H_{\text{ét}}^1(E_L, \mathbb{Z}/p^n(2))) & \longrightarrow & H_{\text{ét}}^1(L, \mathbb{Z}/p^n(1)) & \xrightarrow{(1)} & H_{\text{ét}}^2(L, \mathbb{Z}/p^n(2)) \\ & & \downarrow \nu & & \uparrow (2) \\ H_{\text{ét}}^2(E_L, \mathbb{Z}/p^n(2)) & \xrightarrow{\nu \circ \tau_\infty^{\text{ét}}} & L^\times/p^n & & K_2^M(L)/p^n. \end{array}$$

Here the arrow (2) is a Galois symbol map (2.3.3) (see also Remark 2.3.4). We claim that the arrow (1) maps  $a \in H_{\text{ét}}^1(L, \mathbb{Z}/p^n(1))$  to  $a \cup \{q\}^{\text{ét}}$  up to a sign. Indeed, the connecting homomorphism

$$\mathbb{Z}/p^n = H_{\text{ét}}^0(L, \mathbb{Z}/p^n) \longrightarrow H_{\text{ét}}^1(L, \mathbb{Z}/p^n(1))$$

associated with (2.4.7) sends 1 to  $\{q\}^{\text{ét}}$ , and the claim follows from a straight-forward computation on cup products. The lemma follows from these facts.  $\square$

### 3.4 Proof of Theorem 3.2.3

We prove of Theorem 3.2.3, assuming the diagram (3.3.1). Assume that  $f(q_0) \in \widehat{A}$  lies in the image of  $\text{reg}_{\text{dR}}$ . By the lower square of the diagram (3.3.1), there exists  $h(q_0) \in \widehat{A}^\times$  such that

$$\frac{dh(q_0)}{h(q_0)} = f(q_0) \frac{dq_0}{q_0} \in \Omega_{\widehat{A}}^1. \quad (3.4.1)$$

We fix such an  $h(q_0)$  in what follows. By Lemma 3.3.2 and the upper square of the diagram (3.3.1),  $h(q_0)$  must satisfy

$$\{h(q_0), q_0\} = 0 \in K_2^M(L)/p^n \text{ for any } n \geq 0. \quad (3.4.2)$$

Now expand  $f(q_0)$  into a series of the form (3.2.1):

$$f(q_0) = \sum_{j \leq 0} b_j q_0^j + \sum_{j \geq 1} \left( \frac{a_{1j} \zeta_1 q_0^j}{1 - \zeta_1 q_0^j} + \cdots + \frac{a_{dj} \zeta_d q_0^j}{1 - \zeta_d q_0^j} \right) \quad (a_{ij} \in \mathbb{Z}_p, b_j \in R)$$

and expand  $h(q_0)$  into an infinite product of the following form:

$$h(q_0) = c q_0^m \prod_{j \geq 1} (1 - \zeta_1 q_0^j)^{a'_{1j}} \cdots (1 - \zeta_d q_0^j)^{a'_{dj}} \quad (a'_{ij} \in \mathbb{Z}_p, c \in R^\times, m \in \mathbb{Z}).$$

By (3.4.1), we see that

$$a_{ij} = -j a'_{ij}, \quad (3.4.3)$$

hence  $a_{ij}$  is divisible by  $j$  for any  $j \geq 1$ . We next write down what the equation (3.4.2) yields, using the explicit reciprocity law of higher-dimensional regular local rings due to Kato [Ka4]:

**Proposition 3.4.4**  $a'_{ij}$  is divisible by  $j$  in  $\mathbb{Z}_p$  for any  $j \geq 0$ .

Theorem 3.2.3 immediately follows from this proposition and (3.4.3).

*Proof of Proposition 3.4.4.* Note that the symbol  $\{h(q_0), q_0\}$  is defined in  $K_2^M(\widehat{A})$ . Since  $\Omega_{k((q_0))}^2 = 0$ , we have

$$\{h(q_0), q_0\} = 0 \in K_2^M(\widehat{A})/p^n \text{ for any } n \geq 1, \quad (3.4.5)$$

by (3.4.2) and Proposition 2.3.11. Let  $\phi : \widehat{A} \rightarrow \widehat{A}$  be the Frobenius endomorphism of  $\widehat{A}$  defined by the canonical Frobenius automorphism on  $R$  and the assignment  $q_0 \mapsto q_0^p$ . Define the function  $\ell_\phi : \widehat{A} \rightarrow \widehat{A}$  as

$$\ell_\phi(a) := \frac{1}{p} \log \left( \frac{\phi(a)}{a^p} \right).$$

It follows from (3.4.5) and the explicit reciprocity law ([Ka4] Corollary 2.9), that

$$\ell_\phi(h(q_0)) \frac{dq_0}{q_0} - \ell_\phi(q_0) \frac{dh(q_0)}{h(q_0)} \in d\widehat{A}.$$

Since  $\ell_\phi(q_0) = 0$  by definition, there exists  $\alpha \in \widehat{A}$  satisfying

$$\ell_\phi(h(q_0)) = q_0 \frac{d\alpha}{dq_0}. \quad (3.4.6)$$

Because  $\ell_\phi(fg) = \ell_\phi(f) + \ell_\phi(g)$  and

$$\ell_\phi(1 - rq_0) = \sum_{(n,p)=1} \frac{(rq_0)^n}{n} \quad (r \in R)$$

by definition, we have

$$\ell_\phi(h(q_0)) = \ell_\phi(c) + \sum_{j \geq 1} \sum_{(n,p)=1} \left( a'_{1j} \frac{(\zeta_1 q_0^j)^n}{n} + \cdots + a'_{dj} \frac{(\zeta_d q_0^j)^n}{n} \right).$$

Comparing these coefficients with those in the right hand side of (3.4.6), one can easily check that  $a'_{ij}$  is divisible by  $j$  in  $\mathbb{Z}_p$ .  $\square$

Thus we obtained Theorem 3.2.3, assuming the diagram (3.3.1).

## 4 Construction of the key diagram

The notation remains as in §3.1. In this section, we construct the key homomorphism

$$\tau_\infty^{\text{syn}} : H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \rightarrow H_{\text{syn}}^1(A, \mathbb{Z}_p(1))$$

to establish the commutative diagram (3.3.1).

### 4.1 Preliminary

We define the Frobenius map  $\phi : B \rightarrow B$  as

$$\phi(q_0) = q_0^p \quad \text{and} \quad \phi(a) = \sigma(a) \quad (a \in R), \quad (4.1.1)$$

where  $\sigma$  denotes the canonical Frobenius automorphism of  $R$ . Let  $\mathcal{B}$  (resp.  $\mathcal{R}$ ) be the  $(q_0, p)$ -adic completion of  $B[u, u^{-1}]$  (resp.  $p$ -adic completion of  $R[u, u^{-1}]$ ):

$$\mathcal{B} := \varprojlim_n B[u, u^{-1}]/(q_0, p)^n, \quad \mathcal{R} := \varprojlim_n R[u, u^{-1}]/(p^n).$$

We extend the Frobenius map  $\phi$  on  $B$  to  $\mathcal{B}$  by defining  $\phi(u) := u^p$ . We define the Frobenius map  $\phi$  on  $\mathcal{R}$  as  $\phi(u) := u^p$  and  $\phi(a) = \sigma(a)$  ( $a \in R$ ). The ring homomorphism in Proposition 2.4.1 induces a morphism of schemes

$$\beta : \text{Spec}(\mathcal{B}) \longrightarrow \mathcal{E}, \quad (4.1.2)$$

where  $\mathcal{E}$  is as we defined in §3.1. For a scheme (or a ring)  $Z$  and  $n \geq 1$ , we put

$$Z_n := Z \otimes \mathbb{Z}/(p^n).$$

We have a Frobenius endomorphism  $\varphi : E \rightarrow E$  by (4.1.1) and the construction in §2.5.

**Lemma 4.1.3** *For any integer  $n \geq 1$ , the Frobenius endomorphism  $\varphi_n : E_n \rightarrow E_n$  extends to a Frobenius endomorphism  $\varphi_n : (\mathcal{E}_n, M_n) \rightarrow (\mathcal{E}_n, M_n)$ . Furthermore,  $\varphi_{n+1} \otimes \mathbb{Z}/p^n$  agrees with  $\varphi_n$  for any  $n \geq 1$ .*

This lemma has been used in Lemma 3.1.1.

*Proof.* Let  $Z$  be the singular locus of  $\mathcal{D}$  (see §3.1 for the definition of  $\mathcal{D}$ ). Note that  $X := \mathcal{E} - Z$  is a commutative group scheme over  $\text{Spec}(B)$ . In particular, the multiplication by  $p$  is defined on  $X$  with respect to its group structure, and there is an endomorphism

$$\varphi : X \longrightarrow X$$

defined in a similar way as for  $\varphi$  on  $E$ . By Lemma 2.5.3,  $\varphi$  is a Frobenius endomorphism of  $X$ , i.e.,  $\varphi_1 : X_1 \rightarrow X_1$  induced by  $\varphi$  is the absolute Frobenius of  $X_1$ , because  $X_1$  and

$E_1 = (E_q)_1$  have the same function field. Since  $\mathcal{D}$  is defined by  $q_0$  and  $\varphi$  sends  $q_0$  to  $q_0^p$ , the map  $\varphi : X \rightarrow X$  induces a Frobenius endomorphism

$$\varphi : (X, M_X) \longrightarrow (X, M_X)$$

of log schemes, where  $M_X$  denotes the log structure on  $X$  associated with the regular divisor  $\mathcal{D} - Z$  (see Example 2.2.2).

We start the proof of the lemma. Because a Frobenius endomorphism on  $\mathcal{E}_n$  is the identity map on the underlying topological space, we define an endomorphism  $\psi : \mathcal{O}_{\mathcal{E}_n} \rightarrow \mathcal{O}_{\mathcal{E}_n}$  which lifts the absolute endomorphism of  $\mathcal{O}_{\mathcal{E}_1}$  and which is compatible with  $\varphi_n^*$  on  $\mathcal{O}_{X_n}$ . Let  $\alpha : X \hookrightarrow \mathcal{E}$  and  $\alpha_n : X_n \hookrightarrow \mathcal{E}_n$  be the natural open embeddings. We first show that the natural adjunction map

$$\mathcal{O}_{\mathcal{E}_n} \longrightarrow \alpha_{n*} \mathcal{O}_{X_n} \tag{4.1.4}$$

is bijective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{E}} & \xrightarrow{\times p} & \mathcal{O}_{\mathcal{E}} & \longrightarrow & \mathcal{O}_{\mathcal{E}_1} & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & \alpha_* \mathcal{O}_X & \xrightarrow{\times p} & \alpha_* \mathcal{O}_X & \longrightarrow & \alpha_{1*} \mathcal{O}_{X_1} & \longrightarrow & R^1 \alpha_* \mathcal{O}_X \xrightarrow{\times p} R^1 \alpha_* \mathcal{O}_X. \end{array}$$

Here the vertical arrows are natural adjunction maps, which are bijective by the facts that  $\mathcal{E}$  and  $\mathcal{E}_1$  are regular and that the complement  $Z$  (resp.  $Z_1$ ) has pure codimension 2 in  $\mathcal{E}$  (resp. in  $\mathcal{E}_1$ ). Hence  $R^1 \alpha_* \mathcal{O}_X$  is  $p$ -torsion-free and we obtain the bijectivity of (4.1.4) by repeating a similar argument using  $\mathcal{O}_{\mathcal{E}_n}$  and  $\alpha_{n*} \mathcal{O}_{X_n}$  instead of  $\mathcal{O}_{\mathcal{E}_1}$  and  $\alpha_{1*} \mathcal{O}_{X_1}$ . We define  $\psi : \mathcal{O}_{\mathcal{E}_n} \rightarrow \mathcal{O}_{\mathcal{E}_n}$  as the composite map

$$\mathcal{O}_{\mathcal{E}_n} \simeq \alpha_{n*} \mathcal{O}_{X_n} \xrightarrow{\alpha_{n*}(\varphi_n^*)} \alpha_{n*}(\mathcal{O}_{X_n}) \simeq \mathcal{O}_{\mathcal{E}_n}.$$

It is clear that  $\psi$  lifts the absolute Frobenius endomorphism of  $\mathcal{O}_{\mathcal{E}_1}$ . Thus we obtain a Frobenius endomorphism  $\varphi_n$  on  $\mathcal{E}_n$  which extends  $\varphi_n$  on  $X_n$ , and it is easy to see that this induces a Frobenius endomorphism  $\varphi_n$  on  $(\mathcal{E}_n, M_n)$ . By the construction, this morphism is the only morphism that extends  $\varphi_n$  on  $(X_n, M_{X_n})$ , and the compatibility assertion  $\varphi_{n+1} \otimes \mathbb{Z}/p^n = \varphi_n$  follows from the fact that  $\varphi_n$  on  $(X_n, M_{X_n})$  is induced by  $\varphi$  on  $(X, M_X)$ .  $\square$

## 4.2 Syntomic residue mapping

To construct  $\tau_\infty^{\text{syn}}$ , we first review the definition of residue mappings.

**Definition 4.2.1** *Let  $m$  be a positive interger, and put  $C := B/(q_0, p)^m$  and  $C((u)) := C[[u]][u^{-1}]$ . We define residue mappings*

$$\text{Res}_{u=0} : \Omega_{C((u))}^r \longrightarrow \Omega_C^{r-1}$$

as follows. For  $r = 1$ , we define  $\text{Res}_{u=0}$  as the composite map

$$\Omega_{C((u))}^1 \xrightarrow{\text{can}} \Omega_{C((u))/C}^1 \simeq C((u)) \cdot du \longrightarrow C$$

where the last arrows sends  $c_i u^i du$  ( $c_i \in C$ ) to 0 if  $i \neq -1$ , and to  $c_{-1}$  if  $i = -1$ . For  $r \geq 2$ , we define  $\text{Res}_{u=0}$  as

$$\Omega_{C((u))}^r \xrightarrow{\text{can}} \Omega_C^{r-1} \otimes \Omega_{C((u))/C}^1 \xrightarrow{\text{id} \otimes \text{Res}_{u=0}} \Omega_C^{r-1}.$$

Since  $\mathcal{A}$  has a Frobenius endomorphism  $\phi$  (cf. §4.1), the syntomic cohomology groups  $H_{\text{syn}}^*(\mathcal{A}, \mathcal{S}_n(2))$  are computed by the complex

$$\mathcal{S}_n(2)_{\mathcal{A}, \mathcal{A}} := \text{mapping fiber of } (1 - f_2 : \Omega_{\mathcal{A}_n}^{\bullet \geq 2} \longrightarrow \Omega_{\mathcal{A}_n}^{\bullet})$$

(see Remark 2.1.8). Similarly,  $H_{\text{syn}}^*(A, \mathcal{S}_n(1))$  are computed by the complex

$$\mathcal{S}_n(1)_{A, A} := \text{mapping fiber of } (1 - f_1 : \Omega_{A_n}^{\bullet \geq 1} \longrightarrow \Omega_{A_n}^{\bullet}).$$

We define a residue mapping

$$\varrho_{\mathcal{B}}^r : \Omega_{\mathcal{B}}^r \longrightarrow \varprojlim_m \Omega_{B/(q_0, p)^m}^{r-1} = \Omega_B^{r-1}$$

as the projective limit, with respect to  $m$ , of the composite map:

$$\Omega_{\mathcal{B}}^r \xrightarrow{\text{canonical}} \Omega_{B((u))/(q_0, p)^m}^r \xrightarrow{\text{Res}_{u=0}} \Omega_{B/(q_0, p)^m}^{r-1}.$$

The map  $\varrho_{\mathcal{B}}^r$  induces a map

$$\varrho_{\mathcal{B}, n}^r : \Omega_{\mathcal{B}_n}^r \longrightarrow \Omega_{B_n}^{r-1}. \quad (4.2.2)$$

Inverting  $q_0$ , we get a map

$$\varrho_n^r : \Omega_{\mathcal{A}_n}^r \longrightarrow \Omega_{A_n}^{r-1}. \quad (4.2.3)$$

The following lemma is straight-forward and left to the reader:

**Lemma 4.2.4** *The following square is commutative:*

$$\begin{array}{ccc} \Omega_{\mathcal{A}_n}^r & \xrightarrow{f_2} & \Omega_{\mathcal{A}_n}^r \\ \varrho_n^r \downarrow & & \downarrow \varrho_n^r \\ \Omega_{A_n}^{r-1} & \xrightarrow{f_1} & \Omega_{A_n}^{r-1}. \end{array}$$

By this lemma, the maps  $\varrho_n^{\bullet}$  induce a homomorphism of complexes

$$\mathcal{S}_n(2)_{\mathcal{A}, \mathcal{A}} \longrightarrow \mathcal{S}_n(1)_{A, A}[-1] \quad \text{for } n \geq 1$$

and a residue map of syntomic cohomology

$$\widehat{\tau}_{\infty}^{\text{syn}} : H_{\text{syn}}^2(\mathcal{A}, \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow H_{\text{syn}}^1(A, \mathcal{S}_{\mathbb{Z}_p}(1)). \quad (4.2.5)$$

We define the required arrow  $\tau_{\infty}^{\text{syn}}$  in the diagram (3.3.1) as the composite map

$$\tau_{\infty}^{\text{syn}} : H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\beta^*} H_{\text{syn}}^2(\mathcal{A}, \mathcal{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\widehat{\tau}_{\infty}^{\text{syn}}} H_{\text{syn}}^1(A, \mathcal{S}_{\mathbb{Z}_p}(1)),$$

where  $\beta^*$  denotes the pull-back by  $\beta : \text{Spec}(\mathcal{B}) \rightarrow \mathcal{E}$  in (4.1.2).

**Remark 4.2.5** By (4.2.2), we obtain a residue map

$$\widehat{\tau}_\infty^{\text{syn}} : H_{\text{syn}}^2(\mathcal{B}, \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow H_{\text{syn}}^1(B, \mathcal{S}_{\mathbb{Z}_p}(1)).$$

By a similar construction, we obtain a residue map

$$\widehat{\tau}_\infty^{\text{syn}} : H_{\text{syn}}^2(\mathcal{R}, \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow H_{\text{syn}}^1(R, \mathcal{S}_{\mathbb{Z}_p}(1)).$$

These maps will be used in §4.7 below.

### 4.3 Étale residue mapping

Let  $R[u]_{(p)}$  be the localization of  $R[u]$  at the prime ideal  $(p)$ , and denote its  $p$ -adic completion by  $\mathcal{R}^b$ :

$$\mathcal{R}^b := \varprojlim_n R[u]_{(p)}/(p^n),$$

which is a complete discrete valuation ring with residue field  $k(u)$ . There is a natural embedding of  $R[u, u^{-1}]$ -algebras

$$\mathcal{R} \hookrightarrow \mathcal{R}^b.$$

Let  $K$  be the fraction field of  $R$ , and put

$$\mathcal{R}_K^b := \mathcal{R}^b \otimes_R K = \mathcal{R}^b[p^{-1}].$$

The following lemma will be used in Proposition 4.3.4 below.

**Lemma 4.3.1** For  $n \geq 1$  and  $f \in (\mathcal{R}^b)^\times$  (resp.  $f \in (\mathcal{R}_K^b)^\times$ ), there are  $f_0 \in R[u]_{(p)}^\times$  and  $g \in \mathcal{R}^b$  (resp.  $f_0 \in K(u)^\times$  and  $g \in \mathcal{R}_K^b$ ) with  $f = f_0(1 + p^n g)$ . Consequently the natural maps

$$R[u]_{(p)}^\times/p^n \longrightarrow (\mathcal{R}^b)^\times/p^n, \quad K(u)^\times/p^n \longrightarrow (\mathcal{R}_K^b)^\times/p^n$$

are surjective for any  $n \geq 1$ .

*Proof.* Exercise. □

We construct here an auxiliary residue mapping

$$\widetilde{\tau}_\infty^{\text{ét}} : H_{\text{ét}}^2(\mathcal{R}_K^b, \mathbb{Z}/p^n(2)) \longrightarrow H_{\text{ét}}^1(K, \mathbb{Z}/p^n(1)),$$

which will be useful in §4.7 below. Let  $\overline{K}$  be the algebraic closure of  $K$ , and put  $\mathcal{R}_{\overline{K}}^b := \mathcal{R}_K^b \otimes_K \overline{K}$ . We have a  $G_K$ -equivariant homomorphism

$$(\mathcal{R}_{\overline{K}}^b)^\times/p^n \longrightarrow \mathbb{Z}/p^n, \quad f \longmapsto \text{Res}_{u=0} \frac{df}{f}, \quad (4.3.1)$$

where  $\text{Res}_{u=0}$  denotes a residue map defined in a similar way as for  $\varrho_{\mathcal{B},n}^1$  in (4.2.2). Since  $H_{\text{ét}}^1(\mathcal{R}_{\overline{K}}^b, \mathbb{Z}/p^n(1)) \simeq (\mathcal{R}_{\overline{K}}^b)^\times/p^n$ , this map induces a  $G_K$ -equivariant homomorphism

$$H_{\text{ét}}^1(\mathcal{R}_{\overline{K}}^b, \mathbb{Z}/p^n(1)) \longrightarrow \mathbb{Z}/p^n. \quad (4.3.2)$$

We define the desired map  $\tilde{\tau}_\infty^{\text{ét}}$  by the composition

$$\tilde{\tau}_\infty^{\text{ét}} : H_{\text{ét}}^2(\mathcal{R}_K^b, \mathbb{Z}/p^n(2)) \longrightarrow H_{\text{ét}}^1(K, H_{\text{ét}}^1(\mathcal{R}_{\overline{K}}^b, \mathbb{Z}/p^n(2))) \xrightarrow{(4.3.2)} H_{\text{ét}}^1(K, \mathbb{Z}/p^n(1)),$$

where the first arrow is an edge homomorphism of the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H_{\text{ét}}^a(K, H_{\text{ét}}^b(\mathcal{R}_{\overline{K}}^b, \mathbb{Z}/p^n(2))) \implies H_{\text{ét}}^{a+b}(\mathcal{R}_K^b, \mathbb{Z}/p^n(2))$$

and we have used the following fact:

**Proposition 4.3.4** *The natural restriction map*

$$H_{\text{ét}}^2(\mathcal{R}_K^b, \mathbb{Z}/p^n(2)) \longrightarrow E_2^{0,2} = H_{\text{ét}}^2(\mathcal{R}_{\overline{K}}^b, \mathbb{Z}/p^n(2))^{G_K}$$

is zero for any  $n \geq 1$ .

*Proof.* We show that the natural map

$$K_2^M(\mathcal{R}_K^b)/p^n \longrightarrow K_2^M(\mathcal{R}_{\overline{K}}^b)/p^n$$

is zero, which implies the assertion by Remark 2.3.4 (2). Let  $K(u)$  be the rational function field in  $u$  over  $K$ , which is a subfield of  $\mathcal{R}_K^b$ . By Lemma 4.3.1 the natural map

$$K_2^M(K(u))/p^n \longrightarrow K_2^M(\mathcal{R}_K^b)/p^n$$

is surjective. Hence it is enough to show that the natural map

$$K_2^M(K(u))/p^n \longrightarrow K_2^M(\overline{K}(u))/p^n$$

is zero. One can see  $K_2^M(\overline{K}(u))/p^n = 0$  in the following way. Since

$$\{\overline{K}^\times, \overline{K}(u)^\times\} = 0 \quad \text{in} \quad K_2^M(\overline{K}(u))/p^n,$$

it is enough to show that  $\{u-a, b-u\} = 0$ . If  $a = b$ , it is clear. If  $a \neq b$ , one may replace  $(u-a)$  with  $(u-a)/(b-a)$  and  $(b-u)$  with  $(b-u)/(b-a)$ . Then one has

$$\left\{ \frac{u-a}{b-a}, \frac{b-u}{b-a} \right\} = 0 \quad \text{as} \quad \frac{u-a}{b-a} + \frac{b-u}{b-a} = 1,$$

which shows the assertion. □

The following proposition will be used in §4.7 below.

**Proposition 4.3.5** *Let  $m$  be a positive integer, and let  $s_m : B \rightarrow R$  be the homomorphism of  $R$ -algebras sending  $q_0$  to  $p^m$ . Consider the following cartesian diagram of schemes:*

$$\begin{array}{ccccccc} \text{Spec}(\mathcal{R}^b) & \xrightarrow{\gamma} & \text{Spec}(\mathcal{B}_{(m)}) & \xrightarrow{\alpha_m} & \mathcal{E}_{(m)} & \longrightarrow & \text{Spec}(R) \\ & & \downarrow & \square & \downarrow & \square & \downarrow s_m \\ & & \text{Spec}(\mathcal{B}) & \xrightarrow{\beta} & \mathcal{E} & \longrightarrow & \text{Spec}(B), \end{array}$$



Therefore the commutativity of the outer rectangle is reduced to that of the central square, i.e., we are reduced to the case that  $r \geq 3$  (because we have  $q = q_0^r = q_1^{3r}$  and  $(p, 3r) = 1$  by the assumptions on  $p$  and  $r$ ).

We may thus assume  $r \geq 3$ . Then we introduce an element  $\xi_{\mathcal{E}} \in H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2))$  as follows. Let  $Z_i \subset \mathcal{E}$  be the section defined by  $u = q_0^i$ . Put

$$\mathcal{D}_i := Z_i \cap \mathcal{D} \quad \text{and} \quad Z := \sum_{i=0}^{r-1} Z_i.$$

Noting  $H_{\text{syn}}^0(Z_i(\mathcal{D}_i), \mathcal{S}_{\mathbb{Z}_p}(1)) = 0$ , we obtain, from Proposition 2.2.14, an exact sequence

$$0 \rightarrow H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \rightarrow H_{\text{syn}}^2(\mathcal{E}(\mathcal{D} + Z), \mathcal{S}_{\mathbb{Z}_p}(2)) \xrightarrow{\partial} \bigoplus_{i=0}^{r-1} H_{\text{syn}}^1(Z_i(\mathcal{D}_i), \mathcal{S}_{\mathbb{Z}_p}(1)). \quad (4.4.1)$$

Let  $0 < a < b < r$  be integers. Put

$$f(u) := \frac{\theta(q_0^a u)^r}{\theta(u)^{r-a} \theta(q_0^a)^a} = (-u)^a \frac{\theta(q_0^a u)^r}{\theta(u)^r} \quad \text{and} \quad g(u) := \frac{\theta(q_0^b u)^r}{\theta(u)^{r-b} \theta(q_0^b)^b} = (-u)^b \frac{\theta(q_0^b u)^r}{\theta(u)^r},$$

which are rational functions on  $E = E_{q,A}$  by Proposition 2.4.4. Put

$$\xi'_{\mathcal{E}} := \left\{ \frac{f(u)}{f(q_0^{-b})}, \frac{g(u)}{g(q_0^{-a})} \right\}^{\text{syn}} = \left\{ \frac{f(u)}{f(q_0^{-b})} \right\}^{\text{syn}} \cup \left\{ \frac{g(u)}{g(q_0^{-a})} \right\}^{\text{syn}} \in H_{\text{syn}}^2(\mathcal{E}(\mathcal{D} + Z), \mathcal{S}_{\mathbb{Z}_p}(2)),$$

and the braces  $\{-\}^{\text{syn}}$  denote the syntomic symbol in (2.3.5) and (2.3.6) with  $U := \mathcal{E} - \mathcal{D} - Z$ . One can easily check that the boundary  $\partial(\xi'_{\mathcal{E}})$  agrees with the syntomic symbol of the tame symbol of

$$\left\{ \frac{f(u)}{f(q_0^{-b})}, \frac{g(u)}{g(q_0^{-a})} \right\} \in K_2^M(\mathcal{O}(U)).$$

The tame symbol vanishes at each  $Z_i - D_i$  by a straight-forward computation. Therefore one has  $\partial(\xi'_{\mathcal{E}}) = 0$ , and  $\xi'_{\mathcal{E}}$  defines an element  $\xi_{\mathcal{E}} \in H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2))$  by the exact sequence (4.4.1).

**Proposition 4.4.2**  $H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2))$  is generated by the subgroups

$$H_{\text{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)), \quad R_{\text{dR}} \cup \{q_0\}^{\text{syn}} \quad \text{and} \quad \mathbb{Z}_p \xi_{\mathcal{E}},$$

where  $R_{\text{dR}}$  denotes the image of the natural map

$$R = \varprojlim_{n \geq 1} H_{\text{dR}}^0(\mathcal{E}_n) \longrightarrow H_{\text{syn}}^1(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(1))$$

induced by (2.1.3), and the braces  $\{-\}^{\text{syn}}$  denote the syntomic symbol in (2.3.5):

$$\{-\}^{\text{syn}} : \Gamma(E, \mathcal{O}_E^\times) \longrightarrow H_{\text{syn}}^1(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(1)).$$

This proposition will be proved in §4.5 below. We explain how to prove the commutativity of (3.3.1). It is enough, by Proposition 4.4.2, to check the commutativity for the elements of

$$R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}, \quad \mathbb{Z}_p \xi_{\mathcal{E}} \quad \text{and} \quad H_{\mathrm{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)).$$

The commutativity for  $R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}$  is clear. Indeed, we have

$$\tau_{\infty}^{\mathrm{syn}}(R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}) = 0 \quad \text{in} \quad H_{\mathrm{syn}}^1(A, \mathcal{S}_{\mathbb{Z}_p}(1))$$

by the construction of  $\tau_{\infty}^{\mathrm{syn}}$ . On the other hand, we have

$$\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}} \circ c^2(R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}}) = \tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}(\{1 + pR\}^{\acute{\mathrm{e}}\mathrm{t}} \cup \{q_0\}^{\acute{\mathrm{e}}\mathrm{t}}) = 0 \quad \text{in} \quad H_{\acute{\mathrm{e}}\mathrm{t}}^1(L, \mathbb{Z}_p(1))$$

by Remark 2.3.7 and Theorem 2.3.8. We will show

$$c^1 \circ \tau_{\infty}^{\mathrm{syn}}(\xi_{\mathcal{E}}) = \tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}} \circ c^2(\xi_{\mathcal{E}}) \quad \text{in} \quad H_{\acute{\mathrm{e}}\mathrm{t}}^1(L, \mathbb{Z}_p(1)) \simeq L^{\times}/p^n \quad (4.4.3)$$

in §4.6 below. As for the commutativity for the elements of  $H_{\mathrm{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2))$ , it is equivalent to the commutativity of the following square:

$$\begin{array}{ccc} H_{\mathrm{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{\tau_{\infty}^{\mathrm{syn}}} & H_{\mathrm{syn}}^1(B, \mathcal{S}_{\mathbb{Z}_p}(1)) \\ c^2 \downarrow & & \downarrow c^1 \\ H_{\acute{\mathrm{e}}\mathrm{t}}^2(\mathcal{E}[p^{-1}], \mathbb{Z}_p(2)) & \xrightarrow{\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}} & H_{\acute{\mathrm{e}}\mathrm{t}}^1(B[p^{-1}], \mathbb{Z}_p(1)), \end{array} \quad (4.4.4)$$

where the arrow  $\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}$  is defined by the composite map of sheaves on  $B[p^{-1}]^{\acute{\mathrm{e}}\mathrm{t}}$

$$R^1 f_{\mathcal{E}*} \mathbb{Z}/p^n(2) \longrightarrow j_* R^1 f_* \mathbb{Z}/p^n(2) \xrightarrow{(2.4.7)} j_* \mathbb{Z}/p^n(1) = \mathbb{Z}/p^n(1) \quad (n \geq 1)$$

and similar arguments as for  $\tau_{\infty}^{\acute{\mathrm{e}}\mathrm{t}}$  in the diagram (3.3.1). Here  $f_{\mathcal{E}}$ ,  $f$  and  $j$  are as follows:

$$\begin{array}{ccc} E[p^{-1}] & \hookrightarrow & \mathcal{E}[p^{-1}] \\ f \downarrow & & \downarrow f_{\mathcal{E}} \\ \mathrm{Spec}(A[p^{-1}]) & \xhookrightarrow{j} & \mathrm{Spec}(B[p^{-1}]). \end{array}$$

We will prove the commutativity of (4.4.4) in §4.7 below.

## 4.5 Proof of Proposition 4.4.2

Put

$$\mathcal{E}^{(0)} := \mathcal{E}, \quad \mathcal{E}^{(1)} := \mathcal{D}^{(1)}, \quad \mathcal{E}^{(2)} := \mathcal{D}^{(2)},$$

where  $\mathcal{D}^{(1)}$  denotes the disjoint union of the irreducible components of  $\mathcal{D}$ , and  $\mathcal{D}^{(2)}$  denotes the disjoint union of the intersections of two distinct irreducible components of  $\mathcal{D}$ . Since  $\mathcal{D}$  is the standard Néron  $r$ -gon over  $\mathrm{Spec}(R)$ , we have

$$\mathcal{E}^{(1)} = \prod_{j=1}^r \mathbb{P}_R^1 \quad \text{and} \quad \mathcal{E}^{(2)} = \prod_{j=1}^r \mathrm{Spec}(R).$$

Fix an integer  $n \geq 1$ . One can easily check that  $\mathcal{E}$  and  $\mathcal{E}^{(m)}$  ( $m = 0, 1, 2$ ) satisfy the conditions (i)–(iv) in Proposition 2.2.14, using Lemma 3.1.1 and Remark 3.1.2. Hence there is a spectral sequence

$$E_1^{a,b} = H_{\text{syn}}^{2a+b}(\mathcal{E}^{(-a)}, \mathcal{S}_n(a+2)) \implies H_{\text{syn}}^{a+b}(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(2)). \quad (4.5.1)$$

Since we have

$$E_1^{a,b} = 0 \quad \text{unless } -2 \leq a \leq 0 \text{ and } 2a + b \geq 0, \quad (4.5.2)$$

the quotient group  $C_n := H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(2))/E_{\infty}^{0,2}$  fits into an exact sequence

$$0 \longrightarrow E_2^{-1,3} \longrightarrow C_n \longrightarrow E_2^{-2,4} \longrightarrow E_2^{0,3}.$$

We will prove the following lemma:

**Lemma 4.5.3** (1) Put  $R_n := R/(p^n)$ . Then  $E_2^{-1,3}$  agrees with the diagonal subgroup of

$$E_1^{-1,3} = \bigoplus_{j=1}^r H_{\text{syn}}^1(\mathbb{P}_R^1, \mathcal{S}_n(1)) \simeq \bigoplus_{j=1}^r R_n.$$

(2)  $E_2^{-2,4}$  agrees with the diagonal subgroup of

$$E_1^{-2,4} = \bigoplus_{j=1}^r H_{\text{syn}}^0(R, \mathcal{S}_n(0)) \simeq \bigoplus_{j=1}^r \mathbb{Z}/p^n.$$

Moreover, the edge homomorphism

$$H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(2)) \longrightarrow E_1^{-2,4}$$

sends  $\xi_{\mathcal{E}}$  to  $(r, r, \dots, r)$  up to a sign.

We first prove Proposition 4.4.2, admitting this lemma. Consider the spectral sequence (4.5.1). Since  $H_{\text{syn}}^0(\mathbb{P}_R^1, \mathcal{S}_n(1)) = 0$  by (2.1.9), we have

$$E_1^{0,2} = E_2^{0,2} \xrightarrow{(4.5.2)} E_{\infty}^{0,2}$$

and hence a short exact sequence

$$0 \longrightarrow H_{\text{syn}}^2(\mathcal{E}, \mathcal{S}_n(2)) \longrightarrow H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(2)) \longrightarrow C_n \longrightarrow 0.$$

Taking projective limit with respect to  $n \geq 1$ , we get an exact sequence

$$0 \longrightarrow H_{\text{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow H_{\text{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow \varprojlim_{n \geq 1} C_n.$$

By Lemma 4.5.3, there is a short exact sequence

$$0 \longrightarrow R_n \longrightarrow C_n \longrightarrow \mathbb{Z}/p^n \longrightarrow 0, \quad (4.5.4)$$

which shows that  $\varprojlim_{n \geq 1} C_n$  is generated by elements which lift to either

$$R_{\mathrm{dR}} \cup \{q_0\}^{\mathrm{syn}} \quad \text{or} \quad \mathbb{Z}_p \xi_{\mathcal{E}} \quad (\subset H_{\mathrm{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2))).$$

Thus we obtain the proposition. We prove Lemma 4.5.3 in what follows.

*Proof of Lemma 4.5.3.* The assertion on the the edge homomorphism  $H_{\mathrm{syn}}^2(\mathcal{E}(\mathcal{D}), \mathcal{S}_n(2)) \rightarrow E_1^{-2,4}$  follows from computations on symbols (cf. [A1] Remark 5.5). Thus  $E_2^{-2,4}$  contains the diagonal subgroup of  $E_1^{-2,4}$ , by the assumption that  $p$  is prime to  $r$ . Similarly, it is easy to see that  $E_2^{-1,3}$  contains the diagonal subgroup of  $E_1^{-1,3}$ . It remains to show that  $E_2^{-1,3}$  and  $E_2^{-2,4}$  are contained in the diagonal subgroups of  $E_1^{-1,3}$  and  $E_1^{-2,4}$ , respectively.

Recall that  $\mathcal{D}$  is the standard Néron  $r$ -gon over  $\mathrm{Spec}(R)$ . Let  $\mathcal{D}_1, \dots, \mathcal{D}_r$  be the irreducible components of  $\mathcal{D}$ , which are all isomorphic to  $\mathbb{P}_R^1$ . Changing the ordering of these components if necessary, we suppose that  $\mathcal{D}_1$  meets  $\mathcal{D}_r$  and  $\mathcal{D}_2$ , and that  $\mathcal{D}_j$  meets  $\mathcal{D}_{j-1}$  and  $\mathcal{D}_{j+1}$  for  $j = 2, \dots, r-1$  (hence  $\mathcal{D}_r$  meets  $\mathcal{D}_{r-1}$  and  $\mathcal{D}_1$ ). Put

$$T_j := \mathcal{D}_j \times_{\mathcal{E}} \mathcal{D}_{j+1} \quad (j = 1, \dots, r-1), \quad T_r := \mathcal{D}_r \times_{\mathcal{E}} \mathcal{D}_1,$$

which are all isomorphic to  $\mathrm{Spec}(R)$ . Let

$$i : \mathcal{E}^{(1)} = \prod_{j=1}^r \mathcal{D}_j \longrightarrow \mathcal{E}$$

be the natural finite morphism. Since  $\mathcal{D}_j \simeq \mathbb{P}_R^1$  for each  $1 \leq j \leq r$ , we have isomorphisms

$$\begin{aligned} H_{\mathrm{syn}}^1(\mathcal{D}_j, \mathcal{S}_n(1)) &\simeq H_{\mathrm{dR}}^0(\mathcal{D}_{j,n}) \simeq R_n, \\ H_{\mathrm{syn}}^2(\mathcal{D}_j, \mathcal{S}_n(1)) &\simeq \mathrm{Ker}(1 - \sigma : R_n \rightarrow R_n) = \mathbb{Z}/p^n, \\ H_{\mathrm{syn}}^3(\mathcal{D}_j, \mathcal{S}_n(2)) &\simeq H_{\mathrm{dR}}^2(\mathcal{D}_{j,n}) \simeq R_n \quad (\text{trace isomorphism}), \end{aligned}$$

by (2.1.9). Lemma 4.5.3 follows from the following claims (i) and (ii):

(i) *The composite map*

$$E_1^{-1,3} = \bigoplus_{j=1}^r R_n = \bigoplus_{j=1}^r H_{\mathrm{dR}}^0(\mathcal{D}_{j,n}) \xrightarrow{i_*} H_{\mathrm{dR}}^2(\mathcal{E}_n) \xrightarrow{i^*} \bigoplus_{j=1}^r H_{\mathrm{dR}}^2(\mathcal{D}_{j,n}) \simeq \bigoplus_{j=1}^r R_n$$

is given by the  $r \times r$  matrix

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \ddots & \vdots & 0 \\ 0 & 1 & -2 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

Consequently,  $E_2^{-1,3}$  is contained in the diagonal subgroup of  $E_1^{-1,3}$  by the assumption that  $r$  is prime to  $p$  (note also that the above composite map agrees with the composite map  $E_1^{-1,3} \rightarrow E_1^{0,3} \rightarrow \bigoplus_{j=1}^r H_{\text{syn}}^3(\mathcal{D}_j, \mathcal{S}_n(2)) \simeq \bigoplus_{j=1}^r R_n$ ).

(ii) *The edge homomorphism*

$$E_1^{-2,4} = \bigoplus_{j=1}^r \mathbb{Z}/p^n[T_j] \longrightarrow E_1^{-1,4} = \bigoplus_{j=1}^r H_{\text{syn}}^2(\mathcal{D}_j, \mathcal{S}_n(1)) \simeq \mathbb{Z}/p^n[D_j]$$

sends  $[T_j]$  to

$$\begin{cases} [D_j] - [D_{j+1}] & (1 \leq j \leq r-1) \\ [D_1] - [D_r] & (j = r), \end{cases}$$

where the signs arise from the construction of the spectral sequence (4.5.1). Consequently,  $E_2^{-2,4}$  is contained in the diagonal subgroup of  $E_1^{-2,4}$ .

These claims follows from standard facts on intersection theory of divisors. The details are straight-forward and left to the reader. This completes the proof of Lemma 4.5.3 and Proposition 4.4.2.

## 4.6 Proof of (4.4.3)

This subsection is devoted to showing (4.4.3). More precisely, we will prove

$$\begin{aligned} c^1 \circ \tau_{\infty}^{\text{syn}}(\xi_{\mathcal{E}}) &= (-1)^{a(r-b)} q_0^{a(b-a)(b-r)} \left( \frac{\theta(q_0^b)^b}{\theta(q_0^{b-a})^{b-a} \theta(q_0^a)^a} \right)^r \left( \frac{S(q_0^b)}{S(q_0^{b-a})S(q_0^a)} \right)^{r^2} \\ &= \tau_{\infty}^{\text{ét}} \circ c^2(\xi_{\mathcal{E}}) \end{aligned}$$

in  $L^\times/p^n$ , where we put

$$S(\alpha) := \prod_{k=1}^{\infty} \left( \frac{1 - \alpha q^k}{1 - \alpha^{-1} q^k} \right)^k.$$

Since the equality for  $\tau_{\infty}^{\text{ét}}$  has been shown in [A1] Lemma 7.4, we prove the equality for  $\tau_{\infty}^{\text{syn}}$ . It follows from the construction of the syntomic residue map (§4.2) that we may truncate  $\theta(u)$  with respect to  $q_0$ . Therefore we can calculate  $\tau_{\infty}^{\text{syn}}$  by factorizing the infinite products. It is enough to check the following:

$$\widehat{\tau}_{\infty}^{\text{syn}}\{a, b\} = 1, \quad a, b \in A^\times \quad (4.6.1)$$

$$\widehat{\tau}_{\infty}^{\text{syn}}\{a, u\} = a, \quad a \in A^\times \quad (4.6.2)$$

$$\widehat{\tau}_{\infty}^{\text{syn}}\{a, g(u)\} = 1, \quad a \in A^\times, g(u) \in R[[q_0, u]]^\times \quad (4.6.3)$$

$$\widehat{\tau}_{\infty}^{\text{syn}}\{u, g(u)\} = g(0)^{-1}, \quad g(u) \in R[[q_0, u]]^\times \quad (4.6.4)$$

$$\widehat{\tau}_{\infty}^{\text{syn}}\{u, 1 - bu^{-1}\} = 1, \quad b \in (p, q_0) \quad (4.6.5)$$

$$\widehat{\tau}_{\infty}^{\text{syn}}\{g(u), h(u)\} = 1, \quad g(u), h(u) \in R[[q_0, u]]^\times \quad (4.6.6)$$

$$\widehat{\tau}_\infty^{\text{syn}}\{g(u), 1 - bu^{-1}\} = g(0)^{-1}g(b), \quad g(u) \in R[[q_0, u]]^\times, \quad b \in (p, q_0) \quad (4.6.7)$$

$$\widehat{\tau}_\infty^{\text{syn}}\{1 - bu^{-1}, 1 - cu^{-1}\} = 1, \quad b, c \in (p, q_0) \quad (4.6.8)$$

where  $\widehat{\tau}_\infty^{\text{syn}}$  is the residue map (4.2.5). We prove only (4.6.4) and (4.6.7). The other equalities are simpler and left to the reader. By the definition of syntomic symbols (cf. §2.3), we have

$$\{u\}^{\text{syn}} = \left( \frac{du}{u}, 0 \right), \quad \{g(u)\}^{\text{syn}} = \left( \frac{dg(u)}{g(u)}, \frac{1}{p} \log \frac{\varphi(g(u))}{g(u)^p} \right).$$

By the definition of the product of syntomic cohomology (cf. Definition 2.2.9), we have

$$\begin{aligned} \{u, g(u)\}^{\text{syn}} &= \left( \frac{du}{u} \frac{dg(u)}{g(u)}, -\frac{1}{p} \log \frac{\varphi(g(u))}{g(u)^p} \frac{du}{u} \right) \\ &= \left( -\frac{dg(u)}{g(u)} \frac{du}{u}, -\frac{1}{p} \log \frac{\varphi(g(u))}{g(u)^p} \frac{du}{u} \right). \end{aligned}$$

Therefore

$$\widehat{\tau}_\infty^{\text{syn}}[u, g(u)] = \left( -\frac{dg(0)}{g(0)}, -\frac{1}{p} \log \frac{\varphi(g(0))}{g(0)^p} \right) = -[g(0)],$$

which completes the proof of (4.6.4). One can reduce (4.6.7) to (4.6.4) in the following way. It is enough to show  $\widehat{\tau}_\infty^{\text{syn}}\{g(u), u - b\} = g(b)$ . Consider an endomorphism  $w : \mathcal{A} \rightarrow \mathcal{A}$  of topological ring such that  $w(u) = u + b$ . Then we have  $\widehat{\tau}_\infty^{\text{syn}} \circ w = \widehat{\tau}_\infty^{\text{syn}}$  as  $\varrho_n^r \circ w = \varrho_n^r$  (cf. (4.2.3)). Thus one has

$$\widehat{\tau}_\infty^{\text{syn}}\{g(u), u - b\} = \widehat{\tau}_\infty^{\text{syn}} w \{g(u), u - b\} = \widehat{\tau}_\infty^{\text{syn}}\{g(u + b), u\} = g(b).$$

as required.

## 4.7 Commutativity of (4.4.4)

We prove the commutativity of (4.4.4). Let  $\mathcal{R}^b$  and  $\mathcal{R}_K^b$  be as we defined in §4.3. Since  $B$  is a regular local ring,  $B[p^{-1}]$  is a UFD. By Remark 2.3.4 (1), we have

$$H_{\text{ét}}^1(B[p^{-1}], \mathbb{Z}_p(1)) \simeq \varprojlim_n B[p^{-1}]^\times / p^n.$$

**Lemma 4.7.1** *For  $m \geq 1$ , let  $s_m : B \rightarrow R$  be the  $R$ -homomorphism sending  $q_0$  to  $p^m$ . Then the map*

$$\prod_{m \geq 1} s_m : \varprojlim_n B[p^{-1}]^\times / p^n \longrightarrow \prod_{m \geq 1} \varprojlim_n K^\times / p^n$$

*is injective.*

*Proof.* Exercise (left to the reader). □

We consider the following cartesian squares of schemes for each  $m \geq 1$ :

$$\begin{array}{ccccc} \mathrm{Spec}(\mathcal{R}^b) & \xrightarrow{\gamma} & \mathrm{Spec}(\mathcal{B}_{(m)}) & \xrightarrow{\alpha_m} & \mathcal{E}_{(m)} & \longrightarrow & \mathrm{Spec}(R) \\ & & \sigma_m \downarrow & \square & t_m \downarrow & \square & s_m \downarrow \\ & & \mathrm{Spec}(\mathcal{B}) & \xrightarrow{\beta} & \mathcal{E} & \longrightarrow & \mathrm{Spec}(B), \end{array}$$

where  $\mathcal{B}_{(m)}$  and  $\mathcal{E}_{(m)}$  are defined by this diagram, and  $\gamma$  is induced by the natural inclusion  $\mathcal{R} \hookrightarrow \mathcal{R}^b \simeq \mathcal{B}_{(m)}$ . See §4.1 for  $\mathcal{B}$  and  $\beta$ . We defined  $\mathcal{E}_{(m)}$  (resp.  $\mathrm{Spec}(\mathcal{B}_{(m)})$ ) by the lower (resp. upper) cartesian square. By Lemma 4.7.1, the commutativity of (4.4.4) is reduced to showing that of the following diagram for all  $m \geq 1$ :

$$\begin{array}{ccc} H_{\mathrm{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{t_m^* \circ c^2} & H_{\mathrm{\acute{e}t}}^2(\mathcal{E}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \\ \tau_{\infty}^{\mathrm{syn}} \downarrow & & \downarrow \tau_{\infty}^{\mathrm{\acute{e}t}} \\ H_{\mathrm{syn}}^1(B, \mathcal{S}_{\mathbb{Z}_p}(1)) & \xrightarrow{s_m^* \circ c^1} & H_{\mathrm{\acute{e}t}}^1(K, \mathbb{Z}_p(1)). \end{array} \quad (4.7.2)$$

**Lemma 4.7.3** *The diagram (4.7.2) factors as follows:*

$$\begin{array}{ccccc} H_{\mathrm{syn}}^2(\mathcal{E}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{c^2} & H_{\mathrm{\acute{e}t}}^2(\mathcal{E}[p^{-1}], \mathbb{Z}_p(2)) & \xrightarrow{t_m^*} & H_{\mathrm{\acute{e}t}}^2(\mathcal{E}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \\ \beta^* \downarrow & \circ & \beta^* \downarrow & \circ & \downarrow \alpha_m^* \\ H_{\mathrm{syn}}^2(\mathcal{B}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{c^2} & H_{\mathrm{\acute{e}t}}^2(\mathcal{B}[p^{-1}], \mathbb{Z}_p(2)) & \xrightarrow{\sigma_m^*} & H_{\mathrm{\acute{e}t}}^2(\mathcal{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \\ \widehat{\tau}_{\infty}^{\mathrm{syn}} \downarrow & & & & \downarrow \widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}} \\ H_{\mathrm{syn}}^1(B, \mathcal{S}_{\mathbb{Z}_p}(1)) & \xrightarrow{s_m^* \circ c^1} & & & H_{\mathrm{\acute{e}t}}^1(K, \mathbb{Z}_p(1)), \end{array}$$

and the upper squares are commutative. Here  $\widehat{\tau}_{\infty}^{\mathrm{syn}}$  denotes the residue map constructed in §4.2 and  $\widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}}$  is defined as the projective limit, with respect to  $n \geq 1$ , of the composite map

$$\widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}} : H_{\mathrm{\acute{e}t}}^2(\mathcal{B}_{(m)}[p^{-1}], \mathbb{Z}/p^n(2)) \xrightarrow{\gamma^*} H_{\mathrm{\acute{e}t}}^2(\mathcal{R}_K^b, \mathbb{Z}/p^n(2)) \xrightarrow{\widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}}} H_{\mathrm{\acute{e}t}}^1(K, \mathbb{Z}/p^n(1)).$$

See §4.3 for  $\widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}}$ .

*Proof.* The factorization  $\tau_{\infty}^{\mathrm{syn}} = \widehat{\tau}_{\infty}^{\mathrm{syn}} \circ \beta^*$  follows from the construction of these maps, and the factorization  $\tau_{\infty}^{\mathrm{\acute{e}t}} = \widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}} \circ \alpha_m^*$  follows from Proposition 4.3.5. The second assertion follows from the functoriality of  $c^2$  and that of étale cohomology.  $\square$

**Lemma 4.7.4** *In the diagram of Lemma 4.7.3, the lower square factors as follows:*

$$\begin{array}{ccccc} H_{\mathrm{syn}}^2(\mathcal{B}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{\sigma_m^*} & H_{\mathrm{syn}}^2(\mathcal{B}_{(m)}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{c^2} & H_{\mathrm{\acute{e}t}}^2(\mathcal{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \\ \widehat{\tau}_{\infty}^{\mathrm{syn}} \downarrow & \circ & \widehat{\tau}_{\infty}^{\mathrm{syn}} \downarrow & & \downarrow \widehat{\tau}_{\infty}^{\mathrm{\acute{e}t}} \\ H_{\mathrm{syn}}^1(B, \mathcal{S}_{\mathbb{Z}_p}(1)) & \xrightarrow{s_m^*} & H_{\mathrm{syn}}^1(R, \mathcal{S}_{\mathbb{Z}_p}(1)) & \xrightarrow{c^1} & H_{\mathrm{\acute{e}t}}^1(K, \mathbb{Z}_p(1)), \end{array}$$

and the left square commutes.

*Proof.* The first assertion follows from the functoriality of  $c^2$  and  $c^1$ . The commutativity of the left square follows from the construction of  $\widehat{\tau}_\infty^{\text{syn}}$ 's (see §4.2).  $\square$

**Lemma 4.7.5** *In the diagram of Lemma 4.7.4, the right square factors as follows:*

$$\begin{array}{ccc}
H_{\text{syn}}^2(\mathcal{B}_{(m)}, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{c^2} & H_{\text{ét}}^2(\mathcal{B}_{(m)}[p^{-1}], \mathbb{Z}_p(2)) \\
\gamma^* \downarrow & \circlearrowleft & \downarrow \gamma^* \\
H_{\text{syn}}^2(\mathcal{R}^b, \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{c^2} & H_{\text{ét}}^2(\mathcal{R}_K^b, \mathbb{Z}_p(2)) \\
\widehat{\tau}_\infty^{\text{syn}} \downarrow & & \downarrow \widehat{\tau}_\infty^{\text{ét}} \\
H_{\text{syn}}^1(R, \mathcal{S}_{\mathbb{Z}_p}(1)) & \xrightarrow{c^1} & H_{\text{ét}}^1(K, \mathbb{Z}_p(1)),
\end{array}$$

and the upper square commutes. Here  $\widetilde{\tau}_\infty^{\text{syn}}$  denotes the syntomic residue map defined in a similar way as for  $\widehat{\tau}_\infty^{\text{syn}}$ . Note also that  $\mathcal{R}_K^b = \mathcal{R}^b[p^{-1}]$ .

*Proof.* The first assertion follows from the construction of  $\widehat{\tau}_\infty^{\text{syn}}$  and  $\widehat{\tau}_\infty^{\text{ét}}$ . The commutativity of the upper square follows from the functoriality of  $c^2$  (Theorem 2.2.15).  $\square$

By Lemmas 4.7.3–4.7.5, the commutativity of (4.7.2) is reduced to that of the lower square of the diagram in Lemma 4.7.5, which is further reduced to that of the following square for all  $n \geq 1$ :

$$\begin{array}{ccc}
H_{\text{syn}}^2(\mathcal{R}^b, \mathcal{S}_n(2)) & \xrightarrow{c^2} & H_{\text{ét}}^2(\mathcal{R}_K^b, \mathbb{Z}/p^n(2)) \\
\widetilde{\tau}_\infty^{\text{syn}} \downarrow & & \downarrow \widetilde{\tau}_\infty^{\text{ét}} \\
H_{\text{syn}}^1(R, \mathcal{S}_n(1)) & \xrightarrow{c^1} & H_{\text{ét}}^1(K, \mathbb{Z}/p^n(1)).
\end{array} \tag{4.7.6}$$

By Theorem 2.3.8 and Theorem 2.3.9, the diagram (4.7.6) is written as

$$\begin{array}{ccc}
K_2^M(\mathcal{R}^b)/p^n & \longrightarrow & K_2^M(\mathcal{R}_K^b)/p^n \\
\tau^{\text{syn}} \downarrow & & \downarrow \tau^{\text{ét}} \\
R^\times/p^n & \longrightarrow & K^\times/p^n.
\end{array} \tag{4.7.7}$$

where the horizontal arrows are the natural maps and  $\tau^{\text{syn}}$  (resp.  $\tau^{\text{ét}}$ ) is induced from  $\widetilde{\tau}_\infty^{\text{syn}}$  (resp.  $\widetilde{\tau}_\infty^{\text{ét}}$ ). Thus the commutativity of (4.7.6) is reduced to the explicit calculations of  $\tau^{\text{syn}}$  and  $\tau^{\text{ét}}$ . By Lemma 4.3.1 we may replace  $K_2^M(\mathcal{R}^b)/p^n$  with  $K_2^M(R[u]_{(p)})/p^n$  and  $K_2^M(\mathcal{R}_K^b)/p^n$  with  $K_2^M(K(u))/p^n$ . Then the explicit formula of  $\tau^{\text{ét}}$  has been shown in [A1] Theorem 4.4. Therefore it is enough to show that the formula of  $\tau^{\text{syn}}$  agrees with it. However this follows from (4.6.1)–(4.6.8). This finishes the proof of the commutativity of the diagram (4.7.7) and hence the diagrams (4.7.6), (4.4.4) and (3.3.1). This completes the proof of Theorem 3.2.3.

## 5 Main result on Elliptic surfaces over $p$ -adic fields

We mean by an *elliptic surface* over a commutative ring  $A$  a projective flat morphism  $\pi : X \rightarrow C$  with a section  $e : C \rightarrow X$  such that  $X$  and  $C$  are projective smooth over  $\text{Spec}(A)$  of relative dimension 2 and 1, respectively and such that the general fiber of  $\pi$  is an elliptic curve.

For a field  $F$ , we denote the absolute Galois group  $\text{Gal}(\overline{F}/F)$  by  $G_F$ . We often write  $H^*(F, -)$  for the continuous Galois cohomology  $H_{\text{cont}}^*(G_F, -)$ .

### 5.1 Split multiplicative fiber

Let

$$\widetilde{T}_{\mathbb{Z}} := \coprod \mathbb{P}_{\mathbb{Z}}^1$$

be the disjoint union of copies of  $\mathbb{P}_{\mathbb{Z}}^1$  indexed by  $\mathbb{Z}/m$ . Gluing the section  $0 : \text{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{P}_{\mathbb{Z}}^1$  of the  $i$ -th  $\mathbb{P}_{\mathbb{Z}}^1$  with  $\infty : \text{Spec}(\mathbb{Z}) \hookrightarrow \mathbb{P}_{\mathbb{Z}}^1$  of the  $(i+1)$ -st  $\mathbb{P}_{\mathbb{Z}}^1$ , we obtain a connected proper curve  $T_{\mathbb{Z}}$  over  $\text{Spec}\mathbb{Z}$  whose normalization is  $\widetilde{T}_{\mathbb{Z}} \rightarrow T_{\mathbb{Z}}$ . We call  $T_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$  the *Néron polygon* or the *Néron  $m$ -gon* over a ring  $A$  (cf. [DR] II.1.1). Let  $\pi : X \rightarrow C$  be an elliptic surface over a ring  $A$ . For an  $A$ -valued point  $P \in C(A)$ , we call the fiber  $\pi^{-1}(P)$  *split multiplicative of type  $I_m$*  over  $A$ , if there is a closed subscheme  $D^\dagger \subset \pi^{-1}(P)$  which is isomorphic to the Néron  $m$ -gon  $T_{\mathbb{Z}} \times_{\mathbb{Z}} A$ . We call  $\pi^{-1}(P)$  *multiplicative*, if  $\pi^{-1}(P) \otimes_A \overline{k}$  is split multiplicative over  $\overline{k}$  for any geometric point  $\text{Spec}(\overline{k}) \rightarrow \text{Spec}(A)$ . A singular fiber which is not multiplicative is called an *additive* fiber. When  $A = \overline{F}$  is an algebraically closed field, all the singular fibers are classified by Kodaira and Néron (cf. [Si] IV §8). In particular, we note

$$\pi^{-1}(P) \text{ is multiplicative} \iff H_{\text{ét}}^1(\pi^{-1}(P), \mathbb{Z}_p) \neq 0 \implies H_{\text{ét}}^1(\pi^{-1}(P), \mathbb{Z}_p) = \mathbb{Z}_p.$$

### 5.2 Formal Eisenstein seires

Let  $p \geq 5$  be a prime number. Let  $K$  be a finite *unramified* extension of  $\mathbb{Q}_p$  and let  $R$  be its integer ring. Let  $\pi_R : \mathcal{X} \rightarrow \mathcal{C}$  be an elliptic surface over  $R$ . Let  $\Sigma = \{P_1, \dots, P_s\}$  be the set of all  $R$ -rational points for which the fiber  $\mathcal{D}_i := \pi^{-1}(P_i)$  are split multiplicative fibers of type  $I_{r_i}$ . Put  $\mathcal{D} := \sum_{i=1}^s \mathcal{D}_i$  and  $\mathcal{U} := \mathcal{X} - \mathcal{D}$ . We assume

$$p \nmid 6r_1 \cdots r_s.$$

Let  $t_i$  be a uniformizer of  $\mathcal{O}_{\mathcal{C}, P_i}$ . Let  $\iota_i : \text{Spec}(R((t_i))) \rightarrow \mathcal{C} - \{P_i\}$  be the punctured neighborhood of  $P_i$ , and let  $X_i$  be the fiber:

$$\begin{array}{ccc} X_i & \longrightarrow & \mathcal{U} \\ \downarrow & \square & \downarrow \pi_R \\ \text{Spec}(R((t_i))) & \xrightarrow{\iota_i} & \mathcal{C} - \{P_i\}. \end{array}$$

Then  $X_i$  is isomorphic to a Tate elliptic curve over  $R((t_i))$ . More explicitly, let  $q \in R((t_i))$  be the unique power series such that  $\text{ord}_{t_i}(q) = r_i$  and

$$j(X_i) = \frac{1}{q} + 744 + 196884q + \cdots. \quad (5.2.1)$$

Then there is an isomorphism of  $R((t_i))$ -schemes

$$X_i \simeq E_q, \quad (5.2.2)$$

which is unique up to the translation and the involution  $u \mapsto u^{-1}$  (cf. [DR] VII.2.6). Put

$$a_i := t_i^{r_i} \cdot j(X_i)|_{t_i=0} = \frac{t_i^{r_i}}{q}|_{t_i=0} \in R^\times.$$

Put  $K_i := K(\sqrt[r_i]{a_i})$ , and let  $R_i$  be its integer ring. Note that  $K_i$  is also unramified over  $\mathbb{Q}_p$  as  $r_i$  is prime to  $p$ . There is  $q_i \in t_i \cdot R_i[[t_i]]^\times$  such that  $q_i^{r_i} = q$ , and we have  $R_i((t_i)) = R_i((q_i))$ . Let  $\kappa_i$  be the composition of natural maps

$$\begin{aligned} \kappa_i : \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D})) &\longrightarrow \Gamma(X_i, \Omega_{X_i/R}^2) \simeq \Gamma(E_q, \Omega_{E_q/R}^2) \\ &\longrightarrow R_i((q_i)) \frac{dq_i}{q_i} \frac{dx}{2y+x} \simeq R_i((q_i)), \end{aligned} \quad (5.2.3)$$

where the isomorphism in the middle is induced by (5.2.2). We define a  $\mathbb{Z}_p$ -submodule  $\mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$  of  $\Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D}))$  by

$$\omega \in \mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \iff \kappa_i(\omega) \text{ is a formal power series of Eisenstein type for each } i \quad (\S 3.2)$$

and call it the space of *formal Eisenstein series*. As an immediate consequence of the main result on Tate curve (Theorem 3.2.3), the image of the syntomic cohomology group is contained in the space of formal Eisenstein series:

$$H_{\text{syn}}^2(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \longrightarrow \mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \subset \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D})). \quad (5.2.4)$$

We also introduce a  $\mathbb{Z}_p$ -submodule  $\mathbf{E}^{(n)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \subset \Gamma(\mathcal{X}, \Omega_{\mathcal{X}/R}^2(\log \mathcal{D}))$  such that  $\omega \in \mathbf{E}^{(n)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$  if and only if each  $\kappa_i(\omega)$  ( $1 \leq i \leq s$ ) satisfies (E1) (§3.2) and

$$(E2)^{(n)} \quad a_k^{(j)} \in k^2 \mathbb{Z}_p \text{ for all } j \text{ and } 1 \leq k \leq n.$$

Obviously one has

$$\mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} = \bigcap_{n \geq 1} \mathbf{E}^{(n)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}.$$

### 5.3 Main result on elliptic surfaces

The boundary maps on étale, syntomic and de Rham cohomology induce a commutative diagram

$$\begin{array}{ccc}
H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K} & \xrightarrow{\partial_{\text{ét}}} & \bigoplus_{i=1}^s \mathbb{Z}_p[D_i] \\
\uparrow a & & \parallel \\
H_{\text{syn}}^2(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) & \xrightarrow{\partial_{\text{syn}}} & \bigoplus_{i=1}^s \mathbb{Z}_p[\mathcal{D}_i] \\
\downarrow (5.2.4) & & \parallel \\
\mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} & \xrightarrow{\partial_{\text{dR}}} & \bigoplus_{i=1}^s \mathbb{Z}_p[\mathcal{D}_i].
\end{array} \tag{5.3.1}$$

Here  $\partial_{\text{dR}}$  denotes the the Poincare residue map.  $\partial_{\text{ét}}$  denotes the composite map

$$H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2)) \longrightarrow H_{\overline{D}}^3(\overline{X}, \mathbb{Z}_p(2)) \simeq \bigoplus_{i=1}^s \mathbb{Z}_p[\mathcal{D}_i]$$

where the first arrow is the boundary localization sequence of the étale cohomology and the last isomorphism is obtained from the Poincare-Lefschetz duality. See Proposition 2.2.14 for  $\partial_{\text{syn}}$ . Note that  $\partial_{\text{ét}}$  is injective modulo torsion.

**Lemma 5.3.2** *We have*

$$\partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K}) \subset \partial_{\text{dR}}(\mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}). \tag{5.3.3}$$

*Proof.* It is enough to show that the map  $a$  in (5.3.1) is surjective. Put  $\mathcal{V} := \mathcal{C} - \Sigma$ . Let  $\mathcal{U}_s$  and  $\mathcal{V}_s$  denote the fibers of  $\mathcal{U} \rightarrow \text{Spec}(R)$  and  $\mathcal{V} \rightarrow \text{Spec}(R)$  over the closed point  $s$  of  $\text{Spec}(R)$ . By a result of Tsuji [Ts2] Theorem 5.1, there is an exact sequence

$$H_{\text{syn}}^2(\mathcal{X}(\mathcal{D}), \mathcal{S}_{\mathbb{Z}_p}(2)) \rightarrow H_{\text{ét}}^2(U, \mathbb{Z}_p(2)) \xrightarrow{\delta_U} \varprojlim_n \mathcal{O}(\mathcal{U}_s)^\times / p^n \simeq \varprojlim_n \mathcal{O}(\mathcal{V}_s)^\times / p^n, \tag{5.3.4}$$

where the last isomorphism follows from the fact that  $\mathcal{U}_s \rightarrow \mathcal{V}_s$  is proper with geometrically connected fibers. On the other hand, we assert that there is an exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(S, \mathbb{Z}_p(2)) \xrightarrow{\pi_K^*} H_{\text{ét}}^2(U, \mathbb{Z}_p(2)) \longrightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K} \longrightarrow 0, \tag{5.3.5}$$

where we put  $S := \mathcal{V} \otimes_R K$ . Indeed, the Hochschild-Serre spectral sequences

$$E_{2,U}^{p,q} = H^p(K, H_{\text{ét}}^q(\overline{U}, \mathbb{Z}_p(2))) \implies E_U^{p+q} = H^{p+q}(U, \mathbb{Z}_p(2)) \tag{5.3.6}$$

$$E_{2,S}^{p,q} = H^p(K, H_{\text{ét}}^q(\overline{S}, \mathbb{Z}_p(2))) \implies E_S^{p+q} = H^{p+q}(S, \mathbb{Z}_p(2)) \tag{5.3.7}$$

and the isomorphisms  $H_{\text{ét}}^q(\overline{S}, \mathbb{Z}_p(2)) \simeq H_{\text{ét}}^q(\overline{U}, \mathbb{Z}_p(2))$  for  $q \leq 1$  (cf. Lemma 5.4.1 below) yield a short exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(S, \mathbb{Z}_p(2)) \xrightarrow{\pi_K^*} H_{\text{ét}}^2(U, \mathbb{Z}_p(2)) \longrightarrow E_{\infty,U}^{0,2} \longrightarrow 0.$$

It remains to show  $E_{2,U}^{0,2} = E_{\infty,U}^{0,2}$ . Since  $\text{cd}(K) = 2$ , it is enough to show that the edge homomorphism  $E_{2,U}^{0,2} \rightarrow E_{2,U}^{2,1}$  is zero. There is a commutative diagram with exact rows

$$\begin{array}{ccccc} E_{2,U}^{0,2} & \xrightarrow{\alpha} & E_{2,U}^{2,1} & \xrightarrow{\beta} & E_U^3 \\ \uparrow & & \uparrow & & \uparrow \\ E_{2,S}^{0,2} & \xrightarrow{\gamma} & E_{2,S}^{2,1} & \xrightarrow{\delta} & E_S^3. \end{array}$$

Therefore  $\alpha = 0 \iff \beta$  is injective  $\iff \delta$  is injective  $\iff \gamma = 0$ . The last condition follows from  $E_{2,S}^{0,2} = 0$ , which completes the proof of the exact sequence (5.3.5).

Finally we show that  $a$  is surjective. By (5.3.4) and (5.3.5), it is enough to show

$$\text{Im}(H_{\text{ét}}^2(U, \mathbb{Z}/p^n(2)) \xrightarrow{\delta_U} \mathcal{O}(\mathcal{U}_s)^\times/p^n) = \text{Im}(H_{\text{ét}}^2(S, \mathbb{Z}/p^n(2)) \xrightarrow{\delta_S} \mathcal{O}(\mathcal{V}_s)^\times/p^n).$$

The inclusion ‘ $\supset$ ’ follows from the fact  $\pi_s^* \circ \delta_U = \delta_S \circ \pi_K^*$ , and ‘ $\subset$ ’ follows from the fact  $e_s^* \circ \delta_U = \delta_S \circ e_K^*$ , where  $e_R : \mathcal{C} \rightarrow \mathcal{X}$  denotes the section, and  $\pi_F$  and  $e_F$  ( $F = K, s$ ) denote  $\pi_R \otimes_R F$  and  $e_R \otimes_R F$ , respectively.  $\square$

Since  $\partial_{\text{ét}}$  is injective modulo torsion, the right hand side of (5.3.3) gives an upper bound of the rank of  $H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K}$ . However it is in general impossible to compute it even though one can compute  $E^{(n)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$  for a particular  $n$ . The main result allows us to obtain an “computable” upper bound under some mild conditions.

**Theorem 5.3.8** *Let  $\pi_R : \mathcal{X} \rightarrow \mathcal{C}$  and  $\mathcal{D}$  be as before. Let*

$$f_p : \bigoplus_{i=1}^s \mathbb{Z}_p[\mathcal{D}_i] \longrightarrow \bigoplus_{i=1}^s \mathbb{F}_p[\mathcal{D}_i]$$

*be the natural map taking the residue class modulo  $p$ , and let  $\overline{\partial_{\text{dR}}}$  be the composit map*

$$E(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p} \xrightarrow{\partial_{\text{dR}}} \bigoplus_{i=1}^s \mathbb{Z}_p[\mathcal{D}_i] \xrightarrow{f_p} \bigoplus_{i=1}^s \mathbb{F}_p[\mathcal{D}_i].$$

*Assume  $p \nmid 6r_1 \cdots r_s$  and further the following conditions:*

**(A)**  $H_{\text{ét}}^3(\overline{X}, \mathbb{Z}_p)$  is torsion-free.

**(B)**  $H_{\text{ét}}^2(\overline{X}, \mathbb{Z}/p(2))^{G_K} = 0$ .

*Then we have*

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K} \leq \dim_{\mathbb{F}_p} \overline{\partial_{\text{dR}}}(E(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}). \quad (5.3.9)$$

*More precisely, for  $v \in \bigoplus_i \mathbb{Z}_p[\mathcal{D}_i]$*

$$f_p(v) \notin \overline{\partial_{\text{dR}}}(E(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}) \implies v \notin \partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K}). \quad (5.3.10)$$

*Proof.* Applying  $f_p$  on (5.3.3), we obtain

$$\overline{\partial_{\text{ét}}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K}) \subset \overline{\partial_{\text{dR}}}(\mathbf{E}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}).$$

Therefore all we have to do is to show that the quotient group

$$\mathbb{Z}_p^{\oplus s} / \partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K}) \quad (5.3.11)$$

is torsion-free (note  $H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K} \cong \partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K})$ ).

Put  $V_{\mathbb{Z}_p} := H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2)) / \langle D \rangle \otimes \mathbb{Z}_p(1)$  where  $\langle D \rangle$  denotes the subgroup of the cycle classes of irreducible components of  $\overline{D}$ . There is an exact sequence

$$0 \longrightarrow V_{\mathbb{Z}_p} \longrightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2)) \xrightarrow{\partial_{\text{ét}}} \mathbb{Z}_p^{\oplus s} \longrightarrow 0 \quad (5.3.12)$$

under the condition **(A)** by Lemma 5.4.7 below. Moreover  $H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2))$  is torsion-free by Lemma 5.4.3 below and the natural surjective map

$$H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2)) \longrightarrow V_{\mathbb{Z}_p}$$

has a  $G_K$ -equivariant splitting (cf. Lemma 5.4.4 below). We thus obtain an exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_K} \xrightarrow{\partial_{\text{ét}}} \mathbb{Z}_p^{\oplus s} \longrightarrow H^1(K, H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2))). \quad (5.3.13)$$

The condition **(B)** together with an exact sequence

$$0 \longrightarrow H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2)) \xrightarrow{p} H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2)) \longrightarrow H_{\text{ét}}^2(\overline{X}, \mathbb{Z}/p(2)) \longrightarrow 0$$

implies that  $H^1(K, H_{\text{ét}}^2(\overline{X}, \mathbb{Z}_p(2)))$  is torsion-free. Hence so is the quotient (5.3.11).  $\square$

Finally we give the following useful criterion for the conditions **(A)** and **(B)**.

**Proposition 5.3.14** *Consider the following conditions:*

**(A')** *the elliptic surface  $\pi : \overline{X} \rightarrow \overline{C}$  has at least one additive (singular) fiber,*

**(B')** *Let  $Y = \mathcal{X}_s$  be the special fiber of  $\mathcal{X}$  over the closed point  $s \in \text{Spec}(R)$ , and let  $C : \Gamma(Y, \Omega_Y^2) \rightarrow \Gamma(Y, \Omega_Y^2)$  be the Cartier operator. Then  $C - \text{id}$  is injective.*

*Then **(A')**  $\implies$  **(A)**, and **(B')**  $\implies$  **(B)**.*

*Proof.* See Lem.5.4.6 below for the first assertion. The second assertion follows from the torsion  $p$ -adic Hodge theory by Fontaine and Messing. One has the isomorphism

$$H_{\text{ét}}^2(\overline{X}, \mathbb{Z}/p(2))^{G_K} \cong \text{Ker}(\Gamma(Y, \Omega_{Y/\mathbb{F}}^2) \xrightarrow{\varphi_2 - 1} H_{\text{dR}}^2(Y/\mathbb{F}))$$

where 1 denotes the natural inclusion (cf. [BM] Theorem 3.2.3). Let  $v : H_{\text{dR}}^2(Y/\mathbb{F}) \rightarrow H^0(\Omega_Y^2/d\Omega_Y^1)$  be the natural map and  $C : H^0(\Omega_Y^2/d\Omega_Y^1) \xrightarrow{\sim} H^0(\Omega_Y^2)$  be the Cartier operator. Then  $C \circ v \circ \varphi_2 = \text{id}$  and  $C \circ v \circ 1 = C$ . Hence the injectivity of  $C - \text{id}$  implies that of  $\varphi_2 - 1$ .  $\square$

## 5.4 Auxiliary results on Betti cohomology groups

In the previous section, we used a number of results on étale cohomology groups of elliptic surfaces over  $\overline{K}$ , which we prove here. We take the base-change by a fixed embedding  $\overline{K} \hookrightarrow \mathbb{C}$  and then replace the étale cohomology with the Betti cohomology  $H^* = H_B^*$ . For simplicity, we denote  $X \otimes_K \mathbb{C}$ ,  $S \otimes_K \mathbb{C}$  and  $U \otimes_K \mathbb{C}$  by  $X$ ,  $S$  and  $U$ , respectively.

**Lemma 5.4.1** *Suppose that  $\pi : X \rightarrow C$  is not iso-trivial. Then  $H^q(S, \mathbb{Z}) \simeq H^q(U, \mathbb{Z})$  for  $q \leq 1$ .*

*Proof.* The case  $q = 0$  is clear. As for the case  $q = 1$ , the Leray spectral sequence

$$E_2^{pq} = H^p(S, R^q \pi_* \mathbb{Z}_U) \implies H^{p+q}(U, \mathbb{Z}) \quad (5.4.2)$$

yields an exact sequence

$$0 \longrightarrow H^q(S, \mathbb{Z}) \longrightarrow H^q(U, \mathbb{Z}) \longrightarrow \Gamma(S, R^1 \pi_* \mathbb{Z}).$$

Since  $\Gamma(S, R^1 \pi_* \mathbb{Z}) \hookrightarrow H^1(X_t, \mathbb{Z})^{\pi_1(S)} = 0$ , the assertion follows.  $\square$

**Lemma 5.4.3** *Assume that  $H^3(X, \mathbb{Z})$  is  $p$ -torsion-free. Then  $H^2(X, \mathbb{Z})$  is  $p$ -torsion-free as well.*

*Proof.* By the universal coefficient theorem  $H^1(X, M) \simeq \text{Hom}(H_1(X, \mathbb{Z}), M)$  for an abelian group  $M$ . Therefore if  $H_1(X, \mathbb{Z}) = H^3(X, \mathbb{Z})$  is  $p$ -torsion-free, then the map  $H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z}/p)$  is surjective. Equivalently, the multiplication by  $p$  is injective on  $H^2(X, \mathbb{Z})$ .  $\square$

**Lemma 5.4.4** *Let  $\langle D \rangle \subset H^2(X, \mathbb{Z})$  be the subgroup generated by the cycle classes of irreducible components of  $D = \sum_{i=1}^s D_i$ . Assume  $p \geq 5$ . Then the inclusion map*

$$\langle D \rangle \otimes \mathbb{Z}_p \longrightarrow H^2(X, \mathbb{Z}_p)$$

*has a natural splitting.*

*Proof.* Let us first assume that  $X$  is minimal (hence  $D_i$  is a Néron polygon). Let  $D_i^{(k)}$  ( $k \geq 1$ ) be the irreducible components of  $D_i$ , and let

$$\bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \longrightarrow H^2(X, \mathbb{Z})$$

be the map sending  $[D_i^{(k)}]$  to its cycle class. Let

$$H^2(X, \mathbb{Z}) \longrightarrow \bigoplus_{i,k} H^2(D_i^{(k)}, \mathbb{Z}) \simeq \bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \quad \text{and} \quad H^2(X, \mathbb{Z}) \xrightarrow{e^*} H^2(C, \mathbb{Z}) \simeq \mathbb{Z}[C]$$

be the natural pull-back maps. Then the composite map

$$\bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow \bigoplus_{i,k} \mathbb{Z}[D_i^{(k)}] \oplus \mathbb{Z}[C] \quad (5.4.5)$$

is described as follows

$$[D_i^{(k)}] \longmapsto \sum_{j,l} (D_i^{(k)}, D_j^{(l)})[D_j^{(l)}] + (D_i^{(k)}, e(C))[C]$$

where  $(-, -)$  denotes the intersection pairing. Therefore by an elementary computation on the intersection matrix, we see that the composite map (5.4.5) induces an injective map

$$\langle D \rangle \simeq \bigoplus_{i,k \geq 1} \mathbb{Z}[D_i^{(k)}] \Big/ \langle [D_i] - [D_j]; i < j \rangle \hookrightarrow \bigoplus_{i \geq 1, k \geq 2} \mathbb{Z}[D_i^{(k)}] \oplus \mathbb{Z}[C],$$

whose matrix has determinant prime to 6. Next we consider a general  $X$ . Take the minimal model  $\mu : X \rightarrow X_0$ . Then  $H^2(X, \mathbb{Z})$  is a direct sum of  $H^2(X_0, \mathbb{Z})$  and the cycle classes of exceptional divisors. Since  $\langle \mu(D) \rangle \otimes \mathbb{Z}_p$  is a direct summand of  $H^2(X_0, \mathbb{Z}_p)$ ,  $\langle D \rangle \otimes \mathbb{Z}_p$  is a direct summand of  $H^2(X, \mathbb{Z}_p)$ . This completes the proof.  $\square$

**Lemma 5.4.6** *Assume that there is at least one additive singular fiber  $E$ . Then  $H_1(X, \mathbb{Z}) \simeq H_1(C, \mathbb{Z})$ . In particular  $H_1(X, \mathbb{Z}) = H^3(X, \mathbb{Z})$  is torsion-free.*

*Proof.* It is enough to show that  $e_* : H_1(C, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$  is surjective. Let  $S^0 \subset S$  be a Zariski open (non-empty) subset such that  $U^0 := \pi^{-1}(S^0) \rightarrow S^0$  is a smooth fibration. By the fibration exact sequence  $1 \rightarrow \pi_1(X_t) \rightarrow \pi_1(U^0) \rightarrow \pi_1(S^0) \rightarrow 1$ , which is split by  $e$ , one has a surjection  $\pi_1(X_t) \times \pi_1(S_0)^{\text{ab}} \rightarrow \pi_1(U_0)^{\text{ab}}$ . Since  $\pi_1(U_0)^{\text{ab}} \rightarrow \pi_1(X)^{\text{ab}} = H_1(X, \mathbb{Z})$  is surjective, it is enough to show that the map  $\pi_1(X_t) = H_1(X_t, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$  is zero, which follows from the fact that it factors through  $H_1(E, \mathbb{Z}) = 0$ .  $\square$

**Lemma 5.4.7** *Let  $\partial : H^2(U, \mathbb{Z}) \rightarrow H_D^3(X, \mathbb{Z}) = \mathbb{Z}^{\oplus s}$  be the boundary map on the Betti cohomology arising from the localization exact sequence*

$$\cdots \longrightarrow H^2(U, \mathbb{Z}) \xrightarrow{\partial} \mathbb{Z}^{\oplus s} \longrightarrow H^3(X, \mathbb{Z}) \longrightarrow H^3(U, \mathbb{Z}) \longrightarrow \cdots$$

*Then  $\partial \otimes \mathbb{Q}$  is surjective. In particular, if  $H^3(X, \mathbb{Z})$  is  $p$ -torsion-free, then  $\partial \otimes \mathbb{Z}_p$  is surjective.*

*Proof.* We show that  $H^3(X, \mathbb{Q}) \rightarrow H^3(U, \mathbb{Q})$  is injective. The case  $s = 0$  (i.e.,  $X = U$ ) is obvious. Assume  $s > 0$ . One has  $H^3(U, \mathbb{Q}) = H^1(S, R^2\pi_*\mathbb{Q}_U) = H^1(S, \mathbb{Q})$  from the Leray spectral sequence (5.4.2). Since

$$H^2(C, R^1\pi_*\mathbb{Q}_X) \xleftarrow{\sim} H_c^2(S, R^1\pi_*\mathbb{Q}_U) = H^0(S, R^1\pi_*\mathbb{Q}_U)^\vee = 0,$$

one also has  $H^3(X, \mathbb{Q}) = H^1(C, \mathbb{Q})$  from the Leray spectral sequence for  $\pi : X \rightarrow C$ . Thus the map  $H^3(X, \mathbb{Q}) \rightarrow H^3(U, \mathbb{Q})$  is identified with the natural restriction map  $H^1(C, \mathbb{Q}) \rightarrow H^1(S, \mathbb{Q})$ , which is clearly injective.  $\square$

## 6 Application to Beilinson's Tate conjecture for $K_2$

By the main result on elliptic surfaces (Theorem 5.3.8) we obtain a computable upper bound of the Galois fixed part of étale cohomology group. We can now obtain a number of non-trivial examples of Beilinson's Tate conjecture for  $K_2$  and also non-zero elements of the Selmer group of Bloch-Kato.

### 6.1 Beilinson's Tate conjecture for $K_2$

#### 6.1.1 Example 1

Let  $k \geq 1$  be an integer. Let  $\zeta_k$  be a primitive  $k$ -th root of unity, and put  $F = \mathbb{Q}(\zeta_k)$ . Let  $\mathcal{O}_F$  be the ring of integers in  $F$ . We consider an elliptic surface  $\pi : \mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}}^1$  over  $\mathcal{O} := \mathcal{O}_F[1/6k]$  whose generic fiber over  $F(t)$  is given by

$$3Y^2 + X^3 + (3X + 4t^k)^2 = 0 \quad (6.1.1)$$

(see the beginning of §5 for the notion of elliptic surface over a ring). We assume that  $\overline{X} := \mathcal{X} \otimes_{\mathcal{O}} \overline{F}$  is relatively minimal over  $\mathbb{P}_{\overline{F}}^1$ . The surface  $\overline{X}$  is a rational surface if  $1 \leq k \leq 3$ , a K3 surface if  $4 \leq k \leq 6$  and the Kodaira dimension  $\kappa(\overline{X}) = 1$  if  $k \geq 7$ . Letting  $t = 0 \in \mathbb{P}_{\mathcal{O}}^1(\mathcal{O})$  and  $t = \zeta_k \in \mathbb{P}_{\mathcal{O}}^1(\mathcal{O})$  be the  $\mathcal{O}$ -rational points, we put  $\mathcal{D}_0 := \pi^{-1}(0)$  and  $\mathcal{D}_i := \pi^{-1}(\zeta_k^i)$  ( $1 \leq i \leq k$ ) which are relative normal crossing divisors over  $\mathcal{O}$ . Then  $\mathcal{D}_i$  ( $1 \leq i \leq k$ ) is split multiplicative of type  $I_1$  over  $\mathcal{O}$ , and  $\mathcal{D}_0$  is multiplicative of type  $I_{3k}$  over  $\mathcal{O}$ . If  $\sqrt{-3} \in F$  then  $\mathcal{D}_0$  is split. The singular fibers of  $\overline{X}$  are  $D_i := \mathcal{D}_i \otimes_{\mathcal{O}} \overline{F}$  and possibly  $D_{\infty} := \pi^{-1}(\infty)$ . The type of the fiber  $D_{\infty}$  is as follows:

$k \bmod 3$	1	2	3
$D_{\infty}$	IV*	IV	(smooth)

Put  $\mathcal{U} := \mathcal{X} - (\mathcal{D}_1 + \cdots + \mathcal{D}_k)$ ,  $U_F := \mathcal{U} \otimes_{\mathcal{O}} F$  and  $\overline{U} := U_F \otimes_F \overline{F}$ . We then discuss the Beilinson-Tate conjecture for  $K_2(U_F)$ , namely the surjectivity of the higher Chern class map

$$K_2(U_F \otimes_F L) \otimes \mathbb{Q}_p \longrightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_L} \quad (G_L := \text{Gal}(\overline{L}/L))$$

for a finite extension  $L$  of  $F$ .

**Claim 6.1.2** *Let  $L \supset K \supset F$  and  $[L : K] < \infty$ . Then we have*

$$H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K} = H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_L}.$$

*Proof.* The exact sequence (5.3.12) yields a commutative diagram ( $V := H^2(\overline{X}, \mathbb{Q}_p(2))$ )

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_L} & \xrightarrow{\partial_{\text{ét}}} & \mathbb{Q}_p^k & \longrightarrow & H_{\text{cont}}^1(G_L, V) \\ & & \uparrow & & \parallel & & \uparrow \text{res}_{L/K} \\ 0 & \longrightarrow & H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_K} & \xrightarrow{\partial_{\text{ét}}} & \mathbb{Q}_p^k & \longrightarrow & H_{\text{cont}}^1(G_K, V). \end{array}$$

The assertion follows from the fact that the right vertical arrow is injective.  $\square$

Fix a prime  $q$  of  $F$  above  $p$  which is prime to  $6k$ . Let us denote by  $D_q \subset G_F$  the decomposition group of  $q$ . Let  $K = F_q$  be the completion and  $R \subset K$  the ring of integers. Then  $K$  is a finite unramified extension of  $\mathbb{Q}_p$  and  $D_q \cong G_K := \text{Gal}(\overline{K}/K)$ . We put  $X_R := \mathcal{X} \otimes_{\mathcal{O}} R$ ,  $U_R := \mathcal{U} \otimes_{\mathcal{O}} R$  and  $D_R := \sum_{i=1}^k \mathcal{D}_i \otimes_{\mathcal{O}} R$ .

Let us see the conditions **(A)'** and **(B)'**. As we have seen above, if  $(k, 3) = 1$  then  $\overline{X}$  has the additive fiber  $D_\infty$ , namely **(A)'** is satisfied. Next we see the condition **(B)'**. Let  $\mathbb{F} = R/pR$  be the residue field. Let  $f(X) = -1/3(X^3 + (3X + 4t^k)^2)$  and  $\sum_k a_k t^k$  the coefficient of  $X^{p-1}$  in  $f(X)^{(p-1)/2}$ . Write  $k = 3\ell + a$  with  $1 \leq a \leq 3$ .  $\Gamma(Y, \Omega_{Y/\mathbb{F}}^2)$  has a basis  $\{t^i dt dX/Y; 0 \leq i \leq \ell - 1\}$ . One has

$$\begin{aligned} t^i dt \frac{dX}{Y} &= t^i f(X)^{(p-1)/2} dt \frac{dX}{Y^p} \\ &\equiv \sum_{k=1}^{\ell} a_{kp-i-1} t^{kp-1} dt \frac{X^{p-1} dX}{Y^p} \quad (\text{in } \Gamma(\Omega_{Y/\mathbb{F}}^2/d\Omega_{Y/\mathbb{F}}^1)) \\ &= \sum_{k=1}^{\ell} a_{kp-i-1} C^{-1}(t^{k-1} dt \frac{dX}{Y}) \end{aligned}$$

where  $a_i := 0$  if  $i < 0$ . Thus the Cartier operator  $C$  is described by a matrix

$$A = \begin{pmatrix} a_{p-1} & \cdots & a_{p-\ell} \\ \vdots & & \vdots \\ a_{\ell p-1} & \cdots & a_{\ell p-\ell} \end{pmatrix}.$$

Hence **(B)'** is satisfied if and only if there is no nontrivial solution of

$$A \begin{pmatrix} \alpha_1^p \\ \vdots \\ \alpha_\ell^p \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_\ell \end{pmatrix} \quad (\alpha_1, \dots, \alpha_\ell \in \mathbb{F}).$$

We can now apply Theorem 5.3.8. For example, let  $k = 5$ ,  $p = 11$ ,  $K = \mathbb{Q}_{11}$  and  $R = \mathbb{Z}_{11}$ . Then the conditions **(A)'** and **(B)'** are satisfied. A straightforward calculation (with the aid of computer) shows that  $\overline{\partial_{\text{dR}}}(E^{(99)}(X_R, D_R)_{\mathbb{Z}_{11}})$  is generated by

$$-4[D_1] - 4[D_4] + [D_5], \quad 3[D_1] + 3[D_2] + [D_4], \quad [D_1] - 3[D_2] + [D_3] \quad (6.1.3)$$

in  $\oplus_{i=1}^5 \mathbb{F}_{11}[D_i]$ . We thus have

$$\dim_{\mathbb{Q}_{11}} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_{11}(2))^{G_K} \leq 3. \quad (6.1.4)$$

This remains true if we replace  $K$  with any finite extension  $L$  by Claim 6.1.2. The same result can be checked for several  $(k, p)$ 's.

One has

$$\partial_{\text{ét}}(H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Z}_p(2))^{G_F}) \subset \bigcap_{q|p} \partial_{\text{ét}}(H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Z}_p(2))^{D_q}) \quad (6.1.5)$$

where  $q$  runs over all primes of  $F$  above  $p$ . If each  $\partial_{\text{ét}}(H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Z}_p(2))^{D_q})$  is a direct summand of  $\mathbb{Z}_p^{\oplus k}$  then the right hand side of (6.1.5) is a direct summand as well. Then one has

$$\text{rank}_{\mathbb{Z}_p} \bigcap_{q|p} \partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{D_q}) \leq \dim_{\mathbb{F}_p} \bigcap_{q|p} \overline{\partial}_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{D_q})$$

and it is bounded by using the formal Eisenstein series. For example, in case  $k = 5, p = 11$ , one has

$$\bigcap_{q|p} \overline{\partial}_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{D_q}) \subset \mathbb{F}_p([D_1] + [D_2] + \cdots + [D_5]),$$

hence

$$\text{rank}_{\mathbb{Z}_p} H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{G_F} \leq \text{rank}_{\mathbb{Z}_p} \bigcap_{q|p} \partial_{\text{ét}}(H_{\text{ét}}^2(\overline{U}, \mathbb{Z}_p(2))^{D_q}) \leq 1. \quad (6.1.6)$$

The same thing is true if we replace  $F$  with any finite extension  $L$  by Claim 6.1.2, and same thing can be checked for several  $(k, p)$ 's.

We claim that the equalities hold in (6.1.6). To do this, it is enough to show that  $\partial K_2(U_F) \otimes \mathbb{Q} \neq 0$ . If  $k = 1$  then  $X = X_1$  is the elliptic modular surface for  $\Gamma_1(3)$ . Therefore it follows from Beilinson's theorem that the image of the boundary map on  $K_2(X_1 - D_1)$  is 1-dimensional. Since there is the finite surjective map  $X \rightarrow X_1$  induced from  $t \mapsto t^k$ , one has  $\dim_{\mathbb{Q}} \partial K_2(U_F) \otimes \mathbb{Q} \geq 1$ . (Explicitly, the symbol

$$\left\{ \frac{\sqrt{-3}(Y - X - 4) - 4(t^k - 1)}{\sqrt{-3}(Y + X + 4) - 4(t^k - 1)}, \frac{8(-\sqrt{-3}Y + 3X + 4t^k)(t^k - 1)}{(X + 4)^3} \right\} \quad (6.1.7)$$

in  $\Gamma(\overline{U}, \mathcal{K}_2) \otimes \mathbb{Q}$  has non-zero boundary.)

Finally let  $F' \supset \mathbb{Q}(\sqrt{-3}, \zeta_k)$  and put  $U'_{F'} = U_{F'} - D_0$ . Then one can also show

$$\dim_{\mathbb{Q}} \partial K_2(U'_{F'}) \otimes \mathbb{Q} = \dim_{\mathbb{Q}_p} H_{\text{ét}}^2(\overline{U}', \mathbb{Q}_p(2))^{G_{F'}} = \dim_{\mathbb{Q}_p} \bigcap_{q|p} H_{\text{ét}}^2(\overline{U}', \mathbb{Q}_p(2))^{D_q} = 2$$

where  $q$  runs over all primes of  $F'$  above 11. (The symbol (6.1.11) and

$$\left\{ \frac{\sqrt{-3}Y - 3X - 4t^k}{-8t^k}, \frac{\sqrt{-3}Y + 3X + 4t^k}{8t^k} \right\} \in \Gamma(\overline{U}', \mathcal{K}_2) \otimes \mathbb{Q} \quad (6.1.8)$$

span the 2-dimensional boundary image.) This remains true if we replace  $F'$  with any its finite extension or  $U'_{F'}$  with  $U'_{F'} - (\text{any other fibers})$ . The same results can be checked for several  $(k, p)$ 's.

Summarizing above we obtain

**Theorem 6.1.9** *Let  $F$  be a number field such that  $F \supset \mathbb{Q}(\sqrt{-3}, \zeta_k)$ . Let  $\pi_F : X_F \rightarrow \mathbb{P}_F^1$  be the elliptic surface whose generic fiber is defined by (6.1.1). Put  $U_F := X_F - (D_0 + \cdots + D_k) - (\text{any other fibers})$ . Then  $K_2(U_F) \otimes \mathbb{Q}_p \rightarrow H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F}$  is surjective and*

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F} = \dim_{\mathbb{Q}_p} \bigcap_{q|p} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{D_q} = 2 \quad (6.1.10)$$

if  $(k, p)$  is one of the following cases:

$k$	5	7	11	13
$p$	$7 \leq p \leq 97$	$5 \leq p \leq 109$ and $p \neq 7$	$5 \leq p \leq 61$ and $p \neq 11$	$5 \leq p \leq 73$ and $p \neq 13$

Explicitly  $H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F}$  is spanned by the image of the following symbols

$$\xi_1 = \left\{ \frac{\sqrt{-3}(Y - X - 4) - 4(t^k - 1)}{\sqrt{-3}(Y + X + 4) - 4(t^k - 1)}, \frac{8(-\sqrt{-3}Y + 3X + 4t^k)(t^k - 1)}{(X + 4)^3} \right\} \quad (6.1.11)$$

$$\xi_2 = \left\{ \frac{\sqrt{-3}Y - 3X - 4t^k}{-8t^k}, \frac{\sqrt{-3}Y + 3X + 4t^k}{8t^k} \right\} \quad (6.1.12)$$

(Note  $\partial(\xi_1) = [D_1] + [D_2] + \cdots + [D_k]$  and  $\partial(\xi_2) = k[D_0]$ .)

### 6.1.2 Example 2

Let  $k \geq 1$  be an integer and  $p$  a prime number such that  $(p, 6k) = 1$ . Let  $F$  be a number field such that  $F \supset \mathbb{Q}(\sqrt{-1}, \zeta_k)$ . Let  $\pi_F : X_F \rightarrow \mathbb{P}_F^1$  be the elliptic surface whose generic fiber is defined by

$$3Y^2 = 2X^3 - 3X^2 + t^k. \quad (6.1.13)$$

The surface  $X_{\overline{F}}$  is a rational surface if  $1 \leq k \leq 6$ , a K3 surface if  $7 \leq k \leq 12$  and  $\kappa(X_{\overline{F}}) = 1$  if  $k \geq 13$ . We put  $D_0 := \pi^{-1}(0)$ ,  $D_i := \pi^{-1}(\zeta_k^i)$  ( $1 \leq i \leq k$ ) and  $U_F := X_F - (D_0 + D_1 + \cdots + D_k)$ . Then  $D_i$  are split multiplicative fibers. The other singular fiber of  $X_{\overline{F}}$  is  $\pi^{-1}(\infty)$  whose type is as follows:

$k \bmod 6$	1	2	3	4	5	6
$\pi^{-1}(\infty)$	II*	IV*	I <sub>0</sub> *	IV	II	(smooth)

In the same way as in §6.1.1 one can show the following theorem.

**Theorem 6.1.14** *Let  $F$  be a number field such that  $F \supset \mathbb{Q}(\sqrt{-1}, \zeta_k)$ . Let  $\pi_F : X_F \rightarrow \mathbb{P}_F^1$  and  $D_i$  be as above. Put  $U_F := X_F - (D_0 + \cdots + D_k) - (\text{any other fibers})$ . Then  $K_2(U_F) \otimes \mathbb{Q}_p \rightarrow H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Q}_p(2))^{G_F}$  is surjective and*

$$\dim_{\mathbb{Q}_p} H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Q}_p(2))^{G_F} = \dim_{\mathbb{Q}_p} \bigcap_{q|p} H_{\text{ét}}^2(U_{\overline{F}}, \mathbb{Q}_p(2))^{D_q} = 2 \quad (6.1.15)$$

if  $(k, p)$  is one of the following cases:

$k$	7	8	9	11	13
$p$	$5 \leq p \leq 109$ and $p \neq 7$	$5 \leq p \leq 43$	$5 \leq p \leq 47$	$5 \leq p \leq 61$ and $p \neq 11$	$5 \leq p \leq 73$ and $p \neq 13$

Moreover  $H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_F}$  is generated by the image of symbols

$$\xi_1 = \left\{ \frac{Y - \sqrt{-1}X}{Y + \sqrt{-1}X}, -\frac{t^k}{2X^3} \right\}, \quad \xi_2 = \left\{ \frac{Y - (X - 1)}{Y + (X - 1)}, -\frac{t^k - 1}{2(X - 1)^3} \right\}.$$

(Note  $\partial(\xi_1) = k[D_0]$  and  $\partial(\xi_2) = [D_1] + [D_2] + \cdots + [D_k]$ .)

## 6.2 Bloch-Kato's Selmer group

In the seminal paper [BK2], Bloch and Kato defined the  $f$ -part and the  $g$ -part in the continuous Galois cohomology

$$H_f^1 \subset H_g^1 \subset H_{\text{cont}}^1(G_k, V)$$

for a continuous  $\mathbb{Q}_p$ -representation  $V$  of  $G_k$ , where  $k$  is a  $p$ -adic local field or a number field. In particular, the  $f$ -part is often called the *Selmer group* of Bloch-Kato. Let  $X$  be a projective smooth variety over  $k$ . Then the higher Chern character map on Quillen's  $K$ -theory induces a  $p$ -adic regulator map

$$\varrho_{1,2} : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H_g^1(G_k, H^2(\overline{X}, \mathbb{Q}_p(2))) \quad (6.2.1)$$

([SS1], (3.3.1), Lemma 3.5.2). Let  $\mathcal{X}$  be a regular proper flat scheme over the ring of integers  $\mathcal{O}_k$  which has a finite surjective map  $h : \mathcal{X} \otimes_{\mathcal{O}_k} k \rightarrow X$ . We define the *integral part* of  $K$ -theory as

$$K_1(X)_{\mathbb{Z}}^{(2)} := \text{Im}(K_1(\mathcal{X})^{(2)} \rightarrow K_1(\mathcal{X} \otimes_{\mathcal{O}_k} k)^{(2)} \xrightarrow{h_*} K_1(X)^{(2)}),$$

which is independent of the choice of  $\mathcal{X}$  ([Scho]). Then the image of the integral  $K$ -theory under  $\rho_{1,2}$  is contained in the  $f$ -part ([Ni], [Sa]), and we obtain the following integral regulator map:

$$\varrho_{1,2} : K_1(X)_{\mathbb{Z}}^{(2)} \otimes \mathbb{Q}_p \longrightarrow H_f^1(G_k, H^2(\overline{X}, \mathbb{Q}_p(2))). \quad (6.2.2)$$

When  $k$  is a number field, Bloch and Kato conjecture that (6.2.1) and (6.2.2) are bijective, but it is a widely open problem.

To consider the surjectivity of these regulator maps, we decompose each of them into decomposable part and indecomposable part. The latter is much more difficult to compute in general. For a finite field extension  $L/k$ , let  $\pi_L$  be the composite of the product and the norm map of  $K$ -groups:

$$L^\times \otimes \text{Pic}(X \otimes_k L) \longrightarrow K_1(X \otimes_k L)^{(2)} \longrightarrow K_1(X)^{(2)}.$$

We define  $K_1^{\text{dec}}(X)^{(2)}$ , the *decomposable* part of  $K_1(X)^{(2)}$ , as the  $\mathbb{Q}$ -subspace of  $K_1(X)^{(2)}$  generated by the image of

$$\bigoplus_{L/k} L^\times \otimes \text{Pic}(X \otimes_k L) \xrightarrow{\sum \pi_L} K_1(X)^{(2)},$$

where  $L$  runs through all finite extensions of  $k$ . We put  $K_1^{\text{ind}}(X)^{(2)} := K_1(X)^{(2)} / K_1^{\text{dec}}(X)^{(2)}$  and call it the *indecomposable*  $K_1$  of  $X$ . We put

$$K_1^{\text{dec}}(X)_{\mathbb{Z}}^{(2)} := K_1^{\text{dec}}(X)^{(2)} \cap K_1(X)_{\mathbb{Z}}^{(2)}, \quad K_1^{\text{ind}}(X)_{\mathbb{Z}}^{(2)} := \text{Im}(K_1(X)_{\mathbb{Z}}^{(2)} \rightarrow K_1^{\text{ind}}(X)^{(2)}).$$

One easily sees  $K_1(X)_{\mathbb{Z}}^{(2)} \cong K_1^{\text{dec}}(X)_{\mathbb{Z}}^{(2)} \oplus K_1^{\text{ind}}(X)_{\mathbb{Z}}^{(2)}$ . Put

$$V^{\text{ind}} := H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2)) / \text{NS}(\overline{X}) \otimes \mathbb{Q}_p(1).$$

Then the regulator map  $\varrho = \varrho_{1,2}$  induces a commutative diagram

$$\begin{array}{ccc} K_1^{\text{ind}}(X)^{(2)} \otimes \mathbb{Q}_p & \xrightarrow{\varrho} & H_g^1(G_k, V^{\text{ind}}) \\ \uparrow & & \uparrow \\ K_1^{\text{ind}}(X)_{\mathbb{Z}}^{(2)} \otimes \mathbb{Q}_p & \xrightarrow{\varrho} & H_f^1(G_k, V^{\text{ind}}) \end{array}$$

This is the difficult part of the regulator map  $\varrho$ . It is very difficult even to ask whether it vanishes or not in general. We apply the results in the previous section to show the non-vanishing in case  $X$  is an elliptic surface. The strategy is as follows.

Let  $\pi : X \rightarrow C$  be an elliptic surface over a number field or a  $p$ -adic local field  $k$  and  $D_i = \pi^{-1}(P_i)$  split multiplicative fibers over  $P_i \in C(k)$ . Let  $\tilde{D}_i \rightarrow D_i$  be the normalization. Then the cokernel of  $K_1(\tilde{D}_i)^{(2)} \rightarrow K_1'(D_i)^{(1)}$  is a  $\mathbb{Q}$ -vector space of rank one and the generator is described in the following way. Let  $D_i^\dagger \subset D_i$  be the Néron polygon. Let  $D_i^\dagger = \sum E_j$  be the irreducible decomposition and  $Q_j$  the intersection points in  $E_j$  and  $E_{j+1}$ . Let  $f_j$  be a rational function on  $E_j$  such that  $\text{div}_{E_j}(f_j) = Q_j - Q_{j-1}$ . There is a localization exact sequence

$$0 \longrightarrow K_1'(D_i^\dagger)^{(1)} \longrightarrow K_1(D_i^\dagger - \coprod_j Q_j)^{(1)} \xrightarrow{d} \bigoplus_j K_0(Q_j)$$

and the element

$$(f_j)_j \in \bigoplus_j K_1(E_j - \{Q_j, Q_{j+1}\})^{(1)} \cong K_1(D_i^\dagger - \coprod_j Q_j)^{(1)}$$

lies in the kernel of  $d$ . Hence it defines an element  $\xi_i' \in K_1'(D_i^\dagger)^{(1)}$ . Let  $\xi_i \in K_1'(D_i)^{(1)}$  be the image of  $\xi_i'$  via the push-forward  $K_1'(D_i^\dagger)^{(1)} \rightarrow K_1'(D_i)^{(1)}$ . Then  $\xi_i$  gives a generator of the cokernel of  $K_1'(\tilde{D}_i)^{(1)} \rightarrow K_1'(D_i)^{(1)}$ .

Put  $D = \sum_{i=1}^s D_i$ . Let  $K_1(X)_D$  be the subgroup of  $K_1(X)^{(2)}$  generated by the images of  $K_1(\tilde{D}_i)^{(1)}$ . Let  $\langle D \rangle \subset \text{NS}(X_{\overline{F}})$  be the subgroup generated by irreducible components of  $D$ . Put  $V := H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(2))/\langle D \rangle \otimes \mathbb{Q}_p(1)$ . One has a commutative diagram

$$\begin{array}{ccccc} K_2(U) \otimes \mathbb{Q}_p & \xrightarrow{\partial} & \bigoplus_{i=1}^s \mathbb{Q}_p \cdot \xi_i & \longrightarrow & K_1(X)^{(2)}/K_1(X)_D \otimes \mathbb{Q}_p \\ \downarrow & & \downarrow \cong & & \downarrow \varrho \\ 0 & \longrightarrow & H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_k} & \xrightarrow{\partial_{\text{ét}}} & \bigoplus_{i=1}^s \mathbb{Q}_p[D_i] \longrightarrow H_g^1(G_k, V) \end{array} \quad (6.2.3)$$

with exact rows. We also write the image of  $\xi_i$  in  $K_1(X)^{(2)}/K_1(X)_D$  by  $\xi_i$ . We have

$$\sum a_i \varrho(\xi_i) \neq 0 \text{ in } H_g^1(G_k, V) \iff \sum a_i [D_i] \notin \partial_{\text{ét}} H_{\text{ét}}^2(\overline{U}, \mathbb{Q}_p(2))^{G_k}. \quad (6.2.4)$$

Let  $Z \subset X$  be an irreducible curve and  $\tilde{Z} \rightarrow Z$  the normalization. Let

$$v_Z : H^1(G_k, H^2(\overline{X}, \mathbb{Q}_p(2))) \longrightarrow H^1(G_k, H^2(\tilde{Z}, \mathbb{Q}_p(2))) = \mathbb{Q} \otimes \varprojlim_n k^\times/p^n$$

be the pull-back map. Assume that

$$v_Z(\sum a_i \varrho(\xi_i)) = 0 \quad \text{for each generator } Z \text{ of } \text{NS}(\overline{X})_{\mathbb{Q}}. \quad (6.2.5)$$

Since the intersection form on  $\text{NS}(\overline{X})_{\mathbb{Q}}$  is non-degenerate, one has

$$\text{NS}^{\perp} \oplus \text{NS}(\overline{X})_{\mathbb{Q}_p} \xrightarrow{\sim} H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1))$$

and  $\text{NS}^{\perp} \otimes \mathbb{Q}_p(1) \simeq V^{\text{ind}}$ , where  $\text{NS}^{\perp}$  denotes the orthogonal complement of  $\text{NS}(\overline{X})_{\mathbb{Q}_p}$  in  $H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1))$ . Therefore (6.2.5) implies that  $\varrho(\sum a_i \xi_i)$  is contained in

$$\text{the image of } H^1(\mathbb{Q}_p, \text{NS}^{\perp} \otimes \mathbb{Q}_p(1)) \rightarrow H^1(\mathbb{Q}_p, V)$$

and hence

$$\sum a_i \varrho(\xi_i) \neq 0 \text{ in } H_f^1(G_k, V^{\text{ind}}) \quad (6.2.6)$$

under (6.2.4). If  $\sum a_i \xi_i$  is integral, then we obtain a non-zero element of the Selmer group  $H_f^1(G_k, V^{\text{ind}})$  of Bloch-Kato.

### 6.2.1 Example 1

Recall the elliptic surface (6.1.1) in §6.1.1. If  $k$  is odd, the Néron-Severi group  $\text{NS}(\overline{X}) \otimes \mathbb{Q}$  is generated by irreducible components of singular fibers and the section  $e(\mathbb{P}^1)$  ([St] Example 3). Let  $\xi_i \in K_1(X)^{(2)}$  be the element which is defined from

$$f_i = \frac{Y - (X + 4)}{Y + (X + 4)} \Big|_{D_i} \in K_1(D_i^{\text{reg}})^{(1)}$$

in the above way. It is easy to check that it satisfies (6.2.5). One can also check that if  $(k, 6) = 1$  then each  $\xi_i$  is integral (i.e.  $\xi_i \in K_1(X)_{\mathbb{Z}}^{(2)}$ ). By Theorem 6.1.9, we have

**Theorem 6.2.7** *Let  $\pi_F : X_F \rightarrow \mathbb{P}_F^1$  and  $D_i$  be as in Theorem 6.1.9. Then the composition*

$$\left( \bigoplus_{i=1}^k \mathbb{Q}_p \xi_i \right) / \mathbb{Q}_p(\xi_1 + \cdots + \xi_k) \xrightarrow{\varrho} H_f^1(G_F, V^{\text{ind}}) \longrightarrow \bigoplus_{q|p} H_f^1(G_{F_q}, V^{\text{ind}})$$

is injective if  $(k, p)$  is one of the following cases:

$k$	5	7	11	13
$p$	$7 \leq p \leq 97$	$5 \leq p \leq 109$ and $p \neq 7$	$5 \leq p \leq 61$ and $p \neq 11$	$5 \leq p \leq 73$ and $p \neq 13$

### 6.2.2 Example 2

Recall the elliptic surface (6.1.13) in §6.1.2. If  $(k, 30) = 1$  the Néron-Severi group  $\text{NS}(\overline{X}) \otimes \mathbb{Q}$  is generated by the irreducible components of singular fibers and the section  $e(\mathbb{P}^1)$  ([St] Example 4). Similarly to before, Theorem 6.1.14 implies

**Theorem 6.2.8** *Let  $\pi_F : X_F \rightarrow \mathbb{P}_F^1$  and  $D_i$  be as in Theorem 6.1.14. Then the composition*

$$\left( \bigoplus_{i=1}^k \mathbb{Q}_p \xi_i \right) / \mathbb{Q}_p(\xi_1 + \cdots + \xi_k) \xrightarrow{\varrho} H_f^1(G_F, V^{\text{ind}}) \longrightarrow \bigoplus_{q|p} H_f^1(G_{F_q}, V^{\text{ind}})$$

is injective if  $(k, p)$  is one of the following cases:

$k$	7	11	13
$p$	$5 \leq p \leq 109$ and $p \neq 7$	$5 \leq p \leq 61$ and $p \neq 11$	$5 \leq p \leq 73$ and $p \neq 13$

## 7 An elliptic $K3$ surface over $\mathbb{Q}_p$ with finitely many torsion zero-cycles

In this section, we discuss another application to  $p$ -adic regulator map on  $K_1$  and finiteness of torsion zero-cycles. Hereafter all cohomology groups of schemes are taken over the étale topology.

To state the main result of this section, we start from a numerical condition on a prime number  $p \geq 5$ . Let  $q$  be an indeterminate, and define formal power series  $E_1, E_{3,a}, E_{3,b}$  in  $\mathbb{Z}[[q]]$  as

$$\begin{aligned} E_1 &:= 1 + 6 \sum_{k=1}^{\infty} \left( \frac{q^{3k-2}}{1 - q^{3k-2}} - \frac{q^{3k-1}}{1 - q^{3k-1}} \right), \\ E_{3,a} &:= 1 - 9 \sum_{k=1}^{\infty} \left( \frac{(3k-2)^2 q^{3k-2}}{1 - q^{3k-2}} - \frac{(3k-1)^2 q^{3k-1}}{1 - q^{3k-1}} \right), \\ E_{3,b} &:= \sum_{k=1}^{\infty} \frac{k^2 (q^k - q^{2k})}{1 - q^{3k}}, \end{aligned}$$

which are obtained from the  $q$ -expansion of Eisenstein series for  $\Gamma_1(3)$  at the cusp  $\tau = \infty$ . Let  $t \in \mathbb{Z}[2^{-1}]((q))$  satisfy  $t^4 = E_{3,a}/(E_1)^3$ :

$$t = 1 - \frac{27}{4}q + \frac{1053}{32}q^2 - \frac{23085}{2^7}q^3 + \frac{2130003}{2^{11}}q^4 - \frac{49565277}{2^{13}}q^5 + \dots$$

Put

$$g(q) = -\frac{27t}{4}E_{3,b}, \quad f_1(q) = -\frac{27t}{4(t-1)}E_{3,b}, \quad f_2(q) = -\frac{27t}{4(t+1)}E_{3,b} \quad (7.0.1)$$

and express

$$f_1(q) = 1 + \sum_{i=1}^{\infty} \frac{a_i q^i}{1 - q^i}, \quad f_2(q) = \sum_{i=1}^{\infty} \frac{b_i q^i}{1 - q^i}, \quad g(q) = \sum_{i=1}^{\infty} \frac{c_i q^i}{1 - q^i}$$

with  $a_i, b_i, c_i \in \mathbb{Z}_p$ . Then we say that a prime number  $p \geq 5$  satisfies **C(p)** if the following two conditions are satisfied:

**C(p)-1.** Let  $k_p$  be the coefficient of  $x^{p-1}t^{p-1}$  in the polynomial  $(-3(x^3 + (3x + 4t^4)^2))^{(p-1)/2}$  (cf. §7.1 **Fact 1**). Then  $k_p \not\equiv 1 \pmod{p}$ .

**C(p)-2.** There is no solution  $n \in \mathbb{Z}_p$  which satisfy all of the following congruent relations:

$$\begin{cases} a_{ip} - b_{ip} + nc_{ip} \equiv 0 \pmod{p^2 \mathbb{Z}_p} & (i = 1, 2, \dots, p-1) \\ a_{p^2} - b_{p^2} + nc_{p^2} \equiv 0 \pmod{p^4 \mathbb{Z}_p}. \end{cases} \quad (7.0.2)$$

The following is the main result of this section.

**Theorem 7.0.3** *Let  $\pi : X_{\mathbb{Q}} \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  be the elliptic K3 surface over  $\mathbb{Q}$  whose general fiber  $\pi^{-1}(t)$  is an elliptic curve given by*

$$3Y^2 + X^3 + (3X + 4t^4)^2 = 0.$$

*Let  $p \geq 5$  be a prime number which satisfies  $\mathbf{C}(\mathbf{p})$ , and put  $X := X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$  and  $\overline{X} := X \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}$ . Then the  $p$ -adic regulator*

$$\varrho : K_1(X)^{(2)} \otimes \mathbb{Q}_p \longrightarrow H_g^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2))) \quad (7.0.4)$$

*is surjective.*

The condition  $\mathbf{C}(\mathbf{p})$  is in fact computable, and the authors checked that it holds for  $p = 7, 11, 19, 23, 31$  with the aid of computer. They expect it is true for any  $p$  with  $p \equiv 3 \pmod{4}$ , but do not have any idea to prove it. When  $p \equiv 1 \pmod{4}$ , they do not know any example of  $p$  which satisfies  $\mathbf{C}(\mathbf{p})$ , nor whether the conclusion of Theorem 7.0.3 (or Corollary 7.0.5 below) holds.

Theorem 7.0.3 immediately implies the following finiteness result by [SS1] Theorem 3.1.1.

**Corollary 7.0.5** *The  $p$ -primary torsion subgroup of  $\mathrm{CH}_0(X)$  is finite, if  $p$  satisfies  $\mathbf{C}(\mathbf{p})$  (e.g.  $p = 7, 11, 19, 23, 31$ ).*

See §7.3 below for the finiteness of the full torsion part  $\mathrm{CH}_0(X)_{\mathrm{tors}}$ . The proof of Theorem 7.0.3 is divided into two parts, i.e., the surjectivity onto  $H_g^1/H_f^1$  and  $H_f^1$  (see §7.2). The former one will be reduced to results of Flach and Mildenhall (see §7.2.1 below). To show the latter one, we shall construct a new indecomposable element in  $K_1(\mathcal{X})^{(2)}$  and show the non-vanishing in  $H_f^1$  under the condition  $\mathbf{C}(\mathbf{p})$ , where Theorem 5.3.8 will play an essential role (see Proposition 7.2.5 below).

**Remark 7.0.6** *The elliptic surface defined by  $3Y^2 + X^3 + (3X + 4t)^2 = 0$  ( $t \in \mathbb{P}^1$ ) is the universal family of elliptic curves over the modular curve  $X_1(3)$ . The elliptic K3 in Theorem 7.0.3 is a finite covering of this surface, but it is no longer modular.*

## 7.1 Preliminary facts

Before proving the theorem, we write down some facts on the  $X_{\mathbb{Q}}$  and  $X_{\overline{\mathbb{Q}}} := X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$  which we shall need later (proof is left to the reader).

**Fact 1.**  $X_{\mathbb{Q}}$  has good reduction at  $p \geq 5$ . For  $p \geq 5$ , the reduction  $Y_p$  of  $X_{\mathbb{Q}}$  at  $p$  is ordinary if and only if  $p \equiv 1 \pmod{4}$ , and super-singular if and only if  $p \equiv 3 \pmod{4}$ .

Note that the Cartier operator  $C : H^0(\Omega_{Y_p}^2) \rightarrow H^0(\Omega_{Y_p}^2)$  is given by multiplication by  $k_p$  in  $\mathbf{C}(\mathbf{p})$ -1.  $Y_p$  is ordinary if and only if  $k_p \equiv 0 \pmod{p}$  and otherwise it is super-singular.

**Fact 2.** The functional  $j$ -invariant is  $27(9 - 8t^4)^3 / ((1 - t^4)t^{12})$ . There are 5 multiplicative fibers over  $t = 0, \pm 1, \pm \sqrt{-1}$ , and one additive fiber over  $t = \infty$ :

$$\frac{t}{\pi^{-1}(t)} \mid \begin{array}{c|c} 0 & \pm 1, \pm \sqrt{-1} \\ \hline I_{12} & I_1 \end{array} \mid \infty \\ \hline IV^*$$

**Fact 3.** The Néron-Severi group  $\text{NS}(X_{\overline{\mathbb{Q}}})$  has rank 20. Explicitly it is generated by the following irreducible curves:

- (i) irreducible components of  $\pi^{-1}(0)$  (12-components),
- (ii) irreducible components of  $\pi^{-1}(\infty)$  (7-components),
- (iii) the section at infinity  $e(\mathbb{P}^1) = E$ ,
- (iv) the section  $C$  defined by  $x = -4t^4/3$  and  $y = 8t^6/9$ .

Note that  $\pi^{-1}(0) \equiv \pi^{-1}(\infty)$  is the only relation in  $\text{NS}(X_{\overline{\mathbb{Q}}})$  among the above.

**Fact 4.** Put  $\overline{Y}_p := Y_p \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ . The Néron-Severi group  $\text{NS}(\overline{Y}_p)$  has rank 20 if  $p \equiv 1 \pmod{4}$ , and rank 22 if  $p \equiv 3 \pmod{4}$ .

When  $p \equiv 1 \pmod{4}$ , one has  $\text{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = \text{NS}(\overline{Y}_p) \otimes \mathbb{Q}$ . When  $p \equiv 3 \pmod{4}$ , one has  $\text{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \neq \text{NS}(\overline{Y}_p) \otimes \mathbb{Q}$  and we will describe curves on  $\overline{Y}_p$  which do not come from  $X_{\overline{\mathbb{Q}}}$  in §7.2.1.

## 7.2 Proof of Theorem 7.0.3

Let  $\mathcal{X} \rightarrow \text{Spec}(\mathbb{Z}_p)$  be a projective smooth model of  $X = X_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ , and let  $Y \rightarrow \text{Spec}(\mathbb{F}_p)$  be its special fiber. Since  $K_1^{\text{dec}}(X)^{(2)} \otimes \mathbb{Q}_p$  is onto  $H^1(\mathbb{Q}_p, \text{NS}(\overline{X}) \otimes \mathbb{Q}_p(1))$  (cf. [S] Lemma 3.6), we may replace the target with  $H_g^1(\mathbb{Q}_p, V)$ , where we put

$$V^{\text{ind}} := H^2(\overline{X}, \mathbb{Q}_p(2)) / \text{NS}(\overline{X}) \otimes \mathbb{Q}_p(1).$$

We shall prove it in the following steps.

**Step 1.**  $K_1(X)^{(2)} \otimes \mathbb{Q}_p \rightarrow H_g^1(\mathbb{Q}_p, V^{\text{ind}}) / H_f^1(\mathbb{Q}_p, V^{\text{ind}})$  is surjective for any  $p \geq 5$ .

**Step 2.**  $K_1(\mathcal{X})^{(2)} \otimes \mathbb{Q}_p \rightarrow H_f^1(\mathbb{Q}_p, V^{\text{ind}})$  is surjective for  $p \geq 5$  satisfying **C(p)**.

### 7.2.1 Proof of Step 1

By the Tate conjecture for  $Y$  ([ArSw]) and the same arguments as in [LS] Theorem 5.1, there is a canonical isomorphism

$$H_g^1(\mathbb{Q}_p, V^{\text{ind}}) / H_f^1(\mathbb{Q}_p, V^{\text{ind}}) \simeq \text{NS}(Y) / \text{NS}(X) \otimes \mathbb{Q}_p$$

and the composition

$$K_1(X)^{(2)} \otimes \mathbb{Q}_p \rightarrow H_g^1(\mathbb{Q}_p, V^{\text{ind}}) / H_f^1(\mathbb{Q}_p, V^{\text{ind}}) \simeq \text{NS}(Y) / \text{NS}(X) \otimes \mathbb{Q}_p$$

is given by the boundary map  $K_1(X)^{(2)} \rightarrow \text{NS}(Y) \otimes \mathbb{Q}$  arising from the localization exact sequence in  $K$ -theory.

**Proposition 7.2.1** *Let  $C_i$  ( $i = 1, 2$ ) be elliptic curves over  $\mathbb{Q}$  defined by equations*

$$3y^2 + x^4 + 1 = 0 \quad \text{and} \quad u^2 + 4v^4 + 3 = 0,$$

*respectively. Then there is a dominant rational map*

$$f : C_1 \times C_2 \dashrightarrow X_0$$

*of degree 8 given by  $(x, y) \times (u, v) \mapsto (X, Y, t) = ((ux)^4, u^6x^4y, uvx)$ .*

*Proof.* Straight-forward. □

Note that  $C_1$  and  $C_2$  are isomorphic to each other up to twist. Let  $S \rightarrow C_1 \times C_2$  be a birational transformation such that  $f$  is extended to a morphism  $\bar{f} : S \rightarrow X$ . Put

$$V^{\text{ind}}(-) := H^2(-, \mathbb{Q}_p(2))/\text{NS}(-) \otimes \mathbb{Q}_p(1).$$

Then one has

$$V^{\text{ind}}(\bar{X}) \hookrightarrow V^{\text{ind}}(\bar{S}) \xleftarrow{\sim} V^{\text{ind}}(\overline{C_1 \times C_2}).$$

We show that the first map is bijective. Indeed, we have  $\dim_{\mathbb{Q}_p} V^{\text{ind}}(\bar{X}) = 2$  by **Fact 3**. On the other hand, since  $C_1$  and  $C_2$  are isomorphic to a CM elliptic curve up to twist,  $V^{\text{ind}}(\overline{C_1 \times C_2})$  is also 2-dimensional. Hence  $V^{\text{ind}}(\bar{X}) = V^{\text{ind}}(\bar{S})$ .

In order to show **Step 1** it is enough to show that

$$K_1(C_1 \times C_2)^{(2)} \otimes \mathbb{Q}_p \rightarrow H_g^1(\mathbb{Q}_p, V^{\text{ind}}(\overline{C_1 \times C_2}))/H_f^1(\mathbb{Q}_p, V^{\text{ind}}(\overline{C_1 \times C_2}))$$

is surjective. We may replace  $\mathbb{Q}_p$  with arbitrary finite extension  $K/\mathbb{Q}_p$  by a standard norm argument. Fix  $K$  such that  $C_{1,K} \simeq C_{2,K}(=: C)$  with smooth reduction  $\mathcal{C}_s$  and such that

$$\text{End}(C) = \text{End}(\bar{C}) \quad \text{and} \quad \text{End}(\mathcal{C}_s) = \text{End}(\bar{\mathcal{C}}_s).$$

We show that

$$\begin{aligned} K_1(C \times C)^{(2)} \otimes \mathbb{Q}_p &\longrightarrow H_g^1(K, V^{\text{ind}}(\overline{C \times C}))/H_f^1(K, V^{\text{ind}}(\overline{C \times C})) \\ &\simeq \text{End}(\bar{\mathcal{C}}_s) \otimes \mathbb{Q}_p / \text{End}(\bar{C}) \otimes \mathbb{Q}_p \end{aligned}$$

is surjective. If  $\mathcal{C}_s$  is ordinary (i.e.,  $p \equiv 1 \pmod{4}$ ), the target is zero. If  $\mathcal{C}_s$  is super-singular (i.e.,  $p \equiv 3 \pmod{4}$ ), it is generated by the Frobenius endomorphism and its composition with the CM endomorphism. The surjectivity then follows from [F] Proposition 2.1 or [M] Theorem 5.8. This completes the proof of **Step 1**.

## 7.2.2 Proof of Step 2

This is a crucial step. We first show that  $H_f^1(\mathbb{Q}_p, V^{\text{ind}})$  is 1-dimensional over  $\mathbb{Q}_p$ .

**Lemma 7.2.2** *There is an isomorphism*

$$H_f^1(\mathbb{Q}_p, V^{\text{ind}}) \simeq H^2(X, \mathcal{O}_X).$$

*Proof.* We first note the following isomorphisms:

$$\begin{aligned} H_f^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2))) &\xrightarrow{c^3} H_{\text{syn}}^3(\mathcal{X}, \mathcal{S}_{\mathbb{Q}_p}(2)) \\ &\xleftarrow{\delta^3} \text{Coker}(1 - f_2 : \Gamma(X, \Omega_{X/\mathbb{Q}_p}^2) \rightarrow H_{\text{dR}}^2(X/\mathbb{Q}_p)) \\ &=: H_{\text{dR}}^2(X/\mathbb{Q}_p)/(1 - f_2), \end{aligned}$$

where  $f_2$  denotes the map induced by  $f_2$  in Definition 2.1.6. The first isomorphism, induced by the map  $c^3$  in Theorem 2.2.15, is due to Langer-Saito [LS] Theorem 6.1, a special case of the  $p$ -adic point conjecture raised by Schneider [Sch2] (cf. [Ne] III Theorem (3.2)). The second isomorphism  $\delta^3$  is the connecting map induced by the definition of  $\mathcal{S}_n(2)$  (see [Sch2] p. 242 or [Ne] III (3.1.1) for details). We denote the composite of these isomorphisms by  $\alpha$ .

Let  $L$  be a finite extension of  $\mathbb{Q}_p$  for which  $\text{NS}(X_L)$  has rank 20, i.e.,  $\text{NS}(X_L) \cong \text{NS}(\overline{X})$  (cf. **Fact 3**). We construct a commutative diagram as follows:

$$\begin{array}{ccc} \text{NS}(X_L) \otimes L & \xrightarrow{\varpi_{L/\mathbb{Q}_p}} & H_f^1(\mathbb{Q}_p, \text{NS}(\overline{X}) \otimes \mathbb{Q}_p(1)) \\ \text{tr}_{L/\mathbb{Q}_p} \circ \varrho_{\text{dR}, L} \downarrow & & \downarrow \\ H_{\text{dR}}^2(X/\mathbb{Q}_p)/(1 - f_2) & \xrightarrow{\sim \alpha} & H_f^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2))), \end{array} \quad (7.2.3)$$

where the map  $\varrho_{\text{dR}, L}$  denotes the de Rham Chern class with values in  $H_{\text{dR}}^2(X_L/L)$  and  $\text{tr}_{L/\mathbb{Q}_p}$  denotes the trace map  $H_{\text{dR}}^2(X_L/L) \rightarrow H_{\text{dR}}^2(X/\mathbb{Q}_p)$ . The arrow  $\varpi_{L/\mathbb{Q}_p}$  is defined as the composite ('exp' denotes the exponential map defined in [BK2] Definition 3.10)

$$\begin{aligned} \text{NS}(X_L) \otimes L &\xrightarrow{\sim} \text{NS}(X_L) \otimes H_f^1(L, \mathbb{Q}_p(1)) \\ &\xrightarrow[\text{id} \otimes \text{exp}]{} \text{NS}(X_L) \otimes H_f^1(L, \mathbb{Q}_p(1)) \xrightarrow{\text{Cor}_{L/\mathbb{Q}_p}} H_f^1(\mathbb{Q}_p, \text{NS}(\overline{X}) \otimes \mathbb{Q}_p(1)), \end{aligned}$$

which is surjective by a standard norm argument. In view of the right exactness of  $H_f^1(\mathbb{Q}_p, -)$  (cf. [Ne] III (1.7.3)), once we show the commutativity of the diagram (7.2.3), the assertion of the lemma will follow from the fact that the Hodge Chern class map  $\text{NS}(X_L) \otimes L \rightarrow H^1(X_L, \Omega_{X_L/L}^1)$  is surjective. Therefore it remains to show the commutativity of (7.2.3).

Let  $\mathfrak{o}$  be the integer ring of  $L$ , and put  $\mathcal{X}_{\mathfrak{o}} := \mathcal{X} \otimes_{\mathbb{Z}_p} \mathfrak{o}$ . There is a commutative diagram of pairings

$$\begin{array}{ccccc} H_{\text{dR}}^2(X_L/L) \times L & & \xrightarrow{\cup} & & H_{\text{dR}}^2(X_L/L) \\ \uparrow \iota & & \downarrow \delta^1 & & \downarrow \delta^3 \\ H_{\text{syn}}^2(\mathcal{X}_{\mathfrak{o}}, \mathcal{S}_{\mathbb{Q}_p}(1)) \times H_{\text{syn}}^1(\mathfrak{o}, \mathcal{S}_{\mathbb{Q}_p}(1)) & \xrightarrow{\cup} & & & H_{\text{syn}}^3(\mathcal{X}_{\mathfrak{o}}, \mathcal{S}_{\mathbb{Q}_p}(2)) \\ \downarrow c^2 & & \downarrow c^1 & & \downarrow c^3 \\ H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1))^{G_L} \times H^1(L, \mathbb{Q}_p(1)) & \xrightarrow{\cup} & & & H^1(L, H^2(\overline{X}, \mathbb{Q}_p(2))), \end{array}$$

where  $\iota$  (resp.  $\delta^1$ ) denotes the natural map (resp. connecting map) induced by the definition of  $\mathcal{S}_n(1)$  (cf. Definition 2.1.6, [Ne] III (3.1.1)). This diagram commutes by the definition of the product structure of syntomic cohomology and the compatibility of  $c^i$ 's with product structures (cf. Theorem 2.2.15). Moreover we have

$$c^1 \circ \delta^1 = \exp \quad \text{and} \quad c^3 \circ \delta^3 = \alpha$$

by the definitions of these maps, and we have the following commutative diagram of the 1st Chern class maps by [Ts1] Proposition 3.2.4 (3), Lemma 4.8.9:

$$\begin{array}{ccccc} & & \text{Pic}(\mathcal{X}_o) & & \\ & \swarrow \varrho_{\text{dR}} & \downarrow \varrho_{\text{syn}} & \searrow \varrho_{\text{ét}} & \\ H_{\text{dR}}^2(X_L/L) & \xleftarrow{\iota} & H_{\text{syn}}^2(\mathcal{X}_o, \mathcal{S}_{\mathbb{Q}_p}(1)) & \xrightarrow{c^2} & H_{\text{ét}}^2(\overline{X}, \mathbb{Q}_p(1))^{G_L}. \end{array}$$

One can easily deduce the commutativity of (7.2.3) from these facts.  $\square$

By this lemma, it is enough to show that there is an element  $\xi \in K_1(\mathcal{X})^{(2)}$  such that  $\varrho(\xi) \neq 0$  in  $H_f^1(\mathbb{Q}_p, V^{\text{ind}})$ . Let  $\mathcal{D}_1 := \pi^{-1}(1)$  and  $\mathcal{D}_2 := \pi^{-1}(-1)$  be the multiplicative fiber over  $\mathbb{Z}_p$  which are Néron 1-gon (**Fact 2**). Put  $\mathcal{D} := \mathcal{D}_1 + \mathcal{D}_2$  and  $\mathcal{U} := \mathcal{X} - \mathcal{D}$ . We consider rational functions

$$f_1 := \frac{Y - (X + 4)}{Y + (X + 4)} \Big|_{\mathcal{D}_1}, \quad f_2 := \frac{Y - (X + 4)}{Y + (X + 4)} \Big|_{\mathcal{D}_2}$$

on  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively. They define elements  $\xi'_i \in K'_1(\mathcal{D}_i)^{(1)}$  by Quillen's localization exact sequence

$$0 \longrightarrow K'_1(\mathcal{D}_i)^{(1)} \longrightarrow K'_1(\mathcal{D}_i^{\text{reg}})^{(1)} \xrightarrow{j_* \text{div}} \mathbb{Q}.$$

Here  $\mathcal{D}_i^{\text{reg}}$  denotes the smooth locus of  $\mathcal{D}_i$ , which is isomorphic to  $\mathbb{G}_{m, \mathbb{Z}_p}$ , and  $j : \tilde{\mathcal{D}}_i \rightarrow \mathcal{D}_i$  denotes the normalization. We denote by  $\xi_i \in K_1(\mathcal{X})^{(2)}$  the image of  $\xi'_i$  via the natural map  $K'_1(\mathcal{D}_i)^{(1)} \rightarrow K_1(\mathcal{X})^{(2)}$ . The goal is to prove

$$\varrho(\xi_1 - \xi_2) \neq 0 \text{ in } H_f^1(\mathbb{Q}_p, V^{\text{ind}}). \quad (7.2.4)$$

**Proposition 7.2.5** *If  $p$  satisfies **C(p)**, then we have*

$$[D_1] - [D_2] \notin \partial_{\text{ét}}(H^2(\overline{U}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}})$$

where  $\partial_{\text{ét}}$  is the boundary map in (5.3.1). Hence we have

$$\varrho(\xi_1 - \xi_2) \neq 0 \text{ in } H_f^1(\mathbb{Q}_p, V) \quad (7.2.6)$$

by the commutative diagram (6.2.3) where  $V := H^2(\overline{X}, \mathbb{Q}_p(2)) / \langle D_1 \rangle \otimes \mathbb{Q}_p(1)$ .

We note that one can further show  $H^2(\overline{U}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} = 0$  though we do not need it.

*Proof.* It follows from **C(p)-1** and **Fact 2** that the conditions **(A)'** and **(B)'** in Proposition 5.3.14 and hence **(A)** and **(B)** in Theorem 5.3.8 are satisfied. We apply (5.3.10). It is enough to show

$$[D_1] - [D_2] \notin \overline{\partial_{\text{dR}}}(\mathbf{E}^{(p^2)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p})$$

under **C(p)**. The space  $\Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathbb{Z}_p}^2(\log \mathcal{D}))$  is a free  $\mathbb{Z}_p$ -module of rank 3 generated by

$$\omega_0 := dt \frac{dX}{Y}, \quad \omega_1 := \frac{dt}{t-1} \frac{dX}{Y}, \quad \omega_2 := \frac{dt}{t+1} \frac{dX}{Y},$$

which satisfies

$$\partial_{\text{dR}}(\omega_0) = 0, \quad \partial_{\text{dR}}(\omega_1) = \mathcal{D}_1, \quad \partial_{\text{dR}}(\omega_2) = \mathcal{D}_2.$$

Let  $\mathcal{X}^*$  be the ‘tubular neighborhood’ of  $\mathcal{D}_1$ :

$$\begin{array}{ccc} \mathcal{X}^* & \longrightarrow & \mathcal{X} - \mathcal{D}_1 \\ \downarrow & \square & \downarrow \\ \text{Spec}(\mathbb{Z}_p((t-1))) & \longrightarrow & \mathbb{P}_{\mathbb{Z}_p}^1 - \{1\}. \end{array}$$

We fix an isomorphism between  $\mathcal{X}^*$  and the Tate elliptic curve

$$E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q) \quad \text{over } \mathbb{Z}_p((q))$$

given by

$$\begin{aligned} x + \frac{1}{12} &= -\frac{E_1^2 \cdot (X+3)}{12}, & y + \frac{1}{2}x &= \frac{E_1^3 \cdot Y}{24}, & t^4 &= \frac{E_{3,a}}{(E_1)^3} = \frac{E_{3,a}}{E_{3,a} + 27E_{3,b}}, \\ \left( \implies j(q) &= \frac{27(9-8t^4)^3}{(1-t^4)t^{12}}, \quad j(q^3) = \frac{27(1+8t^4)^3}{(1-t^4)^3 t^4} \right) \end{aligned}$$

where  $E_1, E_{3,a}$  and  $E_{3,b}$  are as in **C(p)**. Let  $\iota$  be the composite map

$$\iota : \Gamma(\mathcal{X}, \Omega_{\mathcal{X}}^2(\log \mathcal{D})) \rightarrow \Gamma(\mathcal{X}^*, \Omega_{\mathcal{X}^*}^2) \simeq \Gamma(E_q, \Omega_{E_q}^2) \rightarrow \mathbb{Z}_p((q)) \frac{du}{u} \frac{dq}{q}$$

where  $u$  denotes the Tate parameter of  $E_q$ . We note

$$\iota \left( \frac{dt}{t} \frac{dX}{Y} \right) = -\frac{27}{4} E_{3,b} \frac{du}{u} \frac{dq}{q}, \quad \iota \left( \frac{d(t^4-1)}{t^4-1} \frac{dX}{Y} \right) = E_{3,a} \frac{du}{u} \frac{dq}{q}$$

and

$$\iota(\omega_1) = f_1(q) \frac{du}{u} \frac{dq}{q}, \quad \iota(\omega_2) = f_2(q) \frac{du}{u} \frac{dq}{q}, \quad \iota(\omega_0) = g(q) \frac{du}{u} \frac{dq}{q},$$

where  $f_i(q)$  and  $g(q)$  are as in (7.0.1). Assume  $[D_1] - [D_2] \in \overline{\partial_{\text{dR}}}(\mathbf{E}^{(p^2)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p})$ , namely there is an  $n \in \mathbb{Z}_p$  such that  $\omega_1 - \omega_2 + n\omega_0 \in \mathbf{E}^{(p^2)}(\mathcal{X}, \mathcal{D})_{\mathbb{Z}_p}$ . This means that

$$a_i - b_i + nc_i \equiv 0 \pmod{i^2 \mathbb{Z}_p} \quad (7.2.7)$$

for all  $i \leq p^2$  with  $p|i$ , which contradicts to **C(p)-2**.  $\square$

It remains to check (6.2.5) for  $\xi_1 - \xi_2$ . If  $Z$  is a curve in (i) or (ii) (see Fact 3 in §7.1), one clearly has  $v_Z(\varrho(\xi_i)) = 1$ . If  $Z = E$  is the section of infinity, then one has

$$\frac{Y - (X + 4)}{Y + (X + 4)} \Big|_{D_i \cap E} = 1$$

and hence  $v_Z(\varrho(\xi_i)) = 1$ . If  $Z = C$  is the section in (iv), then one has

$$\frac{Y - (X + 4)}{Y + (X + 4)} \Big|_{D_i \cap C} = \frac{8/9 - (-4/3 + 4)}{8/9 + (-4/3 + 4)} = -\frac{1}{2}.$$

Hence  $v_Z(\varrho(\xi_1)) = v_Z(\varrho(\xi_2)) = -1/2$  and  $v_Z(\varrho(\xi_1 - \xi_2)) = 1$ . This completes the proof of (7.2.4) and hence **Step 2**.

### 7.3 Finiteness of torsion in $\text{CH}_0(X)$

We end this paper by showing that the torsion part of  $\text{CH}_0(X)$  is finite. Since we have proved the finiteness of the  $p$ -primary torsion part, it remains to show that the  $\ell$ -primary torsion part is finite for any  $\ell \neq p$  and zero for almost all  $\ell \neq p$ . In view of the isomorphism (7.3.3) below, Bloch's exact sequence (cf. [CTR1] (2.1))

$$0 \longrightarrow H_{\text{zar}}^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow N^1 H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) \longrightarrow \text{CH}_0(X)\{\ell\} \longrightarrow 0$$

and the isomorphism  $H_{\text{zar}}^1(X, \mathcal{K}_2) \otimes \mathbb{Q} \simeq K_1(X)^{(2)}$  (cf. [So]), it is enough to show that the regulator map

$$K_1(X)^{(2)} \otimes \mathbb{Q}_\ell \longrightarrow H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell(2))) \quad (7.3.1)$$

is surjective for any  $\ell \neq p$  and that  $H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$  is divisible for almost all  $\ell \neq p$ .

Assume  $\ell \neq p$  in what follows. Put  $\overline{Y} := Y \otimes_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ . There are isomorphisms

$$H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell(2))) \simeq H^0(\mathbb{F}_p, H^2(\overline{Y}, \mathbb{Q}_\ell(1))) \simeq \text{NS}(Y)_{\mathbb{Q}_\ell}$$

by the Tate conjecture for  $Y$  ([ArSw]) and a similar argument as for (7.3.4) below. The map (7.3.1) is identified with the boundary map

$$K_1(X)^{(2)} \otimes \mathbb{Q}_\ell \longrightarrow \text{NS}(Y)_{\mathbb{Q}_\ell}. \quad (7.3.2)$$

This map is surjective by **Step 1**, which shows that (7.3.1) is surjective.

Next we show that  $H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$  is divisible for almost all  $\ell$ . Since  $\overline{X}$  is a  $K3$  surface, we have  $H^1(\overline{X}, \mathbb{Z}_\ell) = H^3(\overline{X}, \mathbb{Z}_\ell) = 0$  and  $H^2(\overline{X}, \mathbb{Z}_\ell)$  is torsion-free for any  $\ell$ . Hence we have

$$H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^2(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \quad \text{and} \quad H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^3(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0,$$

and moreover

$$H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2)) = H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \quad (7.3.3)$$

by a Hochschild-Serre spectral sequence. We compute the right hand side as follows. There is a short exact sequence

$$\begin{aligned} 0 \rightarrow H^1(\mathbb{F}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) &\rightarrow H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \\ &\rightarrow H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}} \rightarrow 0. \end{aligned}$$

By the divisibility of  $H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$  and a standard argument on weights (cf. [CTSS] §2, [D]), the first term is zero. Thus we have

$$H^1(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))) \simeq H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}}, \quad (7.3.4)$$

and we are reduced to showing that the right hand side is divisible for almost all  $\ell$ . Put

$$N_\ell := H^2(\overline{Y}, \mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}},$$

and note the following fact due to Deligne [D]:

(\*) *the characteristic polynomial of the geometric Frobenius  $\varphi$  acting on  $H^2(\overline{Y}, \mathbb{Q}_\ell(1))$  is independent of  $\ell (\neq p)$ .*

Since  $H^2(\overline{Y}, \mathbb{Z}_\ell(1))$  is torsion-free, it is easy to see that  $H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell$  is torsion-free as well, for any  $\ell$ . Hence there is a short exact sequence

$$0 \longrightarrow N_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \longrightarrow (H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0$$

for any  $\ell$ . By (\*), we have

$$((H^2(\overline{Y}, \mathbb{Z}_\ell(1))/N_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^{G_{\mathbb{F}_p}} = 0$$

for almost all  $\ell$ . For such  $\ell$ , we have

$$H^2(\overline{Y}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))^{G_{\mathbb{F}_p}} = N_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell,$$

which is divisible. This completes the proof of the finiteness of  $\text{CH}_0(X)_{\text{tors}}$ .

**Remark 7.3.5** *Here is a more systematic (but essentially the same) proof of the finiteness result in this subsection. By the surjectivity of (7.3.2) and a result of Spiess [Sp] Proposition 4.3 (see also [SS2] for a generalization), we have*

$$\text{CH}_0(X)_{\text{tors}} \simeq \text{CH}_0(Y)_{\text{tors}} \quad \text{up to the } p\text{-primary torsion part.}$$

*The right hand side is finite by Colliot-Thélène–Sansuc–Soulé [CTSS].*

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