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COMPLEXITY OF SOLVING TROPICAL LINEAR SYSTEMS

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Abstract. An algorithm is designed for solving a tropical linear system with complexity polynomial in the size of the system.

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Introduction

A *tropical* or *min-plus semiring* (see e.g. Butkovic 2010) has two operations: min, +. Examples are provided by integers $\mathbb{Z}$, nonnegative integers $\mathbb{Z}^{(\geq 0)}$, real numbers $\mathbb{R}$, or nonnegative real numbers $\mathbb{R}^{(\geq 0)}$. Also the *extended semirings* $\mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$ and respectively, $\mathbb{Z}_\infty^{(\geq 0)}$, $\mathbb{R}_\infty$, $\mathbb{R}_\infty^{(\geq 0)}$ are considered. Studying algorithms, we deal mainly with $\mathbb{Z}$ or $\mathbb{Z}_\infty$.

We say that a *tropical linear system*

$$(0.1) \qquad \min_{1\leq j\leq n}\{a_{i,j}+x_j\}, \quad 1\leq i\leq m$$

(or $(m \times n)$-matrix $A = (a_{i,j})$) has a *tropical solution* $x = (x_1 \ldots, x_n)$ if for every row $1 \leq i \leq m$ there are two columns $1 \leq k < l \leq n$ such that

$$a_{i,k}+x_k = a_{i,l}+x_l = \min_{1\leq j\leq n}\{a_{i,j}+x_j\}$$

(see e.g. Bogart *et al.* 2007, Theobald 2006). Our purpose is to design an algorithm to solve (0.1).

Birkhäuser

In Section 1, we assume that coefficients $a_{i,j} \in \mathbb{Z}$ (we call it the case of *finite coefficients*) and that $0 \leq a_{i,j} \leq M$ for all i, j. We describe an algorithm which yields a solution $x \in \mathbb{Z}^n$ of (0.1) or detects its insolvability with complexity polynomial in M, n, m. The algorithm runs by induction on m and starting with any solution of the first $m-1$ equations of (0.1), the algorithm modifies it in a solution of (0.1) or detects the insolvability of (0.1). One can view the algorithm as a tropical analog of the Gram-Schmidt process with respect to the *tropical norm* introduced in Section 1.

In Section 2, we study the case of the (*extended*) coefficients $a_{i,j} \in \mathbb{Z}_\infty$ and look for a solution $x \in \mathbb{Z}_\infty^n$ of (0.1) with not all its coordinates x_j, $1 \leq j \leq n$ equal ∞. We assume that $0 \leq a_{i,j} \leq M$ for all finite coefficients. We describe an algorithm which solves (0.1) also with complexity polynomial in M, n, m. Reordering the columns and rows of $(m \times n)$-matrix $A := (a_{i,j})$ the algorithm brings it to a block form $(A_{p,q})$, $1 \leq p, q \leq t$ such that each of the first $t-1$ diagonal blocks $A_{i,i}$, $1 \leq i \leq t-1$ has no (tropical) solution, and all upper-triangular blocks $A_{p,q}$, $1 \leq p < q \leq t$ have all entries equal ∞. It would be interesting to solve (0.1) with complexity polynomial in $\log M$, n, m.

In Section 3, we study tropical *non-homogeneous* linear systems

$$(0.2) \qquad \min_{1 \leq j \leq n} \{a_{i,j} + x_j,\, a_i\}, \quad 1 \leq i \leq m$$

and describe an algorithm for their solving relying on the algorithm from Section 2 with a similar complexity bound.

In Section 4 as a consequence of the algorithm from Section 2, we give a characterization of solvability of (0.1) in terms of the *tropical* and *Kapranov ranks* of matrix A (their definitions are reminded in Section 4) and generalize this characterization to the extended real coefficients from $\mathbb{R}_\infty$. For finite coefficients from $\mathbb{R}$, this follows from Develin *et al.* (2005), while for $\mathbb{R}_\infty$, the solvability in terms of the tropical rank was established in Izhakian & Rowen (2009).

In Section 5, we describe an algorithm which tests whether (0.1) has a unique (in the *tropical projective space*) solution also with complexity polynomial in M, n, m (answering a question posed to the author by Thorsten Theobald). On the other hand, in Grigoriev & Podolskii 2012 it is shown that the problem of calculating the dimension of the set of tropical solutions of (0.1) is NP-hard. We mention also that in Bogart *et al.* (2007) an example of a linear polynomial ideal is exhibited with an exponential lower bound on the size of its tropical bases.

In Butkovic & Hevery (1985), it was shown that one can test the tropical singularity of a square matrix in polynomial time. It is known that calculations of the tropical rank (Kim & Roush 2005) and of the Kapranov rank (Kim & Roush 2006) are both NP-hard in general. Moreover, it is proved in Kim & Roush (2006) that the problem of solving systems of polynomial equations over a given infinite field is reducible to the problem of testing whether the Kapranov rank of a matrix over this field equals 3. In Theobald (2006), it was established that solving tropical polynomial systems (already of degrees 2) is NP-complete.

We mention that even in classical algebra, two different notions of a rank of an $(m \times n)$-matrix A over a commutative integral domain K (Grigoriev 1981) are known. Define $Rk(A)$ to be the minimal r such that $A = X_1 \cdot Y_1 + \cdots + X_r \cdot Y_r$ for suitable $(m \times 1)$-matrices $X_1, \ldots, X_r$ and $(1 \times n)$-matrices $Y_1, \ldots, Y_r$ over K. Obviously, $Rk(A)$ is greater or equal to the usual rank $rk(A)$ and can be greater than the latter by a factor up to 2 over polynomial rings K.

1. Solving tropical linear systems with finite coefficients

In this section, we study the case of finite coefficients $a_{i,j} \in \mathbb{Z}$ of system (0.1) and assume that $0 \le a_{i,j} \le M$, $1 \le i \le m$, $1 \le j \le n$. Then w.l.o.g. one can look for a solution $x = (x_1, \ldots, x_n)$ with coefficients $x_j \ge 0$, $1 \le j \le n$ being also integers.

We introduce the notation of the *tropical norm* of a vector $||x|| = \sum_{1 \le j \le n} x_j - n \cdot \min_{1 \le j \le n}\{x_j\}$. Observe that for the coordinatewise operations min, + on vectors x, y, we have $|| \min\{x, y\}|| \le \max\{||x||, ||y||\}$ and $||x+y|| \le ||x|| + ||y||$. A vector x is equivalent in the *tropical projective space* (Develin *et al.* 2005) to a *normalized* vector $x - \min_{1 \le j \le n}\{x_j\} \cdot (1, \ldots, 1)$. For normalized vectors, an inequality $|| \min\{x, y\}|| \le \min\{||x||, ||y||\}$ holds.

Theorem 1.1. *There is an algorithm which for an input (0.1) either finds its solution or detects its insolvability with complexity* $O(M \cdot \log M \cdot n^2 \cdot m^2)$.

Lemma 1.2. *If (0.1) has a solution* $(x_1, \dots, x_n)$, *then (0.1) has a solution* $(x'_1, \dots, x'_n)$ *satisfying* $0 \le x'_j \le M$, $x'_j \le x_j$, $1 \le j \le n$.

Proof (of Lemma 1.2). One can suppose w.l.o.g. that $\min_{1\le j\le n}\{x_j\} = 0$. Therefore, for each row i, $\min_{1\le j\le n}\{a_{i,j}+x_j\} \le M$ holds. Hence, if column j_0 satisfies the property $a_{i,j_0} + x_{j_0} = \min_{1\le j\le n}\{a_{i,j} + x_j\}$ for a suitable row i (we call such a column j_0 *active*), then $x'_{j_0} := x_{j_0} \le M$. For any non-active j_0, one can put $x'_{j_0} := \min\{x_{j_0}, M\}$. □

Proof (of Theorem 1.1). We use induction on m. Assume by the inductive hypothesis that the algorithm has already produced a current solution x for the $((m-1) \times n)$-submatrix A' of the matrix A excluding the first row of A such that $0 \le x_j \le M$, $1 \le j \le n$. Reordering the columns, we suppose that $a_{1,1} + x_1 = \min_{1\le j\le n}\{a_{1,j} + x_j\}$. The algorithm modifies the vector x (keeping the property of being a solution of A') until the modified vector becomes a solution also for the first row or detects that A has no solutions. One can assume that $a_{1,1} + x_1 < a_{1,j} + x_j$, $j \ge 2$, otherwise the algorithm terminates the inductive step.

We construct by recursion a subset J of columns. At the beginning, $J = \{1\}$. For a current $J = \{1, \dots, k\}$ for each $1 \le i \le k$, we have

$$a_{i,i} + x_i = \min_{1\le j\le n}\{a_{i,j} + x_j\} < a_{i,j_1} + x_{j_1},\ j_1 > i. \tag{1.3}$$

Suppose that there exists a row $i = i_{k+1}$ for which there is a unique $j_0 \notin J$ such that $a_{i,j_0} + x_{j_0} = \min_{1\le j\le n}\{a_{i,j} + x_j\}$. Clearly, $i > k$ due to (1.3). Transpose column j_0 with $k+1$ and row i with $k+1$, respectively. Put current $J := \{1, \dots, k+1\}$. Then, (1.3) is fulfilled for the new J.

Now assume that the algorithm fails to augment J. Observe that J does not depend on the order of choosing rows $i = i_{k+1}$ in the above construction.

First, suppose that $J = \{1, \ldots, n\}$. In this case, the $(n \times n)$-submatrix of A induced by its first n rows is tropically non-singular, and consequently, (0.1) has no solution and the algorithm halts.

Now, let $k = |J| < n$. If $k = 1$, we add to x_1 the number $\min_{2 \le j \le n}\{a_{1,j} + x_j\} - (a_{1,1} + x_1) \ge 1$ and obtain a solution of (0.1). Thereupon, we apply Lemma 1.2 to the obtained solution; as a result, the algorithm terminates the inductive step.

Thus, from now on we assume that $k > 1$. We call row i *attracted* if for every j_0 such that $a_{i,j_0} + x_{j_0} = \min_{1 \le j \le n}\{a_{i,j} + x_j\}$, we have $j_0 \in J$. Obviously, the first row is attracted. Reordering the rows, one may suppose that exactly the first l rows are attracted. Note that for any row $i > l$, there are at least two different columns $j_1, j_2 \notin J$ such that $a_{i,j_1} + x_{j_1} = a_{i,j_2} + x_{j_2} = \min_{1 \le j \le n}\{a_{i,j} + x_j\}$.

For $1 \le i \le l$ denote

$$a_i := \min_{k < j \le n}\{a_{i,j} + x_j\} - \min_{1 \le j \le n}\{a_{i,j} + x_j\} \ge 1$$

and $a := \min_{1 \le i \le l}\{a_i\}$. The algorithm modifies the vector $(x_1, \ldots, x_n)$ in such a way that

$$y_j := x_j + a,\ 1 \le j \le k; \quad y_j := x_j,\ j > k.$$

Then, the vector $y := (y_1, \ldots, y_n)$ is still a solution of A' and $a_{1,1} + y_1 = \min_{1 \le j \le n}\{a_{1,j} + y_j\}$. Moreover, the tropical norm

$$||(a_{1,j} + y_1, \ldots, a_{1,n} + y_n)|| = ||(a_{1,j} + x_1, \ldots, a_{1,n} + x_n)|| - a \cdot (n - k)$$

has dropped.

Thereupon, the algorithm applies Lemma 1.2 to vector $(y_1, \ldots, y_n)$. Observe that this does not change the tropical norm since each of the first k columns is active (taking into account that $k > 1$), and hence, every $y_j \le M$ for $j > k$, and thereby, y_j does not change in the course of applying Lemma 1.2.

Thus, we have described a single iteration of the algorithm. The next iteration starts with the modified vector y replacing x. The complexity of the execution of the iteration can be bounded by $O(\log M \cdot m \cdot n)$. The total number of iterations does not exceed the tropical norm $||(a_{1,j} + x_1, \ldots, a_{1,n} + x_n)|| \le 2 \cdot M \cdot (n - 1)$. Since the described induction (considering each time one more row of matrix A) requires m steps, we obtain the complexity bound in Theorem 1.1. □

2. Solving tropical linear systems with coefficients extended by infinity

From now on we assume that entries of (0.1) are $a_{i,j} \in \mathbb{Z}_\infty$ and $0 \leq a_{i,j} \leq M$ when $a_{i,j} \in \mathbb{Z}$. We are looking for solutions $(x_1, \dots, x_n)$ over $\mathbb{Z}_\infty$ with not all the coordinates equal to ∞.

THEOREM 2.1. *There is an algorithm which for a tropical linear system (0.1) over $\mathbb{Z}_\infty$ either finds a solution or detects its insolvability with complexity $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$.*

LEMMA 2.2. *If $(x_1, \dots, x_n) \in (\mathbb{R}_\infty)^n$ is a solution of (0.1), then there exists a solution $(x'_1, \dots, x'_n)$ of (0.1) such that for any $1 \leq j, j_1 \leq n$ it holds:*

- $x'_j = \infty$ *iff* $x_j = \infty$;
- $0 \leq x'_j \leq \min\{x_j, (M+1) \cdot n\}$, *provided that* $x_j \neq \infty$;
- $x_j - x_{j_1} > M$ *iff* $x'_j - x'_{j_1} > M$.

PROOF (of Lemma 2.2). In the course of the proof, we will modify the vector $(x_1, \dots, x_n)$ keeping for it the same notation. One can assume w.l.o.g. that $0 = x_1 = \min_{1 \leq j \leq n}\{x_j\}$. Consider a graph whose vertices are the finite coordinates x_j, and a pair of coordinates x_p, x_q is connected by an edge if for some row i we have $a_{i,p} + x_p = a_{i,q} + x_q = \min_{1 \leq j \leq n}\{a_{i,j} + x_j\} \neq \infty$.

Consider the connected component of the graph which contains x_1. Let the component contain p vertices, and after their reordering, one can assume that it consists of $x_1, \dots, x_p$, hence $x_j \leq M \cdot (p-1)$ for $1 \leq j \leq p$. After reordering the coordinates, one can assume that $x_{p+1} = \min_{j>p}\{x_j\}$. If $x_{p+1} = \infty$ the Lemma is proved. Otherwise, if $x_{p+1} \geq M \cdot p + 1$, then replace x_j with $x_j - x_{p+1} + (M \cdot p + 1)$ for all $j > p$. Take a connected component of the graph which contains x_{p+1}. Let it consist of q vertices $x_{p+1}, \dots, x_{p+q}$. As above we conclude that $x_{p+j} \leq M \cdot (p + q - 1) + 1$, $1 \leq j \leq q$. Continuing in this way, we complete the proof of Lemma. □

PROOF. (of Theorem 2.1) We use induction on m and first formulate the inductive hypothesis. Suppose that the $((m-1) \times n)$-submatrix A' of A (after reordering the rows and columns) has a block structure

$$\begin{pmatrix} A_{1,1} & \infty & \cdots & \infty & \infty \\ A_{2,1} & A_{2,2} & \cdots & \infty & \infty \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{t-1,1} & A_{t-1,2} & \cdots & A_{t-1,t-1} & \overline{\infty} \\ \overline{A_{t,1}} & \overline{A_{t,2}} & \cdots & \overline{A_{t,t-1}} & \overline{A_{t,t}} \end{pmatrix}$$

where $A_{p,q}$ is of size $u_p \times v_q$ for $1 \leq p, q \leq t-1$, while the lowest blocks $\overline{A_{t,q}}$ are of sizes $\overline{u_t} \times v_q$ for $1 \leq q < t$, the rightmost blocks $\overline{A_{p,t}}$ are of sizes $u_p \times \overline{v_t}$ for $1 \leq p < t$, finally the diagonal block $\overline{A_{t,t}}$ is of size $\overline{u_t} \times \overline{v_t}$ where $\overline{u_t} = m-1-u_1-\cdots-u_{t-1}$, $\overline{v_t} = n-v_1-\cdots-v_{t-1}$.

Also a vector $(y_1, \ldots, y_n) \in \mathbb{Z}^n$ with $0 \leq y_j \leq (M+1) \cdot n$ for $1 \leq j \leq n$ is yielded. For each diagonal block $A_{p,p} = (a_{u+i,v+j})$, $1 \leq p < t$, $u := u_1 + \cdots + u_{p-1}$, $1 \leq i \leq u_p$, $v := v_1 + \cdots + v_{p-1}$, $1 \leq j \leq v_p$ (except for the lowest diagonal block $\overline{A_{t,t}}$), we have $a_{u+i,v+i} + y_{v+i} = \min_{1 \leq j \leq v_p}\{a_{u+i,v+j} + y_{v+j}\}$ for $1 \leq i \leq v_p$ and $a_{u+i,v+i} + y_{v+i} < a_{u+i,v+j}$ for $i < j \leq v_p$. Therefore, in particular $u_p \geq v_p$ for $p < t$. It is not excluded that $\overline{u_t} = 0$, while the case $\overline{v_t} = 0$ would mean that the algorithm under description terminates with the output that system (0.1) has no solutions (cf. Lemma 2.6 below).

Every entry of each upper-triangular block $A_{p,q}$ for $p < q$ (as well as of $\overline{A_{p,t}}$ for $p < t$) equals ∞. Moreover, the vector from $\mathbb{Z}^n_\infty$ whose coordinates in the first $t-1$ blocks equal ∞ and in the last t-th block coincide with y_j for $v_1 + \cdots + v_{t-1} < j \leq n$ is a tropical solution of the matrix A'.

For the sake of simplifying notations, define the matrix $B' = \overline{A_{t,t}}$ of size $(r-1 := \overline{u_t}) \times (s := \overline{v_t})$. One can assume that B' has no rows consisting fully of ∞ entries; otherwise, the corresponding row of matrix A' can join the previous $(t-1)$-st block.

We assume that the matrix A' is obtained from A by deleting its $(m-r+1)$-th row. By $B = (b_{i,j})$ for $1 \leq i \leq r$ and $1 \leq j \leq s$, denote the $(r \times s)$-submatrix of A located in its lower right corner. Deleting the first row from B, we obtain B'. Also one can suppose w.l.o.g. that $b_{1,1} + y_{n-s+1} = \min_{1 \leq j \leq s}\{b_{1,j} + y_{n-s+j}\}$.

The algorithm will modify the vector $(y_{n-s+1}, \ldots, y_n)$ (keeping for a current vector the same notation) while preserving the property that $(y_{n-s+1}, \ldots, y_n)$ is a (tropical) solution of B'. One can assume w.l.o.g. that $b_{1,1} + y_{n-s+1} < \min_{2 \leq j \leq s}\{b_{1,j} + y_{n-s+j}\}$, since otherwise the vector from $\mathbb{Z}^n_\infty$ with all coordinates in the first

$t-1$ blocks equal ∞ and coinciding with vector $(y_{n-s+1},\dots,y_n)$ in the last t-th block provides a solution of A which would terminate the inductive step. In this case, in the block structure $u_1,\dots,u_{t-1},v_1,\dots,v_{t-1},\overline{v_t}$ do not change, while $\overline{u_t}$ increases by one. Applying Lemma 2.2, one can assume w.l.o.g. that $0 \le y_j \le (M+1)\cdot n$, $n-s+1 \le j \le n$.

As above in the proof of Theorem 1.1, the algorithm constructs recursively a set $J \subset \{n-s+1,\dots,n\}$ of columns of matrix B, while modifying the vector $(y_{n-s+1},\dots,y_n)$, and we describe a single iteration of this modification. As in the proof of Theorem 1.1, let $(n-s+1) \in J$. Again as above, we introduce the set of attracted rows. For every attracted row, i denotes

$$b_i := \min_{j\notin J}\{b_{i,j-n+s}+y_j\} - \min_{j\in J}\{b_{i,j-n+s}+y_j\} \ge 1.$$

Then, $b := \min\{b_i\} \ge 1$ where min is taken over all the attracted rows. Thus, the algorithm modifies vector $(y_{n-s+1},\dots,y_n)$ by adding b to every y_j for $j \in J$. Thereupon, the algorithm applies Lemma 2.2 to the vector $(y_{n-s+1},\dots,y_n)$ which satisfies B'. Hence, one can assume w.l.o.g. that $0 \le y_j \le (M+1)\cdot s$ for $n-s+1 \le j \le n$.

The algorithm introduces the following *directed* graph G with s vertices $\{n-s+1,\dots,n\}$. There is an edge in G from j_1 to j_2 if $y_{j_1} - y_{j_2} \le M$. Observe that an application of Lemma 2.2 to the vector $(y_{n-s+1},\dots,y_n)$ does not change the graph G. Denote by $S \subset \{n-s+1,\dots,n\}$ the set of all the vertices which can be reached in G starting with vertex $n-s+1$. In the course of executing the algorithm, while modifying J, G, S, we keep for them the same notations. □

LEMMA 2.3. *After any iteration of the algorithm, the set J remains a subset of S. The set S after an iteration becomes a subset of S before the iteration.*

PROOF. At the current iteration, the inclusion $J \subset S$ holds by virtue of construction of J, since for any row i and any pair of columns $j \in S$, $l \notin S$ we have $b_{i,l-n+s}+y_l \ne b_{i,j-n+s}+y_j$, unless $b_{i,l-n+s} = b_{i,j-n+s} = \infty$. Therefore, after the modification of the vector $(y_{n-s+1},\dots,y_n)$, its coordinates y_j for $j \in J$ increase, while the other coordinates do not change. Consequently, the modified S is a subset of the previous S. □

LEMMA 2.4. *For any attracted row i and any $l \notin S$ we have $b_{i,l-n+s} = \infty$.*

PROOF. If $b_{i,l-n+s} < \infty$, then $b_{i,l-n+s} + y_l < b_{i,j-n+s} + y_j$ for any $j \in S$ which contradicts to that row i is attracted and that $J \subset S$ (due to Lemma 2.3). □

Now assume that $J = S$. Denote by $v_t := \#J$ and by u_t the number of attracted rows of B. Reorder the rows and the columns of B (and respectively, of A) in such a way that the set of the first v_t columns of B coincides with J, and the set of the first u_t rows of B coincides with the set of attracted rows of B. Moreover, one can suppose that for any $1 \leq i \leq v_t$ we have

$$(2.5)\quad b_{i,i} + y_{n-s+i} = \min_{1 \leq j \leq s}\{b_{i,j} + y_{n-s+j}\} < \min_{i < l \leq s}\{b_{i,l} + y_{n-s+l}\}.$$

Then, the algorithm constructs a modified block decomposition of A being a refinement of the block decomposition from the inductive hypothesis: the last $\overline{u_t}$ rows (respectively, the last $\overline{v_t}$ columns) of A are partitioned into the first u_t rows and the remaining $\overline{u_{t+1}} := \overline{u_t} - u_t$ rows (respectively, into the first v_t columns and the remaining $\overline{v_{t+1}} := \overline{v_t} - v_t$ columns). Thus, as blocks of A we obtain the new ones $A_{t,q}$ for $q \leq t$; $A_{p,t}$ for $p \leq t$; $\overline{A_{t+1,q}}$ for $q \leq t+1$; $\overline{A_{p,t+1}}$ for $p \leq t+1$. The diagonal block $A_{t,t}$ satisfies the inductive hypothesis by its construction, see (2.5), and each entry of $\overline{A_{t,t+1}}$ equals ∞ due to Lemma 2.4. This completes the inductive step for m rows (i.e., for the matrix A).

The algorithm terminates when it is impossible to continue its work. This can happen when either all the rows of (0.1) or all its columns are exhausted. First, consider the case when all the rows of (0.1) are exhausted (i.e., A contains all the rows of (0.1)), but not all the columns are exhausted. Then, two possibilities can occur. Either a (modified) vector $(y_{n-s+1}, \ldots, y_n)$ is a solution of matrix B, then the algorithm terminates before completing a block decomposition of matrix A (at the inductive step) and outputs a solution of (0.1) (see above). Or the inductive step is completed with all the rows of B being attracted (since all the rows of (0.1) are exhausted) and with $J = S \neq \{n-s+1, \ldots, n\}$ (since not all the columns of (0.1) are exhausted). In the latter case, $\overline{u_t} = u_t$; in other words, the blocks $\overline{A_{t+1,q}}$ are void, and block $\overline{A_{t,t+1}}$ is not empty with each entry equal to ∞. Then, (0.1) has a solution whose coordinates at the first t blocks equal ∞ and at the $(t+1)$-st block equal, say, 0 (or some other arbitrary integer).

Secondly, consider the case when all the columns of (0.1) are exhausted, i.e., $J = \{n-s+1, \ldots, n\}$. Then, observe that $\overline{u_t} = u_t$, $\overline{v_t} = v_t$; thus, the blocks $\overline{A_{t+1,q}}$ and $\overline{A_{p,t+1}}$ for some $1 \leq p, q \leq t+1$ are void. Consider the $(n \times n)$-submatrix $\tilde{C} = (\tilde{c}_{i,j})$ of A consisting of its first v_p rows from each block of decomposition of A for $1 \leq p \leq t$. Denote by $C = (c_{i,j})$ the matrix such that $c_{i,j} := \tilde{c}_{i,j} + y_j$ for $1 \leq i, j \leq n$. Evidently, the tropical linear systems with matrices $\tilde{C}$ and C have solutions simultaneously.

LEMMA 2.6. *Let the $(n \times n)$-matrix C be decomposed into blocks $C_{p,q}$ of sizes $v_p \times v_q$ for $1 \leq p, q \leq t$ with $n = v_1 + \cdots + v_t$. Moreover, for each diagonal block $C_{p,p} = (c_{\overline{v}+i,\overline{v}+j})$ with $1 \leq i, j \leq v_p$ and $1 \leq p \leq t$, where $\overline{v} = v_1 + \cdots + v_{p-1}$, we have*

$$c_{\overline{v}+i,\overline{v}+i} = \min_{1 \leq j \leq v_p} \{c_{\overline{v}+i,\overline{v}+j}\} < \min_{i < l \leq v_p} \{c_{\overline{v}+i,\overline{v}+l}\}$$

for every $1 \leq i \leq v_p$. In addition, any entry of an upper-triangular block $C_{p,q}$, $p < q$ equals ∞. Then a tropical linear system with matrix C has no solution over $\mathbb{Z}_\infty$.

PROOF. Suppose that the vector $(z_1, \ldots, z_n)$ is a tropical solution of matrix C. Let p be the first block $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ of $(z_1, \ldots, z_n)$ which contains a finite coordinate. Then $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ is a tropical solution of the matrix $C_{p,p}$. Take a unique $1 \leq j_0 \leq v_p$ such that

$$z_{\overline{v}+j_0} = \min_{1 \leq j \leq v_p} \{z_{\overline{v}+j}\} < \min_{1 \leq j \leq j_0} \{z_{\overline{v}+j}\}.$$

Then we conclude that $(z_{\overline{v}+1}, \ldots, z_{\overline{v}+v_p})$ is not a tropical solution of the j_0-th row of matrix $C_{p,p}$ because $c_{\overline{v}+j_0,\overline{v}+j_0} + z_{\overline{v}+j_0} < \min_{1 \leq j \leq v_p, j \neq j_0} \{c_{\overline{v}+j_0,\overline{v}+j} + z_{\overline{v}+j}\}$. This contradiction proves the Lemma. ☐

Lemma 2.6 implies the correctness of the described algorithm: it outputs a solution of (0.1) over $\mathbb{Z}_\infty$ iff (0.1) is solvable.

Now we estimate the complexity of the algorithm. We recall that in the course of an iteration modifying the vector $(y_{n-s+1}, \ldots, y_n)$, the modified set S becomes a subset of the previous set S (see Lemma 2.3). First we bound from above the number of iterations in which S does not change. Observe that the integer $N := (s-1) \cdot y_{n-s+1} - y_{n-s+2} - \cdots - y_n$ increases after every iteration because the algorithm adds an integer $b \geq 1$ to each y_j for $j \in J \subset S$ (due to Lemma 2.3), while $n-s+1 \in J$, in addition $J \neq S$ (otherwise, the algorithm completes the inductive step). At the beginning of the inductive step $N \geq -(s-1) \cdot (M+1) \cdot n$ (cf. Lemma 2.2). If N becomes larger than $M \cdot s^2$ then S should change (since not all the vertices of S become reachable in graph G). Therefore, after at most of $O(M \cdot s \cdot n)$ iterations set S changes. Again due to Lemma 2.3, S can be modified at most s times. Thus, the whole number of iterations in the inductive step is less than $O(M \cdot s^2 \cdot n) \leq O(M \cdot n^3)$.

The complexity of executing a single iteration is bounded by $\log(M \cdot n) \cdot m \cdot n$ (cf. Lemma 2.2). The number of inductive steps (augmenting the set of rows of (0.1) under consideration) does not exceed m. Summarizing, this provides the complexity bound $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$ of the algorithm and completes the proof of Theorem 2.1.

When the paper was already submitted, the author learned that a different algorithm for solving tropical linear systems was designed in Akian *et al.* (2010) with a similar complexity bound as in Theorem 2.1 (implying also Corollary 4.2 below). The approach from Akian *et al.* (2010) involves mean payoff games and provides in addition an algorithm for solving *min-linear systems* (Butkovic 2010)

$$(2.7) \quad \min_{1 \leq j \leq n_1} \{a_{i,j} + x_j\} = \min_{1 \leq l \leq n_2} \{b_{i,l} + y_l\}, \quad 1 \leq i \leq m.$$

For the first time, an algorithm for solving system (2.7) with a complexity bound polynomial in n_1, n_2, m, M was proposed in Butkovic & Zimmermann (2006). Bezem *et al.* (2008) produced an example of system (2.7) with sizes $n_1 = n_2 = 2$, $m = 3$ and $a_{1,1} = a_{1,2} = b_{1,2} = 1$, $a_{2,2} = b_{2,2} = M$ (the remaining entries vanish) for which the algorithm from Butkovic & Zimmermann

(2006) runs with the complexity lower bound polynomial in M. In a similar way, the algorithm from Akian *et al.* (2010) runs with the complexity lower bound polynomial in M for 2×3 matrices with $a_{1,1} = a_{1,2} = a_{2,1} = 0, a_{2,2} = 1, a_{1,3} = a_{2,3} = M$ Grigoriev & Podolskii (2012).

Observe that an example of this sort (with matrices of a constant size) for the algorithm from Theorem 2.1 (for a different problem of solving a tropical linear system (0.1)) would be impossible, because the algorithm from Theorem 2.1 runs actually within the complexity polynomial in $\exp(n \cdot m)$, $\log M$. Indeed, for each t and row $1 \leq i \leq r$ of matrix B consider the set of columns $1 \leq j \leq s$ such that $b_{i,j} + y_{n-s+j} = \min_{1 \leq l \leq s}\{b_{i,l} + y_{n-s+l}\}$ (cf. (2.5)). One can verify that the sets of all such pairs i, j are distinct at different steps of the algorithm. Recently, in Davydow (2012) a slightly better upper bound polynomial in $\binom{n+m}{n} \cdot \log M$ on the complexity of the algorithm from Theorem 1.1 was established. On the other hand, a family of tropical linear systems was exhibited in Davydow (2012) for which the algorithm from Theorem 1.1 requires exponential time.

3. Solving tropical non-homogeneous linear systems

Treating (0.1) as a tropical homogeneous linear system, one can consider its non-homogeneous counterpart (0.2). Denote by $\hat{A}$ the matrix of size $m \times (n+1)$ obtained from $A = (a_{i,j})$ by joining as the last $(n+1)$-th column $(a_1, \ldots, a_m)^T$. Then (0.2) has a tropical solution over $\mathbb{Z}_\infty$ iff the homogeneous linear system with the matrix $\hat{A}$ has a tropical solution $(x_1, \ldots, x_n, x_{n+1})$ such that $x_{n+1} \neq \infty$. We describe an algorithm which can test the existence of such a solution.

The algorithm from Theorem 2.1 brings the matrix $\hat{A}$ (after handling all its m rows) to the block form $(A_{p,q})$ with block sizes $u_1, \ldots, u_t$; $v_1, \ldots, v_t$ (possibly $u_t = 0$). We assume that the homogeneous system with the matrix $\hat{A}$ has a tropical solution (which is detected by the algorithm from Theorem 2.1), otherwise (0.2) has no tropical solution.

The proof of Lemma 2.6 entails that any solution of the homogeneous system with the matrix $\hat{A}$ has coordinates equal ∞ in the first $t-1$ blocks of sizes $v_1, \dots, v_{t-1}$ (we recall that the algorithm from Theorem 2.1 reorders the columns and rows of $\hat{A}$). On the other hand, there is a solution with all finite coordinates in the last t-th block of size v_t. Thus, the criterion of solvability of (0.2) is that the last $(n+1)$-th column of $\hat{A}$ belongs to the last t-th block.

Assume that the entries $a_{i,j}$, a_i satisfy the same bounds as $a_{i,j}$ from (0.1). Making use of Theorem 2.1, we get

Corollary 3.1. *There is an algorithm which for an input (0.2) either finds its solution over $\mathbb{Z}_\infty$ or detects its insolvability within complexity $O(M \cdot \log(M \cdot n) \cdot n^4 \cdot m^2)$.*

4. Solvability of tropical linear systems via tropical and Kapranov ranks

As a direct consequence of Theorem 2.1, we get a criterion of solvability of a tropical linear system (0.1) over $\mathbb{Z}_\infty$ in terms of its tropical and Kapranov ranks (Develin *et al.* 2005).

Similar to matrices over $\mathbb{Z}$, we call $(n \times n)$-matrix $A = (a_{i,j})$ *tropically non-singular* if there exists a unique assignment $\{a_{i,\pi(i)}\}_{1 \le i \le n}$ for a permutation $\pi \in Sym(n)$ with a minimal sum $\sum_{1 \le i \le n} a_{i,\pi(i)}$ (in this case, the latter sum is obviously finite). Then, as usually, the *tropical rank* of an $(m \times n)$-matrix is defined as the maximal size of tropically non-singular submatrices.

For an $(m \times n)$-matrix $A = (a_{i,j})$, its *lifting* is defined as an $(m \times n)$-matrix $F = (f_{i,j})$ over the field of Puiseux series $K = \mathbb{C}((t^{1/\infty}))$ such that $\mathrm{ord}(f_{i,j}) = a_{i,j}$ or $f_{i,j} = 0$ when $a_{i,j} = \infty$. Then, the Kapranov rank of A is said to be less or equal to r if there exists a lifting F of A with rank (over K) at most r.

Corollary 4.1. *The following three statements are equivalent:*

i) A tropical linear system (0.1) with $(m \times n)$-matrix A has a solution over $\mathbb{Z}_\infty$;

ii) The tropical rank of A is less than n;

iii) The Kapranov rank of A is less than n.

Proof. The implication iii) $\Rightarrow$ ii) is evident (cf. e.g., Develin *et al.* 2005). In Develin *et al.* (2005), the equivalence of ii) and iii) for matrices over $\mathbb{R}$ (so, with finite coefficients) is also shown. Also the equivalence of i) and ii) was established in Izhakian & Rowen (2009).

The implication ii) $\Rightarrow$ i) follows from Theorem 2.1. Indeed, if (0.1) has no solutions, the algorithm designed in the proof of Theorem 2.1 terminates by exhausting the columns of (0.1). Hence there is an $(n \times n)$-submatrix $\tilde{C} = (\tilde{c}_{i,j})$ of A such that the $(n \times n)$-matrix $C = (c_{i,j})$ for which $c_{i,j} = \tilde{c}_{i,j} + y_j$ for an appropriate vector $(y_1, \dots, y_n) \in \mathbb{Z}^n$ fulfills the properties of Lemma 2.6. Clearly, the matrix C has a unique minimal assignment located on its diagonal and thereby is tropically non-singular, the same holds for $\tilde{C}$ as well.

To establish the remaining implication i) $\Rightarrow$ iii), consider a solution $(x_1, \dots, x_n) \in (\mathbb{Z}_\infty)^n$ of A. We take a vector $z := (z_1, \dots, z_n) \in K^n$ such that $z_j = t^{x_j}$ or $z_j = 0$ when $x_j = \infty$. Our purpose is to produce an $(m \times n)$-matrix $F = (f_{i,j})$ over K such that $F \cdot z = 0$ and $\mathrm{ord}(f_{i,j}) = a_{i,j}$ or $f_{i,j} = 0$ when $a_{i,j} = \infty$ (i.e., F will be a lifting of A).

Fix a row i for the time being. If $\min_{1 \le j \le n}\{a_{i,j} + x_j\} = \infty$, we have $f_{i,j} \cdot z_j = 0$ for $1 \le j \le n$. Now let $a_{i,l} + x_l = \min_{1 \le j \le n}\{a_{i,j} + x_j\} < \infty$ for all l in a certain subset $L \subset \{1, \dots, n\}$ with at least two elements. We look at $f_{i,j} = \sum_{k \ge a_{i,j}} g_{j,k} \cdot t^k$ as polynomials with indeterminate coefficients $g_{j,k} \in \mathbb{Z}$. Fix in an arbitrary way all $f_{i,j} := t^{a_{i,j}}$ (when $a_{i,j} < \infty$) for all j except a single $l_0 \in L$. Expanding the equality $\sum_{1 \le j \le n} f_{i,j} \cdot z_j = 0$ in the powers of t, we obtain in a unique way a polynomial $f_{i,l_0} = -(\#L - 1) \cdot t^{a_{i,l_0}} + \cdots \in \mathbb{Z}[t]$ with $\mathrm{ord}(f_{i,l_0}) = a_{i,l_0}$. Since the rank of F (being a lifting of A) is less than n, we establish iii). $\square$

Clearly, one can detect solvability of (0.1) by verifying the tropical singularity of all $(n \times n)$-submatrices of A (see Corollary 4.1), thus within the complexity polynomial in $\log M$, $\binom{m}{n}$, cf. Butkovic & Hevery (1985).

Corollary 4.2. *The problem of solvability of a tropical linear system belongs to the complexity class $NP \cap coNP$.*

Remark 4.3. *For (extended) rational coefficients* $a_{i,j} \in \mathbb{Q}_\infty$*, Theorem 2.1 and Corollary 4.1 hold literally.*

Remark 4.4. *For (extended) real coefficients* $a_{i,j} \in \mathbb{R}_\infty$*, statements i) and ii) of Corollary 4.1 are equivalent. Indeed, for the implication ii)* $\Rightarrow$ *i) one can in the proof of Theorem 2.1 replace the induction with a transfinite induction, while modifying the vector* $(y_{n-s+1}, \ldots, y_n)$ *and proving existence of a solution of (0.1) (again the matrix* C *from Lemma 2.6 is tropically non-singular).*

To prove the inverse implication i) $\Rightarrow$ *ii), assume that* $(x_1, \ldots, x_n) \in (\mathbb{R}_\infty)^n$ *is a solution of a tropical square linear system (0.1), i.e.,* $m = n$*, and that* A *has a unique minimal assignment. Reordering the rows and the columns of* A*, one can suppose w.l.o.g. that* $x_j = \infty$ *iff* $j > k$ *and in addition that the unique minimal assignment is located on the diagonal of* A*. Then, the vector* $(x_1, \ldots, x_k) \in \mathbb{R}^k$ *is a solution of the* $(k \times k)$*-submatrix* $A_k = (a_{i,j})$ *witk* $1 \le i, j \le k$ *of* A *in its upper left corner. Consider a directed graph* H *with* k *vertices* $x_1, \ldots, x_k$*. For a pair of vertices* x_i, x_j *with* $i \neq j$*, there is an edge* (x_i, x_j) *in* H *if* $a_{i,j} + x_j = \min_{1 \le l \le k}\{a_{i,l} + x_l\}$*. Since* $(x_1, \ldots, x_k)$ *is a solution of* A_k*, for any* $1 \le i \le k$ *there is* $1 \le j \le k$ *such that* H *contains the edge* (x_i, x_j)*. Therefore, there exists a simple cycle* $x_{i_1}, x_{i_2}, \ldots, x_{i_s}$ *in* H*. Then, the assignment of* A *obtained from the diagonal one by means of replacing* $a_{i_1,i_1}, a_{i_2,i_2}, \ldots, a_{i_s,i_s}$ *with* $a_{i_1,i_2}, a_{i_2,i_3}, \ldots, a_{i_s,i_1}$*, has the same sum as the diagonal one. This contradiction to the tropical singularity of* A *proves ii).*

5. Testing uniqueness of a solution of a tropical linear system

Let $y = (y_1, \ldots, y_n) \in \mathbb{Z}_\infty^n$ be a solution of (0.1) (being obtained, say, by the algorithm designed in Theorem 2.1). One can suppose w.l.o.g. that $0 \le y_j \le (M + 1) \cdot n + 1$ when y_j being finite for $1 \le j \le n$, cf. Lemma 2.2. Our purpose is to test whether y is a unique (in the tropical projective space Develin *et al.* 2005) solution of (0.1). We refer to two vectors as different if they are different in the tropical projective space. The set $S_\infty(y) \subset \{1, \ldots, n\}$ of all $1 \le l \le n$ such that $y_l = \infty$ we call the *infinity support* of y.

Lemma 5.1. *Assume that there exists a solution* $z = (z_1, \dots, z_n) \in \mathbb{Z}_\infty^n$ *of (0.1) different from* y*. Then, there exists a solution* $w = (w_1, \dots, w_n) \in \mathbb{Z}_\infty^n$ *of (0.1) and a pair of indices* $1 \le j \ne l \le n$ *with* $\{j, l\} \not\subset S_\infty(y)$ *and* $j, l \notin S_\infty(w)$ *such that*

i) if $j, l \notin S_\infty(y)$*, then* $w_l \le y_l$*,* $w_j \le y_j$*,* $y_l - w_l + y_j - w_j = 1$*;*

ii) if $j \in S_\infty(y)$*,* $l \notin S_\infty(y)$ *then* $w_j - w_l = (M+1) \cdot n$*.*

Moreover, $S_\infty(w) = S_\infty(y) \cap S_\infty(z)$*.*

Proof. One can suppose w.l.o.g. that $0 \le z_j \le (M+1) \cdot n$ for $1 \le j \le n$, cf. Lemma 2.2, and still y and z are different. If among the three sets $S_\infty(y) \setminus S_\infty(z)$, $S_\infty(z) \setminus S_\infty(y)$, $\{1, \dots, n\} \setminus (S_\infty(y) \cup S_\infty(z))$ at least two are nonempty, pick j from one of them and l from another one. Otherwise, $S_\infty(y) \setminus S_\infty(z) = S_\infty(z) \setminus S_\infty(y) = \emptyset$, in this case as j, l pick any two elements from $\{1, \dots, n\} \setminus (S_\infty(y) \cup S_\infty(z))$ with the property that $y_l - z_l \ne y_j - z_j$ (such j, l exist since y, z are different).

First, consider the case when $j, l \notin S_\infty(y)$. The vector $z' := z + (\max\{y_l - z_l, y_j - z_j\} - 1) \cdot (1, \dots, 1)$ is a solution of (0.1) (note that $\max\{y_l - z_l, y_j - z_j\} \in \mathbb{Z}$ because not both z_j, z_l equal ∞ by virtue of the choice of j, l). Put $w := (w_1, \dots, w_n) := \min\{y, z'\}$. Let for definiteness $y_l - z_l > y_j - z_j$. Then, $w_l = y_l - 1$ and $w_j = y_j$, which proves the Lemma in the first case.

Secondly, assume that $j \in S_\infty(y)$, $l \notin S_\infty(y)$. The vector $y' := y + (z_j - (M+1) \cdot n - y_l) \cdot (1, \dots, 1)$ is a solution of (0.1), put $w := (w_1, \dots, w_n) := \min\{y', z\}$. Then, $w_j = z_j$ and $w_l = y'_l = z_j - (M+1) \cdot n \le z_l$ due to the assumption above. □

Now we describe an algorithm which tests whether y is a unique solution of (0.1). For each pair $\{j, l\} \not\subset S_\infty(y)$, the algorithm chooses integers w_j, w_l with $\min\{w_j, w_l\} = 0$ which satisfy either i) or ii) from Lemma 5.1. The algorithm extends (0.1) by a tropical linear equation $\min\{w_l + x_j, w_j + x_l\}$ and applies Theorem 2.1 to the extended system. Lemma 5.1 implies that if y is not a unique solution of (0.1) then the extended system for at least one of the pairs $\{j, l\} \not\subset S_\infty(y)$ will have a solution. Conversely, if the extended system has a solution $x = (x_1, \dots, x_n)$, then either $x_j, x_l \in \mathbb{Z}$, $x_j - x_l = w_j - w_l$ or $x_j = x_l = \infty$. In any of two latter cases, vector x differs from y.

Summarizing and employing Theorem 2.1, we conclude with the following

Corollary 5.2. *There is an algorithm which for a given system (0.1) tests whether it has a unique solution, with complexity* $O(M \cdot \log(M \cdot n) \cdot n^7 \cdot m^2)$.

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References

M. Akian, S. Gaubert & A. Guterman (2010). The correspondence between tropical convexity and mean payoff games. *Proc. 19 Intern. Symp. Math. Theory of Networks and Systems, Budapest*, 1295–1302.

M. Akian, S. Gaubert & A. Guterman (2012). Tropical polyhedra are equivalent to mean payoff games. *Internat. J. Algebra Comput.* **22**(1), 1250001, 43 pp.

M. Bezem, R. Nieuwenhuis & E. Rodriguez Carbonell (2008). Exponential behaviour of the Butkovic-Zimmermann algorithm for solving two-sided linear systems in max-algebra. *Discrete Appl. Math.* **156**, 3506–3509.

T. Bogart, A. N. Jensen, D. Speyer, B. Sturmfels & R. R. Thomas (2007). Computing tropical varieties. *J. Symb. Comput.* **42**, 54–73.

P. Butkovic (2010). *Max-linear systems: theory and algorithms.* Springer.

P. Butkovic & F. Hevery (1985). A condition for the strong regularity of matrices in the minimax algebra. *Discr. Appl. Math.* **11**, 209–222.

P. Butkovic & K. Zimmermann (2006). A strongly polynomial algorithm for solving two-sided linear systems in max-algebra. *Discrete Appl. Math.* **154**, 437–446.

A.P. Davydow (2012). Upper and lower bounds for Grigoriev's algorithm for solving integral tropical linear systems, Combinatorics and graph theory. Part IV, RuFiDiM'11, *Zap. Nauchn. Sem.* POMI, POMI, St. Petersburg, **402**, 69–82.

M. Develin, F. Santos & B. Sturmfels (2005). On the rank of a tropical matrix. *In Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., Cambridge Univ. Press* **52**, 213–242.

D. Grigoriev (1981). Multiplicative complexity of a bilinear form over a commutative ring. *Lect. Notes Comput. Sci.* **118**, 281-286.

D. Grigoriev, V.V. Podolskii (2012). Complexity of tropical and min-plus linear prevarieties. *arXiv:math/1204.4578.*

Z. Izhakian & L. Rowen (2009). The tropical rank of a tropical matrix. *Communic. Algebra* **37**, 3912–3927.

K. H. Kim & F. W. Roush (2005). Factorization of polynomials in one variable over the tropical semiring. *arXiv:math/050116/v2.*

K. H. Kim & F. W. Roush (2006). Kapranov rank vs. tropical rank. *Proc. Amer. Math. Soc.* **134**, 2487–2494.

T. Theobald (2006) On the frontiers of polynomial computations in tropical geometry. *J. Symbolic Comput.* **41**, 1360–1375.

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