PERFECT FORMS AND THE COHOMOLOGY OF MODULAR GROUPS

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Abstract. For $N = 5$, 6 and 7, using the classification of perfect quadratic forms, we compute the homology of the Voronoï cell complexes attached to the modular groups $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$. From this we deduce the rational cohomology of those groups.

1. Introduction

Let $N \geqslant 1$ be an integer and let $SL_N(\mathbb{Z})$ be the modular group of integral matrices with determinant one. Our goal is to compute its cohomology groups with trivial coefficients, i.e. $H^q(SL_N(\mathbb{Z}), \mathbb{Z})$. The case $N = 2$ is well-known and follows from the fact that $SL_2(\mathbb{Z})$ is the amalgamated product of two finite cyclic groups ([19], [4], II.7, Ex.3, p.52). The case $N = 3$ was done in [21]: for any $q > 0$ the group $H^q(SL_3(\mathbb{Z}), \mathbb{Z})$ is killed by 12. The case $N = 4$ has been studied by Lee and Szczarba in [12]: modulo 2, 3 and 5–torsion, the cohomology group $H^q(SL_4(\mathbb{Z}), \mathbb{Z})$ is trivial whenever $q > 0$, except that $H^3(SL_4(\mathbb{Z}), \mathbb{Z}) = \mathbb{Z}$. In Theorem 7.3 below, we solve the cases $N = 5$, 6 and 7.

For these calculations we follow the method of [12], i.e. we use the perfect forms of Voronoï. Recall from [22] and [13] that a perfect form in N variables is a positive definite real quadratic form h on $\mathbb{R}^N$ which is uniquely determined (up to a scalar) by its set of integral minimal vectors. Voronoï proved in [22] that there are finitely many perfect forms of rank N, modulo the action of $SL_N(\mathbb{Z})$. These are known for $N \leqslant 8$ (see §2 below).

Voronoï used perfect forms to define a cell decomposition of the space X_N^* of positive real quadratic forms, the kernel of which is defined over $\mathbb{Q}$. This cell decomposition (cf. §3) is invariant under $SL_N(\mathbb{Z})$, hence it can be used to compute

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the equivariant homology of X_N^* modulo its boundary. On the other hand, this equivariant homology turns out to be isomorphic to the groups $H_q(SL_N(\mathbb{Z}), St_N)$, where St_N is the Steinberg module (see [5] and §3.4 below). Finally, Borel–Serre duality [5] asserts that the homology $H_*(SL_N(\mathbb{Z}), St_N)$ is dual to the cohomology $H^*(SL_N(\mathbb{Z}), \mathbb{Z})$ (modulo torsion).

To perform these computations for $N \leqslant 7$, we needed the help of a computer. The reason is that the Voronoï cell decomposition of X_N^* gets soon very complicated when N increases. For instance, when $N = 7$, there are more than two million orbits of cells of dimension 18, modulo the action of $SL_N(\mathbb{Z})$ (see Figure 2 below). For this purpose, we have developed a C library [16], which uses PARI [15] for some functionalities. The algorithms are based on exact methods. As a result we get the full Voronoï cell decomposition of the spaces X_N^* for $N \leqslant 7$ (with either $GL_N(\mathbb{Z})$ or $SL_N(\mathbb{Z})$ action). Those decompositions are summarized in the figures and tables below. The computations were done on several computers using different processor architectures (which is useful for checking the results) and for $N = 7$ the overall computational time was more than a year.

The paper is organized as follows. In §2, we recall the Voronoï theory of perfect forms. In §3, we introduce a complex of abelian groups that we call the "Voronoï complex" which computes the homology groups $H_q(SL_N(\mathbb{Z}), St_N)$. In §4, we explain how to get an explicit description of the Voronoï complex in rank $N = 5$, 6 or 7, starting from the description of perfect forms available in the literature (especially in the work of Jaquet [11]). In Figures 1 and 2 we display the rank of the groups in the Voronoï complex and in Tables 1–5 we give the elementary divisors of its differentials. The homology of the Voronoï complex (hence the groups $H_q(SL_N(\mathbb{Z}), St_N)$) follows from this. It is given in Theorem 4.3.

We found two methods to test whether our computations are correct. First, checking that the virtual Euler characteristic of $SL_N(\mathbb{Z})$ vanishes leads to a mass formula for the orders of the stabilizers of the cells of X_N^* (cf. §4.5). Second, the identity $d_{n-1} \circ d_n = 0$ for the differentials in the Voronoï complex is a non-trivial equality when these differentials are written as explicit (large) matrices.

In §5 we give an explicit formula for the top homology group of the Voronoï complex (Theorem 5.1). In §6 we prove that the Voronoï complex of $GL_N(\mathbb{Z})$ is a direct factor of the Voronoï complex of $GL_6(\mathbb{Z})$ shifted by one. Finally, in §7 we explain how to compute the cohomology of $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ (modulo torsion) from our results on the homology of the Voronoï complex in §4. Our main result is stated in Theorem 7.3.

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Notation: For any positive integer n we let $\mathcal{S}_n$ be the class of finite abelian groups the order of which has only prime factors less than or equal to n.

2. VORONOÏ'S REDUCTION THEORY

2.1. **Perfect forms.** Let $N \geqslant 2$ be an integer. We let C_N be the set of positive definite real quadratic forms in N variables. Given $h \in C_N$, let $m(h)$ be the finite set of minimal vectors of h, i.e. vectors $v \in \mathbb{Z}^N$, $v \neq 0$, such that $h(v)$ is minimal. A form h is called *perfect* when $m(h)$ determines h up to scalar: if $h' \in C_N$ is such that $m(h') = m(h)$, then h' is proportional to h.

***Example* 2.1.** The form $h(x,y) = x^2 + y^2$ has minimum 1 and precisely 4 minimal vectors $\pm(1,0)$ and $\pm(0,1)$. This form is not perfect, because there is an infinite number of positive definite quadratic forms having these minimal vectors, namely the forms $h(x,y) = x^2 + axy + y^2$ where a is a non-negative real number less than 1. By contrast, the form $h(x,y) = x^2 + xy + y^2$ has also minimum 1 and has exactly 6 minimal vectors, viz. the ones above and $\pm(1,-1)$. This form is perfect, the associated lattice is the "honeycomb lattice".

Denote by C_N^* the set of non-negative real quadratic forms on $\mathbb{R}^N$ the kernel of which is spanned by a proper linear subspace of $\mathbb{Q}^N$, by X_N^* the quotient of C_N^* by positive real homotheties, and by $\pi : C_N^* \to X_N^*$ the projection. Let $X_N = \pi(C_N)$ and $\partial X_N^* = X_N^* - X_N$. Let Γ be either $GL_N(\mathbb{Z})$ or $SL_N(\mathbb{Z})$. The group Γ acts on C_N^* and X_N^* on the right by the formula

$$h \cdot \gamma = \gamma^t \, h \, \gamma \,, \quad \gamma \in \Gamma \,, \ h \in C_N^* \,,$$

where h is viewed as a symmetric matrix and γ^t is the transpose of the matrix γ. Voronoï proved that there are only finitely many perfect forms modulo the action of Γ and multiplication by positive real numbers ([22], Thm. p.110).
The following table gives the current state of the art on the enumeration of perfect forms.

rank	1	2	3	4	5	6	7	8	9
#classes	1	1	1	2	3	7	33	10916	$\geqslant$ 500000

The classification of perfect forms of rank 8 was achieved by Dutour, Schürmann and Vallentin in 2005 [7], [18]. They have also shown that in rank 9 there are at least 500000 classes of perfect forms. The corresponding classification for rank 7 was completed by Jaquet in 1991 [11], for rank 6 by Barnes [2], and by Voronoï for the other dimensions. We refer to the book of Martinet [13] for more details on the results up to rank 7.

2.2. **A cell complex.** Given $v \in \mathbb{Z}^N - \{0\}$ we let $\hat{v} \in C_N^*$ be the form defined by

$$\hat{v}(x) = (v \mid x)^2 \,, \ x \in \mathbb{R}^N \,,$$

where $(v \mid x)$ is the scalar product of v and x. The *convex hull in* X_N^* of a finite subset $B \subset \mathbb{Z}^N - \{\mathbf{0}\}$ is the subset of X_N^* which is the image under π of the quadratic forms $\sum_j \lambda_j \, \hat{v}_j \in C_N^*$, where $v_j \in B$ and $\lambda_j \geqslant 0$. For any perfect form h, we let $\sigma(h) \subset X_N^*$ be the convex hull of the set $m(h)$ of its minimal vectors. Voronoï proved in [22], §§8-15, that the cells $\sigma(h)$ and their intersections, as h runs over all perfect forms, define a cell decomposition of X_N^*, which is invariant under the action of Γ. We endow X_N^* with the corresponding CW-topology. If τ is a closed cell in X_N^* and h a perfect form with $\tau \subset \sigma(h)$, we let $m(\tau)$ be the set of vectors v in

$m(h)$ such that $\hat{v}$ lies in τ. Any closed cell τ is the convex hull of $m(\tau)$, and for any two closed cells τ, τ' in X_N^* we have $m(\tau) \cap m(\tau') = m(\tau \cap \tau')$.

3. The Voronoï complex

3.1. **Definition.** Let $d(N) = N(N+1)/2 - 1$ be the dimension of X_N^* and $n \leqslant d(N)$ a natural integer. We denote by $\Sigma_n^\star = \Sigma_n^\star(\Gamma)$ a set of representatives, modulo the action of Γ, of those cells of dimension n in X_N^* which meet X_N, and by $\Sigma_n = \Sigma_n(\Gamma) \subset \Sigma_n^\star(\Gamma)$ the cells σ for which any element of the stabilizer Γ_σ of σ in Γ preserves the orientation. Let V_n be the free abelian group generated by Σ_n. We define as follows a map

$$d_n : V_n \to V_{n-1}\,.$$

For each closed cell σ in X_N^* we fix an orientation of σ, i.e. an orientation of the real vector space $\mathbb{R}(\sigma)$ of symmetric matrices spanned by the forms $\hat{v}$ with $v \in m(\sigma)$. Let $\sigma \in \Sigma_n$ and let τ' be a face of σ which is equivalent under Γ to an element in Σ_{n-1} (i.e. τ' neither lies on the boundary nor has elements in its stabilizer reversing the orientation). Given a positive basis B' of $\mathbb{R}(\tau')$ we get a basis B of $\mathbb{R}(\sigma) \supset \mathbb{R}(\tau')$ by appending to B' a vector $\hat{v}$, where $v \in m(\sigma) - m(\tau')$. We let $\varepsilon(\tau', \sigma) = \pm 1$ be the sign of the orientation of B in the oriented vector space $\mathbb{R}(\sigma)$ (this sign does not depend on the choice of v).

Next, let $\tau \in \Sigma_{n-1}$ be the (unique) cell equivalent to τ' and let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$. We define $\eta(\tau, \tau') = 1$ (resp. $\eta(\tau, \tau') = -1$) when γ is compatible (resp. incompatible) with the chosen orientations of $\mathbb{R}(\tau)$ and $\mathbb{R}(\tau')$.

Finally we define

$$d_n(\sigma) = \sum_{\tau \in \Sigma_{n-1}} \sum_{\tau'} \eta(\tau, \tau')\, \varepsilon(\tau', \sigma)\, \tau\,, \tag{1}$$

where τ' runs through the set of faces of σ which are equivalent to τ.

3.2. **A spectral sequence.** According to [4], VII.7, there is a spectral sequence E_{pq}^r converging to the equivariant homology groups $H_{p+q}^\Gamma(X_N^*, \partial X_N^*; \mathbb{Z})$ of the homology pair $(X_N^*, \partial X_N^*)$, and such that

$$E_{pq}^1 = \bigoplus_{\sigma \in \Sigma_p^\star} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma)\,,$$

where $\mathbb{Z}_\sigma$ is the orientation module of the cell σ and, as above, $\Sigma_p^\star$ is a set of representatives, modulo Γ, of the p-cells σ in X_N^* which meet X_N. Since σ meets X_N, its stabilizer Γ_σ is finite and, by Lemma 7.1 in §7 below, the order of Γ_σ is divisible only by primes $p \leqslant N+1$. Therefore, when q is positive, the group $H_q(\Gamma_\sigma, \mathbb{Z}_\sigma)$ lies in $\mathcal{S}_{N+1}$.

When Γ_σ happens to contain an element which changes the orientation of σ, the group $H_0(\Gamma_\sigma, \mathbb{Z}_\sigma)$ is killed by 2, otherwise $H_0(\Gamma_\sigma, \mathbb{Z}_\sigma) \cong \mathbb{Z}_\sigma)$. Therefore, modulo $\mathcal{S}_2$, we have

$$E_{n\,0}^1 = \bigoplus_{\sigma \in \Sigma_n} \mathbb{Z}_\sigma\,,$$

and the choice of an orientation for each cell σ gives an isomorphism between $E_{n\,0}^1$ and V_n.

3.3. **Comparison.** We claim that the differential

$$d^1_n : E^1_{n\,0} \to E^1_{n-1,0}$$

coincides, up to sign, with the map d_n defined in 3.1. According to [4], VII, Prop. (8.1), the differential d^1_n can be described as follows.

Let $\sigma \in \Sigma^\star_n$ and let τ' be a face of σ. Consider the group $\Gamma_{\sigma\tau'} = \Gamma_\sigma \cap \Gamma_{\tau'}$ and denote by

$$t_{\sigma\tau'} : H_*(\Gamma_\sigma, \mathbb{Z}_\sigma) \to H_*(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma)$$

the transfer map. Next, let

$$u_{\sigma\tau'} : H_*(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) \to H_*(\Gamma_{\tau'}, \mathbb{Z}_{\tau'})$$

be the map induced by the natural map $\mathbb{Z}_\sigma \to \mathbb{Z}_{\tau'}$, together with the inclusion $\Gamma_{\sigma\tau'} \subset \Gamma_{\tau'}$. Finally, let $\tau \in \Sigma^\star_{n-1}$ be the representative of the Γ-orbit of τ', let $\gamma \in \Gamma$ be such that $\tau' = \tau \cdot \gamma$, and let

$$v_{\tau'\tau} : H_*(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) \to H_*(\Gamma_\tau, \mathbb{Z}_\tau)$$

be the isomorphism induced by γ. Then the restriction of d^1_n to $H_*(\Gamma_\sigma, \mathbb{Z}_\sigma)$ is equal, up to sign, to the sum

$$\sum_{\tau'} v_{\tau'\tau}\, u_{\sigma\tau'}\, t_{\sigma\tau'} \,, \tag{2}$$

where τ' runs over a set of representatives of faces of σ modulo Γ_σ.

To compare d^1_n with d_n we first note that, when $\tau \in \Sigma_{n-1}$,

$$v_{\tau'\tau} : H_0(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) = \mathbb{Z} \to H_0(\Gamma_\tau, \mathbb{Z}_\tau) = \mathbb{Z}$$

is the multiplication by $\eta(\tau, \tau')$, as defined in §3.1. Next, when $\sigma \in \Sigma_n$, the map

$$u_{\sigma\tau'} : H_0(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) = \mathbb{Z}_\sigma = \mathbb{Z} \to H_0(\Gamma_{\tau'}, \mathbb{Z}_{\tau'}) = \mathbb{Z}$$

is the multiplication by $\varepsilon(\tau', \sigma)$, up to a sign depending on n only. Finally, the transfer map

$$t_{\sigma\tau'} : H_0(\Gamma_\sigma, \mathbb{Z}_\sigma) = \mathbb{Z} \to H_0(\Gamma_{\sigma\tau'}, \mathbb{Z}_\sigma) = \mathbb{Z}$$

is the multiplication by $[\Gamma_\sigma : \Gamma_{\sigma\tau'}]$. Multiplying the sum (2) by this number amounts to the same as taking the sum over all faces of σ as in (1). This proves that d_n coincides, up to sign, with d^1_n on $E^1_{n\,0} = V_n$. □

In particular, we get that $d_{n-1} \circ d_n = 0$. Note that this identity will give us a non-trivial test of our explicit computations of the complex.

Notation: The resulting complex $(V_\bullet, d_\bullet)$ will be denoted by Vor_Γ, and we call it the *Voronoï complex*.

3.4. **The Steinberg module.** Let T_N be the spherical Tits building of SL_N over $\mathbb{Q}$, i.e. the simplicial set defined by the ordered set of non-zero proper linear subspaces of $\mathbb{Q}^N$. The reduced homology $\tilde{H}_q(T_N, \mathbb{Z})$ of T_N with integral coefficients is zero except when $q = N - 2$, in which case

$$\tilde{H}_{N-2}(T_N, \mathbb{Z}) = \mathrm{St}_N$$

is by definition the Steinberg module [5]. According to [20], Prop. 1, the relative homology groups $H_q(X^*_N, \partial X^*_N; \mathbb{Z})$ are zero except when $q = N - 1$, and

$$H_{N-1}(X^*_N, \partial X^*_N; \mathbb{Z}) = \mathrm{St}_N \,.$$

From this it follows that, for all $m \in \mathbb{N}$,

$$H_m^{\Gamma}(X_N^*, \partial X_N^*; \mathbb{Z}) = H_{m-N+1}(\Gamma, \mathrm{St}_N)$$

(see e.g. [20], §3.1). Combining this equality with the previous sections we conclude that, modulo $\mathcal{S}_{N+1}$,

$$H_{m-N+1}(\Gamma, \mathrm{St}_N) = H_m(\mathrm{Vor}_\Gamma)\,. \tag{3}$$

4. THE VORONOÏ COMPLEX IN DIMENSIONS 5, 6 AND 7

In this section, we explain how to compute the Voronoï complexes of rank $N \leqslant 7$.

4.1. **Checking the equivalence of cells.** As a preliminary step, we develop an effective method to check whether two cells σ and σ' of the same dimension are equivalent under the action of Γ. The cell σ (resp. σ') is described by its set of minimal vectors $m(\sigma)$ (resp. $m(\sigma')$). We let b (resp. b') be the sum of the forms $\hat{v}$ with $v \in m(\sigma)$ (resp. $m(\sigma')$). If σ and σ' are equivalent under the action of Γ the same is true for b and b', and the converse holds true since two cells of the same dimension are equal when they have an interior point in common.

To compare b and b' we first check whether or not they have the same determinant. In case they do, we let M (resp. M') be the set of numbers $b(x)$ with $x \in m(\sigma)$ (resp. $b'(x)$ with $x \in m(\sigma')$). If b and b' are equivalent, then the sets M and M' must be equal.

Finally, if $M = M'$ we check if b and b' are equivalent by applying an algorithm of Plesken and Souvignier [17] (based on an implementation of Souvignier).

4.2. **Finding generators of the Voronoï complex.** In order to compute Σ_n (and $\Sigma_n^\star$), we proceed as follows. Fix $N \leqslant 7$. Let $\mathcal{P}$ be a set of representatives of the perfect forms of rank N. A choice of $\mathcal{P}$ is provided by Jaquet [11]. Furthermore, for each $h \in \mathcal{P}$, Jaquet gives the list $m(h)$ of its minimal vectors, and the list of all perfect forms $h'\gamma$ (one for each orbit under $\Gamma_{\sigma(h)}$), where $h' \in \mathcal{P}$ and $\gamma \in \Gamma$, such that $\sigma(h)$ and $\sigma(h')\gamma$ share a face of codimension one. This provides a complete list C_h^1 of representatives of codimension one faces in $\sigma(h)$.

From this, one deduces the full list $\mathcal{F}_h^1$ of faces of codimension one in $\sigma(h)$ as follows: first list all the elements in the automorphism group $\Gamma_{\sigma(h)}$; this can be obtained by using a second procedure implemented by Souvignier [17] which gives generators for $\Gamma_{\sigma(h)}$. We represent the latter generators as elements in the symmetric group $\mathfrak{S}_M$, where M is the cardinality of $m(h)$, acting on set $m(h)$ of minimal vectors. Using those generators, we let GAP [9] list all the elements of $\Gamma_{\sigma(h)}$, viewed as elements of the symmetric group above.

The next step is to create a shortlist $\mathcal{F}_h^2$ of codimension 2 facets of $\sigma(h)$ by intersecting all the translates under $\mathfrak{S}_M$ of codimension 1 facets with each member of C_h^1 and only keeping those intersections with the correct rank ($=d(N) - 2$). The resulting shortlist is reasonably small and we apply the procedure of 4.1 to reduce the shortlist to a set of representatives C_h^2 of codimension 2 facets.

We then proceed by induction on the codimension to define a list $\mathcal{F}_h^p$ of cells of codimension $p > 2$ in $\sigma(h)$. Given $\mathcal{F}_h^p$, we let $C_h^p \subset \mathcal{F}_h^p$ be a set of representatives for the action of Γ. We then let $\mathcal{F}_h^{p+1}$ be the set of cells $\varphi \cap \tau$, with $\varphi \in \mathcal{F}_h^2$, and $\tau \in C_h^p$.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Sigma_n^\star(GL_5(\mathbb{Z}))$	2	5	10	16	23	25	23	16	9	4	3						
$\Sigma_n\,(GL_5(\mathbb{Z}))$					1	7	6	1	0	2	3						
$\Sigma_n^\star(GL_6(\mathbb{Z}))$		3	10	28	71	162	329	589	874	1066	1039	775	425	181	57	18	7
$\Sigma_n\,(GL_6(\mathbb{Z}))$						3	46	163	340	544	636	469	200	49	5		
$\Sigma_n^\star(SL_6(\mathbb{Z}))$		3	10	28	71	163	347	691	1152	1532	1551	1134	585	222	62	18	7
$\Sigma_n\,(SL_6(\mathbb{Z}))$			3	10	18	43	169	460	815	1132	1270	970	434	114	27	14	7

FIGURE 1. Cardinality of Σ_n and $\Sigma_n^\star$ for $N = 5, 6$ (empty slots denote zero).

n	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$\Sigma_n^\star$	6	28	115	467	1882	7375	26885	87400	244029	569568	1089356	1683368	2075982	2017914	1523376	876385	374826	115411	24623	3518	352	33
Σ_n				1	60	1019	8899	47271	171375	460261	955128	1548650	1955309	1911130	1437547	822922	349443	105054	21074	2798	305	33

FIGURE 2. Cardinality of Σ_n and $\Sigma_n^\star$ for $GL_7(\mathbb{Z})$.

Next, we let $\Sigma_n^\star$ be a system of representatives modulo Γ in the union of the sets $C_h^{d(N)-n}, h \in \mathcal{P}$. We then compute generators of the stabilizer of each cell in $\Sigma_n^\star$ with the help of another algorithm developed by Plesken and Souvignier in [17], and we check whether all generators preserve the orientation of the cell. This gives us the set Σ_n as the set of those cells which pass that check.

Proposition 4.1. *The cardinality of* Σ_n *and* $\Sigma_n^\star$ *is displayed in Figure 1 for rank* $N = 5, 6$ *and in Figure 2 for rank* $N = 7$.

***Remark* 4.2.** The first line in Figure 1 has already been computed by Batut (cf. [1], p.409, second column of Table 2).

4.3. **The differential.** The next step is to compute the differentials of the Voronoï complex by using formula (1) above. In Table 3, we give information on the differentials in the Voronoï complex of rank 5. For instance the second line, denoted d_{11}, is about the differential from V_{11} to V_{10}. In the bases Σ_{11} and Σ_{10}, this differential is given by a matrix A with $\Omega = 513$ non-zero entries, with $m = 46 = \mathrm{card}(\Sigma_{10})$ rows and $n = 163 = \mathrm{card}(\Sigma_{11})$ columns. The rank of A is 42, and the rank of its kernel is 121. The elementary divisors of A are 1 (multiplicity 40) and 2 (multiplicity 2).

The cases of $SL_4(\mathbb{Z})$, $GL_6(\mathbb{Z})$ and $SL_6(\mathbb{Z})$ are treated in Table 1, Table 3 and Table 4, respectively.

A	Ω	n	m	rank	ker	elementary divisors
d_4	0	1	0	0	1	
d_5	1	1	1	1	0	1(1)
d_6	0	1	1	0	1	
d_7	0	0	1	0	0	
d_8	0	1	0	0	1	
d_9	2	2	1	1	1	2(1)

Table 1. Results on the rank and elementary divisors of the differentials for $SL_4(\mathbb{Z})$.

A	Ω	n	m	rank	ker	elementary divisors
d_8	0	1	0	0	1	
d_9	2	7	1	1	6	1(1)
d_{10}	18	6	7	5	1	1(4), 2(1)
d_{11}	5	1	6	1	0	1(1)
d_{12}	0	0	1	0	0	
d_{13}	0	2	0	0	2	
d_{14}	4	3	2	2	1	5(1), 15(1)

Table 2. Results on the rank and elementary divisors of the differentials for $GL_5(\mathbb{Z})$.

A	Ω	n	m	rank	ker	elementary divisors
d_{10}	17	46	3	3	43	1(3)
d_{11}	513	163	46	42	121	1(40), 2(2)
d_{12}	2053	340	163	120	220	1(120)
d_{13}	4349	544	340	220	324	1(217), 2(3)
d_{14}	6153	636	544	324	312	1(320), 2(1), 6(2), 12(1)
d_{15}	5378	469	636	312	157	1(307), 2(3), 60(2)
d_{16}	2526	200	469	156	44	1(156)
d_{17}	597	49	200	44	5	1(41), 3(1), 6(1), 36(1)
d_{18}	43	5	49	5	0	1(5)

Table 3. Results on the rank and elementary divisors of the differentials for $GL_6(\mathbb{Z})$.

Our results on the differentials in rank 7 are shown in Table 5. While the matrices are sparse, they are not sparse enough for efficient computation. They have a poor conditioning with some dense columns or rows (this is a consequence of the fact that the complex is not simplicial and non-simplicial cells can have a large number of non-trivial intersections with the faces). We have obtained full information on the rank of the differentials. For the computation of the elementary divisors complete results have been obtained in the case of matrices of d_n for $10 \leqslant n \leqslant 14$ and $24 \leqslant n \leqslant 27$ only. See [6] for a detailed description of the computation.

4.4. **The homology of the Voronoï complexes.** From the computation of the differentials, we can determine the homology of Voronoï complex. Recall that if we

A	Ω	n	m	rank	ker	elementary divisors
d_7	12	10	3	3	7	1(3)
d_8	48	18	10	7	11	1(7)
d_9	140	43	18	11	32	1(11)
d_{10}	613	169	43	32	137	1(32)
d_{11}	2952	460	169	136	324	1(129), 2(6), 6(1)
d_{12}	7614	815	460	323	492	1(318), 2(3), 4(2)
d_{13}	12395	1132	815	491	641	1(491)
d_{14}	14966	1270	1132	641	629	1(637), 3(3), 12(1)
d_{15}	12714	970	1270	629	341	1(621), 2(5), 6(1), 60(2)
d_{16}	6491	434	970	339	95	1(338), 2(1)
d_{17}	1832	114	434	95	19	1(92), 3(2), 18(1)
d_{18}	257	27	114	19	8	1(17), 2(2)
d_{19}	62	14	27	8	6	1(7), 10(1)
d_{20}	28	7	14	6	1	1(1), 3(4), 504(1)

TABLE 4. Results on the rank and elementary divisors of the differentials for $SL_6(\mathbb{Z})$.

A	Ω	n	m	rank	ker	elementary divisors
d_{10}	8	60	1	1	59	1
d_{11}	1513	1019	60	59	960	1 (59)
d_{12}	37519	8899	1019	960	7939	1 (958), 2 (2)
d_{13}	356232	47271	8899	7938	39333	1 (7937), 2 (1)
d_{14}	1831183	171375	47271	39332	132043	1 (39300), 2 (29), 4 (3)
d_{15}	6080381	460261	171375	132043	328218	?
d_{16}	14488881	955128	460261	328218	626910	?
d_{17}	25978098	1548650	955128	626910	921740	?
d_{18}	35590540	1955309	1548650	921740	1033569	?
d_{19}	37322725	1911130	1955309	1033568	877562	?
d_{20}	29893084	1437547	1911130	877562	559985	?
d_{21}	18174775	822922	1437547	559985	262937	?
d_{22}	8251000	349443	822922	262937	86506	?
d_{23}	2695430	105054	349443	86505	18549	?
d_{24}	593892	21074	105054	18549	2525	1 (18544), 2 (4), 4 (1)
d_{25}	81671	2798	21074	2525	273	1 (2507), 2 (18)
d_{26}	7412	305	2798	273	32	1 (258), 2 (7), 6 (7), 36 (1)
d_{27}	600	33	305	32	1	1 (23), 2 (4), 28 (3), 168 (1), 2016 (1)

TABLE 5. Results on the rank and elementary divisors of the differentials for $GL_7(\mathbb{Z})$.

have a complex of free abelian groups

$$\cdots \to \mathbb{Z}^{\alpha} \xrightarrow{f} \mathbb{Z}^{\beta} \xrightarrow{g} \mathbb{Z}^{\gamma} \to \cdots$$

with f and g represented by matrices, then the homology is

$$\ker(g)/\mathrm{Im}(f) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_\ell\mathbb{Z} \oplus \mathbb{Z}^{\beta-\mathrm{rank}(f)-\mathrm{rank}(g)},$$

where $d_1, \ldots, d_\ell$ are the elementary divisors of the matrix of f.

We deduce from Tables 1–5 the following result on the homology of the Voronoï complex.

Theorem 4.3. *The non-trivial homology of the Voronoï complexes associated to* $GL_N(\mathbb{Z})$ *with* $N = 5, 6$ *modulo* $\mathcal{S}_5$ *is given by:*

$$\begin{aligned} H_n(\mathrm{Vor}_{GL_5(\mathbb{Z})}) &\cong \mathbb{Z}, \quad \text{if } n = 9, 14\,, \\ H_n(\mathrm{Vor}_{GL_6(\mathbb{Z})}) &\cong \mathbb{Z}, \quad \text{if } n = 10, 11, 15\,, \end{aligned}$$

while in the case $SL_6(\mathbb{Z})$ *we get, modulo* $\mathcal{S}_7$*, that*

$$H_n(\mathrm{Vor}_{SL_6(\mathbb{Z})}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 10, 11, 12, 20\,, \\ \mathbb{Z}^2, & \text{if } n = 15\,. \end{cases}$$

Furthermore, for $N = 7$ *we get*

$$H_n(\mathrm{Vor}_{GL_7(\mathbb{Z})} \otimes \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } n = 12, 13, 18, 22, 27\,, \\ 0 & \text{otherwise}. \end{cases}$$

Notice that, if N is odd, $SL_N(\mathbb{Z})$ and $GL_N(\mathbb{Z})$ have the same homology modulo $\mathcal{S}_2$. Notice also that, for simplicity, in the statement of the theorem we did not use the full information given by the list of elementary divisors in Tables 1–5.

4.5. **Mass formulae for the Voronoï complex.** Let $\chi(SL_N(\mathbb{Z}))$ be the virtual Euler characteristic of the group $SL_N(\mathbb{Z})$. It can be computed in two ways. First, the mass formula in [4] gives

$$\chi(SL_N(\mathbb{Z})) = \sum_{\sigma \in E} (-1)^{\dim(\sigma)} \frac{1}{|\Gamma_\sigma|} = \sum_{n=N}^{d(N)} (-1)^n \sum_{\sigma \in \Sigma_n^\star} \frac{1}{|\Gamma_\sigma|}\,,$$

where E is a family of representatives of the cells of the Voronoï complex of rank N modulo the action of $SL_N(\mathbb{Z})$, and Γ_σ is the stabilizer of σ in $SL_N(\mathbb{Z})$. Second, by a result of Harder [10], we know that

$$\chi(SL_N(\mathbb{Z})) = \prod_{k=2}^{N} \zeta(1-k)\,,$$

hence $\chi(SL_N(\mathbb{Z})) = 0$ if $N \geqslant 3$.

A non-trivial check of our computations is to test the compatibility of these two formulas, and the corresponding check for rank $N = 5$ had been performed by Batut (cf. [1], where a proof of an analogous statement, for any N, but instead pertaining to *well-rounded* forms, which in our case are precisely the ones in $\Sigma_\bullet^\star$, is attributed to Bavard [3]).

If we add together the terms $\frac{1}{|\Gamma_\sigma|}$ for cells σ of the same dimension to a single term, then we get for $N = 6$, starting with the top dimension,

$$\begin{gathered} \frac{45047}{1451520} - \frac{10633}{11520} + \frac{6425}{576} - \frac{12541}{192} \\ + \frac{7438673}{34560} - \frac{3841271}{8640} + \frac{9238}{15} - \frac{266865}{448} + \frac{14205227}{34560} - \frac{14081573}{69120} \\ + \frac{830183}{11520} - \frac{205189}{11520} + \frac{61213}{20736} - \frac{1169}{3840} + \frac{17}{1008} - \frac{1}{2880} \\ = \chi(SL_6(\mathbb{Z})) = 0\,. \end{gathered}$$

For $N = 7$ we obtain similarly

$$
\begin{gathered}
-\frac{290879}{107520}+\frac{13994381}{103680}-\frac{31815503}{13824}+\frac{1362329683}{69120}-\frac{6986939119}{69120}\\
+\frac{7902421301}{23040}-\frac{340039739981}{414720}+\frac{174175928729}{120960}-\frac{132108094091}{69120}\\
+\frac{27016703389}{13824}-\frac{13463035571}{8640}+\frac{14977461287}{15360}-\frac{22103821919}{46080}\\
+\frac{8522164169}{46080}-\frac{17886026827}{322560}+\frac{1764066533}{138240}-\frac{101908213}{46080}+\frac{12961451}{46080}\\
-\frac{10538393}{414720}+\frac{162617}{103680}-\frac{721}{11520}+\frac{43}{32256}\\
=\chi\,(SL_7(\mathbb{Z}))=0\,.
\end{gathered}
$$

5. EXPLICIT HOMOLOGY CLASSES

5.1. Equivariant fundamental classes.

Theorem 5.1. *The top homology group* $H_{d(N)}(\mathrm{Vor}_{SL_N(\mathbb{Z})}\otimes\mathbb{Q})$ *has dimension 1. When* $N=4,5,6$ *or* 7*, it is represented by the cycle*

$$\sum_\sigma \frac{1}{|\Gamma_\sigma|}[\sigma]\,,$$

where σ *runs through the perfect forms of rank* N *and the orientation of each cell is inherited from the one of* X_N/Γ.

Proof. The first assertion is clear since, by (3) above and (6) below we have

$$H_{d(N)}(\mathrm{Vor}_{SL_N(\mathbb{Z})}\otimes\mathbb{Q})\cong H_{d(N)-N+1}(SL_N(\mathbb{Z}),St_N\otimes\mathbb{Q})\cong H^0(SL_N(\mathbb{Z}),\mathbb{Q})\cong\mathbb{Q}\,.$$

In order to prove the second claim, write the differential between codimension 0 and codimension 1 cells as a matrix A of size $n_1\times n_0$, with $n_i=|\Sigma_{d(N)-i}(\Gamma)|$ denoting the number of codimension i cells in the Voronoï cell complex. It can be checked that in each of the n_1 rows of A there are precisely two non-zero entries. Moreover, the absolute value of the (i,j)-th entry of A is equal to the quotient $|\Gamma_{\sigma_j}|/|\Gamma_{\tau_i}|$ (an *integer*), where $\sigma_j\in\Sigma_0(\Gamma)$ and $\tau_i\in\Sigma_1(\Gamma)$. Finally, one can multiply some columns by -1 (which amounts to changing the orientation of the corresponding codimension 0 cell) in such a way that each row has exactly one positive and one negative entry. □

***Example* 5.2.** For $N=5$ the differential matrix d_{14} (cf. Table 2) between codimension 0 and codimension 1 is given by

$$\begin{pmatrix}40 & 0 & -15\\ 40 & -15 & 0\end{pmatrix},$$

so the kernel is generated by $(3,8,8)=11520\,(\frac{1}{3840},\frac{1}{1440},\frac{1}{1440})$, while the orders of the three automorphism groups are 3840, 1440 and 1440, respectively.

***Example* 5.3.** Similarly, the differential $d_{20} : V_{20} \to V_{19}$ for rank $N = 6$ (cf. Table 3) is represented by the matrix

$$\begin{pmatrix} 0 & 0 & 96 & 0 & 0 & 0 & -21 \\ 3240 & 0 & 0 & 0 & -21 & 0 & 0 \\ 0 & 0 & 1440 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 18 & 0 & -6 & 0 \\ -12960 & 0 & 0 & 0 & 0 & 12 & 0 \\ -3240 & 0 & 0 & 9 & 0 & 0 & 0 \\ 0 & -360 & 0 & 1 & 0 & 0 & 0 \\ -4320 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 960 & -6 & 0 & 0 & 0 \\ 0 & -216 & 96 & 0 & 0 & 0 & 0 \\ -45 & 45 & 0 & 0 & 0 & 0 & 0 \\ -2592 & 0 & 1152 & 0 & 0 & 0 & 0 \\ -3240 & 0 & 1440 & 0 & 0 & 0 & 0 \\ -432 & 0 & 192 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its kernel is generated by

$$(28, 28, 63, 10080, 4320, 30240, 288)$$

while the orders of the corresponding automorphism groups are, respectively,

$$103680, 103680, 46080, 288, 672, 96, 10080\,,$$

and we note that $28 \cdot 103680 = 63 \cdot 46080 = 10080 \cdot 288 = 4320 \cdot 672 = 30240 \cdot 96$.

5.2. **An explicit non-trivial homology class for rank $N = 5$.** The kernel of the 6×7-matrix of d_{10} for $GL_5(\mathbb{Z})$, displayed in the proof of Theorem 6.1, equation (4) below, is spanned by $(0, 0, 0, 0, 0, 1, -1)$ together with $(5, 1, -8, 16, 15, 2, 2)$. The latter one provides a non-trivial homology class in $H_{10}(\mathrm{Vor}_{GL_5(\mathbb{Z})}) \cong H^5(GL_5(\mathbb{Z}), \mathbb{Z})$ (modulo $\mathcal{S}_5$), given as a linear combination of cells (in terms of minimal vector indices) as follows:

$$\begin{array}{rl} 5 & \varphi([1, 15, 4, 16, 10, 11, 17, 18, 3, 5]) \\ + & \varphi([1, 15, 4, 16, 10, 11, 17, 18, 2, 5]) \\ -8 & \varphi([1, 15, 4, 10, 11, 17, 18, 3, 2, 5]) \\ +16 & \varphi([1, 6, 15, 4, 16, 10, 11, 17, 18, 2]) \\ +15 & \varphi([1, 6, 15, 4, 16, 10, 11, 17, 18, 5]) \\ +2 & \varphi([1, 6, 7, 4, 16, 10, 11, 17, 2, 5]) \\ +2 & \varphi([1, 6, 7, 19, 13, 20, 15, 10, 11, 2])\,. \end{array}$$

Here the indices refer to the following order for the set $m(P_5^1)$ of minimal vectors

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	1	0	1	1	0	0	0	1	0	0	1	1	0	0	1	0	0	0
0	1	1	0	0	−1	1	1	0	0	1	0	0	0	0	1	0	0	1	0
0	0	0	−1	0	0	0	0	0	0	1	−1	−1	0	0	0	1	1	−1	1
0	0	0	−1	0	0	0	−1	1	1	0	0	0	−1	−1	1	0	0	0	−1
0	−1	−1	1	−1	0	0	0	−1	−1	−1	0	0	0	0	−1	−1	−1	0	0

and $\varphi(u)$ for a vector u of indices is the convex hull of the minimal vectors corresponding to those indices, as in §2.2.

6. Splitting off the Voronoï complex Vor_N from Vor_{N+1} for small N

In this section, we will be concerned with $\Gamma = GL_N(\mathbb{Z})$ only and we adapt the notation $\Sigma_n(N) = \Sigma_n(GL_N(\mathbb{Z}))$ for the sets of representatives.

6.1. **Inflating well-rounded forms.** Let A be the symmetric matrix attached to a form h in C_N^*. Suppose the cell associated to A is *well-rounded*, i.e., its set of minimal vectors $S = S(A)$ spans the underlying vector space $\mathbb{R}^N$. Then we can associate to it a form $\tilde{h}$ with matrix $\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & m(A) \end{pmatrix}$ in C_{N+1}^*, where $m(A)$ denotes the minimum positive value of A on $\mathbb{Z}^N$. The set $\tilde{S}$ of minimal vectors of $\tilde{A}$ contains the ones from S, each vector being extended by an $(N+1)$-th coordinate 0. Furthermore, $\tilde{S}$ contains the additional minimal vectors $\pm e_{N+1} = \pm(0, \ldots, 0, 1)$, and hence it spans $\mathbb{R}^{N+1}$, i.e., $\tilde{A}$ is well-rounded as well. In the following, we will call forms like $\tilde{A}$ as well as their associated cells *inflated*.

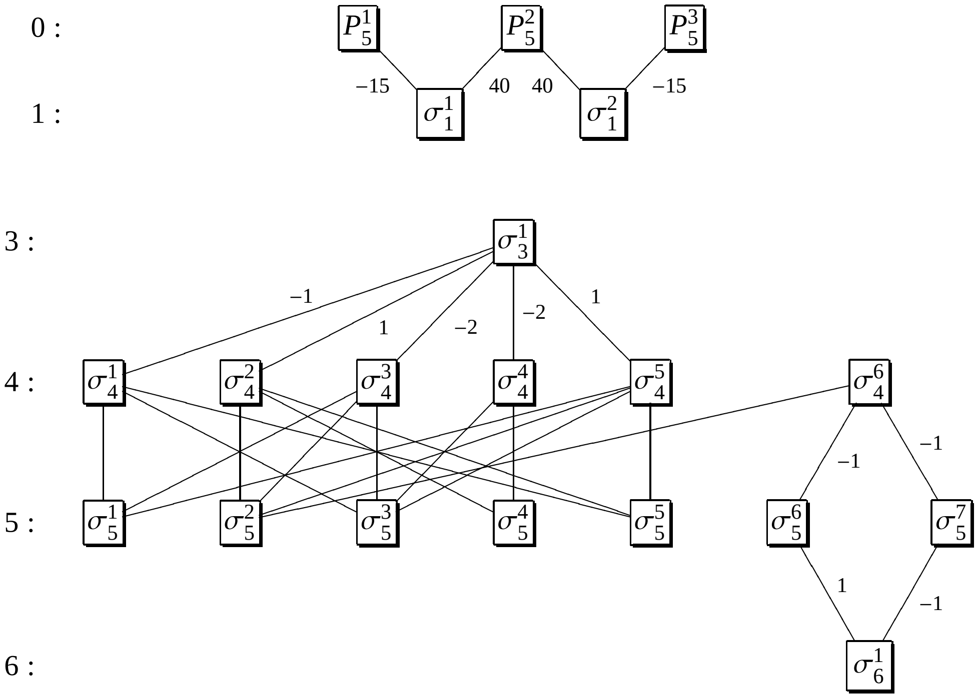
The stabilizer of h in $GL_N(\mathbb{Z})$ thereby embeds into the one of $\tilde{h}$ inside $GL_{N+1}(\mathbb{Z})$ (at least modulo ±Id) under the usual stabilization map.

Note that, by iterating the same argument r times, A induces a well-rounded form also in $\Sigma_\bullet^\star(N+r)$ which, for $r \geqslant 2$, does not belong to $\Sigma_\bullet(N+r)$ since there is an obvious orientation-reversing automorphism of the inflated form, given by the permutation which swaps the last two coordinates.

6.2. **The case $N = 5$.**

Theorem 6.1. *The complex* $\mathrm{Vor}_{GL_5(\mathbb{Z})}$ *is isomorphic to a direct factor of* $\mathrm{Vor}_{GL_6(\mathbb{Z})}$, *with degrees shifted by 1.*

Proof. The Voronoï complex of $GL_5(\mathbb{Z})$ can be represented by the following weighted graph with levels



Here the nodes in line j (marked on the left) represent the elements in $\Sigma_{d(N)-j}(5)$, i.e. we have 3, 2, 0, 1, 6, 7 and 1 cells in codimensions 0, 1, 2, 3, 4, 5 and 6, respectively, and arrows show incidences of those cells, while numbers attached to arrows give the corresponding incidence multiplicities. Since entering the multiplicities relating codimensions 4 and 5 would make the graph rather unwieldy, we give them instead in terms of the matrix corresponding to the differential d_{10} connecting dimension 10 to 9 (columns refer, in this order, to $\sigma_5^1, \dots, \sigma_5^7$, while rows refer to $\sigma_4^1, \dots, \sigma_4^6$)

$$\begin{pmatrix} -5 & 0 & -5 & 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 2 & -2 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ -1 & -2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}. \tag{4}$$

As is apparent from the picture, there are two connected components in that graph. The corresponding graph for $GL_6(\mathbb{Z})$ has three connected components, two of which are "isomorphic" (as weighted graphs with levels) to the one above for $GL_5(\mathbb{Z})$, except for a shift in codimension by 5 (e.g. codimension 0 cells in $\Sigma_\bullet(5)$ correspond to codimension 5 cells in $\Sigma_\bullet(6)$), i.e. a shift in dimension by 1.

In fact, it is possible, after appropriate coordinate transformations, to identify the minimal vectors (viewed up to sign) of any given cell in the two inflated components of $\Sigma_\bullet(6)$ alluded to above with the minimal vectors of another cell which is inflated from one in $\Sigma_\bullet(5)$, except precisely *one* minimal vector (up to sign) which is *fixed* under the stabilizer of the cell.

Let us illustrate this correspondence for the top-dimensional cell σ of the perfect form $P_5^1 \in \Sigma_{14}(5)$, also denoted $P(5,1)$ in [11] and D_5 in [12], with the list $m(P_5^1)$ of minimal vectors given already at the end of §5.2.

Using the algorithm described in §4.1, the corresponding inflated cell $\widetilde{\sigma}$ in $\Sigma_{15}(6)$ can be found to be, in terms of its 21 minimal vectors of the perfect form P_6^1 in Jaquet's notation (see [11] and §5.2 for the full list $m(P_6^1)$),

v_1	v_2	v_4	v_5	v_{10}	v_{12}	v_{13}	v_{14}	v_{16}	v_{17}	v_{18}	v_{22}	v_{24}	v_{25}	v_{26}	v_{27}	v_{29}	v_{33}	v_{34}	v_{35}	v_{36}
1	−1	0	−1	0	0	0	−1	1	1	0	1	1	0	1	0	0	1	0	0	1
0	1	−1	0	0	0	−1	0	1	0	1	1	0	1	0	1	0	0	1	0	1
0	0	1	1	0	−1	0	0	−1	0	0	−1	0	0	−1	−1	0	−1	−1	0	−1
0	0	0	0	1	0	0	0	−1	−1	−1	0	0	0	−1	−1	−1	0	0	0	−1
0	0	0	0	0	1	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	−1
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	2

The transformation

$$\gamma = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 & -1 & 0 \end{pmatrix}$$

sends v_1 to $(0,0,0,0,0,1)$ and sends each of the other vectors to the corresponding one of the form $(v,0)$ where v is the corresponding minimal vector for P_5^1 (in the order given above).

One can verify that the other two perfect forms P_5^2 and P_5^3 (denoted by Voronoï A_5 and φ_2, respectively) give rise to a corresponding inflated cell in $\Sigma_{15}(6)$ in a similar way.

Concerning the cells of *positive* codimension in $\Sigma_\bullet(5)$, it turns out that these all have a representative which is a facet in σ. Furthermore, the matrix γ induces an isomorphism from the subcomplex of $\Sigma_\bullet(6)$ spanned by $\widetilde{\sigma}$ and all its facets to the complex obtained by inflation, as in §6.1 above, from the complex spanned by σ_5 and all its facets. Finally, one can verify that the cells attached to P_5^2 and P_5^3 are conjugate, after inflation, to cells in $\Sigma_{15}(6)$, and that the differentials for Vor_{GL_5} and Vor_{GL_6} agree on these. This ends the proof of the theorem. □

6.3. **Other cases.** A similar situation holds for $\Sigma_\bullet(3)$ and $\Sigma_\bullet(4)$, but as $\Sigma_\bullet(3)$ consists of a single cell only, the picture is far less significant.

For $N = 4$, there is only one cell leftover in $\Sigma_\bullet(4)$, in fact in $\Sigma_6(4)$, and it is already inflated from $\Sigma_5(3)$, as shown in §6.3. Hence its image in $\Sigma_7^\star(5)$ will allow an orientation reversing automorphism and hence will not show up in $\Sigma_7(5)$. This illustrates the remark at the end of 6.1.

Finally, for $N = 6$, the cells in the third component of the incidence graph for $GL_6(\mathbb{Z})$ mentioned in the proof of Theorem 6.1 above appear, in inflated form, in the Voronoï complex for $GL_7(\mathbb{Z})$ which inherits the homology of that component, since in the weighted graph of $GL_7(\mathbb{Z})$, which is connected, there is only one incidence of an inflated cell with a non-inflated one. Therefore we do not have a splitting in this case.

7. The Cohomology of modular groups

7.1. **Preliminaries.** Recall the following simple fact:

Lemma 7.1. *Assume that p is a prime and $g \in \mathrm{GL}_N(\mathbb{R})$ has order p. Then $p \leqslant N+1$.*

Proof. The minimal polynomial of g is the cyclotomic polynomial $x^{p-1} + x^{p-2} + \cdots + 1$. By the Cayley-Hamilton theorem, this polynomial divides the characteristic polynomial of g. Therefore $p - 1 \leqslant N$. □

We shall also need the following result:

Lemma 7.2. *The action of $\mathrm{GL}_N(\mathbb{R})$ on the symmetric space X_N preserves its orientation if and only if N is odd.*

Proof. The subgroup $\mathrm{GL}_N(\mathbb{R})^+ \subset \mathrm{GL}_N(\mathbb{R})$ of elements with positive determinant is the connected component of the identity, therefore it preserves the orientation of X_N. Any $g \in \mathrm{GL}_N(\mathbb{R})$ which is not in $\mathrm{GL}_N(\mathbb{R})^+$ is the product of an element of $\mathrm{GL}_N(\mathbb{R})^+$ with the diagonal matrix $\varepsilon = \mathrm{diag}\,(-1, 1, \ldots, 1)$, so we just need to check when ε preserves the orientation of X_N. The tangent space TX_N of X_N at the origin consists of real symmetric matrices $m = (m_{ij})$ of trace zero. The action of ε is given by $m \cdot \varepsilon = \varepsilon^t\, m\, \varepsilon$ (cf. §2.1) and we get

$$(m \cdot \varepsilon)_{ij} = m_{ij}$$

unless $i = 1$ or $j = 1$ and $i \neq j$, in which case $(m \cdot \varepsilon)_{ij} = -m_{ij}$. Let δ_{ij} be the matrix with entry 1 in row i and column j, and zero elsewhere. A basis of TX_N consists of the matrices $\delta_{ij} + \delta_{ji}$, $i \neq j$, together with $N - 1$ diagonal matrices. For this basis, the action of ε maps $N-1$ vectors v to their opposite $-v$ and fixes the other ones. The lemma follows. □

7.2. **Borel–Serre duality.** According to Borel and Serre ([5], Thm. 11.4.4 and Thm. 11.5.1), the group $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$ is a virtual duality group with dualizing module

$$H^{v(N)}(\Gamma, \mathbb{Z}[\Gamma]) = \mathrm{St}_N \otimes \tilde{\mathbb{Z}},$$

where $v(N) = N(N-1)/2$ is the virtual cohomological dimension of Γ and $\tilde{\mathbb{Z}}$ is the orientation module of X_N. It follows that there is a long exact sequence

$$\cdots \to H_n(\Gamma, \mathrm{St}_N) \to H^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}) \to \hat{H}^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}) \to H_{n-1}(\Gamma, \mathrm{St}_N) \to \cdots \tag{5}$$

where $\hat{H}^*$ is the Farrell cohomology of Γ [8]. From Lemma 7.1 and the Brown spectral sequence ([4], X (4.1)) we deduce that $\hat{H}^*(\Gamma, \tilde{\mathbb{Z}})$ lies in $\mathcal{S}_{N+1}$. Therefore

$$H_n(\Gamma, \mathrm{St}_N) \equiv H^{v(N)-n}(\Gamma, \tilde{\mathbb{Z}}), \text{ modulo } \mathcal{S}_{N+1}. \tag{6}$$

When N is odd, then $\mathrm{GL}_N(\mathbb{Z})$ is the product of $\mathrm{SL}_N(\mathbb{Z})$ by $\mathbb{Z}/2$, therefore

$$H^m(\mathrm{GL}_N(\mathbb{Z}), \mathbb{Z}) \equiv H^m(\mathrm{SL}_N(\mathbb{Z}), \mathbb{Z}), \text{ modulo } \mathcal{S}_2.$$

When N is even, then the action of $\mathrm{GL}_N(\mathbb{Z})$ on $\tilde{\mathbb{Z}}$ is given by the sign of the determinant (see Lemma 7.2) and Shapiro's lemma gives

$$H^m(\mathrm{SL}_N(\mathbb{Z}), \mathbb{Z}) = H^m(\mathrm{GL}_N(\mathbb{Z}), M), \tag{7}$$

with

$$M = \mathrm{Ind}_{\mathrm{SL}_N(\mathbb{Z})}^{\mathrm{GL}_N(\mathbb{Z})} \mathbb{Z} \equiv \mathbb{Z} \oplus \tilde{\mathbb{Z}}, \text{ modulo } \mathcal{S}_2.$$

To summarize: when $\Gamma = SL_N(\mathbb{Z})$ or $GL_N(\mathbb{Z})$, where $N \leqslant 7$, we know $H^m(\Gamma, \tilde{\mathbb{Z}})$ by combining (3) (end of §3.4), Theorem 4.3 and (6). This allows us to compute the cohomology of $GL_N(\mathbb{Z})$. The results are given in Theorem 7.3 below.

7.3. **The cohomology of modular groups.**

Theorem 7.3. (i) *Modulo $\mathcal{S}_5$ we have*

$$H^m(SL_5(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \quad m = 0, 5, \\ 0 & \text{otherwise}. \end{cases}$$

(ii) *Modulo $\mathcal{S}_7$ we have*

$$H^m(GL_6(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } \quad m = 0, 5, 8, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$H^m(SL_6(\mathbb{Z}), \mathbb{Z}) = \begin{cases} \mathbb{Z}^2 & \text{if } \quad m = 5, \\ \mathbb{Z} & \text{if } \quad m = 0, 8, 9, 10, \\ 0 & \text{otherwise}. \end{cases}$$

(iii) *For $N = 7$, we have,*

$$H^m(SL_7(\mathbb{Z}), \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } \quad m = 0, 5, 11, 14, 15, \\ 0 & \text{otherwise}. \end{cases}$$

***Remark* 7.4.** Morita asks in [14] whether the class of infinite order in $H^5(GL_5(\mathbb{Z}), \mathbb{Z})$ survives in the cohomology of the group of outer automorphisms of the free group of rank five.

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