Locally free sheaves on complex supermanifolds[1]

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1. Introduction

An important part of the classical theory of real or complex manifolds is the theory of (smooth, real analytic or complex analytic) vector bundles. With any vector bundle over a manifold $(M, \mathcal{F})$ the sheaf of its (smooth, real analytic or complex analytic) sections is associated which is a locally free sheaf of $\mathcal{F}$-modules, and in this way all the locally free sheaves of $\mathcal{F}$-modules over $(M, \mathcal{F})$ can be obtained. In the present paper, locally free sheaves of $\mathcal{O}$-modules on a complex analytic supermanifold $(M, \mathcal{O})$ (or equivalently sheaves of sections of vector bundles over $(M, \mathcal{O})$) are studied.

It is well-known that any smooth supermanifold $(M, \mathcal{O})$ is split, i.e. $\mathcal{O} \simeq \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is the sheaf of sections of a certain vector bundle over M. In the complex case this statement is false, see [6]. However, we can assign the split supermanifold $(M, \operatorname{gr} \mathcal{O})$ to any complex analytic supermanifold $(M, \mathcal{O})$, which is called the *retract* of $(M, \mathcal{O})$. Given a locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules on a complex analytic supermanifold $(M, \mathcal{O})$, we construct a locally free sheaf $\operatorname{gr} \mathcal{E}$ on the retract $(M, \operatorname{gr} \mathcal{O})$, which is called the *retract* of $\mathcal{E}$. It can be easily shown that $\operatorname{gr} \mathcal{E} \simeq \operatorname{gr} \mathcal{O} \otimes \mathcal{E}_{\mathrm{red}}$, where $\mathcal{E}_{\mathrm{red}}$ is the pullback of $\mathcal{E}$ with respect to the natural embedding of the manifold $(M, \mathcal{F})$ into $(M, \mathcal{O})$. In Section 2 we obtained a classification of locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules which have a given retract $\operatorname{gr} \mathcal{E}$ in terms of non-abelian 1-cohomology, Theorem 2. In the special case $\mathcal{O} \simeq \operatorname{gr} \mathcal{O}$ our classification result can be simplified, Theorem 3.

In Section 3 we study locally free sheaves of modules over projective superspaces. In the case of complex projective spaces, the problem of the (indecomposable) bundle classification is far from being solved, see [10]. There are two cases, however, in which all bundles are known to be direct sums of line bundles — over $\mathbb{CP}^1$ by the classical Birkhoff – Grothendieck Theorem and over $\mathbb{CP}^\infty$ by the Barth – Van de Ven – Tyurin theorem. We study similar question in the super context. In the case of $\mathbb{CP}^{1|m}$, $m > 0$, we showed that the Birkhoff – Grothendieck Theorem does not hold true. (The fact that this theorem is false for some $\mathbb{CP}^{1|m}$ was noticed in [9].) Furthermore,

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we achieved the result similar to the Barth – Van de Ven – Tyurin Theorem for projective superspaces.

Section 4 is devoted to the study of the tangent sheaf $\mathcal{T}$ of a split supermanifold $(M, \bigwedge \mathcal{G})$ in more details. The main result is here the equivalence of the triviality of the 1-cohomology class corresponding to $\mathcal{T}$ and the existence of a holomorphic connection of the bundle corresponding to the locally free sheaf of $\mathcal{F}$-modules $\mathcal{G}$.

In Subsection 5 a spectral sequence which connects the cohomology with values in a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ with the cohomology with values in its retract $\operatorname{gr}\mathcal{E}$ is constructed. This spectral sequence permits to compute the cohomology group $H^*(M, \mathcal{E})$ using the cohomology class corresponding to $\mathcal{E}$ by Theorem 3 and the cohomology group $H^*(M, \operatorname{gr}\mathcal{E})$. Note that $\operatorname{gr}\mathcal{E}$ is a sheaf of sections of a certain vector bundle over M. Hence to compute $H^*(M, \operatorname{gr}\mathcal{E})$ we may use the well elaborated tools of complex analytic geometry. We described the first two terms of the spectral sequence and the first non zero differential.

A classification of locally free sheaves of $\mathcal{O}$-modules over a smooth supermanifold $(M, \mathcal{O})$ was obtained in [14], Section 4.3. It was shown that any locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ is isomorphic to $\operatorname{gr}\mathcal{E}$. The similar result for fibre superbundles was proved in [16]. In [4] the split holomorphic case was studied. In particular it was shown there that there exists a holomorphic locally free sheaf of $\mathcal{O}$-modules over a holomorphic supermanifold $(M, \mathcal{O})$, which is not isomorphic to its retract $\operatorname{gr}\mathcal{E}$. There a classification up to isomorphism of locally free sheaves of $\mathcal{O}$-modules over a (holomorphic) split supermanifold $(M, \mathcal{O})$, $\mathcal{O} \simeq \bigwedge \mathcal{G}$, is obtained in terms of cohomology set $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$. In the present paper we suggest the different approach to the classification of locally free sheaves of $\mathcal{O}$-modules over a split supermanifold, Theorem 3, and more generally over a non-split supermanifold, Theorem 2. Let us explain the difference in more details. Clearly one has a split homomorphism $T : \mathrm{GL}(n, \bigwedge \mathcal{G}) \to \mathrm{GL}(n, \mathbb{C})$ by taking the degree zero part of $\mathrm{GL}(n, \bigwedge \mathcal{G})$. It induces the map $H^1(T) : H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G})) \to H^1(M, \mathrm{GL}(n, \mathbb{C}))$. Denote by $a_{\mathcal{E}}$ the element of $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$, which corresponds to a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$. Then, in our notations, $\mathcal{E}_{\mathrm{red}}$ corresponds to $H^1(T)(a_{\mathcal{E}})$. In our paper we classify all locally free sheaves $\mathcal{E}$ such that $\mathcal{E}_{\mathrm{red}}$ is fixed. Therefore, instead of computing $H^1(M, \mathrm{GL}(n, \bigwedge \mathcal{G}))$, we suggest to use results concerning classification of holomorphic bundles over a manifold, obtained in classical geometry, and consider locally free sheaves with given retract on a split supermanifold. The idea to classify non-split object, more precisely, supermanifolds, using retrcts appeared firstly in [6].

We would like also to mention that, as in the classical case, the line superbundles can be described using the exp-map, see e.g. [2], Chapter VI, Section 2. The Picard groups of generic super-grassmannians were computed in [13].

Notations.

$(M, \mathcal{O})$	supermanifold
$(M, \operatorname{gr}\mathcal{O})$	the retract of $(M, \mathcal{O})$
$\mathcal{T} = \mathcal{D}er\mathcal{O}$	the tangent sheaf of $(M, \mathcal{O})$
$\mathcal{A}ut\mathcal{O}$	the sheaf of automorphisms of the structure sheaf $\mathcal{O}$
$\mathcal{A}ut_0 \operatorname{gr}\mathcal{O}$	the sheaf of automorphisms of $\operatorname{gr}\mathcal{O}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{O}$
$\operatorname{gr}\mathcal{E}$	the retract of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$
$\mathcal{A}ut^{\mathcal{R}}\mathcal{E}$	the sheaf of automorphisms of a sheaf of $\mathcal{R}$-modules $\mathcal{E}$
$\mathcal{A}ut_0^{\mathcal{R}} \operatorname{gr}\mathcal{E}$	the sheaf of automorphisms of a $\mathbb{Z}$-graded sheaf of $\mathcal{R}$-modules $\operatorname{gr}\mathcal{E}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{E}$
$\mathcal{QA}ut\mathcal{E}$	the sheaf of quasi-automorphisms of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$
$\mathcal{QA}ut_0 \operatorname{gr}\mathcal{E}$	the sheaf of quasi-automorphisms of a $\mathbb{Z}$-graded locally free sheaf $\operatorname{gr}\mathcal{E}$ preserving the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{E}$
$\mathcal{A}ut_{\bar{0}}^{\mathcal{F}}\mathcal{S}$	a subsheaf of $\mathcal{A}ut^{\mathcal{F}}\mathcal{S}$ consisting of even automorphisms of a $\mathbb{Z}_2$-graded sheaf S
$\mathcal{E}nd^{\mathcal{O}}\mathcal{E}$	the sheaf of endomorphisms of a sheaf of $\mathcal{O}$-modules $\mathcal{E}$

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2. Main definitions and classification theorems

2.1 Main definitions and classification of complex supermanifolds with a given retract

We consider here complex analytic supermanifolds in the sense of Berezin and Leites (see [3, 8]). Thus, a *supermanifold* $(M, \mathcal{O})$ of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space which is locally isomorphic to a superdomain in $\mathbb{C}^{n|m}$. The underlying complex manifold $(M, \mathcal{F})$ is called the *reduction* of $(M, \mathcal{O})$. Sometime we will denote it by M. A *morphism* $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ between two supermanifolds with reductions $(M, \mathcal{F}_M)$ and $(N, \mathcal{F}_N)$ is a morphism between $\mathbb{Z}_2$-graded ringed spaces, i.e., a pair $F = (F_{red}, F^*)$, where $F_{red} : M \to N$ is a continuous mapping and $F^* : \mathcal{O}_N \to (F_{red})_*\mathcal{O}_M$ is a homomorphism of sheaves of $\mathbb{Z}_2$-graded ringed spaces. A morphism F is called an *isomorphism* if F is invertible.

We consider $\mathbb{Z}_2$-graded sheaves of $\mathcal{O}$-modules $\mathcal{S} = \mathcal{S}_{\bar 0} + \mathcal{S}_{\bar 1}$ on $(M, \mathcal{O})$. Denote by $\Pi(\mathcal{S})$ the same sheaf of $\mathcal{O}$-modules $\mathcal{S}$ supplied with the following $\mathbb{Z}_2$-grading:

$$\Pi(\mathcal{S})_{\bar 0} = \mathcal{S}_{\bar 1},\ \Pi(\mathcal{S})_{\bar 1} = \mathcal{S}_{\bar 0}.$$

A $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules on $(M, \mathcal{O})$ is called *free* (*locally free*) *of rank* $p|q$, $p, q \geq 0$, if it is isomorphic (respectively, locally isomorphic) to the $\mathbb{Z}_2$-graded sheaf of $\mathcal{O}$-modules $\mathcal{O}^p \oplus \Pi(\mathcal{O})^q$. For example, the tangent sheaf $\mathcal{T}$ of a supermanifold $(M, \mathcal{O})$ of dimension $n|m$ is a locally free sheaf of $\mathcal{O}$-modules of rank $n|m$.

The simplest class of supermanifolds constitute the so-called *split supermanifolds*. We recall that a supermanifold $(M, \mathcal{O})$ is called *split* if $\mathcal{O} = \bigwedge_{\mathcal{F}} \mathcal{G}$, where $\mathcal{G}$ is a locally free sheaf of $\mathcal{F}$-modules on M. With any supermanifold $(M, \mathcal{O})$ one can associate a split supermanifold $(M, \operatorname{gr} \mathcal{O})$ of the same dimension which is called the *retract* of $(M, \mathcal{O})$. To construct it, let us consider the $\mathbb{Z}_2$-graded sheaf of ideals $\mathcal{J} = \mathcal{J}_{\bar 0} \oplus \mathcal{J}_{\bar 1} \subset \mathcal{O}$ generated by $\mathcal{O}_{\bar 1}$. The structure sheaf of the retract is defined by

$$\operatorname{gr} \mathcal{O} = \bigoplus_{p \geq 0} \operatorname{gr} \mathcal{O}_p, \ \text{ where } \ \operatorname{gr} \mathcal{O}_p = \mathcal{J}^p / \mathcal{J}^{p+1},\ \ \mathcal{J}^0 := \mathcal{O}.$$

It can be easy shown that $\mathcal{F} \simeq \mathcal{O}/\mathcal{J}$, $\operatorname{gr} \mathcal{O}_1$ is a locally free sheaf of $\mathcal{F}$-modules on M and $\operatorname{gr} \mathcal{O}_p = \bigwedge_{\mathcal{F}}^p \operatorname{gr} \mathcal{O}_1$. We will use the following two locally split exact sequences:

$$\begin{array}{c} 0 \to \mathcal{J}_{\bar 0} \to \mathcal{O}_{\bar 0} \to \mathcal{F} \to 0; \\ 0 \to (\mathcal{J}^2)_{\bar 1} \to \mathcal{O}_{\bar 1} \to (\operatorname{gr} \mathcal{O})_1 \to 0. \end{array} \tag{1}$$

Note that a supermanifold is split iff the sequences (1) are globally split.

Let $(M, \mathcal{O})$ be a split supermanifold. Then any $\mathbb{Z}_2$-graded locally free sheaf $\mathcal{S} = \mathcal{S}_{\bar 0} \oplus \mathcal{S}_{\bar 1}$ of $\mathcal{F}$-modules on M gives rise to a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on $(M, \mathcal{O})$. It is defined in the following way: $\mathcal{E} := \mathcal{O} \otimes_{\mathcal{F}} \mathcal{S}$. Its $\mathbb{Z}_2$-grading is given by

$$\begin{array}{l} \mathcal{E}_{\bar 0} = \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 0} + \mathcal{O}_{\bar 1} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1}, \\ \mathcal{E}_{\bar 1} = \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1} + \mathcal{O}_{\bar 0} \otimes_{\mathcal{F}} \mathcal{S}_{\bar 1}. \end{array} \tag{2}$$

Let now $\mathcal{E} = \mathcal{E}_{\bar 0} \oplus \mathcal{E}_{\bar 1}$ be a locally free sheaf of $\mathcal{O}$-modules of rang $p|q$ on an arbitrary supermanifold $(M, \mathcal{O})$. We are going to construct a locally free sheaf of the same rank on the retract of $(M, \mathcal{O})$. First, we note that

$\mathcal{S} := \mathcal{E}/\mathcal{J}\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M. Moreover, $\mathcal{S}$ admits the $\mathbb{Z}_2$-grading

$$\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$$

by two locally free sheaves of $\mathcal{F}$-modules

$$\mathcal{S}_{\bar{0}} := \mathcal{E}_{\bar{0}}/(\mathcal{J}\mathcal{E}) \cap \mathcal{E}_{\bar{0}}, \ \mathcal{S}_{\bar{1}} := \mathcal{E}_{\bar{1}}/(\mathcal{J}\mathcal{E}) \cap \mathcal{E}_{\bar{1}}$$

of ranks p and q respectively. We have the following two locally split exact sequences:

$$\begin{array}{c} 0 \to \mathcal{J}\mathcal{E} \cap \mathcal{E}_{\bar{0}} \to \mathcal{E}_{(0)\bar{0}} \xrightarrow{\alpha} \mathcal{S}_{\bar{0}} \to 0; \\ 0 \to \mathcal{J}\mathcal{E} \cap \mathcal{E}_{\bar{1}} \to \mathcal{E}_{(0)\bar{1}} \xrightarrow{\beta} \mathcal{S}_{\bar{1}} \to 0, \end{array} \tag{3}$$

where α and β are the natural projection maps. The sheaf $\mathcal{E}$ possesses the filtration:

$$\mathcal{E} = \mathcal{E}_{(0)} \supset \mathcal{E}_{(1)} \supset \mathcal{E}_{(2)} \supset \dots, \tag{4}$$

where

$$\mathcal{E}_{(p)} = \mathcal{J}^p\mathcal{E}, \ p \geq 1.$$

Using this filtration, we can construct the following locally free sheaf of $\operatorname{gr}\mathcal{O}$-modules on the retract $(M, \operatorname{gr}\mathcal{O})$:

$$\begin{array}{c} \operatorname{gr}\mathcal{E} = \bigoplus_p \operatorname{gr}\mathcal{E}_p, \ \text{ where} \\ \operatorname{gr}\mathcal{E}_p = \mathcal{E}_{(p)}/\mathcal{E}_{(p+1)} \simeq \operatorname{gr}\mathcal{O}_p \otimes_{\mathcal{F}} \mathcal{S}. \end{array}$$

From $\operatorname{gr}\mathcal{O} = \bigwedge \operatorname{gr}\mathcal{O}_1$ and $\operatorname{gr}\mathcal{O}_p = \bigwedge^p \operatorname{gr}\mathcal{O}_1$ it follows that

$$\operatorname{gr}\mathcal{E} \simeq \bigwedge \operatorname{gr}\mathcal{O}_1 \otimes_{\mathcal{F}} \mathcal{S}.$$

The sheaf $\operatorname{gr}\mathcal{E}$ we will call the *retract* of $\mathcal{E}$. By definition, the sheaf $\operatorname{gr}\mathcal{E}$ is $\mathbb{Z}$-graded. It possesses also the $\mathbb{Z}_2$-grading given by (2).

Our aim now is to classify locally free sheaves of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O})$ which have a fixed retract. First we formulate the well-known theorem of Green (see [4]) which classifies complex supermanifolds $(M, \mathcal{O}_M)$ with a given retract up to isomorphism, inducing the identical isomorphism of reductions. The main tool used in both classification theorems is the 1-cohomology set $H^1(M, \mathcal{Q})$, where $\mathcal{Q}$ is a sheaf of non-abelian groups on M. We denote by ϵ the unit element of $H^1(M, \mathcal{Q})$ which corresponds to the unit 1-cocycle.

In what follows, we denote by $\mathcal{A}ut\mathcal{O}$ the sheaf of automorphisms of the sheaf of superalgebras $\mathcal{O}$ and by $\mathcal{A}ut^{\mathcal{R}}\mathcal{E}$ the sheaf of automorphisms of a sheaf of $\mathcal{R}$-modules $\mathcal{E}$ on M, where $\mathcal{R}$ is a sheaf of (super)algebras on M. The sheaf $\mathcal{A}ut\mathcal{O}$ possesses the filtration

$$\mathcal{A}ut\mathcal{O} = \mathcal{A}ut_{(0)}\mathcal{O} \supset \mathcal{A}ut_{(2)}\mathcal{O} \supset \dots, \tag{5}$$

where

$$\mathcal{A}ut_{(2p)}\mathcal{O} = \{a \in \mathcal{A}ut\mathcal{O} \mid a(u) \equiv u \operatorname{mod} \mathcal{J}^{2p}\}.$$

Furthermore, the group $H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}) \simeq H^0(M, \mathcal{A}ut^{\mathcal{F}} \operatorname{gr} \mathcal{O}_1)$ acts on the sheaf $\mathcal{A}ut \operatorname{gr} \mathcal{O}$ by $\operatorname{Int} : (a, \delta) \mapsto a \circ \delta \circ a^{-1}$, where $\delta \in \mathcal{A}ut \operatorname{gr} \mathcal{O}$ and $a \in H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O})$. Clearly, the group $H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O})$ leaves invariant the subsheaves of groups $\mathcal{A}ut_{(2p)} \operatorname{gr} \mathcal{O}$. Hence this group acts on the sets $H^1(M, \mathcal{A}ut_{(2p)} \operatorname{gr} \mathcal{O})$, and the unit element ϵ is fixed under this action.

Denote by $[(M, \mathcal{O})]$ the class of supermanifolds which are isomorphic to $(M, \mathcal{O})$. (Here we consider complex supermanifolds up to isomorphisms inducing the identical isomorphism of reductions.)

Theorem 1. [**Green**] *Let $(M, \mathcal{O}_{\operatorname{gr}})$ be a split complex supermanifold. Then*

$$\{[(M, \mathcal{O})] \mid \operatorname{gr} \mathcal{O} = \mathcal{O}_{\operatorname{gr}}\} \xleftrightarrow{1:1} H^1(M, \mathcal{A}ut_{(2)} \operatorname{gr} \mathcal{O})/H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}),$$

where $(M, \mathcal{O}_{\operatorname{gr}})$ corresponds to ϵ.

2.2 Classification theorems for locally free sheaves with a given retract

Let $(M, \mathcal{O})$ and $(M, \mathcal{O}')$ be two supermanifolds, $\mathcal{E}_1$ and $\mathcal{E}_2$ be locally free sheaves of $\mathcal{O}$-modules and $\mathcal{O}'$-modules on M respectively. Suppose that $\Psi : \mathcal{O} \to \mathcal{O}'$ is a homomorphism of sheaves of superalgebras. A homomorphism of $\mathbb{Z}_2$-graded sheaves of vector spaces $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ is called a Ψ-*morphism* if

$$\Phi(fv) = \Psi(f)\Phi(v), \ f \in \mathcal{O}, \ v \in \mathcal{E}_1.$$

In this case we write $\Phi = \Phi_\Psi$. A Ψ-*morphism* $\Phi : \mathcal{E} \to \mathcal{E}$ is called a Ψ-*isomorphism* if Φ is invertible. A Ψ-isomorphism $\Phi : \mathcal{E} \to \mathcal{E}$ we also will call a Ψ-*automorphism* of $\mathcal{E}$. A homomorphism (isomorphism) of $\mathbb{Z}_2$-graded sheaves of vector spaces $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$ will be called a *quasi-morphism* (*quasi-isomorphism*) if it is a Ψ-*morphism* (Ψ-*isomorphism*) for a certain Ψ. The sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ will be called *quasi-isomorphic* if it exists a quasi-isomorphism $\Phi : \mathcal{E}_1 \to \mathcal{E}_2$. A quasi-isomorphism $\mathcal{E} \to \mathcal{E}$ will be called a *quasi-automorphism* of $\mathcal{E}$. We will study the sheaf $\mathcal{QA}ut\mathcal{E}$, where

$$\mathcal{QA}ut\mathcal{E}(U) = \{\Phi \ \mid \ \Phi \ \text{is a quasi-automorphism of} \ \mathcal{E}|_U\} \tag{6}$$

for each open subset $U \subset M$. One verifies easily that $\Phi_\Psi \circ \Theta_\Upsilon$, where $\Phi_\Psi, \Theta_\Upsilon \in \mathcal{QA}ut\mathcal{E}$, is a $\Psi \circ \Upsilon$-morphism. It follows that $\mathcal{QA}ut\mathcal{E}$ is a sheaf of groups. It possesses the double filtration by the subsheaves

$$\mathcal{QA}ut_{(p)(q)}\mathcal{E} := \{\Phi_\Psi \in \mathcal{QA}ut\mathcal{E} \mid \Phi_\Psi(v) \equiv v \operatorname{mod} \mathcal{E}_{(p)}, \ \Psi(f) \equiv f \operatorname{mod} \mathcal{J}^q \\ \text{for } v \in \mathcal{E}, f \in \mathcal{O}\}, \ p, q \geq 0.$$

We also define the following subsheaves:

$$\mathcal{QA}ut_0(\operatorname{gr}\mathcal{E}) := \{\Phi_\Psi \mid \Phi_\Psi \in \mathcal{QA}ut(\operatorname{gr}\mathcal{E}),\ \Phi_\Psi \text{ preserves the } \mathbb{Z}\text{-grading of } \operatorname{gr}\mathcal{E}\}. \tag{7}$$

$$\mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{S} := \{\Phi \mid \Phi \in \mathcal{A}ut^{\mathcal{F}}\mathcal{S},\ \Phi \text{ preserves the } \mathbb{Z}_2\text{-grading of } \mathcal{S}\}, \tag{8}$$

where $\mathcal{S}$ is a $\mathbb{Z}_2$-graded sheaf of $\mathcal{F}$-modules.

Lemma 1. *We have an isomorphism of sheaves of groups*

$$\mathcal{QA}ut_0(\operatorname{gr}\mathcal{E}) \simeq \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1) \times \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}}.$$

Proof. Let us define the mapping

$$\Theta : \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1) \times \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}} \to \mathcal{QA}ut_0(\operatorname{gr}\mathcal{E})$$

by

$$(\psi, \Phi) \mapsto \Phi_{\wedge\psi},\ \psi \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1),\ \Phi \in \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}},$$

where

$$\Phi_{\wedge\psi}(hv) := \wedge\psi(h)\Phi(v)$$

for $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$ and $\wedge\psi$ is the automorphism of the sheaf $\operatorname{gr}\mathcal{O}$ induced by ψ. This is a homomorphism of sheaves of groups. In fact, suppose that another pair (ψ', Φ'), where $\psi' \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1)$, $\Phi' \in \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}}$, is given. Then we have

$$(\Phi_{\wedge\psi} \circ \Phi'_{\wedge\psi'})(hv) = \Phi_{\wedge\psi}(\wedge\psi'(h)\Phi'_{\wedge\psi'}(v)) = \wedge\psi(\wedge\psi'(h))\Phi_{\wedge\psi}(\Phi'_{\wedge\psi'}(v)) = (\Phi \circ \Phi'_{\wedge\psi\circ\wedge\psi'})(hv)$$

for $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$.

Let us prove that $\operatorname{Ker}\Theta = (\mathrm{id}, \mathrm{id})$. Suppose that $\Theta(\psi, \Phi) = \mathrm{id}$. Then $\Phi_{\wedge\psi}(hv) = \wedge\psi(h)\Phi(v) = hv$ for all $h \in \operatorname{gr}\mathcal{O}$, $v \in \mathcal{E}_{\mathrm{red}}$. Putting $h = 1$, we see that $\Phi(v) = v$, i.e., $\Phi = \mathrm{id}$. Since $\mathcal{E}_{\mathrm{red}}$ is locally free, this implies that $\wedge\psi(h) = h$, therefore, $\psi = \mathrm{id}$. Thus, the homomorphism Θ is injective.

Let us now prove that it is surjective. Let $\Phi_\Psi \in \mathcal{QA}ut_0(\operatorname{gr}\mathcal{E})$ be given. Let us show that $\Phi_\Psi \in \operatorname{Im}\Theta$. Since $\Phi_\Psi|_{\mathcal{E}_{\mathrm{red}}} : \mathcal{E}_{\mathrm{red}} \to \mathcal{E}_{\mathrm{red}}$ and Φ_Ψ preserves the $\mathbb{Z}_2$-grading of $\operatorname{gr}\mathcal{E}$, we have $\Phi := \Phi_\Psi|_{\mathcal{E}_{\mathrm{red}}} \in \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}}$. Furthermore, if $h \in \operatorname{gr}\mathcal{O}_p$ and $v \in \mathcal{E}_{\mathrm{red}}$, then

$$\Phi_\Psi(hv) = \Psi(h)\Phi(v) \in \operatorname{gr}\mathcal{E}_p.$$

It follows that $\Psi(h) \in \operatorname{gr}\mathcal{O}_p$, and hence Ψ preserves the $\mathbb{Z}$-grading of $\operatorname{gr}\mathcal{O}$. We have $\psi = \Psi|_{\operatorname{gr}\mathcal{O}_1} \in \mathcal{A}ut^{\mathcal{F}}(\operatorname{gr}\mathcal{O}_1)$ and $\wedge\psi = \Psi$. The proof is complete.$\Box$

We will use the above notation, fixing a split complex supermanifold $(M, \mathcal{O}_{\mathrm{gr}})$ and a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules $\mathcal{S}$ on M. Our aim is to classify locally free sheaves $\mathcal{E}$ of $\mathcal{O}$-modules on complex supermanifolds $(M, \mathcal{O})$ with retract $(M, \mathcal{O}_{\mathrm{gr}})$, whose retract $\operatorname{gr}\mathcal{E}$ coincides with $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_{\mathrm{gr}} \otimes_{\mathcal{F}} \mathcal{S}$.

The group $H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$ acts on the sheaf $\mathcal{QA}ut\mathcal{E}_{\mathrm{gr}}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$ and $\delta \in \mathcal{QA}ut\mathcal{E}_{\mathrm{gr}}$. It is easy to see that this action leaves invariant the subsheaves $\mathcal{QA}ut_{(p)(q)}\mathcal{E}_{\mathrm{gr}}$ and hence induces an action of $H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$ on the cohomology set $H^1(M, \mathcal{QA}ut_{(p)(q)}\mathcal{E}_{\mathrm{gr}})$.

If $\phi : M \to N$ is a holomorphic map of manifolds and $p : \mathbb{E} \to N$ is a vector bundle, we may define the pullback bundle $\phi^*(\mathbb{E})$ on N. The corresponding to $\phi^*(\mathbb{E})$ sheaf is $\mathcal{F}_M \otimes_{\phi^*(\mathcal{F}_N)} \phi^*(\mathcal{E})$, where $\mathcal{E}$ is the sheaf of sections corresponding to $\mathbb{E}$, $\mathcal{F}_M$ and $\mathcal{F}_N$ are the sheaves of holomorphic functions on M and N respectively. Let $\pi : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ be a morphism of two supermanifolds and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_N$-modules on N of rang $p|q$. Similarly, we can define the sheaf $\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{E})$. This sheaf is a locally free sheaf of $\mathcal{O}_M$-modules on M of rang $p|q$, since

$$\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{O}_N) \simeq \mathcal{O}_M.$$

Sometimes we will denote the sheaf $\mathcal{O}_M \otimes_{\pi^*_{\mathrm{red}}(\mathcal{O}_N)} \pi^*_{\mathrm{red}}(\mathcal{E})$ by $\widetilde{\pi}(\mathcal{E})$.

Let us consider the special case $(M, \mathcal{O}_M) = (N, \mathcal{O}_N)$, $\pi = (\mathrm{id}, \pi^*)$ and $\pi^* \in H^0(M, \mathcal{A}ut\mathcal{O}_M)$. We have

$$\widetilde{\pi}(\mathcal{E}) = \mathcal{O}_M \otimes_{\mathrm{id}^*(\mathcal{O}_N)} \mathrm{id}^*(\mathcal{E}) = \mathcal{O}_M \otimes_{\mathcal{O}_N} \mathcal{E}.$$

The sheaves $\widetilde{\pi}(\mathcal{E})$ and $\mathcal{E}$ are $(\pi^*)^{-1}$-isomorphic, the $(\pi^*)^{-1}$-isomorphism is given by $f \otimes s \mapsto (\pi^*)^{-1}(f)s$, where $f \in \mathcal{O}_M$ and $s \in \mathcal{E}$. Let $\Phi_{\Psi^*} : \mathcal{E} \to \mathcal{E}'$ be an Ψ^*-isomorphism of two locally free sheaves of $\mathcal{O}_M$-modules on M. We put $\Psi := (\mathrm{id}, \Psi^*)$. We see that $\widetilde{\Psi}(\mathcal{E})$ and $\mathcal{E}'$ are id-isomorphic.

Furthermore, let us consider the sheaf $\mathcal{A}ut^{\mathcal{O}}\mathcal{E}$ of automorphisms of the $\mathcal{O}$-modules sheaf $\mathcal{E}$. It possesses the filtration:

$$\mathcal{A}ut^{\mathcal{O}}\mathcal{E} = \mathcal{A}ut^{\mathcal{O}}_{(0)}\mathcal{E} \supset \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E} \supset \ldots,$$

where

$$\mathcal{A}ut^{\mathcal{O}}_{(p)}\mathcal{E} := \{a \in \mathcal{A}ut^{\mathcal{O}}\mathcal{E} \mid a(v) \equiv v \operatorname{mod} \mathcal{E}_{(p)}\}, \ p \geq 0.$$

The group $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr}\mathcal{E}) \simeq H^0(M, \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}\mathcal{E}_{\mathrm{red}})$ acts on the sheaf $\mathcal{A}ut^{\mathcal{O}} \operatorname{gr}\mathcal{E}$ by $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr}\mathcal{E})$ and $\delta \in \mathcal{A}ut^{\mathcal{O}} \operatorname{gr}\mathcal{E}$. It is easy to see that this action leaves the subsheaves $\mathcal{A}ut^{\mathcal{O}}_{(p)} \operatorname{gr} \mathcal{E}$ invariant and hence induces an action of $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr} \mathcal{E})$ on the cohomology set $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(p)} \operatorname{gr} \mathcal{E})$.

We have the exact sequence of sheaves of groups

$$\mathrm{id} \to \mathcal{A}ut^{\mathcal{O}}\mathcal{E} \to \mathcal{Q}\mathcal{A}ut\mathcal{E} \to \mathcal{A}ut\mathcal{O} \to \mathrm{id},$$

where the first homomorphism is the natural embedding (an automorphism of $\mathcal{A}ut^{\mathcal{O}}\mathcal{E}$ is regarded as an id-morphism) and the second one, say $F : \mathcal{Q}\mathcal{A}ut\mathcal{E} \to \mathcal{A}ut\mathcal{O}$, is defined by $\Phi_\Psi \mapsto \Psi$. Note that $F(\mathcal{Q}\mathcal{A}ut_{(p)(q)}\mathcal{E}) \subset \mathcal{A}ut_{(q)}\mathcal{O}$ and in the case $\mathcal{E} = \operatorname{gr} \mathcal{E}$ the restriction $F|\mathcal{Q}\mathcal{A}ut_0 \operatorname{gr} \mathcal{E}$ coincides with the natural projection

$$\mathcal{Q}\mathcal{A}ut_0(\mathcal{E}_{\mathrm{gr}}) \simeq \mathcal{A}ut_0 \operatorname{gr} \mathcal{O} \times \mathcal{A}ut^{\mathcal{F}}_{\bar{0}}(\mathcal{E}_{\mathrm{red}}) \to \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}$$

(see Lemma 1).

The homomorphism F commutes with the actions of $H^0(M, \mathcal{Q}\mathcal{A}ut_0 \operatorname{gr} \mathcal{E})$ and $H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O})$ on $\mathcal{Q}\mathcal{A}ut_{(p)(q)}(\operatorname{gr} \mathcal{E})$ and $\mathcal{A}ut_{(q)}(\operatorname{gr} \mathcal{O})$ respectively. More precisely,

$$F(a \circ \delta \circ a^{-1}) = F(a) \circ F(\delta) \circ F(a^{-1}),$$

where $a \in H^0(M, \mathcal{Q}\mathcal{A}ut_0 \operatorname{gr} \mathcal{E})$ and $\delta \in \mathcal{Q}\mathcal{A}ut \operatorname{gr} \mathcal{E}$. It follows that F induces the map of sets

$$\widetilde{F} : H^1(M, \mathcal{Q}\mathcal{A}ut_{(1)(2)} \operatorname{gr} \mathcal{E})/H^0(M, \mathcal{Q}\mathcal{A}ut_0 \operatorname{gr} \mathcal{E}) \to H^1(M, \mathcal{A}ut_{(2)} \operatorname{gr} \mathcal{O})/H^0(M, \mathcal{A}ut_0 \operatorname{gr} \mathcal{O}).$$

Let $\Phi_\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ be a Ψ-morphism of locally free sheaves of $\mathcal{O}$-modules. Since $\Psi(\mathcal{J}^p) \subset \mathcal{J}^p$, we see that $\Phi_\Psi((\mathcal{E}_1)_{(p)}) \subset (\mathcal{E}_2)_{(p)}$, $p \geq 0$. We denote by $\operatorname{gr}(\Phi_\Psi) : \operatorname{gr} \mathcal{E}_1 \to \operatorname{gr} \mathcal{E}_2$ the induced morphism. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote

$$[\mathcal{E}] = \{\mathcal{E}' \mid \mathcal{E}' \text{ is quasi-isomorphic to } \mathcal{E}\}.$$

Theorem 2. *Let $(M, \mathcal{O}_{\mathrm{gr}})$ be a split supermanifold, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules on M and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_{\mathrm{gr}} \otimes_{\mathcal{F}} \mathcal{S}$.*

1) We have a bijection

$$\{[\mathcal{E}] \mid \operatorname{gr} \mathcal{O} = \mathcal{O}_{\mathrm{gr}},\ \operatorname{gr} \mathcal{E} = \mathcal{E}_{\mathrm{gr}}\} \xleftrightarrow{1:1} H^1(M, \mathcal{Q}\mathcal{A}ut_{(1)(2)}\mathcal{E}_{\mathrm{gr}})/H^0(M, \mathcal{Q}\mathcal{A}ut_0\mathcal{E}_{\mathrm{gr}}).$$

The unit $\epsilon \in H^1(M, \mathcal{Q}\mathcal{A}ut_{(1)(2)}\mathcal{E}_{\mathrm{gr}})$ is fixed with respect to the action of the group $H^0(M, \mathcal{Q}\mathcal{A}ut_0\mathcal{E}_{\mathrm{gr}})$.

2) Let $a \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut_0\mathcal{O}_{\mathrm{gr}})$. Then there is a bijection between elements of the set $\widetilde{F}^{-1}(a)$ and classes of isomorphic locally free sheaves on supermanifolds which are contained in $[(M, \mathcal{O})]$.

Proof. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on $(M, \mathcal{O})$ and $\mathcal{U} = \{U_i\}$ be an open covering of M such that (1) and (3) are split over U_i and $\mathcal{E}|_{U_i}$ are free. In this case $(\mathrm{gr}\,\mathcal{E})|_{U_i}$ are free sheaves of $(\mathrm{gr}\,\mathcal{O})$-modules, too. We fix local bases $(\hat{e}^i_j)$ and $(\hat{f}^i_k)$ of the sheaves of $\mathcal{F}$-modules $(\mathcal{E}_{\mathrm{red}})_{\bar{0}}|_{U_i}$ and $(\mathcal{E}_{\mathrm{red}})_{\bar{1}}|_{U_i}$, $U_i \in \mathcal{U}$, respectively.

We are going to define an isomorphism $\delta_i : \mathcal{E}|_{U_i} \to (\mathrm{gr}\,\mathcal{E})|_{U_i}$. Let $e^i_j \in \mathcal{E}_{(0)\bar{0}}$ such that $\alpha(e^i_j) = \hat{e}^i_j$ and $f^i_k \in \mathcal{E}_{(0)\bar{1}}$ such that $\beta(f^i_k) = \hat{f}^i_k$. Then (e^i_j, f^i_k) is a local basis of $\mathcal{E}|_{U_i}$. A splitting of (1) determines local isomorphisms $\sigma_i : \mathcal{O}|_{U_i} \to \mathrm{gr}\,\mathcal{O}|_{U_i}$. We put

$$\delta_i(\sum h_j e^i_j + \sum g_k f^i_k) = \sum \sigma_i(h_j)\hat{e}^i_j + \sum \sigma_i(g_k)\hat{f}^i_k, \ \ h_j, g_k \in \mathcal{O}.$$

Obviously, δ_i is an isomorphism. We put $\gamma_{ij} := \sigma_i \circ \sigma_j^{-1}$ and $(g_{ij})_{\gamma_{ij}} := \delta_i \circ \delta_j^{-1}$. It is clear that $(\gamma_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}(\mathrm{gr}\,\mathcal{O}))$ and

$$((g_{ij})_{\gamma_{ij}}) \in Z^1(\mathcal{U}, \mathcal{QA}ut_{(1)(2)}(\mathrm{gr}\,\mathcal{E})).$$

Conversely, if $((g_{ij})_{\gamma_{ij}}) \in Z^1(\mathcal{U}, \mathcal{QA}ut_{(1)(2)}(\mathrm{gr}\,\mathcal{E}))$, we can construct a locally free sheaf of $\mathcal{O}$-modules on $(M, \mathcal{O}(\gamma_{ij}))$, where $(M, \mathcal{O}(\gamma_{ij}))$ is the supermanifold corresponding to the cocycle $(\gamma_{ij}) \in Z^1(\mathcal{U}, \mathcal{A}ut_{(2)}\,\mathrm{gr}\,\mathcal{O})$ by the Green Theorem. Indeed, we have to identify $\mathrm{gr}\,\mathcal{E}|_{U_i}$ with $\mathrm{gr}\,\mathcal{E}|_{U_j}$ over $U_i \cap U_j$ using $(g_{ij})_{\gamma_{ij}}$.

The standard calculation shows that if two cocycles $((g_{ij})_{\gamma_{ij}})$ and $((g'_{ij})_{\gamma'_{ij}})$ are cohomological, then the corresponding locally free sheaves of $\mathcal{O}$-modules are quasi-isomorphic and this quasi-isomorphism denoted by Φ_Ψ has the property $\mathrm{gr}(\Phi_\Psi) = \mathrm{id}_{\mathrm{id}}$. Conversely, if $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is a quasi-isomorphism of locally free sheaves of $\mathcal{O}$-modules such that $\mathrm{gr}(\Phi_\Psi) = \mathrm{id}_{\mathrm{id}}$, then the corresponding cocycles are cohomological.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two locally free sheaves of $\mathcal{O}$-modules on $(M, \mathcal{O})$ such that $\mathrm{gr}\,\mathcal{E} = \mathrm{gr}\,\mathcal{E}' = \mathcal{E}_{\mathrm{gr}}$. Assume that $\Phi_\Psi : \mathcal{E} \to \mathcal{E}'$ is an isomorphism. Then $\mathrm{gr}(\Phi_\Psi) \in H^0(M, \mathcal{QA}ut_0\,\mathrm{gr}\,\mathcal{E})$. Suppose that $\mathcal{E}$ corresponds to $(g_{ij})_{\gamma_{ij}} = \delta_i \circ \delta_j^{-1}$, where $\gamma_{ij} = \sigma_i \circ \sigma_j^{-1}$, and $\mathcal{E}'$ corresponds to $(g'_{ij})_{\gamma'_{ij}} = \delta'_i \circ (\delta'_j)^{-1}$, where $\gamma'_{ij} = \sigma'_i \circ (\sigma'_j)^{-1}$. There exist isomorphisms $(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} : \mathrm{gr}\,\mathcal{E}|_{U_i} \to \mathrm{gr}\,\mathcal{E}|_{U_i}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{gr}\,\mathcal{E}|_{U_i} & \xrightarrow{(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i}} & \mathrm{gr}\,\mathcal{E}|_{U_i} \\ {\scriptstyle \delta_i}\uparrow & & \uparrow{\scriptstyle \delta'_i} \\ \mathcal{E}|_{U_i} & \xrightarrow{\Phi_\Psi} & \mathcal{E}|_{U_i} \end{array}.$$

Since $\operatorname{gr}\delta_i = \operatorname{gr}\delta'_i$, it follows that $\operatorname{gr}((\widetilde{\Phi}_i)_{\widetilde{\Psi}_i}) = \operatorname{gr}(\Phi_\Psi)$ and hence

$$(\Theta_i)_{\Omega_i} := \operatorname{gr}(\Phi_\Psi)^{-1} \circ (\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \in \mathcal{QA}ut_{(1)(2)} \operatorname{gr}\mathcal{E}.$$

Further, we have

$$(g'_{ij})_{\gamma'_{ij}} = \delta'_i \circ (\delta'_j)^{-1} = (\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \circ \delta_i \circ (\Phi_\Psi)^{-1} \circ \Phi_\Psi \circ \delta_j^{-1} \circ ((\widetilde{\Phi}_j)_{\widetilde{\Psi}_j})^{-1} =$$
$$(\widetilde{\Phi}_i)_{\widetilde{\Psi}_i} \circ (g_{ij})_{\gamma_{ij}} \circ ((\widetilde{\Phi}_j)_{\widetilde{\Psi}_j})^{-1} = \operatorname{gr}(\Phi_\Psi) \circ (\Theta_i)_{\Omega_i} \circ (g_{ij})_{\gamma_{ij}} \circ (\Theta_j^{-1})_{\Omega_j^{-1}} \circ \operatorname{gr}(\Phi_\Psi)^{-1}.$$

Hence, the cohomology classes corresponding to $(g_{ij})_{\gamma_{ij}}$ and $(g'_{ij})_{\gamma'_{ij}}$ belong to the same orbit of the group $H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$.

Conversely, assume that $b \in H^0(M, \mathcal{QA}ut_0\mathcal{E}_{\mathrm{gr}})$ and $(g'_{ij})_{\gamma'_{ij}} = b \circ (g_{ij})_{\gamma_{ij}} \circ b^{-1}$. Then $\delta'_i \circ (\delta'_j)^{-1} = b \circ \delta_i \circ \delta_j^{-1} \circ b^{-1}$ and we can define the isomorphism $\Gamma : \mathcal{E} \to \mathcal{E}'$ by $\Gamma|_{U_i} := (\delta'_i)^{-1} \circ b \circ \delta_i$, where $\mathcal{E}$ and $\mathcal{E}'$ correspond to $(g_{ij})_{\gamma_{ij}}$ and $(g'_{ij})_{\gamma'_{ij}}$ respectively.

Let $a \in H^1(M, \mathcal{A}ut_{(2)}\mathcal{O}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut_0\mathcal{O}_{\mathrm{gr}})$. By Theorem 1 we may assign to each a the class of isomorphic supermanifolds $[(M, \mathcal{O})]$. From the proof of Theorem 2 it follows that there is a bijection between elements of the set $\widetilde{F}^{-1}(a)$ and classes of isomorphic locally free sheaves on supermanifolds which are contained in $[(M, \mathcal{O})]$.□

2.3 A classification theorem for locally free sheaves on a split supermanifold

Denote by $[\mathcal{E}]_{\mathrm{id}}$ the class of id-isomorphic (i.e., isomorphic) to $\mathcal{E}$ locally free sheaves of $\mathcal{O}$-modules on a split complex supermanifold $(M, \mathcal{O})$.

Theorem 3. *Let $(M, \mathcal{O})$ be a split supermanifold, $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded locally free sheaf of $\mathcal{F}$-modules on M and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O} \otimes_{\mathcal{F}} \mathcal{E}_{\mathrm{red}}$. Then*

$$\{[\mathcal{E}]_{\mathrm{id}} \mid \operatorname{gr}\mathcal{E} = \mathcal{E}_{\mathrm{gr}}\} \stackrel{1:1}{\longleftrightarrow} H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})/H^0(M, \mathcal{A}ut^{\mathcal{O}}_0\mathcal{E}_{\mathrm{gr}}).$$

Moreover, the unit $\epsilon \in H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})$ is a fixed point with respect to the action of the group $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0\mathcal{E}_{\mathrm{gr}})$.

Proof. Let us use the notations from the proof of Theorem 2. Since $(M, \mathcal{O})$ is split, we may assume that $\sigma_i = \sigma|_{U_i}$, where σ is determined by a global splitting of (1). It follows that the cocycle (g_{ij}) lies in $Z^1(\mathcal{U}, \mathcal{A}ut^{\mathcal{O}}_{(1)}\mathcal{E}_{\mathrm{gr}})$. The further proof is similar to the proof the Theorem 2. □

3. Locally free sheaves of modules on projective superspaces

In this subsection we will discuss two remarkable theorems about locally free sheaves on projective spaces, proved by Barth – Van de Ven – Tyurin and Birkhoff – Grothendieck, in the super-context.

3.1 Exact sequences corresponding to $\mathcal{A}ut^{\mathcal{O}}\mathcal{E}$

Let $(M, \mathcal{O})$ be a split complex supermanifold and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Denote by $\mathcal{E}nd^{\mathcal{O}}\mathcal{E}$ the sheaf of $\mathcal{O}$-endomorphisms of $\mathcal{E}$. This sheaf possesses the filtration

$$\mathcal{E}nd^{\mathcal{O}}\mathcal{E} = \mathcal{E}nd^{\mathcal{O}}_{(0)}\mathcal{E} \supset \mathcal{E}nd^{\mathcal{O}}_{(1)}\mathcal{E} \supset \dots,$$

$$\mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E} := \{A \in \mathcal{E}nd^{\mathcal{O}}\mathcal{E} \mid A(\mathcal{E}_{(q)}) \subset \mathcal{E}_{(q+p)} \text{ for all } q \geq 0\}.$$

The map

$$\exp : \mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E} \to \mathcal{A}ut^{\mathcal{O}}_{(p)}\mathcal{E},$$

given by the usual exp-series is a bijection of sheaves of sets for all $p \geq 1$ due to the fact that $\log = (\exp)^{-1}$ is well defined. In general it is not a homomorphism of sheaves of groups. We may define the map

$$\lambda_p : \mathcal{A}ut^{\mathcal{O}}_{(p)}\mathcal{E} \to \mathcal{E}nd^{\mathcal{O}}_{(p)}\mathcal{E}/\mathcal{E}nd^{\mathcal{O}}_{(p+1)}\mathcal{E},\ p \geq 1,$$

given by

$$a \mapsto A + \mathcal{E}nd^{\mathcal{O}}_{(p+1)},\ \text{where } a = \exp(A).$$

This map is surjective and $\operatorname{Ker}\lambda_p = \mathcal{A}ut^{\mathcal{O}}_{(p+1)}\mathcal{E}$. Clearly, it is a homomorphism of sheaves of groups. We will also consider the subsheaves of $\mathcal{E}nd^{\mathcal{O}}\operatorname{gr}\mathcal{E}$

$$\mathcal{E}nd^{\mathcal{O}}_{p}\operatorname{gr}\mathcal{E} := \{A \in \mathcal{E}nd^{\mathcal{O}}\operatorname{gr}\mathcal{E} \mid A(\operatorname{gr}\mathcal{E}_q) \subset \operatorname{gr}\mathcal{E}_{p+q}\},\ p \geq 0.$$

Then

$$\mathcal{E}nd^{\mathcal{O}}_{(p)}\operatorname{gr}\mathcal{E} = \bigoplus_{q \geq p} \mathcal{E}nd^{\mathcal{O}}_{q}\operatorname{gr}\mathcal{E}.$$

It follows that

$$\mathcal{E}nd^{\mathcal{O}}_{(p)}\operatorname{gr}\mathcal{E}/\mathcal{E}nd^{\mathcal{O}}_{(p+1)}\operatorname{gr}\mathcal{E} \simeq \mathcal{E}nd^{\mathcal{O}}_{p}\operatorname{gr}\mathcal{E}.$$

Hence, we get the exact sequence

$$0 \to \mathcal{A}ut^{\mathcal{O}}_{(p+1)}\operatorname{gr}\mathcal{E} \to \mathcal{A}ut^{\mathcal{O}}_{(p)}\operatorname{gr}\mathcal{E} \stackrel{\lambda_p}{\to} \mathcal{E}nd^{\mathcal{O}}_{p}\operatorname{gr}\mathcal{E} \to 0,\ p \geq 1. \tag{9}$$

The following lemma gives a description of the sheaf $\mathcal{E}nd_p\operatorname{gr}\mathcal{E}$, $p \geq 1$, in terms of the sheaves $\mathcal{O}$ and $\mathcal{E}_{\mathrm{red}}$.

Lemma 2. *We have*

$$\mathcal{E}nd^{\mathcal{O}}_{p}\operatorname{gr}\mathcal{E} \simeq \begin{cases} \mathcal{O}_p \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar 0} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 1} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar 1} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 0}), & p \text{ is odd;} \\ \mathcal{O}_p \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar 0} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 0} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar 1} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar 1}), & p \text{ is even.} \end{cases}$$

Proof. Firstly, note that an endomorphism $A \in \mathcal{E}nd_p(\operatorname{gr}\mathcal{E})$ is determined by its restriction $A|_{\operatorname{gr}\mathcal{E}_0}$. Secondly, $A|_{\operatorname{gr}\mathcal{E}_0} : \operatorname{gr}\mathcal{E}_0 \to \operatorname{gr}\mathcal{E}_p$ is an $\mathcal{F}$-linear map preserving parity (2). The result follows from the relation $\operatorname{gr}\mathcal{E}_q \simeq \operatorname{gr}\mathcal{O}_q \otimes \mathcal{E}_{\mathrm{red}}$. □

Now we can recover the following well-known result, see [9, 14]:

Proposition 1. *Let $(M, \mathcal{O})$ be a smooth supermanifold and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}$-modules on M. Then $\mathcal{E} \simeq \mathcal{O} \otimes_{\mathcal{F}} \mathcal{E}_{\mathrm{red}}$.*

Proof. Indeed, $(M, \mathcal{O})$ is split by the Batchelor Theorem. In this case

$$H^1(M, \mathcal{E}nd_p^{\mathcal{O}} \operatorname{gr}\mathcal{E}) = \{0\}$$

by Lemma 2. Hence

$$H^1(M, \mathcal{A}ut_{(1)}^{\mathcal{O}} \operatorname{gr}\mathcal{E}) = \{\epsilon\},$$

and our assertion follows from the Theorem 3.□

3.2 The Barth – Van de Ven – Tyurin Theorem for supermanifolds

Let us briefly recall the classical Barth – Van de Ven – Tyurin Theorem. Consider the sequence of complex projective spaces

$$\mathbb{CP}^1 \xrightarrow{\varphi_1} \mathbb{CP}^2 \xrightarrow{\varphi_2} \dots,$$

where φ_i are standard embeddings. (The inductive limit of this sequence is also called the *complex projective ind-space* $\mathbb{CP}^\infty$ (see [5, 17] and more detailed [7].) We consider collections $E = \{E_N\}_{N\geq 1}$ of holomorphic vector bundles E_N of a finite rank over $\mathbb{CP}^N$, $N \geq 1$, such that $\widetilde{\varphi}_N(E_{N+1}) = E_N$. (Such collections are also called *vector bundles over* $\mathbb{CP}^\infty$.) If $E = \{E_N\}_{N\geq 1}$ and $E' = \{E'_N\}_{N\geq 1}$ are two such collections, then the collection $E \oplus E' := \{E_N \oplus E'_N\}_{N\geq 1}$ is called the *direct sum* of E and E'. A *morphism of collections* $f : E \to E'$ is a set $\{f_N : E_N \to E'_N\}_{N\geq 1}$ of morphisms of vector bundles such that $\widetilde{\varphi}_N \circ f_{N+1} = f_N \circ \widetilde{\varphi}_N$. A morphism of two collections $f : E \to E'$ is called an *isomorphism* if it possesses the inverse morphism.

Theorem 4. [Barth – Van de Ven – Tyurin] *Any collection $E = \{E_N\}_{N\geq 1}$ of holomorphic vector bundles E_N of a finite rank over $\mathbb{CP}^N$ is isomorphic to a direct sum of collections $E^i = \{E^i_N\}_{N\geq 1}$ of vector bundles E^i_N of rank 1.*

For collections of rank 2 this result was proved by W. Barth and A. Van de Ven in [1], and for collections of an arbitrary finite rank by A. Tyuirin in [17].

The similar question may be considered in the case of complex supermanifolds. Recall that the *projective superspace* $(M, \mathcal{O}) = \mathbb{CP}^{n|m}$ of dimension

$n|m$ is a complex supermanifold with the reduction $M = \mathbb{CP}^n$ and the structure sheaf $\mathcal{O} = \bigwedge \mathcal{L}(-1)^m$, where $\mathcal{L}(-1)$ is the sheaf of $\mathcal{F}$-modules inverse to the sheaf $\mathcal{L}(1)$, which corresponds to a hyperplane in $\mathbb{CP}^n$. The classical homogeneous coordinates $z_0, ..., z_n$ on $\mathbb{CP}^n$ can be supplemented by odd homogeneous coordinates $\zeta_1, ..., \zeta_m$, giving rise to the system of homogeneous coordinates on $\mathbb{CP}^{n|m}$.

Let us consider the sequence of projective superspaces

$$\mathbb{CP}^{1|k_1} \xrightarrow{\varphi_1} \mathbb{CP}^{2|k_2} \xrightarrow{\varphi_2} \dots,$$

where $k_i \leq k_{i+1}$ and φ_i are standard embeddings, i.e any map $\varphi_i : \mathbb{CP}^{i|k_i} \to \mathbb{CP}^{i+1|k_{i+1}}$ is given in homogeneous coordinates $(z_j,\, \zeta_r)$ and $(z'_s,\, \zeta'_t)$ on $\mathbb{CP}^{i|k_i}$ and $\mathbb{CP}^{i+1|k_{i+1}}$ respectively by

$$\begin{gathered} z'_s = z_s,\ s = 1, \cdots, i,\ z_{i+1} = 0; \\ \zeta'_t = \zeta_t,\ t = 1, \cdots, k_i,\ \zeta'_t = 0,\ t = k_i + 1, \cdots, k_{i+1}. \end{gathered}$$

We study collections $\mathcal{E} = \{\mathcal{E}_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}_n$ of a finite rank over $\mathbb{CP}^{n|k_n}$, $n \geq 1$, such that $\widetilde{\varphi}_n(\mathcal{E}_{n+1}) = \mathcal{E}_n$. A morphism of two collections and their direct sum are defined similarly to the classical case. We are going to prove the following theorem:

Theorem 5. *Any collection $\mathcal{E} = \{\mathcal{E}_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}_n$ of a finite rank over $\mathbb{CP}^{n|k_n}$ is isomorphic to a direct sum of collections $\mathcal{E}^i = \{\mathcal{E}^i_n\}_{n\geq 1}$ of locally free sheaves $\mathcal{E}^i_n$ of rank $1|0$ or $0|1$.*

Proof. Note that $\mathcal{E}_{\rm red} = \{(\mathcal{E}_n)_{\rm red}\}$ is the collection of locally free sheaves such that $\widetilde{(\varphi_i)}_{\rm red}((\mathcal{E}_{i+1})_{\rm red}) = (\mathcal{E}_i)_{\rm red}$ and $(\varphi_i)_{\rm red} : \mathbb{CP}^i \to \mathbb{CP}^{i+1}$ are standard embeddings. By Theorem 4 we have $\mathcal{E}_{\rm red} \simeq \bigoplus_r \mathcal{S}^r$, where $\mathcal{S}^r = \{\mathcal{S}^r_n\}$ is a collection of locally free sheaves of rang 1 (and of super-rank $1|0$ or $0|1$). Hence the collection $\operatorname{gr}\mathcal{E} = \{\operatorname{gr}\mathcal{E}_n\}$, where we identify $\operatorname{gr}\mathcal{E}_n = \mathcal{O}_{\mathbb{CP}^n} \otimes (\mathcal{E}_n)_{\rm red}$, is isomorphic to the collection $\{\mathcal{O}_{\mathbb{CP}^n} \otimes \bigoplus_r \mathcal{S}^r_n\}$.

Our aim is to show that $\mathcal{E} \simeq \operatorname{gr}\mathcal{E}$. Using Lemma 2 and the well-known fact: $H^1(\mathbb{CP}^n, \mathcal{L}(r)) = \{0\}$ for $n > 1$ and any $r \in \mathbb{Z}$, we conclude that $H^1(\mathbb{CP}^n, \mathcal{E}nd^{\mathcal{O}}_p(\operatorname{gr}\mathcal{E}_n)) = \{0\}$ for $p \geq 1$ and $n > 1$. Hence, by the sequence (9) we get

$$H^1(\mathbb{CP}^n, \mathcal{A}ut^{\mathcal{O}}_{(1)}(\operatorname{gr}\mathcal{E}_n)) = \{\epsilon\} \text{ for } n > 1.$$

It follows by Theorem 3 that the following isomorphisms

$$f_n : \mathcal{E}_n \xrightarrow{\sim} \operatorname{gr}\mathcal{E}_n = \sum_r \mathcal{O}_{\mathbb{CP}^n} \otimes \mathcal{S}^r_n.$$

exist. Let us show that we can choose the isomorphisms f_n such that they commute with pullbacks of the bundles. Fix an isomorphism f_n. Let us construct an isomorphism

$$f'_{n+1} : \mathcal{E}_{n+1} \xrightarrow{\sim} \mathcal{O}_{\mathbb{CP}^{n+1}} \otimes (\mathcal{E}_{n+1})_{\mathrm{red}}$$

such that $\widetilde{\varphi}_n \circ f'_{n+1} = f_n \circ \widetilde{\varphi}_n$. Denote by $\mathcal{I}_n$ the sheaf of ideals corresponding to the subsupermanifold $\varphi_n : \mathbb{CP}^{n|k_n} \to \mathbb{CP}^{n+1|k_{n+1}}$. By definition we have

$$\begin{gathered} \mathcal{E}_n = \widetilde{\varphi}_n(\mathcal{E}_{n+1}) = \varphi^*_{\mathrm{red}}(\mathcal{E}_{n+1}/\mathcal{I}_n\mathcal{E}_{n+1}), \\ \operatorname{gr}\mathcal{E}_n = \widetilde{\varphi}_n(\operatorname{gr}\mathcal{E}_{n+1}) = \varphi^*_{\mathrm{red}}(\operatorname{gr}\mathcal{E}_{n+1}/\mathcal{I}_n \operatorname{gr}\mathcal{E}_{n+1}). \end{gathered}$$

Denote by $\mathcal{B}_n$ the sheaf of automorphisms of the sheaf of $\mathcal{O}_{\mathbb{CP}^{n+1}}/\mathcal{I}_n\mathcal{O}_{\mathbb{CP}^{n+1}}$-modules $\operatorname{gr}\mathcal{E}_{n+1}/\mathcal{I}_n \operatorname{gr}\mathcal{E}_{n+1}$ and by $(\mathcal{B}_n)_{(1)}$ the subsheaf of $\mathcal{B}_n$:

$$(\mathcal{B}_n)_{(1)} := \{a \in \mathcal{B}_n \mid a(v) = v \mod(\operatorname{gr}\mathcal{E}_{n+1}/\mathcal{I}_n \operatorname{gr}\mathcal{E}_{n+1})_{(1)}\},$$

where $(\operatorname{gr}\mathcal{E}_{n+1}/\mathcal{I}_n \operatorname{gr}\mathcal{E}_{n+1})_{(1)}$ is the image of $(\operatorname{gr}\mathcal{E}_{n+1})_{(1)}$ by the natural homomorphism. Note that we have $\sup((\mathcal{B}_n)_{(1)}) = \varphi_{\mathrm{red}}(\mathbb{CP}^n)$ and $\varphi^*_{\mathrm{red}}((\mathcal{B}_n)_{(1)}) = \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\operatorname{gr}\mathcal{E}_n)$.

Further, any automorphism from $\mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\operatorname{gr}\mathcal{E}_{n+1})$ preserves $\mathcal{I}_n \operatorname{gr}\mathcal{E}_{n+1}$. Hence, we have the map

$$F_n : \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\operatorname{gr}\mathcal{E}_{n+1}) \to (\mathcal{B}_n)_{(1)},$$

which is surjective as a sheaf homomorphism because we always can find locally preimage of elements from $(\mathcal{B}_n)_{(1)}$. Denote by $\mathcal{A}_n$ the kernel of F_n. Let us choose a Stein cover $\mathcal{U} = \{U_i\}$ of $\mathbb{CP}^{n+1}$ such that

$$0 \to \mathcal{A}_n(U_i) \to \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^{n+1}}}_{(1)}(\operatorname{gr}\mathcal{E}_{n+1})(U_i) \to (\mathcal{B}_n)_{(1)}(U_i) \to 0.$$

is exact for any i. Assume also that $\mathcal{U}$ satisfies conditions of the proof of Theorem 2. Denote by

$$(g^n_{ij}) \in H^1(\mathcal{U}, (\mathcal{B}_n)_{(1)}) \text{ and } (g^{n+1}_{ij}) \in H^1(\mathcal{U}, \mathcal{A}ut^{\mathcal{O}_{\mathbb{CP}^n}}_{(1)}(\operatorname{gr}\mathcal{E}_{n+1}))$$

the cocycles corresponding to $\mathcal{E}_n$ and $\mathcal{E}_{n+1}$ by Theorem 3. Recall that $g^n_{ij} = \delta^n_i \circ (\delta^n_j)^{-1}$, where $\delta^n_i : \mathcal{E}_n|_{U_i} \to \operatorname{gr}\mathcal{E}_n|_{U_i}$ is the isomorphism from Theorem 2 assuming is addition $\sigma_i = \mathrm{id}$ for any i. Similarly, $g^{n+1}_{ij} = \delta^{n+1}_i \circ (\delta^{n+1}_j)^{-1}$. Since $\widetilde{\varphi}(\mathcal{E}_{n+1}) = \mathcal{E}_n$, we may assume that $\widetilde{\varphi}^n \circ \delta^{n+1}_i|_{U_i} = \delta^n_i \circ \widetilde{\varphi}^n|_{U_i}$. Therefore, $F_n(g^{n+1}_{ij}) = g^n_{ij}$.

We have shown that $(g^n_{ij}) \sim \epsilon$ hence there are $\alpha^n_i \in \mathcal{B}_{(1)}(U_i)$ such that $(\alpha^n_i)^{-1} \circ g^n_{ij} \circ \alpha^n_j = \mathrm{id}$. Using the surjectivity of $F_n|_{U_i}$, we may choose $\alpha^{n+1}_i \in$

$F_n^{-1}(\alpha_i^n)$. Then $(h_{ij}) \in H^1(\mathcal{U}, \mathcal{A}_n)$, where $h_{ij} = (\alpha_i^{n+1})^{-1} \circ g_{ij}^{n+1} \circ \alpha_j^{n+1}$. It is easy to see that

$$\mathcal{A}_n = \exp(\begin{array}{l} (\mathcal{I}_n)_{\bar{0}} \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar{0}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{1}} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar{1}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{0}}) \oplus \\ (\mathcal{I}_n)_{\bar{1}} \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar{0}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{0}} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar{1}} \otimes (\mathcal{E}_{\mathrm{red}})^*_{\bar{1}}). \end{array}$$

Therefore, we get as for $\mathcal{A}ut_{(1)}^{\mathcal{O}_{\mathbb{CP}^n}}(\operatorname{gr}\mathcal{E}_n)$ that $H^1(\mathbb{CP}^{n+1}, \mathcal{A}_n) = \{\epsilon\}$. Therefore, there are $\beta_i \in \mathcal{A}_n(U_i)$ such that $h_{ij} = \beta_i \circ \beta_j^{-1}$. Denote

$$f'_{n+1}|_{U_i} := \beta_i^{-1} \circ (\alpha_i^{n+1})^{-1} \circ \delta_i^{n+1}.$$

By construction, we have $\widetilde{\varphi}_n \circ f'_{n+1} = f_n \circ \widetilde{\varphi}_n$. The proof is complete.□

3.3 About the Birkhoff – Grothendieck Theorem for supermanifolds.

In this subsection we will show that the Birkhoff – Grothendieck Theorem: *Any finite rank vector bundle on the complex projective space $\mathbb{CP}^1$ is isomorphic to a direct sum of line bundles*,

does not hold true for the projective superspace $\mathbb{CP}^{1|n}$, where $n \geq 1$. Denote by $\mathcal{O}_n$ the structure sheaf of $\mathbb{CP}^{1|n}$ and by i_n the standard embedding $\mathbb{CP}^{1|1} \to \mathbb{CP}^{1|n}$, $n \geq 1$. Clearly, there is a map $j_n : \mathbb{CP}^{1|n} \to \mathbb{CP}^{1|1}$, $n \geq 1$, such that $j_n^* : \mathcal{O}_1 \to \mathcal{O}_n$ is injective and $j_n \circ i_n = \mathrm{id}$. Let $\mathcal{E}_1$ be a locally free sheaf of $\mathcal{O}_1$-modules. Denote

$$\mathcal{E}_n := \mathcal{O}_n \otimes_{j_n^*(\mathcal{O}_1)} \mathcal{E}_1.$$

Then $\mathcal{E}_n$ is also locally free and $\mathcal{E}_n$ is an extension of $\mathcal{E}_1$. In other words, we have proved that any locally free sheaf on $\mathbb{CP}^{1|1}$ admits an extension to $\mathbb{CP}^{1|n}$. It follows that to prove our assertion it is enough to show that there exists a locally free sheaf of $\mathcal{O}_1$-modules of rank ≥ 2, which is not a direct sum of two lines bundles.

Let us study firstly line bundles on $\mathbb{CP}^{1|1}$. By (9) we get that $\mathcal{A}ut_{(1)}^{\mathcal{O}_1} \operatorname{gr}\mathcal{E} \simeq \mathcal{E}nd_1^{\mathcal{O}} \operatorname{gr}\mathcal{E}$ for any rank and from Lemma 2 it follows that $\mathcal{E}nd_1^{\mathcal{O}} \operatorname{gr}\mathcal{E} = \{0\}$ if $\operatorname{rank}\operatorname{gr}\mathcal{E} = 1|0$ or $0|1$. Therefore, by Theorem 3 any line bundle $\mathcal{E}$ is isomorphic to $\operatorname{gr}\mathcal{E}$.

Further, let $(\mathcal{E}_{\mathrm{red}})_{\bar{0}} = \mathcal{L}(0)$, $(\mathcal{E}_{\mathrm{red}})_{\bar{1}} = \mathcal{L}(-1)$ and $\mathcal{E}_{\mathrm{gr}} = \mathcal{O}_1 \otimes ((\mathcal{E}_{\mathrm{red}})_{\bar{0}} \oplus (\mathcal{E}_{\mathrm{red}})_{\bar{1}})$. Then

$$H^1(\mathbb{CP}^1, \mathcal{E}nd_1^{\mathcal{O}} \mathcal{E}_{\mathrm{gr}}) \simeq H^1(\mathbb{CP}^1, \mathcal{L}(-2)) \simeq \mathbb{C}.$$

Using the fact that the unit 1-cohomology class is a fixed point for the action of $H^0(\mathbb{CP}^1, \mathcal{A}ut_0^{\mathcal{O}_1}\mathcal{E}_{\mathrm{gr}})$ on $H^1(\mathbb{CP}^1, \mathcal{A}ut_{(1)}^{\mathcal{O}_1}\mathcal{E}_{\mathrm{gr}})$, we see that there is a locally free sheaf of $\mathcal{O}_1$-modules $\mathcal{E}$ such that $\operatorname{gr}\mathcal{E} = \mathcal{E}_{\mathrm{gr}}$ but $\mathcal{E}$ is not isomorphic to $\mathcal{E}_{\mathrm{gr}}$.

4. The tangent sheaf of a split supermanifold.

Let us recall some well-known facts about the tangent sheaf $\mathcal{T}$ of a split supermanifold $(M, \mathcal{O}) \simeq (M, \bigwedge \mathcal{G})$. First, the sheaf $\mathcal{T}$ is $\mathbb{Z}$-graded (not only $\mathbb{Z}_2$-graded):

$$\mathcal{T} = \bigoplus_{p \geq -1} \mathcal{T}_p,$$

where

$$\mathcal{T}_p := \{v \in \mathcal{T} \mid v(\mathcal{O}_q) \subset \mathcal{O}_{p+q} \text{ for all } q \geq 0\}, \ p \geq -1.$$

Second, the following sequence

$$0 \to \bigwedge^{p+1} \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{\delta} \mathcal{T}_p \xrightarrow{\gamma} \bigwedge^{p} \mathcal{G} \otimes \Theta \to 0, \ p \geq -1, \qquad (10)$$

where Θ is the tangent sheaf of M, is exact (see [12]). The mapping γ is the restriction of a derivation of degree p onto the subsheaf $\mathcal{F} \subset \mathcal{O}$ and δ identifies any sheaf homomorphism $\mathcal{G} \to \bigwedge^{p+1} \mathcal{G}$ with a derivation of degree p that is zero on $\mathcal{F}$.

Denote by $\mathbb{G}$ the vector bundle corresponding to $\mathcal{G}$. As usual by a *(holomorphic) connection* in a vector bundle $\mathbb{G} \to M$ over a complex manifold M, we mean a bilinear map

$$\nabla : \Theta \times \mathcal{G} \to \mathcal{G}$$

satisfying the following conditions:

- $\nabla_{fX} s = f \nabla_X s,$
- $\nabla_X (fs) = f \nabla_X s + X(f) s,$

where $f \in \mathcal{F}$, $X \in \Theta$ and $s \in \mathcal{G}$. If ∇ and ∇' are connections in $\mathbb{G} \to M$ and $\mathbb{G}' \to M$ respectively, the *tensor product connection* $\nabla \otimes \nabla'$ in $\mathbb{G} \otimes \mathbb{G}'$ is well defined. Recall that

$$(\nabla \otimes \nabla'_X)(s \otimes s') = \nabla_X(s) \otimes s' + s \otimes \nabla'_X(s').$$

It is easy to see that the tensor product connection $\nabla \otimes \cdots \otimes \nabla$ in $\mathbb{G} \otimes \cdots \otimes \mathbb{G}$ (p-times) induces the *wedge product connection* $\wedge^p \nabla$ in $\bigwedge^p \mathbb{G}$, $p > 0$.

Let ∇ be a connection on $\mathbb{G}$. Then to each $X \in \Theta$ we may assign a vector field Y_X on $(M, \mathcal{O}) \simeq (M, \bigwedge \mathcal{G})$ of degree 0 defined by

$$Y_X(f) = X(f), \ f \in \mathcal{F}, \quad Y_X(f) = \wedge^p \nabla(f), \ f \in \bigwedge^p \mathcal{G},$$

The Leibniz rule for Y_X follows from the definitions of a connection and a wedge product connection. Consider the sequence (10) for $p = 0$

$$0 \to \mathcal{G} \otimes \mathcal{G}^* \xrightarrow{\delta} \mathcal{T}_0 \xrightarrow{\gamma} \Theta \to 0. \tag{11}$$

We have just shown that the connection ∇ defines the splitting of (11) by $X \mapsto Y_X$. The converse statement is also true: if we have a splitting i of (11), we may define the connection ∇_i by

$$(\nabla_i)_X(s) := i(X)(s), \ s \in \mathcal{G}.$$

Note that the curvature tensor of $\nabla = \nabla_i$

$$R(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]} = ([i(X), i(Y)] - i([X, Y]))|_{\mathcal{G}}$$

measures the defection of i to be a homomorphism of sheaves of Lie algebras.

Theorem 6. *Let $(M, \mathcal{O}_M) \simeq (M, \bigwedge \mathcal{G})$ be a (holomorphic) split supermanifold and $\mathcal{T}$ the tangent sheaf. The following conditions are equivalent:*

1. *the sheaf $\mathcal{T}$ corresponds to the unit 1-cohomology class with values in $\mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T}$ by the Theorem 3;*
2. *the sequence (11) splits;*
3. *$\mathcal{G}$ possesses a (holomorphic) connection.*

Proof. By the discussion above we have to prove only that $\mathcal{T}$ corresponds to the trivial 1-cocycles of $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T})$ if and only if the sequence (11) splits. Let $\theta_0 : \Theta \to \mathcal{T}_0$ be a splitting of (11). Then the sequence (10) splits for all $p \geq 0$, we may define the splitting $\theta_p : \bigwedge^p \mathcal{G} \otimes \Theta \to \mathcal{T}_p$ by $\theta_p(f \otimes v) = f\theta_0(v)$. It follows that

$$\mathcal{T}_p \simeq \bigwedge^p \mathcal{G} \otimes \Theta + \bigwedge^{p+1} \mathcal{G} \otimes \mathcal{G}^*.$$

Hence,

$$\mathcal{T} \simeq \bigwedge \mathcal{G} \otimes (\mathcal{G}^* + \Theta) \simeq \bigwedge \mathcal{G} \otimes (\mathcal{T}_{\mathrm{red}}) = \operatorname{gr} \mathcal{T}.$$

Conversely, since the unit cocycle of $H^1(M, \mathcal{A}ut^{\mathcal{O}}_{(1)} \operatorname{gr} \mathcal{T})$ is a fixed point with respect to the action of $H^0(M, \mathcal{A}ut^{\mathcal{O}}_0 \operatorname{gr} \mathcal{T})$, there is an isomorphism

$\Phi : \mathcal{T} \to \operatorname{gr}\mathcal{T}$ such that $\operatorname{gr}\Phi = \mathrm{id}$ (see proof of Theorem 2). It follows that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{T}_{\bar 0} & \xrightarrow{\Phi|_{\mathcal{T}_{\bar 0}}} & (\operatorname{gr}\mathcal{T})_{\bar 0} \\ {\scriptstyle\pi}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pr}} \\ \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} & = & \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \end{array},$$

where pr is the projection of

$$\operatorname{gr}\mathcal{T} = \bigoplus_{p\geq 0}(\mathcal{J}^p\mathcal{T})_{\bar 0}/(\mathcal{J}^{p+1}\mathcal{T})_{\bar 0} + \bigoplus_{p\geq 0}(\mathcal{J}^p\mathcal{T})_{\bar 1}/(\mathcal{J}^{p+1}\mathcal{T})_{\bar 1}$$

onto $\mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0}$ and π is the natural projection. Further, by the definitions of all morphisms the following diagram is also commutative

$$\begin{array}{ccc} \mathcal{T}_{\bar 0} & \xrightarrow{\pi} & \mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \\ {\scriptstyle \mathrm{pr}_{\mathcal{T}_0}}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau} \\ \mathcal{T}_0 & \xrightarrow{\gamma} & \Theta \end{array},$$

where τ is an isomorphism defined by $v + (\mathcal{J}\mathcal{T})_{\bar 0} \mapsto \mathrm{pr}_{\mathcal{F}} \circ v|_{\mathcal{F}}$. Denote by i the natural embedding $\mathcal{T}_{\bar 0}/(\mathcal{J}\mathcal{T})_{\bar 0} \hookrightarrow (\operatorname{gr}\mathcal{T})_{\bar 0}$. We may define a splitting of (11) by $\mathrm{pr}_{\mathcal{T}_0} \circ (\Phi|_{\mathcal{T}_{\bar 0}})^{-1} \circ i \circ \tau^{-1}$. The proof is complete.$\square$

5. A spectral sequence

An important problem is to calculate the cohomology group $H^*(M, \mathcal{E})$ of a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on a supermanifold $(M, \mathcal{O})$. If $(M, \mathcal{O})$ is split, then $\mathcal{E}$ is a locally free sheaf of $\mathcal{F}$-modules on M, and its cohomology group can be calculated in many cases using the well elaborated tools of complex analytic geometry. In non-split case these methods cannot be applied directly, but we can use the associated split supermanifold $(M, \operatorname{gr}\mathcal{O})$ and the sheaf $\operatorname{gr}\mathcal{E}$.

5.1 Quasi-derivations.

Let $(M, \mathcal{O})$ be an arbitrary supermanifold and $\mathcal{E}$ a locally free sheaf on $(M, \mathcal{O})$. Let us take an even vector field $\Gamma \in \mathcal{T}_{\bar 0}(U)$ on a superdomain $(U, \mathcal{O}|U) \subset (M, \mathcal{O})$. A $\mathbb{Z}_2$-graded vector spaces sheaf homomorphism $A_\Gamma : \mathcal{E}|U \to \mathcal{E}|U$ is called a Γ-*derivation* if $A_\Gamma(fv) = \Gamma(f)v + fA_\Gamma(v)$, $f \in \mathcal{O}|U$ and $v \in \mathcal{E}|U$. A homomorphism of $\mathbb{Z}_2$-graded sheaf of vector spaces $B : \mathcal{E} \to \mathcal{E}$ will be called a *quasi-derivation* if it is a Γ-derivation for a certain

Γ. Denote by $\mathcal{QD}er\mathcal{E}$ the sheaf of quasi-derivations. It is a sheaf of Lie algebras with respect to the commutator $[A_\Gamma, B_\Upsilon] := A_\Gamma \circ B_\Upsilon - B_\Upsilon \circ A_\Gamma$. The sheaf $\mathcal{QD}er\mathcal{E}$ possesses the double filtration:

$$\mathcal{QD}er_{(p)(q)}\mathcal{E} := \{A_\Gamma \in \mathcal{QD}er\mathcal{E} \mid \; A_\Gamma(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)},\ \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q} \text{ for all } r, s \in \mathbb{Z}\}.$$

The map

$$\exp : \mathcal{QD}er_{(1)(2)}\mathcal{E} \to \mathcal{QA}ut_{(1)(2)}\mathcal{E}$$

is an isomorphism of sheaves of sets. Let us consider the subsheaf $\mathcal{QD}er_{k,k} \operatorname{gr} \mathcal{E}$ of $\mathcal{QD}er_{(k)(k)} \operatorname{gr} \mathcal{E}$ defined by

$$\begin{aligned} \mathcal{QD}er_{k,k} \operatorname{gr} \mathcal{E} := \; & \{A_\Gamma \in \mathcal{QD}er_{(k)(k)} \operatorname{gr} \mathcal{E} \mid A_\Gamma(\operatorname{gr} \mathcal{E}_r) \subset \operatorname{gr} \mathcal{E}_{r+k}, \\ & \Gamma(\operatorname{gr} \mathcal{O}_s) \subset \operatorname{gr} \mathcal{O}_{s+k} \text{ for all } r, s \in \mathbb{Z}\}. \end{aligned}$$

Note that $\mathcal{QD}er_{k,k} \operatorname{gr} \mathcal{E} = \mathcal{E}nd_k^{\operatorname{gr} \mathcal{O}} \operatorname{gr} \mathcal{E}$ if k is odd.

Denote by μ_k, $k \geq 1$, the following mapping:

$$\mu_k : \mathcal{QA}ut_{(k)(2)} \operatorname{gr} \mathcal{E} \to \mathcal{QD}er_{k,k} \operatorname{gr} \mathcal{E},$$

$$\mu_k(a_\gamma) = \bigoplus_q \operatorname{pr}_{q+k} \circ A_\Gamma \circ \operatorname{pr}_q,$$

where $a_\gamma = \exp(A_\Gamma)$ and $\operatorname{pr}_k : \operatorname{gr} \mathcal{E} \to \operatorname{gr} \mathcal{E}_k$ is the natural projection. The kernel of this map is $\mathcal{QA}ut_{(k+1)(2)} \operatorname{gr} \mathcal{E}$. Moreover, the following sequence

$$0 \to \mathcal{QA}ut_{(k+1)(2)} \operatorname{gr} \mathcal{E} \longrightarrow \mathcal{QA}ut_{(k)(2)} \operatorname{gr} \mathcal{E} \xrightarrow{\mu_k} \mathcal{QD}er_{k,k} \operatorname{gr} \mathcal{E} \to 0$$

is exact. Denoting by $H_{(k)}(\operatorname{gr} \mathcal{E})$ the image of the natural mapping

$$H^1(M, \mathcal{QA}ut_{(k)(2)} \operatorname{gr} \mathcal{E}) \to H^1(\mathcal{QA}ut_{(1)(2)} \operatorname{gr} \mathcal{E}),$$

we get the filtration:

$$H^1(M, \mathcal{QA}ut_{(1)(2)} \operatorname{gr} \mathcal{E}) = H_{(1)}(\operatorname{gr} \mathcal{E}) \supset H_{(2)}(\operatorname{gr} \mathcal{E}) \supset \dots.$$

Take $a_\gamma \in H_{(1)}(\operatorname{gr} \mathcal{E})$. We define the order of a_γ the maximal one of the numbers k such that $a_\gamma \in H_{(k)}(\operatorname{gr} \mathcal{E})$. The *order* of a locally free sheaf $\mathcal{E}$ of $\mathcal{O}$-modules on a supermanifold $(M, \mathcal{O}_M)$ is by definition the order of the corresponding cohomology class.

5.2 The spectral sequence.

Let $\mathcal{E}$ be a vector superbundle on a supermanifold $(M, \mathcal{O})$ of dimension $n|m$. Now we will construct a spectral sequence for the cohomology of the sheaf $\mathcal{E}$. We fix an open Stein cover $\mathfrak{U} = (U_i)_{i\in I}$ of M and consider the corresponding Čech cochain complex $C^*(\mathfrak{U},\mathcal{E}) = \bigoplus_{p\geq 0} C^p(\mathfrak{U},\mathcal{E})$.

The $\mathbb{Z}_2$-grading of $\mathcal{E}$ gives rise to the $\mathbb{Z}_2$-gradings in $C^*(\mathfrak{U},\mathcal{E})$ and $H^*(M,\mathcal{E})$ given by

$$\begin{aligned}
C_{\bar 0}(\mathfrak{U},\mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U},\mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U},\mathcal{E}_{\bar 1}),\\
C_{\bar 1}(\mathfrak{U},\mathcal{E}) &= \bigoplus_{q\geq 0} C^{2q}(\mathfrak{U},\mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} C^{2q+1}(\mathfrak{U},\mathcal{E}_{\bar 0}).\\
H_{\bar 0}(M,\mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M,\mathcal{E}_{\bar 0}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M,\mathcal{E}_{\bar 1}),\\
H_{\bar 1}(M,\mathcal{E}) &= \bigoplus_{q\geq 0} H^{2q}(M,\mathcal{E}_{\bar 1}) \oplus \bigoplus_{q\geq 0} H^{2q+1}(M,\mathcal{E}_{\bar 0}).
\end{aligned} \tag{12}$$

The filtration (4) for $\mathcal{E}$ gives rise to the filtration

$$C^*(\mathfrak{U},\mathcal{E}) = C_{(0)} \supset \ldots \supset C_{(p)} \supset \ldots \supset C_{(m+1)} = 0 \tag{13}$$

of this complex by the subcomplexes

$$C_{(p)} = C^*(\mathfrak{U},\mathcal{E}_{(p)}).$$

Denoting by $H(M,\mathcal{E})_{(p)}$ the image of the natural mapping $H^*(M,\mathcal{E}_{(p)}) \to H^*(M,\mathcal{E})$, we get the filtration

$$H^*(M,\mathcal{E}) = H(M,\mathcal{E})_{(0)} \supset \ldots \supset H(M,\mathcal{E})_{(p)} \supset \ldots. \tag{14}$$

Denote by $\operatorname{gr} H^*(M,\mathcal{E})$ the bigraded group associated with the filtration (14); its bigrading is given by

$$\operatorname{gr} H^*(M,\mathcal{E}) = \bigoplus_{p,q\geq 0} \operatorname{gr}_p H^q(M,\mathcal{E}).$$

By the general procedure, invented by Leray, the filtration (13) gives rise to a spectral sequence of bigraded groups E_r converging to $E_\infty \simeq \operatorname{gr} H^*(M,\mathcal{E})$. It is constructed in the following way.

For any $p, r \geq 0$, define the vector spaces

$$C_r^p = \{c \in C_{(p)} \mid dc \in C_{(p+r)}\}.$$

Then, for a fixed p, we have

$$C_{(p)} = C_0^p \supset \ldots \supset C_r^p \supset C_{r+1}^p \supset \ldots.$$

The r-th term of the spectral sequence is defined by

$$E_r = \bigoplus_{p=0}^{m} E_r^p, \ r \geq 0,$$

where

$$E_r^p = C_r^p / C_{r-1}^{p+1} + dC_{r-1}^{p-r+1}.$$

Since $d(C_r^p) \subset C_r^{p+r}$, d induces a derivation d_r of E_r of degree r such that $d_r^2 = 0$. Then E_{r+1} is naturally isomorphic to the homology algebra $H(E_r, d_r)$. Denoting $Z_r = \operatorname{Ker} d_r$, we have the natural mapping $\kappa_{r+1}^r : Z_r \to E^{r+1}$. For any $s > r$, denote $\kappa_s^r = \kappa_s^{s-1} \circ \ldots \circ \kappa_{r+1}^r$ (this composition is not defined on the entire Z_r).

The $\mathbb{Z}_2$-grading (12) in $C^*(\mathfrak{U}, \mathcal{E})$ gives rise to certain $\mathbb{Z}_2$-gradings in C_r^p and E_r^p, turning E_r into a superspace. Clearly, the coboundary operator d in $C^*(\mathfrak{U}, \mathcal{E})$ is odd. It follows that the coboundary d_r is odd for any $r \geq 0$.

The superspaces E_r are also endowed with a second $\mathbb{Z}$-grading. Namely, for any $q \in \mathbb{Z}$, set

$$\begin{aligned} C_r^{p,q} &= C_r^p \cap C^{p+q}(\mathfrak{U}, \mathcal{E}), \\ E_r^{p,q} &= C_r^{p,q} / C_{r-1}^{p+1,q-1} + dC_{r-1}^{p-r+1,q+r-2}. \end{aligned}$$

Then

$$E_r = \bigoplus_{p,q} E_r^{p,q}.$$

Clearly,

$$d_r(E_r^{p,q}) \subset E_r^{p+r,q-r+1} \tag{15}$$

for any r, p, q.

One sees easily that $C_r^{p,q} = 0$ for all p and r if $q \leq -(m+1)$. Therefore, for a fixed q, we have $d(C_r^{p,q}) = 0$ for all $r \geq q + m + 2$. This implies that $\kappa_{r+1}^r : E_r^{p,q} \to E_{r+1}^{p,q}$ is an isomorphism for all p and $r \geq r_0(q) = q + m + 2$. Setting $E_\infty^{p,q} = E_{r_0(q)}^{p,q}$, we get the bigraded superspace

$$E_\infty = \bigoplus_{p,q} E_\infty^{p,q}.$$

Now we prove certain properties of the spectral sequence (E_r). Some of them are well known and are valid in a more general situation.

Proposition 2. *The first two terms of the spectral sequence (E_r) can be identified with the following bigraded spaces:*

$$\begin{aligned} E_0 &= C^*(\mathfrak{U}, \operatorname{gr} \mathcal{E}), \\ E_1 &= H^*(M, \operatorname{gr} \mathcal{E}). \end{aligned}$$

Here

$$E_0^{p,q} = C^{p+q}(\mathfrak{U}, (\operatorname{gr}\mathcal{E})_p),$$
$$E_1^{p,q} = H^{p+q}(M, (\operatorname{gr}\mathcal{E})_p).$$

Proof. By definition, we have

$$E_0^p = C_{(p)}/C_{(p+1)},\ p \geq 0,$$

where the coboundary operator d_0 of degree 0 is induced by $d : C_{(p)} \to C_{(p)}$. On the other hand, the exact sequence

$$0 \to \mathcal{E}_{(p+1)} \to \mathcal{E}_{(p)} \to \operatorname{gr}\mathcal{E}_p \to 0$$

and Theorem B for Stein supermanifolds imply the exact sequence

$$0 \to \mathcal{E}_{(p+1)}(U) \to \mathcal{E}_{(p)}(U) \to \operatorname{gr}\mathcal{E}_p(U) \to 0$$

for any Stein open subset $U \subset M$. Therefore

$$C^*(\mathfrak{U}, (\operatorname{gr}\mathcal{E})_p) \simeq C_{(p)}/C_{(p+1)} = E_0^p,\ p \geq 0.$$

One sees easily that this is an isomorphism of complexes and that the resulting isomorphism $C^*(\mathfrak{U}, \operatorname{gr}\mathcal{E}) \simeq E_0$ is an isomorphism of bigraded spaces. It follows that

$$E_1 \simeq H(E_0, d_0) \simeq H^*(\mathfrak{U}, \operatorname{gr}\mathcal{E}) \simeq H^*(M, \operatorname{gr}\mathcal{E}).\square$$

Proposition 3. *There is the following identification of bigraded algebras:*

$$E_\infty = \operatorname{gr} H^*(M, \mathcal{E}),$$

where

$$E_\infty^{p,q} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).$$

Proof. Clearly, for $r \geq r_0(q)$ we have $C_r^{p,q} = Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)})$. It follows that

$$\begin{aligned} E_\infty^{p,q} &= Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)})/Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)}) + dC^{p+q-1}(\mathfrak{U}, \mathcal{E}) \cap Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p)}) \\ &= H^{p+q}(M, \mathcal{E})_{(p)}/(Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)})/dC^{p+q-1}(\mathfrak{U}, \mathcal{E}) \cap Z^{p+q}(\mathfrak{U}, \mathcal{E}_{(p+1)})) \\ &= H^{p+q}(M, \mathcal{E})_{(p)}/H^{p+q}(M, \mathcal{E})_{(p+1)} = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}).\square \end{aligned}$$

Corollary. *If M is compact, then*

$$\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E_\infty^{p,q}.$$

Proof. In fact, if M is compact, then all cohomology groups with values in a coherent analytic sheaf on $(M,\mathcal{O})$ or M are of finite dimension.$\square$

Now we prove our main result concerning the first non-zero coboundary operators among $d_1,\ d_2,\ldots$. We may suppose that for each $i\in I$ there exists an isomorphism of sheaves $\sigma_i:\mathcal{O}|U_i\to \operatorname{gr}\mathcal{O}|U_i$, inducing the identity isomorphism $\operatorname{gr}\mathcal{O}|U_i\to\operatorname{gr}\mathcal{O}|U_i$.

By Theorem 2, a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}\to(M,\mathcal{O})$ corresponds to the cohomology class a_γ of the 1-cocycle $((a_\gamma)_{ij})\in Z^1(\mathfrak{U},\mathcal{QA}ut_{(1)(2)}\operatorname{gr}\mathcal{E})$, where $(a_\gamma)_{ij}=\delta_i\circ\delta_j^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to k, then we may choose $\delta_i,\ i\in I$, in such a way that $((a_\gamma)_{ij})\in Z^1(\mathfrak{U},\mathcal{QA}ut_{(k)(2)}\operatorname{gr}\mathcal{E})$. We can write $a_\gamma=\exp A_\Gamma$, where $A_\Gamma\in C^1(\mathfrak{U},\mathcal{QD}er_{(1)(2)}\operatorname{gr}\mathcal{E})$.

We will identify the differential spaces (E_0,d_0) and $(C^*(\mathfrak{U},\operatorname{gr}\mathcal{E}),d)$ via the isomorphism of Proposition 2. Clearly, $\delta_i:\mathcal{E}_{(p)}|U_i\to\operatorname{gr}\mathcal{E}_{(p)}|U_i=\sum_{r\geq p}\operatorname{gr}\mathcal{E}_r|U_i$ is an isomorphism of sheaves for any $i\in I,\ p\geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups

$$\psi:C^*(\mathfrak{U},\mathcal{E})\to C^*(\mathfrak{U},\operatorname{gr}\mathcal{E})$$

such that

$$\psi:C^*(\mathfrak{U},\mathcal{E}_{(p)})\to C^*(\mathfrak{U},\operatorname{gr}\mathcal{E}_{(p)}),\ p\geq 0.$$

We give it by

$$\psi(c)_{i_0\ldots i_q}=\delta_{i_0}(c_{i_0\ldots i_q})$$

for any $(i_0,\ldots,i_q)$ such that $U_{i_0}\cap\ldots\cap U_{i_q}\neq\emptyset$. In general, ψ is not an isomorphism of complexes. Nevertheless, we can express explicitly the coboundary d of the complex $C^*(\mathfrak{U},\mathcal{E})$ by means of d_0 and a_γ.

Proposition 4. *For any $c\in C^q(\mathfrak{U},\operatorname{gr}\mathcal{E})=\oplus_p E_0^{q-p,p}$, we have*

$$(\psi(d\psi^{-1}(c)))_{i_0\ldots i_{q+1}}=(d_0c)_{i_0\ldots i_{q+1}}+((a_\gamma)_{i_0i_1}-\mathrm{id})(c_{i_1\ldots i_{q+1}}).$$

Proof. We can write

$$\begin{aligned}(d\psi^{-1}(c))_{i_0\ldots i_{q+1}}&=\sum_{\alpha=0}^{q+1}(-1)^\alpha\psi^{-1}(c)_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}}\\&=\sum_{\alpha=1}^{q+1}(-1)^\alpha\psi^{-1}(c)_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}}+\psi^{-1}(c)_{i_1\ldots i_{q+1}}\\&=\delta_{i_0}^{-1}(\sum_{\alpha=1}^{q+1}(-1)^\alpha c_{i_0\ldots\hat{i}_\alpha\ldots i_{q+1}})+\delta_{i_1}^{-1}(c_{i_1\ldots i_{q+1}})\\&=\delta_{i_0}^{-1}((d_0c)_{i_0\ldots i_{q+1}}-c_{i_1\ldots i_{q+1}})+\delta_{i_1}^{-1}(c_{i_1\ldots i_{q+1}}).\end{aligned}$$

Therefore

$$\begin{aligned}(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} &= \delta_{i_0}(d\psi^{-1}(c))_{i_0\dots i_{q+1}}\\ &= (d_0c)_{i_0\dots i_{q+1}} - c_{i_1\dots i_{q+1}} + (a_\gamma)_{i_0i_1}(c_{i_1\dots i_{q+1}})\\ &= (d_0c)_{i_0\dots i_{q+1}} + ((a_\gamma)_{i_0i_1} - \mathrm{id})(c_{i_1\dots i_{q+1}}).\end{aligned}$$

This implies our assertion.□

This proposition makes it possible to calculate the spectral sequence (E_r) whenever d_0 and the cochain a_γ are known. Now we find the explicit form of certain coboundary operators $d_r,\ r \geq 1$.

Theorem 7. *Suppose that the locally free sheaf of $\mathcal{O}$-modules $\mathcal{E} \to (M, \mathcal{O}_M)$ has order k and denote by a_γ the cohomology class corresponding to $\mathcal{E}$ by Theorem 2. Then $d_r = 0$ for $r = 1, \dots, k-1$, and $d_k = \mu_k(a_\gamma)$.*

Proof. Take a cocycle $c \in E_0^{p,q-p},\ d_0c = 0$, and denote by c^* its cohomology class in $E_1^{p,q-p}$. Clearly, c and c^* are represented by the cochain $\psi^{-1}(c) \in C_0^p$. By Proposition 4,

$$(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} = ((a_\gamma)_{i_0i_1} - \mathrm{id})(c_{i_1\dots i_{q+1}}).$$

Now we see that

$$(\psi(d\psi^{-1}(c)))_{i_0\dots i_{q+1}} = \mu_k(a_\gamma)_{i_0i_1}(c_{i_1\dots i_{q+1}}) + u_{i_0\dots i_{q+1}},$$

where $u \in C_{(p+k+1)}$. This means that

$$\psi(d\psi^{-1}(c)) = \mu_k(a_\gamma)(c) + u,$$

whence $d_1 = d_2 = \dots = d_{(k-1)} = 0$. Identifying E_k with E_1, we also see that d_kc^* is represented by the cochain $\psi^{-1}(\mu_k(a_\gamma)(c))$. It follows that

$$d_{2k}c^* = \mu_k(a_\gamma)(c^*).\square$$

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