ON ADDITIVE INVARIANTS OF ACTIONS OF ADDITIVE AND MULTIPLICATIVE GROUPS

WENCHUAN HU

Abstract. The additive invariants of an algebraic variety is calculated in terms of those of the fixed point set under the action of additive and multiplicative groups, by using Białynicki-Birula's fixed point formula for a projective algebraic set with a $\mathbb{G}_m$-action or $\mathbb{G}_a$-action.

The method is also generalized to calculate certain additive invariants for Chow varieties. As applications, we obtain the Hodge polynomial of Chow varieties in characteristic zero and the number of points for Chow varieties over finite fields.

As applications, we obtain the l-adic Euler-Poincaré characteristic for the Chow varieties of certain projective varieties over an algebraically closed field of arbitrary characteristic. Moreover, we show that the virtual Hodge $(p, 0)$ and $(0, q)$-numbers of the Chow varieties and affine group varieties are zero for all p, q positive.

Contents

1. Introduction

In this paper we generalize a method of Białynicki-Birula (cf. [B-B1]) in studying the fixed point schemes under actions of additive and multiplication group schemes and apply it to calculate additive invariants of projective varieties admitting one of these actions, especially to affine group varieties and Chow varieties.

Recall that an additive invariant λ on the category Var_K of algebraic varieties (a variety means a reduced and irreducible scheme) over a field K with values in a ring R, is a map

$$\lambda : Var_K \to R$$

such that

$$\begin{cases} \lambda(X) = \lambda(X') & \text{for } X \cong X', \\ \lambda(X) = \lambda(Y) + \lambda(X - Y) & \text{for } Y \text{ closed in } X, \\ \lambda(X \times Y) = \lambda(X) \cdot \lambda(Y) & \text{for every } X \text{ and } Y. \end{cases}$$

Examples of additive invariants includes the Euler characteristic, the l-adic Euler-Poincaré characteristic, the Hodge polynomial, counting points, etc. For more examples and details on additive invariants, the reader is referred to Loeser's lecture [Lo].

Our motivation comes from the computation of the Euler characteristic of the Chow variety of complex projective spaces by Lawson and Yau (cf. [LY]). More precisely, it is from the calculation of the Euler characteristic of the complex Chow variety $C_{p,d}(\mathbb{P}^n_{\mathbb{C}})$ (or simply $C_{p,d}(\mathbb{P}^n)$ if there is no confusion) parameterizing effective p-cycles of degree d in the complex projective space $\mathbb{P}^n$. The following formula was shown to hold:

Theorem 1.1 (The Lawson-Yau formula). *For all $n, p, d \geq 0$, one has*

$$\chi(C_{p,d}(\mathbb{P}^n)) = \binom{v_{p,n}+d-1}{d},$$

where $v_{p,n} = \binom{n+1}{p+1}$ and $\chi(M)$ is the Euler characteristic of M.

Lawson and Yau use a fixed point formula of a weakly holomorphic S^1-action in their computation for the Euler characteristic of Chow varieties. We observe that it would work nicely for other interesting additive invariants once we have corresponding fixed point formulas. The basic tool we will use in our proof is a mild generalized version of the following fixed point formula for l-adic Euler-Poincaré characteristic, as proved by Bialynicki-Birula.

Let X be a projective algebraic subset over a field K with a $\mathbb{G}_m$-action. Note that $\mathbb{G}_m \cong \operatorname{Spec} K[t, t^{-1}]$. That is, there is a morphism $\phi : \mathbb{G}_m \times X \to X$ such that $\phi(1, x) := x$ and $\phi(t_1 t_2, x) = \phi(t_1, \phi(t_2, x))$.

Theorem 1.2 (Białynicki-Birula, [B-B1]). *Let X be a projective algebraic subset over an algebraically closed field K with a $\mathbb{G}_m$-action. Then*

$$\chi(X, l) = \chi(F, l),$$

where F is the fixed point set of this action.

The first main result in this paper is the following statement.

Theorem 1.3 (Corollary 2.4). *Let $\lambda : Var_K \to R$ be an additive invariant satisfying $\lambda(\mathbb{G}_m) = 0$ (resp. $\lambda(\mathbb{G}_a) = 0$). Then $\lambda(X) = \lambda(X^{\mathbb{G}_m})$ (resp. $\lambda(X) = \lambda(X^{\mathbb{G}_a})$).*

Note that K is not required to be algebraically closed in this theorem.

By applying this to the Chow variety $C_{p,d}(\mathbb{P}^n)_K$ parametrizing effective p-cycles of degree d in the projective space $\mathbb{P}^n_K$ over an algebraically closed field K, we obtain the following result.

Theorem 1.4. *Let* $\lambda : Var_K \to R$ *be an additive invariant satisfying* $\lambda(\mathbb{G}_m) = 0$ *and* $\lambda(\mathrm{Spec}K) = 1$. *Then*

$$\lambda(C_{p,d}(\mathbb{P}^n)_K) = \binom{v_{p,n}+d-1}{d} \in R.$$

In particular, if $\lambda(-) = \chi(-, l)$ is the l-adic Euler-Poincaré characteristic and $R = \mathbb{Z}$, then we obtain the l-adic Euler-Poincaré characteristic for $C_{p,d}(\mathbb{P}^n)_K$.

Corollary 1.5.

(1) $$\chi(C_{p,d}(\mathbb{P}^n)_K, l) = \binom{v_{p,n}+d-1}{d}.$$

In particular, we obtain the following results on the virtual Hodge numbers and the virtual Betti numbers of the Chow variety $C_{m,d}(\mathbb{P}^n_K)$ parameterizing algebraic m-cycles of degree d in $\mathbb{P}^n_K$.

Theorem 1.6. *Let* K *be an algebraically closed subfield of* $\mathbb{C}$. *For integers* $n \geq m \geq 0$ *and* $d \geq 0$, *the virtual Hodge* $(p,0)$ *and* $(0,q)$-*numbers of the Chow variety* $C_{m,d}(\mathbb{P}^n_K)$ *are zero for all integers* $p, q > 0$. *Moreover, the virtual Hodge* (p,q)-*numbers* $\tilde{h}^{p,q}(C_{m,d}(\mathbb{P}^n_K))$ *of* $C_{m,d}(\mathbb{P}^n_K)$ *satisfies the following equation:*

$$\sum_{p-q=i} \tilde{h}^{p,q}(C_{m,d}(\mathbb{P}^n_K)) = 0$$

for all $i \neq 0$ *and*

$$\sum_{p \geq 0} \tilde{h}^{p,p}(C_{m,d}(\mathbb{P}^n_K)) = \chi(C_{m,d}(\mathbb{P}^n_K)).$$

The method in proving Theorem 1.4 is applied to obtain additive invariants for the Chow varieties of general toric varieties as well as the Chow varieties parametrizing irreducible cycles in the product of arbitrary many projective spaces.

Acknowledgement. I would like to thank professor V. Voevodsky for his interesting and helpful comments on an earlier version of the paper.

2. A GENERALIZATION OF BIAŁYNICKI-BIRULA'S METHOD TO ADDITIVE INVARIANTS

Let A be a fixed algebraic variety over a field K of arbitrary characteristic. An algebraic scheme Y is said to be simply equivalent to an algebraic scheme X if Y is isomorphic to a closed subscheme X' of X and there is an isomorphism $f : X - X' \to Z \times A$ for some algebraic scheme Z. The smallest equivalence relation containing the relation of simple A-equivalence is called the A-equivalence.

Lemma 2.1. *Let* X, Y *and* A *be algebraic varieties over* K *and let* $\lambda : Var_K \to R$ *be an additive invariant. Suppose that* X *is* A-*equivalent to* Y. *If* $\lambda(A) = 0 \in R$, *then* $\lambda(X) = \lambda(Y) \in R$.

Proof. It is enough to consider the case that X is simply A-equivalent to Y since λ is an additive invariant. By definition, there is an open quasi-projective scheme U of X such that $X - U$ is isomorphic to Y and an isomorphism $f : U \to U' \times A$, where U' is an algebraic scheme. In this case, we have $\lambda(X) = \lambda(Y) + \lambda(U) = \lambda(Y) + \lambda(U')\lambda(A) = \lambda(Y)$. This completes the proof of the lemma. $\square$

Theorem 2.2. *Let $G = \mathbb{G}_m$ and suppose that G acts on a reduced and irreducible algebraic scheme X. Then X is A-equivalent to X^G, where X^G denotes the fixed point set of the G-action and $A = \mathrm{Spec}(K[x, x^{-1}])$. Likewise, if X admits the action of the additive group $G = \mathbb{G}_a$ and $A = \mathrm{Spec}(K[x])$, then X is A-equivalent to X^G.*

Proof. In the following G is either $\mathbb{G}_a$ or $\mathbb{G}_m$. The case is clear if $X = X^G$. Suppose that $X \neq X^G$. We will show that there exists a G-invariant non-empty open subscheme U of X isomorphic to $Z \times A$, for some scheme Z. To see this, let U' be a non-empty open irreducible subscheme of X such that the quotient $\phi : U' \to U'/G$ exists. The te generic fiber F of ϕ is an algebraic scheme over the field of rational functions $K(U'/G) = K(U')^G$. Moreover, the fiber F with the action of G is homogeneous. Hence there exists a $K(U'/G)$-rational point in F and F is isomorphic to G/H for some algebraic group subscheme over $K(U')^G$ of G, where the action of G on G/H is induced by translations. By our assumption, the group scheme G/H is isomorphic to G. Hence X is birationally G-equivalent to some product $U'_1 \times G$. Therefore, X contains an open G-invariant subscheme U which is isomorphic to non-empty open subscheme of $U'_1 \times G$. Note that a G-invariant open subscheme of $U'_1 \times G$ is of the form $U_1 \times G$ for some open subscheme U_1 of U'_1. Thus X is A-equivalent to $X - U$. If $X - U = (X - U)^G$, then $X^G = X - U$ and so X is A-equivalent to X^G. Otherwise, $X - U \neq (X - U)^G$ then we repeat the above step where X is replaced by $X - U$. Since X is noetherian, we obtain a closed subscheme X_0 of X such that $X_0 = X_0^G$ and X_0 is A-equivalent to X. From the construction of X_0, we see that $X_0^G = X^G$. Therefore, X^G is A-equivalent to X. $\square$

Remark 2.3. *The proof above follows from Białynicki-Birula's argument, where the base field he considered is an algebraically closed field. However, the proof works for an arbitrary field.*

Corollary 2.4. *Let $\lambda : Var_K \to R$ be an additive invariant satisfying $\lambda(\mathbb{G}_m) = 0$ (resp. $\lambda(\mathbb{G}_a) = 0$). Then $\lambda(X) = \lambda(X^{\mathbb{G}_m})$ (resp. $\lambda(X) = \lambda(X^{\mathbb{G}_a})$).*

For additive invariants λ defined on Var_K to be interesting, we require that $\lambda(\mathrm{Spec}(K)) = 1$. In fact, it follows from the definition of additive invariants that $\lambda(\mathrm{Spec}(K))$ is either 0 or 1. Moreover, if $\lambda(\mathrm{Spec}(K)) = 0$, then it follows from the definition that $\lambda \equiv 0$. So we only consider non-trivial additive invariants λ below, i.e., $\lambda(\mathrm{Spec}(K)) = 1$.

Example 2.5. *Let $\lambda : Var_K \to \mathbb{Z}$ be an additive invariant such that $\lambda(\mathbb{G}_m) = 0$. Then for $X = X(\Delta)$ a (possible singular) toric variety associated to a fan Δ, we have $\lambda(X) = d_n(\Delta)$, where $d_n(\Delta)$ is the number of n-dimensional cones in Δ and n is the dimension of X.*

Proof. For $X = X(\Delta)$ an arbitrary toric variety, we write X as the disjoint union of its orbits O_τ under $\mathbb{G}_m^{\times n}$. Each orbit O_τ is isomorphic to $(\mathbb{G}_m)^{\times i}$. By assumption, $\lambda(\mathbb{G}_m) = 0$. This implies that $\lambda((\mathbb{G}_m)^{\times i}) = 0$ for $i > 0$. The number of 0-dimensional orbits is exactly the number of n-dimensional cones in Δ, i.e., $d_n(\Delta)$. $\square$

Note that the Euler characteristic of $X(\Delta)$ is also $d_n(\Delta)$ (cf. [Fu, Ch.3]). This is not surprising since the Euler characteristic χ is an additive invariant satisfying $\chi(\mathbb{G}_m) = 0$ (see the next section).

If a variety X admits a $\mathbb{G}_m$-action with isolated fixed points, then for any additive invariant $\lambda : Var_K \to \mathbb{Z}$ with $\lambda(\mathbb{G}_m) = 0$, $\lambda(X)$ coincides with the cardinality of the fixed point set. In particular, the fixed point set of $\mathbb{G}_m$-action on an algebraic torus can not have isolated fixed points.

A variety is called **cellular** if there is a filtration $\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_N = Y$ such that $Y_i - Y_{i-1}$ is isomorphic to $\mathbb{C}^{\mu_i}$ for all i (where $0 = \mu_0 \leq \mu_1 \leq \cdots$).

Example 2.6. *Let $\lambda : Var_K \to \mathbb{Z}$ be an additive invariant such that $\lambda(\mathbb{G}_m) = 0$. Then for a cellular variety Y as above, one has $\lambda(Y) = N$.*

Example 2.7. *(cf. [B-B1, Cor.5]) Let $\lambda : Var_K \to \mathbb{Z}$ be an additive invariant such that $\lambda(\mathbb{G}_m) = 0$. For an algebraic connected reduced affine group scheme G, one has $\lambda(G) = 0$ or 1. Moreover, $\lambda(G) = 1$ if and only if G is unipotent.*

Proof. If G is not unipotent , then it contains a subgroup isomorphic to $\mathbb{G}_m$. The action of $\mathbb{G}_m \cong H$ by left translations of G has no fixed point. By Corollary 2.4, we have $\lambda(G) = 0$. If G is unipotent, then G is isomorphic to K^n. Hence $\lambda(G) = 1$. $\square$

3. Examples of additive invariants

3.1. **Euler characteristic.** When K is a subfield of $\mathbb{C}$, the Euler characteristic is given by

$$\chi(X) := \sum_i (-1)^i \mathrm{rank} H^i(X(\mathbb{C}), \mathbb{C}).$$

For more general K and a variety X over K, let $H^i(X, \mathbb{Z}_l)$ be the l-adic cohomology group of X, where l is a positive integer prime to the characteristic $\mathrm{char}(K)$ of K. Set $H^i(X, \mathbb{Q}_l) := H^i(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Denote by $\beta^i(X, l) := \dim_{\mathbb{Q}_l} H^i(X, \mathbb{Q}_l)$ the i-th l-adic Betti number of X. The l-adic Euler characteristic is defined by

$$\chi(X, l) := \sum_i (-1)^i \beta^i(X, l).$$

Similarly, let $H_c^i(X, \mathbb{Z}_l)$ be the l-adic cohomology group of X with compact support. Set $\beta_c^i(X, l) := \dim_{\mathbb{Q}_l} H_c^i(X, \mathbb{Q}_l)$ the i-th l-adic Betti number of X with compact support and

$$\chi_c(X, l) := \sum_i (-1)^i \beta_c^i(X, l)$$

the l-adic Euler-Poincaré characteristic with compact support. Note that $\chi_c(X, l)$ is independent of the choice of l prime to $\mathrm{char}(K)$ (See, e.g., [K] or [I]).

Those $\chi, \chi_c, \chi(-, l)$ and $\chi_c(-, l)$ are additive invariants from Var_k to $\mathbb{Z}$, which follows from the fact that $\chi = \chi_c$ and $\chi(-, l) = \chi_c(-, l)$ (cf. [Fu] for the case over $\mathbb{C}$, [Lau] for general cases).

From Corollary 2.4 and note that both $\chi(\mathbb{G}_m)$ (in the case that k is a subfield of $\mathbb{C}$) and $\chi(\mathbb{G}_m, l)$ are zero. So one gets Białynicki-Birula's result.

Corollary 3.1 ([B-B1]). *Suppose that X admits a $\mathbb{G}_m$-action with the fixed point set $X^{\mathbb{G}_m}$. Then we have*

  (1) $\chi(X) = \chi(X^{\mathbb{G}_m})$ *if K is a subfield of $\mathbb{C}$.*
  (2) $\chi(X, l) = \chi(X^{\mathbb{G}_m}, l)$ *if $\mathrm{char}(K)$ is positive.*

3.2. **Hodge polynomials.** In this subsection, we assume that K is a field of characteristic zero. Then there is an additive invariant $H : Var_K \to \mathbb{Z}[u,v]$, with the properties:

(1) $H_X(u,v) := \sum_{p,q}(-1)^{p+q} \dim H^q(X, \Omega_X^p) u^p v^q$ if X is nonsingular and projective (or complete).
(2) $H_X(u,v) = H_U(u,v) + H_Y(u,v)$ if Y is a closed algebraic subset of X and $U = X - Y$.
(3) If $X = Y \times Z$, then $H_X(u,v) = H_Y(u,v) \cdot H_Z(u,v)$.

The existence and uniqueness of such a polynomial follow from Deligne's Mixed Hodge theory(cf. [D1, D2]). The coefficient of $u^p v^q$ of $H_X(u,v)$ is called the *virtual Hodge (p,q)-number* of X and we denote it by $\tilde{h}^{p,q}(X)$. Note that from the definition, $\tilde{h}^{p,q}(X)$ coincides with the usual Hodge number (p,q)-number $h^{p,q}(X)$ if X is a smooth projective variety. To apply the results in the last section, we need suitable modifications. Since $\mathbb{G}_m \cong \mathrm{Spec}(K[x, x^{-1}])$, we have $H_{\mathbb{G}_m}(u,v) = uv - 1 \neq 0$. So Corollary 2.4 can not be applied directly to the additive invariant H. However, if we take the values of H in the quotient $\mathbb{Z}[u,v]/\langle uv - 1 \rangle$, i.e., the composed map of H with the quotient homomorphism $\mathbb{Z}[u,v] \to \mathbb{Z}[u,v]/\langle uv - 1 \rangle$, then we get a new additive invariant $\widetilde{H} : Var_K \to \mathbb{Z}[u,v]/\langle uv - 1 \rangle \cong \mathbb{Z}[u, u^{-1}]$. This modified additive invariant $\widetilde{H}$ satisfies $\widetilde{H}(\mathbb{G}_m) = 0$. The following result is from Corollary 2.4.

Corollary 3.2. *Suppose that X admits a $\mathbb{G}_m$-action with the fixed point set $X^{\mathbb{G}_m}$. Then*

$$\widetilde{H}_X(u) = \widetilde{H}_{X^{\mathbb{G}_m}}(u) \in \mathbb{Z}[u, u^{-1}].$$

Equivalently, Corollary 3.2 can be written in a different way as the following:

$$(2) \qquad \sum_{p-q=i} \tilde{h}^{p,q}(X) = \sum_{p-q=i} \tilde{h}^{p,q}(X^{\mathbb{G}_m}).$$

In particular, if $X^{\mathbb{G}_m}$ is dimension zero over K, then $\widetilde{H}_{X^{\mathbb{G}_m}}(u)$ is independent of u and so is $\widetilde{H}_X(u)$. In this case, the virtual Hodge (p,q)-numbers of X satisfies the the following relation:

$$(3) \qquad \sum_{p-q=i} \tilde{h}^{p,q}(X) = 0$$

for all $i \neq 0$ and

$$\sum_{p \geq 0} \tilde{h}^{p,p}(X) = \chi(X).$$

More generally, Equation (3) holds for all $|i| > \dim X^{\mathbb{G}_m}$. In the case that X is smooth and projective over k, then $\tilde{h}^{p,q}(X) = h^{p,q}(X) \geq 0$. So we obtain from Equation (2) that $\sum_{p-q=i} h^{p,q}(X) = \sum_{p-q=i} h^{p,q}(X^{\mathbb{G}_m})$. In particular, $h^{p,q}(X) = \tilde{h}^{p,q}(X) = 0$ for all p,q such that $|p-q| > \dim(X^{\mathbb{G}_m})$. This is a pretty simpler proof of a slight weaker version of Białynicki-Birula decomposition theorem (cf. [B-B2]).

For example, if X is nonsingular toric projective variety over $\mathbb{C}$, then one has $h^{p,q}(X) = 0$ for all $p \neq q$ and $h^{p,p}(X) = \beta^{2p}(X)$. This follows from the fact that a nonsingular projective toric variety admits a $\mathbb{G}_m$-action with finite isolated fixed points.

Now we consider algebraic varieties admitting actions of the additive group $\mathbb{G}_a$. Since $\mathbb{G}_a \cong \mathrm{Spec}(K[x])$, we have $H_{\mathbb{G}_a}(u, v) = uv$. To apply Corollary 2.4, we need to take the value in $\mathbb{Z}[u, v]/\langle uv \rangle$. That is, the composed map of H with the quotient map $\mathbb{Z}[u, v] \to \mathbb{Z}[u, v]/\langle uv \rangle$ gives us an additive invariant $\overline{H}_X(u, v)$ such that $\overline{H}_{\mathbb{G}_a}(u, v) = 0$.

Corollary 3.3. *Suppose that X admits a $\mathbb{G}_a$-action with the fixed point set $X^{\mathbb{G}_a}$. Then*

$$\overline{H}_X(u, v) = \overline{H}_{X^{\mathbb{G}_a}}(u, v) \in \mathbb{Z}[u, v]/\langle uv \rangle.$$

Corollary 3.3 implies that the following equations

$$\tilde{h}^{p,0}(X) = \tilde{h}^{p,0}(X^{\mathbb{G}_m})$$

and

$$\tilde{h}^{0,q}(X) = \tilde{h}^{0,q}(X^{\mathbb{G}_m})$$

hold for $p, q \geq 0$.

In the case that X is smooth and projective with a $\mathbb{G}_a$-action, we have $h^{p,0}(X) = \tilde{h}^{p,0}(X) = 0$ and $h^{0,q}(X) = \tilde{h}^{0,q}(X) = 0$ for $p, q > \dim X^{\mathbb{G}_a}$.

By applying Corollary 3.3 to an algebraic connected affine group variety, we have the following result.

Corollary 3.4. *Let G be an algebraic connected affine group variety. Then*

$$\tilde{h}^{p,0}(G) = \tilde{h}^{0,q}(G) = 0$$

for all $p, q > 0$. In particular, the 1st virtual Betti number of G is zero.

Proof. If G is not a torus, then G contain a subgroup H isomorphic to $\mathbb{G}_a$. The action of $\mathbb{G}_a \cong H$ by left translations of G has no fixed point. By Corollary 3.3, we get $\tilde{h}^{p,0}(G) = \tilde{h}^{0,q}(G) = 0$ for all $p, q \geq 0$. If $G = \mathbb{G}_m^{\times n}$, then we get $H_G(u, v) = (uv - 1)^n$ since H is an additive invariant and $H_{\mathbb{G}_m}(u, v) = uv - 1$. From this formula we get immediately that $\tilde{h}^{p,0}(G) = \tilde{h}^{0,q}(G) = 0$ for all $p, q > 0$. $\qquad\square$

3.3. Counting points. Let $\mathbb{F}_q$ be the finite field of q elements and let X be an algebraic scheme defined over $\mathbb{F}_q$. Let $N_n(X)$ denote the number of closed points in $X(\mathbb{F}_{q^n})$. Note that N_n defines an additive invariant on the category $Var_{\mathbb{F}_q}$ of algebraic varieties over the field $\mathbb{F}_q$ with integer values, is a map

$$N_n : Var_{\mathbb{F}_q} \to \mathbb{Z}$$

such that

$$\begin{cases} N_n(X) = N_n(X') & \text{for } X \cong X', \\ N_n(X) = N_n(Y) + N_n(X - Y) & \text{for } Y \text{ closed in } X, \\ N_n(X \times Y) = N_n(X) \cdot N_n(Y) & \text{for every } X \text{ and } Y. \end{cases}$$

For example, if $X = A^1_{\mathbb{F}_q} = \mathrm{Spec}(\mathbb{F}_q[x])$ the affine line over $\mathbb{F}_q$, then $N_n(X) = q^n$; if $X = \mathrm{Spec}(\mathbb{F}_q[x, x^{-1}])$, then $N_n(X) = q^n - 1$.

Lemma 3.5. *Let X, Y and A be algebraic varieties over $\mathbb{F}_q$. Suppose that X is A-equivalent to Y. Then*

   (1) *if $A = \mathrm{Spec}(\mathbb{F}_q[x])$ then $N_n(X) \equiv N_n(Y) \bmod(q)$.*
   (2) *if $A = \mathrm{Spec}(\mathbb{F}_q[x, x^{-1}])$ then $N_n(X) \equiv N_n(Y) \bmod (q - 1)$.*

Proof. It is enough to consider the case that X is simply A-equivalent to Y since N_n is an additive invariant. By definition, there is an open quasi-projective scheme U of X such that $X - U$ is isomorphic to Y and an isomorphism $f : U \to U' \times A$, where U' is an algebraic scheme. In this case, we have $N_n(X) = N_n(Y) + N_n(U) = N_n(Y) + N_n(U')N_n(A)$.

Case (1). If $A = \text{Spec}(\mathbb{F}_q[x])$, then $N_n(A) = q^n \equiv 0 \ mod(q)$. Hence $N_n(X) \equiv N_n(Y) \ \text{mod}(q)$.

Case (2). If $A = \text{Spec}(\mathbb{F}_q[x, x^{-1}])$, then $N_n(A) = q^n - 1 \equiv 0 \ mod(q-1)$. Hence $N_n(X) \equiv N_n(Y) \ \text{mod}(q-1)$.

This completes the proof of the lemma. $\square$

As an application of Theorem 2.2 and Lemma 3.5, we have the following result.

Corollary 3.6. *Suppose that an algebraic scheme X admits G-action.*
  (1) *If $G = \mathbb{G}_a$, then $N_n(X) \equiv N_n(X^G) \ \text{mod}(q)$.*
  (2) *If $G = \mathbb{G}_m$, then $N_n(X) \equiv N_n(X^G) \ \text{mod}(q-1)$.*

Proof. This follows from the combination of Theorem 2.2 and Lemma 3.5. $\square$

4. Additive invariants for Chow varieties

4.1. The Chow variety for projective spaces. In this section we give a direct proof of Corollary 1.5 and Theorem 1.4, which not only is a simplification of Lawson and Yau's proof for Theorem 1.1 but also works for Chow varieties over arbitrary algebraically closed field.

Now we give a proof of Corollary 1.5 by using Białynicki-Birula's result.

The proof of Corollary 1.5. We consider the action of $\mathbb{G}_m$ on $\mathbb{P}_K^{n+1}$ given by setting

$$\Phi_t([z_0, ..., z_n, z_{n+1}]) = [z_0, ..., z_n, tz_{n+1}],$$

where $t \in \mathbb{G}_m$ and $[z_0, ..., z_n, z_{n+1}]$ are homogeneous coordinates for $\mathbb{P}_K^{n+1}$.

This action on $\mathbb{P}_K^{n+1}$ induces an action of $\mathbb{G}_m$ on $C_{p+1,d}(\mathbb{P}^{n+1})_K$. From the definition of the action $\mathbb{G}_m$ on $\mathbb{P}_K^{n+1}$, it is pretty clear that any subvariety V of $\dim V = p + 1$ is invariant under the action $\mathbb{G}_m$ if the support of V is included in the hyperplane $(z_{n+1} = 0) \cong \mathbb{P}_K^n$.

We also observe that if a $(p + 1)$-dimensional irreducible algebraic variety V is defined by a collection of homogeneous polynomials F_λ on $\mathbb{P}_K^{n+1}$, but those polynomials are independent of the last coordinate z_{n+1}, then V is invariant under $\mathbb{G}_m$. Geometrically, such a variety V is a cone of over an algebraic subvariety supported in the hyperplane $(z_{n+1} = 0)$.

Denote $Q = [0 : \cdots : 0 : 1] \in \mathbb{P}_K^{n+1}$ and note that Q is $\mathbb{G}_m$-fixed. Note that only those varieties are irreducible invariant subvarieties of dimension $p+1$ in $\mathbb{P}_K^{n+1}$ under this $\mathbb{G}_m$-action. To see this, we first observe from the definition of the action that if an irreducible variety V contains Q and another fixed point P on $\mathbb{P}_K^n$, then so does the projective line l_{PQ} passing P and Q. Suppose $V \subseteq \mathbb{P}_K^{n+1}$ such that $V \nsubseteq (z_{n+1} = 0) \cong \mathbb{P}_K^n$. Since both V and $\mathbb{P}_K^n$ are $\mathbb{G}_m$-invariant, $V' := V \cap \mathbb{P}_K^n$ is $\mathbb{G}_m$-invariant. The subvariety V corresponds to the fixed point set of the restriction of the $\mathbb{G}_m$-action on V when $t \to 0$. Therefore, the cone $\Sigma_Q V'$ is $\mathbb{G}_m$-invariant. Note that we must have $Q \in V$. The point Q corresponds to the fixed point set of the restriction of the $\mathbb{G}_m$-action on V when $t \to \infty$. Hence we have $\Sigma_Q V' \subseteq V$. Since $\dim \Sigma_Q V' = p + 1 = \dim V$ and V is irreducible, we have $\Sigma_Q V' = V$.

The fixed point set $C_{p+1,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m}$ of the induced action on $C_{p+1,d}(\mathbb{P}^{n+1})_K$ contains cycles c of the form $c = \sum n_k V_k + \sum m_j W_j$ of degree $\deg c := \sum n_k \deg V_k + \sum m_j \deg W_j = d$, where $V_k \subset \mathbb{P}_K^n$ is irreducible and $W_j = \Sigma_Q W_j'$ for some irreducible variety $W_j \subset \mathbb{P}_K^n$ of $\dim W_j' = p$. Therefore, we have

$$(4) \qquad C_{p+1,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m} = \coprod_{i=0}^{d} \{ C_{p+1,i}(\mathbb{P}^n)_K \times \Sigma_Q C_{p,d-i}(\mathbb{P}^n)_K \}.$$

Since $\Sigma : C_{p,d-i}(\mathbb{P}^n)_K \to C_{p,d-i}(\mathbb{P}^{n+1})_K$ induces a homeomorphism onto its image in $C_{p,d-i}(\mathbb{P}^{n+1})_K$, we have

$$(5) \qquad \begin{aligned} \chi(C_{p+1,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m}, l) &= \chi(\coprod_{i=0}^d \{ C_{p+1,i}(\mathbb{P}^n)_K \times \Sigma_Q C_{p,d-i}(\mathbb{P}^n)_K \}, l) \\ &= \sum_{i=0}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)_K \times \Sigma_Q C_{p,d-i}(\mathbb{P}^n)_K, l) \\ &= \sum_{i=0}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)_K, l) \cdot \chi(\Sigma_Q C_{p,d-i}(\mathbb{P}^n)_K, l) \\ &= \sum_{i=0}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)_K, l) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)_K, l), \end{aligned}$$

where the second equality follows from the exclusion-inclusion principle of the Euler-Poincaré characteristic (cf. [Lau], [H]), the third equality follows from the Künneth formula for l-adic cohomology.

From Theorem 1.2, we have

$$(6) \qquad \chi(C_{p+1,d}(\mathbb{P}^{n+1})_K, l) = \chi(C_{p+1,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m}, l).$$

The combination of Equation (5) and (6) gives us a recursive formula

$$(7) \qquad \chi(C_{p+1,d}(\mathbb{P}^{n+1})_K, l) = \sum_{i=0}^{d} \chi(C_{p+1,i}(\mathbb{P}^n)_K, l) \cdot \chi(C_{p,d-i}(\mathbb{P}^n)_K, l).$$

The above idea also can be used to calculate the initial values $\chi(C_{0,d}(\mathbb{P}^n)_K, l)$ as follows. By definition, an element in $C_{0,d}(\mathbb{P}^{n+1})_K$ means an effective cycle c on $\mathbb{P}_K^{n+1}$ such that $\deg c = d$. Since a point P is a fixed point of the $\mathbb{G}_m$-action if and only if $P = Q$ or $P \in (z_{n+1} = 0) \cong \mathbb{P}_K^n$, we get $c \in C_{0,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m}$ if and only if $c = mQ + \sum n_i P_i$, where $n_i \geq 0$ and $\sum n_i = d - m$. Hence

$$(8) \qquad C_{0,d}(\mathbb{P}^{n+1})_K^{\mathbb{G}_m} = \coprod_{m=0}^{d} C_{0,d-m}(\mathbb{P}^n)_K.$$

This together with Theorem 1.2 implies the following formula for the Euler-Poincaré characteristics.

$$\chi(C_{0,d}(\mathbb{P}^{n+1})_K, l) = \sum_{m=0}^{d} \chi(C_{0,d-m}(\mathbb{P}^n)_K, l).$$

From this recursive formula, we get

$$(9) \qquad \chi(C_{0,d}(\mathbb{P}^n)_K, l) = \binom{n+d}{d}.$$

The combination of Equation (7) and (9) completes the alternate proof of Corollary 1.5. $\qquad \square$

If we set

$$Q_{p,n}(t) := \sum_{d=0}^{\infty} \chi(C_{p,d}(\mathbb{P}^n)_K, l) t^d,$$

then Corollary 1.5 may be restated as

$$Q_{p,n}(t) = \left(\frac{1}{1-t}\right)^{\binom{n+1}{p+1}}, \quad \text{where } \chi(C_{p,0}(\mathbb{P}^n)_K) := 1.$$

The proof of Theorem 1.4. Note that Bialynicki-Birula's result in [B-B1, Th.2] and the assumption $\lambda(\mathbb{G}_m) = 0$ imply that if X is a projective algebrac set over K with a $\mathbb{G}_m$-action, then

$$\lambda(X) = \lambda(X^{\mathbb{G}_m}),$$

where $X^{\mathbb{G}_m}$ is the fixed point set of this action.

This together with Equation (4) gives us the following recursive formula

$$(10) \qquad \lambda(C_{p+1,d}(\mathbb{P}^{n+1})_K) = \sum_{i=0}^{d} \lambda(C_{p+1,i}(\mathbb{P}^n)_K) \cdot \lambda(C_{p,d-i}(\mathbb{P}^n)_K).$$

By Equation (8) we get

$$\lambda(C_{0,d}(\mathbb{P}^{n+1})_K) = \sum_{m=0}^{d} \lambda(C_{0,d-m}(\mathbb{P}^n)_K).$$

This recursive formula together with the assumption $\lambda(\mathrm{Spec}K) = 1$ implies that

$$(11) \qquad\qquad\qquad \lambda(C_{0,d}(\mathbb{P}^n)_K) = \binom{n+d}{d}.$$

By the same argument as in the proof of Corollary 1.5, we obtain the formula in the theorem from Equation (10) and (11).

$$\square$$

In the subsection below we will compute the virtual Hodge polynomial and numbers of the Chow varieties for projective spaces over an algebraically closed subfield of $\mathbb{C}$.

Corollary 4.1. *Assume that* $\mathrm{char}(K) = 0$ *and let* $\widetilde{H} : Var_K \to \mathbb{Z}[u, u^{-1}]$ *be given as above. Then we have*

$$\widetilde{H}(C_{p,d}(\mathbb{P}^n)_K) = \binom{v_{p,n}+d-1}{d} \in \mathbb{Z}[u, u^{-1}].$$

Proof. It is easy to check from the definition of $\widetilde{H}$ that $\widetilde{H}(\mathrm{Spec}K) = 1$ and $H(\mathbb{G}_m) = uv - 1$. So $\widetilde{H}(\mathbb{G}_m) = 0$. Now the corollary follows from Theorem 1.4. $\square$

Remark 4.2. *From Corollary 4.1,* $\widetilde{H}(C_{p,d}(\mathbb{P}^n)_K)$ *is independent of* u*. This implies that the coefficients of both* u *and* v *in the Hodge polynomial* $H(C_{p,d}(\mathbb{P}^n)_K)$ *vanish.*

By applying Corollary 3.2 to the Chow variety $C_{m,d}(\mathbb{P}^n_K)$ parameterizing algebraic m-cycles of degree d in the projective space $\mathbb{P}^n_K$, we have the following result.

Corollary 4.3. *For integers* $n \geq m \geq 0$ *and* $d \geq 0$*, the virtual Hodge* (p, q)*-number of the Chow variety* $C_{m,d}(\mathbb{P}^n_K)$ *satisfies the following equations:*

$$\sum_{p-q=i} \tilde{h}^{p,q}(C_{m,d}(\mathbb{P}^n_K)) = 0$$

for all $i \neq 0$ *and*

$$\sum_{p \geq 0} \tilde{h}^{p,p}(C_{m,d}(\mathbb{P}^n_K)) = \chi(C_{m,d}(\mathbb{P}^n_K)).$$

Proof. Instead of proving that there is a $\mathbb{G}_m$-action on $C_{p,d}(\mathbb{P}_K^n)$ such that the fixed point set consists of isolated points, we show that there is a $(\mathbb{G}_m)^{\times n}$-action on $C_{p,d}(\mathbb{P}_K^n)$ such that whose fixed point set is finite. This is shown in the proof of Corollary 1.5 by constructing a sequence of $\mathbb{G}_m$-action on $C_{p,d}(\mathbb{P}_K^n)$ and its fixed points set. The initial idea for such a construction over the complex number field is from Lawson and Yau (cf. [LY]). Now the equation follows from Equation (12). $\square$

By applying Corollary 3.3 to the Chow variety $C_{m,d}(\mathbb{P}_K^n)$, we get the following result.

Corollary 4.4. *For integers $n \geq m \geq 0$, $d \geq 0$ and $p, q > 0$, the virtual Hodge $(p,0)$-number and $(0,q)$-number of the Chow variety $C_{m,d}(\mathbb{P}_K^n)$ vanish.*

Proof. Note that $C_{m,d}(\mathbb{P}_K^n)$ admits an action of a unitriangular group J with exact one fixed point (cf. [Ho, §7]). The general fact is that the group J contains a subgroup isomorphic to $\mathbb{G}_a$. Therefore, $\overline{H}_{C_{m,d}(\mathbb{P}_K^n)}(u,v) = 1 \in \mathbb{Z}[u,v]/\langle uv \rangle$. This implies that
$$H_{C_{m,d}(\mathbb{P}_K^n)}(u,v) = 1 + uvg(u,v) \in \mathbb{Z}[u,v],$$
where $g(u,v) \in \mathbb{Z}[u,v]$. Since $\tilde{h}_{C_{m,d}(\mathbb{P}_K^n)}^{p,0}$ is the coefficient of u^p in the Hodge polynomial $H_{C_{m,d}(\mathbb{P}_K^n)}(u,v)$, we obtain $\tilde{h}_{C_{m,d}(\mathbb{P}_K^n)}^{p,0} = 0$ from the above explicit formula for $H_{C_{m,d}(\mathbb{P}_K^n)}(u,v)$. By the same reason, $\tilde{h}_{C_{m,d}(\mathbb{P}_K^n)}^{0,q} = 0$. $\square$

Note that for an algebraic variety over k, if we set $\tilde{\beta}^i(X) := \sum_{p+q=i} \tilde{h}^{p,q}(X)$, then we get the virtual Poincaré polynomial $\widetilde{P}_X(t) = \sum_i \tilde{\beta}^i(X)t^i$ (cf. [Fu, p.92]) and $\tilde{\beta}^i(X)$ is called the ith *virtual Betti number* of X.

From Corollary 4.4, we get $\tilde{\beta}^1(C_{m,d}(\mathbb{P}_K^n)) = 0$. It can not be obtained by the vanishing of the usual 1st Betti number of $C_{m,d}(\mathbb{P}_{\mathbb{C}}^n)$.

From Corollary 4.3, we have

(12)
$$\sum_{i \geq 1} \tilde{\beta}^{2i-1}(C_{m,d}(\mathbb{P}_K^n)) = 0$$

and

$$\sum_{i \geq 0} \tilde{\beta}^{2i}(C_{m,d}(\mathbb{P}_K^n)) = \chi(C_{m,d}(\mathbb{P}_K^n)).$$

In particular, if one could verify that $\tilde{\beta}^{2i-1}(C_{m,d}(\mathbb{P}_K^n))$ are nonnegative for all i, then Equation (12) would imply that the vanishing of all odd virtual Betti numbers.

Remark 4.5. *Note that the usual 1st Betti number of $C_{m,d}(\mathbb{P}_{\mathbb{C}}^n)$ is zero. This is implied by the fact that $C_{m,d}(\mathbb{P}_{\mathbb{C}}^n)$ is simply connected (cf. [Ho] or [Law, Lemma 2.6]). However, The simply connectedness of an algebraically closed set X does **not** imply the vanishing of the 1st Betti number of X. For example, let $X = C(E) \cup \mathbb{P}_{\mathbb{C}}^2$, where $C(E)$ is a projective cone of a smooth plane cubic E in the plane $\mathbb{P}_{\mathbb{C}}^2$ and $E = C(E) \cap \mathbb{P}_{\mathbb{C}}^2$. The fact that $\pi_1(X) = 0$ follows from a direct application of Van Kampen theorem. An elementary calculation yields $\widetilde{H}_X(u,v) = 1 + u + v + uv - u^2v - uv^2 + 2u^2v^2$ and so $\tilde{\beta}^1(X) = 2 \neq 0$.*

The next result, as an application of the above theorem, we count points of $C_{p,d}(\mathbb{P}^n)(\mathbb{F}_{q^m})$ modulo $(q-1)$. Recall that the proof of Theorem 2 in [B-B1] does not require K to be an algebraically closed field. When $K = \mathbb{F}_q$, the map $N_m : X \to |X(\mathbb{F}_{q^m})|$ gives rise to an additive invariant $N_m : Var_K \to \mathbb{Z}$.

Corollary 4.6. *Let $N_m : X \to |X(\mathbb{F}_{q^m})|$ be given as above. For any integer $m \geq 1$, we have*

$$N_m(C_{p,d}(\mathbb{P}^n)_{\mathbb{F}_{q^m}}) \equiv \binom{v_{p,n}+d-1}{d} \mod(q-1).$$

Proof. Since $N_m(\mathbb{G}_m) = q^m - 1 \equiv 0 \mod(q-1)$ and $N_m(\mathrm{Spec}\mathbb{F}_q) = 1$, the corollary follows from Theorem 1.4. □

In particular, when X is the Chow variety $C_{m,d}(\mathbb{P}^n_K)$ over k, we get

(1)
$$N_n(C_{m,d}(\mathbb{P}^n_K)) \equiv 1 \mod(q).$$

(2)
$$N_n(C_{m,d}(\mathbb{P}^n_K)) \equiv \binom{v_{p,n}+d-1}{d} \mod(q-1), \; where \; v_{p,n} = \binom{n+1}{p+1}.$$

Proof. (1) follows from the fact that there is a sequence of $\mathbb{G}_a$-actions on $C_{m,d}(\mathbb{P}^n_K)$ and the last one has exactly one fixed point (cf. [Ho]). (2) follows from the fact that there is a sequence of $\mathbb{G}_m$-actions on $C_{m,d}(\mathbb{P}^n_K)$ while the last one has exactly $\binom{v_{p,n}+d-1}{d}$ isolated fixed points (cf. the proof of Corollary 1.5). □

For example, the number of points on k-point on $C_{m,d}(\mathbb{P}^n_K)$ for $k = \mathbb{F}_2$ is always odd.

4.2. **The Chow variety for the product of projective spaces.** In this section, we deal with more general cases. Let X_K be a projective variety over K (we omit the subscript K below).

Let $\mathcal{C}_p(X)$ be the topological monoid of all effective p-cycles on X and let $\Pi_p(X)$ be the monoid $\pi_0(\mathcal{C}_p(X))$ of connected component of $\mathcal{C}_p(X)$. For $\alpha \in \Pi_p(X)$, let $C_\alpha(X)$ be the space of effective algebraic cycles c on X which are in the same connected component α.

Under this setting, if we consider the $\mathbb{G}_m$-action on $\mathbb{P}^{n+1} \times X$ by

$$\Phi_t([z_0, ..., z_n, z_{n+1}], x) = ([z_0, ..., z_n, tz_{n+1}], x),$$

then for any $\alpha \in \Pi_{p+1}(\mathbb{P}^{n+1} \times X)$, the fixed point set of the induced $\mathbb{G}_m$ on $C_\alpha(\mathbb{P}^{n+1} \times X)$ contains effective cycles of the form

$$c = \sum n_k V_k + \sum m_j W_j + \sum l_i U_i, \quad n_k, m_j, l_i \geq 0$$

whose class in $\Pi_{p+1}(\mathbb{P}^{n+1} \times X)$ is α, where $V_k \subset \mathbb{P}^n \times X$ is irreducible, $W_j = \Sigma_Q W_j'$ for some irreducible variety $W_j \subset \mathbb{P}^n \times X$ of $\dim W_j' = p$ and $U_k \subset X$ is irreducible of $\dim U_k = p+1$. Therefore, we have

$$C_\alpha(\mathbb{P}^{n+1} \times X)^{\mathbb{G}_m} = \coprod_{\alpha = \beta + \Sigma_Q \gamma + \gamma'} \{C_\beta(\mathbb{P}^n \times X) \times \Sigma_Q C_\gamma(\mathbb{P}^n \times X) \times C_{\gamma'}(X)\},$$

where $\beta \in \Pi_{p+1}(\mathbb{P}^n \times X)$, $\gamma \in \Pi_p(\mathbb{P}^n \times X)$ and $\gamma' \in \Pi_{p+1}(X)$. Hence we have

(13)
$$\chi(C_\alpha(\mathbb{P}^{n+1} \times X)^{\mathbb{G}_m}, l) = \sum_{\alpha = \beta + \Sigma_Q \gamma + \gamma'} \chi(C_\beta(\mathbb{P}^n \times X), l) \cdot \chi(C_\gamma(\mathbb{P}^n \times X), l) \cdot \chi(C_{\gamma'}(X), l).$$

By Theorem 1.2, we have

$$\chi(C_\alpha(\mathbb{P}^{n+1} \times X), l) = \chi(C_\alpha(\mathbb{P}^{n+1} \times X)^{\mathbb{G}_m}, l)$$

Therefore, by Equation (13) we have the following recursive formula

$$(14) \quad \chi(C_\alpha(\mathbb{P}^{n+1} \times X) = \sum_{\alpha = \beta + \Sigma_Q \gamma + \gamma'} \chi(C_\beta(\mathbb{P}^n \times X), l) \cdot \chi(C_\gamma(\mathbb{P}^n \times X), l) \cdot \chi(C_{\gamma'}(X), l).$$

From this we recover the Euler-Poincaré characteristic of $C_\alpha(\mathbb{P}^{n+1} \times X)$ from those of X. Therefore, we can obtain the Euler-Poincaré characteristic for arbitray many products of projective spaces (cf. [H] for a computation without group actions).

4.3. **Toric varieties.** The result in this subsection is a formula for the Euler-Poincaré characteristic for the Chow variety of general toric varieties, which is inspired by Elizondo [E]. For background on toric varieties, the reader is referred to Fulton's book [Fu].

Recall that a toric variety over K is an irreducible variety X containing the algebraic group $T = \mathbb{G}_m^{\times n}$ as a Zariski open subset such that the action of $\mathbb{G}_m^{\times n}$ on itself extends to an action on X.

The p-**th Euler series of** X is defined by the following formal power series

$$E_p(X) := \sum_{\alpha \in \Pi_p(X)} \chi(C_\alpha(X), l)\alpha.$$

Since its simplicity, the proof of Theorem 4.7 is given below, which is almost word by word translated from the case over complex number field (cf. [E, Th. 2.1]).

Theorem 4.7. *Denote by* $V_1, ..., V_N$ *the p-dimensional invariant irreducible subvarieties of* X. *Let* $e_{[V_i]}$ *be the characteristic function of the subset* $\{[V_i], i = 1, 2, ..., N\}$ *of* $\Pi_p(X)$, *where* $[V]$ *denotes its class in* $\Pi_p(X)$. *Then*

$$E_p(X) = \prod_{1 \leq i \leq N} \left(\frac{1}{1 - e_{[V_i]}} \right).$$

Proof. Note first we have $\chi(C_\alpha(X), l) = \chi(C_\alpha(X)^T, l)$ by applying Theorem 1.2 inductively for n-times. Then $E_p(X) = \prod_{1 \leq i \leq N} f_i$, where $f_i(\alpha) = 1$ if $\alpha = n \cdot [V_i]$ and 0 otherwise. Note that $1 = (1 - e_{[V_i]}) \cdot \bar{f}_i$ since, by definition, $f_i(\alpha) = (1 + e_{[V_i]} + e_{[V_i]}^2 + \cdots)(\alpha)$ for all $\alpha \in \Pi_p(X)$. $\square$

One needs to know which irreducible subvariety V of a toric variety X is invariant under the action of algebraic torus T. This has been answered in [E], i.e., the closure of an orbit under the action T. Therefore, theoretically one can obtain the Euler-Poincaré characteristic for any toric variety. One may need additional work to get an explicit formula for $E_p(X)$ in terms of the generators of Π_p. Elizondo has illuminated how to apply Theorem 4.7 to particular examples in complex case. His methods also works for the algebraic case. Those examples include projective spaces, the product of two projective spaces, Hirzebruch surfaces, the blowing up of a projective space a point, etc.

Here we give a remark on the Euler-Chow series of certain projective bundles. Let E_1 and E_2 be two algebraic vector bundle over a projective variety X. Let $\mathbb{P}(E_1)$ (resp. $\mathbb{P}(E_2)$) be the projectivization of E_1 (resp. E_2). Then, in complex case, the Euler-Chow series $E_p(\mathbb{P}(E_1 \oplus E_2))$ can be computed in terms of that of $\mathbb{P}(E_1)$, $\mathbb{P}(E_2)$ and $\mathbb{P}(E_1) \times_X \mathbb{P}(E_2)$, where the last one is the fiber product of $\mathbb{P}(E_1)$

and $\mathbb{P}(E_2)$ over X (cf. [EL]). The proof there word for word works for the algebraic analogue, except that the fixed point formula there is replaced by Theorem 1.2. As an application, one can obtain the Euler-Chow series for Grassmannians and Flag varieties over K.

The calculation of Euler-Poincaré characteristic for product of projective spaces, or more generally, of toric varieties X works well for an additive invariant $\lambda :$ $Var_K \to R$ satisfying $\lambda(\mathbb{G}_m) = 0$ and $\lambda(\mathrm{Spec} K) = 1$. For such a λ, the p-**th** λ-**series of** X is defined to be the following formal power series

$$\Lambda_p(X) := \sum_{\alpha \in \Pi_p(X)} \lambda(C_\alpha(X))\alpha.$$

The same formula in Theorem 4.7 holds for Λ_p on any toric variety X.

Theorem 4.8. *For a toric variety X in Theorem 4.7, we have*

$$\Lambda_p(X) = \prod_{1 \leq i \leq N} \left(\frac{1}{1 - e_{[V_i]}} \right).$$

4.4. **Chow varieties parameterizing irreducible varieties.** In this subsection, we compute the l-adic Euler-Poincaré characteristic of the Chow varieties parameterizing irreducible subvarieties of a given dimension and degree in projective spaces. Let $I_{p,d}(\mathbb{P}^n)_K \subset C_{p,d}(\mathbb{P}^n)_K$ be the subset contains p-dimensional subvarieties of $\mathbb{P}_K^n$ of degree d, i.e.,

$$I_{p,d}(\mathbb{P}^n)_K = \{V \in C_{p,d}(\mathbb{P}^n)_K | V \text{ is irreducible of } \deg V = d \}.$$

Note that $I_{p,1}(\mathbb{P}^n)_K$ is the Grassmannian of $p+1$-plane in $\mathbb{P}_K^n$, i.e., $I_{p,1}(\mathbb{P}^n)_K = G(p+1, n+1)$. For $d > 1$ each $I_{p,d}(\mathbb{P}^n)_K$ is a finite union of quasi-projective varieties. The following result is about the l-adic Euler-Poincaré characteristic of $I_{p,d}(\mathbb{P}^n)_K$.

Theorem 4.9 ([H]). *For $(l, \mathrm{char}(K)) = 1$, we have*

$$\chi(I_{p,d}(\mathbb{P}^n)_K, l) = \begin{cases} \binom{n+1}{p+1} & \text{for } d = 1, \\ 0 & \text{for } d > 1. \end{cases}$$

Proof. The proof here is similar to the case over the complex number field. Recall that the action of the algebraic n-torus $T^n := \mathbb{G}_m^{n+1}/\mathbb{G}_m$ is given by

$$\Phi_t([z_0, z_1, ..., z_n]) = [t_0 z_0, t_1 z_1, ..., t_n z_n]$$

where $t = (t_0, ..., t_n)$ and $[z_0, z_1, ..., z_n]$ are homogeneous coordinate for $\mathbb{P}_K^n$. This action on $\mathbb{P}^n$ induces an action of T on $I_{p,d}(\mathbb{P}^n)_K$ and hence on its closure $\bar{I}_{p,d}(\mathbb{P}^n)_K$ in $C_{p,d}(\mathbb{P}^n)_K$ and $\bar{I}_{p,d}(\mathbb{P}^n) - I_{p,d}(\mathbb{P}^n)$. By Theorem 1.2, we have

(15) $$\chi(F_{p,d}(\mathbb{P}^n)_K, l) = \chi(\bar{I}_{p,d}(\mathbb{P}^n)_K).$$

By induction on Theorem 1.2, we obtain that a p-dimensional T-invariant cycles is a linear combination of p-planes, we get $F_{p,d}(\mathbb{P}^n)_K \subset \bar{I}_{p,d}(\mathbb{P}^n)_K - I_{p,d}(\mathbb{P}^n)_K$ for $d > 1$, where $F_{p,d}(\mathbb{P}^n)_K$ is the fixed point set of T-action on $\bar{I}_{p,d}(\mathbb{P}^n)_K$. This together with Theorem 1.2 implies that

(16) $$\chi(F_{p,d}(\mathbb{P}^n)_K, l) = \chi(\bar{I}_{p,d}(\mathbb{P}^n)_K - I_{p,d}(\mathbb{P}^n)_K, l).$$

By the inclusion-exclusion property for l-adic Euler-Poincaré characteristic (cf. [Lau]), we have $\chi(\bar{I}_{p,d}(\mathbb{P}^n)_K - I_{p,d}(\mathbb{P}^n)_K, l) = \chi(\bar{I}_{p,d}(\mathbb{P}^n)_K, l) - \chi(I_{p,d}(\mathbb{P}^n)_K, l)$. This

together with Equation (15) and (16) implies $I_{p,d}(\mathbb{P}^n)_K, l) = 0$ for $d > 1$. This case that $d = 1$ follows from the fact $\chi(I_{p,1}(\mathbb{P}^n), l) = \chi(G(p+1, n+1), l) = \binom{n+1}{p+1}$. $\quad\square$

Similarly, we have an alternative shorter proof of the following result.

Proposition 4.10. *For* $(l, \mathrm{char}(K)) = 1$, *we have*

$$\chi(I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l) = \begin{cases} \binom{n+1}{k+1}\binom{m+1}{l+1}, & \textit{if } \alpha = [\mathbb{P}^k_K \times \mathbb{P}^l_K], \textit{ where } k + l = p, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The action of the algebraic torus $T := T^n \times T^m$ on $\mathbb{P}^n_K \times \mathbb{P}^m_K$ is given as the product of the actions on each factor defined in the above proof of Theorem 4.9. For each $\alpha \in \mathrm{Ch}_p(\mathbb{P}^n_K \times \mathbb{P}^m_K)$, where $\mathrm{Ch}_p(X)$ denotes the Chow group of p-cycles on X, the action of T on $\mathbb{P}^n_K \times \mathbb{P}^m_K$ induces an action on $C_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$ and $I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$ since the rational equivalent class of an irreducible variety is preserved by this action. Since the action is algebraic, it extends to the closure $\bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$ of $I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$ in $C_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$.

By Theorem 1.2, we have

$$\chi(F_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l) = \chi(\bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l),$$

where $F_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$ the fixed point set of this action in $\bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K$.

The T-invariant cycles in α are exactly finite sum of products of k-planes in $\mathbb{P}^n_K$ and $(p-k)$-planes in $\mathbb{P}^m_K$, where $0 \le k \le p$. Hence if $\alpha \ne e_{k,l}$ for all $k+l = p, k, l \ge 0$, then $F_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K \subset \bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m))_K - I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K)$. Applying Theorem 1.2 to $\bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m))_K - I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K)$, we have

$$\chi(F_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l) = \chi(\bar{I}_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K - I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l).$$

These two equations together the inclusion-exclusion property for l-adic Euler-Poincaré characteristic imply that

$$\chi(I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l) = 0.$$

If $\alpha = e_{k,l}$ for some $k, l \ge 0, k + l = p$, then $I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K = G(k+1, n+1) \times G(l+1, m+1)$ and so $\chi(I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K, l) = \binom{n+1}{k+1}\binom{m+1}{l+1}$.

The completes the proof of Proposition 4.10. $\quad\square$

From the proof of Theorem 4.9 and Proposition 4.10, we observe that it works nicely for general additive invariants $\lambda : Var_K \to R$ satisfying $\lambda(\mathbb{G}_m) = 0$ and $\lambda(\mathrm{Spec}K) = 1$. That is, the following statement holds.

Proposition 4.11. *For additive invariants* $\lambda : Var_K \to R$ *satisfying* $\lambda(\mathbb{G}_m) = 0$ *and* $\lambda(\mathrm{Spec}K) = 1$, *we have*

$$\lambda(I_{p,d}(\mathbb{P}^n)_K) = \begin{cases} \binom{n+1}{p+1} & \textit{for } d = 1, \\ 0 & \textit{for } d > 1. \end{cases}$$

and

$$\lambda(I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K) = \begin{cases} \binom{n+1}{k+1}\binom{m+1}{l+1}, & \textit{if } \alpha = [\mathbb{P}^k_K \times \mathbb{P}^l_K], \textit{ where } k + l = p, \\ 0, & \textit{otherwise.} \end{cases}$$

This proposition has the following immediate corollary.

Corollary 4.12. *The Hodge polynomial* $H(I_{p,d}(\mathbb{P}^n)_K) \in \mathbb{Z}[u, v]$ *(resp.* $H(I_\alpha(\mathbb{P}^n \times \mathbb{P}^m)_K))$ *is in the ideal* $\langle uv - 1 \rangle$ *generated by* $uv - 1$ *for* $d > 1$ *(resp.* $\alpha \ne [\mathbb{P}^k_K \times \mathbb{P}^l_K])$.

References

[B-B1]   A. Białynicki-Birula, *On fixed point schemes of actions of multiplicative and additive groups.* Topology 12 (1973), 99–103.

[B-B2]   A. Białynicki-Birula, *Some theorems on actions of algebraic groups.* Ann. of Math. (2) 98 (1973), 480–497.

[Ca]   J. B. Carrell, *Holomorphic C^* actions and vector fields on projective varieties.* Topics in the theory of algebraic groups, 1–37, Notre Dame Math. Lectures, 10, Univ. Notre Dame Press, Notre Dame, IN, 1982.

[D1]   P. Deligne, *Théorie de Hodge. II*, Inst. Hautes Études Sci. Publ. Math. No. 40 (1971), 5–57.

[D2]   P. Deligne, *Théorie de Hodge. III*, Inst. Hautes Études Sci. Publ. Math. No. 44 (1974), 5–77.

[E]   J. Elizondo, *The Euler series of restricted Chow varieties.* Compositio Math. 94 (1994), no. 3, 297–310.

[EL]   J. Elizondo and P. Lima-Filho, P. *Chow quotients and projective bundle formulas for Euler-Chow series.* J. Algebraic Geom. 7 (1998), no. 4, 695–729.

[Fu]   W. Fulton, *Introduction to toric varieties.* Annals of Mathematics Studies, 131. The William H. Roever Lectures in Geometry. Princeton University Press, Princeton, NJ, 1993. xii+157 pp. ISBN: 0-691-00049-2

[Ho]   G. Horrocks, *Fixed point schemes of additive group actions.* Topology 8 1969 233–242.

[H]   W. Hu, *Lawson-Yau Formula and its generalization.* arXiv:0809.2988.

[I]   L. Illusie, *Miscellany on traces in ℓ-adic cohomology: a survey.* (English summary) Jpn. J. Math. 1 (2006), no. 1, 107–136.

[K]   N. M. Katz, *Review of l-adic cohomology.* Motives (Seattle, WA, 1991), 21–30, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994. 14F2

[Lau]   G. Laumon, *Comparaison de caractéristiques d'Euler-Poincaré en cohomologie l-adique.* (French. English summary) C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 3, 209–212.

[Law]   H. B. Lawson, *Algebraic cycles and homotopy theory.*, Ann. of Math. **129**(1989), 253-291.

[LY]   H. B. Lawson and Stephen S. T. Yau, *Holomorphic symmetries.* Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 4, 557–577.

[Lo]   F. Loeser, *Seattle lectures on motivic integration.* Algebraic geometry—Seattle 2005. Part 2, 745–784, Proc. Sympos. Pure Math., 80, Part 2, Amer. Math. Soc., Providence, RI, 2009.

School of Mathematics, Sichuan University, Chengdu 610064, P. R. China

E-mail address: huwenchuan@gmail.com