NILPOTENT OPERATORS AND WEIGHTED PROJECTIVE LINES

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Abstract. We show a surprising link between singularity theory and the invariant subspace problem of nilpotent operators as recently studied by C. M. Ringel and M. Schmidmeier, a problem with a longstanding history going back to G. Birkhoff. The link is established via weighted projective lines and (stable) categories of vector bundles on those.

The setup yields a new approach to attack the subspace problem. In particular, we deduce the main results of Ringel and Schmidmeier for nilpotency degree p from properties of the category of vector bundles on the weighted projective line of weight type $(2,3,p)$, obtained by Serre construction from the triangle singularity $x^2+y^3+z^p$. For $p=6$ the Ringel-Schmidmeier classification is thus covered by the classification of vector bundles for tubular type $(2,3,6)$, and then is closely related to Atiyah's classification of vector bundles on a smooth elliptic curve.

Returning to the general case, we establish that the stable categories associated to vector bundles or invariant subspaces of nilpotent operators may be naturally identified as triangulated categories. They satisfy Serre duality and also have tilting objects whose endomorphism rings play a role in singularity theory. In fact, we thus obtain a whole sequence of triangulated (fractional) Calabi-Yau categories, indexed by p, which naturally form an ADE-chain.

1. Introduction and main results

In recent work Ringel and Schmidmeier thoroughly studied the classification problem for invariant subspaces of nilpotent linear operators (in a graded and ungraded version) [22–24]. This problem has a long history and can actually be traced back to Birkhoff's problem [2], dealing with the classification of subgroups of finite abelian p-groups. We note that Simson [26] determined the complexity for the classification of indecomposable objects, depending on the nilpotency degree. Even more generally, Simson considered the classification problem for chains of invariant subspaces, without however attempting an explicit classification. For additional information on the history of the problem we refer to [23, 26]. The main achievement of [23] is such an explicit classification for $p \leq 6$ where the case $p=6$, yielding tubular type, is the most difficult one and very much related to the representation theory of so-called tubular algebras, a problem initiated and accomplished by Ringel in [21]. In the present paper we describe an unexpected access to the invariant subspace problem for graded nilpotent operators through the theory of weighted projective lines.

Our approach focusses on the categorical structure and the global aspects (Serre duality, tilting, Calabi-Yau dimension) of the invariant subspace problem, and yields complete and satisfying results making use of the knowledge of the structure of vector bundles on a weighted projective line. The study complements the treatment from [23], which is in the spirit linear algebra and more explicit concerning the structure of individual representations. Our study further links the problem with

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other mathematical subjects (singularities, vector bundles, Cohen-Macaulay modules, Calabi-Yau categories) and largely enhances our knowledge about the original problem. In particular, our treatment yields a uniform treatment of three formerly unrelated problems, each forming a socalled ADE-chain: the study of triangle singularities of type $(2,3,p)$, the invariant subspace problem of linear operators which are nilpotent of degree p, and finally the representation theory of an equioriented quiver of Dynkin type $\mathbb{A}_{2(p-1)}$ equipped with all nilpotency relations of degree 3.

Let $\mathbb{X} = \mathbb{X}(2,3,p)$ denote the weighted projective line of weight type $(2,3,p)$, where the integer p is at least 2. Following [7], the category coh-$\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ is obtained by applying Serre's construction [25] to the (suitably graded) *triangle singularity* $x_1^2 + x_2^3 + x_3^p$.

The properties of coh-$\mathbb{X}$ are very similar to the properties of the category of (algebraically) coherent sheaves on a smooth projective curve or of the category of (analytically) coherent sheaves on a compact Riemann surface. Common to the three categories is that they are hereditary, that is, extensions $\mathrm{Ext}^i(-,-)$ for degree $i \geq 2$ vanish. Further each coherent sheaf X decomposes into a direct sum of a locally free sheaf E (a vector bundle) and a sheaf having finite support. This yields the concept of *rank* of X defined as the rank of the locally free sheaf E. Another property shared by these categories is that the Euler characteristic of the geometric object is a measure for the complexity of the category of coherent sheaves, especially for the category of vector bundles vect-$\mathbb{X}$. On the other hand, a property characteristic for weighted projective lines is the existence of a tilting object T for the category of coherent sheaves on a weighted projective line [13]. This means an object T where all self-extensions $\mathrm{Ext}^1(T,T)$ vanish and which, moreover, generates the category by forming extensions, kernels of monomorphisms and cokernels of epimorphisms. This in turn implies a close relationship to the representation theory of the finite dimensional endomorphism algebra A of T, formally expressed in an equivalence of the bounded derived categories $\mathrm{D}^b(\text{coh-}\mathbb{X})$ and $\mathrm{D}^b(\text{mod-}A)$ of coherent sheaves on $\mathbb{X}$ and finite dimensional modules over A, respectively. In the non-weighted situations such a tilting object only exists in the case of the projective line or, in the case of Riemann surfaces, for the Riemann sphere.

We recall that the Picard group of $\mathbb{X}$ is naturally isomorphic to the rank one abelian group $\mathbb{L} = \mathbb{L}(2,3,p)$ on three generators $\vec{x}_1, \vec{x}_2, \vec{x}_3$ subject to the relations $2\vec{x}_1 = 3\vec{x}_2 = p\vec{x}_3$. Up to isomorphism the line bundles are therefore given by the system $\mathcal{L}$ of twisted structure sheaves $\mathcal{O}(\vec{x})$ with $\vec{x} \in \mathbb{L}$. A key aspect of our paper is a properly chosen subdivision of the system $\mathcal{L}$ of all line bundles into two disjoint classes $\mathcal{P}$ and $\mathcal{F}$ of line bundles, called *persistent* and *fading*, respectively. This subdivision arises from the partition of $\mathbb{L}$ into the subsets $\mathbb{P} = \mathbb{Z}\vec{x}_3 \sqcup (\vec{x}_2 + \mathbb{Z}\vec{x}_3)$ and $\mathbb{F} = \mathbb{L} \setminus \mathbb{P}$, each consisting of cosets modulo $\mathbb{Z}\vec{x}_3$.

Let $[\mathcal{F}]$ denote the ideal of all morphisms in the category vect-$\mathbb{X}$ of vector bundles which factor through a finite direct sum of fading line bundles. We recall that a *Frobenius category* is an exact category (Quillen's sense) which has enough projectives and injectives, and where the projective and the injective objects coincide.

Theorem A. *Assume $\mathbb{X}$ has weight type $(2,3,p)$ with $p \geq 2$. Then the following holds.*

(1) *The category* vect-$\mathbb{X}$ *is a Frobenius category with the system $\mathcal{L}$ of all line bundles as the indecomposable projective-injective objects.*
(2) *The factor category* vect-$\mathbb{X}/[\mathcal{F}]$ *is a Frobenius category with the system $\mathcal{P}$ of persistent line bundles forming the full subcategory $\mathcal{P}/[\mathcal{F}] = \mathcal{L}/[\mathcal{F}]$ of indecomposable projective-injective objects; we use the notation $\underline{\mathcal{P}}$.*
(3) *The stable categories* vect-$\mathbb{X}/[\mathcal{L}]$ *and* $(\text{vect-}\mathbb{X}/[\mathcal{F}])/(\mathcal{P}/[\mathcal{F}])$ *are naturally equivalent as triangulated categories; we use the notation* $\underline{\text{vect}}$-$\mathbb{X}$.

Lemma B. *The category $\underline{\mathcal{P}}$ is equivalent to the path category of the quiver*

$$(1.1)\qquad \begin{array}{ccccccccccccc} \cdots \longrightarrow & \circ & \xrightarrow{x} & \circ & \xrightarrow{x} & \circ & \xrightarrow{x} \cdots \xrightarrow{x} & \circ & \xrightarrow{x} & \circ & \longrightarrow \cdots \\ & {\scriptstyle y}\downarrow & & {\scriptstyle y}\downarrow & & {\scriptstyle y}\downarrow & & {\scriptstyle y}\downarrow & & {\scriptstyle y}\downarrow & \\ \cdots \longrightarrow & \circ & \xrightarrow[x]{} & \circ & \xrightarrow[x]{} & \circ & \xrightarrow[x]{} \cdots \xrightarrow[x]{} & \circ & \xrightarrow[x]{} & \circ & \longrightarrow \cdots \end{array}$$

modulo the ideal given by all commutativities $xy = yx$ and all nilpotency relations $x^p = 0$.

Hence the category of finite dimensional contravariant k-linear representations of the above quiver with relations is naturally isomorphic to the category $\operatorname{mod}\underline{\mathcal{P}}$ of finitely presented right modules over $\underline{\mathcal{P}}$. We call $\underline{\mathcal{P}}$ (and by abuse of language sometimes also the quiver (1.1)) the (infinite) *p-ladder*. Clearly a right $\underline{\mathcal{P}}$-module is exactly a morphism $U \to M$ between two $\mathbb{Z}$-graded $k[x]/(x^p)$-modules, where x gets degree 1. The category $\tilde{\mathcal{S}}(p)$ consists of all those morphisms which are monomorphisms. As a full subcategory $\tilde{\mathcal{S}}(p)$ is extension-closed in $\operatorname{mod}\underline{\mathcal{P}}$, hence $\tilde{\mathcal{S}}(p)$ inherits an exact structure which is actually Frobenius with the projectives from $\operatorname{mod}\underline{\mathcal{P}}$ as the projective-injective objects. Note further that $\mathbb{Z}$ acts on $\tilde{\mathcal{S}}(p)$ by degree shift, denoted by s.

Theorem C. *Assume $\mathbb{X}$ has weight type $(2, 3, p)$. Then the functor*

$$\Phi\colon \text{vect-}\mathbb{X} \longrightarrow \operatorname{mod}\underline{\mathcal{P}}, \quad E \mapsto \underline{\mathcal{P}}(-, E)$$

induces equivalences $\Phi\colon \text{vect-}\mathbb{X}/[\mathcal{F}] \longrightarrow \tilde{\mathcal{S}}(p)$ of Frobenius categories and $\underline{\Phi}\colon \underline{\text{vect}}\text{-}\mathbb{X} \longrightarrow \underline{\tilde{\mathcal{S}}}(p)$ of triangulated categories, respectively. Moreover, under the functor Φ the degree shift by $\vec{x}_3 \in \mathbb{L}$ on vect-$\mathbb{X}$ corresponds to the degree shift s by $1 \in \mathbb{Z}$ on $\operatorname{mod}\underline{\mathcal{P}}$ and its subcategory $\tilde{\mathcal{S}}(p)$.

This theorem implies (most of) the results from [23] from results on the categories vect-$\mathbb{X}$; it further has a significant number of additional consequences. For instance, we show that the triangulated category $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$, has Serre duality and admits a tilting object. Indeed, we give explicit constructions for two tilting objects T and T' in vect-$\mathbb{X}$ with non-isomorphic endomorphism rings, yielding by Theorem C explicit tilting objects for $\underline{\tilde{\mathcal{S}}}(p)$. The two tilting objects have $2(p-1)$ pairwise indecomposable summands. The endomorphism algebra of T is the representation-finite Nakayama algebra $A(2(p-1), 3)$ given by the quiver

$$(1.2)\qquad 1 \xrightarrow{x} 2 \xrightarrow{x} 3 \xrightarrow{x} 4 \xrightarrow{x} \cdots \xrightarrow{x} 2p-3 \xrightarrow{x} 2(p-1)$$

with all nilpotency relations $x^3 = 0$. Let $[1, n]$ denote the linearly ordered set $\{1, 2, \ldots, n\}$. Then the endomorphism ring of T' is given as the incidence algebra $B(2, p-1)$ of the poset $[1, 2] \times [1, p-1]$, that is the algebra given by the fully commutative quiver

$$(1.3)\qquad \begin{array}{ccccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow \cdots \longrightarrow & p-2 & \longrightarrow & p-1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1' & \longrightarrow & 2' & \longrightarrow & 3' & \longrightarrow \cdots \longrightarrow & (p-2)' & \to & (p-1)'. \end{array}$$

It is well-known that rectangular diagrams of this shape appear in singularity theory, see for instance [5, 6].

Algebraically, an established method to investigate the complexity of a singularity is due to R. Buchweitz [3], later revived by D. Orlov [19] who primarily deals with the graded situation. Given the $\mathbb{L}$-graded triangle singularity $S = k[x_1, x_2, x_3]/(x_1^2 + x_2^3 + x_3^p)$ this amounts to consider the Frobenius category $\mathrm{CM}^{\mathbb{L}}$-S of $\mathbb{L}$-graded maximal Cohen-Macaulay modules, its associated stable category $\underline{\mathrm{CM}}^{\mathbb{L}}$-$S$ and the *singularity category of S* defined as the quotient

$\mathrm{D}^{\mathbb{L}}_{\mathrm{Sg}}(S) = \mathrm{D}^b(\mathrm{mod}^{\mathbb{L}}\text{-}S)/\mathrm{D}^b(\mathrm{proj}^{\mathbb{L}}\text{-}S)$. It is shown in [3, 19] that the two constructions yield naturally equivalent triangulated categories $\underline{\mathrm{CM}}^{\mathbb{L}}\text{-}S = \mathrm{D}^{\mathbb{L}}_{\mathrm{Sg}}(S)$. It further follows from [7] that sheafification yields natural equivalences $\mathrm{CM}^{\mathbb{L}}\text{-}S \xrightarrow{\sim} \mathrm{vect}\text{-}\mathbb{X}$ with the indecomposable projective $\mathbb{L}$-graded S-modules corresponding to the line bundles on $\mathbb{X}$, and then inducing natural identifications

$$\mathrm{D}^{\mathbb{L}}_{\mathrm{Sg}}(S) = \underline{\mathrm{CM}}^{\mathbb{L}}\text{-}S = \underline{\mathrm{vect}}\text{-}\mathbb{X}, \text{ where } \mathbb{X} = \mathbb{X}(2,3,p). \tag{1.4}$$

In particular, comparing the sizes of the triangulated categories $\mathrm{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ and $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ by the ranks of their Grothendieck groups yields

$$\mathrm{rk}(\mathrm{K}_0(\underline{\mathrm{vect}}\text{-}\mathbb{X})) - \mathrm{rk}(\mathrm{K}_0(\mathrm{coh}\text{-}\mathbb{X})) = p - 6,$$

a formula nicely illustrating the effects of (an $\mathbb{L}$-graded version of) Orlov's theorem [19]. Moreover, for each p the triangulated categories from (1.4) are fractional Calabi-Yau where, up to cancelation, the Calabi-Yau dimension equals $1 - 2\chi_{\mathbb{X}}$. Here $\chi_{\mathbb{X}} = 1/p - 1/6$ is the orbifold Euler characteristic of $\mathbb{X}$. By Theorem C all these assertions transfer to properties of $\underline{\tilde{\mathcal{S}}}(p)$. For details and further applications we refer to Section 5.

The structure of the paper is as follows. In Section 2 we recall some basic properties of weighted projective lines. In Section 3 we survey fundamental properties of projective covers and injective hulls in vect-$\mathbb{X}$. There we also introduce an important class of vector bundles of rank two, called Auslander bundles. These will play a key role for the proofs of the main results given in Section 4. Section 5 is devoted to applications concerning the categories $\tilde{\mathcal{S}}(p) = \mathrm{vect}\text{-}\mathbb{X}/[\mathcal{F}]$ and $\underline{\tilde{\mathcal{S}}}(p) = \underline{\mathrm{vect}}\text{-}\mathbb{X}$. In Appendix A we present a tilting object for the category $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ which is important for the applications discussed in Section 5.

2. Definitions and basic properties

We recall some basic notions and facts about weighted projective lines. We restrict our treatment to the case of three weights. So let p_1, p_2, $p_3 \geq 2$ be integers, called weights. Denote by S the commutative algebra

$$S = \frac{k[X_1, X_2, X_3]}{(X_1^{p_1} + X_2^{p_2} + X_3^{p_3})} = k[x_1, x_2, x_3].$$

Let $\mathbb{L} = \mathbb{L}(p_1, p_2, p_3)$ be the abelian group given by generators $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ and defining relations $p_1\vec{x}_1 = p_2\vec{x}_2 = p_3\vec{x}_3 =: \vec{c}$. The $\mathbb{L}$-graded algebra S is the appropriate object to study the triangle singularity $x_1^{p_1} + x_2^{p_2} + x_3^{p_3}$. The element $\vec{c}$ is called the *canonical element*. Each element $\vec{x} \in \mathbb{L}$ can be written in canonical form

$$\vec{x} = n_1\vec{x}_1 + n_2\vec{x}_2 + n_3\vec{x}_3 + m\vec{c} \tag{2.1}$$

with unique n_i, $m \in \mathbb{Z}$, $0 \leq n_i < p_i$.

The algebra S is $\mathbb{L}$-graded by setting $\deg x_i = \vec{x}_i$ $(i = 1,2,3)$, hence $S = \bigoplus_{\vec{x}\in\mathbb{L}} S_{\vec{x}}$. By an $\mathbb{L}$-graded version of the Serre construction [25], the weighted projective line $\mathbb{X} = \mathbb{X}(p_1, p_2, p_3)$ of weight type (p_1, p_2, p_3) is given by its category of coherent sheaves $\mathrm{coh}\text{-}\mathbb{X} = \mathrm{mod}^{\mathbb{L}}(S)/\mathrm{mod}^{\mathbb{L}}_0(S)$, the quotient category of finitely generated $\mathbb{L}$-graded modules modulo the Serre subcategory of graded modules of finite length. The abelian group $\mathbb{L}$ is ordered by defining the positive cone $\{\vec{x} \in \mathbb{L} \mid \vec{x} \geq \vec{0}\}$ to consist of the elements of the form $n_1\vec{x}_1 + n_2\vec{x}_2 + n_3\vec{x}_3$, where n_1, n_2, $n_3 \geq 0$. Then $\vec{x} \geq \vec{0}$ if and only if the homogeneous component $S_{\vec{x}}$ is non-zero, and equivalently, if in the normal form (2.1) of $\vec{x}$ we have $m \geq 0$.

The image $\mathcal{O}$ of S in $\mathrm{mod}^{\mathbb{L}}(S)/\mathrm{mod}^{\mathbb{L}}_0(S)$ serves as the structure sheaf of coh-$\mathbb{X}$, and $\mathbb{L}$ acts on the above data, in particular on coh-$\mathbb{X}$, by degree shift. Each line bundle has the form $\mathcal{O}(\vec{x})$ for a uniquely determined $\vec{x}$ in $\mathbb{L}$, and we have natural isomorphisms

$$\mathrm{Hom}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}.$$

Defining the *dualizing element* from $\mathbb{L}$ as $\vec{\omega} = \vec{c} - (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)$, the category coh-$\mathbb{X}$ satisfies Serre duality in the form

$$\mathrm{D}\,\mathrm{Ext}^1(X, Y) = \mathrm{Hom}(Y, X(\vec{\omega}))$$

functorially in X and Y. Moreover, Serre duality implies the existence of almost split sequences for coh-$\mathbb{X}$ with the Auslander-Reiten translation τ given by the shift with $\vec{\omega}$.

It is now established that an algebraic analysis of an $\mathbb{L}$-graded singularity S focusses on the *singularity category* $\mathrm{D}^{\mathbb{L}}_{Sg}(S) = \mathrm{D}^b(\mathrm{mod}^{\mathbb{L}}\text{-}S)/\mathrm{D}^b(\mathrm{proj}^{\mathbb{L}}\text{-}S)$. The singularity category is a good measure for the complexity of a singularity; in particular, $\mathrm{D}^{\mathbb{L}}_{Sg}(S) = 0$ if and only if S has finite (graded) global dimension, see [3] for the ungraded and [19] for the $\mathbb{Z}$-graded case.

An often more accessible incarnation of $\mathrm{D}^{\mathbb{L}}_{Sg}(S)$ is given by the stable category $\underline{\mathrm{CM}}^{\mathbb{L}}\text{-}S$ of $\mathbb{L}$-graded Cohen-Macaulay modules (provided S is graded Gorenstein, a hypothesis satisfied in our case.) In our case a more tractable version is available in the form of the stable category $\underline{\mathrm{MF}}^{\mathbb{L}}(S)$ of *matrix factorizations*. A competing representation is provided by the fact that $\mathrm{CM}^{\mathbb{L}}\text{-}S = \text{vect-}\mathbb{X}$, yielding finally the most convenient access: the stable category $\underline{\text{vect}}\text{-}\mathbb{X}$ of vector bundles on $\mathbb{X}$. An introduction to CM-modules and matrix factorizations is found in [27]. As an introduction to singularity theory we recommend the book of Ebeling [5]. Concerning links between representation theory and singularities we refer to the introductory article [20] by I. Reiten.

The category vect-$\mathbb{X}$ carries the structure of a Frobenius category such that the system $\mathcal{L}$ of all line bundles is the system of all indecomposable projective-injectives, see [11]: A sequence $\eta\colon 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$ in vect-$\mathbb{X}$ is *distinguished exact* if all the sequences $\mathrm{Hom}(L, \eta)$ with L a line bundle are exact (equivalently all the sequences $\mathrm{Hom}(\eta, L)$ are exact). In this case we say that α (resp. β) is a *distinguished monomorphism* (resp. *distinguished epimorphism*). Each distinguished exact sequence is further exact in the abelian category coh-$\mathbb{X}$.

By [8], the stable category

$$\underline{\text{vect}}\text{-}\mathbb{X} = \text{vect-}\mathbb{X}/[\mathcal{L}]$$

therefore is triangulated. It is shown in [11] that the triangulated category $\underline{\text{vect}}\text{-}\mathbb{X}$ is Krull-Schmidt with Serre duality induced from the Serre duality of coh-$\mathbb{X}$. The triangulated category $\underline{\text{vect}}\text{-}\mathbb{X}$ is homologically finite. Moreover, we will see in Appendix A that $\underline{\text{vect}}\text{-}\mathbb{X}$ has a tilting object.

It is shown in [7] that the quotient functor $q\colon \mathrm{mod}^{\mathbb{L}}(S) \to \text{coh-}\mathbb{X}$ induces an equivalence $\mathrm{CM}^{\mathbb{L}}(S) \xrightarrow{\sim} \text{vect-}\mathbb{X}$, where $\mathrm{CM}^{\mathbb{L}}(S)$ denotes the category of $\mathbb{L}$-graded (maximal) Cohen-Macaulay modules over S. Under this equivalence indecomposable graded projective modules over S correspond to line bundles in vect-$\mathbb{X}$, resulting in a natural equivalence

$$\underline{\mathrm{CM}}^{\mathbb{L}}(S) \simeq \underline{\text{vect}}\text{-}\mathbb{X}$$

used from now on as an identification. Stable categories of (graded) Cohen-Macaulay modules play an important role in the analysis of singularities, see [3, 9, 10, 19].

Each coherent sheaf on $\mathbb{X}$ has the form $X = E \oplus X_0$ where X_0 is the largest subobject of finite length (then having finite support in $\mathbb{X}$) and E is a vector bundle. By vect-$\mathbb{X}$ we denote the full subcategory of coh-$\mathbb{X}$ given by all vector bundles. Let $\delta : \mathbb{L} \to \mathbb{Z}$ be the homomorphism given on the generators $\vec{x}_i$, $i = 1, 2, 3$, by $\delta(\vec{x}_i) = \bar{p}/p_i$, where $\bar{p} = \mathrm{lcm}(p_1, p_2, p_3)$. There are two important linear forms on the

Grothendieck group $K_0(\text{coh-}\mathbb{X})$ of coh-$\mathbb{X}$, called *rank* and *degree*. The rank (degree) is characterized by the fact that $\operatorname{rk}\mathcal{O}(\vec{x}) = 1$ (resp. $\deg \mathcal{O}(\vec{x}) = \delta(\vec{x})$) for all $\vec{x}$ from $\mathbb{L}$, see [7]. The rank is zero on all sheaves of finite length and > 0 on non-zero vector bundles; further the degree is > 0 on each simple sheaf. These basic properties show that any non-zero X from coh-$\mathbb{X}$ has non-zero rank or non-zero degree, implying that the *slope* $\mu X = \deg X / \operatorname{rk} X$ is a properly defined member of $\mathbb{Q} \cup \{\infty\}$. The slope μX gives a rough information on the position of X in the category coh-$\mathbb{X}$ since $\operatorname{Hom}(X, Y)$ is non-zero (resp. zero) if $\mu Y - \mu X$ (resp. $\mu X - \mu Y$) is large. Further we will need a refinement of the degree, the *determinant* homomorphism $\det\colon K_0(\text{coh-}\mathbb{X}) \to \mathbb{L}$ characterized by $\det \mathcal{O}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{L}$, see [17]. Note that $\deg X = \delta(\det X)$.

3. Projective covers and Auslander bundles

Since S is $\mathbb{L}$-graded local with maximal graded ideal (x_1, x_2, x_3), the category $\text{mod}^{\mathbb{L}}\text{-}S$, hence also $\text{CM}^{\mathbb{L}}\text{-}S = \text{vect-}\mathbb{X}$ has projective covers. Translating from Cohen-Macaulay modules to vector bundles, we have to exhibit an *irredundant* system of *generators for the functor* $\mathcal{L}(-, E)$. That is, we have to find morphisms $L_i \xrightarrow{u_i} E$, $L_i \in \mathcal{L}$, $i = 1, \ldots, n$ such that each morphism $f\colon L \to E$, with $L \in \mathcal{L}$, has the form $f = \sum_{i=1}^n u_i \alpha_i$ for some $\alpha_i\colon L \to L_i$ and, moreover, no proper subsystem of $u_1, \ldots, u_n$ has this property. In this case $u = (u_1, \ldots, u_n)\colon \bigoplus L_i \to E$ is the projective cover of E in vect-$\mathbb{X}$. Existence for and properties of injective hulls follow from their projective counterpart by applying vector bundle duality $\text{vect-}\mathbb{X}^{op} \to \text{vect-}\mathbb{X}$, $X \mapsto \mathcal{H}om(X, \mathcal{O})$.

Proposition 3.1. *Assume E is a vector bundle, and $u_i\colon L_i \to E$, $i = 1, \ldots, n$ are morphisms with line bundles L_i. We put $u = (u_1, \ldots, u_n) : \bigoplus_{i=1}^n L_i \to E$, and denote by F the kernel of U and by $v = (v_1, \ldots, v_n)^t$ the inclusion of U into $\bigoplus_{i=1}^n L_i$. Then $(u_1, \ldots, u_n)$ is a generating system for $\mathcal{L}(-, E)$ if and only if the the sequence*

$$\mu : 0 \longrightarrow F \xrightarrow{v} \bigoplus_{i=1}^{n} L_i \xrightarrow{u} E \longrightarrow 0 \tag{3.1}$$

is distinguished exact. In this case, the following assertions are equivalent:
(i) E has no line bundle summand, and $(u_1, \ldots, u_n)$ is irredundant.
(ii) F has no line bundle summand, and $(v_1, \ldots, v_n)$ is irredundant.

Proof. The first claim follows immediately from the definitions. Next we show $(i) \Rightarrow (ii)$. Assume that $(v_1, \ldots, v_n)$ is not irredundant and, say, $(v_1, \ldots, v_{n-1})$ is already generating $\mathcal{L}(F, -)$. With $P' = \bigoplus_{i=1}^{n-1} L_i$ there results a commutative diagram

$$\begin{array}{cccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
\mu' : & 0 & \longrightarrow & F & \xrightarrow{v'} & P' & \xrightarrow{u'} & E'' & \longrightarrow 0 \\
 & & & \| & & \uparrow{\scriptstyle \pi_n} & & \uparrow{\scriptstyle \pi''_n} & \\
\mu : & 0 & \longrightarrow & F & \xrightarrow{(v', v_n)^t} & P' \oplus L_n & \xrightarrow{(u', u_n)} & E & \longrightarrow 0 \\
 & & & & & \uparrow{\scriptstyle \iota_n} & & \uparrow{\scriptstyle \iota''_n} & \\
 & & & & & L_n & = & L_n & \\
 & & & & & \uparrow & & \uparrow & \\
 & & & & & 0 & & 0 &
\end{array}$$

where the rows are distinguished exact, and where the central column splits. It follows that the right column is distinguished and then splits, since L_n is relative injective in vect-$\mathbb{X}$. Hence L_n is a line bundle summand of E, contradicting assumption (i).

Assume next that L is a line bundle such that $F = F' \oplus L$, and write $v = (v', w)$. As a composition of two distinguished monomorphisms then $w = (w_1, \dots, w_n)$, where $w_i : L \to L_i$, is also a distinguished monomorphism, hence splits since L is injective in vect-$\mathbb{X}$. Writing $P = \bigoplus_{i=1}^n L_i$ there exists $p : P \to L$, where $p = (p_1, \dots, p_n)$, with $1_L = \sum_{i=1}^n p_i w_i$. One of the summands, say $p_n w_n$ must be non-zero. Thus composition $L \xrightarrow{w_n} L_n \xrightarrow{p_n} L$ is an isomorphism implying that $pr_n \circ w = w_n : L \to L_n$ is an isomorphism where $pr_n : \bigoplus_{i=1}^n L_i \to L_n$ denotes the nth projection. To simplify notation, we identify L with the direct summand $w(L)$ of P. We have shown that $pr_n : P \to L_n$ restricts to an isomorphism $L_n \to L$, yielding a splitting $P = L \oplus \bigoplus_{i=1}^{n-1} L_i$. We may then rewrite μ as

$$0 \longrightarrow F' \oplus L \xrightarrow{\begin{pmatrix} * & 0 \\ * & 1_L \end{pmatrix}} \left(\bigoplus_{i=1}^{n-1} L_i\right) \oplus L \xrightarrow{(u_1, \dots, u_{n-1}, u_n)} E \longrightarrow 0$$

where $u' = (u_1, \dots, u_{n-1})$. Exactness now implies $u_n = 0$, hence $(u_1, \dots, u_{n-1})$ generates $\mathcal{L}(-, E)$ thus yielding a contradiction. This finishes the proof of $(i) \Rightarrow (ii)$. The implication $(ii) \to (i)$ follows from the previous one by vector bundle duality vect-$\mathbb{X} \to$ vect-$\mathbb{X}$, $X \mapsto \check{X} = \mathcal{H}om(X, \mathcal{O})$, by observing that the duality preserves line bundles and exact sequences. □

A morphism $u \colon P \to E$ with $P = \bigoplus_{i=1}^n L_i$, $L_i \in \mathcal{L}$, projective is called a *projective cover* or hull of E if u is a distinguished epimorphism, and each morphism $h : X \to E$ such that $u \circ h$ is a distinguished epimorphism is itself a distinguished epimorphism. The projective cover of E is unique up to isomorphism. The concept of an *injective hull* is dual.

Corollary 3.2. *In the absence of line bundle summands for E or F in (3.3), the injective hull of E equals the projective hull of F. Moreover, assume E and F have rank ≥ 2. Then E is indecomposable if and only if F is indecomposable.*

Proof. We only need to show the last assertion. Assume for instance that E is indecomposable, and F has a non-trivial decomposition $F = F' \oplus F''$. Then, using $\Im$ for the injective hull, we have $E = \Im(F)/F = \Im(F')/F' \oplus \Im(F'')/F''$ implying that E is decomposable, contrary to our assumption. □

For later use we note the following surprising result.

Proposition 3.3. *Let F be a vector bundle with injective hull $F \xrightarrow{(v_1, \dots, v_n)} \bigoplus_{i=1}^n L_i$. Then each $v_i \colon F \to L_i$ is an epimorphism in* coh-$\mathbb{X}$.

Proof. For $i = 1, \dots, n$ let L_i' be the image of v_i in L_i. Note that L_i' is again a line bundle. Let $v_i' \colon F \to L_i'$ be the morphism induced by v_i, and put $v' = (v_1', \dots, v_n')$. Clearly, $F \xrightarrow{v'} \bigoplus_{i=1}^n L_i'$ is a distinguished monomorphism, then yielding a distinguished monomorphism $h : \bigoplus_{i=1}^n L_i \to \bigoplus_{i=1}^n L_i'$ by the defining property of the injective hull $\Im(F)$. Passing to degrees we obtain

$$\sum_{i=1}^n \deg L_i \leq \sum_{i=1}^n \deg L_i' \leq \sum_{i=1}^n \deg L_i.$$

The first inequality uses that the cokernel of h has rank zero, and then finite length. The second inequality uses the inclusions $L_i' \hookrightarrow L_i$. We hence obtain $\deg L_i' = \deg L_i$ for each $i = 1, \dots, n$ and then $L_i' = L_i$, proving our claim. □

Nearly all information one has on the category vect-$\mathbb{X}$ is obtained by using the existence of line bundle filtrations for vector bundles, and then invoking the very explicit knowledge one has for morphism and extension spaces between line bundles [7]. Recall in this context that $\mathrm{Hom}(\mathcal{O}(\vec{x}),\mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}$ and $\mathrm{Ext}^1(\mathcal{O}(\vec{x}),\mathcal{O}(\vec{y})) = \mathrm{D}\, S_{\vec{x}-\vec{y}+\vec{\omega}}$. Passage to the stable category $\underline{\mathrm{vect}}$-$\mathbb{X}$ kills the carriers of all this information. A (partial) replacement is found in the Auslander bundles. By definition, the *Auslander bundle* $E = E(L)$ is obtained as the extension term of the almost split sequence

$$\eta:\ 0 \longrightarrow L(\vec{\omega}) \xrightarrow{\alpha} E(L) \xrightarrow{\beta} L \longrightarrow 0 \tag{3.2}$$

where L denotes a line bundle. Note for this that $\mathrm{D}\,\mathrm{Ext}^1(L, L(\vec{\omega})) = \mathrm{End}(L) = k$, such that E is uniquely determined up to isomorphism. The next result points to the importance of having exactly three weights. (For the proof of this statement, as in [7] we temporarily allow an arbitrary number of weights.)

Proposition 3.4. *Let $t = t_{\mathbb{X}}$ denote the number of weights of $\mathbb{X}$. Then the Auslander bundle $E = E(L)$ has trivial endomorphism ring $\mathrm{End}(E) = k$ if and only if $t \geq 3$. Moreover, E is exceptional if and only if $t = 3$. In this case , E is determined by its class $[E]$ in the Grothendieck group $\mathrm{K}_0(\mathrm{coh}\text{-}\mathbb{X})$.*

Proof. From the exact Hom-Ext sequence (L, η) we first obtain $\mathrm{Hom}(L, E) = 0 = \mathrm{Ext}^1(L, E)$ since the connecting homomorphism is an isomorphism. Next, from the exact Hom-Ext sequence $(L(\vec{\omega}), \eta)$ we deduce $\mathrm{Hom}(L(\vec{\omega}), L) = k^{4-t}$ and $\mathrm{Ext}^1(L(\vec{\omega}), L) = k^{t-3}$ with the convention that $k^n = 0$ for $n < 0$. The above uses the normal form expressions $-\vec{\omega} = \sum_{i=1}^t \vec{x}_i + (2-t)\vec{c}$ and $2\vec{\omega} = \sum_{i=1}^t (p_i - 2)\vec{x}_i + (t-4)\vec{c}$. Finally, application of $(-, E)$ to η yields exactness of $0 \to (L(\vec{\omega}), E) \to (E, E) \to (L, E) \to {}^1(L(\vec{\omega}), E) \to {}^1(E, E) \to {}^1(L, E) \to 0$, hence $\mathrm{End}(E) = k^{4-t}$ and $\mathrm{Ext}^1(E, E) = k^{t-3}$. For the last assertion we refer to [18, Prop. 4.4.1]. □

Except in cases $(2, 2, n)$, the assignment $L \mapsto E(L)$ yields a natural bijection between line bundles and Auslander bundles. We will show later (Corollary 3.8) that for three weights, all Auslander bundles are also exceptional in $\underline{\mathrm{vect}}$-$\mathbb{X}$.

Lemma 3.5. *We assume a weight triple. Let X be an indecomposable bundle of rank ≥ 2. Then there exists an Auslander bundle E and a morphism $u\colon E \to X$ such that $u \notin [\mathcal{L}]$.*

Proof. Choose a line bundle L' of maximal degree (=slope) such that there is a morphism $0 \neq h'\colon L' \to X$. The almost split sequence $\eta\colon 0 \to L' \xrightarrow{\alpha} E \xrightarrow{\beta} L'(-\vec{\omega}) \to 0$ yields a morphism $h\colon E \to X$ with $h\alpha = h' \neq 0$. We show $h \notin [\mathcal{L}]$: Otherwise there would be a factorization $h = \sum_{i=1}^n b_i a_i$ with morphisms $E \xrightarrow{a_i} L_i \xrightarrow{b_i} X$ and line bundles L_i. Then we have $0 \neq h\alpha = \sum_{i=1}^n b_i a_i \alpha$, yielding an index i with non-zero composition $b_i a_i \alpha$. In particular $\mu\, L' \leq \mu\, L_i$ and $\mathrm{Hom}(L_i, X) \neq 0$. By the choice of L' we get $\mu L' = \mu\, L$, thus $a_i\alpha$ is an isomorphism. This implies that η splits, a contradiction. □

Proposition 3.6. *We assume a weight triple. Let $E = E(L)$ be the Auslander bundle given by the sequence $0 \to L(\vec{\omega}) \xrightarrow{\alpha} E \xrightarrow{\beta} L \to 0$. Then there is a distinguished exact sequence*

$$\mu:\ 0 \longrightarrow F \xrightarrow{(v_0,v_1,v_2,v_3)^t} L(\vec{\omega}) \oplus \bigoplus_{i=1}^3 L(-\vec{x}_i) \xrightarrow{(u_0,u_1,u_2,u_3)} E \longrightarrow 0 \tag{3.3}$$

in vect-$\mathbb{X}$ *which defines the projective cover of E, where F is a bundle of rank two, and exceptional in* vect-$\mathbb{X}$. *Moreover, for weight type $(2, a, b)$ we have $F = E(-\vec{x}_1)$.*

We note that already for weight type $(3,3,3)$ the bundle F is *not* an Auslander bundle.

Proof. Step 1: Let E be the Auslander bundle given by the almost-split sequence $\eta: \ 0 \longrightarrow L(\vec{\omega}) \xrightarrow{\alpha} E \xrightarrow{\beta} L \longrightarrow 0$. By the almost-split property, the maps $x_i\colon L(-\vec{x}_i) \to L$ lift to maps $u_i\colon L(-\vec{x}_i) \to E$ (in fact unique since $\operatorname{Ext}^1(L(-\vec{x}_i), L(\vec{\omega})) \simeq \operatorname{D}\operatorname{Hom}(L, L(-\vec{x}_i)) = 0$). We claim that the maps $u_0 = \alpha\colon \overline{L}(\vec{\omega}) \to E$, $u_i\colon L(-\vec{x}_i) \to E$ $(i = 1, 2, 3)$ form an irredundant system of generators of $\mathcal{L}(-, E)$. Write $\overline{L} = L(-\vec{y})$, $\vec{y} \in \mathbb{L}$. Let $f\colon \overline{L} \to E$ be a map. The map $\beta \circ f\colon L(-\vec{y}) \to L$ cannot be an isomorphism since η does not split. Hence involving $\operatorname{Hom}(L(-\vec{y}), L) = S_{\vec{y}} \subseteq (x_1, x_2, x_3)$ we see $\beta \circ f = \sum_{i=1}^{3} x_i f_i$ for some f_1, f_2, f_3. Now, consider $g = f - \sum_{i=1}^{3} u_i f_i$; we obviously get $\beta \circ g = 0$, so that g has the form $g = \alpha \circ f_0 = u_0 \circ f_0$. We have shown $f = \sum_{i=0}^{3} u_i f_i$ establishing that $u_0, \dots, u_3$ generate $\mathcal{L}(-, E)$. That the system is irredundant follows from the fact that the line bundle summands of the central term of (3.3) are pairwise Hom-orthogonal.

Step 2: We put $L_0 = L(\vec{\omega})$ and $L_i = L(-\vec{x}_i)$ for $i = 1, 2, 3$. By Proposition 3.1 the sequence μ is distinguished exact and then also exact in coh-$\mathbb{X}$, implying that F has rank 2. We are going to show that $F \simeq E(-\vec{x}_1)$. Note that F is indecomposable; otherwise F would be the direct sum of two line bundles, hence 0 in $\underline{\text{vect}}$-$\mathbb{X}$, contradicting $F \simeq E[-1] \neq 0$ in $\underline{\text{vect}}$-$\mathbb{X}$. We claim, moreover, that F is exceptional in coh-$\mathbb{X}$, that is, satisfies

$$\text{a) } \operatorname{End}(F) = k \quad \text{and} \quad b) \operatorname{Ext}^1(F, F) = 0.$$

From $F \simeq E[-1] \in \underline{\text{vect}}$-$\mathbb{X}$ we infer that $\underline{\operatorname{End}}(F) = k$. To prove a) it thus suffices to show that each morphism $h\colon F \to F$ having a factorization $h = \alpha \circ v$ with $\alpha\colon \bigoplus_{i=0}^{3} L_i \to F$, is already zero. Indeed, otherwise $v \circ \alpha\colon \bigoplus L_i \to \bigoplus L_i$ is non-zero. Since $L_0, \dots, L_3$ are pairwise Hom-orthogonal, we obtain $\operatorname{End}(\bigoplus L_i) = \bigoplus \operatorname{End}(L_i)$, and deduce that there exists an $i \in \{0, \dots, 3\}$ such that $v_i \circ \alpha_i\colon L_i \to L_i$ is non-zero, hence an isomorphism. Therefore F decomposes, yielding a contradiction.

Next we assume weight type $(2, a, b)$ and show $[F] = [E(-\vec{x}_1)]$ holds in $\operatorname{K}_0(\text{coh-}\mathbb{X})$. There is a unique simple sheaf S_2 in the second exceptional point such that there is an exact sequence (1) $0 \to L(-\vec{x}_2) \xrightarrow{x_2} L \to S_2 \to 0$, see [7]. By the assumption on the weight type we have $\vec{\omega} - \vec{x}_1 = -\vec{x}_2 - \vec{x}_3$. Hence, applying the shift by $-\vec{x}_3$ to (1) yields an exact sequence (2) $0 \to L(\vec{\omega} - \vec{x}_1) \xrightarrow{x_2} L(-\vec{x}_3) \to S_2 \to 0$. Passing to classes in the Grothendieck group, we then obtain $[L] - [L(-\vec{x}_2)] = [L(-\vec{x}_3)] - [L(\vec{\omega} - \vec{x}_1)]$, and then invoking exactness of (3.3) we obtain $[E(-\vec{x}_1) = [L(\vec{\omega} - \vec{x}_1)] = \sum_{i=1}^{3} [L(\vec{x}_i)] - [L] = [F]$. Since E is exceptional and $[F] = [E(-\vec{x}_1]$, then

$$1 = \langle F, F \rangle = \dim \operatorname{End}(F) - \dim \operatorname{Ext}^1(F, F) = 1 - \dim \operatorname{Ext}^1(F, F),$$

and $\operatorname{Ext}^1(F, F) = 0$ follows. Thus F is exceptional. Since exceptional objects X in coh-$\mathbb{X}$ are determined by their class $[X]$, see [18, Prop. 4.4.1] it follows that $F \simeq E(-\vec{x}_1)$. □

To show exceptionality of Auslander bundles in the triangulated category $\underline{\text{vect}}$-$\mathbb{X}$ the following argument is useful. For distinction we use the notation $\underline{\operatorname{Hom}}(X, Y)$ to denote morphism spaces in $\underline{\text{vect}}$-$\mathbb{X}$.

Lemma 3.7. *Assume E and F are vector bundles of rank two, and $u\colon E \to F$ is nonzero in $\underline{\operatorname{Hom}}(E, F) \neq 0$. Then $\det(E) \leq \det(F)$.*

Proof. Observe first that u is a monomorphism, since otherwise the image of u would be a line bundle. We thus obtain an exact sequence $0 \to E \to F \to C \to 0$

with C of finite length. Passage to determinants proves the claim since $\det S > 0$ for each simple sheaf S, [17]. □

By [8] the suspension [-1] in the stable category $\underline{\mathrm{vect}}$-$\mathbb{X}$ is induced by taking the projective hull. With the above notations we thus obtain $E[-1] = F$.

Proposition 3.8. *Assume a weight triple. Then the following holds:*

(i) Each Auslander bundle is exceptional in $\underline{\mathrm{vect}}$-$\mathbb{X}$*.*

(ii) The suspension functor $[2]$ *and the degree shift* σ_0 *by* $\vec{c}$ *are isomorphic as functors on* $\underline{\mathrm{vect}}$-$\mathbb{X}$*.*

(iii) For each Auslander bundle E *we have* $\det E[n] - \det E = n\vec{c}$ *for each integer* n*.*

(iv) Let $\underline{\mathcal{A}}$ *be the full subcategory of* $\underline{\mathrm{vect}}$-$\mathbb{X}$ *formed by the Auslander bundles. For weight type* $(2, a, b)$ *the suspension functor* $[1]$ *and the degree shift* σ_1 *by* $\vec{x}_1$ *are isomorphic as functors on* $\underline{\mathcal{A}}$*.*

Proof. The proof of assertion (ii) invokes the theory of matrix factorizations for a hypersurface singularity f. It uses that the stable category of $\mathbb{L}$-graded matrix-factorizations is naturally equivalent to the category $\underline{\mathrm{CM}}^{\mathbb{L}}$-$S$, and yields that the second suspension is isomorphic to the shift by the degree of the singularity f, so in our case yields the degree shift by $\vec{c}$, compare [9, Theorem 2.14] for the $\mathbb{Z}$-graded case. Assertion (iii) is obtained by applying the determinant to the sequence (3.3). Finally, assertion (iv) follows by observing that the explicit construction of $F = E[-1]$ by means of the sequencef (3.3) is functorial in E.

Concerning (i) we know already that $\mathrm{End}(E) = k$ such that $\underline{\mathrm{End}}(E) = k$ follows. By Serre duality we further have $\underline{\mathrm{Hom}}(E, E[n]) = \mathrm{D}\,\underline{\mathrm{Hom}}(E[n-1], E(\vec{\omega}))$, and we have to prove that this expression is zero for each non-zero integer n. Assume for contradiction that it is non-zero for some integer $n \neq 0$. Applying the determinant and using (iii) implies that for an integer n as above, the inequalities (a) $n\vec{c} \geq 0$ and (b) $(n-1)\vec{c} \leq 2\vec{\omega}$ hold. Now, (a) is violated for $n < 0$ and (b) is violated for $n > 0$, thus proving the claim. □

4. Proofs

From now on the weight type is always the triple $(p_1, p_2, p_3) = (2, 3, p)$ with $p \geq 2$. In this Section we provide the proofs for Theorem A, Theorem C and Lemma B. Note that only the proof for Lemma B is straightforward. By contrast the proofs for Theorems A and C are far from obvious. Additionally they behave quite sensitive with respect to a (re)arrangement of the steps involved.

Proof of Lemma B. To prepare the proof of Lemma B we observe that the category $\underline{\mathcal{P}}$ has the shape of an infinite ladder:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathcal{O}(-\vec{x}_3) & \xrightarrow{x_3} & \mathcal{O} & \xrightarrow{x_3} & \mathcal{O}(\vec{x}_3) & \xrightarrow{x_3} & \mathcal{O}(2\vec{x}_3) & \xrightarrow{x_3} \cdots \\
& \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle x_2} & \\
\cdots \longrightarrow & \mathcal{O}(\vec{x}_2 - \vec{x}_3) & \xrightarrow{x_3} & \mathcal{O}(\vec{x}_2) & \xrightarrow{x_3} & \mathcal{O}(\vec{x}_2 + \vec{x}_3) & \xrightarrow{x_3} & \mathcal{O}(\vec{x}_2 + 2\vec{x}_3) & \xrightarrow{x_3} \cdots
\end{array}$$

where the *upper bar* (resp. *lower bar*) is formed by all line bundles $\mathcal{O}(n\vec{x}_3)$, (resp. $\mathcal{O}(\vec{x}_2 + n\vec{x}_3)$) for an arbitrary integer n.

Commutativity of the diagram (1.1) follows from the commutativity of S. Applying $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{y})) = S_{\vec{y}-\vec{x}}$ it follows that each morphism in $\mathcal{P}$, viewed as a full subcategory of vect-$\mathbb{X}$, is a linear combination of powers of x_2 and x_3. Next we observe that $\underline{\mathrm{Hom}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x} + \vec{c})) = 0$ holds for each $\vec{x} \in \mathbb{P}$. Indeed $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{x} + \vec{c}))$ is generated by x_2^3 and x_1^2, moreover each of the two morphisms factors through a fading line bundle ($\mathcal{O}(\vec{x}+2\vec{x}_2)$ and $\mathcal{O}(\vec{x}+\vec{x}_1)$, respectively).

Finally, we have $x_2x_3^{p-1} \neq 0$ (and hence $x_3^{p-1} \neq 0$) in vect-$\mathbb{X}/[\mathcal{F}]$ since there are no morphisms from $\mathcal{O}(\vec{x})$ to $\mathcal{O}(\vec{x}+\vec{x}_2+(p-1)\vec{x}_3)$ factoring through a fading line bundle. Indeed, every $\vec{y} \in \mathbb{L}$ with $\vec{0} \leq \vec{y} \leq \vec{x}_2+(p-1)\vec{x}_3$ is of the form $\vec{y} = a\vec{x}_2+b\vec{x}_3$ with $a = 0, 1$ and $b = 0, \dots, p-1$, implying that $\vec{y}$ belongs to $\mathbb{P}$. □

Lemma 4.1. *Let L be a line bundle. Then for each integer $n \geq 1$ the following sequence is exact in* coh-$\mathbb{X}$*.*

$$\eta_n : 0 \longrightarrow L \xrightarrow{(x_1,x_2^n)^t} L(\vec{x}_1) \oplus L(n\vec{x}_2) \xrightarrow{(-x_2^n,x_1)} L(\vec{x}_1+n\vec{x}_2) \longrightarrow 0. \tag{4.1}$$

Proof. The exact sequence is obtained from the following pushout diagram in coh-$\mathbb{X}$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \xrightarrow{x_2^n} & L(n\vec{x}_2) & \longrightarrow & S_2^{(n)} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle x_1} & & \| & & \\
0 & \longrightarrow & L(\vec{x}_1) & \xrightarrow{x_2^n} & L(\vec{x}_1+n\vec{x}_2) & \longrightarrow & S_2^{(n)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & S_1 & = & S_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4.2}$$

with S_1 a simple sheaf concentrated in x_1 and $S_2^{(n)}$ a sheaf of length n concentrated in x_2. □

Denote by $\check{E} = \mathcal{H}om(E, \mathcal{O})$ the dual vector bundle, and note that $\mathcal{O}(\vec{x})\check{} = \mathcal{O}(-\vec{x})$ for $\vec{x} \in \mathbb{L}$.

Lemma 4.2. *The functor $d : (\text{vect-}\mathbb{X})^{op} \longrightarrow \text{vect-}\mathbb{X}$, $E \mapsto \check{E}(\vec{x}_2)$, defines a self-duality preserving the partition $\mathcal{L} = \mathcal{P} \sqcup \mathcal{F}$. In particular, d induces self-dualities of* vect-$\mathbb{X}/[\mathcal{F}]$ *and $\underline{\mathcal{P}}$.* □

For the further discussion our next result is of central importance. It expresses a fundamental property of the partition $\mathcal{L} = \mathcal{P} \sqcup \mathcal{F}$.

Proposition 4.3. *Let L be a persistent line bundle. Then the following holds:*

(1) *The functor $\mathcal{F}(L,-) = \mathrm{Hom}(L,-)_{|\mathcal{F}}$ is generated by x_1, x_2^2 if L belongs to the upper bar of $\mathcal{P}$ and by x_1, x_2 if L belongs to the lower bar of $\mathcal{P}$. With the notation from (4.1) put $\eta = \eta_2$ (resp. $\eta = \eta_1$) if L belongs to the upper (resp. lower) bar. With the exception of L, all terms of η then belong to* add$(\mathcal{F})$*, and for each $F \in \mathrm{add}(\mathcal{F})$ the sequence $\mathrm{Hom}(\eta, F)$ is exact.*

(2) *The functor $\mathcal{F}(-,L)$ is generated by x_1, x_2 if L belongs to the upper bar of $\mathcal{P}$ and is generated by x_1, x_2^2 if L belongs to the lower bar of $\mathcal{P}$.*

Proof. By the preceding lemma it suffices to show assertion (1). Applying a suitable shift with $\vec{x} \in \mathbb{Z}\vec{x}_3$ we can assume that $L = \mathcal{O}$ or $L = \mathcal{O}(\vec{x}_2)$. In the first case each morphism $\mathcal{O} \to \mathcal{O}(\vec{y})$ with $\vec{y} \in \mathbb{F}$ factors through $\mathcal{O} \xrightarrow{(x_1,x_2^2)^t} \mathcal{O}(\vec{x}_1) \oplus \mathcal{O}(2\vec{x}_2)$. In case $L = \mathcal{O}(\vec{x}_2)$ each such morphism $\mathcal{O}(\vec{x}_2) \to \mathcal{O}(\vec{y})$ factors through $\mathcal{O}(\vec{x}_2) \xrightarrow{(x_1,x_2)^t} \mathcal{O}(\vec{x}_1+\vec{x}_2) \oplus \mathcal{O}(2\vec{x}_2)$. The preceding lemma now yields the short exact sequences η_2 and η_1, respectively, whose middle and end terms are clearly fading. The exactness of the sequences $\mathrm{Hom}(\eta_i, F)$, $i = 1, 2$, then immediately follows. □

Proposition 4.4. *Let* $\eta\colon 0 \to X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to 0$ *be a distinguished exact sequence in* vect-$\mathbb{X}$*. Then the sequence*

$$\Phi(\eta)\colon 0 \longrightarrow \Phi(X') \xrightarrow{\alpha_*} \Phi(X) \xrightarrow{\beta_*} \Phi(X'') \longrightarrow 0$$

is an exact sequence in mod-$\underline{\mathcal{P}}$.

Proof. *α_* is injective:* Let L be a persistent line bundle and $f'\colon L \to X'$ a morphism with $\alpha f' \in [\mathcal{F}](L, X)$. Using Proposition 4.3 we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \eta\colon 0 & \longrightarrow & X' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & X'' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle f'} & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & L & \longrightarrow & L_1 \oplus L_2 & \longrightarrow & L_3 & \longrightarrow & 0 \end{array} \tag{4.3}$$

where L_1, L_2 and L_3 belong to $\mathcal{F}$. Since the sequence η is distinguished exact in vect-$\mathbb{X}$, the morphism f'' lifts via β, so equivalently f' extends to $L_1 \oplus L_2$. Hence $f' \in [\mathcal{F}](L, X')$, as claimed.

$\ker \beta_* \subseteq \operatorname{im}(\alpha_*)$*:* Assume $L \in \mathcal{P}$ and $f\colon L \to X$ satisfies $\beta f \in [\mathcal{F}](L, X'')$. This yields a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{\alpha} & X & \xrightarrow{\beta} & X'' & \longrightarrow & 0 \\ & & & & \uparrow{\scriptstyle f} & \nwarrow^{\bar{b}} \circlearrowleft & \uparrow{\scriptstyle b} & & \\ & & & & L & \xrightarrow[a]{} & L_1 \oplus L_2 & & \end{array} \tag{4.4}$$

with $L_1, L_2 \in \mathcal{F}$. Now b lifts via β to a morphism $\bar{b}\colon L_1 \oplus L_2 \to X$. It follows $\beta(f - \bar{b}a) = 0$ and hence there exists $f'\colon L \to X'$ with $\alpha f' = f - \bar{b}a$ implying $\alpha_*(f') = f$ in $\underline{\mathcal{P}}(L, X)$.

β_* is surjective: This is obvious since η is distinguished exact, and then already the mapping $\operatorname{Hom}(L, \beta)\colon \operatorname{Hom}_{\mathbb{X}}(L, X) \to \operatorname{Hom}_{\mathbb{X}}(L, X'')$ is surjective. □

Proposition 4.5. *For each E from* vect-$\mathbb{X}$ *the right $\underline{\mathcal{P}}$-module $\Phi(E) = \underline{\mathcal{P}}(-, E)$ is finitely presented, indeed finite dimensional. Moreover, for each persistent line bundle L from the upper bar the morphism $x_2^*\colon \underline{\mathcal{P}}(L(\vec{x}_2), E) \to \underline{\mathcal{P}}(L, E)$, induced by $x_2\colon L \to L(\vec{x}_2)$, is a monomorphism.*

Proof. In vect-$\mathbb{X}$ we choose a distinguished exact sequence $0 \to E' \to P \to E \to 0$ with P from $\operatorname{add}(\mathcal{L})$. By Proposition 4.4 the induced mapping $\Phi(P) \to \Phi(E)$ is surjective. Moreover, $\Phi(P)$ is a finitely generated projective module over $\underline{\mathcal{P}}$, hence finite dimensional. This implies that $\Phi(E)$ is finite dimensional and finitely presented.

Next we show that all maps $x_2^*\colon \underline{\mathcal{P}}(L(\vec{x}_2), E) \to \underline{\mathcal{P}}(L, E)$ induced by $L \xrightarrow{x_2} L(\vec{x}_2)$, where L is persistent from the upper bar, are injective. Let $f\colon L(\vec{x}_2) \to E$ be a morphism such that $fx_2 \in [\mathcal{F}]$. By Proposition 4.3 we get a commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{x_2} & L(\vec{x}_2) \\ {\scriptstyle (x_1, x_2^2)^t}\downarrow & & \downarrow{\scriptstyle f} \\ L(\vec{x}_1) \oplus L(2\vec{x}_2) & \xrightarrow[(g,h)]{} & E, \end{array}$$

and hence $fx_2 = gx_1 + hx_2^2$, that is, $(f - hx_2)x_2 = gx_1$. Using the pushout property of diagram (4.2) (with $n = 1$) we obtain a morphism $\ell\colon L(\vec{x}_1 + 2\vec{x}_2) \to E$ such that $\ell x_1 = f - hx_2$, and $f \in [\mathcal{F}]$ follows. □

Together with Proposition 4.4 we get

Corollary 4.6. *Viewing Φ as a functor from the Frobenius category* vect-$\mathbb{X}$ *to the Frobenius category $\tilde{\mathcal{S}}(p)$ the functor is exact, that is, Φ sends distinguished exact sequences to distinguished exact sequences.* □

The kernel of Φ. Next, we are going to show that the kernel of Φ agrees with the ideal $[\mathcal{F}]$ of morphisms factoring through finite direct sums of fading line bundles.

Lemma 4.7. *Let $\mathbb{X} = \mathbb{X}(2, 3, p)$ with $p \geq 2$. Then the factor group $\mathbb{L}/\mathbb{Z}\vec{x}_3$ is cyclic of order 6 and generated by the class of $\vec{\omega}$. Moreover both in τ- and τ^--direction, the τ-orbit of any line bundle in* vect-$\mathbb{X}$ *consists of persistent and fading bundles according to the 6-periodic pattern $+ - + - - -$, where $+$ and $-$ stand for persistent and fading, respectively.*

Proof. By construction $\mathbb{L}/\mathbb{Z}\vec{x}_3$ is the abelian group on generators $\dot{x}_1, \dot{x}_2$ with relations $2\dot{x}_1 = 3\dot{x}_2 = 0$, hence $\mathbb{L}/\mathbb{Z}\vec{x}_3$ is cyclic of order 6. Further we have the following congruences modulo $\mathbb{Z}\vec{x}_3$:

$$0\vec{\omega} \equiv 0, \quad 1\vec{\omega} \equiv \vec{x}_1 + 2\vec{x}_2, \quad 2\vec{\omega} \equiv \vec{x}_2, \quad 3\vec{\omega} \equiv \vec{x}_1, \quad 4\vec{\omega} \equiv 2\vec{x}_2, \quad 5\vec{\omega} \equiv \vec{x}_1 + \vec{x}_2 \tag{4.5}$$

which immediately implies the last claim. □

Lemma 4.8. *Let E be an Auslander bundle. Then there exists a persistent line bundle which is a direct summand of the projective cover $P(E)$ of E, equivalently we have $\Phi E \neq 0$.*

Proof. Let $L = \mathcal{O}$, then by Proposition 3.6 the projective cover of the Auslander bundle $E(\mathcal{O})$ in vect-$\mathbb{X}$ is given by the expression $P(E(\mathcal{O})) = \mathcal{O}(\vec{\omega}) \oplus \bigoplus_{i=1}^{3} \mathcal{O}(-\vec{x}_i)$, and $\mathcal{O}(-\vec{x}_3)$ is persistent. The assertion clearly also holds for $L = \mathcal{O}(n\vec{x}_3)$ $(n \in \mathbb{Z})$. By the preceding lemma it then suffices to show that after twisting the expression for $P(E(\mathcal{O}))$ with $i\vec{\omega}$ (for $i = 1, \ldots, 5$) there will always exist a persistent line bundle on the right hand side. This follows from Table 1 with entries from $\mathbb{L}$ modulo $\mathbb{Z}\vec{x}_3$,

$\vec{x}$	$\vec{x} + \vec{\omega}$	$\vec{x} - \vec{x}_1$	$\vec{x} - \vec{x}_2$	$\vec{x} - \vec{x}_3$
$0\vec{\omega}$	$\vec{x}_1 + 2\vec{x}_2$	$\vec{x}_1$	$2\vec{x}_2$	$\boxed{0}$
$1\vec{\omega}$	$\boxed{\vec{x}_2}$	$2\vec{x}_2$	$\vec{x}_1 + \vec{x}_2$	$\vec{x}_1 + 2\vec{x}_2$
$2\vec{\omega}$	$\vec{x}_1$	$\vec{x}_1 + \vec{x}_2$	$\boxed{0}$	$\boxed{\vec{x}_2}$
$3\vec{\omega}$	$2\vec{x}_2$	$\boxed{0}$	$\vec{x}_1 + 2\vec{x}_2$	$\vec{x}_1$
$4\vec{\omega}$	$\vec{x}_1 + \vec{x}_2$	$\vec{x}_1 + 2\vec{x}_2$	$\boxed{\vec{x}_2}$	$2\vec{x}_2$
$5\vec{\omega}$	$\boxed{0}$	$\boxed{\vec{x}_2}$	$\vec{x}_1$	$\vec{x}_1 + \vec{x}_2$

TABLE 1. Persistent direct summands of $P(E)$

where elements from $\mathbb{P}$ are boxed. Since each row in the table contains an element from $\mathbb{P}$ the claim follows. □

Lemma 4.9. *Let E be an Auslander bundle and $u \colon E \to X$ be a morphism in* vect-$\mathbb{X}$ *with $\Phi u = 0$. Then $u \in [\mathcal{F}]$.*

Proof. We divide the proof into several steps. Note that step (1) and (2) hold for general weight triples, and only step (3) requests weight type $(2, 3, p)$.

(1) Let $E = E(L)$ as in (3.2). By (3.3) the projective cover of E is given by the expression $P(E) = L(\vec{\omega}) \oplus \bigoplus_{i=1}^{3} L(-\vec{x}_i)$. By Lemma 4.8 at least one of the line bundles $L(\vec{\omega})$, $L(-\vec{x}_1)$, $L(-\vec{x}_2)$, $L(-\vec{x}_3)$ is persistent.

(2) *Claim.* Let $\vec{y}\in\{-\vec{\omega},\vec{x}_1,\vec{x}_2,\vec{x}_3\}$. Then there is an exact sequence

$$0\longrightarrow L(-\vec{y})\xrightarrow{\alpha'} E\xrightarrow{\beta'} L(\vec{y}+\vec{\omega})\longrightarrow 0. \tag{4.6}$$

Indeed, if $\vec{y}=-\vec{\omega}$, we take the almost split sequence (3.2). If $\vec{y}=\vec{x}_i$, then we are going to show that there is an exact sequence

$$0\longrightarrow L(-\vec{x}_i)\xrightarrow{\pi_i} E\xrightarrow{\kappa_i} L(\vec{x}_i+\vec{\omega})\longrightarrow 0, \tag{4.7}$$

where π_i is induced by $x_i\colon L(-\vec{x}_i)\to L$ such that $\beta\pi_i=x_i$, and similarly κ_i is such that $\kappa_i\alpha=x_i\colon L(\vec{\omega})\to L(\vec{\omega}+\vec{x}_i)$. Indeed we have $\kappa_i\pi_i=0$, since $\operatorname{Hom}(L(-\vec{x}_i),L(\vec{x}_i+\vec{\omega}))=0$. By Proposition 3.3 the morphism κ_i is an epimorphism. As a non-zero map from a line bundle the map π_i is a monomorphism. We put $U=\ker(\kappa_i)/\operatorname{im}(\pi_i)$. Since rank and degree are additive on the sequence (4.7) we obtain $U=0$ and hence the exactness of (4.7).

(3) Let $\vec{y}\in\{-\vec{\omega},\vec{x}_1,\vec{x}_2,\vec{x}_3\}$ be such that $L(-\vec{y})$ is persistent, by step (1). It follows that there is a short exact sequence

$$0\longrightarrow L(-\vec{y})\xrightarrow{a} L_1\oplus L_2\xrightarrow{b} L_3\longrightarrow 0$$

with fading line bundles L_1,L_2,L_3, and satisfying the properties of Proposition 4.3. Since $\Phi u=0$ and thus $u\alpha'\in[\mathcal{F}]$ we obtain a commutative square

$$\begin{CD} E @>u>> X \\ @A{\alpha'}AA @AA{c}A \\ L(-\vec{y}) @>a>> L_1\oplus L_2. \end{CD}$$

We next form the pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & L(\vec{y}+\vec{\omega}) & = & L(\vec{y}+\vec{\omega}) & & & & \\
 & & {\scriptstyle\beta'}\uparrow & & \uparrow & & & & \\
0 & \longrightarrow & E & \longrightarrow & \overline{E} & \longrightarrow & L_3 & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha'}\uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & L(-\vec{y}) & \xrightarrow{a} & L_1\oplus L_2 & \xrightarrow{b} & L_3 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since u factors through $\overline{E}$, it is sufficient to show that $\overline{E}\in\operatorname{add}(\mathcal{F})$.

One checks easily that $L(\vec{y}+\vec{\omega})$ is fading. (For $\vec{y}=-\vec{\omega}$ this follows from the 6-periodic pattern in Lemma 4.7.) Therefore it is sufficient to show, that $\operatorname{Ext}^1(L(\vec{y}+\vec{\omega}),L_i)=0$, by Serre duality equivalently, that $\operatorname{Hom}(L_i,L(\vec{y}+2\vec{\omega}))=0$ (for $i=1,2$).

By Proposition 4.3 we can assume that $L_1=L(-\vec{y}+\vec{x}_1)$, and $L_2=L(-\vec{y}+2\vec{x}_2)$ if $L(-\vec{y})$ is from the upper bar, and $L_2=L(-\vec{y}+\vec{x}_2)$ if $L(-\vec{y})$ is from the lower bar. Therefore, one has to check whether $\operatorname{Hom}(\mathcal{O},\mathcal{O}(2\vec{y}+2\vec{\omega}-\vec{x}))$ is zero, that is, whether $2\vec{y}+2\vec{\omega}-\vec{x}\not\geq 0$ for $\vec{x}\in\{\vec{x}_1,\vec{x}_2,2\vec{x}_2\}$. There are two cases:

1. case. Assume that $P(E)$ admits a direct summand $L(-\vec{y})$ which is a persistent line bundle from the upper bar. In this case $\vec{x}\in\{\vec{x}_1,2\vec{x}_2\}$. Table 1 shows that we can assume $L=\mathcal{O}(i\vec{\omega})$ for $i=0$, 2, 3, 5, and the value of $\vec{y}$ can also extracted from that table. In all these cases it is easy to see that the condition $2\vec{y}+2\vec{\omega}-\vec{x}\not\geq 0$ is satisfied.

2. case. Assume that each persistent line bundle summand of $P(E)$ is from the lower bar. In this case $\vec{x}\in\{\vec{x}_1,\vec{x}_2\}$. Table 1 shows that we can assume $L=\mathcal{O}(\vec{\omega})$

and $\vec{y} = -\vec{\omega}$, or $L = \mathcal{O}(4\vec{\omega})$ and $\vec{y} = \vec{x}_2$. In these cases again the condition $2\vec{y} + 2\vec{\omega} - \vec{x} \not\geq 0$ holds. □

Proposition 4.10. *Let $X \in \text{vect-}\mathbb{X}$ be indecomposable such that $\Phi X = 0$. Then $X \in \mathcal{F}$.*

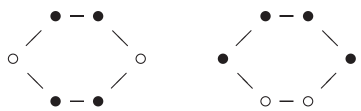
Proof. If X is a line bundle this is clear. Assume $\operatorname{rk} X \geq 2$. By Lemma 3.5 we obtain a morphism $u\colon E \to X$ where E is an Auslander bundle and $u \notin [\mathcal{L}]$. By Lemma 4.9 we get $0 \neq \Phi u\colon \Phi E \to \Phi X$, in particular $\Phi X \neq 0$. □

Let σ_i, $i = 1, 2, 3$, denote the degree shift (line bundle twist) $E \mapsto E(\vec{x}_i)$. Then the isomorphism classes of line bundles decompose into 6 orbits under the action of the group $\langle \sigma_3 \rangle$.

Corollary 4.11. *Let X be an indecomposable bundle of $\operatorname{rk} X \geq 2$. Then the projective cover $P(X)$ of X admits a persistent line bundle as a direct summand. Moreover $P(X)$ admits line bundle summands from at least four pairwise distinct $\langle \sigma_3 \rangle$-orbits.*

Proof. Since $\Phi(X) \neq 0$ if and only if $P(X)$ contains a persistent line bundle, the first claim immediately follows. Concerning the last claim we recall that the class of $\vec{\omega}$ generates $\mathbb{L}/\mathbb{Z}\vec{x}_3$ which is cyclic of order 6 and that the classes of 0 and $2\vec{\omega}$ represent the persistent members of $\mathbb{L}$. For each integer n, then also $P(X)(n\vec{\omega}) = P(X(n\vec{\omega}))$ contains a persistent line bundle. Let U be the subset of $\mathbb{L}/\mathbb{Z}\vec{x}_3$ corresponding to the $\langle \sigma_3 \rangle$-orbits of line bundles in $P(X)$. By the above, for each integer n the set U must contain n or $n + 2$. As is easily checked this implies $|U| \geq 4$, proving the claim. □

Actually there are, up to cyclic permutation, just two possibilities for a four-element subset U as above, given by the two following patterns, where a black dot indicates membership in U.



Lemma 4.12. *Assume $P \in \operatorname{add}(\mathcal{L})$. There exists an exact sequence $\eta\colon 0 \to P \xrightarrow{\alpha} P_0 \xrightarrow{\beta} P_1 \to 0$ in $\text{coh-}\mathbb{X}$ with P_0, P_1 from $\operatorname{add}(\mathcal{F})$ such that:*

1. *for each persistent line bundle L' the sequence $\operatorname{Hom}(L', \eta)$ is exact;*
2. *For each fading line bundle L' the sequence $\operatorname{Hom}(\eta, L')$ is exact.*

Proof. It suffices to show the statement if $P = L$ is indecomposable. If $L \in \mathcal{F}$, then one can take $\eta\colon 0 \to L \to L \to 0 \to 0$. Let now $L \in \mathcal{P}$. Then let

$$\eta = \eta_n\colon 0 \longrightarrow L \xrightarrow{(x_1, x_2^n)^t} L(\vec{x}_1) \oplus L(n\vec{x}_2) \xrightarrow{(-x_2^n, x_1)} L(\vec{x}_1 + n\vec{x}_2) \longrightarrow 0$$

be one of the sequences from Proposition 4.3, where $n = 2$ if L is from the upper bar and $n = 1$ if L is from the lower bar. Condition (2) follows from 4.3. Let L' be a persistent line bundle and $h\colon L' \to L(\vec{x}_1 + n\vec{x}_2)$ a morphism. Without loss of generality assume that $h \neq 0$. By considering the four possible cases $n = 1$ or $n = 2$ and L' from the upper or from the lower bar, respectively, one shows that $h \in \operatorname{Hom}(L', L(\vec{x}_1 + n\vec{x}_2)) = x_1 \operatorname{Hom}(L', L(n\vec{x}_2))$, in particular h factors through the middle term of η. This shows condition (1). □

The next result constitutes a key step in our proof of Theorems A and C.

Proposition 4.13. *Each morphism $h\colon E \to F$ in $\text{vect-}\mathbb{X}$ with $\Phi(h) = 0$ belongs to the ideal $[\mathcal{F}]$, that is, h factors through a member of $\operatorname{add}(\mathcal{F})$.*

Proof. Let $P \xrightarrow{\pi} E \to 0$ be a distinguished epimorphism with $P \in \mathrm{add}(\mathcal{L})$. Since $\Phi h = 0$, the composition $h\pi$ factors through an object of $\mathrm{add}(\mathcal{F})$, and by (2) from the preceding lemma we obtain a commutative diagram

$$\begin{array}{l} \eta\colon 0 \longrightarrow P \xrightarrow{\alpha} P_0 \xrightarrow{\beta} P_1 \longrightarrow 0 \\ \qquad\qquad\; h\pi \downarrow \;\circlearrowleft\; \swarrow \gamma \\ \qquad\qquad\quad\; F \end{array}$$

where η is an exact sequence in coh-$\mathbb{X}$ with $P_0,\ P_1 \in \mathrm{add}(\mathcal{F})$. From this we get a commutative diagram

$$\begin{array}{l} \mu\colon 0 \longrightarrow P \xrightarrow{(\pi,\alpha)^t} E \oplus P_0 \xrightarrow{(\sigma_1,\sigma_2)} C \longrightarrow 0 \\ \qquad\qquad\qquad (h,-\gamma) \downarrow \;\circlearrowleft\; \swarrow \delta \\ \qquad\qquad\qquad\qquad\;\; F \end{array}$$

in coh-$\mathbb{X}$ whose row is exact. It suffices to show that the cokernel C lies in $\mathrm{add}(\mathcal{F})$. To prove this consider the following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \eta\colon 0 & \longrightarrow & P & \xrightarrow{\alpha} & P_0 & \xrightarrow{\beta} & P_1 & \longrightarrow & 0 \\ & & \| & & \uparrow{\scriptstyle (0,1)} & & \uparrow{\scriptstyle p} & & \\ \mu\colon 0 & \longrightarrow & P & \xrightarrow{\varepsilon=(\pi,\alpha)^t} & E \oplus P_0 & \xrightarrow{\sigma=(\sigma_1,\sigma_2)} & C & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle (1,0)^t} & & \uparrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & E & = & E & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns. For each persistent line bundle L, applying the functor $\mathrm{Hom}(L,-)$ the first (compare part (1) of the preceding lemma) and the third row, and the first and the second column stay exact. It follows that also the third column stays exact implying that $\mathrm{Hom}(L,\mu)$ is exact for each $L \in \mathcal{P}$, in particular $\mathrm{Hom}(L,\sigma)$ is an epimorphism for $L \in \mathcal{P}$. We conclude that $\Phi\sigma\colon \underline{\mathcal{P}}(-, E \oplus P_0) \to \underline{\mathcal{P}}(-, C)$ is an epimorphism. Since $\Phi\varepsilon = \Phi\pi$ is also an epimorphism, the composition $0 = \Phi(\sigma\varepsilon) = \Phi(\sigma)\Phi(\varepsilon)$ is an epimorphism as well, yielding $\Phi C = 0$, equivalently $C \in \mathrm{add}(\mathcal{F})$. □

The functor Φ is full. Our next lemma plays a key role in order to show that the functor $\Phi\colon \text{vect-}\mathbb{X} \to \text{mod-}\underline{\mathcal{P}}$ is full.

Lemma 4.14. *We assume that* $\eta\colon 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$ *is a distinguished exact sequence in* vect-$\mathbb{X}$*. Then* $\beta = \mathrm{coker}(\alpha)$ *holds in* vect-$\mathbb{X}/[\mathcal{F}]$.

Proof. (1) β is an epimorphism in vect-$\mathbb{X}/[\mathcal{F}]$: To this end let $f\colon E'' \to X$ be a morphism in vect-$\mathbb{X}$ such that $f\beta \in \ker\Phi$. Then $(\Phi f)(\Phi\beta) = 0$. By Proposition 4.4, $\Phi\beta$ is an epimorphism, thus $\Phi f = 0$, that is, $f \in \ker\Phi$, hence $f \in [\mathcal{F}]$ by Proposition 4.13.

(2) Let $h\colon E \to X$ be a morphism in vect-$\mathbb{X}$ such that $h\alpha \in [\mathcal{F}]$. Hence there is $P \in \mathrm{add}(\mathcal{F})$ such that $h\alpha = [E' \xrightarrow{a} P \xrightarrow{b} X]$. Since η is distinguished exact there is a morphism $a'\colon E \to P$ such that $a'\alpha = a$. We obtain $(h - ba')\alpha = 0$. Since η is

exact, there is a morphism $h'\colon E'' \to X$ with $h - ba' = h'\beta$, which leads to $h = h'\beta$ modulo $[\mathcal{F}]$. □

Proposition 4.15. *The functor* $\Phi\colon \text{vect-}\mathbb{X} \to \text{mod-}\underline{\mathcal{P}}$ *is full, and induces a full embedding* $\text{vect-}\mathbb{X}/[\mathcal{F}] \hookrightarrow \tilde{\mathcal{S}}(p)$.

Proof. Let $h\colon \Phi E \to \Phi F$ be a morphism in mod-$\underline{\mathcal{P}}$. Consider projective covers in vect-$\mathbb{X}$:

$$0 \longrightarrow E' \xrightarrow{\alpha} P \xrightarrow{\beta} E \longrightarrow 0,$$
$$0 \longrightarrow F' \xrightarrow{\gamma} Q \xrightarrow{\delta} F \longrightarrow 0.$$

By Proposition 4.4 we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi E' & \xrightarrow{\Phi\alpha} & \Phi P & \xrightarrow{\Phi\beta} & \Phi E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle \bar{h}=\Phi u} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & \Phi F' & \xrightarrow{\Phi\gamma} & \Phi Q & \xrightarrow{\Phi\delta} & \Phi F & \longrightarrow & 0. \end{array}$$

We have $\Phi(\delta u\alpha) = \Phi(\delta)\Phi(u)\Phi(\alpha) = 0$, hence $(\delta u)\alpha$ belongs to $[\mathcal{F}]$. By the preceding lemma there is a morphism $v\colon E \to F$ with $v\beta = \delta u$ in vect-$\mathbb{X}/[\mathcal{F}]$. Applying Φ we get $(h - \Phi v)\Phi\beta = 0$. Since $\Phi\beta$ is an epimorphism we get $h = \Phi(v)$. □

Reflecting exactness. The next proposition turns out to be crucial in comparing the exact structures of vect-$\mathbb{X}$, vect-$\mathbb{X}/[\mathcal{F}]$, mod $\underline{\mathcal{P}}$ and $\underline{\tilde{\mathcal{S}}}(p)$.

Proposition 4.16. *Let* $\eta\colon 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$ *be a sequence in* vect-$\mathbb{X}$ *such that* $\Phi(\eta)$ *is exact in* mod-$\underline{\mathcal{P}}$. *Modifying terms by adding suitable summands from* $\text{add}(\mathcal{F})$, *we can change* η *to a distinguished exact sequence* $\hat{\eta}$ *in* vect-$\mathbb{X}$ *such that* $\Phi(\eta)$ *and* $\Phi(\hat{\eta})$ *are isomorphic in* mod-$\underline{\mathcal{P}}$ *and, accordingly,* η *and* $\hat{\eta}$ *are isomorphic in* vect-$\mathbb{X}/[\mathcal{F}]$.

Proof. Let $P \xrightarrow{\pi} E'' \to 0$ in vect-$\mathbb{X}$ be a projective cover. Since $\Phi\beta$ is an epimorphism in vect-$\mathbb{X}/[\mathcal{F}]$ the morphism π can be lifted to E. That is, there is a morphism $\bar{\pi}\colon P \to E$ such that $\beta\bar{\pi} - \pi$ factors through a bundle P_f which is a direct sum of fading line bundles, say $\beta\bar{\pi} - \pi = [P \xrightarrow{\bar{h}} P_f \xrightarrow{h} E'']$. It then follows that $E \oplus P_f \xrightarrow{(\beta,-h)} E''$ is a distinguished epimorphism. If K denotes its kernel we obtain a commutative diagram of distinguished exact sequences in vect-$\mathbb{X}$:

$$\begin{array}{rcccccccc} \hat{\eta}\colon 0 & \longrightarrow & K & \xrightarrow{\hat{\alpha}} & E \oplus P_f & \xrightarrow{(\beta,-h)} & E'' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle (1,0)^t} & & \| & & \\ \eta\colon 0 & \longrightarrow & E' & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & E'' & \longrightarrow & 0. \end{array}$$

Applying Φ to this diagram we obtain that $\Phi((1,0)^t)$ and hence $\Phi(\gamma)$ are isomorphisms. Since $\Phi\colon \text{vect-}\mathbb{X}/[\mathcal{F}] \to \text{mod}\,\underline{\mathcal{P}}$ is a full embedding, γ becomes an isomorphism in the factor category vect-$\mathbb{X}/[\mathcal{F}]$. □

The functor Φ is dense. The next lemma will serve as an induction step to prove that the functor $\Phi\colon \text{vect-}\mathbb{X} \to \tilde{\mathcal{S}}(p)$ is dense.

Lemma 4.17. *Let* L *be a persistent line bundle and* $\eta\colon 0 \to L(\vec{\omega}) \xrightarrow{\alpha} E \xrightarrow{\beta} L \to 0$ *the corresponding almost split sequence in* vect-$\mathbb{X}$. *Application of* Φ *yields an exact sequence*

$$0 \longrightarrow \Phi(E) \xrightarrow{\Phi(\beta)} \Phi(L) \xrightarrow{\pi} S \longrightarrow 0 \tag{4.8}$$

in mod-$\underline{\mathcal{P}}$*, where* S *is a simple module (not necessarily lying in* $\tilde{\mathcal{S}}(p)$*).*

Proof. By assumption $\mathbb{X}$ has exactly three weights, therefore the Auslander bundle E is indecomposable and hence $\Phi(\beta)\colon \Phi(E) \to \Phi(L)$ is not an isomorphism since Φ induces a full embedding vect-$\mathbb{X}/[\mathcal{F}] \hookrightarrow \operatorname{mod}\underline{\mathcal{P}}$. The modules $\Phi(E)$ and $\Phi(L)$ have local endomorphism rings; moreover, $\Phi(L)$ is indecomposable projective. Denote by $\pi\colon \Phi(L) \to S$ the natural projection on the simple top. Since the mapping $\Phi(\beta)$ belongs to the radical of mod-$\underline{\mathcal{P}}$ we obtain $\pi\beta = 0$.

We claim that the map $\Phi(\beta)\colon \Phi(E) \to \Phi(L)$ is injective. Indeed let L_1 be a persistent line bundle and $f\colon L_1 \to E$ such that $\Phi(\beta f) = 0$. This yields a factorization $\beta f = [L_1 \xrightarrow{a} P \xrightarrow{b} L]$ with P from add$(\mathcal{F})$. As a radical morphism b then lifts via β, thus $b = \beta\bar{b}$ for some morphism $\bar{b}\colon P \to E$. We obtain $\beta(f - \bar{b}a) = 0$ such that $f - \bar{b}a$ factors (via α) over $L(\omega)$. Since $L \in \mathcal{P}$, equation (4.5) from Lemma 4.7 shows that $L(\vec{\omega})$ is fading. It follows that f belongs to $[\mathcal{F}]$, proving the claim.

By the preceding argument we obtain an exact sequence $0 \to \Phi(E) \to \Phi(L) \to C \to 0$. We claim that the cokernel term C is a simple $\underline{\mathcal{P}}$-module. We first show that C — viewed as a representation of $\underline{\mathcal{P}}$ — has support $\{L\}$, and hence is semisimple. For each persistent line bundle L_1, not isomorphic to L, each morphism $\gamma\colon \Phi(L_1) \to C$ lifts by projectivity of $\Phi(L_1)$ to a morphism $\Phi(u)\colon \Phi(L_1) \to \Phi(L)$. Since η is almost split the non-isomorphism $u\colon L_1 \to L$ lifts via β, then implying that $\gamma = 0$. We have shown that $C \cong S^n$ where $S = S_L$ denotes the simple module concentrated in L. Moreover, $n \geq 1$ since $C \neq 0$. As an indecomposable projective module $\Phi(L)$ is local, and we conclude that $n = 1$, implying that C is simple. $\square$

Proposition 4.18. *For each module* M *in* $\tilde{\mathcal{S}}(p)$ *there exists a bundle* X *such that* $\Phi(X)$ *is isomorphic to* M*.*

Proof. We argue by induction on the (finite) dimension n of M. If $n = 0$, the assertion is evident. So assume that $n > 0$. Then we obtain an exact sequence $0 \to M' \to M \to S \to 0$ in mod-$\underline{\mathcal{P}}$, where S is simple and M' belongs to $\tilde{\mathcal{S}}(p)$. Invoking Lemma 4.17 we obtain an Auslander bundle $E = E(L)$ and a commutative diagram in $\operatorname{mod}\underline{\mathcal{P}}$ with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Phi E & = & \Phi E & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \Phi\beta} & & \\
\mu\colon 0 & \longrightarrow & M' & \longrightarrow & \overline{M} & \longrightarrow & \Phi L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & S & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Since ΦL is projective the sequence μ splits yielding $\overline{M} = M' \oplus \Phi(L)$. By induction M' belongs to the image of Φ, say $M' = \Phi(F')$. Summarizing we obtain an exact sequence

$$0 \to \Phi E \xrightarrow{(\Phi\beta, \Phi u')^t} \Phi L \oplus \Phi F' \longrightarrow M \to 0 \tag{4.9}$$

in mod-$\underline{\mathcal{P}}$. We put $\bar{x}_i = \vec{x}_i + \vec{\omega}$ and form in vect-$\mathbb{X}$ the injective hull

$$0 \to E \xrightarrow{a} L \oplus \bigoplus_{i=1}^{3} L(\bar{x}_i) \longrightarrow E(\vec{x}_1) \to 0$$

of E (compare (3.3)), where $a = (\beta, \kappa_1, \kappa_2, \kappa_3)^t$, with β from the almost split sequence (3.2) and the κ_i like in (4.7). We obtain in coh-$\mathbb{X}$ the exact sequence

$$\gamma\colon 0 \to E \xrightarrow{(\beta,(\bar{x}_i),u')^t} L \oplus \bigoplus_{i=1}^{3} L(\bar{x}_i) \oplus F' \longrightarrow C \to 0,$$

which is distinguished exact in vect-$\mathbb{X}$. To prove this we simplify notation, and write γ as the exact sequence $0 \to E \xrightarrow{u} F \xrightarrow{v} C \to 0$. By construction of γ each morphism of E into a line bundle L extends to F, hence the sequence $\mathrm{Hom}(\gamma, L)$ is exact. We show that C is a vector bundle. Let C_0 denote the torsion part of C, then the natural morphism $n : C \to C/C_0$ induces an isomorphism $\mathrm{Hom}(C/C_0, L) \to \mathrm{Hom}(C, L)$ for each line bundle L. This implies that the sequence $\bar{\gamma} : 0 \to E \xrightarrow{u} F \xrightarrow{n \circ v} C/C_0 \to 0$ also has the property that the sequence $\mathrm{Hom}(\bar{\gamma}, L)$ is exact for each line bundle L. Additionally $\bar{\gamma}$ consists of vector bundles, which implies that $\bar{\gamma}$ is distinguished exact, in particular exact in coh-$\mathbb{X}$. Comparison of γ and $\bar{\gamma}$ now shows that C and C/C_0 are isomorphic, hence C is a vector bundle and γ is distinguished exact as claimed. There are two cases:

1. case. L belongs to the upper bar. Then all line bundles $L(\bar{x}_i)$ are fading $(i = 1, 2, 3)$, and

$$\Phi\gamma\colon 0 \to \Phi E \xrightarrow{(\Phi\beta,\Phi u')^t} \Phi L \oplus \Phi F' \to \Phi\big(C\big) \to 0$$

is exact. Comparing this with (4.9) we obtain $\Phi\big(C\big) \simeq M$.

2. case. L belongs to the lower bar. Then $L(\bar{x}_1)$ is persistent, and $L(\bar{x}_2)$, $L(\bar{x}_3)$ are fading. We obtain the diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Phi E & \xrightarrow{(\Phi\beta,\Phi u')^t} & \Phi(L) \oplus \Phi(F') & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle proj.} & & \uparrow & & \\
0 & \longrightarrow & \Phi E & \xrightarrow{(\Phi\beta,\Phi\bar{x}_1,\Phi u')^t} & \Phi(L) \oplus \Phi(L(\bar{x}_1)) \oplus \Phi(F') & \longrightarrow & \Phi(C) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \Phi(L(\bar{x}_1)) & = & \Phi(L(\bar{x}_1)) & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The sequence

$$0 \longrightarrow \Phi L(\bar{x}_1) \longrightarrow \Phi C \longrightarrow M \longrightarrow 0$$

is exact with all terms lying in $\tilde{\mathcal{S}}(p)$. This sequence splits since $\Phi L(\bar{x}_1)$ is injective in $\tilde{\mathcal{S}}(p)$. We get $\Phi C = M \oplus \Phi L(\bar{x}_1)$. Write $C = \bigoplus_{i=1}^{n} C_i$ with all $C_i \in$ vect-$\mathbb{X}$ indecomposable. Since Φ is full the $\Phi C_i \neq 0$ have local endomorphism rings. Because the category $\underline{\tilde{\mathcal{S}}}(p)$ is Krull-Schmidt, it follows that M is the direct sum of some of the ΦC_i, hence M lies in the image of Φ. □

For later applications we need a related result:

Proposition 4.19. *Let L be a persistent line bundle from the upper bar and let S_L be the simple right $\underline{\mathcal{P}}$-module concentrated in L. Then S_L belongs to $\tilde{\mathcal{S}}(p)$ and has the form $\Phi\big(E(L)(\vec{x}_1)\big)$, where $E(L)$ denotes the Auslander bundle attached to L.*

Moreover, each simple $\underline{\mathcal{P}}$-module belonging to $\tilde{\mathcal{S}}(p)$ has the above form.

Proof. This follows from the proof (1. case) of Proposition 4.18 (with $M = S$ and hence $F' = 0$). □

Frobenius structure and proof of Theorems A and C. Define a sequence $0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$ in vect-$\mathbb{X}/[\mathcal{F}]$ to be distinguished exact if it is isomorphic to a sequence which is induced by a distinguished exact sequence in vect-$\mathbb{X}$.

We will prove now Theorems A and C. Part (1) from Theorem A was already shown before, part (3) is trivial.

By Propositions 4.5, 4.13, 4.15 and 4.18 the assignment $E \mapsto \underline{\mathcal{P}}(-, E)$ induces an equivalence of categories $\Phi\colon$ vect-$\mathbb{X}/[\mathcal{F}] \longrightarrow \tilde{\mathcal{S}}(p)$. It follows from Propositions 4.4 and 4.16 that a sequence $\eta\colon 0 \to E' \xrightarrow{\alpha} E \xrightarrow{\beta} E'' \to 0$ in vect-$\mathbb{X}/[\mathcal{F}]$ is distinguished exact if and only if $\Phi(\eta)$ is exact in $\tilde{\mathcal{S}}(p)$. It follows (via Φ) that the distinguished exact sequences give vect-$\mathbb{X}/[\mathcal{F}]$ the structure of a Frobenius category, and moreover, such that the indecomposable projective-injective objects are given by the objects of $\underline{\mathcal{P}}$. This proves part (2) from Theorem A. Hence $\Phi\colon$ vect-$\mathbb{X}/[\mathcal{F}] \longrightarrow \tilde{\mathcal{S}}(p)$ is even an equivalence of Frobenius categories, which shows the first statement of Theorem C. (We note that it is possible to establish directly that the distinguished exact sequence define on $\underline{\text{vect}}$-$\mathbb{X}/[\mathcal{F}]$ the structure of a Frobenius category, without involving the functor Φ.) The second statement of Theorem C is an immediate consequence of the first together with Theorem A (3). The last assertion of Theorem C on the shift-commutation of Φ follows by construction.

5. Applications

Theorem C allows to obtain the main results from [23] and further properties as direct consequences of properties from the theory of weighted projective lines. Indeed, as a general rule, we will prove results first for the category vect-$\mathbb{X}$ or the stable category $\underline{\text{vect}}$-$\mathbb{X}$ of vector bundles, and then export such results to $\tilde{\mathcal{S}}(p)$ or $\underline{\tilde{\mathcal{S}}}(p)$. In particular, the difficult classification for the tubular case $\tilde{\mathcal{S}}(6)$ thus appears as a consequence of the classification of indecomposable bundles on vect-$\mathbb{X}(2,3,6)$ from [17] which is analogous to Atiyah's classification [1] of vector bundles on a smooth elliptic curve. Of course, we also use the approach to establish additional properties of $\underline{\tilde{\mathcal{S}}}(p)$ among them the existence of various types of tilting objects and establish that the categories are Calabi-Yau.

Action of the Picard group. Obviously, the $\mathbb{L}$-action on vect-$\mathbb{X}$ by line bundle twist (= degree shift) induces an $\mathbb{L}$-action on $\underline{\text{vect}}$-$\mathbb{X}$. By transport of structure, Theorem C then induces an $\mathbb{L}$-action on $\underline{\tilde{\mathcal{S}}}(p)$. This action of the Picard group of $\mathbb{X}$ on $\underline{\text{vect}}$-$\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ reveals a certain amount of symmetry of $\underline{\text{vect}}$-$\mathbb{X}$ which is instrumental in proving most of the properties to follow. (An important example is the Calabi-Yau property to be discussed later. By contrast the treatment of the Fuchsian singularities in [10], [16] lacks this amount of symmetry and only yields a finite number of categories which are fractionally Calabi-Yau.)

Proposition 5.1. *The Picard group $\mathbb{L} = \mathbb{L}(2,3,p)$ acts on $\underline{\tilde{\mathcal{S}}}(p)$. Let s denote the automorphism induced by the degree shift of* mod $\underline{\mathcal{P}}$*, then the generators $\vec{x}_i$ of $\mathbb{L}$ act as follows on $\underline{\tilde{\mathcal{S}}}(p)$*

(i) *$\vec{x}_1$ acts as $\tau^3 s^3$,*
(ii) *$\vec{x}_2$ acts as $\tau^2 s^2$,*
(iii) *$\vec{x}_3$ acts as s.*

The proof immediately follows from the next lemma.

Lemma 5.2. *Let $\mathbb{L} = \mathbb{L}(2,3,p)$. Then $\mathbb{L}$ is generated by $\vec{x}_3$ and $\vec{\omega}$. Moreover, we have with $\bar{x}_i = \vec{x}_i + \vec{\omega}$*

(i) $\bar{x}_1 = \bar{x}_2 + \bar{x}_3$,
(ii) $\vec{x}_2 = 2\bar{x}_3$,
(iii) $\vec{x}_1 = 3\bar{x}_3$.

Proof. (i) We have $\bar{x}_2 + \bar{x}_3 = \vec{\omega} + (\vec{\omega} + \vec{x}_2 + \vec{x}_3) = \vec{\omega} + \vec{x}_1 = \bar{x}_1$.
(ii) $2\bar{x}_3 = 2\vec{c} - 2\vec{x}_1 - 2\vec{x}_2 = \vec{c} - 2\vec{x}_2 = \vec{x}_2$.
(iii) $3\bar{x}_3 = 3\vec{c} - 3\vec{x}_1 - 3\vec{x}_2 = \vec{x}_1$. □

The next result is not used otherwise in the paper; it follows using [11].

Corollary 5.3. *By means of the equivalence* Φ*, the suspension functor* $[1]$ *of* $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ *corresponds to the functor* $\tau^3 s^3$ *on* $\underline{\tilde{\mathcal{S}}}(p)$*.*

Proof. For weight type $(2, p, q)$ it is shown in [11] that the shift with $\vec{x}_1$ serves as the suspension functor for $\underline{\mathrm{vect}}\text{-}\mathbb{X}$. □

Tilting objects and Orlov's trichotomy. First we establish two tilting objects in $\underline{\mathrm{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ with non-isomorphic endomorphism rings. We recall that an object T in a triangulated category $\mathcal{T}$ is a *tilting object* if first it has no self-extensions, i.e. $\mathrm{Hom}(T, T[n]) = 0$ for each non-zero integer n and, secondly, it generates $\mathcal{T}$ homologically, i.e. the condition $\mathrm{Hom}(T, X[n]) = 0$ for each integer n forces that $X = 0$.

Proposition 5.4. *Assume* U *is a simple right* $\underline{\mathcal{P}}$*-module lying in* $\tilde{\mathcal{S}}(p)$*. Then*

$$\bigoplus_{a=0,\ldots,p-2,\ b=0,1} \tau^{4a+b} s^{3a+b}(U)$$

is a tilting object in $\underline{\tilde{\mathcal{S}}}(p)$ *with endomorphism ring* $A(2(p-1), 3)$*; see* (1.2).

Proof. Let E be an Auslander bundle. With $\bar{x}_i = \vec{x}_i + \vec{\omega}$ we put

$$M = \{a\bar{x}_1 + b\bar{x}_3 \mid a = 0, \ldots, p-2,\ b = 0, 1\}$$

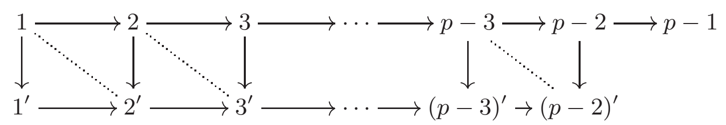
and define T as the direct sum of all $E(\vec{x})$, with $\vec{x}$ in M. It is shown in Theorem A.3 that T is a tilting object of $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ with endomorphism ring $\mathrm{End}(T) = A(2(p-1), 3)$. Transferred to $\underline{\tilde{\mathcal{S}}}(p)$ this yields the claim by Proposition 4.19. □

Independently, and by different methods the derived equivalence of the algebras $A(2(p-1), 3)$ and $B(2, p-1)$ was shown by S. Ladkani [12].

To obtain further interesting tilting objects in $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ we need to enlarge the class of Auslander bundles. For $\vec{x} \in M := \{b\vec{x}_2 + c\vec{x}_3 \mid b = 0, 1;\ c = 0, \ldots, p-2\}$ we define the *extension bundle* $E\langle\vec{x}\rangle$ as the extension term of the unique non-split exact sequence $0 \to \mathcal{O}(\vec{\omega}) \to E\langle\vec{x}\rangle \to \mathcal{O}(\vec{x}) \to 0$. Note for this that $\mathrm{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{\omega})) = k$. It follows from [11] that each extension bundle $E\langle\vec{x}\rangle$ (with $\vec{x}$ in M) is exceptional in $\mathrm{coh}\text{-}\mathbb{X}$ and in $\underline{\mathrm{vect}}\text{-}\mathbb{X}$. More is true, by [11] the system $T = \bigoplus_{\vec{x} \in M} E\langle\vec{x}\rangle$ is a tilting object in $\underline{\mathrm{vect}}\text{-}\mathbb{X}$ with $\mathrm{End}(T) = B(2, p-1)$, the incidence algebra of the poset (1.3) that is the $2 \times (p-1)$-rectangle with all commutativities. Note that such diagrams appear in singularity theory. By applying Theorem C we thus obtain the following result

Proposition 5.5. *The category* $\underline{\tilde{\mathcal{S}}}(p) = \underline{\mathrm{vect}}\text{-}\mathbb{X}$ *with* $\mathbb{X} = \mathbb{X}(2, 3, p)$ *has a tilting object* T *whose endomorphism ring is the algebra* $B(2, p-1)$*. In particular, the algebras* $A(2(p-1), 3)$ *and* $B(2, p-1)$ *are derived equivalent.* □

By $B'(2, p-1)$ we denote the incidence algebra of the poset (fully commutative quiver)



Corollary 5.6. *Let S be a simple $\underline{\mathcal{P}}$-module belonging to $\underline{\tilde{\mathcal{S}}}(p)$. Then the right perpendicular category $S^{\perp}$, consisting of all objects X from $\underline{\tilde{\mathcal{S}}}(p)$ satisfying* $\mathrm{Hom}(S, X[n]) = 0$ *for each integer n, is triangulated with Serre duality. Moreover, the category $S^{\perp}$ has tilting objects U and U' such that* $\mathrm{End}(U) \cong A(2p-3, 3)$ *and* $\mathrm{End}(U') \cong B'(2, p-1)$*. In particular, the algebras $A(2p-3,3)$ and $B'(2,p-1)$ are derived equivalent.*

Proof. We switch to the category $\underline{\mathrm{vect}}\text{-}\mathbb{X}$, where we have to calculate the category $E^{\perp}$ for an Auslander bundle E. The first claim follows from Proposition 5.4 and its proof. For the second claim we use Proposition 5.5 and the fact that $E\langle \vec{x}_2 + (p-2)\vec{x}_3\rangle$ is an Auslander bundle. □

Returning to the context of Proposition 5.5, we want to give the tilting object $\Phi(T)$ of $\underline{\tilde{\mathcal{S}}}(p)$ a more concrete shape. We briefly point out what the $\Phi(E\langle\vec{x}\rangle)$ are in the language of the category $\underline{\tilde{\mathcal{S}}}(p)$. For this let $\underline{\mathcal{P}}^{\mathrm{up}}$, resp. $\underline{\mathcal{P}}^{\mathrm{low}}$, denote the full subcategory of $\underline{\mathcal{P}}$ formed by the objects of the upper (resp. lower) bar. Moreover, we identify mod-$\underline{\mathcal{P}}^{\mathrm{up}}$ with the full subcategory of mod-$\underline{\mathcal{P}}$ of all modules whose support is contained in $\underline{\mathcal{P}}^{\mathrm{up}}$. Further we identify mod-$\underline{\mathcal{P}}$ with the category $\mathrm{mod}^{\mathbb{Z}}\text{-}A$ of finitely generated $\mathbb{Z}$-graded modules over the algebra $A = k[x]/(x^p)$ with x having degree one.

Lemma 5.7. *(a) The restriction functor $\rho\colon \mathrm{mod}\text{-}\underline{\mathcal{P}} \to \mathrm{mod}\text{-}\underline{\mathcal{P}}^{\mathrm{up}}$ has an exact left adjoint λ sending the indecomposable $\underline{\mathcal{P}}^{\mathrm{up}}$-projective $P(\vec{x})$ to the indecomposable $\underline{\mathcal{P}}$-projective $P(\vec{x}_2 + \vec{x})$.*

(b) Putting $T^{\mathrm{up}} = \bigoplus_{j=0}^{p-2} E\langle j\vec{x}_3\rangle$ and $T^{\mathrm{low}} = \bigoplus_{j=0}^{p-2} E\langle \vec{x}_2 + j\vec{x}_3\rangle$, the tilting object T from the above proposition has the form $T = T^{\mathrm{up}} \oplus T^{\mathrm{low}}$. Moreover, with the above identifications this yields:

$$\Phi(T^{\mathrm{up}}) = \bigoplus_{j=0}^{p-2} x^{j+1}A(j) \quad \textit{and} \quad \Phi(T^{\mathrm{low}}) = \bigoplus_{j=0}^{p-2} \lambda(x^{j+1}A(j)). \quad \square$$

Independently this tilting object in $\underline{\tilde{\mathcal{S}}}(p)$ was constructed by Xiao-Wu Chen [4] with a direct argument not relying on Theorem C.

Since the Grothendieck group $\mathrm{K}_0(\mathrm{coh}\text{-}\mathbb{X})$ is free abelian of rank $p+4$, see [7], we obtain from Proposition 5.4 or Proposition 5.5 the next result.

Corollary 5.8. *The Grothendieck group of $\underline{\mathrm{vect}}\text{-}\mathbb{X}(2,3,p) = \underline{\tilde{\mathcal{S}}}(p)$ is free abelian of rank $2(p-1)$. Moreover, we have $\mathrm{rk}\,\mathrm{K}_0(\underline{\tilde{\mathcal{S}}}(p)) - \mathrm{rk}\,\mathrm{K}_0(\mathrm{coh}\text{-}\mathbb{X}) = p-6$.* □

This result serves as a nice illustration of an $\mathbb{L}$-graded version of Orlov's theorem [19]. For this we recall from the Introduction, (1.4) that there are natural equivalences

$$\mathcal{T} := \mathrm{D}^{\mathbb{L}}_{\mathrm{Sg}}(S) = \underline{\mathrm{CM}}^{\mathbb{L}}\text{-}S = \underline{\mathrm{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p), \text{ where } \mathbb{X} = \mathbb{X}(2,3,p) \tag{5.1}$$

where we have used Theorem C for the last identification. It follows from an $\mathbb{L}$-graded version of Orlov's theorem [19] that the comparison between $\mathrm{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ and any of the four triangulated categories above follows a trichotomy determined by the *Gorenstein parameter* of the singularity. In the present $\mathbb{L}$-graded setting this index equals $6-p$, compare Corollary 5.8. (We have normalized the sign in order to make it equal to the sign of the Euler characteristic.)

Proposition 5.9 (Orlov's trichotomy)**.** *Let $\mathbb{X} = \mathbb{X}(2,3,p)$. Then the categories $\mathrm{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ and $\mathcal{T}$ are related as follows.*

1. *For $\chi_{\mathbb{X}} > 0$ the category $\mathcal{T}$ is triangle-equivalent to the right perpendicular category in $\mathrm{D}^b(\mathrm{coh}\text{-}\mathbb{X})$ with respect to an exceptional sequence of $6-p$ members;*

(2) *For* $\chi_{\mathbb{X}} = 0$ *the category* $\mathcal{T}$ *is triangle-equivalent to* $\mathrm{D}^b(\text{coh-}\mathbb{X})$*;*
(3) *For* $\chi_{\mathbb{X}} < 0$ *the category* $\mathrm{D}^b(\text{coh-}\mathbb{X})$ *is triangle equivalent to the right perpendicular category in* $\mathcal{T}$ *with respect to an exceptional sequence of* $p-6$ *members.*

Calabi-Yau dimension and Euler characteristic. Let $\mathcal{T}$ be a triangulated category with Serre duality. Let S denote the Serre functor of $\mathcal{T}$. Assume the existence of a smallest integer $n \geq 1$ such that we have an isomorphism $S^n \cong [m]$ of functors for some integer m. (Here, $[m]$ denotes the m-fold suspension of $\mathcal{T}$.) Then $\mathcal{T}$ is called *Calabi-Yau* of fractional CY-dimension $\frac{m}{n}$. Note that the "fraction" $\frac{m}{n}$ is kept in uncanceled format. The bounded derived category $\mathrm{D}^b(\text{coh-}\mathbb{X})$ of coherent sheaves on $\mathbb{X}(2,3,p)$ is almost never Calabi-Yau, the only exception being the tubular case $p = 6$, where we have fractional CY-dimension $6/6$. It is therefore remarkable that the category $\underline{\tilde{\mathcal{S}}}(p) = \underline{\text{vect}}\text{-}\mathbb{X}(2,3,p)$ is always fractional Calabi-Yau. Moreover, the CY-dimension only depends on the Euler characteristic of $\mathbb{X}$. To show this the next lemma is useful.

Lemma 5.10. *Let* $\mathbb{L} = \mathbb{L}(2,3,p)$*. The class of* $\vec{\omega}$ *in* $\mathbb{L}/\mathbb{Z}\vec{x}_1$ *is of order* $\mathrm{lcm}(3,p)$*. Moreover, the equality*

$$\mathrm{lcm}(3,p) \cdot \vec{\omega} = \Big(\mathrm{lcm}(3,p) \cdot \frac{p-6}{3p}\Big) \cdot \vec{x}_1$$

holds in $\mathbb{L}$*.*

Proof. Write $n \geq 1$ as $n = a \cdot p + b$ with a, $b \in \mathbb{Z}$ and $0 \leq b < p$. Then we have $n\vec{\omega} = n(\vec{x}_1 - \vec{x}_2 - \vec{x}_3) = n\vec{x}_1 - n\vec{x}_2 - n\vec{x}_3$, hence $n\vec{\omega} \in \mathbb{Z}\vec{x}_1$ if and only if $3 \mid n$ and $p \mid n$, which is equivalent to $\mathrm{lcm}(3,p) \mid n$. This shows the first claim. The second follows from $\mathrm{lcm}(3,p) \cdot \vec{\omega} = \mathrm{lcm}(3,p) \cdot \Big(1 - 2/3 - 2/p\Big) \cdot \vec{x}_1$. □

Proposition 5.11. *The category* $\underline{\tilde{\mathcal{S}}}(p)$ *is Calabi-Yau of fractional Calabi-Yau dimension* d_p *given as follows:*

$$d_2 = \frac{1}{3} \quad (= 1 - 2 \cdot \chi_{\mathbb{X}}),$$

$$d_p = \frac{\mathrm{lcm}(3,p) \cdot (1 - 2 \cdot \chi_{\mathbb{X}})}{\mathrm{lcm}(3,p)}, \quad \textit{for } p \geq 3.$$

Here, $\chi_{\mathbb{X}} = 1/p - 1/6 = (6-p)/6p$ *is the Euler characteristic of* $\mathbb{X}(2,3,p)$*, and* $1 - 2 \cdot \chi_{\mathbb{X}} = (4p-6)/3p$*.*

Note that the nominator of d_p is always an integer.

Proof. Assume first that $p \geq 3$. Then the Picard group $\mathbb{L} = \mathbb{L}(2,3,p)$ acts faithfully on $\underline{\text{vect}}\text{-}\mathbb{X}$. Indeed, if E is an Auslander bundle with $E(\vec{x}) \simeq E$ in $\underline{\text{vect}}\text{-}\mathbb{X}$, then $p \geq 3$ implies $\vec{x} = 0$. (In case $p \geq 3$ inspection of the AR components shows that for two line bundles L, L' the corresponding Auslander bundles $E(L)$, $E(L')$ are isomorphic if and only if L and L' are.) Since shift by $\vec{x}_1$ serves as suspension [1] and the Serre functor on $\underline{\text{vect}}\text{-}\mathbb{X}$ is given by $S = \tau[1]$, it follows from the preceding lemma that the fractional Calabi-Yau dimension of $\underline{\text{vect}}\text{-}\mathbb{X}$ is given by $\frac{m+n}{n}$ with $n = \mathrm{lcm}(3,p)$ and $m = \mathrm{lcm}(3,p) \cdot \frac{p-6}{3p}$. For $p = 2$ we have a similar formula, but in the resulting fraction $2/6$ the factor 2 can be canceled, since in this case for two line bundles L, L' the corresponding Auslander bundles $E(L)$, $E(L')$ are isomorphic if and only if $L' \simeq L$ or $L' \simeq L(\vec{x}_1 - \vec{x}_3)$. □

Corollary 5.12. *The category* $\underline{\tilde{\mathcal{S}}}(p)$ *determines* $\text{coh-}\mathbb{X}$*.* □

For distinction, we denote the class of a vector bundle X in $\mathrm{K}_0(\underline{\text{vect}}\text{-}\mathbb{X})$ by $[[X]]$, and use $\langle\langle [[X]], [[Y]] \rangle\rangle = \sum_{n \in \mathbb{Z}} \dim \underline{\mathrm{Hom}}(X, Y[n])$ for the *Euler form.*

Lemma 5.13. *We assume weight type* $(2,3,p)$ *with* $p \geq 3$. *Let* E *be an Auslander bundle. Let* $h = \operatorname{lcm}(6,p)$ *and* $A = \{0, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$. *Then the following holds:*

(i) *We have* $\langle\langle [[E]], [[E(j\vec{\omega})]] \rangle\rangle = 1$ *if and only if* $j\vec{\omega} \in A$ *modulo* $\mathbb{Z}\vec{c}$.
(ii) *We have* $\langle\langle [[E]], [[E(j\vec{\omega})]] \rangle\rangle = -1$ *if and only if* $j\vec{\omega} - \vec{x}_1 \in A$ *modulo* $\mathbb{Z}\vec{c}$.
(iii) *Assume* $1 \leq j < h$, *then* $[[E(j\vec{\omega})]] \neq [[E]]$.
(iv) *Assume* $p \geq 4$ *is even. Then* $[[E(\frac{h}{2}\vec{\omega})]] \neq -[[E]]$.

Proof. Concerning (i) assume first that $j\vec{\omega}$ is congruent to $\vec{y} \in A$ modulo $\mathbb{Z}\vec{c}$. Then $[[E]] = [[E(\vec{y})]]$ and $\langle\langle [[E]], [[E\vec{y}]] \rangle\rangle = 1$ by Proposition A.2. Conversely, assume $\langle\langle [[E]], [[E(j\vec{\omega})]] \rangle\rangle > 0$. By Proposition A.2 there exists an integer n_0 such that $\vec{y} = j\vec{\omega} + n_0\vec{x}_1$ belongs to A and, moreover, $j\vec{\omega} + n\vec{x}_1 \notin A$ for all integers $n \neq n_0$. Since $E(n\vec{x}_1) = E[n]$, we obtain $\langle\langle [[E]], [[E(j\vec{\omega})]] \rangle\rangle = \langle\langle [[E]], [[E(\vec{y})]] \rangle\rangle = 1$. The proof of (ii) is similar to the previous one. Passing to claim (iii), assume $[[E(j\vec{\omega})]] = [[E]]$ and then $\langle\langle [[E]], [[E(j\vec{\omega})]] \rangle\rangle = \langle\langle [[E]], [[E]] \rangle\rangle = 1$. By (i) we obtain that $j\vec{\omega} \in A$ modulo $\mathbb{Z}\vec{c}$. By the assumption on j this excludes the possibility $j\vec{\omega} = 0$ modulo $\mathbb{Z}\vec{c}$, hence $j\vec{\omega} = \bar{x}_i + n\vec{c}$ for some $i = 1,2,3$ and $n \in \mathbb{Z}$. Since up to a common degree shift E and $E(\bar{x}_i)$ belong to the tilting object from Theorem A.3 we conclude that $[[E(\bar{x}_i)]] \neq [[E]]$, thus contradicting our assumption. □

Recall that the *Coxeter transformation* of a triangulated category $\mathcal{T}$ with Serre duality is the automorphism of the Grothendieck group of $\mathcal{T}$ induced by the Auslander-Reiten translation $\tau = S[-1]$, where S denotes the Serre functor for $\mathcal{T}$. From Lemma 5.10 we then deduce the following, see [11].

Proposition 5.14. *The Coxeter transformation* ϕ *of* $\underline{\tilde{\mathcal{S}}}(p) = \underline{\text{vect}}\text{-}\mathbb{X}(2,3,p)$ *has order* $h = 3$ *for* $p = 2$ *and order* $h = \operatorname{lcm}(6,p)$ *otherwise. Moreover, assuming* $p \geq 3$ *we have* $\phi^{h/2} = -1$ *if and only if* p *is odd.* □

Note that, in classical situations, h is called the *Coxeter number*, a nomination which we extend to the present context.

Proof. Assume $p \geq 3$ such that $h = \operatorname{lcm}(6,p)$. We have $h\vec{\omega} = \delta(\vec{\omega})\vec{c}$ such that $\tau^h = [2\delta(\vec{\omega})]\vec{c}$. Passing to the Grothendieck group of $\underline{\text{vect}}\text{-}\mathbb{X}$ we obtain $\phi^h = 1$. By Theorem A.3 there is a system of Auslander bundles whose classes form a $\mathbb{Z}$-basis of $\mathrm{K}_0(\underline{\text{vect}}\text{-}\mathbb{X})$. Then Lemma 5.13 (iii) implies that h is the precise order of ϕ. Assuming p odd, then Lemma 5.10 implies that $\phi^{h/2} = -1$. Moreover, Lemma 5.13 (iv) shows that this is not the case for p even. □

Shape of the categories $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$. In this subsection we show how the structural results of [23] for $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$, in particular the assertions of the shape of Auslander-Reiten components, follow from Theorem C. The shape of the results depends sensibly on the (orbifold) Euler characteristic $\chi_{\mathbb{X}} = 1/p - 1/6$ of $\mathbb{X}(2,3,p)$. Note that $\chi_{\mathbb{X}}$ is $> 0, = 0, < 0$ if and only if $p < 6$, $p = 6$ or $p > 6$, respectively.

The Auslander-Reiten components of $\text{vect-}\mathbb{X}/[\mathcal{F}]$ or of $\underline{\text{vect}}\text{-}\mathbb{X} = \text{vect-}\mathbb{X}/[\mathcal{L}]$ are those from vect-$\mathbb{X}$ with all line bundles from $\mathcal{F}$ (resp. $\mathcal{L}$) removed. By transport of structure this allows to determine the Auslander-Reiten structure of $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$, thus obtaining the corresponding results of [23]. We remark that it may be deduced from Proposition 4.19 under the functor Φ the Auslander bundles correspond exactly to the boundary modules from [23, Section 5.1].

Fundamental domain under shift. It is shown in [23] that identification $E = E(\vec{x}_3)$ yields for $p \leq 5$ the ungraded invariant subspace problem $\mathcal{S}(p)$. More explicitly, with $\mathbb{X} = \mathbb{X}(2,3,p)$ we have a covering functor $\text{vect-}\mathbb{X}/[\mathcal{F}] = \tilde{\mathcal{S}}(p) \to \mathcal{S}(p)$ with infinite cyclic covering group G generated by the degree shift $\sigma_3 = \sigma(\vec{x}_3)$ with $\vec{x}_3$. In the next proposition we describe explicitly a fundamental domain in $\text{vect-}\mathbb{X}/[\mathcal{F}]$ with respect to this G-action. From [23] one obtains a full embedding of the orbit category $(\text{vect-}\mathbb{X}/[\mathcal{F}])/G \hookrightarrow \tilde{\mathcal{S}}(p)$. It is shown in [23] that for $p \leq 6$ this embedding is actually an equivalence. It is conjectured that for $p \geq 7$ the above embedding is not dense.

To describe a fundamental domain with respect to the G-action, we recall from [7] that the slope of a vector bundle X is defined by $\mu X = \deg X / \operatorname{rk} X$, where the degree deg is the linear form on $\mathrm{K}_0(\text{coh-}\mathbb{X})$ which is uniquely determined by $\deg \mathcal{O}(\vec{x}) = \delta(\vec{x})$ and where $\delta \colon \mathbb{L} \to \mathbb{Z}$ is the homomorphism sending $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$ to $\operatorname{lcm}(6,p)/2$, $\operatorname{lcm}(6,p)/3$, $\operatorname{lcm}(6,p)/p$, respectively.

Proposition 5.15. *Let $\mathbb{X}$ be of weight type $(2,3,p)$, $p \geq 3$. Then the following holds:*

(i) *The indecomposable bundles X not in $\mathcal{F}$ having slope in the range $0 \leq \mu X < \delta(\vec{x}_3)$ form a fundamental domain $\mathcal{D}$ of $\text{vect-}\mathbb{X}/[\mathcal{F}]$ with respect to the $\langle \sigma_3 \rangle$-action.*
(ii) *There are exactly 6 line bundles L with slope in the range $0 \leq \mu L < \delta(\vec{x}_3)$.*
(iii) *$\mathcal{D}$ contains exactly two (persistent) line bundles, one of them from the upper bar the other one from the lower bar.*
(iv) *$\mathcal{D}$ contains exactly 6 Auslander bundles.*

Proof. Assertion (i) follows from the formula $\mu(E(\vec{x}_3)) = \mu E + \delta(\vec{x}_3)$. For (ii) we recall that $\mathbb{Z}\vec{x}_3$ has index 6 in $\mathbb{L}$. Moreover, each $\langle \sigma_3 \rangle$-orbit $\{L(n\vec{x}_3) \mid n \in \mathbb{Z}\}$ contains exactly one line bundle in the given slope range. Assertion (iii) amounts to determine all $\vec{x}$ of shape $n\vec{x}_3$ or $\vec{x}_2 + n\vec{x}_3$ satisfying $0 \leq \delta(\vec{x}) < \delta(\vec{x}_3)$. Claim (iv) is a direct consequence of (ii). □

Positive Euler characteristic. This deals with the cases $p = 2, 3, 4$ and 5. Note that the treatment is related to [9], but except for $p = 5$ deals with a different situation.

Proposition 5.16. *For $2 \leq p \leq 5$ let $\Delta = [2,3,p]$ (resp. $\tilde{\Delta}$) be the attached Dynkin (resp. extended Dynkin) diagram.*

(1) *The Auslander-Reiten quiver of $\text{vect-}\mathbb{X}$ consists of a single standard component. The category of indecomposable vector bundles on $\mathbb{X}$ is equivalent to the mesh category of the Auslander-Reiten component $\mathbb{Z}\tilde{\Delta}$. (The vertices corresponding to persistent (fading) vector bundles will be called persistent (fading).)*
(2) *The Auslander-Reiten quiver Γ of $\text{vect-}\mathbb{X}/[\mathcal{F}] = \tilde{\mathcal{S}}(p)$ consists of a single component. It is obtained from the translation quiver $\mathbb{Z}\tilde{\Delta}$ by deleting the fading vertices and adjacent arrows. The category of indecomposable objects of $\tilde{\mathcal{S}}(p)$ is equivalent to the mesh-category of Γ.*
(3) *The category $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ is equivalent to $\mathrm{D}^b(\text{mod-}K\vec{\Delta}_p)$ for some quiver $\vec{\Delta}_p$ with underlying Dynkin graph $\Delta_2 = \mathbb{A}_2$, $\Delta_3 = \mathbb{D}_4$, $\Delta_4 = \mathbb{E}_6$ and $\Delta_5 = \mathbb{E}_8$.*

Proof. We only sketch the argument, for further details we refer to [11]. One first shows that the direct sum of all indecomposable bundles E with slope in the range $0 \leq \mu E < -\delta(\vec{\omega})$ yields a tilting object T for $\text{coh-}\mathbb{X}$. This allows to prove assertion (1). Assertion (2) then follows from (1) using Theorem C. For (3) we use that the indecomposable summands of T which are not line bundles yield a tilting object T' for $\underline{\text{vect}}\text{-}\mathbb{X}$ whose endomorphism ring is as described in (3). □

By way of example we treat the cases $\tilde{\mathcal{S}}(4)$ and $\tilde{\mathcal{S}}(5)$. In Figure 1 we illustrate a fundamental domain in the Auslander-Reiten quiver of $\tilde{\mathcal{S}}(4)$ modulo the shift action by $\mathbb{Z}\vec{x}_3$. The line bundles are the objects at the upper and lower boundary of the graph. In the following figures the fading line bundles are indicated by circles and the adjacent (fading) arrows are dotted. All other objects (in particular the persistent line bundles) are marked by fat points.

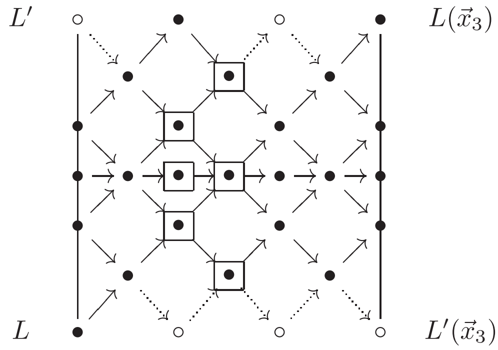


FIGURE 1. Fundamental domain for $\underline{\tilde{\mathcal{S}}}(4)$

We have marked the indecomposable summands of a tilting object for $\underline{\tilde{\mathcal{S}}}(4)$ with endomorphism ring of Dynkin type $\mathbb{E}_6$.

In Figure 2 we illustrate a fundamental domain in the Auslander-Reiten quiver of $\tilde{\mathcal{S}}(5)$ modulo the shift action by $\mathbb{Z}\vec{x}_3$. Here the line bundles are the objects at the lower boundary of the quiver.

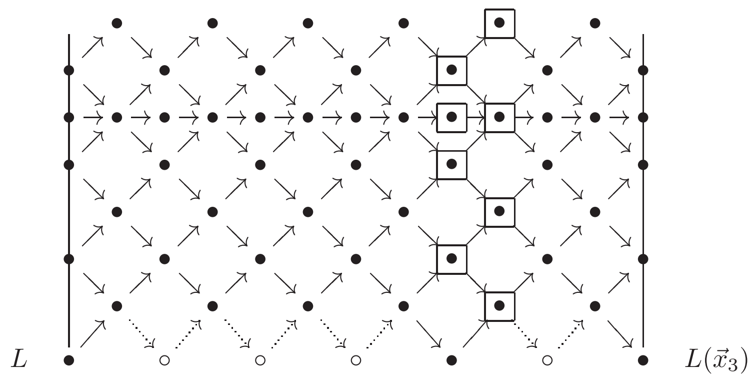


FIGURE 2. Fundamental domain for $\tilde{\mathcal{S}}(5)$

We have marked a tilting object for the stable category $\underline{\text{vect}}$-5 $= \underline{\tilde{\mathcal{S}}}(5)$ with endomorphism ring of Dynkin type $\mathbb{E}_8$.

Euler characteristic zero, the case $p = 6$. For $\chi_{\mathbb{X}} = 0$, that is $p = 6$, the category coh-$\mathbb{X}$ is tubular of type $(2, 3, 6)$. Hence the line bundles are exactly the objects in the tubes of integral slope and of τ-period 6, see [17]. Passing to the factor category vect-$\mathbb{X}/[\mathcal{F}] = \tilde{\mathcal{S}}(p)$ all other Auslander-Reiten components remain unchanged, while the "line bundle components" get the shape from Figure 3

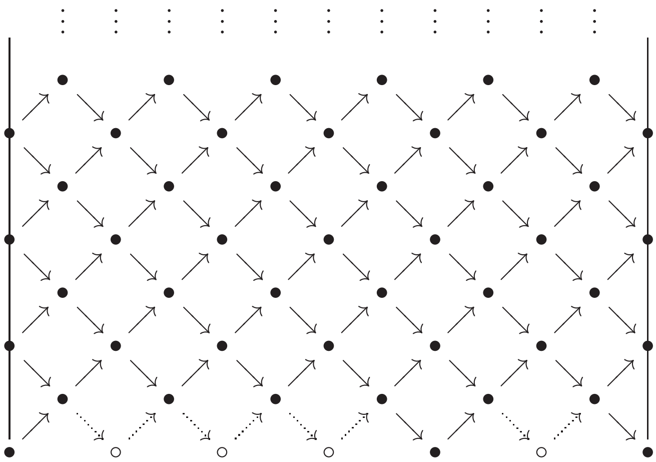


FIGURE 3. Tube for $\tilde{\mathcal{S}}(6)$ containing line bundles.

(line bundles at the lower boundary). Here the identification yields standard (non-stable) tubes of τ-period 6. Concerning the stable category $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(6)$, we have the following result.

Proposition 5.17. *Assume* $\mathbb{X}$ *has weight type* $(2, 3, 6)$. *Then there exists a tilting object in the stable category* $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(6)$ *whose endomorphism ring is the canonical algebra* $\Lambda = \Lambda(2, 3, 6)$. *In particular, we have triangle equivalences* $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(6) \cong \mathrm{D}^b(\text{coh-}\mathbb{X})$.

Proof. We sketch the argument, leaving details to [11]. As shown in [7], the direct sum T of all line bundles $\mathcal{O}(\vec{x}_3 + \vec{x})$ with $\vec{x}$ in the range $0 \le \vec{x} \le \vec{c}$ is a tilting object for coh-$\mathbb{X}$ and $\mathrm{D}^b(\text{coh-}\mathbb{X})$. By [17] there is an auto-equivalence ρ of $\mathrm{D}^b(\text{coh-}\mathbb{X})$ acting on slopes q by $q \mapsto 1/(1 + q)$. It follows that ρT is a bundle whose indecomposable summands have slopes q in the range $1/2 < q < 1$. It follows from this property that ρT is a tilting object for $\underline{\text{vect}}$-$\mathbb{X}$ having all the claimed properties. □

Recall in this context that the category $\mathcal{H} = \text{coh-}\mathbb{X}$ is hereditary, yielding the very concrete description of $\mathrm{D}^b(\text{coh-}\mathbb{X})$ as the *repetitive category* $\bigvee_{n\in\mathbb{Z}} \mathcal{H}[n]$, where each $\mathcal{H}[n]$ is a copy of $\mathcal{H}$ (objects written $X[n]$ with $X \in \mathcal{H}$) and where morphisms are given by $\mathrm{Hom}(X[n], Y[m]) = \mathrm{Ext}^{m-n}_{\mathcal{H}}(X, Y)$ and composition is given by the Yoneda product.

Remark 5.18. The classification of indecomposable bundles over the weighted projective line $\mathbb{X} = \mathbb{X}(2, 3, 6)$ is very similar to Atiyah's classification of vector bundles on a smooth elliptic curve, compare [1] and [17]. Indeed the relationship is very close: Assume the base field is algebraically closed of characteristic different from 2 and 3. If E is a smooth elliptic curve of j-invariant 0, it admits an action of the cyclic group G of order 6 such that the category $\mathrm{coh}_G(E)$ of G-equivariant coherent sheaves on E is equivalent to coh-$\mathbb{X}$. Thus $\underline{\tilde{\mathcal{S}}}(6)$ has the additional description as stable category $\underline{\text{vect}}_G$-$E$ of G-equivariant vector bundles on E.

Negative Euler characteristic. Let $\chi_{\mathbb{X}} < 0$, accordingly $p \ge 7$. Here, the classification problem for $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ is wild. The study of these categories is related to the investigation of Fuchsian singularities in [10, 16] but, with the single exception $p = 7$ yields a different stable category of vector bundles (since only one τ-orbit of line bundles is factored out in the Fuchsian case).

It is shown in [14] that, for $\mathbb{X} = \mathbb{X}(2,3,p)$ and $p \geq 7$, all Auslander-Reiten components for vect-$\mathbb{X}$ have the shape $\mathbb{Z}A_\infty$, and we have exactly $|\mathbb{L}/\mathbb{Z}\vec{\omega}| = p - 6$ components containing a line bundle. Only the shape of these components is affected when passing to the factor category vect-$\mathbb{X}/[\mathcal{F}] = \tilde{\mathcal{S}}(p)$,

Proposition 5.19. *For $p \geq 7$ each Auslander-Reiten component of $\underline{\text{vect}}$-$\mathbb{X}(2,3,p) = \underline{\tilde{\mathcal{S}}}(p)$ is of shape $\mathbb{Z}\mathbb{A}_\infty$. Moreover there is a natural bijection between the set of all Auslander-Reiten components to the set of all regular Auslander-Reiten components over the wild path algebra Λ_0 over the star $[2,3,p]$.*

Proof. Invoking stability arguments, all line bundles lie at the border of their Auslander-Reiten component in vect-$\mathbb{X}$ [14]. Passage to the stable category then shows that all components in $\underline{\text{vect}}$-$\mathbb{X}$ have shape $\mathbb{Z}\mathbb{A}_\infty$. The argument implies, moreover, that there is a natural bijection between the set of Auslander-Reiten components in vect-$\mathbb{X}$ and in $\underline{\text{vect}}$-$\mathbb{X}$, respectively. The claim then follows from [14]. □

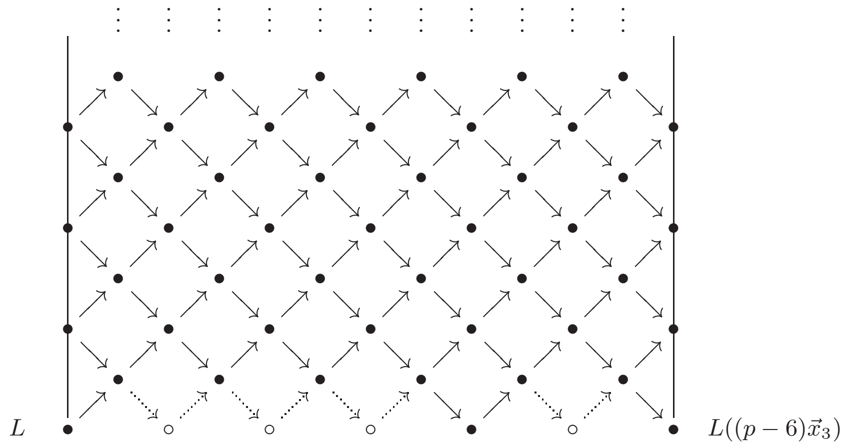


FIGURE 4. Case $p \geq 7$. Fundamental domain for the "distinguished" components

The picture is a nice illustration for Proposition 5.15. For $p \geq 7$ the class of $\vec{x}_3$ is a generator of $\mathbb{L}/\mathbb{Z}\vec{\omega}$ having order $p-6$. Accordingly shift with $\vec{x}_3$ acts on the $(p-6)$-element set of "distinguished" components by cyclic permutation. Figure 4 therefore shows a fundamental domain $\mathcal{D}$ for the $p-6$ "distinguished" components.

ADE-chain. The table below summarizes previous results and displays for $\underline{\text{vect}}$-$\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ with $\mathbb{X}$ of type $(2,3,p)$ the fractional Calabi-Yau dimension, the Euler characteristic $\chi_{\mathbb{X}}$, the Coxeter number h, the representation type, and the derived type of $\underline{\tilde{\mathcal{S}}}(p)$ for small values of p.

p	2	3	4	5	6	7	8	9	p
CY-dim	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{10}{12}$	$\frac{14}{15}$	$\frac{6}{6}$	$\frac{22}{21}$	$\frac{26}{24}$	$\frac{10}{9}$	$\frac{\mathrm{lcm}(3,p)\cdot(1-2\cdot\chi_{\mathbb{X}})}{\mathrm{lcm}(3,p)}$
$\chi_{\mathbb{X}}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{30}$	0	$-\frac{1}{42}$	$-\frac{1}{24}$	$-\frac{1}{18}$	$\frac{1}{p}-\frac{1}{6}$
h	3	6	24	30	6	42	24	18	$\mathrm{lcm}(6,p)$
type	$\mathbb{A}_2$	$\mathbb{D}_4$	$\mathbb{E}_6$	$\mathbb{E}_8$	$(2,3,6)$	$\langle 2,3,7\rangle$	$\langle 2,3,8\rangle$	$\langle 2,3,9\rangle$	$\langle 2,3,p\rangle$
repr. type	repr.-finite				tubular	wild, new type			

TABLE 2. An ADE-chain

The table expresses an interesting property of the sequence of triangulated categories $\underline{\mathrm{vect}}\text{-}\mathbb{X}(2,3,p) = \underline{\tilde{\mathcal{S}}}(p)$. For small values of p, the category $\underline{\mathrm{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(p)$ yields Dynkin type. For $p = 6$ the sequence passes the 'borderline' of tubular type and then continues with wild type. While such situations occur frequently, it is quite rare that one gets an infinite sequence of categories $\mathcal{T}_n$ which all are fractional Calabi-Yau and where the size of $\mathcal{T}_n$, measured in terms of the Grothendieck group, is increasing with n.

Returning to the particular case $\mathcal{T}_p = \underline{\mathrm{vect}}\text{-}\mathbb{X}(2,3,p)$ we know from Theorem A.3 that $\mathcal{T}_p$ has a tilting object T consisting of the Auslander bundles $E \to E(\bar{x}_3) \to E(\bar{x}_1) \to \cdots \to E((p-2)\bar{x}_1) \to E((p-2)\bar{x}_1 + \bar{x}_3)$ and whose endomorphism ring is $A(n,3)$ with $n = 2(p-1)$. This implies that the right perpendicular category formed in $\mathcal{T}_p$ with respect to the exceptional pair consisting of the 'last two' members $E((p-2)\bar{x}_1), E((p-2)\bar{x}_1 + \bar{x}_3)$ of the tilting object T is equivalent to $\mathcal{T}_{p_1}$, implying that $\mathcal{T}_{p_1}$ can be viewed as a triangulated subcategory of $\mathcal{T}_p$ for each $p \geq 3$.

This allows the following attempt in to 'define' the notion of an ADE-chain, by requesting the three properties below:

(1) The triangulated categories $(\mathcal{T}_n)$ form an infinite chain $\mathcal{T}_1 \subset \mathcal{T}_2 \subset \mathcal{T}_2 \subset \cdots$ of triangulated categories with Serre duality which are fractionally Calabi-Yau;
(2) Each category $\mathcal{T}_n$ has a tilting object T_n, hence a Grothendieck group which is finitely generated free of rank n;
(3) The sequence formed by the endomorphism rings $A_p = \mathrm{End}(T_p)$ is a subsequence of a sequence of algebras (B_n) that form an accessible chain of finite dimensional algebras in the sense of [15], that is, $A_1 = k$ and for each integer n the algebra A_{n+1} is a one-point extension or coextension of A_n with an exceptional A_n-module.

In our example the request (3) can be satisfied by means of the algebras $B_n = A(n,3)$. Note, however, that $\mathrm{D}^b(\mathrm{mod}\, A(n,3))$ may fail to be fractionally Calabi-Yau if n is odd.

The special role of the number 6. The numbers 6 and $p-6$ play a special role in dealing with $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$. We advise the reader in this context to check the paper [24] for the ubiquitous appearance of the number 6. Of course this ubiquity of the number 6 has its roots in the relationship between $\tilde{\mathcal{S}}(p)$ and vect-$\mathbb{X}$, where $\mathbb{X}$ has weight type $(2,3,p)$. The following list displays a number of appearances of the two numbers:

(1) The group $\mathbb{L}/\mathbb{Z}\vec{x}_3$ is cyclic of order 6 generated by the class of $\vec{\omega}$.
(2) We have $6\vec{\omega} = (p-6)\vec{x}_3$, thus $\tau^6 = \sigma_3^{p-6}$ holds in vect-$\mathbb{X}$, $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$.
(3) The partition of line bundles into persistent and fading ones obeys the 6-periodic pattern $+-+---$ in each τ-orbit, where $+$ and $-$ stand for persistent and fading, respectively.

(4) Euler characteristic $\chi_{\mathbb{X}}$ of $\mathbb{X}$ and fractional Calabi-Yau dimension d_p of $\underline{\text{vect}}\text{-}\mathbb{X} = \underline{\tilde{\mathcal{S}}}(6)$ are given by $\chi_{\mathbb{X}} = 1/p - 1/6$ and $d_p = (4p-6)/3p$ (up to cancelation), respectively.
(5) The borderline between (derived) representation-finiteness and wildness for $\tilde{\mathcal{S}}(p)$ and $\underline{\tilde{\mathcal{S}}}(p)$ is marked by $p = 6$.
(6) If p and 6 are coprime, then the Auslander-Reiten translation τ on $\underline{\tilde{\mathcal{S}}}(p)$ has a unique $(p-6)$th root in the Picard group.

Additionally we refer to Proposition 5.15 for further occurrences of the number 6.

Appendix A. Existence of a tilting object

The existence of a tilting object T is central for many applications discussed in the previous section. We prove the absence of self-extensions of T by a combination of two methods: (a) the determinant argument from Lemma A.1 and (b) the use of line bundle filtrations for the indecomposable summands of T. Recall that $\bar{x}_i = \vec{x}_i + \vec{\omega}$ for $i = 1, 2, 3$.

Lemma A.1. *Assume weight type $(2, a, b)$, and let E be an Auslander bundle. Then for any $\vec{x} \in \mathbb{L}$ we have $\underline{\text{Hom}}(E, E(\vec{x})) = \text{D}\,\underline{\text{Hom}}(E, E(\bar{x}_1 - \vec{x}))$. In particular $\underline{\text{Hom}}(E, E(\vec{x})) \neq 0$ implies $2\vec{x} \geq 0$ and $2(\bar{x}_1 - \vec{x}) \geq 0$.*

Proof. By Serre duality we obtain that $\underline{\text{Hom}}(E, E(\vec{x})) = \underline{\text{Hom}}(E, E(\vec{x} - \vec{x}_1)[1]) = \text{D}\,\underline{\text{Hom}}(E(\vec{x} - \vec{x}_1), E(\vec{\omega})) = \text{D}\,\underline{\text{Hom}}(E, E(\bar{x}_1 - \vec{x}))$, yielding the first claim. Assume now that $u\colon E \to E(\vec{x})$ becomes non-zero in $\underline{\text{vect}}\text{-}\mathbb{X}$. Then Lemma 3.7 implies the second claim using that $\det(E(\vec{x})) - \det(E) = 2\vec{x}$. □

Our next result is fundamental to establish a tilting bundle consisting of Auslander bundles. We provide an elementary proof. For a more conceptual treatment of the topic we refer to [11].

Proposition A.2. *Assume weight type $(2, 3, p)$. For each Auslander bundle E we then have*

$$\underline{\text{Hom}}(E, E(\vec{x})) \neq 0 \quad \Longleftrightarrow \quad \vec{x} \in \{0, \bar{x}_1, \bar{x}_2, \bar{x}_3\}.$$

Moreover, there exist monomorphisms $u_i\colon E \to E(\bar{x}_i)$ such that

$$\underline{\text{Hom}}(E, E(\bar{x}_i)) = \text{Hom}(E, E(\bar{x}_i)) = k\, u_i \quad i = 1, 2, 3.$$

Further, the morphisms u_1, $u_2 u_3$ and $u_3 u_2$ agree, up to multiplication by a non-zero scalar.

Proof. Step 1: We show first that $\underline{\text{Hom}}(E, E(\bar{x}_i)) = \text{Hom}(E, E(\bar{x}_i))$. Assume that we have a factorization $[E \xrightarrow{\alpha} L(\vec{x}) \xrightarrow{\beta} E\bar{x}_i] \neq 0$ for some $\vec{x} \in \mathbb{L}$, and some $i = 1, 2, 3$. Then $(0 \leq \vec{x}$ or $\vec{\omega} \leq \vec{x})$ and $(\vec{x} \leq \bar{x}_i$ or $\vec{x} \leq \bar{x}_i + \vec{\omega})$. This only leaves the possibilities $\vec{x} = \vec{\omega}$ or $\vec{x} = \bar{x}_i$. The choice $\vec{x} = \vec{\omega}$ yields a non-zero member $E \xrightarrow{\alpha} L(\vec{\omega}) \xrightarrow{j} E$ of $\text{End}(E) = k$ resulting in a splitting of E. The second choice $\vec{x} = \bar{x}_i$ similarly yields a splitting of $E(\bar{x}_i)$ which again is impossible.

Step 2: We know already that $\underline{\text{End}}(E) = k$. Applying $\text{Hom}(-, L(\bar{x}_i))$ to $\eta\colon 0 \to L(\vec{\omega}) \xrightarrow{\iota} E \xrightarrow{\pi} L \to 0$ we obtain an isomorphism

$$(1) \quad \text{Hom}(E, L(\bar{x}_i)) \xrightarrow{-\circ\iota} \text{Hom}(L(\vec{\omega}), L(\bar{x}_i)).$$

Then applying $\text{Hom}(E, -)$ to $\eta(\bar{x}_i)$, we obtain another isomorphism

$$(2) \quad \text{Hom}(E, E(\bar{x}_i)) \xrightarrow{\pi(\bar{x}_i)\circ -} \text{Hom}(E, L(\bar{x}_i)).$$

Combining (1) and (2) we see that $\text{Hom}(E, E(\bar{x}_i)) \cong \text{Hom}(L, L(\vec{x}_i)) = k$. This shows the existence of the u_i. Assume that there is a non-zero composition $E \xrightarrow{\alpha}$

$L(\vec{x}) \xrightarrow{\beta} E(\bar{x}_i)$ for some line bundle $L(\vec{x})$, then $\vec{x} = \omega$ or $\vec{x} = \bar{x}_i$. In either case there results a non-trivial splitting of E or $E(\bar{x}_i)$ which is impossible. Hence $\underline{\mathrm{Hom}}(E, E(\vec{x}_i)) = \mathrm{Hom}(E, E(\vec{x}_i)) = ku_i$ for $i = 1, 2, 3$. By Step 1 the u_i are monomorphisms, hence the second claim follows from $\bar{x}_1 = \bar{x}_2 + \bar{x}_3$.

Step 3: We have (a) $\underline{\mathrm{Hom}}(E, E(\vec{x})) = 0$ for all $\vec{x} > 0$, (b) $\underline{\mathrm{Hom}}(E, E(\bar{x}_2 - a\vec{x}_3)) = 0$ for all $a \geq 1$, and (c) $\underline{\mathrm{Hom}}(E, E(\bar{x}_3 + a\vec{x}_3)) = 0$ for all $a \geq 1$.

ad (a): By Lemma A.1 it is equivalent to show that $\underline{\mathrm{Hom}}(E(\vec{x}), E(\bar{x}_1)) = 0$. By Proposition 3.8 we have $\mathrm{D}\,\mathrm{Hom}(E(\vec{x}_1), E(\bar{x}_1)) = \underline{\mathrm{Hom}}(E(\vec{x}_1), (E(\vec{x}_1))[1]) = 0$, we may hence assume that $0 < \vec{x} \neq \vec{x}_1$. Now we have $\vec{x}_1 - \vec{x} \not\geq 0$ and $(\bar{x}_1 + \vec{\omega}) - \vec{x} \geq 0$, since otherwise $\bar{x}_1 \geq 0$, respectively $\vec{x}_1 + \vec{\omega} \geq 0$ which both is impossible. Finally $\bar{x}_1 - (\vec{x} + \vec{\omega}) = \vec{x}_1 - \vec{x} \not\geq 0$ since otherwise $0 < \vec{x} < \vec{x}_1$ which again is impossible. We conclude that $\mathrm{Hom}(E(\vec{x}), E(\bar{x}_1)) = 0$ for $\vec{x} > 0$.

ad (b) and (c): Since $\bar{x}_1 - (\bar{x}_3 + a\vec{x}_3) = \bar{x}_2 - a\vec{x}_3$, it suffices to show property (b). As for the proof of (a) we check the existence of non-zero morphisms between the line bundle factors $L, L(\vec{\omega})$ for E and $\bar{x}_2 - a\vec{x}_3 + \vec{\omega}$ for $E(\bar{x}_2 - a\vec{x}_3)$. That is, for $\vec{y} = \bar{x}_2 - a\vec{x}_3$ we need to show that none of $\vec{y}, \vec{y} \pm \vec{\omega}$ is ≥ 0. Indeed, $\vec{y} = \bar{x}_2 - a\vec{x}_3 = \vec{c} - \vec{x}_1 - (a+1)\vec{x}_3 \not\geq 0$, $\vec{y} - \vec{\omega} = \vec{x}_2 - a\vec{x}_3 \not\geq 0$, and $\vec{y} + \vec{\omega} = \vec{c} - \vec{x}_2 - (a+2)\vec{x}_3 \not\geq 0$.

Step 4: We are now in a position to prove the first claim of the proposition. Assume $\vec{x} \notin \{0, \vec{x}_1, \vec{x}_2, \vec{x}_3\}$ and $\underline{\mathrm{Hom}}(E, E(\vec{x})) \neq 0$. By Step 3 we may assume that $2\vec{x} \not\geq 0$, $\bar{x}_1 - \vec{x} \not\geq 0$, and (*) $0 < 2\vec{x} < 2\bar{x}_1 = \vec{x}_2 + (p-2)\vec{x}_3$. Therefore the normal form of $\vec{x}$ has the form (**) $\vec{x} = \ell_1\vec{x}_1 + \ell_2\vec{x}_2 + \ell_3\vec{x}_3 - \vec{c}$ where $0 \leq \ell_1 \leq 1$, $0 \leq \ell_2 \leq 2$ and $0 \leq \ell_3 \leq p-1$. Comparison of (*) and (**) then only allows the following two cases: 1) $2\vec{x} = a\vec{x}_3$ and 2) $2\vec{x} = \vec{x}_2 + a\vec{x}_3$ with $0 \leq a \leq p-2$.

Case 1: Here we obtain $\ell_2 = 0$, implying that $\ell_1 = 1$ and $p/2 \leq \ell_3 \leq p-1$. We then get $\vec{x} = \vec{x}_1 + \ell_3\vec{x}_3 - \vec{c} = \bar{x}_2 - a\vec{x}_3$ with $a = p - (\ell_3 + 1) > 0$ since the possibility $\vec{x} = \bar{x}_2$ was excluded. Now Step 3(b) yields $\underline{\mathrm{Hom}}(E, E(\vec{x})) = 0$, contradicting our assumption.

Case 2: Here we obtain $\ell_2 = 2$, leaving just two possibilities for (ℓ_1, ℓ_3):

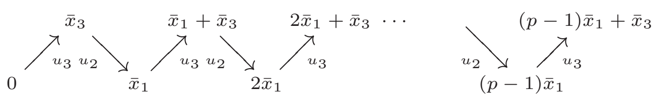
a) $(\ell_1, \ell_3) = (0, \ell_3)$ with $p/2 \leq \ell_3 \leq p-1$, and

b) $(\ell_1, \ell_3) = (1, \ell_3)$ with $0 \leq \ell_3 \leq (p-2)/2$.

In case a) we have $\vec{x} = 2\vec{x}_2 + \ell_3\vec{x}_3 - \vec{c}$ with $p/2 \leq \ell_3 \leq p-1$. Since $\bar{x}_1 - \vec{x} = a\vec{x}_3$ with $a = p - (\ell_3 + 1) \geq 0$, then Step 3(a) implies in view of Lemma A.1 that $\underline{\mathrm{Hom}}(E, E(\vec{x})) = 0$, contradicting our assumption. Finally in case b) we have $\vec{x} = \vec{x}_1 + 2\vec{x}_2 + \ell_3\vec{x}_3 - \vec{c} = \vec{x}_1 - \vec{x}_2 + \ell_3\vec{x}_3 = \bar{x}_3 + \ell_3\vec{x}_3$ with $\ell_3 > 0$. Then Step 3(c) implies the contradiction $\underline{\mathrm{Hom}}(E, E(\vec{x})) = 0$. □

Theorem A.3. *Assume $\mathbb{X}$ has weight type $(2, 3, p)$, E is an Auslander bundle on $\mathbb{X}$, and $M = \{a\bar{x}_1 + b\bar{x}_3 \mid a = 0, \ldots, p-2$ and $b = 0, 1\}$. Then $T = \bigoplus_{\vec{x} \in M} E(\vec{x})$ is a tilting object in $\underline{\mathrm{vect}}$-$\mathbb{X}$ with endomorphism ring isomorphic to $A(2(p-1), 3)$.*

Proof. Endomorphism ring: To calculate $\underline{\mathrm{End}}(T)$ we arrange the summands $E(\vec{x})$ according to the scheme



Then Proposition A.2 implies the claim on the endomorphism ring of T.

(A) T is extensionfree: We show that $\underline{\mathrm{Hom}}(E(\vec{x}), E(\vec{y})[n]) \neq 0$ with $\vec{x}, \vec{y} \in M$ and $n \in \mathbb{Z}$ implies that $n = 0$. From the assumption we obtain the existence of $\vec{z} \in \{0, \bar{x}_1, \bar{x}_2, \bar{x}_3\}$ where $\vec{c} = \vec{y} - \vec{x} + n\vec{x}_1$, and then $\vec{z} = \alpha\bar{x}_1 + \beta\bar{x}_3 + n\vec{x}_1$ with $|\alpha| \leq p-2$, $|\beta| \leq 1$ and $n \in \mathbb{Z}$. Passing to congruences in $\mathbb{L}$ modulo the subgroup

$\langle \vec{x}_1, \vec{x}_2 \rangle$, generated by $\vec{x}_1$ and $\vec{x}_2$, we obtain

$$-\alpha\vec{x}_3 \equiv \vec{z} \equiv \begin{cases} 0 & \text{if } \vec{z} \in \{0, \bar{x}_3\} \\ -\vec{x}_3 & \text{if } \vec{z} \in \{\bar{x}_1, \bar{x}_2\}. \end{cases}$$

Since $\mathbb{L}/\langle \vec{x}_1, \vec{x}_2 \rangle$ is cyclic of order p and generated by the class of $\vec{x}_3$, we deduce from $|\alpha| \leq p-2$ that $\alpha = 0$ for $\vec{z} \in \{0, \bar{x}_3\}$ (case a) or $\alpha = 1$ for $\vec{z} \in \{\bar{x}_1, \bar{x}_2\}$ (case b).

case a: If $\vec{z} = 0$, then $\beta\bar{x}_3 = -n\vec{x}_1$ with $|\beta| \leq 1$. This is only possible for $\beta = 0$ which then implies $n = 0$. If $\vec{z} = \vec{x}_3$, then $(\beta - 1)\bar{x}_3 = -n\vec{x}_1$ with $\beta \in \{-2, -1, 0\}$. This is only possible for $\beta = 0$, again implying $n = 0$.

case b: Again, we have to deal with two cases: If $\vec{z} = \bar{x}_1$ then $\alpha = 1$, and $\beta\bar{x}_3 = -n\vec{x}_1$ with $|\beta| \leq 1$. As in case a this implies $n = 0$. If $\vec{z} = \bar{x}_2$, then $\alpha = 1$, and $(\bar{x}_1 - \bar{x}_3) + (\beta - 1)\bar{x}_3 = \bar{x}_2 - n\vec{x}_1$. Since $\bar{x}_1 - \bar{x}_3 = \bar{x}_2$, we conclude that $n = 0$ as in the second part of case a.

We have thus shown that T has no self-extensions in $\underline{\text{vect}}$-$\mathbb{X}$. It then follows from an $\mathbb{L}$-graded version of Orlov's theorem, compare [11], that T has the correct number $2(p-1)$ of indecomposable summands to make T tilting in $\underline{\text{vect}}$-$\mathbb{X}$. □

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