

COUPLED EQUATIONS FOR KÄHLER METRICS AND YANG–MILLS CONNECTIONS

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ABSTRACT. We study equations on a principal bundle over a compact complex manifold coupling a connection on the bundle with a Kähler structure on the base. These equations generalize the conditions of constant scalar curvature for a Kähler metric and Hermite–Yang–Mills for a connection. We provide a moment map interpretation of the equations and study obstructions for the existence of solutions, generalizing the Futaki invariant, the Mabuchi K-energy and geodesic stability. We finish by giving some examples of solutions.

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INTRODUCTION

In this paper we consider a system of partial differential equations coupling a Kähler metric on a compact complex manifold and a connection on a principal bundle over it. These equations, inspired by the Hitchin–Kobayashi correspondence for bundles and the Yau–Tian–Donaldson conjecture for constant scalar curvature Kähler (cscK) metrics, intertwine the curvature of a Hermitian–Yang–Mills (HYM) connection on the bundle and the scalar curvature of a Kähler metric on the manifold.

To write our equations explicitly, let X be a smooth compact manifold and let G be a compact real Lie group with Lie algebra \mathfrak{g} . Let E be a principal G -bundle over X . We

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fix a positive definite inner product (\cdot, \cdot) on \mathfrak{g} invariant under the adjoint representation. Let Ω^k be the space of smooth k -forms on X . Considering the space $\Omega^k(\text{ad } E)$ of smooth k -forms on X with values in the adjoint bundle $\text{ad } E$, the inner product of \mathfrak{g} induces a pairing

$$\Omega^p(\text{ad } E) \times \Omega^q(\text{ad } E) \longrightarrow \Omega^{p+q}, \quad (0.1)$$

that we write simply as $a_p \wedge a_q$ for any $a_j \in \Omega^j(\text{ad } E)$, $j = p, q$. The unknown variables of the equations are a Kähler structure (g, ω, J) on the base X and a connection A on E , where g , ω and J are respectively the metric, the symplectic form and the complex structure. We will say that a Kähler structure (g, ω, J) on X and a connection A on E satisfy the *coupled Kähler–Yang–Mills equations* with coupling constants $\alpha_0, \alpha_1 \in \mathbb{R}$ if

$$\left. \begin{aligned} \Lambda F_A &= z \\ \alpha_0 S_g + \alpha_1 \Lambda^2(F_A \wedge F_A) &= c \end{aligned} \right\}. \quad (0.2)$$

Here S_g is the scalar curvature of g , F_A is the curvature of A , z is an element of \mathfrak{g} which is invariant under the adjoint G -action and c is a real number. The precise values of z and c are determined by the topology of E , the cohomology class of ω and the coupling constants α_0, α_1 (see Remark 1.2 and (2.11)). The map $\Lambda: \Omega^{p,q}(\text{ad } E) \rightarrow \Omega^{p-1,q-1}(\text{ad } E)$ is the contraction operator acting on (p, q) -type valued forms determined by the Kähler structure. In the sequel, we will refer to (0.2) simply as *the coupled equations*.

A link with holomorphic geometry is provided by the additional integrability condition

$$F_A^{0,2} = 0, \quad (0.3)$$

between the complex structure J on the base and the connection A . Here $F_A^{0,2}$ denotes the $(0, 2)$ part of the curvature, regarded as an $(\text{ad } E)$ -valued smooth form on X . Let G^c be the complexification of the group G . When (0.3) holds, the pair (J, A) endows the associated principal G^c -bundle $E^c = E \times_G G^c$ with a structure of holomorphic principal bundle over the complex manifold (X, J) .

The moment map interpretations of the constant scalar curvature equation for a Kähler metric (cscK) and the HYM equation provide a guiding principle, leading to (0.2). Indeed, equations (0.2) have an interpretation in terms of a moment map. This is the subject of §1 and §2. As observed by Fujiki [21] and Donaldson [17], the cscK equation has a moment map interpretation in terms of a symplectic form ω on the smooth compact manifold X . The group of symmetries of the theory for cscK metrics is the group \mathcal{H} of Hamiltonian symplectomorphisms. This group acts on the space \mathcal{J}^i of integrable almost complex structures on X which are compatible with ω , and this action is Hamiltonian for a natural symplectic form $\omega_{\mathcal{J}}$ on \mathcal{J}^i . The moment map interpretation of the HYM equation was pointed out first by Atiyah and Bott [4] for the case of Riemann surfaces and generalized by Donaldson [15] to higher dimensions. Here one considers the symplectic action of the gauge group \mathcal{G} of the bundle E on the space of connections \mathcal{A} endowed with a natural symplectic form $\omega_{\mathcal{A}}$. Relying on these two previous cases, the phase space for our theory is provided by the subspace of the product

$$\mathcal{P} \subset \mathcal{J}^i \times \mathcal{A} \quad (0.4)$$

defined by the condition (0.3). Our choice of symplectic structure is the restriction to \mathcal{P} of the symplectic form

$$\omega_{\alpha} = \alpha_0 \omega_{\mathcal{J}} + 4\alpha_1 \omega_{\mathcal{A}}, \quad (0.5)$$

for a pair of non-zero coupling constants $\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}^2$.

Consider now the *extended gauge group* $\tilde{\mathcal{G}}$ defined as the group of automorphisms of the bundle E covering Hamiltonian symplectomorphisms of X . This is a non trivial extension

$$1 \rightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \longrightarrow \mathcal{H} \rightarrow 1, \quad (0.6)$$

where \mathcal{G} is the group of automorphisms of E covering the identity on X , and \mathcal{H} , as above, is the group of Hamiltonian symplectomorphisms of X . The group $\tilde{\mathcal{G}}$ acts on \mathcal{P} and in Proposition 2.1 we show that this action is Hamiltonian for any value of the coupling constants, we compute a moment map μ_α , and show that its zero locus corresponds to solutions of (0.2). The coupling between the metric and the connection occurs as a direct consequence of the structure of $\tilde{\mathcal{G}}$. So, away from its singularities, the moduli space of solutions is given by the *symplectic quotient*

$$\mathcal{M}_\alpha = \mu_\alpha^{-1}(0)/\tilde{\mathcal{G}}. \quad (0.7)$$

Furthermore, ω_α is a Kähler form on \mathcal{P} when $\alpha_1/\alpha_0 > 0$, for a natural $\tilde{\mathcal{G}}$ -invariant complex structure on \mathcal{P} . Hence under this condition on the coupling constants, the smooth locus of the moduli space of solutions (0.7) inherits a Kähler structure.

We see that our problem merges the well-studied theories of Hermitian–Yang–Mills connections (obtained for $\alpha_1/\alpha_0 > 0$) and constant scalar curvature Kähler metrics (which correspond to $\alpha_1/\alpha_0 = 0$) into a unique theory. We thus expect the Kähler moduli spaces obtained in our symplectic reduction process to have a rich geometry and topology. In §2.3 we prove that (0.2) arise also as absolute minima of a purely Riemannian functional for G -invariant Riemannian metrics on the total space of E , providing a link to the classical Kaluza–Klein theory.

In §3, which is in some sense the heart of the paper, we undertake the study of obstructions for the existence of solutions to (0.2), generalizing the Futaki invariant, the Mabuchi K-energy and geodesic stability that appear in the cscK theory [22, 41, 11]. We do this geometrically, by considering the following framework. We first fix a cohomology class $\Omega \in H^2(X, \mathbb{R})$ and a smooth principal G^c -bundle $E^c \rightarrow X$. Let Γ be the Lie group given by those G^c -equivariant diffeomorphisms of E^c which cover an element in the connected component of the identity of the diffeomorphism group of X . In §3.1, we associate an infinite dimensional canonical Γ -equivariant double fibration

$$\begin{array}{ccc} & \mathcal{C} & \\ \pi_{\mathcal{B}} \swarrow & & \searrow \pi_{\mathcal{Z}} \\ \mathcal{B} & & \mathcal{Z} \end{array} \quad (0.8)$$

to the data (X, Ω, E^c) . Here \mathcal{B} is the space of pairs (ω, H) , where $\omega \in \Omega$ is a symplectic form on X and H is a reduction of E^c to the maximal compact subgroup $G \subset G^c$, and \mathcal{Z} is a space parameterizing holomorphic structures I on E^c inducing a complex structure J on X . The space of *compatible pairs* $\mathcal{C} \subset \mathcal{B} \times \mathcal{Z}$ is defined as those elements of the product which induce a Kähler structure on X . Using the results of §2, in §3.1 we prove that the fibres of $\pi_{\mathcal{B}}$ are (formally) Kähler manifolds endowed with Hamiltonian group actions.

As a preliminary step for the study of obstructions in §3.3, we prove in §3.2 that the fibres of $\pi_{\mathcal{Z}}$ are infinite dimensional symmetric spaces (that is, each fibre has a canonical torsion-free affine connection ∇ with covariantly constant curvature), with holonomy group contained in the extended gauge group. Note that the fibre B_I of $\pi_{\mathcal{Z}}$ over $I \in \mathcal{Z}$ is

$$B_I = \mathcal{K}_J \times \mathcal{R}, \quad (0.9)$$

where \mathcal{K}_J is the space of Kähler forms on (X, J) in the class Ω and \mathcal{R} is the space of G -reductions of E^c . When specialized to the case of trivial G^c , we recover the symmetric space structure constructed by Mabuchi [42] and rediscovered by Semmes [51] and Donaldson [18]. Our construction follows closely Donaldson's in [18, §2]. A special feature of the symmetric space structure on B_I is that in general it does not carry any canonical compatible Riemannian structure (see Remark 4.4). A technical assumption in our construction is that the G -invariant metric in \mathfrak{g} used to define (0.1) extends to a G^c -invariant symmetric bilinear pairing

$$\mathfrak{g}^c \otimes \mathfrak{g}^c \longrightarrow \mathbb{C},$$

where \mathfrak{g}^c is the complexification of \mathfrak{g} .

In §§3.3 and 3.4 we construct an α -Futaki character

$$\mathcal{F}_I: \text{Lie Aut}(E^c, I) \longrightarrow \mathbb{C},$$

which is a complex character of the Lie algebra of the automorphism group of the holomorphic principal bundle (E^c, I) and which vanishes when (0.2) is satisfied, and an α -K-energy

$$\mathcal{M}_I: B_I \longrightarrow \mathbb{R},$$

which is convex along geodesics on B_I and bounded from below when (0.2) is satisfied, provided that the symmetric space B_I is geodesically convex. Furthermore, we motivate a definition of *geodesic stability* of the orbit $\Gamma \cdot I$ and conjecture a link with (0.2) when Γ_I is finite. We give explicit formulae for the character \mathcal{F}_I , the functional \mathcal{M}_I and the geodesic equation on B_I . When specialized to the case in which G^c is trivial, we recover the Futaki character [22], the Mabuchi K-energy [41] and the notion of geodesic stability [11, 18] used in the study of the cscK equation for Kähler metrics. The contents of §3.4 will be used in Example 5.9 to provide an explicit situation in which there cannot exist solutions to the coupled equations.

We would like to point out that the framework developed in §3 is rather general and may be applied to other situations, in particular, to equations with a further coupling with Higgs fields.

In §4, we establish sufficient conditions for the existence of solutions to the coupled equations near a given solution, when the coupling constants and the Kähler cohomology class are deformed while the complex structure of the base manifold remains fixed. Our approach is based on a generalization of techniques developed by LeBrun & Simanca [38, 39] for the corresponding problem in the cscK theory. We fix a complex structure on X and a structure I of a holomorphic principal G^c -bundle on E^c , and consider the space of solutions (ω, H) of (0.2) with ω in a fixed cohomology class Ω and fixed α_0, α_1 . Then we study the behaviour of this space with respect to deformations of the coupling constants and the Kähler class in a parameter space:

$$(\alpha_0, \alpha_1, \Omega) \in \mathbb{R}^2 \times H^{1,1}(X, \mathbb{R}). \quad (0.10)$$

Before doing this, in §4.1 we introduce the notion of *extremal pairs* (ω, H) . They are analogues in our theory of Calabi's extremal metrics in Kähler geometry. In particular, an extremal pair (ω, H) , with $\omega \in \Omega$, is a solution of (0.2) if and only if the α -Futaki character associated to I and Ω vanishes (Proposition 4.2). In §4.3 we study the linearization of (0.2) and in §4.4 we prove that when $\alpha_1/\alpha_0 > 0$, any solution of the coupled equations (0.2) can be deformed into an extremal pair, for small deformations in the parameter space (0.10)

(Theorem 4.10). In §4.5 we obtain a criterion for the existence of solutions of (0.2) in the weak coupling limit $\alpha_1 \rightarrow 0$, i.e. for $0 < |\alpha_1/\alpha_0| \ll 1$ (Theorem 4.18).

In §5 we discuss some examples of solutions of (0.2) and explain how the existence of solutions to the limit case $\alpha_0 = 0$ can be applied, using results of Y. J. Hong in [30], to obtain cscK metrics on ruled manifolds. As for the examples, in §5.1 we deal with the case of vector bundles over Riemann surfaces and projectively flat bundles over Kähler manifolds satisfying a topological constraint. In both situations, the coupled system (0.2) reduces to the limit case $\alpha_1 = 0$ (cscK equation and HYM equation). When $\dim_{\mathbb{C}} X > 1$, we use Theorem 4.10 to deform the Kähler class and provide non-trivial examples of solutions. In §5.2 we consider homogeneous Hermitian holomorphic vector bundles over homogeneous Kähler manifolds. In §5.3 we discuss some (well known) examples of stable bundles over Kähler–Einstein manifolds where Theorem 4.18 applies. Section 5.3 provides examples of solutions in which the Kähler metric is not cscK and also examples where the invariant \mathcal{F}_I obstructs the existence of solutions for small ratio of the coupling constants.

Coupled equations for metrics and connections have of course been studied for a long time in the context of unified field theories in physics and more recently in string theory (see e.g. [40]). They have also been considered in the context of Riemannian geometry, like the Einstein–Maxwell equations on 4-manifolds studied in [37]. Our motivation, however, for this work has been to find a Kähler analogue of these situations. Another important motivation for us comes from the relation with algebraic geometry, in particular with the moduli problem for pairs consisting of a polarised manifold and a holomorphic bundle over it. Despite its intrinsic mathematical interest and its relevance in theoretical physics, the latter problem has been little explored, probably due to the hard technical difficulties which arise in the algebro-geometric approach as soon as the complex dimension of the base is greater than 1 (see [27, 9, 47] for the case of curves, and [31, 50] for higher dimensions). Throughout this paper we hope to show that the study of our coupled equations provides a reasonable differential-geometric approach to the moduli problem for bundles and varieties, giving compelling evidence of the existence of a Hitchin–Kobayashi correspondence for the coupled equations as has been conjectured in [25].

Since this paper was finished, there have been several developments in the theory of the coupled Kähler–Yang–Mills equations. Keller and Tønnesen-Friedman [32] have found solutions on line bundles over complex threefolds (with positive ratio $\alpha > 0$) that do not admit any cscK metric in the class of the polarization. The second author jointly with C. Tipler have recently found new examples of solutions [26], by deformation of the holomorphic structure on a homogeneous bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. More remarkably, an interesting relation between the coupled equations and physical equations, describing gravitating vortices over a Riemann surface, has been recently found by the authors [2]. These vortices represent the coupling of gravity and a condensed matter system and are known in the physics literature as cosmic strings (or topological defects) in the Abelian Higgs model. Based on classical results by Y. Yang [60], this relation provides a plethora of solutions of the coupled equations in $\mathbb{P}^1 \times \mathbb{P}^1$ and a explicit (conjectural) description of the moduli space (0.7) and the stability condition for this particular case.

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1. HAMILTONIAN ACTION OF THE EXTENDED GAUGE GROUP

In this section we define the *extended gauge group* $\tilde{\mathcal{G}}$ of a bundle over a compact symplectic manifold, an extension of the infinite dimensional Lie groups involved in the moment map problems for the HYM and the cscK equation. We show that the action of $\tilde{\mathcal{G}}$ on the space of connections of the bundle is Hamiltonian and compute an equivariant moment map. Symplectic reductions by Lie group extensions have been studied in the literature in various degrees of generality (see [43] and references therein). Previous work includes split group extensions and more general ones, although it seems that the moment map calculations of §1.3, based on Proposition 1.3, have not been previously made (cf. [43, §3.2]).

1.1. The Hermitian–Yang–Mills equation. First we set out some notation in order to review the moment map interpretations of the HYM equation. Let X be a compact symplectic manifold of dimension $2n$, with symplectic form ω , G a real compact Lie group with Lie algebra \mathfrak{g} , and E a smooth principal G -bundle over X , with the G -action on the right. In the sequel $\omega^{[k]}$ will denote $\frac{\omega^k}{k!}$. The spaces of smooth k -forms on X and smooth k -forms with values in any given vector bundle F on X are denoted by Ω^k and $\Omega^k(F)$, respectively. Fix a positive definite inner product on \mathfrak{g} , invariant under the adjoint action, denoted

$$(\cdot, \cdot) : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathbb{R}.$$

This product induces a metric on the adjoint bundle $\text{ad } E = E \times_G \mathfrak{g}$, which extends to a bilinear map on $(\text{ad } E)$ -valued differential forms (we use the same notation as in [4, §3])

$$\begin{aligned} \Omega^p(\text{ad } E) \times \Omega^q(\text{ad } E) &\longrightarrow \Omega^{p+q} \\ (a_p, a_q) &\longmapsto a_p \wedge a_q. \end{aligned} \tag{1.1}$$

We consider the operator

$$\begin{aligned} \Lambda = \Lambda_\omega : \Omega^k &\longrightarrow \Omega^{k-2} \\ \psi &\longmapsto \omega^\sharp \lrcorner \psi, \end{aligned} \tag{1.2}$$

where \sharp is the operator acting on k -forms induced by the symplectic duality $\sharp : T^*X \rightarrow TX$ and \lrcorner denotes the contraction operator. Its linear extension to $\Omega^k(\text{ad } E)$ is also denoted $\Lambda : \Omega^k(\text{ad } E) \rightarrow \Omega^{k-2}(\text{ad } E)$ (we use the same notation as, e.g., in [15]).

Let \mathcal{A} be the set of connections on E . This is an affine space modelled on $\Omega^1(\text{ad } E)$, with a left action of the gauge group \mathcal{G} of E , i.e. the group of G -equivariant diffeomorphisms of E covering the identity map on X . The 2-form on \mathcal{A} defined by

$$\omega_{\mathcal{A}}(a, b) = \int_X a \wedge b \wedge \omega^{[n-1]} \tag{1.3}$$

for $a, b \in T_A \mathcal{A} = \Omega^1(\text{ad } E)$, $A \in \mathcal{A}$, is a \mathcal{G} -invariant symplectic form. As observed by Atiyah and Bott [4] when X is a Riemann surface and by Donaldson [15, 16] in higher dimensions,

the \mathcal{G} -action on \mathcal{A} is Hamiltonian, with equivariant moment map $\mu_{\mathcal{G}}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^*$ given by

$$\langle \mu_{\mathcal{G}}(A), \zeta \rangle = \int_X \zeta \wedge (\Lambda F_A - z)\omega^{[n]}, \quad (1.4)$$

for $A \in \mathcal{A}$, $\zeta \in \text{Lie } \mathcal{G} = \Omega^0(\text{ad } E)$, where $F_A \in \Omega^2(\text{ad } E)$ is the curvature of $A \in \mathcal{A}$ and z is an element of the space

$$\mathfrak{z} = \mathfrak{g}^G \quad (1.5)$$

of elements of \mathfrak{g} which are invariant under the adjoint G -action, that we identify with sections of $\text{ad } E$. Recall that the moment map satisfies

$$d\langle \mu_{\mathcal{G}}, \zeta \rangle = Y_{\zeta} \lrcorner \omega_{\mathcal{A}}$$

for all $\zeta \in \text{Lie } \mathcal{G}$, where Y_{ζ} is the vector field on \mathcal{A} generated by the infinitesimal action of ζ , and equivariance means that, for all $g \in \mathcal{G}$ and $A \in \mathcal{A}$,

$$\mu_{\mathcal{G}}(g \cdot A) = \text{Ad}(g^{-1})^* \mu_{\mathcal{G}}(A).$$

Suppose now that X is a Kähler manifold, with Kähler form ω and complex structure J . Consider the complexification G^c of G and the associated principal G^c -bundle $E^c = E \times_G G^c$, where G acts on G^c by left multiplication. There is a distinguished \mathcal{G} -invariant subspace

$$\mathcal{A}_J^{1,1} \subset \mathcal{A} \quad (1.6)$$

consisting of connections A with $F_A \in \Omega_J^{1,1}(\text{ad } E)$, or equivalently satisfying $F_A^{0,2} = 0$, where $\Omega_J^{p,q}(\text{ad } E)$ denotes the space of $(\text{ad } E)$ -valued smooth (p, q) -forms with respect to J and $F_A^{0,2}$ is the projection of F_A into $\Omega_J^{0,2}(\text{ad } E)$. This space is in bijection with the space of holomorphic structures on the principal G^c -bundle E^c over the complex manifold (X, J) (see [52]).

Definition 1.1. A connection $A \in \mathcal{A}_J^{1,1}$ is called *Hermitian–Yang–Mills* if it satisfies the *Hermitian–Yang–Mills equation*

$$\Lambda F_A = z. \quad (1.7)$$

Remark 1.2. The element $z \in \mathfrak{z}$ in the right-hand side of (1.7) is determined by the cohomology class $\Omega := [\omega] \in H^2(X)$ and the topology of the principal bundle E . This follows after applying (z_j, \cdot) to (1.7), for an orthonormal basis $\{z_j\}$ of $\mathfrak{z} \subset \mathfrak{g}$, and then integrating over X , we obtain

$$z = \sum_j \frac{\langle z_j(E) \cup \Omega^{[n-1]}, [X] \rangle}{\text{Vol}_{\Omega}} z_j. \quad (1.8)$$

Here, $\Omega^{[k]} := \Omega^k/k!$, $\text{Vol}_{\Omega} := \int_X \omega^{[n]} = \langle \Omega^{[n]}, [X] \rangle$ and $z_j(E) := [z_j \wedge F_A] \in H^2(X)$ is the Chern–Weil class associated to the G -invariant linear form (z_j, \cdot) on \mathfrak{g} , which only depends on the topology of the bundle E (see [36, Ch XII, §1]).

The moduli space of Hermitian–Yang–Mills connections is defined as the set of classes of gauge equivalent solutions to (1.7). This coincides with the quotient

$$\mu_{\mathcal{G}}^{-1}(0)/\mathcal{G}, \quad (1.9)$$

where $\mu_{\mathcal{G}}$ is now the restriction of the moment map to $\mathcal{A}_J^{1,1}$. Away from its singularities, $\mathcal{A}_J^{1,1}$ inherits a complex structure compatible with $\omega_{\mathcal{A}}$ and hence a Kähler structure. Thus the smooth locus of $\mathcal{A}_J^{1,1}$ is a Kähler manifold endowed with a Hamiltonian \mathcal{G} -action and hence, away from singularities, the moduli space of Hermitian–Yang–Mills connections can

be constructed as a Kähler reduction, which, if non-empty, is a finite-dimensional Kähler manifold.

1.2. Hamiltonian actions of extended Lie groups. Consider a general extension of Lie groups

$$1 \rightarrow \mathcal{G} \xrightarrow{\iota} \tilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \rightarrow 1. \quad (1.10)$$

We will describe now, under certain assumptions, the Hamiltonian action of $\tilde{\mathcal{G}}$ on a symplectic manifold, in terms of \mathcal{G} and \mathcal{H} . In the next section we will apply this general set up to the case in which the symplectic manifold is the space of connections of a bundle and $\tilde{\mathcal{G}}$ is the extended gauge group mentioned in the introduction — this may explain the notation.

The extension (1.10) determines an extension of Lie algebras

$$0 \rightarrow \text{Lie } \mathcal{G} \xrightarrow{\iota} \text{Lie } \tilde{\mathcal{G}} \xrightarrow{p} \text{Lie } \mathcal{H} \rightarrow 0, \quad (1.11)$$

where the use of the same symbols ι and p should lead to no confusion. Note that the short exact sequence (1.11) does not generally split as a sequence of Lie algebras, but it always does as a short exact sequence of vector spaces. Let $W \subset \text{Hom}(\text{Lie } \tilde{\mathcal{G}}, \text{Lie } \mathcal{G})$ be the affine space of vector space splittings. Since $\mathcal{G} \subset \tilde{\mathcal{G}}$ is a normal subgroup, there is a well-defined $\tilde{\mathcal{G}}$ -action on W , given by

$$g \cdot \theta := \text{Ad}(g) \circ \theta \circ \text{Ad}(g^{-1}), \text{ for } g \in \tilde{\mathcal{G}}, \theta \in W.$$

Let \mathcal{A} be a manifold with an action of the ‘extended’ Lie group $\tilde{\mathcal{G}}$. Suppose that there exists a $\tilde{\mathcal{G}}$ -equivariant smooth map $\theta: \mathcal{A} \rightarrow W$. Let $\omega_{\mathcal{A}}$ be a symplectic form on \mathcal{A} preserved by the $\tilde{\mathcal{G}}$ -action. Using θ , we will characterise the existence of a $\tilde{\mathcal{G}}$ -equivariant moment map for this action in terms of \mathcal{G} and \mathcal{H} . The case considered in this paper (see §1.3) is an example where such a θ exists. Observe that if \mathcal{A} is a point, then θ determines an isomorphism $\text{Lie } \tilde{\mathcal{G}} \cong \text{Lie } \mathcal{G} \times \text{Lie } \mathcal{H}$, which shows that in this case the existence of θ is a very strong condition.

Suppose that the $\tilde{\mathcal{G}}$ -action is Hamiltonian, with $\tilde{\mathcal{G}}$ -equivariant moment map $\mu_{\tilde{\mathcal{G}}}: \mathcal{A} \rightarrow (\text{Lie } \tilde{\mathcal{G}})^*$. We can use θ to decompose this map into two pieces corresponding to $\text{Lie } \mathcal{G}$ and $\text{Lie } \mathcal{H}$. Consider θ^{\perp} uniquely defined by $\text{Id} - \iota \circ \theta = \theta^{\perp} \circ p$, where ι and p given in (1.10). Then the map

$$\begin{aligned} W &\longrightarrow \text{Hom}(\text{Lie } \mathcal{H}, \text{Lie } \tilde{\mathcal{G}}) \\ \theta &\longmapsto \theta^{\perp} \end{aligned}$$

is $\tilde{\mathcal{G}}$ -equivariant, where the $\tilde{\mathcal{G}}$ -action on $\text{Hom}(\text{Lie } \mathcal{H}, \text{Lie } \tilde{\mathcal{G}})$ given by

$$g \cdot \theta^{\perp} = \text{Ad}(g) \circ \theta^{\perp} \circ \text{Ad}(p(g^{-1}))$$

for $g \in \tilde{\mathcal{G}}$. Moreover, the map

$$\theta^{\perp}: \mathcal{A} \longrightarrow \text{Hom}(\text{Lie } \mathcal{H}, \text{Lie } \tilde{\mathcal{G}})$$

is $\tilde{\mathcal{G}}$ -equivariant and we can decompose the moment map as

$$\langle \mu_{\tilde{\mathcal{G}}}, \zeta \rangle = \langle \mu_{\tilde{\mathcal{G}}}, \iota \theta \zeta \rangle + \langle \mu_{\tilde{\mathcal{G}}}, \theta^{\perp} p(\zeta) \rangle, \quad (1.12)$$

for all $\zeta \in \text{Lie } \tilde{\mathcal{G}}$, where the summands in the right hand side define a pair of $\tilde{\mathcal{G}}$ -equivariant maps $\mu_{\mathcal{G}}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^*$, $\sigma_{\theta}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{H})^*$, given by

$$\begin{aligned}\langle \mu_{\mathcal{G}}, \zeta \rangle &:= \langle \mu_{\tilde{\mathcal{G}}}, \iota \zeta \rangle, \text{ for all } \zeta \in \text{Lie } \mathcal{G}, \\ \langle \sigma_{\theta}, \eta \rangle &:= \langle \mu_{\tilde{\mathcal{G}}}, \theta^{\perp} \eta \rangle, \text{ for all } \eta \in \text{Lie } \mathcal{H}.\end{aligned}$$

Note that since \mathcal{G} is a normal subgroup of $\tilde{\mathcal{G}}$, we can require the map $\mu_{\mathcal{G}}$ to be $\tilde{\mathcal{G}}$ -equivariant. It is now straightforward from the moment map condition for $\mu_{\tilde{\mathcal{G}}}$ to check that $\mu_{\mathcal{G}}$ is a moment map for the \mathcal{G} -action on \mathcal{A} , i.e. $d\langle \mu_{\mathcal{G}}, \zeta \rangle = Y_{\zeta} \lrcorner \omega_{\mathcal{A}}$ for all $\zeta \in \text{Lie } \mathcal{G}$. In order to see that σ_{θ} satisfies a similar infinitesimal condition, giving our characterization of Hamiltonian $\tilde{\mathcal{G}}$ -action, we first introduce some notation. Given a smooth map $\zeta: \mathcal{A} \rightarrow \text{Lie } \tilde{\mathcal{G}}$, Y_{ζ} denotes the vector field on \mathcal{A} given by

$$Y_{\zeta|A} := \left. \frac{d}{dt} \right|_{t=0} \exp(t\zeta_A) \cdot A, \quad (1.13)$$

for all $A \in \mathcal{A}$. In particular, $\theta: \mathcal{A} \rightarrow W$ induces a map

$$\begin{aligned}Y_{\theta^{\perp}}: \text{Lie } \mathcal{H} &\longrightarrow \Omega^0(T\mathcal{A}) \\ \eta &\longmapsto Y_{\theta^{\perp}\eta}.\end{aligned}$$

Note also that, by definition, $d\theta$ is a $\tilde{\mathcal{G}}$ -invariant $\text{Hom}(\text{Lie } \mathcal{H}, \text{Lie } \mathcal{G})$ -valued 1-form on \mathcal{A} .

Proposition 1.3. *The $\tilde{\mathcal{G}}$ -action on \mathcal{A} is Hamiltonian if and only if the action of $\mathcal{G} \subset \tilde{\mathcal{G}}$ on \mathcal{A} is Hamiltonian, with a $\tilde{\mathcal{G}}$ -equivariant moment map $\mu_{\mathcal{G}}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^*$, and there exists a smooth $\tilde{\mathcal{G}}$ -equivariant map $\sigma_{\theta}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{H})^*$ satisfying*

$$Y_{\theta^{\perp}\eta} \lrcorner \omega_{\mathcal{A}} = \langle \mu_{\mathcal{G}}, \langle d\theta, \eta \rangle \rangle + d\langle \sigma_{\theta}, \eta \rangle, \quad (1.14)$$

for all $\eta \in \text{Lie } \mathcal{H}$. In this case, a $\tilde{\mathcal{G}}$ -equivariant moment map $\mu_{\tilde{\mathcal{G}}}: \mathcal{A} \rightarrow (\text{Lie } \tilde{\mathcal{G}})^*$ is given by

$$\langle \mu_{\tilde{\mathcal{G}}}, \zeta \rangle = \langle \mu_{\mathcal{G}}, \theta \zeta \rangle + \langle \sigma_{\theta}, p(\zeta) \rangle, \text{ for all } \zeta \in \text{Lie } \tilde{\mathcal{G}}. \quad (1.15)$$

Proof. To prove the “only if” part it remains to check (1.14). This follows by definition, differentiating in (1.12) and using that

$$\begin{aligned}d\langle \mu_{\mathcal{G}}, \theta \zeta \rangle &= \langle d\mu_{\mathcal{G}}, \theta \zeta \rangle + \langle \mu_{\mathcal{G}}, \langle d\theta, \eta \rangle \rangle \text{ and} \\ Y_{\zeta} \lrcorner \omega &= Y_{\theta \zeta} \lrcorner \omega + Y_{\theta^{\perp} p(\zeta)} \lrcorner \omega, \text{ with } \eta := p(\zeta),\end{aligned}$$

where the first equation is obtained applying the chain rule, and the second one holds because $\zeta = \theta \zeta + \theta^{\perp} p(\zeta)$ and Y_{ζ} is linear in ζ . The “if” part is straightforward from the statement and is left to the reader. \square

Note that condition (1.14) for σ_{θ} generalizes the usual infinitesimal condition $Y_{\eta} \lrcorner \omega_{\mathcal{A}} = d\langle \mu_{\mathcal{H}}, \eta \rangle$ ($\eta \in \mathcal{H}$) for moment maps $\mu_{\mathcal{H}}$ for the induced \mathcal{H} -action on \mathcal{A} when the Lie group extension (1.10) splits.

1.3. The extended gauge group action on the space of connections. We apply now the general theory developed in §1.2 to compute the moment map for the action of the *extended gauge group* of a bundle over a compact symplectic manifold, on the space of connections.

Let X be a compact symplectic manifold of dimension $2n$, with symplectic form ω . Let G be a Lie group and E be a smooth principal G -bundle on X , with projection

map $\pi: E \rightarrow X$. Let \mathcal{H} be the group of Hamiltonian symplectomorphisms of (X, ω) and $\text{Aut } E$ be the group of automorphisms of the bundle E . Recall that an *automorphism* of E is a G -equivariant diffeomorphism $g: E \rightarrow E$. Any such automorphism covers a unique diffeomorphism $\check{g}: X \rightarrow X$, i.e. a unique \check{g} such that $\pi \circ g = \check{g} \circ \pi$. We define the *Hamiltonian extended gauge group* (to which we will simply refer as extended gauge group) of E ,

$$\tilde{\mathcal{G}} \subset \text{Aut } E,$$

as the group of automorphisms which cover elements of \mathcal{H} . Then the gauge group of E , already defined in §1.1, is the normal subgroup $\mathcal{G} \subset \tilde{\mathcal{G}}$ of automorphisms covering the identity.

The map $\tilde{\mathcal{G}} \xrightarrow{p} \mathcal{H}$ assigning to each automorphism g the Hamiltonian symplectomorphism \check{g} that it covers is surjective. To show this, let $h \in \mathcal{H}$. By definition there exists a Hamiltonian isotopy $[0, 1] \times X \rightarrow X: (t, x) \mapsto h_t(x)$ from $h_0 = \text{Id}$ to $h_1 = h$, which is the flow of a smooth family of vector fields $\eta_t \in \text{Lie } \mathcal{H}$, i.e. with $dh_t/dt = \eta_t \circ h_t$ (see e.g. [45, §3.2]). Choose a connection A on E . Let $\zeta_t \in \text{Lie } \tilde{\mathcal{G}}$ be the horizontal lift to E of η_t given by A . The vector fields ζ_t are G -invariant so their time-dependent flow g_t exists for all $t \in [0, 1]$ and the $g_t: E \rightarrow E$ are G -equivariant. Since ζ_t is a lift of η_t to E , its flow g_t covers h_t (i.e. $h_t = \check{g}_t$), so in particular $g_t \in \tilde{\mathcal{G}}$ for all t and $g_1 \in \tilde{\mathcal{G}}$ covers $h = h_1$. Thus p is surjective. We thus have an exact sequence of Lie groups

$$1 \rightarrow \mathcal{G} \xrightarrow{\iota} \tilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \rightarrow 1, \quad (1.16)$$

where ι is the inclusion map.

Remark 1.4. Note that the sequence (1.16) is exact even when the structure group G and the base manifold X are non-compact. The crucial fact is that \mathcal{H} lies in the identity component of the diffeomorphism group $\text{Diff } X$ of X (see [1] for further details).

There is an action of $\text{Aut } E$, and hence of the extended gauge group, on the space \mathcal{A} of connections on E . To define this action, we view the elements of \mathcal{A} as G -equivariant splittings $A: TE \rightarrow VE$ of the short exact sequence

$$0 \rightarrow VE \longrightarrow TE \longrightarrow \pi^*TX \rightarrow 0, \quad (1.17)$$

where $VE = \ker d\pi$ is the vertical bundle. Using the action of $g \in \text{Aut } E$ on TE , its action on \mathcal{A} is given by $g \cdot A := g \circ A \circ g^{-1}$. Any such splitting A induces a vector space splitting of the Atiyah short exact sequence

$$0 \rightarrow \text{Lie } \mathcal{G} \xrightarrow{\iota} \text{Lie}(\text{Aut } E) \xrightarrow{p} \text{Lie}(\text{Diff } X) \rightarrow 0 \quad (1.18)$$

(cf. [4, equation (3.4)]), where $\text{Lie}(\text{Diff } X)$ is the Lie algebra of vector fields on X and $\text{Lie}(\text{Aut } E)$ is the Lie algebra of G -invariant vector fields on E . This splitting is given by maps

$$\theta_A: \text{Lie}(\text{Aut } E) \longrightarrow \text{Lie } \mathcal{G}, \quad \theta_A^\perp: \text{Lie}(\text{Diff } X) \longrightarrow \text{Lie}(\text{Aut } E) \quad (1.19)$$

such that $\iota \circ \theta_A + \theta_A^\perp \circ p = \text{Id}$, where θ_A is the vertical projection given by A and θ_A^\perp the horizontal lift of vector fields on X to vector fields on E given by A .

Lemma 1.5. *Let $A \in \mathcal{A}$, $\zeta \in \text{Lie}(\text{Aut } E)$ and $\check{\zeta} := p(\zeta) \in \text{Lie}(\text{Diff } X)$. Then the infinitesimal action $Y_{\check{\zeta}|A} \in T_A\mathcal{A} = \Omega^1(\text{ad } E)$ of $\check{\zeta}$ on A is given by*

$$Y_{\check{\zeta}|A} = -d_A(\theta_A\zeta) - \check{\zeta} \lrcorner F_A, \quad (1.20)$$

where $d_A: \Omega^k(\text{ad } E) \rightarrow \Omega^{k+1}(\text{ad } E)$ is the covariant derivative associated to A .

Proof. By the Leibniz rule, for all $v \in \Omega^0(TE)$,

$$\frac{d}{dt}\Big|_{t=0} (e^{t\zeta} \circ A \circ e^{-t\zeta}(v)) = \theta_A[\zeta, v] - [\zeta, \theta_A v] = \theta_A[\zeta, v - \theta_A v],$$

where in the second equality we have used the fact that ζ covers a vector field $\check{\zeta}$ on X , so that the vector field $[\zeta, \theta_A v]$ is vertical. It is easy to see that this expression is tensorial in v , so at each point of E it only depends on its projection $\pi_* v$. Hence the vector $Y_{\zeta|A} \in T_A \mathcal{A}$, regarded as an element of $\Omega^1(\text{ad } E)$, is given by

$$\begin{aligned} Y_{\zeta|A}(y) &= \theta_A[\zeta, \theta_A^\perp y] = [\theta_A \zeta, \theta_A^\perp y] + \theta_A[\theta_A^\perp \check{\zeta}, \theta_A^\perp y] \\ &= (-d_A(\theta_A \zeta) - \check{\zeta} \lrcorner F_A)(y), \end{aligned}$$

for any $y \in \Omega^0(TX)$, where we have used the formulae

$$y \lrcorner d_A \zeta = [\theta_A^\perp y, \zeta], \quad F_A(y, y') = -\theta_A[\theta_A^\perp y, \theta_A^\perp y'] \quad (1.21)$$

(see the equation before (4.2) and the equation after (3.4) in [4] and note that we are using a different sign convention for the curvature). \square

The splitting (1.19) restricts to a splitting of the exact sequence

$$0 \rightarrow \text{Lie } \mathcal{G} \xrightarrow{\iota} \text{Lie } \tilde{\mathcal{G}} \xrightarrow{p} \text{Lie } \mathcal{H} \rightarrow 0 \quad (1.22)$$

induced by (1.16). Following the notation of §1.2, it is easy to see that the map

$$\begin{aligned} \theta: \mathcal{A} &\longrightarrow W \\ A &\longmapsto \theta_A \end{aligned} \quad (1.23)$$

is a $\tilde{\mathcal{G}}$ -equivariant smooth map. It is also clear that the $\tilde{\mathcal{G}}$ -action on \mathcal{A} is symplectic, for the symplectic form (1.3). The methods of §1.2 apply here to provide a moment map. To see this, we use the isomorphism of Lie algebras

$$\text{Lie } \mathcal{H} \cong C_0^\infty(X), \quad (1.24)$$

where $\text{Lie } \mathcal{H}$ is the Lie algebra of Hamiltonian vector fields on X and $C_0^\infty(X)$ is the Lie algebra of smooth real functions on X with zero integral over X with respect to $\omega^{[n]}$, with the Poisson bracket. This isomorphism is induced by the map $C^\infty(X) \rightarrow \text{Lie } \mathcal{H}: \phi \mapsto \eta_\phi$, which to each function ϕ assigns its Hamiltonian vector field η_ϕ , defined by

$$d\phi = \eta_\phi \lrcorner \omega. \quad (1.25)$$

Proposition 1.6. *The $\tilde{\mathcal{G}}$ -action on \mathcal{A} is Hamiltonian, with equivariant moment map $\mu_{\tilde{\mathcal{G}}}: \mathcal{A} \rightarrow (\text{Lie } \tilde{\mathcal{G}})^*$ given by*

$$\langle \mu_{\tilde{\mathcal{G}}}, \zeta \rangle = \langle \mu_{\mathcal{G}}, \theta \zeta \rangle + \langle \sigma, p(\zeta) \rangle, \quad \text{for all } \zeta \in \text{Lie } \tilde{\mathcal{G}}, \quad (1.26)$$

where $\mu_{\mathcal{G}}: \mathcal{A} \rightarrow (\text{Lie } \mathcal{G})^*$ and $\sigma: \mathcal{A} \rightarrow (\text{Lie } \mathcal{H})^*$ are given by

$$\begin{aligned} \langle \mu_{\mathcal{G}}, \theta \zeta \rangle(A) &= \int_X \theta_A \zeta \wedge (\Lambda F_A - z) \omega^{[n]}, \\ \langle \sigma, \eta_\phi \rangle(A) &= -\frac{1}{4} \int_X \phi (\Lambda^2(F_A \wedge F_A) - 4\Lambda F_A \wedge z) \omega^{[n]}, \end{aligned} \quad (1.27)$$

for all $A \in \mathcal{A}$, $\phi \in C_0^\infty(X)$.

Proof. The result follows, by Proposition 1.3, from the facts that $\mu_{\mathcal{G}}$ and σ are $\tilde{\mathcal{G}}$ -equivariant, which is immediate from (1.27) by the change of variable theorem, and the map σ defined by (1.27) satisfies (1.14). To show this, let $\zeta \in \text{Lie}(\text{Aut } E)$, $A \in \mathcal{A}$ and note that (1.20) also applies to maps $\zeta: \mathcal{A} \rightarrow \text{Lie}(\text{Aut } E)$ (with $Y_{\zeta|A}$ defined by (1.13)). In particular,

$$Y_{\theta_{\perp A} \eta}(A) = -\eta \lrcorner F_A, \text{ for } \eta \in \text{Lie } \mathcal{H}.$$

The $\text{Hom}(\text{Lie } \mathcal{H}, \text{Lie } \mathcal{G})$ -valued 1-form $d\theta$ on \mathcal{A} is given by

$$\begin{aligned} d\theta(a): \text{Lie } \mathcal{H} &\longrightarrow \text{Lie } \mathcal{G} \\ \eta &\longmapsto \langle d\theta(a), \eta \rangle = a(\eta), \end{aligned}$$

for $A \in \mathcal{A}$ and $a \in T_A \mathcal{A} = \Omega^1(\text{ad } E)$. Observe that the quantity

$$\mu'(A) = \int_X (\theta_A \zeta \wedge z - \phi \Lambda F_A \wedge z) \omega^{[n]}$$

is locally constant on \mathcal{A} , so it is enough to assume $z = 0$. To see this, we use the path $A_t = A + ta$ to calculate

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \mu'(A + ta) &= \int_X a(\eta_\phi) \wedge z \omega^{[n]} - \phi d_A a \wedge z \omega^{[n-1]} \\ &= \int_X a \wedge z d\phi \wedge \omega^{[n-1]} - \phi d_A a \wedge z \omega^{[n-1]} = 0. \end{aligned}$$

Here we have used the identity $dF_{A_t}/dt = d_A a$ for $t = 0$ and integration by parts, combined with the equality

$$a \wedge z d\phi \wedge \omega^{[n-1]} = a(\eta_\phi) \wedge z \omega^{[n]}.$$

Assuming $z = 0$, for the last term of the right hand side of (1.14), we have

$$\Lambda^2(F_A \wedge F_A) \omega^{[n]} = 2F_A \wedge F_A \wedge \omega^{[n-2]}.$$

Using now the Bianchi identity $d_A F_A = 0$, a similar calculation as before shows that

$$\begin{aligned} d\langle \sigma, \eta \rangle(a) &= -\frac{1}{2} \frac{d}{dt} \Big|_{t=0} \int_X \phi (F_{A_t} \wedge F_{A_t}) \wedge \omega^{[n-2]} \\ &= - \int_X \phi d_A a \wedge F_A \wedge \omega^{[n-2]} \\ &= \int_X (\eta \lrcorner \omega) \wedge a \wedge F_A \wedge \omega^{[n-2]}. \end{aligned} \tag{1.28}$$

To compute the integral in the last equality, note that $(a \wedge F_A) \wedge \omega^{n-1} = 0$, so contracting with η we obtain

$$a \wedge F_A \wedge (\eta \lrcorner \omega) \wedge \omega^{[n-2]} = a(\eta) \wedge \Lambda F_A \omega^{[n]} - a \wedge (\eta \lrcorner F_A) \wedge \omega^{[n-1]},$$

using the identity $F_A \wedge \omega^{[n-1]} = \Lambda F_A \omega^{[n]}$. Combined with (1.28), we thus obtain (1.14):

$$\begin{aligned} d\langle \sigma, \eta \rangle(a) &= \int_X a \wedge (\eta \lrcorner F_A) \wedge \omega^{n-1} - \int_X a(\eta) \wedge (\Lambda F_A - z) \omega^{[n]} \\ &= (Y_{\theta_{\perp A} \eta} \lrcorner \omega_{\mathcal{A}})(a) - \langle \mu_{\mathcal{G}}, \langle d\theta(a), \eta \rangle \rangle. \end{aligned} \quad \square$$

2. THE COUPLED EQUATIONS

In this section we give a moment map interpretation of the coupled equations (0.2) for the action of the extended gauge group, introduced in §1. We also define a purely Riemannian functional, the Calabi–Yang–Mills functional, whose absolute minimum over the phase space are precisely the solutions of the coupled equations, that we interpret in terms of the Kaluza–Klein theory for G -invariant metrics on the total space of the bundle. With this purpose we first recall the moment map interpretation of the cscK equation given by Fujiki and Donaldson.

2.1. The Hermitian scalar curvature. The moment map interpretation of the scalar curvature was first given by Fujiki [21] for the Riemannian scalar curvature of Kähler manifolds and generalized independently by Donaldson [17] for the Hermitian scalar curvature of almost Kähler manifolds. Here we follow closely Donaldson’s approach.

First we recall the notion of Hermitian scalar curvature of an almost Kähler manifold. Fix a compact symplectic manifold X of dimension $2n$, with symplectic form ω . An almost complex structure J on X is called compatible with ω if the bilinear form $g_J(\cdot, \cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric on X . Any almost complex structure J on X which is compatible with ω defines a Hermitian metric on T^*X and there is a unique unitary connection on T^*X whose $(0,1)$ component is the operator $\bar{\partial}_J: \Omega_J^{1,0} \rightarrow \Omega_J^{1,1}$ induced by J . The real 2-form ρ_J is defined as $-\mathbf{i}$ times the curvature of the induced connection on the canonical line bundle $K_X = \Lambda_{\mathbb{C}}^n T^*X$, where \mathbf{i} is the imaginary unit $\sqrt{-1}$. The Hermitian scalar curvature S_J is the real function on X defined by

$$S_J \omega^{[n]} = 2\rho_J \wedge \omega^{[n-1]}. \quad (2.1)$$

The normalization is chosen so that S_J coincides with the Riemannian scalar curvature when J is integrable. The space \mathcal{J} of almost complex structures J on X which are compatible with ω is an infinite dimensional Kähler manifold, with complex structure $\mathbf{J}: T_J \mathcal{J} \rightarrow T_J \mathcal{J}$ and Kähler form $\omega_{\mathcal{J}}$ given by

$$\mathbf{J}\Phi := J\Phi \text{ and } \omega_{\mathcal{J}}(\Psi, \Phi) := \frac{1}{2} \int_X \text{tr}(J\Psi\Phi)\omega^{[n]}, \quad (2.2)$$

for $\Phi, \Psi \in T_J \mathcal{J}$, respectively. Here we identify $T_J \mathcal{J}$ with the space of endomorphisms $\Phi: TX \rightarrow TX$ such that Φ is symmetric with respect to the induced metric $\omega(\cdot, J\cdot)$ and satisfies $\Phi J = -J\Phi$.

The group \mathcal{H} of Hamiltonian symplectomorphisms $h: X \rightarrow X$ acts on \mathcal{J} by push-forward, i.e. $h \cdot J := h_* \circ J \circ h_*^{-1}$, preserving the Kähler form. As proved by Donaldson [17, Proposition 9], the \mathcal{H} -action on \mathcal{J} is Hamiltonian with equivariant moment map $\mu_{\mathcal{H}}: \mathcal{J} \rightarrow (\text{Lie } \mathcal{H})^*$ given by

$$\langle \mu_{\mathcal{H}}(J), \eta_{\phi} \rangle = - \int_X \phi S_J \omega^{[n]}, \quad (2.3)$$

for $\phi \in C_0^{\infty}(X)$, identified with an element η_{ϕ} in $\text{Lie } \mathcal{H}$ by (1.24) and (1.25). The \mathcal{H} -invariant subspace $\mathcal{J}^i \subset \mathcal{J}$ of integrable almost complex structures is a complex submanifold (away from its singularities), and therefore inherits a Kähler structure. Over \mathcal{J}^i , the Hermitian scalar curvature S_J is the Riemannian scalar curvature of the Kähler metric determined by J and ω . Hence the quotient

$$\mu_{\mathcal{H}}^{-1}(0)/\mathcal{H}, \quad (2.4)$$

where $\mu_{\mathcal{H}}$ is now the restriction of the moment map to \mathcal{J}^i , is the moduli space of Kähler metrics with fixed Kähler form ω and constant scalar curvature. Away from singularities, this moduli space can thus be constructed as a Kähler reduction (see [21] and references therein for details).

2.2. The coupled equations as a moment map condition. Fix a compact symplectic manifold X of dimension $2n$ with symplectic form ω , a compact Lie group G and a smooth principal G -bundle E on X . Let \mathcal{J} be the space of almost complex structures compatible with ω and \mathcal{A} the space of connections on E . Using the symplectic forms on \mathcal{A} and \mathcal{J} induced by ω (see (1.3) and (2.2)), we define a symplectic form on the product $\mathcal{J} \times \mathcal{A}$, for each pair of non-zero real constants $\alpha = (\alpha_0, \alpha_1)$, as the weighted sum

$$\omega_\alpha = \alpha_0 \omega_{\mathcal{J}} + 4\alpha_1 \omega_{\mathcal{A}} \quad (2.5)$$

(we omit pullbacks to $\mathcal{J} \times \mathcal{A}$). The extended gauge group $\tilde{\mathcal{G}}$ has a canonical action on $\mathcal{J} \times \mathcal{A}$ and this action is symplectic for any ω_α . Following the notation of §1.3, this action is given by

$$g \cdot (J, A) = (p(g) \cdot J, g \cdot A),$$

for $g \in \tilde{\mathcal{G}}$ and $(J, A) \in \mathcal{J} \times \mathcal{A}$, with p as in (1.16). Using the moment maps $\mu_{\mathcal{H}}$ and $\mu_{\tilde{\mathcal{G}}}$ given by (2.3) and Proposition 1.6, we obtain the following.

Proposition 2.1. *The $\tilde{\mathcal{G}}$ -action on $\mathcal{J} \times \mathcal{A}$ is Hamiltonian with respect to ω_α , with equivariant moment map $\mu_\alpha: \mathcal{J} \times \mathcal{A} \rightarrow (\text{Lie } \tilde{\mathcal{G}})^*$ given by*

$$\begin{aligned} \langle \mu_\alpha(J, A), \zeta \rangle &= 4\alpha_1 \int_X \theta_A \zeta \wedge (\Lambda F_A - z) \omega^{[n]} \\ &\quad - \int_X \phi (\alpha_0 S_J + \alpha_1 \Lambda^2(F_A \wedge F_A) - 4\alpha_1 \Lambda F_A \wedge z) \omega^{[n]}, \end{aligned} \quad (2.6)$$

for all $(J, A) \in \mathcal{J} \times \mathcal{A}$, $\zeta \in \text{Lie } \tilde{\mathcal{G}}$, and $p(\zeta) = \eta_\phi$ with $\phi \in C_0^\infty(X)$.

The $\tilde{\mathcal{G}}$ -action also preserves the almost complex structure \mathbf{I} on $\mathcal{J} \times \mathcal{A}$ given by

$$\mathbf{I}(\dot{J}, a) = (J\dot{J}, -a(J\cdot)), \quad (2.7)$$

for all $(\dot{J}, a) \in T_J \mathcal{J} \times T_A \mathcal{A}$. Using the complex structure \mathbf{J} on \mathcal{J} given by (2.2), the canonical projection $\mathcal{J} \times \mathcal{A} \rightarrow \mathcal{J}$ becomes now a holomorphic submersion. It is easy to see that, for α_0, α_1 positive, the complex structure \mathbf{I} is compatible with the family of symplectic structures (2.5). The formal integrability of the almost complex structure \mathbf{I} is not obvious *a priori*, so we now provide a proof of this fact. By “formal integrability” here, we mean, as in [18], that the associated Nijenhuis tensor vanishes.

Proposition 2.2. *The almost complex structure \mathbf{I} is formally integrable.*

Proof. Since the complex structure \mathbf{J} on the base \mathcal{J} and the one on each fibre are integrable, the integrability condition for \mathbf{I} reduces to the vanishing condition for the value of the Nijenhuis tensor $N_{\mathbf{I}}$ on each pair of vectors $\dot{J} \in T_J \mathcal{J}$, $a \in T_A \mathcal{A}$, for $(J, A) \in \mathcal{J} \times \mathcal{A}$. Now, a and \dot{J} extend to vector fields on \mathcal{A} and \mathcal{J} , respectively, and hence to $\mathcal{J} \times \mathcal{A}$ (a extends to a constant vector field on the affine space \mathcal{A} and \dot{J} extends to a vector field on \mathcal{J} given by $\dot{J}_{|_{\mathcal{J}'}} = (1/2)(J\dot{J}J' - J'J\dot{J})$). Furthermore,

$$N_{\mathbf{I}}(\dot{J}, a) = [\mathbf{I}\dot{J}, \mathbf{I}a] - \mathbf{I}[\dot{J}, a] - \mathbf{I}[\dot{J}, \mathbf{I}a] - [\dot{J}, a] = [\mathbf{I}\dot{J}, \mathbf{I}a] - \mathbf{I}[\dot{J}, \mathbf{I}a],$$

where the brackets denote the Lie brackets between vector fields on $\mathcal{J} \times \mathcal{A}$ and we have used the fact that $[\mathbf{I}\dot{J}, a] = [\dot{J}, a] = 0$ because the flow of a covers the identity on \mathcal{J} . To compute the remaining terms, we denote by $J_t(\dot{J})$ the flow of any vector field \dot{J} on \mathcal{J} , viewed as a vector field on $\mathcal{J} \times \mathcal{A}$. Then $J_t(\dot{J})$ induces the identity on \mathcal{A} , and hence

$$\begin{aligned} N_{\mathbf{I}}(\dot{J}, a) &= \frac{d}{dt}\Big|_{t=0} \mathbf{I}a|_{J_t(\mathbf{I}\dot{J})} - \mathbf{I}|_J \frac{d}{dt}\Big|_{t=0} \mathbf{I}a|_{J_t(\dot{J})} \\ &= -\frac{d}{dt}\Big|_{t=0} a(J_t(\mathbf{I}\dot{J})\cdot) + \mathbf{I}|_J \frac{d}{dt}\Big|_{t=0} a(J_t(\dot{J})\cdot) \\ &= -a(J\dot{J}\cdot) - a(\dot{J}J\cdot) = -a(J\dot{J} + \dot{J}J\cdot) = 0, \end{aligned}$$

where a is now viewed as an element of $\Omega^1(\text{ad } E)$.

Note that the vanishing of $N_{\mathbf{I}}(\dot{J}, a)$ does not require any compatibility condition between J and ω . \square

Remark 2.3. There is another $\tilde{\mathcal{G}}$ -invariant almost complex structure on $\mathcal{J} \times \mathcal{A}$ which is given by $\mathbf{I}'(\dot{J}, a) = (J\dot{J}, a(J\cdot))$. This is compatible with ω_α for $\alpha_0 > 0 > \alpha_1$, and the projection $\mathcal{J} \times \mathcal{A} \rightarrow \mathcal{J}$ is pseudoholomorphic for this \mathbf{I}' , but one can modify the proof of Proposition 2.2 to show that \mathbf{I}' is not formally integrable.

Suppose now that X has Kähler structures with Kähler form ω . In the notation of §2.1, this means that the subspace $\mathcal{J}^i \subset \mathcal{J}$ of integrable almost complex structures compatible with ω is not empty. Define

$$\mathcal{P} \subset \mathcal{J} \times \mathcal{A} \tag{2.8}$$

as the space of pairs (J, A) with $J \in \mathcal{J}^i$ and $A \in \mathcal{A}_J^{1,1}$, where $\mathcal{A}_J^{1,1} \subset \mathcal{A}$ is the space of connections defined in (1.6). Then $\mathcal{P} \subset \mathcal{J} \times \mathcal{A}$ is a $\tilde{\mathcal{G}}$ -invariant complex and hence Kähler subspace by construction (see also Lemma 3.1).

We say that a pair $(J, A) \in \mathcal{P}$ satisfies the *coupled Kähler–Yang–Mills equations* if

$$\left. \begin{aligned} \Lambda F_A &= z \\ \alpha_0 S_J + \alpha_1 \Lambda^2(F_A \wedge F_A) &= c \end{aligned} \right\}, \tag{2.9}$$

where S_J is the scalar curvature of the metric $g_J = \omega(\cdot, J\cdot)$ on X and $c \in \mathbb{R}$. These equations are the central subject of this paper. The set of solutions to the coupled equations is invariant under the action of $\tilde{\mathcal{G}}$ and we define the moduli space of solutions as the set of all solutions modulo the action of $\tilde{\mathcal{G}}$. We have the following.

Proposition 2.4. *The subset $\mu_\alpha^{-1}(0) \subset \mathcal{P}$ coincides with the set of pairs $(J, A) \in \mathcal{P}$ satisfying equations (2.9).*

Proof. Suppose that $(J, A) \in \mu_\alpha^{-1}(0)$. First, evaluating $\mu_\alpha(J, A)$ on elements of the form $\theta_A^\perp \eta$ with $\eta \in \text{Lie } \mathcal{H}$, we see that there exists a $c' \in \mathbb{R}$ such that

$$\begin{aligned} \frac{c' - \alpha_0 S_J}{\alpha_1} &= \Lambda^2(F_A \wedge F_A) - 4\Lambda F_A \wedge z \\ &= 2|\Lambda F_A|^2 - 2|F_A|^2 + 8|F_A^{0,2}|^2 - 4\Lambda F_A \wedge z, \end{aligned} \tag{2.10}$$

where the last equality follows from a pointwise computation (cf. [46, proof of Lemma 7.9]). Here, the pointwise norms are defined using the metric $g_J = \omega(\cdot, J\cdot)$ and the inner product

(\cdot, \cdot) on \mathfrak{g} and $F_A^{0,2}$ denotes the $(0, 2)$ part of F_A with respect to J . Second, as $\langle \mu_\alpha(J, A), \zeta \rangle = 0$ for all $\zeta \in \text{Lie } \mathcal{G}$, we have $\Lambda F_A = z$ and hence it is straightforward to see from (2.10) that

$$\alpha_0 S_J + \alpha_1 \Lambda^2(F_A \wedge F_A) = c' + 4\alpha_1 |z|^2 \in \mathbb{R}.$$

The converse follows also from (2.10). \square

Note that we have not used the fact that $(J, A) \in \mathcal{P}$. Observe also that c is a ‘topological constant’, i.e. it only depends on the cohomology class $\Omega := [\omega] \in H^2(X)$, the topology of the bundle E and the coupling constants α_0, α_1 (cf. Remark 1.2). This follows by integrating the second equation in (2.9) over X , obtaining

$$c = \alpha_0 \hat{S} + 2\alpha_1 \hat{c}, \quad (2.11)$$

where \hat{S} is the average of the Hermitian scalar curvature,

$$\hat{S} := \frac{\int_X S_J \omega^{[n]}}{\int_X \omega^{[n]}} = 2\pi \frac{\langle c_1(X) \cup \Omega^{[n-1]}, [X] \rangle}{\text{Vol}_\Omega}, \quad (2.12)$$

which only depends on the cohomology class $\Omega \in H^2(X)$, and

$$\hat{c} := \frac{\int_X F_A \wedge F_A \wedge \omega^{[n-2]}}{\int_X \omega^{[n]}} = \frac{\langle c(E) \cup \Omega^{[n-2]}, [X] \rangle}{\text{Vol}_\Omega}, \quad (2.13)$$

where $c(E) := [F_A \wedge F_A] \in H^4(X)$ is the Chern–Weil class associated to the G -invariant symmetric bilinear form (\cdot, \cdot) on \mathfrak{g} , and so \hat{c} only depends on Ω and the topology of E (see [36, Ch XII, §1]).

From Proposition 2.4, we can identify the moduli space of solutions to the coupled equations with the quotient

$$\mu_\alpha^{-1}(0)/\tilde{\mathcal{G}}, \quad (2.14)$$

where μ_α denotes now the restriction of the moment map to \mathcal{P} . Away from singularities, this is a Kähler quotient for the action of $\tilde{\mathcal{G}}$ on the smooth part of $\mathcal{P} \subset \mathcal{J} \times \mathcal{A}$ equipped with the Kähler form obtained by the restriction of ω_α .

Remark 2.5. The coupled equations (2.9) can also be written as

$$\left. \begin{aligned} \Lambda F_A &= z \\ \alpha_0 S_g - 2\alpha_1 |F_A|_g^2 &= c - 2\alpha_1 |z|^2 \end{aligned} \right\}. \quad (2.15)$$

Here S_g is the scalar curvature of the metric $g = \omega(\cdot, J\cdot)$, $|F_A|_g^2$ is the pointwise norm of F_A defined using g and the inner product (\cdot, \cdot) on \mathfrak{g} , and $z \in \mathfrak{z}$, $c \in \mathbb{R}$ are as in (2.9). The purely Riemannian nature of the second (scalar) equation in (2.15) will be used in §2.3. The equivalence of (2.9) and (2.15) follows from (2.10) using that $A \in \mathcal{A}_J^{1,1}$ (i.e. $F_A^{0,2} = 0$).

2.3. The Calabi–Yang–Mills functional. Kähler metrics of constant scalar curvature arise as the absolute minima of the Calabi functional [8], which is defined as the L^2 -norm of the scalar curvature for Kähler metrics running over a fixed Kähler class on a compact complex manifold. Alternatively, we can see the cscK metrics as the absolute minima of the L^2 -norm of the scalar curvature defined over the space \mathcal{J}^i of complex structures compatible with a fixed symplectic form ω (see e.g. [24]). As a further step in Calabi’s programme, in this section we define the Calabi–Yang–Mills (CYM) functional CYM_α . This is a purely Riemannian functional that intertwines the Yang–Mills functional for connections with the L^2 -norm of the scalar curvature of invariant metrics in the total space of the principal bundle E . Interpreting the elements of $\mathcal{J} \times \mathcal{A}$ as invariant Riemannian metrics g_α on

E , we prove that the absolute minima of CYM_α over $\mathcal{J}^i \times \mathcal{A}$ are precisely the solutions $(J, A) \in \mathcal{P}$ of (2.9). We will also see that the coupled equations (2.9) can be formulated in terms of the Ricci tensor and the scalar curvature of g_α , when it is defined by an element of a suitable subspace $\mathcal{P}^* \subset \mathcal{P}$.

We start with a principal G -bundle E over a compact manifold X and a fixed G -invariant inner product (\cdot, \cdot) on \mathfrak{g} . Consider the G -invariant metric g_V on the vertical bundle $VE \subset TE$ induced by (\cdot, \cdot) via the identification of VE with the trivial bundle $E \times \mathfrak{g}$. Using a connection A on E and a scaling constant $\alpha > 0$, each Riemannian metric g on X lifts to a G -invariant Riemannian metric g_α on E , given by

$$g_\alpha = \pi^*g + \alpha g_V(\theta_A \cdot, \theta_A \cdot), \quad (2.16)$$

where $\pi: E \rightarrow X$ is the canonical projection and $\theta_A: TE \rightarrow VE$ is the vertical projection determined by A . Given positive constants $\alpha_0, \alpha_1 \in \mathbb{R}$, we denote respectively by S_{g_α} , vol_α and $\text{Vol}_\alpha(E)$ the scalar curvature and the volume form of the metric g_α and the volume of E with respect to g_α , where $\alpha = 2\alpha_1/\alpha_0$. We also denote by vol_g and $\text{Vol}_g(X)$ the volume form of the metric g and the corresponding volume of X , respectively. We define the Calabi–Yang–Mills functional by the formula

$$\text{CYM}_\alpha(g, A) := \frac{1}{\text{Vol}_\alpha(E)} \int_E S_{g_\alpha}^2 \text{vol}_\alpha + \frac{\alpha_1}{\text{Vol}_g(X)} \int_X |F_A|_g^2 \text{vol}_g, \quad (2.17)$$

for pairs (g, A) , where g is a Riemannian metric on X , A is a connection on E and $|F_A|_g^2$ is as in (2.15). Note that (2.17) is a weighted sum of the Calabi functional [8] for metrics on E and the Yang–Mills functional for E (see e.g. [20, §2.1.4]).

Fix now a symplectic form ω on X so that $\text{vol}_{g_J} = \omega^{[n]}$ for all $J \in \mathcal{J}$, where $g_J = \omega(\cdot, J \cdot)$ and $\dim X = 2n$. Although the functional (2.17) is well defined for arbitrary Riemannian metrics on X and connections on E , the solutions of the coupled equations (2.9) are the absolute minima of CYM_α only when this functional is restricted to metrics of the form $g = g_J$, where J is in the space \mathcal{J}^i of integrable almost complex structures on X which are compatible with ω . In other words, we consider the functional

$$\begin{aligned} \mathcal{J}^i \times \mathcal{A}: \Omega^k &\longrightarrow \mathbb{R} \\ (J, A) &\longmapsto \text{CYM}_\alpha(g_J, A). \end{aligned} \quad (2.18)$$

Proposition 2.6. *If $(J, A) \in \mathcal{P}$ satisfies the coupled equations (2.9), then the pair (J, A) is an absolute minimum of the functional (2.18), provided that α_0 and α_1 are positive and*

$$\alpha_1 > 2\alpha\hat{S} + \alpha^2(\hat{c} - |z|^2) + 2s, \quad (2.19)$$

where $\alpha = 2\alpha_1/\alpha_0$, s is the (constant) scalar curvature of the biinvariant metric induced by (\cdot, \cdot) on G , z is given by (1.8) and \hat{S} , \hat{c} are as in (2.11), with $\Omega = [\omega]$.

Proof. Note first that for any metric as in (2.16), $\pi: (E, g_\alpha) \rightarrow (X, g)$ is a Riemannian submersion with totally geodesic fibres (see [6, Theorem 9.59], where the G -Riemannian manifold playing the role of the typical fibre is G itself with its biinvariant metric). Then g_α has scalar curvature $S_{g_\alpha} = S_\alpha \circ \pi$, where

$$S_\alpha = S_g - \alpha|F_A|_g^2 + s/\alpha \in C^\infty(X), \quad (2.20)$$

S_g being the scalar curvature of g (see [6, Proposition 9.70]). Here, the group is identified with the fibre E_x over $x \in X$. Since the volume of E_x is independent of x , we have

$$\frac{1}{\text{Vol}_\alpha(E)} \int_E S_{g_\alpha}^2 \text{vol}_\alpha = \frac{1}{\text{Vol}_g(X)} \int_X S_\alpha^2 \text{vol}_g.$$

In particular, for $g = g_J$, with $J \in \mathcal{J}^i$, and $c'' = \alpha_0 \hat{S} + 2\alpha_1(\hat{c} - |z|^2)$, we obtain

$$\begin{aligned} \text{CYM}_\alpha(g, A) &= \frac{\alpha_0^{-2}}{\text{Vol}_\Omega} \left\| \alpha_0 S_g - 2\alpha_1 |F_A|_g^2 - c'' \right\|_{L^2}^2 + \frac{\alpha_1}{\text{Vol}_\Omega} \|F_A\|_{L^2}^2 \\ &\quad + \frac{2(c''/\alpha_0 + s/\alpha)}{\text{Vol}_\Omega} \int_X (S_g - \alpha |F_A|_g^2 - c''/\alpha_0) \text{vol}_g \\ &\quad + (c''/\alpha_0 + s/\alpha)^2 \\ &= \frac{\alpha_0^{-2}}{\text{Vol}_\Omega} \left\| \alpha_0 S_g - 2\alpha_1 |F_A|_g^2 - c'' \right\|_{L^2}^2 \\ &\quad + \frac{\alpha_1 - 2\alpha \hat{S} - \alpha^2(\hat{c} - |z|^2) - 2s}{\text{Vol}_\Omega} \|F_A\|_{L^2}^2 \\ &\quad + \left(\hat{S} + \alpha(\hat{c} - |z|^2) + s/\alpha \right) \left(\hat{S} - \alpha(\hat{c} - |z|^2 + s/\alpha) \right), \end{aligned}$$

where the L^2 -norms are defined using g , $\omega^{[n]}$ and the inner product on \mathfrak{g} . Note that the last summand in the right-hand side of the last equation only depends on α , s , the cohomology class $\Omega := [\omega]$ and the topology of the bundle E . The inequality (2.19) implies that the factor multiplying the Yang–Mills functional is positive, and the result follows from the alternative formulation (2.15) of the coupled equations combined with (2.10), which gives

$$\begin{aligned} \|F_A\|_{L^2}^2 &= \|\Lambda F_A\|_{L^2}^2 + 4\|F_A^{0,2}\|_{L^2}^2 - \hat{c} \text{Vol}_\Omega \\ &= \|\Lambda F_A - z\|_{L^2}^2 + 4\|F_A^{0,2}\|_{L^2}^2 \\ &\quad + 2\langle z(E) \cup \Omega^{[n-1]}, [X] \rangle - (|z|^2 + \hat{c}) \text{Vol}_\Omega. \end{aligned}$$

Here, $z(E) := [z \wedge F_A] \in H^2(X)$ is the Chern–Weil class associated to the G -invariant linear form (z, \cdot) on \mathfrak{g} , with z given by (1.8), so the last line in the previous equation only depends on $\Omega \in H^2(X)$ and the topology of the bundle E (see [36, Ch XII, §1]). \square

Remark 2.7. The inequality (2.19) imposes no restrictions on the solutions (J, A) of (2.9), because any solution (J, A) of (2.9) for some (α_0, α_1) is also a solution for the constants $(t\alpha_0, t\alpha_1)$, for all $t \in \mathbb{R}$. The claim follows from the fact the RHS on (2.19) is invariant by this scaling procedure.

Remark 2.8. Fixing a complex structure on X , we can view CYM_α as a functional on the pairs (ω, A) , with ω as in the second part of Remark 2.5. Exactly as in Proposition 2.6, in this case a solution of the coupled equations is always an absolute minima of this functional.

The coupled equations (2.9) can also be interpreted in purely Riemannian terms, considering the G -invariant metrics g_α on E defined by (2.16). To explain this, note that given such a metric its Ricci tensor R_{g_α} decomposes as $R_{g_\alpha} = (R_{g_\alpha})_{hh} + (R_{g_\alpha})_{vv} + (R_{g_\alpha})_{hv}$, where the indices “ h ” and “ v ” denote the horizontal and vertical directions in TE defined by the connection A , respectively. Let $\mathcal{P}^* \subset \mathcal{P}$ be the open subset of pairs (J, A) with $A \in \mathcal{A}^*$ — the open subset of \mathcal{A} consisting of irreducible connections. By irreducible connection $A \in \mathcal{A}$ we mean, as in [20, §4.2.2], that its isotropy group \mathcal{G}_A inside the gauge

group of E is minimal—the centre of G . Then a pair $(J, A) \in \mathcal{P}^*$ satisfies (2.9) if and only if the associated metric g_α satisfies the following equations.

$$\left. \begin{aligned} (R_{g_\alpha})_{hv} &= 0 \\ S_{g_\alpha} &= \text{const.} \end{aligned} \right\} \quad (2.21)$$

We thus have an interpretation of the Kähler quotient (2.14) (with μ_α restricted to the open subset $\mathcal{P}^* \subset \mathcal{P}$) as a moduli space of G -invariant metrics on the total space of E satisfying (2.21). An interesting fact here is that the condition $\alpha_1/\alpha_0 > 0$ is needed both to have a Kähler form ω_α on \mathcal{P} given by (2.5) (see the explanation before (2.14)) and G -invariant Riemannian metrics g_α on E , as given in (2.16).

To prove the equivalence of (2.9) and (2.21) for a pair $(J, A) \in \mathcal{P}^*$, note that J defines a structure of Kähler manifold on (X, ω) . The Hermitian–Yang–Mills equation $\Lambda F_A = 0$ for an *irreducible* $A \in \mathcal{A}_J^{1,1}$ is equivalent to the *a priori* weaker Yang–Mills equation $d_A^* F_A = 0$ (see [15, Proposition 3]). This follows because if $A \in \mathcal{A}_J^{1,1}$ is an irreducible Yang–Mills connection then, by the Kähler identities,

$$d_A \Lambda F_A = 0 \implies \Lambda F_A \in \text{Lie } \mathcal{G}_A = \mathfrak{z}.$$

Therefore the first equations in (2.15) and (2.21) are equivalent because the Yang–Mills equation is equivalent to the equation $(R_{g_t})_{hv} = 0$ (see [6, Proposition 9.61]). Finally, the second equations in (2.15) and (2.21) are equivalent by (2.20).

Note that the system (2.21) is half way between the Einstein equation and the constant scalar curvature equation, in the sense that

$$g_\alpha \text{ is an Einstein metric} \implies g_\alpha \text{ satisfies (2.21)} \implies S_{g_\alpha} = \text{const.}, \quad (2.22)$$

for all $(J, A) \in \mathcal{P}^*$, as any metric g_α satisfying the Einstein equation $R_{g_\alpha} = \lambda g_\alpha$ (with $\lambda \in \mathbb{R}$) has constant scalar curvature.

3. THE α -FUTAKI CHARACTER AND THE α -K-ENERGY

In §3 we construct obstructions to the existence of solution of the coupled equations, generalizing the Futaki character [22], the Mabuchi K-energy [41, 42] and the notion of geodesic stability [11, 18] used in the cscK Theory. For this, in §§3.1, 3.2, 3.3, we develop an abstract framework that we apply in §3.4 to the study of the coupled equations.

Throughout §3, we fix a compact real manifold X , a cohomology class $\Omega \in H^2(X, \mathbb{R})$, a reductive complex Lie group G^c with Lie algebra \mathfrak{g}^c , a maximal compact Lie subgroup $G \subset G^c$ with Lie algebra \mathfrak{g} and a smooth principal G^c -bundle $\pi: E^c \rightarrow X$. We also fix $z \in \mathfrak{z}$ as in (1.8). We assume that the space of Kähler forms in Ω is non-empty.

3.1. Invariant Hamiltonian Kähler fibrations. In §3.1, we will associate to the data (X, Ω, E^c) a canonical infinite-dimensional double fibration

$$\mathcal{B} \xleftarrow{\pi_{\mathcal{B}}} \mathcal{C} \xrightarrow{\pi_{\mathcal{Z}}} \mathcal{Z},$$

equivariant for the action of an infinite-dimensional Lie group Γ , and show that the fibres of $\pi_{\mathcal{B}}$ are (formally) Kähler manifolds with Hamiltonian group actions. The fibres of $\pi_{\mathcal{Z}}$ will be studied in §3.2.

Let $\text{Diff}_0 X$ be the identity component of the diffeomorphism group of X and $\text{Aut } E^c$ the group of automorphisms of E^c , that is, the G^c -equivariant diffeomorphisms $g: E^c \rightarrow E^c$.

Any such g determines a unique diffeomorphism $\check{g}: X \rightarrow X$ such that $\pi \circ g = \check{g} \circ \pi$. Define the real Lie group

$$\Gamma := \{g \in \text{Aut } E^c \mid \check{g} \in \text{Diff}_0 X\}.$$

Note that the Lie bracket in the Lie algebra $\text{Lie } \Gamma$ of Γ is

$$[y, y']_\Gamma = -[y, y'] \quad (3.1)$$

for $y, y' \in \text{Lie } \Gamma \subset \Omega^0(TE^c)$, where $[\cdot, \cdot]$ is the Lie bracket of vector fields on E^c (cf. [45, Remark 3.3]).

Let \mathcal{Z} be the space of holomorphic structures on the principal G^c -bundle E^c , i.e. the integrable G^c -equivariant almost complex structures I on the total space of E^c which preserve the vertical bundle VE^c and whose restriction to VE^c equals multiplication by $\sqrt{-1}$, via its identification with $E^c \times \mathfrak{g}^c$. By G^c -equivariance, any such I determines a unique integrable almost complex structure \check{I} on X such that $\check{I} \circ d\pi = d\pi \circ I$. The group Γ has a left action on \mathcal{Z} by push-forward, preserving the canonical almost complex structure \mathbf{I} on \mathcal{Z} given by

$$\mathbf{I}\dot{I} = I\dot{I}, \text{ for all } I \in \mathcal{Z}, \dot{I} \in T_I\mathcal{Z} \quad (3.2)$$

(cf. (2.2)), where \dot{I} is viewed as a G^c -equivariant endomorphism of TE^c .

Recall that the space $\mathcal{R} = \Omega^0(E^c/G)$ of smooth sections H of the bundle $E^c/G \rightarrow X$ is in bijection with the set of reductions of E^c to principal G -bundles $E_H \subset E^c$, via the map $H \mapsto E_H := p_G^{-1}(H(X))$, where p_G is the projection $E^c \rightarrow E^c/G$. Let \mathcal{B} be the space of pairs (ω, H) , where $\omega \in \Omega$ is a symplectic form and $H \in \mathcal{R}$. The group Γ has a left action on \mathcal{B} given by

$$g \cdot (\omega, H) = (\check{g}_*\omega, g \cdot H),$$

where $(g \cdot H)(x) := g(x) \cdot H(\check{g}^{-1}(x))$ for $x \in X$ and $\check{g}_*\omega \in \Omega$ by the homotopy invariance of the de Rham cohomology, as $\check{g} \in \text{Diff}_0 X$.

We define the space of *compatible pairs* as

$$\mathcal{C} := \{((\omega, H), I) \mid (X, \check{I}, \omega) \text{ is a Kähler manifold}\} \subset \mathcal{B} \times \mathcal{Z}.$$

Note that this space is invariant under the diagonal Γ -action on $\mathcal{B} \times \mathcal{Z}$. The canonical maps

$$\begin{array}{ccc} & \mathcal{C} & \\ \pi_{\mathcal{B}} \swarrow & & \searrow \pi_{\mathcal{Z}} \\ \mathcal{B} & & \mathcal{Z} \end{array} \quad (3.3)$$

will be viewed as two fibrations with total space \mathcal{C} , whose fibres are

$$Z_b := \pi_{\mathcal{B}}^{-1}(b) \text{ and } B_I := \pi_{\mathcal{Z}}^{-1}(I) \text{ for all } b \in \mathcal{B}, I \in \mathcal{Z}.$$

Since $\mathcal{C} \subset \mathcal{B} \times \mathcal{Z}$ is Γ -invariant, the fibres B_I and Z_b are invariant under the actions of the isotropy groups $\Gamma_I \subset \Gamma$ and $\Gamma_b \subset \Gamma$, respectively.

In more concrete terms, for any $I \in \mathcal{Z}$, the isotropy group Γ_I is the group of automorphisms g of the holomorphic principal G^c -bundle (E^c, I) such that $\check{g} \in \text{Diff}_0 X$ is an automorphism of the complex manifold (X, \check{I}) . Similarly, for any $b = (\omega, H) \in \mathcal{B}$, the isotropy group Γ_b is the group of automorphisms g of the principal G -bundle E_H such that $\check{g} \in \text{Diff}_0 X$ is a symplectomorphism of (X, ω) . Hence the extended gauge group $\tilde{\mathcal{G}}_b$ of E_H on (X, ω) (defined in §1.3) is a subgroup of Γ_b , which is normal because the group of

Hamiltonian symplectomorphisms is a normal subgroup of the symplectomorphism group (see e.g. [45, Proposition 10.2]). Note also that the fibre B_I is a contractible space, as it is

$$B_I = \mathcal{K}_{\check{I}} \times \mathcal{R}, \quad (3.4)$$

where $\mathcal{K}_{\check{I}}$ is the space of Kähler forms in Ω on the complex manifold (X, \check{I}) . The fibre Z_b has a gauge-theoretic description. Let \mathcal{J}_ω be the space of almost complex structures on X compatible with ω and \mathcal{A}_H the space of connections on E_H . Given $b = (\omega, H) \in \mathcal{B}$, define

$$\mathcal{P}_b \subset \mathcal{J}_\omega \times \mathcal{A}_H \quad (3.5)$$

as in (2.8), i.e. as the space of pairs (J, A) such that J is integrable and $F_A \in \Omega_J^{1,1}(\text{ad } E_H)$. This subspace is clearly Γ_b -invariant and has an almost complex structure \mathbf{I} given by (2.7), which is formally integrable by Proposition 2.2. Note also that for all $H \in \mathcal{R}$, each connection $A \in \mathcal{A}_H$ induces canonically a connection on E^c , given by G^c -equivariant maps

$$\theta_A: TE^c \rightarrow VE^c, \quad \theta_A^\perp: \pi^*TX \rightarrow TE^c, \quad (3.6)$$

where $\pi: E^c \rightarrow X$ is the canonical projection (cf. (1.17)), via the canonical isomorphism

$$E^c \cong E_H \times_G G^c \quad (3.7)$$

of principal G^c -bundles (with G acting on G^c by left multiplication).

Lemma 3.1. *The map $\pi_{\mathcal{B}}: \mathcal{C} \rightarrow \mathcal{B}$ is a Γ -invariant almost-complex fibration', that is, its fibres $Z_b \subset \mathcal{Z}$ are preserved by \mathbf{I} and their induced almost complex structures are exchanged by the Γ -action. Furthermore, the map*

$$\begin{aligned} \mathbb{I}: \mathcal{P}_b &\longrightarrow \mathcal{Z} \\ (J, A) &\longmapsto \mathbf{i}\theta_A + \theta_A^\perp \circ \pi^*J \circ d\pi, \end{aligned} \quad (3.8)$$

is a well-defined Γ_b -equivariant holomorphic embedding whose image is Z_b , for all $b = (\omega, H) \in \mathcal{B}$.

Proof. The first assertion follows immediately from (3.2). For the second, note that (3.8) is well defined by direct computation of the Nijenhuis tensor of $\mathbb{I}(J, A)$. Using the classical construction [52] of the Chern connection $\theta_{H,I}$ of $I \in Z_b$ on E_H , we see that the map (3.8) is injective with image Z_b , as

$$I = \mathbb{I}(\check{I}, \theta_{H,I}) \quad (3.9)$$

for all $I \in Z_b$. Furthermore, (3.8) is clearly Γ_b -equivariant. Another direct computation shows now that (3.8) is a holomorphic embedding, i.e. its differential is also injective and exchanges the almost complex structures on \mathcal{P}_b and \mathcal{Z} . \square

As an immediate consequence, $Z_b \cong \mathcal{P}_b$ equipped with the restriction of \mathbf{I} is a formally integrable complex manifold, by Proposition 2.2. Using Lemma 3.1, we can now transfer the constructions of §2.2 to the fibres

$$Z_b = \mathbb{I}(\mathcal{P}_b), \quad (3.10)$$

obtaining the following theorem, where the Lie groups $\Gamma_b \subset \Gamma$ and their normal subgroups $\mathcal{G}_b \subset \Gamma_b$, parameterized by $b \in \mathcal{B}$, are viewed as the fibres of two Lie group subbundles

$$\tilde{\mathcal{G}}_{\mathcal{B}} \subset \Gamma_{\mathcal{B}} \subset \mathcal{B} \times \Gamma \quad (3.11)$$

over \mathcal{B} . Their associated Lie algebra bundles are denoted $\text{Lie } \tilde{\mathcal{G}}_{\mathcal{B}} \subset \text{Lie } \Gamma_{\mathcal{B}} \subset \mathcal{B} \times \text{Lie } \Gamma$.

Theorem 3.2. *Each pair of positive real numbers α_0, α_1 determines a structure of Γ -invariant Hamiltonian Kähler fibration' on $\pi_{\mathcal{B}}: \mathcal{C} \rightarrow \mathcal{B}$, that is, a smooth family $\omega_{\mathcal{C}}$ of Kähler forms ω_b on the fibres Z_b , parameterized by $b \in \mathcal{B}$, which are exchanged by the Γ -action, and a morphism*

$$\mu_{\mathcal{C}}: \mathcal{C} \longrightarrow (\mathrm{Lie} \tilde{\mathcal{G}}_{\mathcal{B}})^* \quad (3.12)$$

of fibrations over \mathcal{B} , whose fibre $\mu_b: Z_b \rightarrow (\mathrm{Lie} \tilde{\mathcal{G}}_b)^*$ is a moment map for the $\tilde{\mathcal{G}}_b$ -action on Z_b , and such that

$$\langle \mu_{g \cdot b}(g \cdot I), \zeta \rangle = \langle \mu_b(I), \mathrm{Ad}(g^{-1})\zeta \rangle \quad (3.13)$$

for all $(b, I) \in \mathcal{C}$, $g \in \Gamma$, $\zeta \in \mathrm{Lie} \tilde{\mathcal{G}}_{g \cdot b}$.

Proof. As in §2, we fix a G -invariant positive definite inner product on \mathfrak{g} . Suppose that it extends to a G^c -invariant symmetric bilinear form $(\cdot, \cdot): \mathfrak{g}^c \otimes \mathfrak{g}^c \rightarrow \mathbb{C}$ (e.g., we can use $(\cdot, \cdot) := -\mathrm{tr}(\rho(\cdot) \circ \rho(\cdot))$ for a faithful representation $\rho: G^c \rightarrow \mathrm{GL}(r, \mathbb{C})$ such that $\rho(G) \subset \mathrm{U}(r)$). This form induces another one on the adjoint bundle $\mathrm{ad} E^c = E^c \times_{G^c} \mathfrak{g}^c$, which extends to a \mathbb{C} -bilinear map

$$\begin{aligned} \Omega^p(\mathrm{ad} E^c) \times \Omega^q(\mathrm{ad} E^c) &\longrightarrow \Omega^{p+q} \otimes \mathbb{C} \\ (a_p, a_q) &\longmapsto a_p \wedge a_q \end{aligned} \quad (3.14)$$

(cf. (1.1)), which clearly is equivariant under the action of $\mathrm{Aut} E^c$ given by pull-back.

Fix $\alpha_0, \alpha_1 > 0$. By the results of §2, for each $b = (\omega, H) \in \mathcal{B}$ we have a Kähler manifold

$$(Z_b, \mathbf{I}, \omega_b), \quad (3.15)$$

where \mathbf{I} is the restriction of (3.2) and ω_b corresponds to (2.5) via the isomorphism $Z_b \cong \mathcal{P}_b$ of Lemma 3.1. Furthermore, the $\tilde{\mathcal{G}}_b$ -action on (Z_b, ω_b) is Hamiltonian, with moment map

$$\mu_b: Z_b \longrightarrow (\mathrm{Lie} \tilde{\mathcal{G}}_b)^* \quad (3.16)$$

which corresponds to the moment map in Proposition 2.1 via the isomorphism $Z_b \cong \mathcal{P}_b$ of Lemma 3.1. Using now the $(\mathrm{Aut} E^c)$ -equivariance of (3.14), it is easy to see that ω_b and μ_b are the fibres of a family $\omega_{\mathcal{C}}$ defining a Γ -invariant Kähler fibration and a morphism of bundles as in (3.12), respectively.

To prove (3.13), note that the actions of $\mathrm{Aut} E^c$ on the Chern connection $\theta_{H,I}$ of $H \in \mathcal{R}$ and $I \in Z_b$, regarded as a connection on E^c , and on its curvature $F_{H,I} \in \Omega^2(\mathrm{ad} E^c)$, satisfy

$$g \cdot \theta_{H,I} = \theta_{g \cdot H, g \cdot I}, \quad g \cdot F_{H,I} = F_{g \cdot H, g \cdot I}, \quad (3.17)$$

for all $g \in \mathrm{Aut} E^c$, $H \in \mathcal{R}$, $I \in \mathcal{Z}$ (cf. [15, §1.1]). Given $(b, I) \in \mathcal{C}$, we define

$$S_{\alpha}(b, I) := -\alpha_0 S_{\omega, \check{I}} - \alpha_1 \Lambda_{\omega}^2 (F_{H,I} \wedge F_{H,I}) + 4\alpha_1 \Lambda_{\omega} F_{H,I} \wedge z \in C^{\infty}(X), \quad (3.18)$$

where $b = (\omega, H)$ and $S_{\omega, \check{I}}$ is the scalar curvature of (X, \check{I}, ω) . By the equivariance of (3.14) and the second identity in (3.17),

$$S_{\alpha}(gb, gI) = S_{\alpha}(b, I) \circ \check{g}^{-1}, \quad (3.19)$$

for all $g \in \mathrm{Aut} E^c$. Combining now (3.9), (3.17) and (3.19), and making a change of variable in (2.6), we obtain (3.13), as required. \square

Remark 3.3. The two fibrations (3.3) can be compared with those in [28, §2.C], used to see that the spaces of tamed and compatible complex structures on a symplectic vector space are contractible (cf. [45, Proposition 2.51]).

3.2. Invariant fibration by symmetric spaces. Throughout §3.2, we will use the framework introduced in §3.1 and in particular the first part of Lemma 3.1 (however, the isomorphism $\mathcal{P}_b \cong Z_b$ of Lemma 3.1 and the families $\omega_{\mathcal{C}}$ and $\mu_{\mathcal{C}}$ of Theorem 3.2 will not be used until §3.3). Our task now is to construct a canonical structure of ‘ Γ -invariant symmetric space fibration’ on $\pi_{\mathcal{Z}}: \mathcal{C} \rightarrow \mathcal{Z}$, that is, symmetric space structures on the fibres B_I that are exchanged by the Γ -action. As in §1.2, the Lie groups and manifolds considered here are infinite dimensional, so one has to be careful with many standard results in finite dimensions. In particular, the Newlander–Nirenberg theorem fails in general, so we use the notion of formally integrable complex structure, as in Proposition 2.2.

Let W be the space of complex structures on the real vector space underlying the Lie algebra $\text{Lie } \Gamma$ (i.e. linear maps whose square is $-\text{Id}$). Consider the tautological Γ -equivariant map

$$\mathcal{Z} \longrightarrow W \tag{3.20}$$

which assigns to each I the endomorphism $\text{Lie } \Gamma \rightarrow \text{Lie } \Gamma: y \mapsto Iy$. Then, since any $I \in \mathcal{Z}$ is integrable, (3.20) satisfies the conditions

$$Y_{Iy|I} = \mathbf{I}Y_{y|I}, \quad [y, y']_{\Gamma} + I[y, Iy']_{\Gamma} + I[Iy, y']_{\Gamma} - [Iy, Iy']_{\Gamma} = 0, \tag{3.21}$$

for all $y, y' \in \text{Lie } \Gamma$ (with $[\cdot, \cdot]_{\Gamma}$ as in (3.1)), where

$$Y_{y|I} \in T_I \mathcal{Z} \tag{3.22}$$

is the infinitesimal action of $y \in \text{Lie } \Gamma$ on $I \in \mathcal{Z}$, given by the Lie derivative $-L_y I$.

To construct the symmetric space fibration, we first prove that \mathcal{Z} parameterizes right-invariant formally integrable complex structures on the group Γ . Given $g \in \Gamma$, define

$$\begin{aligned} L_g: \Gamma &\longrightarrow \Gamma & R_g: \Gamma &\longrightarrow \Gamma \\ h &\longmapsto gh, & h &\longmapsto hg, \end{aligned} \tag{3.23}$$

as the left and right multiplication by g , respectively. To each $I \in \mathcal{Z}$, we associate a right-invariant almost complex structure \mathbf{I} on Γ , defined for $v \in T_g \Gamma$, $g \in \Gamma$ by

$$\mathbf{I}v = (R_g)_* I (R_g)_*^{-1} v. \tag{3.24}$$

Proposition 3.4. *The almost complex structure \mathbf{I} is formally integrable, for all $I \in \mathcal{Z}$.*

Proof. The statement follows from the second equation in (3.21), evaluating the Nijenhuis tensor $N_{\mathbf{I}}$ of \mathbf{I} on right invariant vector fields. \square

The next step in the construction of our symmetric space fibration relies on the following condition for all $I \in \mathcal{Z}$ such that B_I is non-empty (this property will be proved in Proposition 3.16):

(\star) There exists a well-defined isomorphism of vector bundles

$$\begin{aligned} \text{Lie } \tilde{\mathcal{G}}_{B|B_I} &\xrightarrow{\cong} TB_I \\ (b, \zeta) &\longmapsto Y_{I\zeta|b} \end{aligned} \tag{3.25}$$

provided by the infinitesimal action of $I \text{Lie } \tilde{\mathcal{G}}_b \subset \text{Lie } \Gamma$ on B_I .

In the sequel, the inverse of (3.25) is denoted

$$\zeta_I: TB_I \longrightarrow \text{Lie } \tilde{\mathcal{G}}_{B|B_I}. \quad (3.26)$$

Given a compatible pair $(b, I) \in \mathcal{C}$, we define a space

$$\mathcal{Y} = \mathcal{Y}_{b,I} := \{g \in \Gamma \mid g \cdot b \in B_I\}, \quad (3.27)$$

a map $\pi = \pi_{b,I}: \mathcal{Y} \rightarrow B_I$ given by $\pi(g) = g \cdot b$ and a right Γ_b -action on \mathcal{Y} given by right multiplication in Γ .

Proposition 3.5. *For any $(b, I) \in \mathcal{C}$, the following properties hold:*

- (1) \mathcal{Y} is principal Γ_b -bundle over B_I .
- (2) There exists a canonical connection \mathbb{A} on \mathcal{Y} , with horizontal lift

$$\begin{aligned} \theta_{\mathbb{A}}^\perp: \pi^*TB_I &\longrightarrow T\mathcal{Y} \\ (g, v) &\longmapsto (R_g)_*I\zeta_I(v) \end{aligned} \quad (3.28)$$

and curvature given by

$$F_{\mathbb{A}}(v_0, v_1) = (R_g)_*[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma, \quad (3.29)$$

for all $g \in \mathcal{Y}$ and $v_0, v_1 \in T_{g \cdot b}B_I$.

Proof. The Γ_b -action on \mathcal{Y} is clearly free, so leaving aside global topological questions, to prove part (1), it suffices to show that π is surjective and induces $\mathcal{Y}/\Gamma_b \cong B_I$, that is, for all $b' \in B_I$, there exists $g \in \Gamma$ such that $b' = g \cdot b$. Since B_I is contractible (see (3.4)), there exists a smooth curve b_t on B_I with $b_0 = b$, $b_1 = b'$. Let

$$y_t = I\zeta_I(\dot{b}_t) \in \text{Lie } \Gamma, \quad (3.30)$$

with ζ_I given by (3.26). Let $g_t \in \Gamma$ be the flow of y_t , defined by

$$\dot{g}_t \cdot g_t^{-1} = y_t, \quad (3.31)$$

with initial condition $g_0 = 1$. Note that the flow g_t exists for all t because y_t is G^c -invariant, so it covers a vector field \check{y}_t on X , whose flow $\check{g}_t \in \text{Diff}_0 X$ exists for all t because X is compact (cf. (1.16) and Remark 1.4). Now, by the Leibniz rule,

$$\frac{d}{dt}(g_t^{-1} \cdot b_t) = g_t^{-1} \cdot \left(-Y_{y_t|b_t} + \dot{b}_t\right) = 0,$$

because ζ_I inverts the infinitesimal action of $I \text{Lie } \tilde{\mathcal{G}}_b \subset \text{Lie } \Gamma$ on B_I (cf. [18, p. 17]). Thus $g_t^{-1} \cdot b_t$ is independent of t , so $b' = g_1 \cdot b$, as required.

For (2), note that the horizontal lift of curves on B_I to \mathcal{Y} determined by the flow of (3.30) defines a canonical connection \mathbb{A} on \mathcal{Y} . To obtain (3.28), let b_t be a curve on B_I with $\dot{b}_0 = v$ and $g \in \Gamma$ such that $g \cdot b = b_0$. By definition, the horizontal lift g_t of b_t through g is the flow of (3.30) with $g_0 = g$ (recall that it exists because y_t is G^c -invariant). Hence

$$\theta_{\mathbb{A}}^\perp(g, v) = \frac{d}{dt}\Big|_{t=0} g_t g^{-1} = (R_g)_*(I\zeta_I(v)).$$

To check (3.29), given $y \in \text{Lie } \Gamma$ we denote by \mathcal{X}_y the associated left-invariant vector field on Γ , given by

$$\mathcal{X}_{y|g} := (L_g)_*y. \quad (3.32)$$

Since \mathbf{I} is right invariant, $[\mathcal{X}_y, \mathbf{I}\cdot] = \mathbf{I}[\mathcal{X}_y, \cdot]$ for any $y \in \text{Lie } \Gamma$, which implies that

$$[\mathbf{I}\mathcal{X}_{y_0}, \mathbf{I}\mathcal{X}_{y_1}]|_1 = -[y_0, y_1]_\Gamma$$

for any $y_0, y_1 \in \text{Lie } \Gamma$, by Proposition 3.4. Note also that

$$\theta_{\mathbb{A}}^\perp(g, v) = \mathbf{I}(L_g)_*(\text{Ad}(g^{-1})\zeta_I(v)) = \mathbf{I}\mathcal{X}_{\text{Ad}(g^{-1})\zeta_I(v)|g} = ((R_g)_*(\mathbf{I}\mathcal{X}_{\zeta_I(v)}))|g,$$

for any $g \in \mathcal{Y}$ and $v \in T_{gb}B_I$. Hence given $v_0, v_1 \in T_{gb}B_I$,

$$\begin{aligned} F_{\mathbb{A}}(v_0, v_1) &= -\theta_{\mathbb{A}}(R_g)_*[\mathbf{I}\mathcal{X}_{\zeta_I(v_0)}, \mathbf{I}\mathcal{X}_{\zeta_I(v_1)}]|_1 \\ &= \theta_{\mathbb{A}}(R_g)_*[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma = (R_g)_*[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma, \end{aligned}$$

where the first equality follows from (1.21) and the third because

$$(R_g)_*[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma = (L_g)_*\text{Ad}(g^{-1})[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma$$

is a vertical vector field on \mathcal{Y} . □

Given $b, b' \in B_I$, $b' = g \cdot b$ for any g in the fibre of $\mathcal{Y}_{b,I}$ over b' , by Proposition 3.5. Then we have an isomorphism of principal bundles

$$\begin{aligned} \mathcal{Y}_{b,I} &\xrightarrow{\cong} \mathcal{Y}_{b',I} \\ g' &\longmapsto g'g^{-1}, \end{aligned} \tag{3.33}$$

with corresponding isomorphism $\Gamma_b \xrightarrow{\cong} \Gamma_{b'}: g' \longmapsto \text{Ad}(g)g'$ between their structure groups. It follows from the definition of the canonical connection in terms of (3.30), or from (3.28), that this isomorphism exchanges the canonical connections on these principal bundles.

We are now in a position to construct the promised canonical structure of ‘ Γ -invariant symmetric space fibration’ on $\pi_{\mathcal{Z}}: \mathcal{C} \rightarrow \mathcal{Z}$. Observe first that the connection (3.28) induces a canonical affine connection

$$\nabla: \Omega_B^0(TB) \longrightarrow \Omega_B^1(TB) \tag{3.34}$$

on B_I , obtained using the canonical isomorphism

$$TB_I \cong \mathcal{Y} \times_{\Gamma_b} \text{Lie } \tilde{\mathcal{G}}_b \subset \text{ad } \mathcal{Y}, \tag{3.35}$$

which follows from the canonical isomorphism $TB_I \cong \pi^*TB_I/\Gamma_b$ and the Γ_b -equivariant isomorphism of vector bundles

$$\begin{aligned} \mathcal{Y} \times \text{Lie } \tilde{\mathcal{G}}_b &\xrightarrow{\cong} \pi^*TB_I \\ (g, \zeta) &\longmapsto (g, Y_{I\text{Ad}(g)\zeta|g \cdot b}). \end{aligned} \tag{3.36}$$

Note also that the parallel transport $\tau_t(v)$ of a tangent vector $v \in T_{b_0}B_I$ along a curve b_t on B_I , and hence the affine connection ∇ , do not depend on the choice of the base point $b \in B_I$ used implicitly in the right-hand side of (3.35). In fact, it is given by the curve on TB_I defined as

$$\tau_t(v) = Y_{I\zeta_t|b_t}, \text{ where } \zeta_t := \text{Ad}(g_t)\zeta_I(v). \tag{3.37}$$

Here, g_t is the flow of (3.30) with $g_0 = 1$. This follows from (3.35), (3.36) and standard properties about horizontal lifts [35, p. 114].

Note that the canonical connections (3.28) and (3.34) are constructed exactly as for any finite-dimensional symmetric space (cf. e.g. [36, Ch XI, §3]) and that they are exchanged

by the Γ -actions. In fact, our next result shows that (B_I, ∇) is a symmetric space, in a similar sense to [18, §4, Proposition 2].

Theorem 3.6. *Let $I \in \mathcal{Z}$ be such that B_I is non-empty. Then B_I is a symmetric space, i.e. it has a torsion-free affine connection ∇ , with holonomy group contained in $\tilde{\mathcal{G}}_b$ and covariantly constant curvature R_∇ , given by*

$$\zeta_I(R_\nabla(v_0, v_1)v_2) = [[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma, \zeta_I(v_2)]_\Gamma, \quad (3.38)$$

for any $b \in B_I$ and $v_0, v_1, v_2 \in T_b B_I$.

Proof. To prove this, we relate the torsion T_∇ of ∇ with the Nijenhuis tensor N_I of (Γ, \mathbf{I}) and its curvature R_∇ with the curvature $F_\mathbb{A}$ of \mathbb{A} .

Let V_1 and V_2 be two vector fields on B_I . Then

$$T_\nabla(V_1, V_2) := \nabla_{V_1} V_2 - \nabla_{V_2} V_1 - [V_1, V_2].$$

Consider the principal Γ_b -bundle $\pi: \mathcal{Y} \rightarrow B_I$ associated to a fixed $b \in B_I$. By (3.35), $T B_I$ is a subbundle of $\text{ad } \mathcal{Y}$, so V_j induces a Γ_b -invariant vertical vector fields \hat{V}_j on \mathcal{Y} , given by

$$\hat{V}_j(g) = (R_g)_* \zeta_I(V_j(gb)),$$

for $g \in \mathcal{Y}$, $j = 0, 1$. We claim that

$$T_\nabla(V_1, V_2) = -d\pi(N_I(\hat{V}_1, \hat{V}_2)), \quad (3.39)$$

and so $T_\nabla = 0$ by Proposition 3.4. To see this, note first that

$$\mathbf{I}\hat{V}_j = \theta_\mathbb{A}^\perp V_j \quad \text{and} \quad F_\mathbb{A}(V_1, V_2) = -[\hat{V}_1, \hat{V}_2],$$

by (3.28) and (3.29). Moreover, by the construction of ∇ and the definition of the covariant derivative $d_\mathbb{A}$ induced by \mathbb{A} on $\text{ad } \mathcal{Y}$ (see (1.21)),

$$\widehat{\nabla V}_j = d_\mathbb{A} \hat{V}_j := [\theta_\mathbb{A}^\perp(\cdot), \hat{V}_j] = [\mathbf{I}(\cdot), \hat{V}_j].$$

It follows then that

$$\begin{aligned} N_I(\hat{V}_1, \hat{V}_2) &:= [\mathbf{I}\hat{V}_1, \mathbf{I}\hat{V}_2] - \mathbf{I}[\mathbf{I}\hat{V}_1, \hat{V}_2] - \mathbf{I}[\hat{V}_1, \mathbf{I}\hat{V}_2] - [\hat{V}_1, \hat{V}_2] \\ &= [\theta_\mathbb{A}^\perp V_1, \theta_\mathbb{A}^\perp V_2] - \mathbf{I}\widehat{\nabla_{V_1} V_2} + \mathbf{I}\widehat{\nabla_{V_2} V_1} + F_\mathbb{A}(V_1, V_2) \\ &= \theta_\mathbb{A}^\perp([V_1, V_2] - \nabla_{V_1} V_2 + \nabla_{V_2} V_1) \\ &= -\theta_\mathbb{A}^\perp T_\nabla(V_1, V_2), \end{aligned}$$

and so (3.39) holds.

Since the curvature R_∇ is induced by $F_\mathbb{A}$ via the adjoint representation, it follows from (3.29), (3.35) and the fact that $\text{Lie } \tilde{\mathcal{G}}_b \subset \text{Lie } \Gamma$ is a Lie subalgebra, that

$$R_\nabla(v_0, v_1)v_2 = Y_{I[[\zeta_I(v_0), \zeta_I(v_1)]_\Gamma, \zeta_I(v_2)]_\Gamma|b}, \quad (3.40)$$

for $v_0, v_1, v_2 \in T_b B_I$, which implies (3.38), by condition (\star) . Hence, since the group $\tilde{\mathcal{G}}_b$ is normal in Γ_b and B_I is contractible, it follows from (3.40) that the holonomy group of ∇ is contained in $\tilde{\mathcal{G}}_b$ (see [35, Theorem 8.1]). Using (3.40) and the formula (3.37) for the parallel transport τ_t of a curve on B_I , it is now straightforward that $\tau_t^* R_\nabla = R_\nabla$, so $\nabla R_\nabla = 0$. \square

Remark 3.7. When $H^1(X, \mathbb{R}) = 0$, so $\text{Lie } \tilde{\mathcal{G}}_b = \text{Lie } \Gamma_b$, it follows from Proposition 3.5(1) that the bundle \mathcal{Y} , endowed with the restriction of the formally integrable almost complex structure of Proposition 3.4, is an infinitesimal complexification of Γ_b in the sense of Donaldson [18, §4]. If in addition Γ_I is trivial, then there is an alternative proof of Theorem 3.6 which does not use Proposition 3.4. In this case, the almost complex structure on $\mathcal{Y}_{b,I}$ can be defined as the pull-back of the formally integrable almost complex structure on Z_b by the holomorphic map

$$\begin{aligned} \mathcal{Y}_{b,I} &\longrightarrow Z_b \\ g &\longmapsto g^{-1}I. \end{aligned} \tag{3.41}$$

3.3. The uniqueness and existence problem for the coupled equations. We apply now the framework of §§3.1, 3.2 to construct obstructions to the existence of solutions to the coupled equations (2.9).

Fix coupling constants $\alpha_0, \alpha_1 > 0$. It follows from Proposition 2.4, Lemma 3.1 and the construction of μ_b in Theorem 3.2 for each $b = (\omega, H) \in \mathcal{B}$, that the existence of a solution $(J, A) \in \mathcal{P}_b$ of the coupled equations (2.9) (for the symplectic manifold (X, ω) and the principal G -bundle E_H) is equivalent to the condition $\mu_b(I) = 0$ for some $I \in Z_b$. By the equivariance (3.13) of $\mu_{\mathcal{C}}$, this is equivalent to the condition

$$\pi_{\mathcal{Z}}^{-1}(\Gamma \cdot I) \cap \mu_{\mathcal{C}}^{-1}(0) \neq \emptyset, \tag{3.42}$$

where $\Gamma \cdot I \subset \mathcal{Z}$ is the orbit of I . Given such an orbit, in §3.3 we construct a complex character \mathcal{F}_I of the complex Lie algebra $\text{Lie } \Gamma_I$, which vanishes when (3.42) is satisfied, and an ‘integral of the moment map’ $\mathcal{M}_I: B_I \rightarrow \mathbb{R}$, which is bounded from below when (3.42) is satisfied, provided that the symmetric space B_I is geodesically convex. Furthermore, we motivate a definition of ‘geodesic stability’ of the orbit $\Gamma \cdot I$ and conjecture a link with (3.42) when Γ_I is finite.

We first reformulate condition (3.42) in terms of a Γ -invariant family σ of 1-forms σ_I on the fibres B_I of $\pi_{\mathcal{Z}}: \mathcal{C} \rightarrow \mathcal{Z}$, defined by the formula

$$\sigma_I(v) := -\langle \mu_b(I), \zeta_I(v) \rangle, \tag{3.43}$$

for all $(b, I) \in \mathcal{C}$, $v \in T_b B_I$, with ζ_I defined as in (3.26). Here, the Γ -invariance of σ means

$$\sigma_{g \cdot I}(gv) = \sigma_I(v) \tag{3.44}$$

for all $(b, I) \in \mathcal{C}$, $v \in T_b B_I$, $g \in \Gamma$. Note that (3.44) follows from (3.13) and the fact that

$$\text{Ad}(g)\zeta_I(v) = \zeta_{gI}(v)$$

for all $g \in \Gamma$, which is immediate from the definition of ζ_I . Observe also that

$$\pi_{\mathcal{Z}}^{-1}(\Gamma \cdot I) \cap \mu_{\mathcal{C}}^{-1}(0) \neq \emptyset \iff \sigma_I \in \Omega^1(B_I) \text{ has a zero.} \tag{3.45}$$

Now, since B_I is contractible (see (3.4)), it suffices to study σ_I along curves on B_I . Let V_t be a vector field on B_I along a curve b_t on B_I , i.e. a curve on TB_I with $V_t \in T_{b_t} B_I$ for all t . We use the standard notation $\nabla_{\dot{b}_t} V_t$ for the covariant derivative of V_t in the direction of \dot{b}_t on the symmetric space (B_I, ∇) (see (3.34) and Theorem 3.6).

Proposition 3.8.

- (1) $\frac{d}{dt} \sigma_I(V_t) = \omega_{b_t}(Y_{\zeta_I(V_t)|I}, \mathbf{I}Y_{\zeta_I(\dot{b}_t)|I}) + \sigma_I(\nabla_{\dot{b}_t} V_t)$.
- (2) σ_I is closed.

Proof. To prove (1), let g_t the horizontal lift of b_t to $\mathcal{Y}_{b_0, I}$ prescribed by the connection (3.28), with $g_0 = 1$. Then $b_t = g_t \cdot b$ (see Proposition 3.5), so (3.13) implies

$$\sigma_I(V_t) = -\langle \mu_b(I_t), \zeta_t \rangle, \quad (3.46)$$

where $I_t := g_t^{-1} \cdot I$ and $\zeta_t := \text{Ad}(g_t)^{-1} \zeta_I(V_t)$. Using (3.21), we obtain

$$\dot{I}_t = -g_t^{-1} \dot{g}_t g_t^{-1} I = -g_t^{-1} Y_{I_{\zeta_I(\dot{b}_t)|I}} = -g_t^{-1} \mathbf{I} Y_{\zeta_I(\dot{b}_t)|I},$$

so using formula (3.37) for the parallel transport $\tau_{t,s}: T_{b_t} B_I \rightarrow T_{b_s} B_I$ and the definition of covariant derivative (see e.g. [35, p. 114]),

$$\begin{aligned} \nabla_{\dot{b}_t} V_t &:= \frac{d}{ds} \Big|_{s=t} \tau_{t,s}^{-1}(V_t) = \frac{d}{ds} \Big|_{s=t} Y_{I_{\text{Ad}(g_s g_t^{-1})^{-1} \zeta_I(V_t)|b_t}} \\ &= Y_{I_{\text{Ad}(g_t) \zeta_t|b_t}} = g_t Y_{I_{\zeta_t|b}}. \end{aligned}$$

Formula (1) follows now from this equation and the Γ -invariance of σ_I , as they imply $\sigma_{I_t}(Y_{I_t \zeta_t|b}) = \sigma_I(\nabla_{\dot{b}_t} V_t)$, that combined with (3.46) imply

$$\begin{aligned} \frac{d}{dt} \sigma_I(V_t) &= -\langle d\mu_b(\dot{I}_t), \zeta_t \rangle - \langle \mu_b(I_t), \dot{\zeta}_t \rangle \\ &= \omega_b(g_t^{-1} Y_{\zeta_I(V_t)|I}, g_t^{-1} \mathbf{I} Y_{\zeta_I(\dot{b}_t)|I}) + \sigma_{I_t}(Y_{I_t \zeta_t|b}) \\ &= \omega_{b_t}(Y_{\zeta_I(V_t)|I}, \mathbf{I} Y_{\zeta_I(\dot{b}_t)|I}) + \sigma_I(\nabla_{\dot{b}_t} V_t), \end{aligned}$$

since μ_b is a moment map and $\omega_{\mathcal{C}}$ is Γ -invariant.

To prove (2), let V_1 and V_2 be two vector fields on B_I . Then

$$d\sigma_I(V_1, V_2) = V_1(\sigma_I(V_2)) - V_2(\sigma_I(V_1)) - \sigma_I([V_1, V_2]), \quad (3.47)$$

so, using (1) and the fact that $\omega_B(\cdot, \mathbf{I} \cdot)$ is a family of symmetric bilinear forms, we see that

$$d\sigma_I(V_1, V_2) = \sigma_I(T_{\nabla}(V_1, V_2)),$$

which vanishes because ∇ is torsion-free, by Theorem 3.6. \square

To define our first obstruction to (3.42), note that $\text{Lie } \Gamma_I$ is a complex Lie algebra for all $I \in \mathcal{Z}$, by (3.21) and the equivariance of (3.20). Given $I \in \mathcal{Z}$ and $b \in B_I$, combining the 1-form σ_I and (3.20), we obtain a \mathbb{C} -linear map

$$\begin{aligned} \mathcal{F}_I: \text{Lie } \Gamma_I &\longrightarrow \mathbb{C} \\ \zeta &\longmapsto \langle \mathcal{F}_I, \zeta \rangle := \mathbf{i} \sigma_I(Y_{\zeta|b}) + \sigma_I(Y_{I_{\zeta|b}}). \end{aligned} \quad (3.48)$$

By the Γ -invariance of σ (see (3.44)), this map is also Γ -invariant, i.e.

$$\langle \mathcal{F}_{g \cdot I}, \text{Ad}(g)\zeta \rangle = \langle \mathcal{F}_I, \zeta \rangle, \quad (3.49)$$

for all $\zeta \in \text{Lie } \Gamma_I$, $g \in \Gamma$.

Theorem 3.9. *The map (3.48) is independent of $b \in B_I$. It defines a character*

$$\mathcal{F}_I: \text{Lie } \Gamma_I \longrightarrow \mathbb{C}$$

of $\text{Lie } \Gamma_I$ that vanishes if σ_I has a zero.

Proof. The proof essentially follows a previous one by Bourguignon [7]. For the first part, it is enough to prove that $\sigma_I(Y_\zeta)$ is a constant function on B_I , for all $\zeta \in \text{Lie } \Gamma_I$. Now, $\sigma_I \in \Omega^1(B_I)$ is closed (by Proposition (3.8)) and Γ_I -invariant (since σ is Γ -invariant), so

$$d(\sigma_I(Y_\zeta)) = -Y_\zeta \lrcorner d\sigma_I + L_{Y_\zeta} \sigma_I = 0,$$

and hence $\sigma_I(Y_\zeta)$ is constant, because B_I is contractible. The second part follows because \mathcal{F}_I is \mathbb{C} -linear and Γ_I -invariant, by (3.49). \square

To obtain the second obstruction, note that, by Proposition 3.8 and the contractibility of B_I , σ_I is exact and so there exists a functional

$$\mathcal{M}_I: B_I \times B_I \rightarrow \mathbb{R} \tag{3.50}$$

such that $d\mathcal{M}_I(\cdot, b) = \sigma_I$ and $\mathcal{M}_I(b, b) = 0$ for all $b \in B_I$. Along a curve b_t on B_I ,

$$\mathcal{M}_I(b_t, b) = \mathcal{M}_I(b_0, b) + \int_0^t \sigma_I(\dot{b}_s) ds. \tag{3.51}$$

Moreover, the Γ -invariance of σ implies that

$$\mathcal{M}_I(gb', b) = \mathcal{M}_{g^{-1}I}(b', b) + \mathcal{M}_I(b', b), \tag{3.52}$$

for all $g \in \Gamma$ such that $gb' \in B_I$ (i.e. $g \in \mathcal{Y}_{b', I}$).

Proposition 3.10. *The functional $\mathcal{M}_I(\cdot, b): B_I \rightarrow \mathbb{R}$ is convex along geodesics on (B_I, ∇) . If B_I is geodesically convex and σ_I has a zero, then $\mathcal{M}_I(\cdot, b)$ is bounded from below, for all $b \in B_I$.*

Proof. The first part follows because (3.51) and Proposition 3.8(1) imply

$$\frac{d^2}{dt^2} \mathcal{M}_I(b_t, b) = \frac{d}{dt} \sigma_I(\dot{b}_t) = \left\| Y_{\zeta_I(\dot{b}_t)|I} \right\|^2 \geq 0, \tag{3.53}$$

for any geodesic b_t on (B_I, ∇) , where $\| \cdot \|$ is the L^2 -norm with respect to the metric on Z_{b_t} .

For the second part, suppose $b' \in B_I$ is a zero of σ_I . We can suppose $b' = b$, because using (3.51) along a curve joining b and b' , we see that

$$\mathcal{M}_I(\cdot, b') = \mathcal{M}_I(b, b') + \mathcal{M}_I(\cdot, b).$$

Now, given $b'' \in B_I$, by hypothesis there exists a geodesic b_t with $b_0 = b$ and $b_1 = b''$. Hence

$$\mathcal{M}_I(b'', b) = \int_0^1 \int_0^t \left\| Y_{\zeta_I(\dot{b}_s)|I} \right\|^2 ds \wedge dt \geq 0,$$

and so $\mathcal{M}_I(\cdot, b)$ is bounded from below by $0 \in \mathbb{R}$. \square

Corollary 3.11. *If B_I is geodesically convex, then σ_I has at most one zero on B_I modulo the action of Γ_I .*

Proof. Given zeros $b, b' \in B_I$ of σ_I , let b_t a geodesic joining them. Then

$$\left\| Y_{\zeta_I(\dot{b}_t)|I} \right\|^2 = 0$$

for all t , because (3.53) implies that

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \sigma(\dot{b}_t) \end{array} \tag{3.54}$$

is an increasing function which vanishes for $t = 0$ and $t = 1$. Hence the flow g_t of $I\zeta_I(\dot{b}_t)$ lies in Γ_I for all t and $g_t b = b_t$. In particular, $g_1 b = b'$. \square

Remark 3.12. Proposition 3.8 and Theorem 3.9 hold even when α_0, α_1 are not positive (their proofs depend only on the condition that ω_b is of type (1,1) with respect to \mathbf{I}). In §4, we will use these facts about σ_I and \mathcal{F}_I for arbitrary α_0, α_1 . However, Proposition 3.10, Corollary 3.11 and the remainder of §3.3 depend on the assumption that α_0, α_1 are positive, although Proposition 3.14 also holds in the degenerate case $\alpha_0 \alpha_1 = 0$.

If Z_b and Γ_b are finite-dimensional manifolds and $\tilde{\mathcal{G}}_b = \Gamma_b$ is compact, there is a well-known numerical condition, called the Hilbert–Mumford criterion, which characterizes (3.42) (see the example at the end of §3.3). In this case, the principal bundle \mathcal{Y} of Proposition 3.5 is the complexification of Γ_b (by the observations about infinitesimal complexifications at the end of §3.2, as formally integrable almost complex structures are integrable in finite dimensions), and the criterion is formulated in terms of 1-parameter subgroups of \mathcal{Y} . In the generality of §3, the Lie group Γ_b may have no complexification, but the geodesics of the symmetric space (B_I, ∇) are a substitute for the 1-parameter subgroups, and we have the following generalization of this condition (cf. [11], [18, §8]).

Definition 3.13. A point $I \in \mathcal{Z}$ is *geodesically semistable* if

$$\lim_{t \rightarrow \infty} \sigma_I(\dot{b}_t) \geq 0 \quad (3.55)$$

for any infinite geodesic ray b_t , $t \in [0, \infty[$, in (B_I, ∇) . It is *geodesically stable* if the inequality (3.55) is strict whenever b_t is non-constant.

Observe that the limit (3.55) always exists, because (3.54) is an increasing function for geodesic rays, by (3.53). Note also that the geodesic stability and semistability conditions only depend on the Γ -orbit of $I \in \mathcal{Z}$, because σ is Γ -invariant and the connections on the fibres of $\pi_{\mathcal{Z}}$ are exchanged by the Γ -action and hence so are their geodesic rays.

In the finite-dimensional case, by the Kempf–Ness Theorem [33], an orbit $\Gamma \cdot I \in \mathcal{Z}$ is geodesically stable if and only if (3.42) holds and Γ_I is finite (see the example at the end of §3.3). The following result provides some evidence that a sensible question is whether this equivalence also holds in the generality of §3, at least when B_I is geodesically convex.

Proposition 3.14. *Let $(b, I) \in \mathcal{C}$. Then*

- (1) *If $\Gamma_{b,I} := \Gamma_b \cap \Gamma_I$ is not finite, then $\Gamma \cdot I$ is not geodesically stable.*
- (2) *Suppose that B_I is geodesically convex. If (3.42) is satisfied, then $\Gamma \cdot I$ is geodesically semistable.*

Proof. For part (1), let $\zeta \in \text{Lie } \Gamma_I$ be non-zero. Let g_t the flow of $I\zeta$. Then $b_t = g_t b$ is an infinite geodesic ray starting at b , because

$$\dot{b}_t = g_t Y_{I\zeta|b} = Y_{\text{Ad}(g_t)I\zeta|b_t} = Y_{I \text{Ad}(g_t)\zeta|b_t} = \tau_t \dot{b}_0.$$

Furthermore, if $\zeta \in \text{Lie } \Gamma_{b,I}$, then $\dot{b}_0 = Y_{I\zeta|b} \neq 0$, by (3.25), so b_t is non-constant. Then

$$\sigma_I(\dot{b}_t) = \sigma_I(g_t \dot{b}_0) = \sigma_{g_t^{-1}}(\dot{b}_0) = \sigma_I(\dot{b}_0)$$

and so

$$\lim_{t \rightarrow \infty} \sigma_I(\dot{b}_t) = \sigma_I(\dot{b}_0) = \sigma_I(Y_{I\zeta|b}) = \langle \mathcal{F}_I, \zeta \rangle.$$

There are three possibilities. If $\langle \mathcal{F}_I, \zeta \rangle < 0$, then part (1) is obvious. The case $\langle \mathcal{F}_I, \zeta \rangle > 0$ reduces to the previous one by taking the non-trivial geodesic corresponding to $-I\zeta$. Finally, if $\langle \mathcal{F}_I, \zeta \rangle = 0$, since b_t is non-trivial, then by definition I is not geodesically stable.

For part (2), suppose that B_I is geodesically convex and $\Gamma \cdot I$ is not geodesically semistable. Then there exists an infinite geodesic ray b_t such that

$$C := \lim_{t \rightarrow \infty} \sigma_I(\dot{b}_t) < 0,$$

where $\sigma_I(\dot{\sigma}_t) \leq C$ for all t , as (3.54) is an increasing function, so $\mathcal{M}_I(b_t, b_0) \leq Ct$, by (3.51). Therefore $\mathcal{M}_I(\cdot, b_0)$ is not bounded from below, so (3.42) cannot be satisfied, by Proposition 3.10. \square

We would like to point out that the framework developed in §§3.2, 3.3 is rather general, as it relies only on formal properties of the double fibration (3.3), and may be applied to other situations (in particular, to equations with a further coupling with Higgs fields). The basic ingredients are a real Lie group Γ , a Γ -equivariant double fibration (3.3), where $(\mathcal{Z}, \mathbf{I})$ is an almost complex manifold, and a Γ -equivariant map (3.20) satisfying (3.21). It is crucial that $\pi_{\mathcal{Z}}$ satisfies condition (\star) of §3.2, all its fibres are contractible and $\pi_{\mathcal{B}}$ satisfies the properties of Theorem 3.2 for a fibration of normal subgroups as in (3.11) (note that the formal integrability of the almost complex structures on the fibres of $\pi_{\mathcal{B}}$ was never used).

To see how this general framework works, we conclude §3.3 by explaining how it applies to the standard theory of finite-dimensional Kähler quotients (as presented e.g. in [46, §5]) and its relation with Geometric Invariant Theory (GIT). Suppose that \mathcal{Z} is a finite-dimensional Kähler manifold with a left action of a complex reductive Lie group G^c preserving its complex structure. Suppose also that this action restricts to a Hamiltonian action of a maximal compact subgroup $G \subset G^c$, with G -equivariant moment map

$$\mu: \mathcal{Z} \longrightarrow \mathfrak{g}^*,$$

where \mathfrak{g} is the Lie algebra of G . To compare with §§3.2, 3.3, we define:

- Γ is the real Lie group underlying G^c ,
- $\mathcal{B} = G^c/G$ is the orbit space for the action by right multiplication of G on G^c ,
- the map $\mathcal{Z} \rightarrow W$ of (3.20) is the constant map given by the complex structure on the Lie algebra \mathfrak{g}^c of G^c ,
- $\mathcal{C} = \mathcal{B} \times \mathcal{Z}$ and $\tilde{\mathcal{G}}_{\mathcal{B}} = \Gamma_{\mathcal{B}}$.

Then the isotropy group of any G -orbit $b = [g] := gG \in \mathcal{B}$ is

$$\tilde{\mathcal{G}}_b = \Gamma_b = \text{Ad}(g)G$$

and the fibre of the morphism (3.25) over a point $b = [g]$ is

$$\begin{aligned} \text{Ad}(g)\mathfrak{g} &\xrightarrow{\cong} T_g G^c / T_g(gG) \\ \zeta &\longmapsto [(R_g)_*(\mathbf{i}\zeta)]. \end{aligned} \tag{3.56}$$

Therefore (3.25) is an isomorphism and condition (\star) of §3.2 is satisfied, and hence so are the conclusions of Proposition 3.2 and Theorem 3.3. In this finite-dimensional case, the construction of the connections (3.28) and (3.34) reduce to the classical constructions of the canonical connections on finite-dimensional symmetric spaces (see e.g. [36, Ch XI, §3]).

Hence, by [36, Ch XI, Theorem 3.2(3)], the infinite geodesic rays on G^c/G starting at $[g]$ are the curves

$$\begin{aligned} [0, \infty[&\longrightarrow G^c/G \\ t &\longmapsto [e^{t\mathbf{i}}\zeta g], \end{aligned} \tag{3.57}$$

with $\zeta \in \text{Ad}(g)\mathfrak{g}$. Note that the canonical projection

$$\pi_{\mathcal{B}}: G^c/G \times \mathcal{Z} \longrightarrow G^c/G$$

is a ‘trivial’ G^c -invariant complex fibration. However, since G^c does not necessarily preserve $\omega_{\mathcal{Z}}$, to view $\pi_{\mathcal{B}}$ as a G^c -invariant Hamiltonian fibration, we endow this map with the non-trivial family $\omega_{\mathcal{C}}$ of symplectic 2-forms $\omega_b := g_*\omega_{\mathcal{Z}}$ on the fibres $Z_b = \mathcal{Z}$, for $b = [g] \in G^c/G$. Indeed, the isotropy group $\text{Ad}(g)G$ preserves ω_b and has moment map given by

$$\langle \mu_b(I), \zeta \rangle := \langle \mu(g^{-1}I), \text{Ad}(g^{-1})\zeta \rangle, \tag{3.58}$$

for $b = [g] \in G^c/G$, and (3.58) defines the morphism (3.12) of fibrations over G^c/G . Using the isomorphism (3.56), we obtain the formula

$$\langle \mathcal{F}_I, \zeta \rangle = -\langle \mu(I), \zeta_0 \rangle - \mathbf{i}\langle \mu(I), \zeta_1 \rangle,$$

for all $I \in \mathcal{Z}$, $\zeta = \zeta_0 + \mathbf{i}\zeta_1 \in \mathfrak{g}_I^c$, where $\zeta_0, \zeta_1 \in \mathfrak{g}^c$. Hence Theorem 3.9 reduces to [59, Proposition 6 and Corollary 8].

Suppose now that \mathcal{Z} is a G^c -linearised projective manifold, i.e. there is a G^c -equivariant closed embedding $\mathcal{Z} \subset \mathbb{C}\mathbb{P}^N$ and $\omega_{\mathcal{Z}}$ is the restriction of the Fubini–Study Kähler form. Then geodesic stability/semistability coincide with GIT stability/semistability, by the Hilbert–Mumford criterion. This essentially follows because any 1-parameter subgroup

$$\lambda: \mathbb{C}^* \rightarrow G^c$$

restricts to a group homomorphism $\lambda: S^1 \rightarrow \text{Ad}(g)G$ for some $g \in G^c$, which induces an infinite geodesic ray (3.57) starting at $[g]$ and because the Hilbert–Mumford weight for λ at a point $I \in \mathcal{Z}$ is precisely the left-hand side of (3.55). Furthermore, the functional (3.50) is the Kempf–Ness functional [33], which provides the key tool to prove the Kempf–Ness theorem relating the symplectic and GIT quotients:

$$\mu^{-1}(0)/G \cong \mathcal{Z} // G^c.$$

Finally, we should remark that this theorem has been extended to non-projective manifolds (see e.g. [46, §5], [54]). In this case, the functional (3.50) is the integral of the moment map in [46, §5] and geodesic stability coincides with analytic stability (by [46, Corollary 5.3]).

3.4. The α -Futaki character, the α -K-energy and the geodesic equation. We now prove that condition (\star) of §3.2 is satisfied and give explicit formulae for the character \mathcal{F}_I , the functional \mathcal{M}_I and the geodesic equation on B_I introduced in §3.3.

Fix a complex structure on X for which $\Omega \in H^2(X, \mathbb{R})$ is a Kähler class (i.e. it contains a Kähler form) and a holomorphic structure on the principal G^c -bundle $\pi: E^c \rightarrow X$. These data determine a point $I \in \mathcal{Z}$. As explained in §3.3, condition (3.42) for the orbit $\Gamma \cdot I$ is equivalent to the existence of a pair $b = (\omega, H) \in B_I$ such that the point $(J, A) \in \mathcal{P}_b$ corresponding to I via Lemma 3.1 satisfies the coupled equations (2.9). In other words, condition (3.42) for the orbit $\Gamma \cdot I$ is equivalent to the existence of a solution $b = (\omega, H) \in B_I$

to the following coupled equations, where S_ω is the scalar curvature of the Riemannian metric $g_{\check{I}} = \omega(\cdot, \check{I}\cdot)$ and F_H is the curvature of the Chern connection of H and I :

$$\left. \begin{aligned} \Lambda_\omega F_H &= z \\ \alpha_0 S_\omega + \alpha_1 \Lambda_\omega^2(F_H \wedge F_H) &= c \end{aligned} \right\}. \quad (3.59)$$

By (3.45), these equations are satisfied if and only if the 1-form σ_I on B_I has a zero. Now, the definition of σ_I in (3.43), and in fact the whole of §§3.2, 3.3, depend on condition (\star) of §3.2. To prove this condition, note first that by (3.4), there is a canonical isomorphism

$$T_b B_I \cong dd^c C^\infty(X) \oplus \mathbf{i} \Omega^0(\text{ad } E_H), \quad (3.60)$$

for all $b = (\omega, H) \in B_I$, obtained from the dd^c -lemma and from the pointwise isomorphism $\mathbf{i} \text{Lie } G \cong G^c/G$ induced by the exponential. Define now $\text{Lie } \mathcal{G}^c = \Omega^0(\text{ad } E^c)$ and $\text{Lie } \mathcal{G}_H = \Omega^0(\text{ad } E_H)$ as the Lie algebras of the gauge group \mathcal{G}^c of E^c and the gauge group \mathcal{G}_H of E_H , respectively. Consider the projection maps onto the real and imaginary parts associated to $H \in \mathcal{R}$,

$$\text{Re}_H, \text{Im}_H: \text{Lie } \mathcal{G}^c \longrightarrow \text{Lie } \mathcal{G}_H, \quad (3.61)$$

defined by $y = \text{Re}_H y + \mathbf{i} \text{Im}_H y$ for all $y \in \text{Lie } \mathcal{G}^c$, where we are using the canonical isomorphism

$$\text{Lie } \mathcal{G}^c \cong \text{Lie } \mathcal{G}_H \oplus \mathbf{i} \text{Lie } \mathcal{G}_H$$

induced by (3.7) and $\mathfrak{g}^c = \mathfrak{g} \oplus \mathbf{i} \mathfrak{g}$.

Lemma 3.15. *The infinitesimal action of $y \in \text{Lie}(\text{Aut } E^c)$ on $H \in \mathcal{R}$ is*

$$Y_{y|H} = \mathbf{i} \text{Im}_H(\theta_A y),$$

where $\theta_A: TE^c \rightarrow VE^c$ is the vertical projection induced by any connection A on E_H .

Proof. Using the maps θ_A, θ_A^\perp in (3.6), any $y \in \text{Lie}(\text{Aut } E^c)$ can be decomposed as

$$y = \mathbf{i} \text{Im}_H(\theta_A y) + \text{Re}_H(\theta_A y) + \theta_A^\perp \check{y},$$

where \check{y} is the vector field on X covered by y . Hence the flow g_{-t} of $-y$ can be written as

$$g_{-t} = f_{-t} \circ s_t$$

where $f_t \in \text{Aut } E_H$ is the flow of $\text{Re}_H(\theta_A y) + \theta_A^\perp \check{y}$ and s_t is the flow of the time-dependent vector field $-f_{t*}(\mathbf{i} \text{Im}_H(\theta_A y))$. Therefore, using the isomorphism $T_H \mathcal{R} \cong \mathbf{i} \Omega^0(\text{ad } E_H)$ (also used in (3.60)) and the fact that f_t^{-1} preserves H , we see that the flow g_t of y satisfies

$$Y_{y|H} = \frac{d}{dt} \Big|_{t=0} g_t \cdot H = \frac{d}{dt} \Big|_{t=0} s_t^{-1} \cdot H = f_{t*}(\mathbf{i} \text{Im}_H(\theta_A y))|_{t=0} = \mathbf{i} \text{Im}_H(\theta_A y). \quad \square$$

Proposition 3.16. *Condition (\star) is satisfied. The inverse of (3.25) is given by*

$$\zeta_I(v) = -\mathbf{i} \dot{H} - \theta_H^\perp \eta_\phi \in \text{Lie } \tilde{\mathcal{G}}_b, \quad (3.62)$$

where $v \in T_b B_I$ corresponds to $(dd^c \phi, \dot{H}) \in dd^c C^\infty(X) \oplus \mathbf{i} \Omega^0(\text{ad } E_H)$ via (3.60).

Proof. Fix $b = (\omega, H) \in B_I$. Given $\zeta \in \text{Lie } \Gamma$ covering a vector field $\check{\zeta}$ on X , we have

$$Y_{I\zeta|b} = (-d(\check{I}\check{\zeta} \lrcorner \omega), \mathbf{i} \text{Re}_H(\theta_H \zeta)) \quad (3.63)$$

by Lemma 3.15, as $I\zeta$ covers $\check{I}\check{\zeta}$ and $\theta_H \circ I = \mathbf{i} \theta_H$, where $\theta_H: TE^c \rightarrow VE^c$ is the vertical projection in (3.6) induced by the Chern connection of I on E_H . In particular, when $\zeta \in \text{Lie } \tilde{\mathcal{G}}_b$, $\check{\zeta} = \eta_\phi$ is the Hamiltonian vector field of some $\phi \in C^\infty(X)$ and (3.63) becomes

$$Y_{I\zeta|b} = (-dd^c \phi, \mathbf{i} \theta_H \zeta). \quad (3.64)$$

Hence, by (3.60) the infinitesimal action (3.64) is in $T_b B_I$ and so the morphism (3.25) is well defined. Furthermore, (3.64) easily implies that (3.25) has an inverse given by (3.62). \square

Using the formula (3.62), the 1-form σ_I on B_I is given by

$$\sigma_I(v) = 4\alpha_1 \int_X \mathbf{i} \dot{H} \wedge (\Lambda_\omega F_H - z)\omega^{[n]} + \int_X \phi S_\alpha(b, I)\omega^{[n]}, \quad (3.65)$$

for all $v = (dd^c \phi, \dot{H}) \in T_b B_I$, where $\phi\omega^{[n]}$ has zero integral on X and $S_\alpha(b, I)$ is given by (3.18).

The complex character $\mathcal{F}_I: \text{Lie } \Gamma_I \rightarrow \mathbb{C}$ defined by (3.48) provides our first obstruction to the existence of solutions to (3.59). To give an explicit expression for \mathcal{F}_I , note first that

$$\text{Lie } \Gamma_I = \text{Lie Aut}(E^c, I)$$

is the Lie algebra of the automorphism group of the holomorphic bundle (E^c, I) , so each $\zeta \in \text{Lie } \Gamma_I$ covers a real holomorphic vector field $\check{\zeta}$ on (X, \check{I}) . Now, we can write

$$\check{\zeta} = \eta_{\phi_1} + \check{I}\eta_{\phi_2} + \beta,$$

for any given a Kähler form $\omega \in \mathcal{K}_{\check{I}}$, where η_{ϕ_j} is the Hamiltonian vector field of $\phi_j \in C_0^\infty(X)$ on (X, ω) , for $j = 1, 2$, and β is the dual of a 1-form which is harmonic with respect to the Kähler metric $\omega(\cdot, \check{I}\cdot)$ (see e.g. [39]). Using this decomposition in (3.63), we see that the infinitesimal action of $\zeta \in \text{Lie Aut}(E^c, I)$ on $b = (\omega, H) \in B_I$ is

$$Y_{\zeta|b} = (-dd^c \phi_2, \mathbf{i} \text{Im}_H \theta_H \zeta),$$

hence defining the complex-valued function $\phi := \phi_1 + \mathbf{i}\phi_2$,

$$\langle \mathcal{F}_I, \zeta \rangle = -4\alpha_1 \int_X \theta_H \zeta \wedge (\Lambda_\omega F_H - z)\omega^{[n]} - \int_X \phi S_\alpha(b, I)\omega^{[n]}, \quad (3.66)$$

which must vanish if (3.59) has a solution, by Theorem 3.9.

It is now clear from formula (3.66) that for trivial G^c , \mathcal{F}_I is the Futaki invariant [22] of the Kähler class Ω on (X, \check{I}) , up to a multiplicative factor. For non-trivial G^c and $\alpha_0 = 0$, the character \mathcal{F}_I , restricted to the Lie subalgebra of $\text{Lie } \Gamma_I$ consisting of vector fields covering holomorphic complex Hamiltonian vector fields (i.e. vector fields that vanish somewhere on X), has already been constructed by Futaki (see [23, Theorem 1.1]).

Using now (3.51), the α -K-energy can be written explicitly along a curve $b_t = (\omega_t, H_t)$ on B_I , with $\omega_t = \omega_0 + dd^c \phi_t$ and $\dot{\phi}_t \omega_t^{[n]}$ with zero integral on X , as

$$\begin{aligned} \mathcal{M}_I(b_t, b) &= \mathcal{M}_I(b_0, b) + 4\alpha_1 \int_0^t \int_X \mathbf{i} \dot{H}_s \wedge (\Lambda_{\omega_s} F_{H_s} - z)\omega_s^{[n]} \wedge ds \\ &\quad + \int_0^t \int_X \dot{\phi}_s S_\alpha(b_s, I)\omega_s^{[n]} \wedge ds. \end{aligned} \quad (3.67)$$

By Proposition 3.10, $\mathcal{M}_I(\cdot, b)$ is convex along geodesics on the symmetric space (B_I, ∇) . The explicit expression of the affine connection ∇ and its geodesic equation in the coordinates provided by the canonical isomorphism (3.60) are the content of the following proposition.

For the next result, given $b = (\omega, H) \in B_I$, we denote by $(\cdot, \cdot)_\omega$ the metric on T^*X associated to $\omega(\cdot, \check{I}\cdot)$ and by d_H the covariant derivative associated to the Chern connection of H and I .

Proposition 3.17. (1) *The Christoffel symbol $\Gamma: T_b B_I \times T_b B_I \rightarrow T_b B_I$ is*

$$\Gamma(\dot{b}_1, \dot{b}_2) = \left(-dd^c(d\phi_1, d\phi_2)_\omega, -\check{I}\eta_{\phi_1} \lrcorner d_H \dot{H}_2 - \check{I}\eta_{\phi_2} \lrcorner d_H \dot{H}_1 + \mathbf{i}F_H(\eta_{\phi_1}, \check{I}\eta_{\phi_2}) \right),$$

for all $\dot{b}_j = (dd^c\phi_j, \dot{H}_j) \in T_b B_I$, with $j = 1, 2$.

(2) *A curve $b_t = (\omega_t, H_t)$ on B_I , with $\omega_t = \omega + dd^c\phi_t$, is a geodesic if and only if*

$$\left. \begin{aligned} dd^c(\dot{\phi}_t - (d\dot{\phi}_t, d\dot{\phi}_t)_{\omega_t}) &= 0 \\ \ddot{H}_t - 2\check{I}\eta_{\dot{\phi}_t} \lrcorner d_{H_t} \dot{H}_t + \mathbf{i}F_{H_t}(\eta_{\dot{\phi}_t}, \check{I}\eta_{\dot{\phi}_t}) &= 0 \end{aligned} \right\}, \quad (3.68)$$

where $\eta_{\dot{\phi}_t}$ is the Hamiltonian vector field of $\dot{\phi}_t$ over (X, ω_t) .

Proof. The proof of part (1) is a computation of the covariant derivative of a vector field $v_t = (dd^c\psi_t, \mathbf{i}\xi_t)$ along a curve $b_t = (\omega_t, H_t)$ on B_I , i.e. a curve v_t on $T B_I$ with $v_t \in T_{b_t} B_I$ for all t . Recall that the covariant derivative of v_t along b_t is (see e.g. [35, p. 114])

$$\nabla_{\dot{b}_t} v_t := \frac{d}{ds} \Big|_{s=t} \tau_{t,s}^{-1}(v_s) = \dot{v}_t + \Gamma(\dot{b}_t, v_t), \quad (3.69)$$

where $\dot{v}_t = (dd^c\dot{\psi}_t, \mathbf{i}\dot{\xi}_t)$ and $\tau_{t,s}: T_{b_t} B_I \rightarrow T_{b_s} B_I$ is the parallel transport along b_t . To calculate (3.69) we compute the parallel transport $\tau_{0,t}(v)$ of any $v = (dd^c\phi, \dot{H}) \in T_b B_I$ along b_t using (3.37). Let $\zeta_t = \text{Ad}(g_t)\zeta_I(v)$, where g_t is the horizontal lift of b_t to $\mathcal{Y}_{b_0, I}$ (i.e. the flow of $I\zeta_I(\dot{b}_t)$) with $g_0 = 1$ (see (3.27) and Proposition 3.5). By (3.62),

$$\zeta_t = (g_t)_* (-\mathbf{i}\dot{H} - \theta_{H_t, I}^\perp \eta_\phi) = -\mathbf{i}(g_t)_* \dot{H} - \theta_{H_t, I_t}^\perp ((\check{g}_t)_* \eta_\phi) = -\mathbf{i}(g_t)_* \dot{H} - \theta_{H_t, I_t}^\perp \eta_t,$$

where η_t is the Hamiltonian vector field of $\phi \circ \check{g}_t^{-1}$ over (X, ω_t) , so by (3.37) and (3.63),

$$\begin{aligned} \tau_{0,t}(\dot{b}) &= Y_{I\zeta_t|_{b_t}} = (dd^c(\phi \circ \check{g}_t^{-1}), \mathbf{i}\text{Re}_{H_t} \theta_{H_t, I} (-\mathbf{i}(g_t)_* \dot{H} - \theta_{H_t, I_t}^\perp \eta_t)) \\ &= (dd^c(\phi \circ \check{g}_t^{-1}), \text{Im}_{H_t}((g_t)_* \dot{H}) + \eta_t \lrcorner \mathbf{i}(\theta_{H_t, I_t} - \theta_{H_t, I})). \end{aligned} \quad (3.70)$$

Hence we obtain

$$\tau_{t,s}^{-1}(v_s) = (dd^c(\psi_s \circ \check{g}_s), \text{Im}_{H_t}(\mathbf{i}(g_{t,s})^* \xi_s) + \eta_{t,s} \lrcorner \mathbf{i}(\theta_{H_t, I_s} - \theta_{H_t, I})),$$

where $v_s = (dd^c\psi_s, \mathbf{i}\xi_s) \in T_{b_s} B_I$, $g_{t,s}$ is the flow of $I\zeta_I(\dot{b}_s)$ with $g_{t,t} = 1$, $\eta_{t,s}$ is the Hamiltonian vector field of $\psi_s \circ \check{g}_{t,s}$ over (X, ω_t) and $I_s = g_{t,s}^{-1} \cdot I$. Thus denoting $\nabla_{\dot{b}_t} v_t = (w_t^1, w_t^2)$, we conclude that

$$\begin{aligned} w_t^1 &= dd^c(\dot{\psi}_t - d\psi_t(\check{I}\eta_{\dot{\phi}_t, \omega_t})) = dd^c\dot{\psi}_t - dd^c(d\psi_t, d\dot{\phi}_t)_{\omega_t}, \\ w_t^2 &= \text{Im}_{H_t}(\mathbf{i}\dot{\xi}_t + [I\zeta_I(\dot{b}_t), \mathbf{i}\xi_t]) - \mathbf{i}\theta_{H_t, I}[I\zeta_I(\dot{b}_t), \theta_{H_t, I}^\perp \eta_{\psi_t}] \\ &= \mathbf{i}\dot{\xi}_t + \text{Im}_{H_t}[I\zeta_I(\dot{b}_t), \mathbf{i}\xi_t] + \mathbf{i}\theta_{H_t, I}[\zeta_I(\dot{b}_t), \theta_{H_t, I}^\perp(\check{I}\eta_{\psi_t})] \\ &= \dot{v}_t^2 + \text{Im}_{H_t}[\dot{H}_t, \mathbf{i}\xi_t] - [\theta_{H_t, I}^\perp(\check{I}\eta_{\dot{\phi}_t}), \mathbf{i}\xi_t] - [\theta_{H_t, I}^\perp(\check{I}\eta_{\psi_t}), \dot{H}_t] \\ &\quad - \mathbf{i}\theta_{H_t, I}[\theta_{H_t, I}^\perp \eta_{\dot{\phi}_t}, \theta_{H_t, I}^\perp(\check{I}\eta_{\psi_t})] \\ &= \mathbf{i}\dot{\xi}_t - \check{I}\eta_{\dot{\phi}_t} \lrcorner d_{H_t}(\mathbf{i}\xi_t) - \check{I}\eta_{\psi_t} \lrcorner d_{H_t} \dot{H}_t + \mathbf{i}F_{H_t}(\eta_{\dot{\phi}_t}, \check{I}\eta_{\psi_t}). \end{aligned}$$

This proves (1). Note that F_H and ω are of type $(1, 1)$, so the torsion is $T_\nabla = 0$ (cf. Theorem 3.6) and the geodesic equation is (3.68). This proves (2). \square

Remark 3.18. When G^c is the trivial group, so $E^c = X$, Theorem 3.6 and Proposition 3.17 reduce to the corresponding results for the space of Kähler metrics $\mathcal{K}_{\check{I}}$ already studied by Mabuchi [41, 42] and Donaldson [18]. More precisely, we recover the Levi–Civita connection of the Mabuchi metric on the Riemannian symmetric space $\mathcal{K}_{\check{I}}$, the functional $\mathcal{M}_I(\cdot, b)$ is

the Mabuchi K-energy [41, 42] on the space of Kähler metrics, by formula (3.67), and (3.68) reduces to the geodesic equation on the space of Kähler metrics [42]

$$\ddot{\phi}_t - (d\dot{\phi}_t, d\dot{\phi}_t)_{\omega_t} = 0. \quad (3.71)$$

It seems plausible that the methods used by Chen & Tian [10, 13] in their study of (3.71) could be adapted to equation (3.68) and to the existence and uniqueness problem for the coupled equations. As in the case of (3.71), this would require a reformulation of (3.68) as a complex Monge–Ampère equation.

Note that the explicit formula for the Christoffel symbols in Proposition 3.17 provides a direct proof of the vanishing of the torsion T_{∇} (cf. Theorem 3.6). Observe also that the two factors of $B_I = \mathcal{K}_{\tilde{J}} \times \mathcal{R}$ are Riemannian symmetric spaces with holonomy groups contained in \mathcal{H}_{ω} (see [42] and [18, §4]) and \mathcal{G}_H , and that the holonomy group of B_I is contained in their group extension $\tilde{\mathcal{G}}_b$ (see (1.16)). Here, the structure of Riemannian symmetric space on \mathcal{R} depends on the choice of an element $\omega \in \mathcal{K}_{\tilde{J}}$. However, Proposition 3.17 implies that the symmetric space structure of B_I is not the product structure. In fact, it is an open question whether B_I carries a Riemannian metric compatible with ∇ (see Remark 4.4 for details).

4. EXTREMAL PAIRS AND DEFORMATION OF SOLUTIONS

Following the approach of LeBrun & Simanca [38, 39], in this section we define an extremality condition for pairs (ω, H) and also an extremality condition in the weak coupling limit (see (4.1) and (4.52)). We establish existence results for extremal pairs (ω, H) near solutions to the coupled equations under deformations of the coupling constants and the Kähler class (Theorems 4.10 and 4.17) and find sufficient conditions for the existence of solutions to the coupled equations (Theorems 4.11 and 4.18).

In §4 we fix a complex reductive Lie group G^c , an n -dimensional compact complex manifold (X, J) , with underlying real manifold X and complex structure J , and a holomorphic principal G^c -bundle (E^c, I) over (X, J) with underlying real principal G^c -bundle E^c and complex structure I (so $\tilde{I} = J$ in the notation of §3.1). We also fix a maximal compact Lie subgroup $G \subset G^c$. The Lie algebras of $G \subset G^c$ are denoted $\mathfrak{g} \subset \mathfrak{g}^c$, respectively. As in Theorem 3.2, we fix a G^c -invariant symmetric bilinear form $(\cdot, \cdot): \mathfrak{g}^c \otimes \mathfrak{g}^c \rightarrow \mathbb{C}$ which restricts to a G -invariant positive definite inner product on \mathfrak{g} . Finally, $\mathfrak{z} = \mathfrak{g}^G$ and $\mathfrak{z}^c = (\mathfrak{g}^c)^{G^c}$ denote the subsets of elements of \mathfrak{g} and \mathfrak{g}^c which are invariant under the adjoint actions of G and G^c , respectively (cf. (1.5)).

4.1. Extremal pairs. We start studying an extremality condition which will be useful to prove Theorem 4.11. Throughout §4.1, we fix $\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}^2$ such that $\alpha_1 \neq 0$, and a Kähler class Ω on (X, J) . Note that we will not assume $\alpha_0, \alpha_1 > 0$, but that we can still apply Proposition 3.8 and Theorem 3.9 (see Remark 3.12). We define

$$B_{\Omega} := \mathcal{K}_{\Omega} \times \mathcal{R},$$

where \mathcal{K}_{Ω} is the space of Kähler forms in Ω compatible with the complex structure J and $\mathcal{R} = \Omega^0(E^c/G)$ (cf. (3.4)).

The following definition is closely related to the vanishing condition for the linearisation at a solution of the coupled equations (see Proposition 4.7 and Lemma 4.8).

Definition 4.1. A pair $b = (\omega, H) \in B_\Omega$ is *extremal* if it satisfies the equations

$$\left. \begin{aligned} 4\alpha_1 d_H \Lambda_\omega F_H + \eta_\alpha(b) \lrcorner F_H &= 0 \\ L_{\eta_\alpha(b)} J &= 0 \end{aligned} \right\}, \quad (4.1)$$

where $\eta_\alpha(b)$ is the Hamiltonian vector field on (X, ω) of the function

$$S_\alpha(b) := -\alpha_0 S_\omega - \alpha_1 \Lambda_\omega^2 (F_H \wedge F_H) + 4\alpha_1 \Lambda_\omega F_H \wedge z_\Omega \in C^\infty(X), \quad (4.2)$$

(cf. (3.18)). Here S_ω is the scalar curvature of the metric $\omega(\cdot, J\cdot)$ and z_Ω is the element of $\mathfrak{z} \subset \mathfrak{g}$ given by (1.8).

Extremal Kähler metrics in Ω , introduced by Calabi in [8, §1], can be characterized as those $\omega \in \mathcal{K}_\Omega$ such that the Hamiltonian vector field of S_ω over (X, ω) is in $\text{Lie } \mathcal{H}_J$ [8, §2]. In particular, all cscK metrics are extremal. Similarly, extremal pairs admit a description in terms of real-holomorphic vector fields on the total space of E^c . To see this, recall that each $H \in \mathcal{R}$ induces a reduction of E^c to a principal G -bundle $E_H \subset E^c$ (see §3.1) and each $b = (\omega, H) \in B_\Omega$ determines a short exact sequence of Lie groups (see §1.2)

$$1 \rightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \xrightarrow{p} \mathcal{H} \rightarrow 1, \quad (4.3)$$

given from left to right by the gauge group of E_H , the extended gauge group of E_H over (X, ω) and the group of Hamiltonian symplectomorphisms of (X, ω) . This exact sequence induces another one

$$1 \rightarrow \mathcal{G}_I \longrightarrow \tilde{\mathcal{G}}_I \xrightarrow{p} \mathcal{H}_J, \quad (4.4)$$

where $\mathcal{G}_I = \mathcal{G} \cap \text{Aut}(E^c, I)$, $\tilde{\mathcal{G}}_I = \tilde{\mathcal{G}} \cap \text{Aut}(E^c, I)$ and $\mathcal{H}_J = \mathcal{H} \cap \text{Aut}(X, J)$ are finite dimensional complex Lie groups (see e.g. [6, §2.120]). Note that the Lie algebra $\text{Lie } \tilde{\mathcal{G}}_I$ is given by G^c -invariant real-holomorphic vector fields on the total space of (E^c, I) covering Hamiltonian (real-holomorphic) vector fields on (X, J, ω) .

Using the horizontal lift $\theta_H^\perp: \text{Lie } \mathcal{H} \rightarrow \text{Lie } \tilde{\mathcal{G}}_b$, of the Chern connection associated to H and I (cf. (3.6)), we define

$$\zeta_\alpha(b) := -4\alpha_1 (\Lambda_\omega F_H - z_\Omega) - \theta_H^\perp \eta_\alpha(b) \in \text{Lie } \tilde{\mathcal{G}}_b, \quad (4.5)$$

for each $b = (H, \omega) \in B_\Omega$. Then it follows from (1.20) and (3.10) that

$$b \in B_\Omega \text{ is extremal} \iff \zeta_\alpha(b) \in \text{Lie } \tilde{\mathcal{G}}_I. \quad (4.6)$$

The following link between extremal pairs and the coupled equations is a generalization of the corresponding link for Kähler metrics (see e.g. [39, Lemma 1]). To establish this, note that each $\omega \in \mathcal{K}_\Omega$ induces L^2 -inner products on $C^\infty(X)$ and $\Omega^0(\text{ad } E_H)$, given by

$$\langle \phi_0, \phi_1 \rangle_\omega := \int_X \phi_0 \phi_1 \omega^{[n]}, \quad (4.7a)$$

$$\langle \xi_0, \xi_1 \rangle_\omega := \int_X (\xi_0 \wedge \xi_1) \omega^{[n]}, \quad (4.7b)$$

for $\phi_j \in C^\infty(X)$, $\xi_j \in \Omega^0(\text{ad } E_H)$ ($j = 0, 1$). Their associated L^2 -norms are denoted $\|\cdot\|_\omega$.

Proposition 4.2. *A pair $b \in B_\Omega$ is a solution to the coupled equations (3.59) if and only if it is an extremal pair and $\mathcal{F}_{\alpha, \Omega} = 0$.*

Proof. If there exists a solution $b = (\omega, H) \in B_\Omega$ to the coupled equations (3.59), then $\mathcal{F}_{\alpha, \Omega} = 0$, by (3.45) (or (3.66)) and Theorem 3.9, and furthermore, b is obviously an extremal pair, since $d_H \Lambda_\omega F_H = 0$ and $\eta_\alpha(b) = 0$. Conversely, if $b = (\omega, H)$ is extremal, i.e. $\zeta_\alpha(b) \in \text{Lie } \tilde{\mathcal{G}}_I$, then

$$\mathcal{F}_{\alpha, \Omega}(\zeta_\alpha(b)) = \|S_\alpha(b) - \hat{S}_\alpha\|_\omega^2 + 16\alpha_1^2 \|\Lambda_\omega F_H - z_\Omega\|_\omega^2 \geq 0,$$

by (3.66), where $\hat{S}_\alpha = \int_X S_\alpha(b) \omega^{[n]} / \text{Vol}_\Omega$, so $\mathcal{F}_{\alpha, \Omega} = 0$ implies that b satisfies (3.59). \square

Extremal pairs enjoy good regularity properties, similar to those of extremal Kähler metrics [38, Proposition 4]:

Lemma 4.3. *Let (ω, H) be an extremal pair such that ω is a Kähler form of class C^2 on (X, J) and H is a section of E^c/G of class C^4 . Then both ω and H are smooth.*

Proof. We will show by induction on $l \in \mathbb{N}$ that ω and H are Hölder of class $C^{2l-1, \beta}$ and $C^{2l+1, \beta}$ respectively, for all $\beta \in (0, 1)$ and $l \in \mathbb{N}$. By assumption, ω and H are of class $C^{1, \beta}$ and $C^{3, \beta}$, respectively. Suppose now that ω and H are of class $C^{2l-1, \beta}$ and $C^{2l+1, \beta}$, respectively. As $\eta_\alpha(b)$ is a real holomorphic vector field by (4.6), it is real analytic, so $dS_\alpha(b)$ is of class $C^{2l-1, \beta}$, i.e. $S_\alpha(b) \in C^{2l, \beta}$, and hence it follows from (4.2) that the scalar curvature S_ω is of class $C^{2l-1, \beta}$, because

$$\Lambda_\omega^2(F_H \wedge F_H) - 4\Lambda_\omega F_H \wedge z_\Omega$$

is of class $C^{2l-1, \beta}$. Arguing as in [38, Proposition 4], it follows from the regularity theory for the Laplacian and for the Monge–Ampère equation that ω is of class $C^{2l+1, \beta}$ (recall that the scalar curvature can be written in holomorphic coordinates as $\Delta_\omega \log \det(\omega)$). Since $\zeta_\alpha(b)$, defined by (4.5), is a real-holomorphic vector field on (E^c, I) by assumption (see (4.6)), it is real analytic and so $\Lambda_\omega F_H$ is of class $C^{2l, \beta}$, because $\alpha_1 \neq 0$. Identifying H locally with a function on the base with values in $\exp(\mathfrak{ig}) \subset G^c$ and using holomorphic coordinates for the bundle E^c , we can write

$$\Delta_{\bar{\partial}} H = H(\Lambda_\omega F_H - \Lambda_\omega(\bar{\partial}(H^{-1}) \wedge \partial H)), \quad (4.8)$$

where the right-hand side is of class $C^{2l, \beta}$ and $\Delta_{\bar{\partial}}$ is elliptic with $C^{2l+1, \beta}$ coefficients. By the regularity theory of linear elliptic differential operators, H is of class $C^{2l+2, \beta}$ (see e.g. [5, Theorem 3.55]). Applying this argument again to (4.5), we see that the right-hand side of (4.8) is of class $C^{2l+1, \beta}$ and so H is of class $C^{2l+3, \beta}$, as required. \square

Remark 4.4. Note that $B_\Omega = \mathcal{K}_\Omega \times \mathcal{R}$ has a Riemannian metric g_Ω , given by

$$g_\Omega(v_0, v_1) = \int_X \phi_0 \phi_1 \omega^{[n]} + \int_X (\xi_0 \wedge \xi_1) \omega^{[n]} \quad (4.9)$$

for $b = (\omega, H)$, $v_j = (\phi_j, \xi_j) \in T_b B_\Omega \cong C_0^\infty(X) \times \Omega^0(\text{ad } E_H)$ (cf. (4.10)), with $\dot{H} = \mathfrak{ig}$ in the notation of §3.4. Although this metric is rather canonical, it does not endow the symmetric space (B_Ω, ∇) of Theorem 3.6 with a structure of Riemannian symmetric space, since g_Ω is not preserved in general by the canonical affine connection ∇ on B_Ω constructed in §3.2. In fact, by a straightforward calculation using formula (3.70) for the parallel transport,

$$\begin{aligned} (\nabla_{v_0} g_\Omega)(v_1, v_2) &= - \int_X \left(\dot{\xi}_1 \wedge (\theta_H[\zeta_I(v_0), \theta_H^\perp J\eta_{\phi_2}]) \right. \\ &\quad \left. + (\theta_H[\zeta_I(v_0), \theta_H^\perp J\eta_{\phi_1}]) \wedge \dot{\xi}_2 \right) \omega^{[n]}. \end{aligned}$$

However, if the group G^c is trivial, so $B_\Omega = \mathcal{K}_\Omega$, then g_Ω is precisely the Mabuchi metric and $\nabla g_\Omega = 0$, by the previous formula, so we recover the known fact [18, 42] that \mathcal{K}_Ω is a Riemannian symmetric space with Levi–Civita connection ∇ , by Theorem 3.6.

4.2. Holomorphic vector fields on the principal bundle. Given $b = (\omega, H) \in B_\Omega$, we now relate the Lie algebra $\text{Lie } \tilde{\mathcal{G}}_I$ (see (4.4)) to the space of solutions to a fourth-order elliptic differential equation which is closely related to the linearization of our coupled equations. We will use the inner product on $C^\infty(X) \times \Omega^0(\text{ad } E_H)$ induced by (4.7), i.e. given by

$$\langle v_0, v_1 \rangle_\omega := \langle \phi_0, \phi_1 \rangle_\omega + \langle \xi_0, \xi_1 \rangle_\omega, \quad (4.10)$$

for $v_j = (\phi_j, \xi_j) \in C^\infty(X) \times \Omega^0(\text{ad } E_H)$ ($j = 0, 1$).

We define an operator

$$\begin{aligned} P = P_\omega: C^\infty(X) &\longrightarrow \Omega^0(\text{End } TX) \\ \phi &\longmapsto -L_{\eta_\phi} J. \end{aligned} \quad (4.11)$$

In other words, P is induced by the infinitesimal action of $\text{Lie } \mathcal{H}$ on \mathcal{J}_ω . Let P^* be the formal adjoint of P with respect to the L^2 -inner products on $C^\infty(X)$ and $\Omega^0(\text{End } TX)$ induced by $\omega(\cdot, J\cdot)$, with the L^2 -inner product on $\Omega^0(\text{End } TX)$ multiplied by a factor of $1/2$, so that its restriction to $T_J \mathcal{J}$ coincides with $\omega_{\mathcal{J}}(\cdot, \mathbf{J}\cdot)$ (defined by (2.2)). Then $P^* P$ is, up to a multiplicative constant factor, the Lichnerowicz operator of the compact Kähler manifold (X, J, ω) . This is an elliptic self-adjoint semipositive differential operator of order 4, whose kernel is the set of functions ϕ such that $\eta_\phi \in \text{Lie } \mathcal{H}_J$, and which may be interpreted as the linearization of the cscK equation at ω (see e.g. [38]).

We define now an operator which is closely related to the linearization of the coupled equations (see Proposition 4.7) and which will play the role of the Lichnerowicz operator in our study. The operator is

$$\mathbf{L}_{\alpha,b} = (\mathbf{L}_{\alpha,b}^0, \mathbf{L}_{\alpha,b}^1): C^\infty(X) \times \Omega^0(\text{ad } E_H) \longrightarrow C^\infty(X) \times \Omega^0(\text{ad } E_H), \quad (4.12)$$

where $\mathbf{L}_{\alpha,b}^0$ and $\mathbf{L}_{\alpha,b}^1$ are defined by

$$\begin{aligned} \mathbf{L}_{\alpha,b}^0(\phi, \xi) &= \alpha_0 P^* P \phi - 2\alpha_1 \Lambda_\omega^2(F_H \wedge d_H J(d_H \xi + \eta_\phi \lrcorner F_H)) + \mathbf{L}_{\alpha,b}^1(\phi, \xi) \wedge z_\Omega, \\ \mathbf{L}_{\alpha,b}^1(\phi, \xi) &= 4\alpha_1 \Lambda_\omega d_H J(d_H \xi + \eta_\phi \lrcorner F_H) = 4\alpha_1 d_H^*(d_H \xi + \eta_\phi \lrcorner F_H). \end{aligned} \quad (4.13)$$

Here, J is the endomorphism of $\Omega^1(\text{ad } E_H)$ induced by the complex structure J (see [6, (2.8)]), $d_H: \Omega^0(\text{ad } E_H) \rightarrow \Omega^1(\text{ad } E_H)$ is the covariant derivative of the Chern connection of H and I and, by the Kähler identities, $d_H^* = \Lambda_\omega d_H J$ is its formal adjoint.

Recall that the Chern connection associated to any $H \in \mathcal{R}$ and I induces a vector space isomorphism (see §1.3)

$$C^\infty(X)/\mathbb{R} \times \Omega^0(\text{ad } E_H) \xrightarrow{\cong} \text{Lie } \tilde{\mathcal{G}}_b: ([\phi], \xi) \longmapsto \xi + \theta_H^\perp \eta_\phi. \quad (4.14)$$

Let $(Z_b, \mathbf{I}_b, \omega_b)$ be the Kähler manifold constructed in Lemma 3.1 and Theorem 3.2.

Lemma 4.5. *Let $v_j = (\phi_j, \xi_j) \in C^\infty(X) \times \Omega^0(\text{ad } E_H)$, for $j = 0, 1$. Then*

$$\langle v_0, \mathbf{L}_{\alpha,b} v_1 \rangle_\omega = \omega_b(Y_{\zeta_0|I}, \mathbf{I}Y_{\zeta_1|I}) + 4\alpha_1 \langle J\eta_{\phi_0} \lrcorner (d_H \xi_1 + \eta_{\phi_1} \lrcorner F_H), \Lambda_\omega F_H - z_\Omega \rangle_\omega, \quad (4.15)$$

where Y_{ζ_j} is the infinitesimal action of $\zeta_j = \xi_j + \theta_H^\perp \eta_{\phi_j} \in \text{Lie } \tilde{\mathcal{G}}_b$ on Z_b , for $j = 0, 1$.

Proof. By the moment map interpretation of scalar curvature of the Kähler metric $\omega(\cdot, J\cdot)$, its derivative $\delta_J S_\omega: T_J \mathcal{J}_\omega \rightarrow C^\infty(X)$ with respect to $J \in \mathcal{J}_\omega$ satisfies

$$\delta_J S_\omega \circ \mathbf{J} \circ \mathbf{P} = -\mathbf{P}^* \mathbf{P} \quad (4.16)$$

(see §2.1 and [19, equation (26)]). By (3.21) and Lemma 3.1, $\mathbf{I}Y_{\zeta_0|I}$ is identified with the infinitesimal action of $I\zeta_0$ on $(J, A) \in \mathcal{P}_b$, where A is the Chern connection of H and I , so

$$\mathbf{I}Y_{\zeta_1|I} = -L_{I\zeta_1} I = (-JL_{\eta_{\phi_1}} J, J(d_H \xi_1 + \eta_{\phi_1} \lrcorner F_H)), \quad (4.17)$$

by Lemma 1.5. Hence (4.15) follows from formula (2.6) for the moment map μ_b . \square

Given a pair $b = (\omega, H) \in B_\Omega$, an element $\zeta \in \text{Lie } \tilde{\mathcal{G}}_b$ is in $\text{Lie } \tilde{\mathcal{G}}_I$ if and only if $Y_{\zeta|I} = 0$. Using (4.17), we see that $\text{Lie } \tilde{\mathcal{G}}_I \subset \text{Lie } \tilde{\mathcal{G}}_b$ is the subset of elements $\zeta = \xi + \theta_H^\perp \eta_\phi$ such that

$$\mathbf{P} \phi = 0, \quad d_H \xi + \eta_\phi \lrcorner F_H = 0. \quad (4.18)$$

Hence if $(\phi, \xi) \in C^\infty(X) \times \Omega^0(\text{ad } E_H)$ satisfies $\xi + \theta_H^\perp \eta_\phi \in \text{Lie } \tilde{\mathcal{G}}_I$, then it is in $\ker \mathbf{L}_{\alpha,b}$ (see (4.13)). We provide now sufficient conditions to obtain the converse implication.

Given a pair $b = (\omega, H) \in B_\Omega$, H is a *Hermitian–Yang–Mills* reduction (HYM) on (E^c, I) with respect to ω if it satisfies

$$\Lambda_\omega F_H = z_\Omega. \quad (4.19)$$

Note that if H is HYM on (E^c, I) with respect to ω , then (4.15) becomes simply

$$\langle v_0, \mathbf{L}_{\alpha,b} v_1 \rangle_\omega = \omega_b(Y_{\zeta_0|I}, \mathbf{I}Y_{\zeta_1|I}). \quad (4.20)$$

Proposition 4.6. *The operator $\mathbf{L}_{\alpha,b}$ is elliptic. If H is HYM with respect to ω , then $\mathbf{L}_{\alpha,b}$ is also self-adjoint. If furthermore $\alpha_0 \alpha_1 > 0$, then*

$$\ker \mathbf{L}_{\alpha,b} = \{(\phi, \xi) \in C^\infty(X) \times \Omega^0(\text{ad } E_H) \mid \xi + \theta_H^\perp \eta_\phi \in \text{Lie } \tilde{\mathcal{G}}_I\}. \quad (4.21)$$

Proof. The operator $\mathbf{L}_{\alpha,b}$ is elliptic because so are $\mathbf{P}^* \mathbf{P}$ and $d_H^* d_H$. If H is HYM, then we can apply (4.20), where $\omega_b(\cdot, \mathbf{I}\cdot)$ is symmetric, so $\mathbf{L}_{\alpha,b}$ is self-adjoint. We have already seen that the right-hand side of (4.21) is contained in $\ker \mathbf{L}_{\alpha,b}$. If in addition $\alpha_0 \alpha_1 > 0$, then ω_b is compatible with either \mathbf{I} or $-\mathbf{I}$ (see §2.2), so if $v = (\phi, \xi)$ satisfies $\zeta := \xi + \theta_H^\perp \eta_\phi \notin \text{Lie } \tilde{\mathcal{G}}_I$, then $\langle v, \mathbf{L}_{\alpha,b} v \rangle_\omega = \omega_b(Y_{\zeta|I}, \mathbf{I}Y_{\zeta|I}) \neq 0$ by (4.20), and hence $v \notin \ker \mathbf{L}_{\alpha,b}$. This implies (4.21). \square

Observe that, although $\mathbf{L}_{\alpha,b}$ is an analogue in our context of the Lichnerowicz operator, there is an important difference between these two operators, since by Proposition 4.6, we can ensure that $\mathbf{L}_{\alpha,b}$ is self-adjoint and its kernel corresponds to $\text{Lie } \tilde{\mathcal{G}}_I$ via (4.14) only when $b = (\omega, H)$ satisfies the Hermitian–Yang–Mills equation (4.19) and $\alpha_0 \alpha_1 > 0$.

4.3. The linearized coupled equations. Throughout §4.3, we fix a coupling constant $\alpha \in \mathbb{R}^2$, a holomorphic structure I on E^c over (X, J) , a Kähler class Ω on (X, J) and $b = (\omega, H) \in B_\Omega$. Let $H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$ be the vector subspace of those de Rham classes which are representable by real closed $(1, 1)$ -forms on (X, J) . Recall that $H^{1,1}(X, \mathbb{R})$ is identified by Hodge theory with the space $\mathcal{H}^{1,1}(X)$ of real harmonic $(1, 1)$ -forms on (X, J, ω) .

In §4.3, we will compute the first-order deformations of the moment map μ_b constructed in Theorem 3.2 under deformations given by a new Kähler form $\tilde{\omega}$ and a new holomorphic structure \tilde{I} on the principal bundle E^c over (X, J) , given by

$$\tilde{\omega} := \omega + \gamma + dd^c \phi, \quad (4.22a)$$

$$\tilde{I} := e^{i\xi} \cdot I, \quad (4.22b)$$

parametrised by a triple

$$(\gamma, \phi, \xi) \in \mathcal{H}^{1,1}(X) \times C^\infty(X) \times \Omega^0(\text{ad } E_H).$$

We will also consider the deformed pair

$$\tilde{b} = (\tilde{\omega}, \tilde{H}) \in B_{\tilde{\Omega}}, \text{ with } \tilde{H} := e^{-i\xi} \cdot H \in \mathcal{R}, \quad (4.23)$$

where $\tilde{\Omega}$ is the cohomology class of $\tilde{\omega}$. Note that (3.17) implies

$$F_{H, \tilde{I}} = e^{i\xi} \cdot F_{\tilde{H}, I}, \quad (4.24)$$

where $F_{H, I}$ is the curvature of the Chern connection $\theta_H = \theta_{H, I}$ associated to H and I .

In fact, to prove Theorems 4.11 and 4.18, we will need to apply the implicit function theorem, so we will work in Sobolev spaces. Let $L_k^2(X)$ and $L_k^2(\text{ad } E_H)$ be the Sobolev spaces of real-valued functions on X and sections of the bundle $\text{ad } E_H$, respectively, whose distributional derivatives up to order k are square integrable. These are real Hilbert spaces that, by the Sobolev embedding theorem, have natural bounded inclusion maps $L_k^2(X) \subset C^l(X)$ and $L_k^2(\text{ad } E_H) \subset C^l(\text{ad } E_H)$ into the Banach spaces of l -times continuously differentiable functions and sections of $\text{ad } E_H$, respectively, provided $k > n + l$. Moreover, if $k > n$, then $L_k^2(X)$ is a Banach algebra. Fix $k > n$. Let

$$\mathcal{U} = \hat{\mathcal{U}} \times L_{k+4}^2(\text{ad } E_H) \subset \mathcal{H}^{1,1}(X) \times L_{k+4}^2(X) \times L_{k+4}^2(\text{ad } E_H), \quad (4.25)$$

where $\hat{\mathcal{U}} \subset \mathcal{H}^{1,1}(X) \times L_{k+4}^2(X)$ is the open neighbourhood of $(0, 0)$ consisting of pairs (γ, ϕ) such that $\tilde{\omega}(\cdot, J \cdot)$ is a Kähler metric of class C^2 , with $\tilde{\omega}$ defined by (4.22a). Define the *moment map operator*

$$\begin{aligned} T_\alpha &= (T_\alpha^0, T_\alpha^1): \mathcal{U} \longrightarrow L_k^2(X) \times L_{k+2}^2(\text{ad } E_H) \\ &(\gamma, \phi, \xi) \longmapsto \left(S_\alpha(\tilde{b}, I), 4\alpha_1(\Lambda_{\tilde{\omega}} F_{H, \tilde{I}} - z_{\tilde{\Omega}}) \right), \end{aligned} \quad (4.26)$$

where $\tilde{\omega}, \tilde{I}$ and \tilde{b} are defined by (4.22) and (4.23), while $S_\alpha(\tilde{b}, I)$ and $z_{\tilde{\Omega}}$ are given by the formulae (4.2) and (4.19), using the Kähler class

$$\tilde{\Omega} := [\tilde{\omega}] \in H^{1,1}(X, \mathbb{R}).$$

Observe that T_α is a variant for Sobolev spaces of the families of moment maps μ_b .

The following proposition can be compared with [39, Proposition 5].

Proposition 4.7. *For $k > n$, T_α is a well-defined C^1 map whose Fréchet derivative δT_α at the origin $(0, 0, 0)$ is given by*

$$\begin{aligned} \delta T_\alpha(\dot{\gamma}, \dot{\phi}, \dot{\xi}) &= \mathbf{L}_{\alpha, b}(\dot{\phi}, \dot{\xi}) + \left((d(S_\alpha(b, I)), d\dot{\phi})_\omega, 4\alpha_1 J \eta_{\dot{\phi}} \lrcorner d_H \Lambda_\omega F_H \right) \\ &+ \delta_{\dot{\gamma}} T_\alpha, \end{aligned} \quad (4.27)$$

for all $(\dot{\gamma}, \dot{\phi}, \dot{\xi}) \in \mathcal{H}^{1,1}(X) \times L_{k+4}^2(X) \times L_k^2(\text{ad } E_H)$, where

$$\mathbf{L}_{\alpha, b}: L_{k+4}^2(X) \times L_{k+4}^2(\text{ad } E_H) \longrightarrow L_k^2(X) \times L_{k+2}^2(\text{ad } E_H)$$

is given by (4.13), $(\cdot, \cdot)_\omega$ is the inner product on T^*X induced by $\omega(\cdot, J\cdot)$, $\eta_{\dot{\phi}}$ is the Hamiltonian vector field of $\dot{\phi}$ on (X, ω) and $\delta_{\dot{\gamma}}T_\alpha$ is the directional derivative of T_α at the origin in the direction $(\dot{\gamma}, 0, 0)$.

Proof. The operator T_α is well-defined because $L_k^2(X)$ is a Banach algebra for $k > n$, T_α^0 is a non-linear differential operator of order 4 in ϕ and order 2 in γ and ξ , while T_α^1 is a non-linear differential operator of order 2 in ϕ and ξ and order 0 in γ .

To prove that T_α is C^1 , we will calculate its directional derivatives $\delta_{(\dot{\phi}, \dot{\xi})}T_\alpha(\gamma, \phi, \xi)$ and $\delta_{\dot{\gamma}}T_\alpha(\gamma, \phi, \xi)$ at (γ, ϕ, ξ) in the directions $(0, \dot{\phi}, \dot{\xi})$ and $(\dot{\gamma}, 0, 0)$, respectively, for $(\gamma, \phi, \xi) \in \mathcal{U}$, $(\dot{\gamma}, \dot{\phi}, \dot{\xi}) \in \mathcal{H}^{1,1}(X) \times L_{k+4}^2(X) \times L_k^2(\text{ad } E_H)$.

To compute $\delta_{(\dot{\phi}, \dot{\xi})}T_\alpha(\gamma, \phi, \xi)$, we define a curve (on an appropriate Sobolev completion of B_Ω and for $|t|$ small), given by

$$b_t = (\tilde{\omega}_t, \tilde{H}_t) := (\tilde{\omega} + tdd^c\dot{\phi}, e^{-i(\xi+t\dot{\xi})} \cdot H).$$

Let η_t be the Hamiltonian vector field of $\dot{\phi}$ over $(X, \tilde{\omega}_t)$ and g_t the flow of

$$y_t := I\zeta_I(\dot{b}_t) = -I(\dot{\xi} + \theta_{H_t}^\perp \eta_t), \quad (4.28)$$

i.e. the curve of G^c -equivariant automorphisms of E^c satisfying $\dot{g}_t \cdot g_t^{-1} = y_t$, with initial condition $g_0 = \text{Id}$. Since the Kähler class $\tilde{\Omega}$ of $\tilde{\omega}_t$ is constant along the curve b_t , we can apply the constructions in the proof of Proposition 3.5(1), so the flow g_t exists and satisfies

$$b_t = g_t \cdot \tilde{b} \quad (4.29)$$

(as $b_0 = \tilde{b}$). Note that the identity (4.29) holds in a strong sense, as $k > n$, so the Kähler metrics $\tilde{\omega}_t$ are of class C^2 and the G -reductions \tilde{H}_t are of class C^4 . Define another curve

$$I_t := g_t^{-1} \cdot I$$

in (an appropriate Sobolev completion of) the space $Z_{\tilde{b}}$ of holomorphic structures on the principal G^c -bundle E^c which are compatible with \tilde{b} (see §3.1). Using the dependence of $S_\alpha(b_t, I)$ on the holomorphic structure I on E^c , we obtain

$$T_t^0 := T_\alpha^0(\gamma, \phi + t\dot{\phi}, \xi + t\dot{\xi}) = S_\alpha(b_t, I) = S_\alpha(\tilde{b}, I_t) \circ \check{g}_t^{-1}$$

by (3.19) and (4.29). Since $\frac{d}{dt}|_{t=0} I_t = L_{y_0} I$, this implies

$$\begin{aligned} \delta_{(\dot{\phi}, \dot{\xi})} T_\alpha^0(\gamma, \phi, \xi) &= \frac{d}{dt}|_{t=0} T_t^0 = (\delta_I S_\alpha)|_{(\tilde{b}, I)}(L_{y_0} I) + J\eta_{\dot{\phi} \lrcorner} d(S_\alpha(\tilde{b}, I)) \\ &= (\delta_I S_\alpha)|_{(\tilde{b}, I)}(L_{y_0} I) + (d(S_\alpha(\tilde{b}, I)), d\dot{\phi})_{\tilde{\omega}}, \end{aligned} \quad (4.30)$$

where $\delta_I S_\alpha: T_I Z_{\tilde{b}} \rightarrow C^\infty(X)$ is the derivative of S_α with respect to I . Now, by (4.17)

$$L_{y_0} I = (-JL_{\eta_{\dot{\phi}}} J, J(d_{\tilde{H}} \dot{\xi} + \eta_{\dot{\phi} \lrcorner} F_{\tilde{H}}))$$

and from this formula, (4.16) and (4.13), we obtain

$$\begin{aligned} (\delta_I S_\alpha)|_{(\tilde{b}, I)}(\mathbf{I}Y_{\zeta_b}) &= \alpha_0 \mathbf{P}^* \mathbf{P} \dot{\phi} - 2\alpha_1 \Lambda_{\tilde{\omega}}^2(F_{\tilde{H}} \wedge d_{\tilde{H}} J(d_{\tilde{H}} \dot{\xi} + \eta_{\dot{\phi} \lrcorner} F_{\tilde{H}})) + \mathbf{L}_{\alpha, \tilde{b}}^1(\dot{\phi}, \dot{\xi}) \wedge z_{\tilde{\Omega}} \\ &= \mathbf{L}_{\alpha, \tilde{b}}^0(\dot{\phi}, \dot{\xi}), \end{aligned}$$

where $z_{\tilde{\Omega}}$ is defined as in (4.2) using the Kähler class $\tilde{\Omega}$, so the right hand side of (4.30) is

$$\delta_{(\gamma, \phi, \xi)} T_\alpha^0(0, \dot{\phi}, \dot{\xi}) = \mathbf{L}_{\alpha, \tilde{b}}^0(\dot{\phi}, \dot{\xi}) + (d(S_\alpha(\tilde{b}, I)), d\dot{\phi})_{\tilde{\omega}}. \quad (4.31)$$

By (3.17), we also have

$$\begin{aligned} \mathbb{T}_t^1 &:= \mathbb{T}_\alpha^1(\gamma, \phi + \dot{\phi}_t, \xi + t\dot{\xi}) \\ &= 4\alpha_1 \left(e^{i(\xi+t\dot{\xi})} g_t \right) \cdot (\Lambda_{\tilde{\omega}_t} F_{H, I_t} - z_{\tilde{\Omega}}), \end{aligned}$$

and a straightforward calculation shows that

$$\begin{aligned} \delta_{(\dot{\phi}, \dot{\xi})} \mathbb{T}_\alpha^1(\gamma, \phi, \xi) &= \frac{d}{dt} \Big|_{t=0} \mathbb{T}_t^1 = 4\alpha_1 \Lambda_{\tilde{\omega}} d_{\tilde{H}} J(d_{\tilde{H}} \dot{\xi} + \eta_{\dot{\phi}} \lrcorner F_{\tilde{H}}) + 4\alpha_1 J \eta_{\dot{\phi}} \lrcorner d_{\tilde{H}} \Lambda_{\tilde{\omega}} F_{\tilde{H}} \\ &= \mathbf{L}_{\alpha, b}^1(\dot{\phi}, \dot{\xi}) + 4\alpha_1 J \eta_{\dot{\phi}} \lrcorner d_{\tilde{H}} \Lambda_{\tilde{\omega}} F_{\tilde{H}}. \end{aligned} \quad (4.32)$$

To compute $\delta_{\dot{\gamma}} \mathbb{T}_\alpha(\gamma, \phi, \xi)$, for $(\gamma, \phi, \xi) \in \mathcal{U}$ and $\dot{\gamma} \in \mathcal{H}^{1,1}(X)$, we define a curve

$$b_t = (\omega_t, \tilde{H}) = (\tilde{\omega} + t\dot{\gamma}, \tilde{H})$$

(for $t \in \mathbb{R}$ small). Let

$$\begin{aligned} \mathbb{T}_t^0 &:= \mathbb{T}_\alpha^0(\gamma + t\dot{\gamma}, \phi, \xi) = S_\alpha(b_t, I) \\ &= -\alpha_0 S_{\omega_t} - \alpha_1 \Lambda_{\omega_t}^2(F_{\tilde{H}} \wedge F_{\tilde{H}}) + 4\alpha_1 \Lambda_{\omega_t} F_{\tilde{H}} \wedge z_{\Omega_t}, \\ \mathbb{T}_t^1 &:= \mathbb{T}_\alpha^1(\gamma + t\dot{\gamma}, \phi, \xi) = 4\alpha_1 (\Lambda_{\omega_t} F_{H, \tilde{I}} - z_{\Omega_t}), \end{aligned}$$

where $\Omega_t = [\omega_t] \in H^{1,1}(X, \mathbb{R})$. As shown by LeBrun & Simanca (see [39, Proposition 5] and [38, Proposition 6]), the derivative of the first term of \mathbb{T}_t^0 is given by

$$\delta_{\dot{\gamma}} S_{\tilde{\omega}} := \frac{d}{dt} \Big|_{t=0} S_{\omega_t} = \Delta_{\tilde{\omega}}(\tilde{\omega}, \dot{\gamma})_{\tilde{\omega}} - 2(\rho_{\tilde{\omega}}, \dot{\gamma})_{\tilde{\omega}},$$

where $\Delta_{\tilde{\omega}}$ and $\rho_{\tilde{\omega}}$ are the Laplacian and the Ricci curvature of $\tilde{\omega}(\cdot, J\cdot)$, respectively. To calculate the derivatives of \mathbb{T}_t^1 and of the second term of \mathbb{T}_t^0 , we use the equality

$$\frac{d}{dt} \Big|_{t=0} \omega_t^{[n]} = \dot{\gamma} \wedge \tilde{\omega}^{[n-1]} = (\Lambda_{\tilde{\omega}} \dot{\gamma}) \tilde{\omega}^{[n]}$$

and the following computations:

$$\begin{aligned} &\frac{d}{dt} \Big|_{t=0} \left(\Lambda_{\omega_t} F_{H, \tilde{I}} \omega_t^{[n]} \right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(F_{H, \tilde{I}} \wedge \omega_t^{[n-1]} \right) = F_{H, \tilde{I}} \wedge \dot{\gamma} \wedge \tilde{\omega}^{[n-2]}, \\ &\frac{d}{dt} \Big|_{t=0} \left((\Lambda_{\omega_t}^2(F_{\tilde{H}} \wedge F_{\tilde{H}}) - 4\Lambda_{\omega_t} F_{\tilde{H}} \wedge z_{\Omega_t}) \omega_t^{[n]} \right) \\ &= \frac{d}{dt} \Big|_{t=0} \left(2F_{\tilde{H}} \wedge F_{\tilde{H}} \wedge \omega_t^{[n-2]} - 4F_{\tilde{H}} \wedge z_{\Omega_t} \omega_t^{[n-1]} \right) \\ &= 2F_{\tilde{H}} \wedge F_{\tilde{H}} \wedge \dot{\gamma} \wedge \tilde{\omega}^{[n-3]} - 4F_{\tilde{H}} \wedge \left(z_{\tilde{\Omega}} \dot{\gamma} + \frac{\delta_{\dot{\gamma}} z_{\tilde{\Omega}} \tilde{\omega}}{n-1} \right) \wedge \tilde{\omega}^{[n-2]}. \end{aligned}$$

Here, (4.19) implies

$$\delta_{\dot{\gamma}} z_{\tilde{\Omega}} := \frac{d}{dt} \Big|_{t=0} z_{\Omega_t} = \sum_j \beta_j z_j \quad (4.33)$$

for an orthonormal basis $\{z_j\}$ of \mathfrak{z} , with

$$\begin{aligned} \beta_j &:= \frac{d}{dt}|_{t=0} \frac{\langle z_j(E) \cup \Omega_t^{[n-1]}, [X] \rangle}{\text{Vol}_{\Omega_t}} \\ &= \frac{\langle z_j(E) \cup [\dot{\gamma}] \cup \tilde{\Omega}^{[n-2]}, [X] \rangle}{\text{Vol}_{\tilde{\Omega}}} - \frac{\langle z_j(E) \cup \tilde{\Omega}^{[n-1]}, [X] \rangle \langle [\dot{\gamma}] \cup \tilde{\Omega}^{[n-1]}, [X] \rangle}{\text{Vol}_{\tilde{\Omega}}^2}. \end{aligned}$$

From these equalities, we obtain the directional derivatives

$$\delta_{\dot{\gamma}} T_{\alpha}^0(\gamma, \phi, \xi) = \frac{d}{dt}|_{t=0} T_t^0 = \alpha_0 (2(\rho_{\tilde{\omega}}, \dot{\gamma})_{\tilde{\omega}} - \Delta_{\tilde{\omega}}(\tilde{\omega}, \dot{\gamma})_{\tilde{\omega}}) \quad (4.34a)$$

$$\begin{aligned} &- \frac{\alpha_1}{3} \Lambda_{\tilde{\omega}}^3 \left(F_{H, \tilde{I}} \wedge F_{H, \tilde{I}} \wedge \dot{\gamma} \right) + 2\alpha_1 \Lambda_{\tilde{\omega}}^2 \left(F_{H, \tilde{I}} \wedge \left(z_{\tilde{\Omega}} \dot{\gamma} + \frac{\delta_{\dot{\gamma}} z_{\tilde{\Omega}} \tilde{\omega}}{n-1} \right) \right) \\ &+ \alpha_1 \left(\Lambda_{\tilde{\omega}}^2 (F_{H, \tilde{I}} \wedge F_{H, \tilde{I}}) - 4\Lambda_{\tilde{\omega}} F_{\tilde{H}} \wedge z_{\tilde{\Omega}} \right) (\Lambda_{\tilde{\omega}} \dot{\gamma}), \end{aligned}$$

$$\delta_{\dot{\gamma}} T_{\alpha}^1(\gamma, \phi, \xi) = \frac{d}{dt}|_{t=0} T_t^1 = 4\alpha_1 \left((F_{H, \tilde{I}}, \dot{\gamma})_{\tilde{\omega}} - \delta_{\dot{\gamma}} z_{\tilde{\Omega}} \right). \quad (4.34b)$$

It now follows from (4.30), (4.32), (4.34) and the formula (4.33) for $\delta_{\dot{\gamma}} z_{\tilde{\Omega}}$ that the directional derivatives are continuous. Therefore, T_{α} is C^1 and its Fréchet derivative given by (4.27) (by (4.30) and (4.32)). \square

Note that an explicit formula for the directional derivative $\delta_{\dot{\gamma}} T_{\alpha}$ has been calculated in (4.34), although it has not been recorded in (4.27), as it is not needed in this paper.

4.4. Deformation of solutions. As in §4.3, we now fix a holomorphic structure I on E^c over (X, J) , a Kähler class Ω on (X, J) and $b = (\omega, H) \in B_{\Omega}$. Note that \mathcal{H}_J acts trivially on the space $\mathcal{H}^{1,1}(X) \subset \Omega^2(X)$ of real harmonic $(1, 1)$ -forms for the metric $\omega(\cdot, J \cdot)$. Let

$$L_k^2(X)^{\mathcal{H}_J} \subset L_k^2(X) \quad \text{and} \quad L_k^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \subset L_k^2(\text{ad } E_H) \quad (4.35)$$

be the closed subspaces of \mathcal{H}_J -invariant functions and $\tilde{\mathcal{G}}_I$ -invariant sections, respectively. Let $\hat{\mathcal{V}} = \hat{\mathcal{U}} \cap (\mathcal{H}^{1,1}(X) \times L_k^2(X)^{\mathcal{H}_J})$ and

$$\mathcal{V} = \hat{\mathcal{V}} \times L_k^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} = \mathcal{U} \cap \left(\mathcal{H}^{1,1}(X) \times L_{k+4}^2(X)^{\mathcal{H}_J} \times L_{k+4}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \right). \quad (4.36)$$

Given coupling constants $\alpha \in \mathbb{R}^2$, by restriction of the maps of Proposition 4.7, for $k > n$, we obtain well-defined maps

$$\hat{T}_{\alpha}: \mathcal{V} \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}, \quad (4.37a)$$

$$\hat{\mathbf{L}}_{\alpha, b}: L_{k+4}^2(X)^{\mathcal{H}_J} \times L_{k+4}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \quad (4.37b)$$

(cf. [38, (5.1)]), where \hat{T}_{α} is C^1 with Fréchet derivative given by (4.27), and $\hat{\mathbf{L}}_{\alpha, b}$ is a linear elliptic operator.

Let d^* and \mathbf{G} be the formal adjoint of the de Rham differential and the Green operator of the Laplacian for the fixed metric $\omega(\cdot, J \cdot)$, respectively. Then for any symplectic form $\tilde{\omega}$ and any $\tilde{\eta}$ in the Lie algebra $\text{Lie } \mathcal{H}_{\tilde{\omega}}$ of Hamiltonian vector fields over $(X, \tilde{\omega})$ we have

$$d(\mathbf{G}d^*(\tilde{\eta} \lrcorner \tilde{\omega})) = \tilde{\eta} \lrcorner \tilde{\omega}. \quad (4.38)$$

As the image of the Green operator is perpendicular to the constants, the Hamiltonian function $f = \mathbf{G}d^*(\tilde{\eta} \lrcorner \tilde{\omega})$ is ‘normalized’ for the volume form $\omega^{[n]}$, that is, $\int_X f \omega^{[n]} = 0$.

For each $(\gamma, \phi, \xi) \in \mathcal{V}$, we define a linear map

$$\begin{aligned} \mathbf{P}_{(\gamma, \phi, \xi)} &= (\mathbf{P}_{(\gamma, \phi)}^0, \mathbf{P}_{\xi}^1): \mathbb{R} \times \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I) \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \\ (t, v) &\longmapsto \left(\mathbf{G}d^*(p(v) \lrcorner \tilde{\omega}) + t, \theta_{H, \tilde{I}} v \right), \end{aligned} \quad (4.39)$$

where

$$\mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I) := (\text{Lie } \tilde{\mathcal{G}}_I)^{\tilde{\mathcal{G}}_I}$$

is the centre of $\text{Lie } \tilde{\mathcal{G}}_I$ (cf. (1.5)) and $p: \tilde{\mathcal{G}}_I \rightarrow \mathcal{H}_J$ is the map in (4.4), while $\tilde{\omega}$ and \tilde{H} are defined by (4.22a) and (4.23). The map $\mathbf{P}_{(\gamma, \phi, \xi)}$ attaches to a vector field $v \in \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I)$ its vertical part $\theta_{H, \tilde{I}} v$, calculates the normalized Hamiltonian function of the vector field $p(v)$ over $(X, \tilde{\omega})$, and adds an extra parameter t which accounts for the fact that Hamiltonian functions are only determined up to a constant (cf. (4.14), [38, §5], [39, Proposition 2]).

Here is the key link between extremal pairs and the linearization of the coupled equations.

Lemma 4.8. *Let $(\gamma, \phi, \xi) \in \mathcal{V}$.*

- (1) $\mathbf{P}_{(\gamma, \phi, \xi)}$ is injective.
- (2) If $\hat{\mathbf{T}}_{\alpha}(\gamma, \phi, \xi) \in \text{Im } \mathbf{P}_{(\gamma, \phi, \xi)}$, then $\tilde{b} = (\tilde{\omega}, \tilde{H})$ is an extremal pair.
- (3) $\text{Im } \mathbf{P}_0 \subset \ker \hat{\mathbf{L}}_{\alpha, b}$, with equality if $\alpha_0 \alpha_1 > 0$ and H is HYM with respect to ω .

Proof. We first prove that, given $(t, v) \in \mathbb{R} \times \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I)$ and $(f, \chi) := \mathbf{P}_{(\gamma, \phi, \xi)}(t, v)$, we have

$$t = \int_X f \omega^{[n]} / \text{Vol}_{\Omega}, \quad v = \chi + \theta_{H, \tilde{I}}^{\perp} \tilde{\eta}_f, \quad (4.40)$$

where $\tilde{\eta}_f$ is the Hamiltonian vector field associated to $f \in C^{\infty}(X)$ and $\tilde{\omega}$. To see this, note that, since $p(v)$ is holomorphic and preserves $\tilde{\omega}$, it can be written as

$$p(v) = \tilde{\eta}_{\psi} + \beta,$$

where $\tilde{\eta}_{\psi}$ is the real-holomorphic Hamiltonian vector field associated to $\psi \in C^{\infty}(X)$ and $\tilde{\omega}$ and β is a parallel vector field with respect to $\tilde{\omega}$ (see [39, §2]). Then, since $p(v)$ and $\tilde{\eta}_{\psi}$ vanish somewhere on X , we have that $\beta = 0$ and therefore

$$d\psi = p(v) \lrcorner \tilde{\omega} = df.$$

Formula (4.40) follows from the decomposition of v into its vertical and horizontal parts with respect to $\theta_{H, \tilde{I}}$.

Now, (1) follows from (4.40). To prove (2), suppose $\hat{\mathbf{T}}_{\alpha}(\gamma, \phi, \xi) \in \text{Im } \mathbf{P}_{(\gamma, \phi, \xi)}$, i.e.

$$f = S_{\alpha}(\tilde{b}), \quad \chi = \theta_{h, \tilde{I}} v = 4\alpha_1 (\Lambda_{\tilde{\omega}} F_{H, \tilde{I}} - z_{\tilde{\Omega}}). \quad (4.41)$$

From (4.40), it follows that

$$4\alpha_1 d_{\tilde{H}} \Lambda_{\tilde{\omega}} F_{\tilde{H}} = -\tilde{\eta}_f \lrcorner F_{\tilde{H}}, \quad P_{\tilde{\omega}} f = -L_{\tilde{\eta}_f} J = 0, \quad (4.42)$$

where we have used (3.17) to obtain the first equation, while the other identity follows because $\tilde{\eta}_{\psi} = \tilde{\eta}_f$ is real-holomorphic. Therefore $\tilde{b} = (\tilde{\omega}, \tilde{H})$ is an extremal pair.

To prove (3), note first that the inclusion $\text{Im } \mathbf{P}_0 \subset \ker \hat{\mathbf{L}}_{\alpha, b}$ is a straightforward consequence of (4.40). Suppose now that $\alpha_0 \alpha_1 > 0$ and H is HYM with respect to ω . Let $(f, \chi) \in \ker \hat{\mathbf{L}}_{\alpha, b}$. By Proposition 4.6, $v := \chi + \theta_{H, \tilde{I}}^{\perp} \eta_f$ is in $\text{Lie } \tilde{\mathcal{G}}_I$. In fact, $v \in \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I)$, as

f is \mathcal{H}_J -invariant and χ is $\tilde{\mathcal{G}}_I$ -invariant by assumption (see (4.37b)). Therefore $\mathbf{P}_0(v, t) = (f, \chi)$, where $t := \int_X f\omega^{[n]}/\text{Vol}_\Omega$. \square

Let $\langle \cdot, \cdot \rangle_\omega$ be the L^2 -inner product on $L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}$ given by (4.10). We claim that the orthogonal projectors onto $\text{Im } \mathbf{P}_{(\gamma, \phi, \xi)}$, denoted

$$\Pi_{(\gamma, \phi, \xi)}: L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I} \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}$$

vary smoothly with $(\gamma, \phi, \xi) \in \mathcal{V}$. To prove this, note that the map

$$\mathbf{P}: \mathcal{V} \times \mathbb{R} \times \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I) \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}: (\gamma, \phi, \xi, t, v) \longmapsto \mathbf{P}_{(\gamma, \phi, \xi)}(t, v)$$

is C^1 , as $\mathbf{P}_{(\gamma, \phi)}^0(t, v)$ is linear in (γ, ϕ, t, v) and $\mathbf{P}_\xi^1(v)$ depends linearly on v and smoothly on ξ . Moreover, $\mathbf{P}_{(\gamma, \phi, \xi)}$ is an isomorphism onto its image for all $(\gamma, \phi, \xi) \in \mathcal{V}$, by Lemma 4.8. Let $\{w_j\}$ be a basis of the vector space $\mathbb{R} \oplus \mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I)$ and $\{\zeta_j(\gamma, \phi, \xi)\}$ be the orthonormal basis of $\text{Im } \mathbf{P}_{(\gamma, \phi, \xi)}$ extracted from $\{\mathbf{P}_{(\gamma, \phi, \xi)} w_j\}$ by the Gram-Schmidt orthogonalization process. Then the claim follows by the above observations and the fact that

$$\Pi_{(\gamma, \phi, \xi)} = \sum_j \langle \zeta_j(\gamma, \phi, \xi), \cdot \rangle_\omega \zeta_j. \quad (4.43)$$

Furthermore, since $\langle \zeta_j, \zeta_k \rangle_\omega$ are continuous functions on \mathcal{V} , the origin has an open neighbourhood $\mathcal{V}_0 \subset \mathcal{V}$ such that for all $(\gamma, \phi, \xi) \in \mathcal{V}_0$, the following holds (cf. [39, (5.3)]):

$$\ker(\text{Id} - \Pi_{(\gamma, \phi, \xi)}) = \ker(\text{Id} - \Pi_0) \circ (\text{Id} - \Pi_{(\gamma, \phi, \xi)}). \quad (4.44)$$

For any pair of non-negative integers (l, m) , let $I_{l, m} \subset L_l^2(X)^{\mathcal{H}_J} \times L_m^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}$ be the orthogonal complement of $\text{Im } \mathbf{P}_0$. Define

$$\mathcal{W} = \mathcal{V}_0 \cap (\mathcal{H}^{1,1}(X) \times I_{k+4, k+4}).$$

Note that, under the assumptions in the last part of Lemma 4.8, the subspace \mathcal{W} is perpendicular to $\ker \mathbf{L}_{\alpha, b}$. We will use this fact to obtain existence results about deformations of extremal pairs. Define a LeBrun–Simanca map [38, §5]

$$\begin{aligned} \mathbf{T}_\alpha: \quad \mathcal{W} &\longrightarrow I_{k, k+2} \\ (\gamma, \phi, \xi) &\longmapsto (\text{Id} - \Pi_0) \circ (\text{Id} - \Pi_{(\gamma, \phi, \xi)}) \circ \hat{\mathbf{T}}_\alpha(\gamma, \phi, \xi). \end{aligned} \quad (4.45)$$

Then \mathbf{T}_α is C^1 , because it is the composition of C^1 maps.

Given $(\dot{\phi}, \dot{\xi}) \in I_{k+4, k+4}$, to calculate the directional derivative $\delta_{(\dot{\phi}, \dot{\xi})} \mathbf{T}_\alpha$ of \mathbf{T}_α at the origin in the direction $(0, \dot{\phi}, \dot{\xi})$, we define the curve $b_t = (0, t\dot{\phi}, t\dot{\xi})$. Using (4.27), we obtain

$$\begin{aligned} \delta_{(\dot{\phi}, \dot{\xi})} \mathbf{T}_\alpha &= \frac{d}{dt} \mathbf{T}_\alpha(b_t)|_{t=0} = (\text{Id} - \Pi_0) \mathbf{L}_{\alpha, b}(\dot{\phi}, \dot{\xi}) \\ &\quad + (\text{Id} - \Pi_0) \left((d(S_\alpha(b)), d\dot{\phi})_\omega, 4\alpha_1 J\eta_{\dot{\phi}} \lrcorner d_H(\Lambda_\omega F_H) \right) \\ &\quad - (\text{Id} - \Pi_0) \frac{d}{dt} (\Pi_{b_t} \mathbf{T}_\alpha(0))|_{t=0}. \end{aligned}$$

Now, if $b = (\omega, H)$ is a solution to the coupled equations (3.59), then the second summand of the right-hand side vanishes and $\Pi_{b_t} \mathbf{T}_\alpha(0) = \mathbf{T}_\alpha(0)$ for all t , so the third summand of the right-hand side vanishes too and hence, under this assumption, we conclude that

$$\delta_{(\dot{\phi}, \dot{\xi})} \mathbf{T}_\alpha = (\text{Id} - \Pi_0) \circ \hat{\mathbf{L}}_{\alpha, b}(\dot{\phi}, \dot{\xi}). \quad (4.46)$$

Remark 4.9. It is at this point that one runs into technical difficulties if one attempts to apply the approach of LeBrun & Simanca [38] to obtain deformations of an extremal pair which is not a solution of the coupled equations. The problem is that for an arbitrary extremal pair $b = (\omega, H)$, if one proceeds as in [38, Lemma 1], then one obtains

$$\delta_{(\dot{\phi}, \dot{\xi})} \mathbf{T}_\alpha = (\text{Id} - \Pi_0) \left(\hat{\mathbf{L}}_{\alpha, b}(\dot{\phi}, \dot{\xi}) + (0, -J\eta_\alpha(b, I) \lrcorner (d_H \dot{\xi} + \eta_{\dot{\phi}} \lrcorner F_H)) \right),$$

and to construct deformations of b which are also extremal pairs using the approach of [38], we need know that (4.46) is satisfied. A natural condition which implies that (4.46) holds is that $S_\alpha(\omega, H)$ is constant. Furthermore, in the approach of [38], we need to know that $\hat{\mathbf{L}}_{\alpha, b}$ is self-adjoint, with kernel $\text{Im } \mathbf{P}_0$, so another natural condition is that the Hermitian–Yang–Mills equation is satisfied, by Proposition 4.6 and Lemma 4.8(2). In other words, to get a direct generalization of the method of [38], it is natural to impose the condition that b is a solution of the coupled equations, as we will do below.

We can now prove the two main results of §4.4. For this, given $\alpha \in \mathbb{R}^2$, we call $b \in B_\Omega$ an *extremal pair with coupling constants* α if it satisfies (4.1).

Theorem 4.10. *Suppose (ω, H) is a solution to the coupled equations (3.59) with coupling constant α and $[\omega] = \Omega$, where $\alpha = (\alpha_0, \alpha_1) \in \mathbb{R}^2$ satisfies $\alpha_0 \alpha_1 > 0$. Then (α, Ω) has an open neighbourhood $U \subset \mathbb{R}^2 \times H^{1,1}(X, \mathbb{R})$ such that for all $(\tilde{\alpha}, \tilde{\Omega}) \in U$ there exists an extremal pair $(\tilde{\omega}, \tilde{H})$ with coupling constants $\tilde{\alpha}$ and such that $[\tilde{\omega}] = \tilde{\Omega}$.*

Proof. Note that $\eta_\alpha(b) = 0$, as $b = (\omega, H)$ is a solution of the coupled equations (3.59). Since the map \mathbf{T}_α depends linearly on $\alpha = (\alpha_0, \alpha_1)$, it can be viewed as a C^1 map $\mathbf{T}: \mathbb{R}^2 \times \mathcal{W} \rightarrow I_{k, k+2}$, whose the Fréchet derivative at the origin with respect to ϕ and ξ is $\delta \mathbf{T}_\alpha = (\text{Id} - \Pi_0) \circ \hat{\mathbf{L}}_{\alpha, b}$, by (4.46). Since H is HYM with respect to ω and $\alpha_0 \alpha_1 > 0$, Lemma 4.8 applies and $(\text{Id} - \Pi_0) \circ \hat{\mathbf{L}}_{\alpha, b}$ is an isomorphism. Therefore, by the implicit function theorem, there exists an open neighbourhood $U \subset \mathbb{R}^2 \times \mathcal{H}^{1,1}(X)$ of (α, Ω) such that for all $(\tilde{\alpha}, \gamma) \in U$ there exists a pair $(\phi, \xi) \in I_{k+4, k+4}$ such that

$$\mathbf{T}_{\tilde{\alpha}}(\gamma, \phi, \xi) \in \ker((\text{Id} - \Pi_0)(\text{Id} - \Pi_{\gamma, \phi, \xi})),$$

so $\mathbf{T}_{\tilde{\alpha}}(\gamma, \phi, \xi) \in \text{Im } \mathbf{P}_{(\gamma, \phi, \xi)}$ by (4.44). Hence the pair $(\tilde{\omega}, \tilde{H})$ determined by (γ, ϕ, ξ) is extremal with coupling constant $\tilde{\alpha}$, by Lemma 4.8(1), and smooth by Lemma 4.3. \square

Let $H^{1,1}(X, \mathbb{R})^+ \subset H^{1,1}(X, \mathbb{R})$ denote the ‘Kähler cone’ of (X, J) , i.e. the open subset of elements $\Omega \in H^{1,1}(X, \mathbb{R})$ such that \mathcal{K}_Ω is non-empty. Given $(\alpha, \Omega) \in \mathbb{R}_{>0}^2 \times H^{1,1}(X, \mathbb{R})^+$, consider the α -Futaki character $\mathcal{F}_{\alpha, \Omega}: \text{Lie } \Gamma_I \rightarrow \mathbb{C}$ defined in (3.48) (or (3.66)). Denote

$$V(\mathcal{F}) := \{(\alpha, \Omega) \mid \mathcal{F}_{\alpha, \Omega} = 0\} \subset \mathbb{R}_{>0}^2 \times H^{1,1}(X, \mathbb{R})^+.$$

Theorem 4.11. *Let S be the set of pairs $(\alpha, \Omega) \in \mathbb{R}_{>0}^2 \times H^{1,1}(X, \mathbb{R})^+$ for which there exists a solution $(\omega, H) \in B_\Omega$ to the coupled equations (3.59).*

- (1) *Then $S \cap V(\mathcal{F})$ is open in $V(\mathcal{F})$.*
- (2) *If $\text{Aut}(E^c, I)$ is finite, then $S \subset \mathbb{R}^2 \times H^{1,1}(X, \mathbb{R})$ is open.*

Proof. The proof is immediate from Theorem 4.10, together with Proposition 4.2 for part (1) and (4.6) for part (2). \square

4.5. Deformations of solutions in the weak coupling limit. We will obtain now solutions to the coupled equations (3.59) in ‘weak coupling limit’ $0 < |\alpha_1/\alpha_0| \ll 1$ by deforming solutions $(\omega, H) \in B_\Omega$ with coupling constants $\alpha_0 \neq 0, \alpha_1 = 0$. Since we will study these equations for coupling constants in a small open neighbourhood of a pair $(\alpha_0, \alpha_1) \in \mathbb{R}^2$ satisfying $\alpha_0 \neq 0, \alpha_1 = 0$, we can divide the second equation in (3.59) by α_0 . Hence in the sequel we will normalize to $\alpha_0 = 1$ and $\alpha := \alpha_1$ will be called *the* coupling constant.

Note that for $\alpha = 0$, the coupled equations (3.59) are the condition that ω is a cscK metric on (X, J) and H is a Hermitian–Yang–Mills reduction of (E^c, I) with respect to ω , so in particular the pair (ω, H) satisfies the following equations:

$$\left. \begin{aligned} d_H^* F_H &= 0 \\ L_{\eta_{S_\omega}} J &= 0 \end{aligned} \right\} \quad (4.47)$$

Here, $d_H^* F_H = 0$ is the Yang–Mills equation, which is equivalent to

$$d_H \Lambda_\omega F_H = 0 \quad (4.48)$$

by the Kähler identities (see e.g. [15, Proposition 3]), and η_{S_ω} is the Hamiltonian vector field of the scalar curvature S_ω over (X, ω) , so $L_{\eta_{S_\omega}} J = 0$ is the condition that ω is an extremal metric on (X, J) .

If one attempts to generalize Theorem 4.10 to the weak coupling limit, one observes that Proposition 4.2 cannot be used for $\alpha = 0$, but the system of equations (4.47) can be viewed as an adiabatic limit of equation (4.1). In fact, a pair $b_\lambda := (\lambda\omega, H)$ satisfies (4.1) with coupling constant α , for a real number $\lambda > 0$, if and only if

$$4\alpha d_H \Lambda_\omega F_H + \lambda^{-1} \eta_{\lambda \lrcorner} F_H = 0, \quad L_{\eta_\lambda} J = 0, \quad (4.49)$$

where η_λ is the Hamiltonian vector field of $S_{\alpha/\lambda}(\omega, H)$ over (X, ω) , and (4.47) is the formal limit of (4.49) when $\lambda \rightarrow \infty$. Hence a strategy to obtain a solution to the coupled equations (3.59) for $0 < |\alpha_1/\alpha_0| \ll 1$ (equivalently, for $\lambda \gg 0$) could be to deform a solution to (3.59) for $\alpha = 0$ (which is therefore a solution to (4.47)) to obtain a solution of (4.49). The problem is that the kernel of the operator $\mathbf{L}_{\alpha, b_\lambda}$ determined by a solution b_λ to the coupled equations (3.59) has a discontinuity in the limit $\lambda \rightarrow \infty$. More precisely, this kernel for finite $\lambda > 0$ can be identified with $\text{Lie } \tilde{\mathcal{G}}_I$ (see Proposition 4.6), whereas the kernel of $\mathbf{L}_{\alpha, b}$ in the limit $\lambda \rightarrow \infty$ is

$$\{(\phi, \xi) \in C^\infty(X) \times \Omega^0(\text{ad } E_H) \mid \eta_\phi \in \text{Lie } \mathcal{H}_J, d_H^*(d_H \xi + \eta_\phi \lrcorner F_H) = 0\} \quad (4.50)$$

(this follows directly from (4.13)). This discontinuity causes serious technical problems when one attempts to use this strategy within the approach of LeBrun & Simanca.

The source of this difficulty is related to the vanishing of the factor $4\alpha_1$ multiplying the HYM term in the moment maps μ_b when $\alpha_1 = 0$ (see (2.6)). One way to get around this problem is to apply the approach of LeBrun & Simanca to the operator obtained by dropping this factor in the moment map operator T_α . Fix an integer $k > n$ and keep the notation of §§4.3, 4.4. Then the resulting modified moment map operator is

$$\begin{aligned} B_\alpha: \quad \mathcal{U} &\longrightarrow L_k^2(X) \times L_{k+2}^2(\text{ad } E_H) \\ (\gamma, \phi, \xi) &\longmapsto \left(S_\alpha(\tilde{b}), \Lambda_{\tilde{\omega}} F_{H, \tilde{I}} - z_{\tilde{\Omega}} \right), \end{aligned} \quad (4.51)$$

where \mathcal{U} is the open set in (4.25) and $\tilde{\omega}, \tilde{I}$ and \tilde{b} are given by (4.22) and (4.23).

As we will see below, this modification on the moment map operator within the approach of LeBrun & Simanca produces the following modified extremality condition (cf. (4.1)).

Definition 4.12. A pair $b = (\omega, H) \in B_\Omega$ is called *extremal* with coupling constant α in the weak coupling limit if it satisfies the equations

$$\left. \begin{aligned} d_H^* F_H &= 0 \\ L_{\eta_\alpha(b)} J &= 0 \end{aligned} \right\}, \quad (4.52)$$

where $\eta_\alpha(b)$ is the Hamiltonian vector field of $S_\alpha(b)$ over (X, ω) .

Note that the system of equations (4.52) becomes (4.47) when $\alpha = 0$, while for arbitrary α any solution to the coupled equations (3.59) is an extremal pair in the weak coupling limit (see (4.48)). To obtain a partial converse, define the characters

$$\mathcal{F}_{0,\Omega}, \mathcal{F}_{\infty,\Omega}: \text{Lie Aut}(E^c, I) \longrightarrow \mathbb{C}, \quad (4.53)$$

as the α -Futaki characters of the Kähler class Ω for (α_0, α_1) equal to $(1, 0)$ and $(0, 1)$ in (3.66), respectively. By (3.66), up to a multiplicative factor, $\langle \mathcal{F}_{0,\Omega}, \zeta \rangle$ is the Futaki character [22] of the Kähler class Ω on (X, J) evaluated at $p(\zeta)$, where p is the map in (4.4). It is also clear from (3.66) that the existence of a solution to the coupled equations (3.59) does not necessarily imply the vanishing of $\mathcal{F}_{0,\Omega}$ or $\mathcal{F}_{\infty,\Omega}$.

Proposition 4.13. *A solution $b \in B_\Omega$ of (4.52) is a solution to the coupled equations (3.59) if $\mathcal{F}_{0,\Omega} = \mathcal{F}_{\infty,\Omega} = 0$ and the vector field $\eta_\alpha(b)$ over X can be lifted to a holomorphic vector field over the total space of (E^c, I) .*

Proof. By (4.48), $\Lambda_\omega F_H$ is a vertical holomorphic vector field on the total space of (E^c, I) , i.e. $\Lambda_\omega F_H \in \text{Lie } \mathcal{G}_I$. Now, if $\mathcal{F}_{\infty,\Omega} = 0$, then H is HYM with respect to ω , because in this case, by (3.66) we obtain

$$\|\Lambda_\omega F_H - z_\Omega\|_\omega^2 = -\langle \mathcal{F}_{\infty,\Omega}, \Lambda_\omega F_H - z_\Omega \rangle = 0.$$

Moreover, if $\mathcal{F}_{0,\Omega} = \mathcal{F}_{\infty,\Omega} = 0$ and $\eta_\alpha(b) = p(\zeta)$ for a holomorphic vector field ζ on (E^c, I) , then by a straightforward computation using (3.66), we obtain

$$\|S_\alpha(b) - \hat{S}_\alpha\|_\omega^2 = \langle \mathcal{F}_{0,\Omega}, \eta_\alpha(b) \rangle + \alpha \langle \mathcal{F}_{\infty,\Omega}, \zeta \rangle + \alpha \langle \theta_H \zeta, \Lambda_\omega F_H - z_\Omega \rangle = 0,$$

where $\hat{S}_\alpha = \int_X S_\alpha(b) \omega^{[n]} / \text{Vol}_\Omega$, so b is a solution to the coupled equations (3.59). \square

Extremal pairs in the weak coupling limit enjoy the same good regularity properties:

Lemma 4.14. *Let (ω, H) be a solution of (4.52) such that ω is a Kähler form of class C^2 on (X, J) and H is a section of E^c/G of class C^4 . Then both ω and H are smooth.*

Proof. This follows exactly as Lemma 4.3. \square

We define now a linear differential operator which is closely related to the linearization of B_α (see (4.51)) when $\alpha = 0$ and which will play the role in the weak coupling limit of the Lichnerowicz operator (4.11) in the study of the cscK equation or the operator $\mathbf{L}_{\alpha,b}$ defined in §4.4 away from the weak coupling limit. This linear differential operator is

$$\begin{aligned} \mathbf{C}: L_{k+4}^2(X) \times L_{k+4}^2(\text{ad } E_H) &\longrightarrow L_k^2(X) \times L_{k+2}^2(\text{ad } E_H) \\ (\phi, \xi) &\longmapsto (\mathbf{P}^* \mathbf{P} \phi, d_H^*(d_H \xi + \eta_\phi \lrcorner F_H)), \end{aligned} \quad (4.54)$$

where \mathbf{P} is defined as in (4.11). It is easy to see (cf. Proposition 4.6) that the operator \mathbf{C} is elliptic and self-adjoint with respect to the L^2 -inner product $\langle \cdot, \cdot \rangle_\omega$ given by (4.10).

It can be shown as in the proof of Proposition 4.7 that B_α is well-defined and C^1 and that its Fréchet derivative at the origin $(0, 0, 0)$ when $\alpha = 0$ is given by

$$\delta B_0(\dot{\gamma}, \dot{\phi}, \dot{\xi}) = \mathbf{C}(\dot{\phi}, \dot{\xi}) + ((dS_\omega, d\dot{\phi})_\omega, 0) + \delta_\gamma B_0, \quad (4.55)$$

where $\delta_\gamma B_0$ is the directional derivative of B_0 at the origin in the direction $(\dot{\gamma}, 0, 0)$ (cf. (4.27)).

To proceed as in §4.5 following the approach of LeBrun & Simanca, we need to consider the restriction of B_α and \mathbf{C} to suitable subspaces of the Sobolev spaces. Let

$$L_k^2(X)^{\mathcal{H}_J} \subset L_k^2(X) \text{ and } L_k^2(\text{ad } E_H)^{\mathcal{G}_I} \subset L_k^2(\text{ad } E_H)$$

be the closed subspaces consisting of \mathcal{H}_J -invariant functions and \mathcal{G}_I -invariant sections, respectively (cf. (4.35)) and

$$\mathcal{V}' = \mathcal{U} \cap (\mathcal{H}^{1,1}(X) \times L_{k+4}^2(X)^{\mathcal{H}_J} \times L_{k+4}^2(\text{ad } E_H)^{\mathcal{G}_I})$$

(cf. (4.36)). By restriction of (4.51) and (4.54), we obtain well-defined maps

$$\hat{B}_\alpha: \mathcal{V}' \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I}, \quad (4.56a)$$

$$\hat{\mathbf{C}}: L_{k+4}^2(X)^{\mathcal{H}_J} \times L_{k+4}^2(\text{ad } E_H)^{\mathcal{G}_I} \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I}, \quad (4.56b)$$

where \hat{B}_α is C^1 and $\hat{\mathbf{C}}$ is a linear elliptic operator (cf. (4.37)).

Note that in the constructions (4.56) we have used the subspace $L_k^2(\text{ad } E_H)^{\mathcal{G}_I} \subset L_k^2(\text{ad } E_H)$ rather than the possibly smaller subspace $L_k^2(\text{ad } E_H)^{\tilde{\mathcal{G}}_I}$ which appeared in (4.37). In practice, we could say that the exact sequence (4.4) in §4.4 degenerates to the trivial extension

$$1 \rightarrow \mathcal{G}_I \longrightarrow \mathcal{H}_J \times \mathcal{G}_I \longrightarrow \mathcal{H}_J \rightarrow 1 \quad (4.57)$$

in the weak coupling limit $\alpha \rightarrow 0$. In particular, the centre $\mathfrak{z}(\text{Lie } \tilde{\mathcal{G}}_I)$ of $\text{Lie } \tilde{\mathcal{G}}_I$ (see (4.39)) is now replaced by the centre $\mathfrak{z}(\text{Lie } \mathcal{H}_J) \oplus \mathfrak{z}(\text{Lie } \mathcal{G}_I)$ of the Lie algebra of $\mathcal{H}_J \times \mathcal{G}_I$ and $\mathbf{P}_{(\gamma, \phi, \xi)}$ (see (4.39)) is replaced by

$$\begin{aligned} \mathbf{Q}_{(\gamma, \phi, \xi)}: \mathbb{R} \times \mathfrak{z}(\text{Lie } \mathcal{H}_J) \oplus \mathfrak{z}(\text{Lie } \mathcal{G}_I) &\longrightarrow L_{k+3}^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I} \\ (t, w, v) &\longmapsto (\mathbf{G}d^*(w \lrcorner \tilde{\omega}) + t, v), \end{aligned} \quad (4.58)$$

with $(\gamma, \phi, \xi) \in \mathcal{V}'$.

Lemma 4.15. *Let $(\gamma, \phi, \xi) \in \mathcal{V}'$. If $\hat{B}_\alpha(\gamma, \phi, \xi) \in \text{Im } \mathbf{Q}_{(\gamma, \phi, \xi)}$, then $\tilde{b} = (\tilde{\omega}, \tilde{H})$ is a solution of (4.52).*

Proof. This follows exactly as part (2) of Lemma 4.8. \square

Since \mathbf{C} has kernel (4.50) by elliptic regularity, part (3) of Lemma 4.8 has no direct analogue in the weak coupling limit. Lemma 4.16 will provide a suitable replacement of this part of the lemma. Let $\langle \cdot, \cdot \rangle_\omega$ be the L^2 -inner product on $L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I}$ given by (4.10). One can prove as in §4.4 that the orthogonal projector

$$\Pi'_{(\gamma, \phi, \xi)}: L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I} \longrightarrow L_k^2(X)^{\mathcal{H}_J} \times L_{k+2}^2(\text{ad } E_H)^{\mathcal{G}_I}$$

onto $\text{Im } \mathbf{Q}_{(\gamma, \phi, \xi)}$ varies smoothly with $(\gamma, \phi, \xi) \in \mathcal{V}'$ and, by continuity, there exists an open neighbourhood $\mathcal{V}'_0 \subset \mathcal{V}'$ of the origin such that

$$\ker(\text{Id} - \Pi'_{(\gamma, \phi, \xi)}) = \ker(\text{Id} - \Pi_0) \circ (\text{Id} - \Pi'_{(\gamma, \phi, \xi)})$$

for any $(\gamma, \phi, \xi) \in \mathcal{V}'_0$ (cf. (4.44)).

For any pair of non-negative integers (l, m) , let $I'_{l,m} \subset L^2_l(X)^{\mathcal{H}, J} \times L^2_m(\text{ad } E_H)^{\mathcal{G}_I}$ be the orthogonal complement of $\text{Im } \mathbf{Q}_0$. Define

$$\mathcal{W}' = \mathcal{V}'_0 \cap (\mathcal{H}^{1,1}(X) \times I'_{k+4, k+4}).$$

Lemma 4.16. *The induced map $\hat{\mathbf{C}}: I'_{k+4, k+4} \longrightarrow I'_{k, k+2}$ is an isomorphism.*

Proof. This map is well-defined because $\text{Im } \mathbf{Q}_0 \subset \ker(\text{P} \oplus d_H)$. If $\hat{\mathbf{C}}(\phi, \xi) = 0$ for some $(\phi, \xi) \in I'_{k+4, k+4}$, then $\text{P}^* \text{P} \phi = 0$, so $\text{P} \phi = 0$, which implies $\phi = 0$, and $\hat{\mathbf{C}}(\phi, \xi) = 0$ means $d_H^* d_H \xi = 0$, so $d_H \xi = 0$, which implies $\xi = 0$. Thus $\hat{\mathbf{C}}$ is injective. Finally, $\hat{\mathbf{C}}$ is surjective because so is $\text{P}^* \text{P} \oplus d_H^* d_H$. \square

Define now a LeBrun–Simanca map [38, §5]

$$\begin{aligned} \mathbf{B}_\alpha: \mathcal{W}' &\longrightarrow I'_{k, k+2} \\ (\gamma, \phi, \xi) &\longmapsto (\text{Id} - \Pi'_0) \circ (\text{Id} - \Pi'_{(\gamma, \phi, \xi)}) \circ \hat{\mathbf{B}}_\alpha(\gamma, \phi, \xi). \end{aligned} \quad (4.59)$$

As \mathbf{B}_α is the composition of C^1 -maps, it is C^1 . Using Lemma 4.16 and [38, Lemma 1], we can see that its directional derivative at the origin in the direction $(0, \dot{\phi}, \dot{\xi})$ for $\alpha = 0$ is

$$\delta_{(\dot{\phi}, \dot{\xi})} \mathbf{B}_0 = (\text{Id} - \Pi'_0) \hat{\mathbf{C}}(\dot{\phi}, \dot{\xi}) = \hat{\mathbf{C}}(\dot{\phi}, \dot{\xi}), \quad (4.60)$$

for all $(\dot{\phi}, \dot{\xi}) \in I'_{k+4, k+4}$.

We can now prove the two main results of §4.5.

Theorem 4.17. *Suppose that ω is an extremal Kähler metric on (X, J) with $\Omega = [\omega]$ and H is a Yang–Mills reduction of (E^c, I) with respect to ω . Then $(0, \Omega)$ has an open neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ such that for all $(\tilde{\alpha}, \tilde{\Omega}) \in U$ there exists an extremal pair $(\tilde{\omega}, \tilde{H})$ with coupling constant $\tilde{\alpha}$ in the weak coupling limit such that $[\tilde{\omega}] = \tilde{\Omega}$.*

Proof. This follows as Theorem 4.10, combining (4.60) with Lemma 4.16 and the implicit function theorem, and then using Lemmas 4.15 and 4.14. \square

In the following theorem, we say that a reduction $H \in \mathcal{R}$ is irreducible if its Chern connection is irreducible, that is, if its isotropy group inside the gauge group \mathcal{G}_H of E_H is minimal—the centre of G (see §2.3 and also [20, §4.2.2]).

Theorem 4.18. *Assume that there is a cscK metric ω on (X, J) with cohomology class Ω and there are no non-zero Hamiltonian Killing vector fields on X . Then*

- (1) *If (E^c, I) admits an irreducible HYM reduction H with respect to ω , then $(0, \Omega)$ has an open neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ such that for all $(\tilde{\alpha}_1, \tilde{\Omega}) \in U$, there exists a solution $(\tilde{\omega}, \tilde{H})$ to the coupled equations (3.59) with coupling constant $\tilde{\alpha} = (1, \tilde{\alpha}_1)$ and $\tilde{\omega} \in \tilde{\Omega}$.*

- (2) If (E^c, I) admits a HYM reduction H with respect to ω , then there exists $\epsilon > 0$ such that for all $\tilde{\alpha}_1 \in \mathbb{R}$ with $-\epsilon < \tilde{\alpha}_1 < \epsilon$, there exists a solution $(\tilde{\omega}, \tilde{H})$ to the coupled equations (3.59) with coupling constants $(1, \tilde{\alpha}_1)$ and $\tilde{\omega} \in \tilde{\Omega}$.

Proof. Since HYM reductions are Yang–Mills, Theorem 4.17 implies that for all $(\tilde{\alpha}, \tilde{\Omega})$ in a neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ of $(0, \Omega)$, there exists an extremal pair $(\tilde{\omega}, \tilde{H})$ with coupling constant $\tilde{\alpha}$ in the weak coupling limit with $[\tilde{\omega}] = \tilde{\Omega}$ and \tilde{H} irreducible.

Part (1) follows now since the function $S_\alpha(\tilde{\omega}, \tilde{H})$ defined by (4.2) is constant on X for any extremal pair $(\tilde{\omega}, \tilde{H})$, as $\text{Lie } \mathcal{H}_J = 0$ and, furthermore, the vertical real-holomorphic vector field on (E^c, I) defined by $\Lambda_{\tilde{\omega}} F_{\tilde{H}}$ is in \mathfrak{z} , as \tilde{H} is irreducible.

Part (2) follows from Theorem 4.17 and Proposition 4.13, because $\mathcal{F}_{0,\Omega} = \mathcal{F}_{\infty,\Omega} = 0$ by (3.66), as $\text{Lie } \mathcal{H}_J = 0$ and (E^c, I) admits a HYM reduction H with respect to ω . \square

5. EXAMPLES AND CSCK METRICS ON RULED MANIFOLDS

This section contains some examples of solutions to the coupled equations (0.2). In §5.4 we also discuss how the existence of solutions in the limit case $\alpha_0 = 0$ can be applied, using results of Y. J. Hong in [30], to obtain cscK metrics on ruled manifolds.

5.1. Projectively flat bundles. Let (E^c, I) be a holomorphic principal G^c -bundle over a compact complex manifold X . We fix a maximal compact subgroup $G \subset G^c$ and a G -invariant metric (\cdot, \cdot) on \mathfrak{g} . Suppose that there exists a G -reduction H on E^c and a Kähler metric ω on X satisfying

$$\left. \begin{aligned} F_H &= z\omega \\ S_\omega &= \hat{S} \end{aligned} \right\}, \quad (5.1)$$

where F_H is the curvature of the Chern connection of H , z is the element of \mathfrak{z} (see (1.5)) given by (1.8) and $\hat{S} \in \mathbb{R}$. It is then straightforward that the pair (ω, H) provides a solution of the coupled equations (0.2). Note that the first equation in (5.1) implies that the G -bundle E_H corresponding to H is projectively flat, i.e. it is given by a representation $\pi_1(X) \rightarrow G/Z(G)$, where $Z(G)$ denotes the centre of G . Moreover, it implies the following topological constraint

$$[z \wedge F_H] = |z|^2[\omega] \in H^2(X, \mathbb{R}), \quad (5.2)$$

where $[F_H \wedge z]$ is the Chern–Weil class associated to the G -invariant linear form (\cdot, z) on \mathfrak{g} . We discuss now some examples of solutions of (5.1). We apply Theorem 4.18(1) to perturb the Kähler class of the given solution in order to obtain new solutions that do not satisfy the topological constraint (5.2).

Example 5.1. Let X be a compact Riemann surface. Then the coupled equations (0.2), for a G -reduction H on E^c and a Kähler metric ω on X , split into the system in separated variables (5.1), since $\dim_{\mathbb{C}} X = 1$ and the term $(F_H \wedge F_H)$ vanishes. Then the solutions of the coupled equations (0.2) are given by pairs (ω, H) , where ω is a cscK metric and H is a G -reduction such that its Chern connection is Hermitian–Yang–Mills (1.7). Due to the Narasimhan and Seshadri Theorem [14], and Ramanathan’s generalization [48], examples of solutions of the coupled equations (0.2) are given by polystable G^c -bundles over X .

Remark 5.2. In [47], Pandharipande used Geometric Invariant Theory to compactify the moduli space of pairs (X, F) consisting of a smooth algebraic curve X of genus $g > 1$,

polarised by a multiple of its canonical bundle, and a semistable vector bundle F over the curve. By [47, Proposition 8.2.1], such a pair is GIT stable if and only if E is Mumford stable. An interesting issue is that this decoupling phenomenon for the stability condition of a pair (X, F) is reflected in the decoupling of the equations (0.2), as already observed in Example 5.1. In fact, combining the Narasimhan–Seshadri Theorem with the uniformization Theorem on Riemann surfaces, it follows that any GIT stable pair (X, F) in Pandharipande’s construction, with X smooth, admits one and only one (irreducible) solution of (0.2) with Kähler class equal to the class of the polarisation. This gives some evidence to the claim that a Hitchin–Kobayashi correspondence for equations (0.2) exists in arbitrary dimensions, as conjectured by the authors [25]. An important difference with the curve case is that of course in higher dimensions one expects that the stability condition equivalent to the existence of solutions will involve conditions on the base manifold as well. In [25], a new notion of stability for degree zero bundles and polarised varieties has been defined. We hope to address the relation between this stability condition and the existence of solutions of (0.2) in future work.

Let (X, L) be a compact polarised manifold of complex dimension n . Suppose that there exists a cscK metric

$$\omega = \frac{\mathbf{i}}{2\pi} F_H \in c_1(L),$$

where F_H is the curvature of a Hermitian metric H on L . Then (ω, H) is a solution of (5.1), and hence a solution of (0.2). Since H is trivially an irreducible HYM metric with respect to ω , if there are no non-zero Hamiltonian holomorphic vector fields on X , we can apply Theorem 4.18 (1) obtaining solutions of (0.2) with non-zero ratio of the coupling constants and Kähler class close to $[\omega]$ in $H^{1,1}(X, \mathbb{R})$.

Example 5.3. Let X be a degree four hypersurface of \mathbb{P}^3 and set $L = \mathcal{O}_X(1)$. Then X is a K3 surface and, by Yau’s solution [61] of the Calabi conjecture (see e.g. [6]), there exists a unique Kähler Ricci flat metric $\omega \in c_1(L)$. Since (X, ω) is Kähler Ricci flat, any holomorphic vector field on X is ω -parallel and so $\text{Lie}(\text{Aut } X)$ contains no non-zero Hamiltonian holomorphic vector fields. Therefore, applying Theorem 4.18(1), we obtain solutions of (0.2) with non-zero ratio of the coupling constants α_1/α_0 and Kähler class $\tilde{\Omega}$ close to $\Omega = [\omega]$ in $H^{1,1}(X, \mathbb{R})$. As the dimension of $H^{1,1}(X, \mathbb{R})$ is 20, we can assume that $\tilde{\Omega}$ is not contained in the real line spanned by Ω , and so it is not obvious *a priori* that such a Kähler class contains a solution of (0.2) for our choice of manifold X and bundle L .

When (X, ω) is a flat Kähler torus, we can relax condition (5.1) and assume that E is an arbitrary projectively flat Hermitian bundle over X .

Example 5.4. Let $X \cong \mathbb{C}^n/\Lambda_X$ be a complex torus given by a lattice Λ_X in \mathbb{Z}^{2n} and endowed with a flat Kähler metric ω . Examples of holomorphic vector bundles E over X admitting a projectively flat Hermitian metric H are given by representations of a central extension of Λ_X into $U(r) \subset \text{GL}(r, \mathbb{C})$. Suppose that E is given by an irreducible representation of Λ_X and take a projectively flat Hermitian metric H on E , with curvature τId . By a conformal change on H , we can assume that τ is harmonic, and hence it is constant with respect to the natural coordinates in the torus. Then (ω, H) is a solution to the coupled equations (0.2) for arbitrary value of the coupling constants α_0 and α_1 .

Remark 5.5. In [50], Schumacher & Toma constructed a moduli space of (non-uniruled) polarised Kähler manifolds equipped with stable vector bundles, using versal deformations.

This moduli space is endowed with a Kähler metric, provided that the cohomological constraint (5.2) is satisfied, the base manifold X is Kähler–Einstein and the bundle is projectively flat. The gauge-theoretic equations corresponding to this moduli construction are therefore equivalent to (5.1), whose solutions are in particular solutions to the coupled equations (0.2). Note here that the cscK equation and the Kähler–Einstein equation are equivalent, by Hodge theory, if the class of the polarisation is a multiple of $c_1(X)$.

In the examples of §5.1, the coupled equations (0.2) admit decoupled solutions arising from the system in separated variables (5.1). There is a geometric interpretation for this in terms of the extended gauge group $\tilde{\mathcal{G}}$ in (1.10) associated to a solution (ω, H) to $F_H = z\omega$ and the moment map interpretation of (0.2) in §2. Namely, the Chern connection A of H determines a Lie algebra splitting of the short exact sequence

$$0 \rightarrow \text{Lie } \mathcal{G} \longrightarrow \text{Lie } \tilde{\mathcal{G}} \longrightarrow \text{Lie } \mathcal{H} \rightarrow 0$$

(see (1.22)). The splitting is given by the Lie algebra homomorphism

$$\begin{aligned} \Phi: \text{Lie } \mathcal{H} \cong C_0^\infty(X) &\longrightarrow \text{Lie } \tilde{\mathcal{G}} \\ \phi &\longmapsto \theta_A^\perp \eta_\phi - \phi z, \end{aligned} \quad (5.3)$$

(see (1.24)), where $\eta_\phi \lrcorner \omega = d\phi$ and θ_A^\perp is the horizontal lift with respect to the connection A . To see this, note that

$$\begin{aligned} [\Phi(\phi_1), \Phi(\phi_2)] &= [\theta_A^\perp \eta_{\phi_1} - \phi_1 z, \theta_A^\perp \eta_{\phi_2} - \phi_2 z] \\ &= \theta_A^\perp [\eta_{\phi_1}, \eta_{\phi_2}] - \{\phi_1, \phi_2\} z + (F_A - z\omega)(\eta_{\phi_1}, \eta_{\phi_2}) \\ &= \Phi(\{\phi_1, \phi_2\}) + (F_A - \omega z)(\eta_{\phi_1}, \eta_{\phi_2}), \end{aligned}$$

where $\{\phi_1, \phi_2\}$ is the Poisson bracket in $C_0^\infty(X)$ given by ω . Note that this homomorphism does not extend in general to the Lie algebra of the group of diffeomorphisms of X . Therefore, when $\dim_{\mathbb{C}} X = 1$ or E is projectively flat, the coupled system (0.2) may have ‘decoupled’ solutions due to the fact that $\text{Lie } \tilde{\mathcal{G}}$ is a semidirect product of $\text{Lie } \mathcal{G}$ and $\text{Lie } \mathcal{H}$.

5.2. Homogeneous bundles over homogeneous Kähler manifolds. For the basic material on this topic we refer to [6] and [34]. Let X be a compact homogeneous Kählerian manifold (i.e. admitting a Kähler metric) of a compact group G . In other words, $X = G/G_o$, for a closed subgroup $G_o \subset G$, equipped with the canonical G -invariant complex structure (see [6, Remark 8.99]). Then homogeneous holomorphic vector bundles E of rank r over X are in one-to-one correspondence with representations of G_o in $\text{GL}(r, \mathbb{C})$. For any invariant Kähler metric ω on X , there exists a unique G -invariant Hermitian–Yang–Mills unitary connection A , provided that the representation inducing E is irreducible (see [34, Proposition 6.1]). Moreover, for any such choice of invariant metric and connection, the scalar curvature S_ω and the function $\Lambda_\omega^2 \text{tr}(F_A \wedge F_A)$ on X are G -invariant and hence constant. It hence turns out that A satisfies the system of equations

$$\left. \begin{aligned} \Lambda_\omega F_A &= \mathbf{i} \lambda \text{Id} \\ \Lambda_\omega^2 \text{tr}(F_A \wedge F_A) &= -\frac{4\hat{c}}{(n-1)!} \end{aligned} \right\}, \quad (5.4)$$

where $\hat{c} \in \mathbb{R}$ is as in (2.13) and $\lambda \in \mathbb{R}$ is determined by the first Chern class of E and $[\omega]$. Equations (5.4) corresponds to the limit

$$\alpha_0 \rightarrow 0$$

in (0.2). Fix a pair of arbitrary coupling constants $\alpha_0, \alpha_1 > 0$ and a homogeneous holomorphic vector bundle E over X associated to an irreducible representation. Then any Kähler class on X determines a unique G -invariant solution (ω, A) to the coupled equations with coupling constants α_0 and α_1 . To see this, note that each de Rham class on X (in particular, each Kähler class) contains a unique G -invariant representative, obtained from an arbitrary representative by averaging. Trivially, the scalar curvature of any G -invariant Kähler metric is constant. Therefore, the unique G -invariant solution of (0.2) arises as a simultaneous solution of the cscK equation and (5.4), corresponding to the limit cases $\alpha_0 = 0$, and $\alpha_1 = 0$.

Example 5.6. Let (X, ω) be a compact homogeneous Kähler–Einstein surface G/G_0 . By [6, Corollary 8.98], this means that X is a complex torus or it is simply connected. Let E be a homogeneous vector bundle on X induced by an irreducible representation of G_0 in $SU(r)$, with induced G -invariant Hermitian metric H and G -invariant unitary connection A . Then the pair (ω, A) satisfies the system of equations

$$\left. \begin{aligned} F_A^+ &= 0 \\ \alpha_0(\rho_\omega - c'\omega) &= \alpha_1(2(\Lambda_\omega F_A) \wedge F_A - \Lambda_\omega(F_A \wedge F_A) - c''\omega) \end{aligned} \right\}, \quad (5.5)$$

for real numbers c', c'' , where ρ_ω is the Ricci form of ω and $F_A^+ = 0$ is the Anti-Self-Duality equation for the connection A . To prove this, note that A is HYM and

$$\begin{aligned} 2(\Lambda_\omega F_A) \wedge F_A - \Lambda_\omega(F_A \wedge F_A) &= -\Lambda_\omega(F_A \wedge F_A) \\ &= \Lambda_\omega(|F_A|^2 \omega^2) \\ &= |F_A|^2 \omega, \end{aligned}$$

(see (2.10)), where $|\cdot|$ is the pointwise norm with respect to ω . Hence (ω, A) satisfies (5.5) because the function $|F_A|^2$ is constant over X by invariance. Observe that the system (5.5) is stronger than (0.2). Indeed, it can be readily checked from [6, Proposition 9.61] that if (ω, A) satisfies (5.5), then the associated invariant Riemannian metric on the total space of the frame $U(r)$ -bundle of (E, H) over X , constructed as in §2.3, is Einstein, and therefore (ω, A) satisfies (0.2), by (2.22).

5.3. Stable bundles and cscK manifolds. We supply now some cases where Theorem 4.18 can be applied, obtaining examples of solutions with non-zero ratio of the coupling constants and fixed Kähler class. Starting with a cscK metric, we check that the new Kähler metrics that we obtain are not cscK. Using the contents of §3.4, we also give an explicit Example 5.9 in which there cannot exist solutions to the coupled equations.

Example 5.7. Let X be a high degree hypersurface of \mathbb{P}^3 . By theorems of Aubin and Yau (see e.g. [6, Theorem 11.7]), there exists a unique Kähler–Einstein metric $\omega \in c_1(X)$ with negative (constant) scalar curvature. Moreover, $c_1(X) < 0$ implies that the group of automorphisms of the complex manifold X is discrete (see [6, Proposition 2.138]). Let E be a smooth $SU(2)$ -principal bundle over X with second Chern number

$$k = \frac{1}{8\pi^2} \int_X \text{tr}(F_A \wedge F_A) \in \mathbb{Z},$$

where A is a connection on E . When k is sufficiently large, the moduli space M_k of Anti-Self-Dual (ASD) connections A on E with respect to ω is non-empty (see [20, Sec. 10.1.14]). Moreover, if k is large enough, M_k is non-compact but admits a compactification. Let A be a connection that determines a point in M_k . Then A is irreducible and so we can apply

Theorem 4.18(1), obtaining solutions $(\omega_\alpha, A_\alpha)$ of (0.2) with $[\omega_\alpha] = [\omega]$, nonzero values of the coupling constants α_0, α_1 and small ratio

$$\alpha = \frac{\alpha_1}{\alpha_0}.$$

We claim that if the pointwise norm

$$|F_{A_0}|_{\omega_0}^2 : X \longrightarrow \mathbb{R} \quad (5.6)$$

of the initial HYM connection $A_0 = A$ with respect to the Kähler–Einstein metric $\omega_0 = \omega$ is not constant, then ω_α is not cscK for $0 < \alpha \ll 1$. To see this, note that $(\omega_\alpha, A_\alpha)$ approaches uniformly to (ω_0, A_0) as $\alpha \rightarrow 0$ (see Theorem 4.17) and so

$$\lim_{\alpha \rightarrow 0} \left\| |F_{A_\alpha}|_{\omega_\alpha}^2 - |F_{A_0}|_{\omega_0}^2 \right\|_{L^\infty} = 0.$$

Hence if (5.6) is not constant, then $|F_{A_\alpha}|_{\omega_\alpha}^2$ is not constant for small α , so the claim follows from

$$S_{\omega_\alpha} = \frac{c}{\alpha_0} - \alpha \Lambda_{\omega_\alpha}^2 (F_{A_\alpha} \wedge F_{A_\alpha}) = \frac{c}{\alpha_0} + \alpha |F_{A_\alpha}|_{\omega_\alpha}^2,$$

where $c \in \mathbb{R}$. This last equation is satisfied because $(\omega_\alpha, A_\alpha)$ is a solution to (0.2). To choose an ASD connection for which (5.6) is not a constant, we consider a sequence of ASD connections $\{A^l\}_{l=0}^\infty$ defining points of M_k and approaching a point on the boundary of the compactification. When $l \gg 0$, the connections A_l start bubbling. This bubbling is reflected in the fact that the function (5.6) becomes more and more concentrated in a finite number of points of the manifold. Therefore, eventually, we obtain an ASD irreducible connection for which (5.6) is not a constant.

To be more precise, recall that any point on the boundary of the compactification of M_k is given by an ideal connection (see [20, Definition 4.4.1]), i.e. an unordered d -tuple (p_1, \dots, p_d) of points on X and a connection A_∞ on M_{k-d} , the moduli space of ASD connections on a suitable smooth $SU(2)$ -bundle E_{k-d} with second Chern number $k-d$. If $[A_l] \rightarrow [A_\infty]$ as $l \rightarrow \infty$, then for any continuous function f on X (see [20, Theorem 4.4.4]),

$$\lim_{l \rightarrow \infty} \int_X f \operatorname{tr}(F_{A_l} \wedge F_{A_l}) = \int_X f \operatorname{tr}(F_{A_\infty} \wedge F_{A_\infty}) + 8\pi^2 \sum_{m=1}^d f(p_m). \quad (5.7)$$

Take A_∞ in M_{k-d} with $d > 0$. If $|F_{A_l}|_\omega^2$ is constant for all l , using (5.7) and the equality

$$|F_{A_l}|_\omega^2 \omega^2 = \operatorname{tr}(F_{A_l} \wedge F_{A_l}),$$

we obtain that $d = 0$ and hence a contradiction (e.g., in (5.7), take a sequence $\{f_j\}_{j=1}^\infty$ of test functions approaching the delta function of a point p_i on X).

The hypothesis of Theorem 4.18 hold in much more generality. By the Donaldson–Uhlenbeck–Yau Theorem [16, 58], which admits a generalization to principal bundles (see [3, 49]), a family of examples generalizing Example 5.7 is provided by polystable holomorphic principal bundles over cscK manifolds with no non-zero Hamiltonian Killing vector fields. Recall that this theorem states that if a holomorphic principal G^c -bundle (E^c, I) is (Mumford–Takemoto) polystable with respect to a Kähler class Ω on a compact complex manifold X , then for any Kähler form $\omega \in \Omega$ there exists a reduction H of (E^c, I) to G which is HYM with respect to ω .

Let (X, L) be a compact polarised manifold whose first Chern class $c_1(X)$ satisfies

$$c_1(X) = \lambda c_1(L)$$

for some $\lambda \in \mathbb{Z}$. When $\lambda < 0$ (e.g. if X is a high degree hypersurface of \mathbb{P}^m), X has finite group of automorphisms and by the above result of Aubin and Yau, there exists a unique Kähler–Einstein metric $\omega \in c_1(L)$. If $\lambda = 0$, then by Yau’s a solution to Calabi’s Conjecture (see e.g. [6, Theorem 11.7]), there exists a unique Ricci flat metric on $c_1(L)$. As the dimension of the group of automorphisms of such manifolds is equal to its first Betti number (see [6, Remark 11.22]), the simply connected ones (e.g. K3 surfaces) are complex Ricci flat manifolds with finite group of automorphisms. If $\lambda > 0$, it has been recently proved [12, 56] that $c_1(L)$ admits a Kähler–Einstein metric if and only if (X, L) is K-stable. Let us restrict to the case

$$X = \mathbb{P}^2 \# m\overline{\mathbb{P}}^2,$$

the complex surface obtained by blowing up \mathbb{P}^2 at m generic points (see [57]). If we take m such that $3 < m < 8$ then $c_1(X) > 0$, X has finite automorphism group (see [55, Remark 3.12]) and it was proved in [57] that X admits a Kähler–Einstein metric.

On the other hand, given a polarised projective manifold (X, L) (without any assumption on $c_1(X)$), an asymptotic result of Maruyama [44] states that there exist $c_1(L)$ -stable vector bundles E over X of rank r , provided that $r > \dim X > 2$ and

$$c_2(E) \cdot c_1(L)^{n-2} \gg 0. \quad (5.8)$$

If X has finite group of automorphisms and it is endowed with a Kähler–Einstein metric $\omega \in c_1(L)$ as before, then we can apply Theorem 4.18.

Example 5.8. Let (X, ω) be a Kähler–Einstein manifold. Then ω is a cscK metric, which determines a Hermitian–Yang–Mills metric H on the tangent bundle $E^c = TX$. The pair (ω, H) is a solution to (3.59) with $\alpha_1 = 0$, but it is not a solution with $\alpha_1 \neq 0$ unless the Chern connection of H is flat. If $c_1(X) \leq 0$, then there are no non-zero Hamiltonian holomorphic vector fields over X , so $\mathcal{F}_{0,\Omega} = \mathcal{F}_{\infty,\Omega} = 0$ and as in Theorem 4.18, $(0, \Omega)$ has an open neighbourhood $U \subset \mathbb{R} \times H^{1,1}(X, \mathbb{R})$ such that for all $(\tilde{\alpha}, \tilde{\Omega}) \in U$, there exists a solution $(\tilde{\omega}, \tilde{H})$ to the coupled equations (3.59) with coupling constants satisfying $\alpha_1/\alpha_0 = \tilde{\alpha}$ and $[\tilde{\omega}] = \tilde{\Omega}$.

We will now construct an example where the α -Futaki character \mathcal{F}_I obstructs the existence of solutions to the coupled equations for small ratio of the coupling constants.

Example 5.9. Let (X, ω) be a Kähler manifold such that ω is not a cscK metric but it is extremal (e.g. $\mathbb{C}\mathbb{P}^2$ blown up at one point [8]). Recall from §4.1 that the extremality condition is equivalent to the condition that S_ω is the Hamiltonian function of a real holomorphic Killing vector field η . Since ω is not a cscK metric, it follows from (3.66) and (4.53) that the classical Futaki character of the Kähler class $\Omega = [\omega]$ evaluated at η is

$$\langle \mathcal{F}_{0,\Omega}, \eta \rangle = \int_X (S_\omega - \hat{S})^2 \omega^{[n]} > 0.$$

Note that η lifts to a holomorphic vector field $\zeta \in \text{Lie Aut}(TX)$ on the holomorphic tangent bundle $E^c = TX$ of X . It follows from (3.66) that the α -Futaki character \mathcal{F}_I evaluated at ζ is positive for sufficiently small values of $\alpha_1/\alpha_0 > 0$. Hence the pair (X, TX) does not admit a solution (ω, H) to (3.59) with $\omega \in \Omega$ and these values of the coupling constants.

Given an arbitrary holomorphic principal G^c -bundle E^c over X , the obstruction to lift a holomorphic vector field on X to a G^c -invariant holomorphic vector field on E^c lies in $H^1(X, \text{ad } E^c)$ (cf. (1.20)). Note that when $G^c = \mathbb{C}^*$, the previous argument always applies.

Let E^c be a stable holomorphic principal G^c -bundle over a polarised manifold (X, L) . In this situation, the Donaldson-Uhlenbeck-Yau Theorem allows us to think of the coupled equations as a generalization of the constant scalar curvature equation for a Kähler metric $\omega \in c_1(L)$. More precisely, given such ω there exists a unique HYM reduction H on E^c with respect to ω and therefore (3.59) can be interpreted as a single scalar equation for the Kähler metric. Although this approach may not be very useful in general, it becomes very explicit for the case of a line bundle E^c . In this case, a solution of the coupled equations is equivalent to a pair (ω, β) , where β is a harmonic $(1, 1)$ -form with $[\beta]/\sqrt{2} = 2\pi c_1(E^c)$ and satisfying

$$S_\omega - \alpha|\beta|_\omega^2 = c' \quad (5.9)$$

for a real constant $c' \in \mathbb{R}$, where $\alpha = \alpha_1/\alpha_0$. As for this, we simply note that H is a HYM Hermitian metric on E^c with respect to ω if and only if iF_H is harmonic. Therefore, for line bundles, the coupled equations provide a deformation of the constant scalar curvature equation by a harmonic $(1, 1)$ -form (cf. [53]). This point of view has been recently used by Keller and Tønnesen-Friedman to find solutions of the coupled equations on polarised complex 3-folds that do not admit any cscK metric [32]. We should stress that in general equation (5.9) is as difficult as the cscK equation, which has been completely solved only in the Kähler–Einstein case [12, 56]. It would be interesting to study these deformations in terms of K-stability.

5.4. CscK metrics on ruled manifolds. We now briefly discuss the relation between equation (5.4), given by the limit

$$\alpha_0 \rightarrow 0$$

in (0.2), and the existence of solutions to the cscK equation on ruled manifolds. We will use existence results of Y. J. Hong [29, 30].

Let (X, J, ω) be a compact Kähler manifold with constant scalar curvature and E a holomorphic stable vector bundle of degree zero over X (examples of this type were already provided in §5.3). Let H be a Hermitian metric on E whose Chern connection A is HYM (it exists by the Donaldson–Uhlenbeck–Yau Theorem [16, 58]). Let L be the tautological bundle over the projectivised bundle $\mathbb{P}(E)$ of E and $F_{A_{L^*}}$ the curvature of the connection induced by A on L^* . Then the 2-form

$$\frac{i}{2\pi} F_{A_{L^*}}$$

is non-degenerate on the fibres and in fact it induces the Fubini–Study metric, so

$$\widehat{\omega}_k = \frac{i}{2\pi} F_{A_{L^*}} + k\pi^*\omega$$

is a Kähler metric on $\mathbb{P}(E)$ for k large enough. When the automorphism group of (X, J) is finite, Y.J. Hong [29] used a deformation argument to prove that the cohomology class $[\widehat{\omega}_k]$ contains a cscK metric for $k \gg 0$. Let $\widetilde{\mathcal{G}}$ be the extended gauge group of the frame $\text{PU}(r)$ -bundle of the Hermitian vector bundle (E, H) and $\widetilde{\mathcal{G}}_I \subset \widetilde{\mathcal{G}}$ the stabilizer of the connection A . The assumption on $\text{Aut } X$ was removed in [30] (see [30, Definition I.A]), under the additional conditions that the subgroup

$$\widetilde{\mathcal{G}}_I \subset \text{Aut } \mathbb{P}(E)$$

is finite and

$$\Lambda_\omega^2(\text{tr } F_A \wedge \text{tr } F_A + \text{tr } F_A \wedge \rho_\omega + F_A \wedge F_A) = \text{const.} \quad (5.10)$$

Since $c_1(E) = 0$, this second condition reduces to

$$\Lambda_\omega^2 \operatorname{tr}(F_A \wedge F_A) = -\frac{4\hat{c}}{(n-1)!} \in \mathbb{R}.$$

The condition (5.10) appears when one splits the linearization of the cscK equation on $\mathbb{P}(E)$ into vertical and horizontal parts with respect to the connection A .

Hence we conclude that when $c_1(E) = 0$ and \mathcal{G}_I is finite, the existence of a solution to (5.4) is a sufficient condition for the existence of a cscK metric in the cohomology class $[\hat{\omega}_k]$ for $k \gg 0$ (see [30, Theorem III.A]). It would be interesting to study further this relation, trying to prove that the existence of solutions to the coupled equations for small $\frac{\alpha_1}{\alpha_0} > 0$ implies the existence of constant scalar curvature Kähler metrics on $\mathbb{P}(E)$ with Kähler class $kc_1(L)$ for large k . This would provide a generalization of Hong's results in [30].

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