LOCAL-GLOBAL INVARIANTS
OF FINITE AND INFINITE GROUPS:
AROUND BURNSIDE FROM ANOTHER SIDE

BORIS KUNYAVSKIĬ

ABSTRACT. This expository essay is focused on the Shafarevich–Tate set of a group G. Since its introduction for a finite group by Burnside, it has been rediscovered and redefined more than once. We discuss its various incarnations and properties as well as relationships (some of them conjectural) with other local-global invariants of groups.

PREFACE

These notes focus on a local-global invariant $\Sha(G)$ of a group G, its various incarnations, applications and possible generalizations. The cyrillic Ш refers to one of the names of this invariant, the Shafarevich–Tate set. At a glance, this text is a bulk of definitions, vague questions and conjectures, mostly compiled from numerous sources. However, some striking parallels with other invariants of G give a hope that something more sensible is extractable from this eclectic material.

For a finite group G, the invariant we are talking about was introduced by Burnside (naturally, under a different name) as early as in 1911, in the second edition of his famous book [Bur1]. Soon enough, in the paper [Bur2], which was published in 1913, he constructed the first example of G with nontrivial $\Sha(G)$, so the present article can be viewed as a modest commemoration of the 100th anniversary of this event. Since then, $\Sha(G)$ has been rediscovered more than once, each time revealing some new features. The main goal of this paper is to try to put some order in its numerous avatars and attract the attention of experts in both finite and infinite groups, as well as in geometric group theory (and maybe of those whose main interests lie outside group theory), to intriguing interrelations and applications. Towards this end, an attempt was made to make the bibliography as comprehensive as possible, which resulted in disproportionately unbalanced structure of the text, with a modestly sized body followed by a long tail. However, one can hope that the reader will find here something beyond a mere list of references.

As a precaution, it must be said that one should not be deceived by a misleading hint in the title: the paper has nothing in common with the well-known monograph by Kostrikin, where Burnside's name is identified (by default, as a sort of common practice in certain circles) with a notoriously hard group-theoretic problem (which, by the way, also carries a distinctive local-global flavour). We are indeed looking from another side, focusing on the development of a far less advertised contribution of Burnside's.

---

1. Main object

Definition 1.1. [On1] Let G be a group acting on itself by conjugation, $(g, x) \mapsto gxg^{-1}$, and let $H^1(G, G)$ denote the first cohomology pointed set. The set of cohomology classes becoming trivial after restriction to every cyclic subgroup of G is denoted $\text{Ш}(G)$ and called the Shafarevich–Tate set of G.

Definition 1.2. For the lack of a better term, we say that a group G with one-element Shafarevich–Tate set is Ш-rigid.

This term will be explained later, after clarifying relationships with some rigidity phenomena.

Observation 1.3. Ш-rigidity is often a crucial step to establishing important properties of G, or of a whole class of groups. On the other hand, groups with nontrivial $\text{Ш}(G)$ often provide interesting examples (or even allow one to refute long-standing conjectures). Some instances will be given below.

Remark 1.4. The terminology of Definition 1.1 is originated in the prototype of $\text{Ш}(G)$, dating back to the 1950's when it appeared in the context of a high-brow approach to diophantine equations and has been remaining since then one of the favourite objects of arithmetic geometers: given an algebraic group A defined over a number field k, $\text{Ш}(A)$ is defined as the set of cohomology classes $H^1(\Gamma, A(\bar{k}))$ (where the absolute Galois group $\Gamma = \text{Gal}(\bar{k}/k)$ acts naturally on $\bar{k}$-points of A) that become trivial after restriction to every $\Gamma_v = \text{Gal}(\bar{k}_v/k_v)$, where v runs over all places of k. In the purely group-theoretic setting as above, a much more down-to-earth description is available.

The following important remark is due to M. Mazur (see [On3]).

Observation 1.5. A map $f \colon G \to G$ is a cocycle if and only if the map $g \colon G \to G$ defined by $g(x) = f(x)x$ is an endomorphism. Furthermore, f is a coboundary if and only if g is an inner automorphism, and the restriction of f to the cyclic subgroup generated by $x \in G$ is a coboundary if and only if $g(x)$ is conjugate to x. Denote by $\text{End}_c(G)$ (resp. $\text{Aut}_c(G)$) the set of endomorphisms (resp. automorphisms) g of G such that $g(x)$ is conjugate to x for all $x \in G$.

We see that G is Ш-rigid if and only if it satisfies the following condition:

$$(1.1) \qquad\qquad\qquad \text{End}_c(G) = \text{Inn}(G).$$

Remark 1.6. Endomorphisms (or automorphisms) g with the property that $g(x)$ is conjugate to x for all $x \in G$ appear in literature under different names: pointwise inner, conjugating, class-preserving, etc. In this text they will be called *locally inner*. Note that any locally inner endomorphism is injective.

Condition (1.1) is sometimes called Property E (see, e.g., [AKT3]).

Since $\text{Inn}(G) \subseteq \text{Aut}_c(G) \subseteq \text{End}_c(G)$, it is convenient to subdivide the property of being Ш-rigid into two weaker ones: 1) $\text{Inn}(G) = \text{Aut}_c(G)$; 2) $\text{Aut}_c(G) = \text{End}_c(G)$. The first property is sometimes referred to as property A (see, e.g., [Gros]). It can be written down as $\text{Out}_c(G) = 1$ meaning that G has no locally inner outer automorphisms. In this text, a group G satisfying 1) will be called A-rigid, and a group satisfying 2) will be called B-rigid. In these terms, any Ш-rigid group is both A-rigid and B-rigid, and *vice versa*.

2. Zoo of rigid groups

2.1. **B-rigid groups.**

Observation 2.1. The following groups are B-rigid:
  (i) finite;
  (ii) profinite;
  (iii) solvable;
  (iv) cohopfian.

(i) is obvious because every locally inner endomorphism is injective and hence, as G is finite, surjective. (ii) is proved in [On3]. (iii) is proved in [AE]. (iv) is obvious (recall that the property of G to be cohopfian means that G contains no proper subgroups isomorphic to G, or, equivalently, that every injective endomorphism of G is surjective). Note that this property is related to other rigidity properties: for example, it is satisfied by rigid hyperbolic groups [RS] (see [Sel, Theorem 4.4] for a generalization), and irreducible lattices in semisimple Lie groups, except for free groups (because of their Mostow rigidity) [Pr]. (Note that free groups are also B-rigid and, moreover, III-rigid, see Observation 2.4(ii) below.) The cohopfian property also holds for some Kleinian [DP], [OP], [WZ], 3-manifold [PV], [WaWu], and braid groups [BM], [Belli], as well as for some torsion-free nilpotent groups [Beleg].

2.2. **A-rigid groups.** Our first observation is obvious.

Observation 2.2. All complete groups are A-rigid.

Recall that a group G is complete if it is centreless and all its automorphisms are inner. See [Rob] for a survey of finite complete groups. As to infinite groups, typical examples of complete groups arise as groups of automorphisms $G = \mathrm{Aut}\,(F)$, where F is a free group (or a group that is "not so far" from free). The cases where F is a free group, free nilpotent group of class two, or the quotient of a free group by an appropriate characteristic subgroup were treated, respectively, in [DF1]–[DF3] (for groups of finite rank) and [To1]–[To3] (for groups of infinite rank). The cases $F = \mathrm{GL}(n, \mathbb{Z})$ (n odd), $F = \mathrm{PGL}(2, \mathbb{Z})$ ($n \geq 2$), $F = \mathrm{SL}(n, \mathbb{Z})$ ($n \geq 3$) were considered in [Dy]. The cases $F = B_n$ (Artin braid group), $n \geq 4$, and $F = \mathrm{Aut}\,(B_3)$ were established in [DG]. All infinite symmetric groups are complete [DM].

Observation 2.3. The following groups are A-rigid:
  I) Finite groups:
    (i) symmetric groups [OW2];
    (ii) simple groups [FS];
    (iii) p-groups of order at most p^4 [KV1];
    (iv) p-groups having a maximal cyclic subgroup [KV2];
    (v) extraspecial [KV2] and almost extraspecial p-groups;
    (vi) p-groups having a cyclic subgroup of index p^2 [KV3], [FN];
    (vii) groups such that the Sylow p-subgroups are cyclic for odd p, and either cyclic, or dihedral, or generalized quaternion for $p = 2$ [He1] (see [Su], [WalC] for a classification of such groups);
    (viii) Blackburn groups [He4], [HL];
    (ix) abelian-by-cyclic groups [HJ];

    (x) primitive supersolvable groups [La];

   (xi) unitriangular matrix groups over $\mathbb{F}_p$ and the quotients of their lower central series [BVY];

  (xii) central products of A-rigid groups [KV2].

The only new case here is that of almost extraspecial groups (see, e.g., [CT] for the definition and classification). In particular, every such group is a central product of an extraspecial group and a cyclic group, so the result follows from (xii).

See [Ya4] for a survey and some details.

II) Infinite groups:

    (i) the absolute Galois group of $\mathbb{Q}$ [Ik];

   (ii) the absolute Galois group of $\mathbb{Q}_p$ [Ik] (or, more generally, of any its finite extension (Ikeda (unpublished), [JR]);

  (iii) non-abelian free groups [Gros, Lemma 1]; goes back to [Ni];

  (iv) non-abelian free profinite groups [Ja];

   (v) so-called pseudo-p-free profinite groups [JR];

  (vi) free nilpotent groups [En];

 (vii) non-abelian free solvable groups [Rom];

(viii) nontrivial free products [Nes1];

  (ix) one-relator groups of the form $\langle a, b | [a^m, b^n] = 1 \rangle$, $m, n > 1$ [TM];

   (x) non-abelian free Burnside groups of large odd exponent [Ch], [At];

  (xi) fundamental groups of compact orientable surfaces [Gros];

  (xii) Artin braid groups B_n and pure braid groups P_n [DG], [Nes2];

 (xiii) connected compact topological groups [McM];

 (xiv) fundamental groups of closed surfaces with negative Euler characteristic [BKZ];

  (xv) non-elementary subgroups H of hyperbolic groups G such that H does not normalize any nontrivial finite subgroup of G [MO, Corollary 5.4];

 (xvi) some groups of automorphisms and birational automorphisms of the plane and space [De1]–[De4], [KS];

(xvii) unitriangular matrix groups over $\mathbb{Q}$ and the quotients of their lower central series, as well unitriangular matrix groups over $\mathbb{Z}$ [BVY];

(xviii) all finitely generated Coxeter groups [CMi].

2.3. Ш-rigid groups.

Observation 2.4. The following groups are Ш-rigid:

    (i) groups appearing on the lists of both Observations 2.1 and 2.3;

   (ii) free groups ([OW1] for finitely generated free groups and [AKT1] in general);

  (iii) groups $\mathrm{SL}(n, R)$, $\mathrm{PSL}(n, R)$ and $\mathrm{GL}(n, R)$ where R is a euclidean domain [On4], [Wad1], [Wad2];

  (iv) free products of at least two nontrivial groups [AKT3];

   (v) amalgamated products $A *_H B$ where H is a maximal cyclic subgroup of A and B [AKT3];

  (vi) all Fuchsian groups $G(n, r, s)$ except, possibly, triangle groups $G(0, 0, 3)$ [AKT3];

 (vii) almost all orientable Seifert groups, except possibly $G_1(0, 3)$ and $G_1(1, 1)$ [AKT4];

(viii) some "polygonal" [Ki] and "tree" products [KT];

(ix) the Cremona group of birational automorphisms of the complex projective plane [De3];

 (x) all torsion-free hyperbolic groups;

(xi) right angled Artin groups [Mi2, Proposition 6.9 and Remark 6.10].

The lists presented above look scattered, so some remarks and questions are in order.

Remark 2.5. In cases (i)–(v), (viii)–(x), (xii), (xiv), (xv) of Observation 2.3.II, the groups under consideration satisfy a stronger rigidity property: every normal automorphism (i.e., an automorphism preserving normal subgroups) is inner (in case (iii) this is proved in [Lub], [Lue], in case (xii) in [Nes2], and in case (xv) in [MO]; results of the latter paper were generalized in [MarMin]). Note that item (x) provides a "counter-example" to the statement made in the last paragraph of the preface. Groups in (i) are even more rigid: according to [Neu], every automorphism of $\mathrm{Gal}(\mathbb{Q})$ is normal, and therefore it is inner.

Remark 2.6. The connectedness assumption in (xiii) is essential in light of the existence of finite groups that are not A-rigid.

Remark 2.7. For topological groups G, another rigidity property is sometimes used. In [CDG] it is proved that if G is a connected linear real reductive Lie group, then every automorphism of G preserving unitary equivalence classes of unitary representations is inner. For compact groups this property is equivalent to be locally inner so one recovers A-rigidity, as in (xiii). Does A-rigidity hold in the noncompact case?

Remark 2.8. A-rigidity of a finitely generated group G, together with its so-called conjugacy separability, implies that the group $\mathrm{Out}(G)$ is residually finite [Gros]. This observation provides an important source of residually finite groups, see [CMi] and references therein for details.

Remark 2.9. Regarding case (xvi) of Observation 2.3.II, let G be one of the following groups: $\mathrm{Aut}\,(\mathbb{A}^2_{\mathbb{C}})$, the affine Cremona group of the polynomial automorphisms of the affine complex plane; $\mathrm{TAut}\,(\mathbb{A}^n_{\mathbb{C}})$, the subgroup of the affine Cremona group consisting of the tame polynomial automorphisms (for $n = 2$ it coincides with the previous one); $\mathrm{Bir}\,(\mathbb{P}^2_{\mathbb{C}})$, the Cremona group of birational automorphisms of the complex projective plane. Then every automorphism of G is inner, up to composition with an automorphism of $\mathbb{C}$ (see [De1], [KS], [De2], respectively). Thus the corresponding groups are all A-rigid. Moreover, in [De3] it was shown that every endomorphism of $\mathrm{Bir}\,(\mathbb{P}^2_{\mathbb{C}})$ is a composition of an inner automorphism with an endomorphism of $\mathbb{C}$, and hence $\mathrm{Bir}\,(\mathbb{P}^2_{\mathbb{C}})$ is III-rigid (a nontrivial endomorphism of $\mathbb{C}$ cannot be locally inner). See [De4] for a survey of other rigidity properties of the Cremona group.

Remark 2.10. Case (x) of Observation 2.4 is a consequence of the cohopfian property for freely indecomposable torsion-free hyperbolic groups (proved in [Sel]) and of [MO, Corollary 5.4]. If the hyperbolic group is freely decomposable, then one can use Observation 2.4(iv). The statement also follows from more general results of [BV]. (I thank the referee for this remark, as well as for pointing out case (xi) of Observation 2.4.)

Remark 2.11. Unitriangular matrix groups over prime fields and the quotients of their lower central series are all Ш-rigid, in view of Observations 2.1(iii), 2.3.I(xi) and 2.3.II(xvii). By [BVY], over a field that is not prime, none of these groups is A-rigid.

Question 2.12. Let $\mathcal{G}$ be a split simple Chevalley group over a prime field k, $\mathcal{B}$ be a Borel subgroup of $\mathcal{G}$, $\mathcal{U}$ be the unipotent radical of $\mathcal{B}$. Let $G = \mathcal{U}(k)$ be the group of k-rational points of $\mathcal{U}$. Is it Ш-rigid? A similar question can be asked in the case where k is replaced with $\mathbb{Z}$.

Remark 2.13. The case of symmetric groups $\mathrm{Sym}(n)$ in Observation 2.3.I(i) is obvious because they all (except for $n = 2$ and $n = 6$) fall into the class of complete groups in view of a classical theorem of Hölder [Ho], and the exceptional cases are easily treated separately. We put this case as a separate item because of the following natural questions:

Question 2.14.   (i) Are all Weyl groups Ш-rigid?
   (ii) What about other Coxeter groups?
   (iii) What about other reflection groups?

A partial answer to (ii) is provided by the paper [CMi], generalizing some earlier results [FH], [HRT], see Observation 2.3.II(xviii). It is applicable to finite Coxeter groups and thus answers (i) in the affirmative. (It would be instructive to get a more elementary independent proof of (i), using results of [Ba].) As to (iii), the case of finite complex reflection groups, where there is a classification due to Shephard and Todd (see, e.g., [Co]) and a nice description of automorphisms [MarMic], also looks tractable.

Remark 2.15. In view of Observations 2.3.I(ii), 2.3.II(xiii) and 2.4(iii), it is natural to ask what can happen if G is the group of points of an arbitrary simple linear algebraic group (or some generalization of such). Note that the proofs of the results recorded above do not admit straightforward generalizations (some of them are purely computational, as in Observation 2.4(iii), and others use methods of functional analysis, as in Observation 2.3.II(xiii), in order to deduce the needed property from a stronger one; the proof of Observation 2.3.I(ii), relying on the classification of automorphisms, can apparently be generalized).

Here are several cases where one can expect Ш-rigidity and that seem tractable.

Conjecture 2.16. *The following groups G are* Ш*-rigid:*
   (i) *$G = \mathcal{G}(k)$, the group of k-points of a split simple Chevalley group $\mathcal{G}$ defined over a sufficiently large field k;*
   (ii) *the same as in (i), with k replaced with some "good" ring;*
   (iii) *the same as in (i), with any isotropic k-group $\mathcal{G}$;*
   (iv) *the same as in (i), with $\mathcal{G}$ an anisotropic group splitting over a quadratic extension of k;*
   (v) *$G \subset \mathcal{G}(k)$ is a "big" subgroup possessing some rigidity properties in the sense of Mostow, Margulis, and others;*
   (vi) *G is a split Kac–Moody group over a sufficiently large field k.*

Here are some comments. Let us start with (i). In this case, here is a sketch of a possible proof of A-rigidity: according to Steinberg, every automorphism is a composition of inner, graph, field and diagonal automorphisms (see [St], [Hum]);

consider graph, field and diagonal automorphisms separately; the first two types should move some semisimple conjugacy class, and the third moves some unipotent class. In case (ii), we have a Steinberg-like classification of automorphisms [Ab], [Bun], [Klj], and can proceed as in case (i). In cases (iii) and (iv), one can use [BT] and [We1], respectively. Groups appearing in (v) were discussed in [We2]. For groups in (vi), a classification of automorphisms is available [Ca], see also [CMu1], [CMu2]; the case of finite ground fields should be similar to [FS]. Perhaps one can also treat unitary forms of Kac–Moody groups over $\mathbb{C}$. As to B-rigidity, one can also use the approach of [BT] (apparently, to treat (iv)–(vi), it is to be generalized in an appropriate way).

Remark 2.17. Trying to extend A-, B-, or Ш-rigidity to other simple (or almost simple) groups, one should not be overoptimistic. The finitary symmetric group $\mathrm{FSym}(\Omega)$, where Ω is an infinite countable set, is not B-rigid [AE]. Similar arguments can be used to show that the finitary alternating group $\mathrm{FAlt}(\Omega)$ (which is simple) is not B-rigid either. The group of automorphisms of each of these two groups is isomorphic to the infinite symmetric group $\mathrm{Sym}(\Omega)$ (see, e.g., [DM]), it contains locally inner outer automorphisms (conjugation by any element of $\mathrm{Sym}(\Omega)$ applied to a finitary permutation gives the same result as conjugation by some finitary permutation), hence none of these groups is A-rigid.

2.4. **Genericity of rigid groups.** After walking around the zoo of rigid groups in the previous sections, we would like to address the following question: are the species we described there rare or common? How probable is to meet any of them in wildlife? We will formulate an answer as a vague principle.

Principle 2.18. A reasonable property of a reasonable mathematical object lying inside a reasonable class of objects may not hold but it will hold at least for an object in general position (if not always), provided the class under consideration is enlarged or restricted, if necessary, in an appropriate way.

In loose terms, this means that the probability of the event that a randomly picked animal is a penguin, will be much higher if the samples allowed to test are restrictively placed within the Antarctic region.

Anyone can find lots of examples confirming this principle, looking at his/her favourite area of mathematics. Here is an example from number theory. Consider the following property of finite Galois extensions L/K of number fields: every element of K which is a norm everywhere locally is a norm globally (the Hasse principle). This property fails to hold for most Galois extensions. However, if we restrict our attention to *cyclic* extensions, it always holds (Hasse). In another direction, if we allow L/K to vary among all finite extensions (not necessarily normal), the Hasse principle holds in general position, i.e., for an extension L/K of degree n such that the Galois group $\mathrm{Gal}(M/K)$ of the normal closure M of L is the symmetric group $\mathrm{Sym}(n)$ [VK]. More generally, the Hasse principle for principal homogeneous spaces of algebraic tori holds for generic maximal tori in simple algebraic groups [VK], [Kl].

Some instances of Principle 2.18 in the context of Ш-rigidity will be given below. Given a class of groups G, it is usually a challenging conceptual task to enlarge or restrict it in the spirit of this principle. Needless to add that to get a meaningful mathematical statement, one has to convert expressions such as "general position", "random", "generic", "typical", and similar euphemisms, into a precise definition.

Such a goal is far beyond the framework of the present article, and the interested reader is referred to relevant literature (see, e.g., [Grom1], [Grom2], [Ols], [Oll], [KS1], [KS2] and references therein) for different approaches to genericity.

Let us now go over to more concrete considerations.

Examples 2.19.   (i) Let $\mathcal{FP}$ denote the class of finite primitive permutation groups. Then the class of Ш-rigid groups is generic within $\mathcal{FP}$. Indeed, according to [LP], a random element of the symmetric group $\mathrm{Sym}(n)$ is contained in a primitive group other than $\mathrm{Sym}(n)$ or $\mathrm{Alt}(n)$ with probability tending to zero as $n \to \infty$, and it remains to refer to Observations 2.3.I(i) and (ii). According to the same paper [LP], this statement remains true if one considers the class of finite transitive permutation groups.

(ii) Let $\mathcal{L}$ denote the class of linear groups over fields. Then the class of Ш-rigid groups is generic within $\mathcal{L}$. This follows from Observation 2.4(ii) because a random linear group is free [Ao].

(iii) In a similar spirit, in [KS2] it was shown that "random" quotients of the modular group $PSL(2, \mathbb{Z})$ are complete and cohopfian, hence Ш-rigid.

(iv) In view of overwhelming genericity of torsion-free hyperbolic groups (see, e.g., [Ols], [Oll, Chapter I]), Observation 2.4(x) shows that the class of Ш-rigid groups is generic within the class of all groups.

We finish this series of examples by a quote from [KS2]:

'It seems likely that "endomorphism rigidity" is another general aspect of "randomness": A random structure should not have any endomorphisms except those absolutely required by the nature of the structure.'

This quote can be rephrased in the context of rigidity properties considered above. Namely, the classes of A-, B-, and even Ш-rigid groups are much broader than most classically known classes of rigid groups: say, there are A-rigid groups that are not complete (and may be very far from complete, for example, free groups) and B-rigid groups that are not cohopfian. However, Examples 2.19 show that Ш-rigidity is another general aspect of "randomness": A random group should not have any endomorphisms that behave locally as conjugations except those that behave in such a way globally. In other words, the group-theoretic "Hasse principle", introduced in Definitions 1.1 and 1.2, should hold generically.

3. Properties of Ш(G)

Case 1. Finite groups

Recall that in this case Ш(G) is a group. It coincides with $\mathrm{Out}_c(G)$, the group of locally inner outer automorphisms.

Observation 3.1.   (i) The group Ш(G) is solvable [Sah].

(ii) The group Ш(G) may be non-abelian [Sah].

(iii) If G is solvable and each chief factor is complemented, then Ш(G) is supersolvable [La].

(iv) If G is supersolvable and its Frattini subgroup is trivial, then Ш(G) is nilpotent [La].

(v) If G is nilpotent of class c, then Ш(G) is nilpotent of class at most $c - 1$ [Sah].

(vi) Ш(G) does not depend of the isoclinism class of G [Ya3].

Remark 3.2. Observation 3.1(ii) disproves an assertion made in [Bur1]. The smallest known counter-example is of order 2^{15} (in general, the construction gives, for each prime-power q, a group of order $q^{5m}, m \geq 3$).

Remark 3.3.   (i) First examples of groups that are not Ш-rigid appeared among p-groups, they go back to Burnside [Bur2]. The smallest group that is not Ш-rigid is of order 32 (see [WalG]); a classification of groups of order p^5 with nontrivial $\text{Ш}(G)$ was obtained in [Ya3] (and made more precise in [Ya4]). More examples of non-Ш-rigid p-groups were found in [Ya1], [Ya4] among so-called Camina groups. See [Ya2] for some bounds for the order of $\text{Ш}(G)$.
   (ii) Looking beyond p-groups, a classification of "minimal" groups G with $\text{Ш}(G) \neq 1$ in the class of solvable groups all of whose Sylow subgroups are abelian was obtained in [He1].
  (iii) See [BVY] for some matrix counter-examples. Some general constructions of a similar spirit can be found in [Sah], [Sz].

Case 2. Infinite groups

Here much less is known.

The group $\text{Out}_c(G)$ is finite if G is hyperbolic [MS1], or, more generally, relatively hyperbolic [MS2].

The following observation attributed to Passman (see the introduction to [Sah]) implies that $\text{Out}_c(G)$ may be an infinite simple group: this happens for $G = \text{FSym}(\Omega)$, in which case $\text{Out}_c(G)$ is isomorphic to the quotient $\text{Sym}(\Omega)/\text{FSym}(\Omega)$. (Note that this is in sharp contrast with Observation 3.1(i).)

In general, from [Mi1] it follows that $\text{Out}_c(G)$ can be any countable group. Hence the set $\text{Ш}(G)$ can be as large as possible. In the examples of [Mi1], G is finitely generated (in fact, 2-generated), satisfies Kazhdan's property (T) and has exactly two conjugacy classes (including 1).

A possible conceptually interesting question is to understand the situation within some natural classes of groups where such pathologies do not arise. Perhaps, the first class to be considered is that of compact groups.

4. Shafarevich–Tate set vs. Bogomolov multiplier

Throughout this section, unless otherwise stated, G is a finite group and k is an algebraically closed field of characteritic zero. Recall that the Bogomolov multiplier $B_0(G)$ is defined as the subgroup of the Schur multiplier $H^2(G, \mathbb{Q}/\mathbb{Z})$ consisting of the cocycles becoming trivial after restricting to all abelian (or, equivalently, bicyclic) subgroup of G [Bo]. This group coincides with the so-called unramified Brauer group of the quotient V/G, where V is a vector k-space equipped with a faithful, linear, generically free action of G. The latter group is an important birational invariant of V/G, in particular, it equals zero whenever the variety V/G is k-rational (or even retract k-rational). It was introduced by Saltman and used in constructing a counter-example to Noether's problem [Sal]. The Bogomolov multiplier allows one to compute this group solely in terms of G.

Observation 4.1.   (i) In many cases, the Bogomolov multiplier of Ш-rigid groups is zero. This is true at least for the groups listed in items (i)–(ix) of Observation 2.3.I. Moreover, for some of these groups the corresponding varieties V/G are rational (or at least, retract rational). Indeed, in case (i)

rationality is classically known (E. Noether), it follows from the theorem
on elementary symmetric functions. This is also known in cases (iii) [CK],
(iv) [HuKa], (vi) [Ka1]. In case (ix) one knows retract rationality [Ka2].
The Bogomolov multiplier is zero in case (ii) [Ku]. Cases (v), (vii), (viii)
were treated in [KK].

(ii) $\mathrm{III}(G)$ is invariant under isoclinism, and so is $B_0(G)$: if G_1 and G_2 are
isoclinic (i.e., have isomorphic quotients $G_i/Z(G_i)$ and derived subgroups
$[G_i, G_i]$, and these isomorphisms are compatible), then we have $\mathrm{III}(G_1) \cong$
$\mathrm{III}(G_2)$ and $B_0(G_1) \cong B_0(G_2)$. For $\mathrm{III}(G)$ we recall Observation 3.1(vi),
and for $B_0(G)$ this was proved in [Mo2]; moreover, in [BB] it was proved
that if G_1 and G_2 are isoclinic, then the corresponding linear generically free
quotients V/G_1 and V/G_2 are stably birationally equivalent, thus answering
questions posed in [HKK] in the affirmative.

(iii) There are examples of groups G that are not III-rigid but $B_0(G) = 0$. Such
examples can be found among groups of order p^5: for $p = 2$ one always has
$B_0(G) = 0$ [CHKP] but there is a group with $\mathrm{III}(G) \neq 1$ [WalG]; for every
$p \geq 3$ there is an isoclinism family (denoted Φ_7) for which $\mathrm{III}(G) \neq 1$ [Ya4]
but $B_0(G) = 0$ [HKK], [Mo1]. Note, however, that although $B_0(G) = 0$ for
all groups G of order 32, such a group can give rise to a homogeneous space
$X = SL_n/G$ defined over a *finite* field k so that the unramified Brauer
group of X is not zero [BDH]. Thus one may pursue in giving $\mathrm{III}(G)$ some
birational flavour.

One can try to proceed along this empiric line even further, asking the following
questions.

Question 4.2.   (i) Do the groups listed in Observations 2.3.I(x),(xi) satisfy
$B_0(G) = 0$?

(ii) Let $G = G_1 * G_2$ be a central product of groups such that $B_0(G_1) =$
$B_0(G_2) = 0$. Is it true that $B_0(G) = 0$?

(ii′) Let $G = G_1 * G_2$ be a central product of groups such that the corresponding
generically free linear quotients V_1/G_1 and V_2/G_2 are stably rational. Is it
true that so is V/G?

(iii) Do there exist III-rigid groups with nonzero Bogomolov multiplier?

It is a tempting task to find a conceptual explanation of the experimental data
presented above. One can try to use a hint given in [GK], where the group
$\mathrm{III}(G)$ was naturally embedded into the so-called second lazy cohomology group
$H^2_\ell(O_k(G))$ of the Hopf algebra of k-valued functions on G. This object can be
viewed as a far-reaching noncommutative generalization of the Schur multiplier of
G (and coincides with it in the case where G is abelian). The construction of the
embedding is far from obvious: first, the lazy cohomology is identified with the
group of equivalence classes of invariant Drinfeld twists on $k[G]$, and then this lat-
ter group is mapped to a pointed set $\mathcal{B}(G)$ consisting of the pairs (A, b), where
A runs over normal abelian subgroups of G and b is a k^*-valued G-invariant non-
degenerate alternating bilinear form on the Pontryagin dual $\hat{A}$. The fibre at the
pointed element is then identified with $\mathrm{III}(G)$.

Of course, this construction cannot directly be used to reveal an eventual re-
lationship between $\mathrm{III}(G)$ and $B_0(G)$ (say, because the lazy cohomology is not

invariant under isoclinism). However, a more thorough exploration does not seem completely hopeless.

Moreover, one can go beyond finite groups and try to embed an appropriate part of $Ш(G)$ (say, the group $\mathrm{Out_c}(G)$ of locally inner outer automorphisms) into an appropriate cohomology group. In the case where G is compact, one can use an approach presented in [NT].

5. Miscellaneous applications, ramifications, and generalizations

As mentioned in the preface, nonrigid groups often lead to interesting counter-examples. We list several instances of such a phenomenon. Throughout, unless otherwise stated, G is a finite group.

Observation 5.1.   (i) Formanek showed [Fo] that if $Ш(G) \neq 1$ and φ is an appropriately chosen locally inner outer automorphism of G, then the semidirect product $G_1 := G \rtimes \langle \varphi \rangle$ provides a counter-example to a conjecture by Roth [Rot], asserting that each irreducible representation of $G/Z(G)$ is a subrepresentation of the conjugation representation $G/Z(G)$ on $\mathbb{C}[G]$; see [KM] for minimal counter-examples in the class of p-groups. On the other hand, the Roth property turned out to be related to a recently developed "Lie theory" and "noncommutative differential geometry" on finite groups [LMR]. Does there exist any direct connection between $Ш(G)$ and these new theories?

  (ii) The smallest non-$Ш$-rigid group of order 32 constructed in [WalG], as well as more complicated examples in [He1], played a crucial role in refuting some long-standing conjectures in the theory of integral group rings, including Higman's isomorphism problem for integral group rings of finite groups (see [He2], [He4]). Some other interesting examples, based on [Sah], were obtained in [PW].

(iii) Examples of nilpotent Lie groups with $Ш(G) \neq 1$ were used in [GW] to construct compact Riemannian manifolds that are isospectral but not isometric.

(iv) Apart from $Ш(G)$, there are other local-global invariants of a similar flavour. One of such is related to the notion of Coleman automorphism [He3], [HeKi]: this is an automorphism of G that becomes inner after restricting to each Sylow subgroup of G. An argument attributed to Ph. Gille (see [Pa]) shows that any group admitting a non-inner Coleman automorphism provides an example of a principal homogeneous space defined over a number field that has a rational zero-cycle of degree one (=has rational points in extensions of coprime degrees) but has no rational points. Similar arguments were used in [GGHZ] to construct non-isomorphic curves becoming isomorphic over extensions of coprime degrees. The existence of a principal homogeneous space of a *connected* linear algebraic group over an arbitrary field with the same property as above (with a rational zero-cycle of degree one and without rational points) is an open problem (known as Serre's conjecture).

Remark 5.2. As shown in the papers [CMi], [BV] cited above, some classes of $Ш$-rigid groups G are rigid in an even stronger sense: any endomorphism of G which preserves the conjugacy classes of all elements of *short length* (in some suitable metric) must be inner. This observation gives rise to a number of testability

problems of the following flavour. We say that a subset S of a Ш-rigid group G is *a test subset* if the following holds: an endomorphism φ of G is inner if and only if $\varphi(s)$ is conjugate to s for all $s \in S$. Then one can ask about the existence of "small" test subsets. Can one choose S computable in some reasonable sense? finite? one-element? See [BV] for relevant discussions.

Finally, one can introduce a local-global invariant more general than Ш(G).

Definition 5.3. Let G be a group, let $\Gamma \leq G$ be a subgroup acting on G by conjugation, and let $H^1(\Gamma, G)$ denote the first cohomology pointed set. Let $\mathcal{F}$ be a family of subgroups of Γ. We define

$$\text{Ш}_{\mathcal{F}}(\Gamma, G) := \ker[H^1(\Gamma, G) \to \prod_{\Gamma' \in \mathcal{F}} H^1(\Gamma', G)]$$

and call it the Shafarevich–Tate set with respect to $\mathcal{F}$.

In particular, if $\Gamma = G$, we abbreviate Ш$_{\mathcal{F}}(G, G)$ to Ш$_{\mathcal{F}}(G)$.

With this terminology, if G is finite then the solvable group Ш(G) has several natural subgroups: Ш$_{\mathcal{A}}(G)$, Ш$_{\mathcal{N}}(G)$, Ш$_{\mathcal{S}}(G)$, where $\mathcal{A}$, $\mathcal{N}$ and $\mathcal{S}$ stand for the family of all abelian, nilpotent and Sylow subgroups of G respectively. The first one is nilpotent (of class at most 2) [Da3], the second one is abelian [Da3], and the third one is abelian too [HeKi] (earlier Dade [Da1] proved that it is solvable [Da1] and then that it is nilpotent [Da4]); in all known examples the first group is abelian too.

Using a set-up similar to Definition 5.3, D. Segal treated an arithmetic local-global problem of equivalence of binary forms [Seg], and T. Ono studied twists of hyperelliptic curves [On5], [On6].

The reader is welcome to provide more applications and interrelations.

Bibliographical remark. In the list of references below, M. Kumar in [KV1]–[KV3] and M. K. Yadav in [BVY], [Ya1]–[Ya4] refer to the same person.

References

[AE]   A. Abdollahi, G. Endimioni, On conditions for an endomorphism to be an automorphism, Algebra Colloq. 12 (2005), 709–714.

[Ab]   E. Abe, Automorphisms of Chevalley groups over commutative rings, Algebra i Analiz 5 (1993), no. 2, 74–90; English original in St. Petersburg Math. J. 5 (1994), 287–300.

[AKT1]   R. B. J. T. Allenby, G. Kim, C. Y. Tang, Residual finiteness of outer automorphism groups of certain pinched 1-relator groups, J. Algebra 246 (2001), 849–858.

[AKT2]   R. B. J. T. Allenby, G. Kim, C. Y. Tang, On the residual finiteness of Out($\pi_1(M)$) of certain Seifert manifolds, Algebra Colloq. 10 (2003), 121–126.

[AKT3]   R. B. J. T. Allenby, G. Kim, C. Y. Tang, Residual finiteness of outer automorphism groups of finitely generated non-triangle Fuchsian groups, Internat. J. Algebra Computation 15 (2005), 59–72.

[AKT4]   R. B. J. T. Allenby, G. Kim, C. Y. Tang, Outer automorphism groups of certain ori-
         entable Seifert 3-manifold groups, in: "Combinatorial group theory, discrete groups,
         and number theory", Contemp. Math., vol. 421, Amer. Math. Soc., Providence, RI,
         2006, pp. 15–22.

[Ao]   F. Aoun, Random subgroups of linear groups are free, Duke Math. J. 160 (2011), 117–
         173.

[At]   V. S. Atabekyan, Normal automorphisms of free Burnside groups, Izv. Ross. Akad.
         Nauk Ser. Mat. 75 (2011), 3–18; English transl. in Izv. Math. 75 (2011), 223–237.

[Ba]   E. Bannai, Automorphisms of irreducible Weyl groups, J. Fac. Sci. Univ. Tokyo Sect. I
         16 (1969), 273–286.

[BVY]   V. Bardakov, A. Vesnin, M. K. Yadav, Class preserving automorphisms of unitriangular
         groups, Internat. J. Algebra Computation 22 (2012), no. 3, 12500233, 17 pp.

[Beleg] I. Belegradek, On co-Hopfian nilpotent groups, Bull. London Math. Soc. 35 (2003),
         805–811.

[BM]   R. W. Bell, D. Margalit, Braid groups and the co-Hopfian property, J. Algebra 303
         (2006), 275–294.

[Belli] P. Bellingeri, On automorphisms of surface braid groups, J. Knot Theory Ramifications
         17 (2008), 1–11.

[Bo]   F. A. Bogomolov, The Brauer group of quotient spaces by linear group actions, Izv.
         Akad. Nauk SSSR Ser. Mat. 51 (1987), 485–516; English transl. in Math. USSR Izv. 30
         (1988), 455–485.

[BB]   F. Bogomolov, C. Böhning, Isoclinism and stable cohomology of wreath products,
         arXiv:1204.4747.

[BKZ]   O. Bogopolski, E. Kudryavtseva, H. Zieschang, Simple curves on surfaces and an analog
         of a theorem of Magnus for surface groups, Math. Z. 247 (2004), 595–609.

[BV]   O. Bogopolski, E. Ventura, On endomorphisms of torsion-free hyperbolic groups, Inter-
         nat. J. Algebra Comput. 21 (2011), 1415–1446.

[BT]   A. Borel, J. Tits, Homomorphismes "abstraits" de groupes algébriques simples, Ann.
         Math. 97 (1973), 499–571.

[BDH]   M. Borovoi, C. Demarche, D. Harari, Complexes de groupes de type multiplicatif et
         groupe de Brauer non ramifié des espaces homogènes, to appear in Ann. Sci. École
         Norm. Sup.

[Bun]   E. Bunina, Automorphisms of Chevalley groups of different types over commutative
         rings, arXiv:1108.0529.

[Bur1]   W. Burnside, Theory of groups of finite order, 2nd ed., Cambridge Univ. Press, Cam-
         bridge, 1911; reprinted by Dover Publications, New York, 1955.

[Bur2]   W. Burnside, On the outer automorphisms of a group, Proc. London Math. Soc. (2) 11
         (1913), 40–42.

[Ca]   P.-E. Caprace, "Abstract" isomorphisms of split Kac–Moody groups, Mem. Amer.
         Math. Soc., vol. 198, no. 924, Amer. Math. Soc., Providence, RI, 2009.

[CMi]   P.-E. Caprace, A. Minasyan, On conjugacy separability of some Coxeter groups and
         parabolic-preserving automorphisms, arXiv:1210.4328.

[CMu1]   P.-E. Caprace, B. Mühlherr, Isomorphisms of Kac–Moody groups, Invent. Math. 161
         (2005), 361–388.

[CMu2]   P.-E. Caprace, B. Mühlherr, Isomorphisms of Kac–Moody groups which preserve
         bounded subgroups, Adv. Math. 206 (2006), 250–278.

[CT]   J. F. Carlson, J. Thévenaz, Torsion endo-trivial modules, Algebr. Represent. Theory 3
         (2000), 303–335.

[Ch]   E. A. Cherepanov, Normal automorphisms of free Burnside groups of large odd expo-
         nents, Internat. J. Algebra Computation 16 (2006), 839–847.

[CHKP]   H. Chu, S.-J. Hu, M. Kang, Yu. G. Prokhorov, Noether's problem for groups of order
         32, J. Algebra 320 (2008), 3022–3035.

[CK]   H. Chu, M. Kang, Rationality of p-group actions, J. Algebra 237 (2001), 673–690.

[Co]   A. M. Cohen, Finite complex reflection groups, Ann. Sci. École Norm. Sup. 9 (1976),
         379–436; 11 (1978), 613.

[CDG]   R. Conti, C. D'Antoni, L. Geatti, Group automorphisms preserving equivalence classes
         of unitary representations, Forum Math. 16 (2004), 483–503.

[Da1]   E. C. Dade, Automorphismes extérieurs centralisant tout sous-groupe de Sylow, Math.
          Z. 117 (1970), 35–40.
[Da2]   E. C. Dade, Outer automorphisms centralizing every nilpotent subgroup of a finite
          group, Math. Z. 130 (1973), 1–18.
[Da3]   E. C. Dade, Outer automorphisms centralizing every abelian subgroup of a finite group,
          Math. Z. 137 (1974), 93–127.
[Da4]   E. C. Dade, Sylow-centralizing sections of outer automorphism groups of finite groups
          are nilpotent, Math. Z. 141 (1975), 57–76.
[DP]   T. Delzant, L. Potyagailo, Endomorphisms of Kleinian groups, Geom. Funct. Anal. 13
          (2003), 396–436.
[De1]   J. Déserti, Sur le groupe des automorphismes polynomiaux du plan affine, J. Algebra
          297 (2006), 584–599.
[De2]   J. Déserti, Sur les automorphismes du groupe de Cremona, Compos. Math. 142 (2006),
          1459–1478.
[De3]   J. Déserti, Le groupe de Cremona est hopfien, C. R. Acad. Sci. Paris Sér. Math. 344
          (2007), 153–156.
[De4]   J. Déserti, Some properties of the Cremona group, Ensaios Matemáticos, vol. 21, Soc.
          Brasileira Mat., Rio de Janeiro, 2012.
[DM]   J. D. Dixon, B. Mortimer, Permutation groups, Grad. Texts Math., vol. 163, Springer-
          Verlag, New York, 1996.
[Dy]   J. L. Dyer, Automorphism sequences of integer unimodular groups, Illinois J. Math. 22
          (1978), 1–30.
[DF1]   J. L. Dyer, E. Formanek, The automorphism group of a free group is complete, J.
          London Math. Soc. 11 (1975), 181–190.
[DF2]   J. L. Dyer, E. Formanek, Automorphism sequences of free nilpotent groups of class two,
          Math. Proc. Cambridge Phil. Soc. 79 (1976), 271–279.
[DF3]   J. L. Dyer, E. Formanek, Characteristic subgroups and complete automorphism groups,
          Amer. J. Math. 99 (1977), 713–753.
[DG]   J. L. Dyer, E. K. Grossman, The automorphism groups of the braid groups, Amer. J.
          Math. 103 (1981), 1151–1169.
[En]   G. Endimioni, Pointwise inner automorphisms in a free nilpotent group, Quart. J. Math.
          53 (2002), 397–402.
[FS]   W. Feit, G. M. Seitz, On finite rational groups and related topics, Illinois J. Math. 33
          (1989), 103–131.
[Fo]   E. Formanek, The conjugation representation and fusionless extensions, Proc. Amer.
          Math. Soc. 30 (1971), 73–74.
[FH]   W. N. Franzsen, R. B. Howlett, Automorphisms of nearly finite Coxeter groups, Adv.
          Geom. 3 (2003), 301–338.
[FN]   M. Fuma, Y. Ninomiya, "Hasse principle" for finite p-groups with cyclic subgroups of
          index p^2, Math. J. Okayama Univ. 46 (2004), 31–38.
[GGHZ]   D. Goldstein, R. M. Guralnick, E. W. Howe, M. E. Zieve, Nonisomorphic curves that
          become isomorphic over extensions of coprime degrees, J. Algebra 320 (2008), 2526–
          2558.
[GW]   C. S. Gordon, E. N. Wilson, Isospectral deformations of compact solvmanifolds, J. Diff.
          Geom. 19 (1984), 241–256.
[Grom1]   M. Gromov, Hyperbolic groups, in: "Essays in group theory", Math. Sci. Res. Inst.
          Publ., vol. 8, Springer, New York, 1987, pp. 75–263.
[Grom2]   M. Gromov, Random walk in random groups, Geom. Funct. Anal. 13 (2003), 73–146.
[Gros]   E. K. Grossman, On the residual finiteness of certain mapping class groups, J. London
          Math. Soc. 9 (1974), 160–164.
[GK]   P. Guillot, C. Kassel, Cohomology of invariant Drinfeld twists on group algebras, Intern.
          Math. Res. Notices 2010 (2010), 1894–1939.
[HL]   A. Herman, Y. Li, Class preserving automorphisms of Blackburn groups, J. Australian
          Math. Soc. 80 (2006), 351–358.
[He1]   M. Hertweck, Class-preserving automorphisms of finite groups, J. Algebra 241 (2001),
          1–26.
[He2]   M. Hertweck, A counterexample to the isomorphism problem for integral group rings,
          Ann. Math. 154 (2001), 115–138.

[He3]   M. Hertweck, Class-preserving Coleman automorphisms of finite groups, Monatsh. Math. 136 (2002), 1–7.

[He4]   M. Hertweck, Contributions to the integral representation theory of groups, Habilitationsschrift, Univ. Stuttgart, 2004, available online at `http://elib.unistuttgart.de/opus/volltexte/2004/1638/2003`.

[HJ]   M. Hertweck, E. Jespers, Class-preserving automorphisms and the normalizer property for Blackburn groups, J. Group Theory 12 (2009), 157–169.

[HeKi]   M. Hertweck, W. Kimmerle, Coleman automorphisms of finite groups, Math. Z. 242 (2002), 203–215.

[HKK]   A. Hoshi, M. Kang, B. E. Kunyavskii, Noether's problem and unramified Brauer groups, to appear in Asian J. Math.

[HRT]   R. B. Howlett, P. J. Rowley, D. E. Taylor, On outer automorphism groups of Coxeter groups, Manuscripta Math. 93 (1997), 499–513.

[Ho]   O. Hölder, Bildung zusammengesetzer Gruppen, Math. Ann. 46 (1895), 321–422.

[HuKa]   S.-J. Hu, M. Kang, Noether's problem for some p-groups, in: "Cohomological and geometric approaches to rationality problems", Progr. Math., vol. 282, Birkhäuser, Boston, MA, 2010, pp. 149–162.

[Hum]   J. E. Humphreys, On the automorphisms of infinite Chevalley groups, Canad. J. Math. 21 (1969), 908–911.

[Ik]   M. Ikeda, Completeness of the absolute Galois group of the rational number field, J. reine angew. Math. 291 (1977), 1–22.

[Ja]   M. Jarden, Normal automorphisms of free profinite groups, J. Algebra 62 (1980), 118–123.

[JR]   M. Jarden, J. Ritter, Normal automorphisms of absolute Galois groups of $\mathfrak{p}$-adic fields, Duke Math. J. 47 (1980), 47–56.

[Ka1]   M. Kang, Noether's problem for p-groups with a cyclic subgroup of index p^2, Adv. Math. 226 (2011), 218–234.

[Ka2]   M. Kang, Retract rational fields, J. Algebra 349 (2012), 22–37.

[KK]   M. Kang, B. Kunyavskiĭ, The Bogomolov multiplier of rigid finite groups, `arXiv:1304.2691`.

[KM]   E. Kaniuth, A. Markfort, Irreducible subrepresentations of the conjugation representation of finite p-groups, Manuscripta Math. 74 (1992), 161–175.

[KS1]   I. Kapovich, P. Schupp, On group-theoretic models of randomness and genericity, Groups Geom. Dyn. 2 (2008), 383–404.

[KS2]   I. Kapovich, P. Schupp, Random quotients of the modular group are rigid and essentially incompressible, J. reine angew. Math. 628 (2009), 91–119.

[Ki]   G. Kim, Outer automorphism groups of certain polygonal products of groups, Bull. Korean Math. Soc. 45 (2008), 45–52.

[KT]   G. Kim, C. Y. Tang, Outer automorphism groups of polygonal products of certain conjugacy separable groups, J. Korean Math. Soc. 45 (2008), 1741–1752.

[Kl]   Al. A. Klyachko, Tori without affect in semisimple groups, in: "Arithmetic and geometry of varieties", Kuĭbyshev State Univ., Kuĭbyshev, 1989, pp. 67–78. (Russian)

[Klj]   An. A. Klyachko, Automorphisms and isomorphisms of Chevalley groups and algebras, J. Algebra 324 (2010), 2608–2619.

[KS]   H. Kraft, I. Stampfli, On automorphisms of affine Cremona groups, to appear in Ann. Inst. Fourier.

[KV1]   M. Kumar, L. R. Vermani, "Hasse principle" for extraspecial p-groups, Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), 123–125.

[KV2]   M. Kumar, L. R. Vermani, "Hasse principle" for groups of order p^4, Proc. Japan Acad. Ser. A Math. Sci. 77 (2001), 95–98.

[KV3]   M. Kumar, L. R. Vermani, On automorphisms of some p-groups, Proc. Japan Acad. Ser. A Math. Sci. 78 (2002), 46–50.

[Ku]   B. E. Kunyavskii, The Bogomolov multiplier of finite simple groups, in: "Cohomological and geometric approaches to rationality problems", Progr. Math., vol. 282, Birkhäuser, Boston, MA, 2010, pp. 209–217.

[La]   R. Laue, On outer automorphism groups, Math. Z. 148 (1976), 177–188.

[LMR]   J. López Peña, S. Majid, K. Rietsch, Lie theory of finite simple groups and the Roth property, `arXiv:1003.5611`.

[Lub]   A. Lubotzky, Normal automorphisms of free groups, J. Algebra 63 (1980), 494–498.

[LP]   T. Łuczak, L. Pyber, On random generation of the symmetric group, Combin. Probab. Comput. 2 (1993), 505–512.

[Lue]   A. S.-T. Lue, Normal automorphisms of free groups, J. Algebra 64 (1980), 52–53.

[MarMic]   I. Marin, J. Michel, Automorphisms of complex reflection groups, Represent. Theory 14 (2010), 747–788.

[MarMin]   A. Martino, A. Minasyan, Conjugacy in normal subgroups of hyperbolic groups, Forum Math. 24 (2012), 889–909.

[McM]   J. R. McMullen, On the dual object of a compact connected group, Math. Z. 185 (1984), 539–552.

[MS1]   V. Metaftsis, M. Sykiotis, On the residual finiteness of outer automorphisms of hyperbolic groups, Geom. Dedicata 117 (2006), 125–131.

[MS2]   V. Metaftsis, M. Sykiotis, On the residual finiteness of outer automorphisms of relatively hyperbolic groups, J. Pure Appl. Algebra 214 (2010), 1301–1305.

[Mi1]   A. Minasyan, Groups with finitely many conjugacy classes and their automorphisms, Comment. Math. Helv. 84 (2009), 259–296.

[Mi2]   A. Minasyan, Hereditary conjugacy separability of right angled Artin groups and its applications, Groups Geom. Dyn. 6 (2012), 335–388.

[MO]   A. Minasyan, D. Osin, Normal automorphisms of relatively hyperbolic groups, Trans. Amer. Math. Soc. 362 (2010), 6079–6103.

[Mo1]   P. Moravec, Groups of order p^5 and their unramified Brauer groups, J. Algebra 372 (2012), 320–327.

[Mo2]   P. Moravec, Unramified Brauer groups and isoclinism, `arXiv:1203.2422`.

[Nes1]   M. V. Neshchadim, Free products of groups that do not have outer normal automorphisms, Algebra i Logika 35 (1996), 562–566; English transl. in Algebra and Logic 35 (1996), 316–318.

[Nes2]   M. V. Neshchadim, On absence of outer normal automorphisms in some groups of automorphisms, in: "Group and metric properties of mappings", Novosibirsk, NGU, 1995, pp. 48–62. (Russian)

[NT]   S. Neshveyev, L. Tuset, On second cohomology of duals of compact groups, Internat. J. Math. 22 (2011), 1231–1260.

[Neu]   J. Neukirch, Kennzeichnung der p-adischen und der endlichen algebraischen Zahlkörper, Invent. Math. 6 (1969), 296–314.

[Ni]   J. Nielsen, Die Isomorphismengruppe der freien Gruppen, Math. Ann. 91 (1924), 169–209.

[OP]   K. Ohshika, L. Potyagailo, Self-embeddings of Kleinian groups, Ann. Sci. École Norm. Sup. 31 (1998), 329–343.

[Oll]   Y. Ollivier, A January 2005 invitation to random groups, Ensaios Matemáticos, vol. 10, Soc. Brasileira Mat., Rio de Janeiro, 2005.

[Ols]   A. Yu. Ol'shanskiĭ, Almost every group is hyperbolic, Internat. J. Algebra Comput. 2 (1992), 1–17.

[On1]   T. Ono, A note on Shafarevich–Tate sets for finite groups, Proc. Japan Acad. Ser. A Math. Sci. 74 (1998), 77–79.

[On2]   T. Ono, "Hasse principle" for $\mathrm{PSL}_2(\mathbf{Z})$ and $\mathrm{PSL}_2(\mathbf{F}_p)$, Proc. Japan Acad. Ser. A Math. Sci. 74 (1998), 130–131.

[On3]   T. Ono, "Shafarevich–Tate sets" for profinite groups, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 96–97.

[On4]   T. Ono, "Hasse principle" for $\mathrm{GL}_2(D)$, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 141–142.

[On5]   T. Ono, Shafarevich–Tate set for $y^4 = x^4 - l^2$, Turkish J. Math. 23 (1999), 557–573.

[On6]   T. Ono, On Shafarevich–Tate sets, in: "Class field theory—its centenary and prospect (Tokyo, 1998)", Adv. Stud. Pure Math., vol. 30, Math. Soc. Japan, Tokyo, 2001, pp. 537–547.

[OW1]   T. Ono, H. Wada, "Hasse principle" for free groups, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 1–2.

[OW2]   T. Ono, H. Wada, "Hasse principle" for symmetric and alternating groups, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 61–62.

[Pa]   R. Parimala, Homogeneous varieties—zero-cycles of degree one versus rational points, Asian J. Math. 9 (2005), 251–256.

[PW]   D. S. Passman, P. Wauters, Semi-invariants and weights of group algebras of finite groups, Proc. Amer. Math. Soc. 127 (1999), 1323–1329.

[PV]   L. Potyagaĭlo, Sh. Van, On the co-Hopficity of 3-manifold groups, Algebra i Analiz 11 (1999), no. 5, 194–220; English transl. in St. Petersburg Math. J. 11 (2000), 861–881.

[Pr]   G. Prasad, Discrete subgroups isomorphic to lattices in semisimple Lie groups, Amer. J. Math. 98 (1976), 241–261.

[RS]   E. Rips, Z. Sela, Structure and rigidity in hyperbolic groups I, Geom. Funct. Anal. 4 (1994), 337–371.

[Rob]   D. J. S. Robinson, Recent results on finite complete groups, in: "Algebra Carbondale 1980", Lecture Notes Math., vol. 848, Springer, Berlin, 1981, pp. 178–185.

[Rom]   V. A. Roman'kov, Normal automorphisms of discrete groups, Sibirsk. Mat. Zh. 24 (1983), no. 4, 138–149; English transl. in Siberian Math. J. 24 (1983), 604–614.

[Rot]   R. L. Roth, On the conjugating representation of a finite group, Pacific J. Math. 36 (1971), 515–521.

[Sah]   C. Sah, Automorphisms of finite groups, J. Algebra 10 (1968), 47–68; 44 (1977), 573–575.

[Sal]   D. J. Saltman, Noether's problem over an algebraically closed field, Invent. Math. 77 (1984), 71–84.

[Seg]   D. Segal, Local and global equivalence of binary forms. I. Quartics, Quart. J. Math. Oxford 37 (1986), 483–493; II. Odd degree forms, *ibid.* 38 (1987), 103–125.

[Sel]   Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II, Geom. Funct. Anal. 7 (1997), 561–593.

[St]   R. Steinberg, Automorphisms of finite linear groups, Canad. J. Math. 12 (1960), 606–615.

[Su]   M. Suzuki, On finite groups with cyclic Sylow subgroups for all odd primes, Amer. J. Math. 77 (1955), 657–691.

[Sz]   F. Szechtman, n-inner automorphisms of finite groups, Proc. Amer. Math. Soc. 131 (2003), 3657–3664.

[TM]   D. Tieudjo, D. I. Moldavanskii, On the automorphisms of some one-relator groups, Comm. Algebra 34 (2006), 3975–3983.

[To1]   V. Tolstykh, The automorphism tower of a free group, J. London Math. Soc. 61 (2000), 423–440.

[To2]   V. Tolstykh, Infinitely generated free nilpotent groups: completeness of the automorphism groups, Math. Proc. Cambridge Phil. Soc. 147 (2009), 541–566.

[To3]   V. Tolstykh, Small conjugacy classes in the automorphism groups of relatively free groups, J. Pure Appl. Algebra 215 (2011), 2086–2098.

[VK]   V. E. Voskresenskiĭ, B. È. Kunyavskiĭ, Maximal tori in semisimple algebraic groups, manuscript deposited at VINITI 15.03.84, no. 1269-84, 28 pp. (Russian)

[Wad1]   H. Wada, "Hasse principle" for $\mathrm{SL}_n(D)$, Proc. Japan Acad. Ser. A Math. Sci. 75 (1999), 67–69.

[Wad2]   H. Wada, "Hasse principle" for $\mathrm{GL}_n(D)$, Proc. Japan Acad. Ser. A Math. Sci. 76 (2000), 44–46.

[WalC]   C. T. C. Wall, On the structure of finite groups with periodic cohomology, preprint, 2010, available at `www.liv.ac.uk/~ctcw/FGPCnew.pdf`.

[WalG]   G. E. Wall, Finite groups with class-preserving outer automorphisms, J. London Math. Soc. 22 (1947), 315–320.

[WaWu]   S. C. Wang, Y. Q. Wu, Covering invariants and co-Hopficity of 3-manifold groups, Proc. London Math. Soc. 68 (1994), 203–224.

[WZ]   S. C. Wang, Q. Zhou, Embeddings of Kleinian groups with torsion, Acta Math. Sin. (Engl. Ser.) 17 (2001), 21–34.

[We1]   B. Weisfeiler, Abstract isomorphisms of simple algebraic groups split by quadratic extensions, J. Algebra 68 (1981), 335–368.

[We2]   B. Weisfeiler, Abstract homomorphisms of big subgroups of algebraic groups, in: "Topics in the theory of algebraic groups", Notre Dame Math. Lectures, no. 10, Univ. Notre Dame Press, South Bend, Ind.-London, 1982., pp. 135–181.

[WoWo]   P. C. Wong, K. B. Wong, Residual finiteness of outer automorphism groups of certain tree products, J. Group Theory 10 (2007), 389–400.

[Ya1]   M. K. Yadav, On automorphisms of finite p-groups, J. Group Theory 10 (2007), 859–866.

[Ya2]   M. K. Yadav, Class preserving automorphisms of finite p-groups, J. Lond. Math. Soc. 75 (2007), 755–772.

[Ya3]   M. K. Yadav, On automorphisms of some finite p-groups, Proc. Indian Acad. Sci. (Math. Sci.) 118 (2008), 1–11.

[Ya4]   M. K. Yadav, Class preserving automorphisms of finite p-groups: A survey, in: "Groups St Andrews 2009 in Bath., vol. 2", London Math. Soc. Lecture Note Ser., vol. 388, Cambridge Univ. Press, Cambridge *et al.*, 2011, pp. 569–579.

Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, ISRAEL
E-mail address: kunyav@macs.biu.ac.il